1.2 Review of the scalar wave equation

~~1.3 Dispersion~~

~~Let's recall the equation to Maxwell's Equations for a plane wave;~~

Let's recall the vector HelmHoltz eqn. in vacuum

$$\nabla \times (\nabla \times \vec{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \quad (1)$$

We assumed an electric field $\vec{E}$ that was uniform in x and y and was always pointing in the $\hat{x}$ direction, and propagation was in the z-direction.

i.e. $\vec{E} = \hat{x} E_x(z,t)$ This satisfies Eq. (1) It gives:

$$\boxed{\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}} \quad (2) \Rightarrow \text{1-D wave equation}$$

This is a plane wave.

Today we will study the FDTD method on Eq (2) in order to learn a few fundamental things about FDTD

Solution

$$E_x(z,t) = K_1 f\left(t - \frac{z}{V_p}\right) + K_2 g\left(t + \frac{z}{V_p}\right) \quad (3)$$

where $V_p = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \cong 3E8 \left(\frac{m}{s}\right)$

1.3 Dispersion

Because $E_x(z,t)$ can be broken down spectrally by Fourier analysis, we can ask ourselves : what is the relationship between the wave vector and the frequency.

recall that

(4) $$\omega t - \beta z = \omega\left(t - \frac{z}{V_p}\right)$$

(5) so $$V_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$$

giving us $\omega = c\beta$

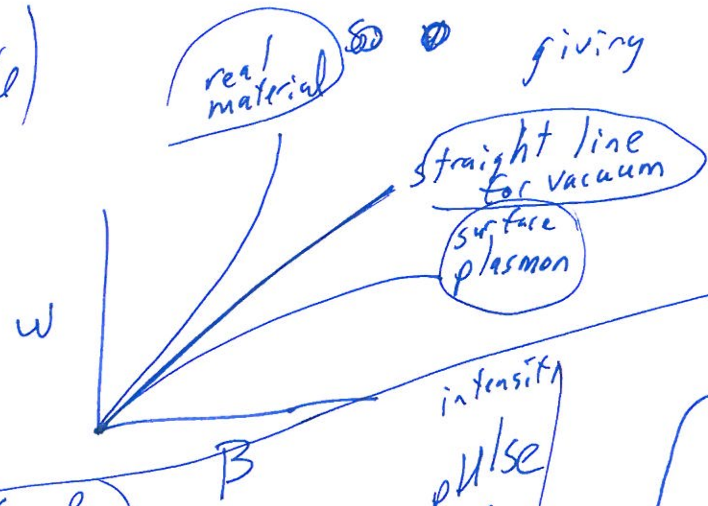
sometimes written as

$\omega = cK$

for a rightward going wave

$$\omega t + \beta z = \omega\left(t + \frac{z}{V_p}\right)$$ for leftward going wave

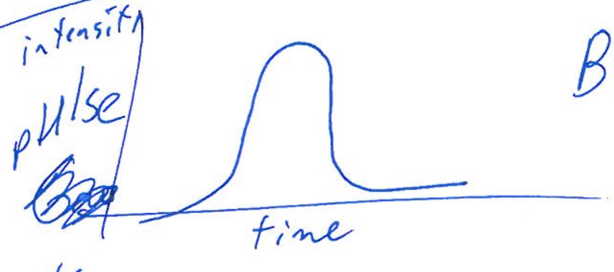
(6) giving $\underline{\omega = -cK}$

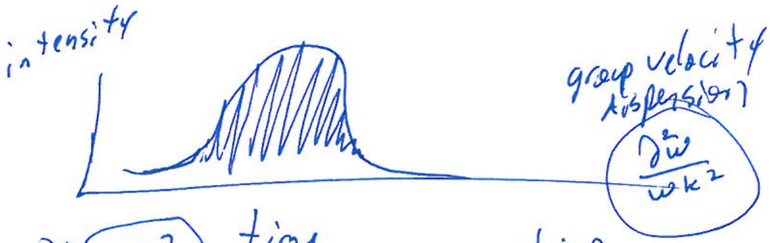


Baseband

1.3.2 (Group Velocity)

If we take a





group velocity

$$V_g = \frac{\partial \omega}{\partial K} = \underline{\underline{c \text{ in vacuum}}}$$

$\neq c$ in glass or other dispersive media

group velocity dispersion

$$\frac{\partial^2 \omega}{\partial k^2}$$

???

$\Longrightarrow$ discuss $\underline{\text{chirp}}$

dispersion of Gaussian wave packet.

Sellmeier

$$\epsilon(\omega) = 1 + \sum_{j=1}^{m} \frac{B_j \omega_j^2}{\omega_j^2 - \omega^2}$$

2 Finite differences: a numerical approximation of a derivative

- Discretize our electric field in space (z only) and time.

Write Taylor Series expansion with respect to space

$$E_x(z_i + \Delta z) = E_x(z, t_n) + \Delta z \frac{\partial E}{\partial z}\bigg|_{z_i, t_n} + \frac{(\Delta z)^2}{2} \frac{\partial^2 E}{\partial z^2}\bigg|_{z_i, t_n} + \frac{(\Delta z)^3}{6} \cdot \frac{\partial^3 E}{\partial z^3}\bigg|_{z_i, t_n} + \underbrace{\frac{(\Delta z)^4}{24} \cdot \frac{\partial^4 E}{\partial z^4}\bigg|_{\xi_1, t_n}}_{\text{error term}} \quad (7)$$

ξ_1: there is such a place to make equality.

let's do same thing to have a formula for $E_x(z_i - \Delta z)$

$$= E_x(z, t_n) - \quad + \quad - \quad + \quad \_\_\_\Big|_{\xi_2} \quad (8)$$

Adding (7) and (8) we get

$$E(z_i + \Delta z)\Big|_{t_n} + E(z_i - \Delta z)\Big|_{t_n} = 2E_{z_i, t_n} + (\Delta z)^2 \cdot \frac{\partial^2 E}{\partial z^2}\bigg|_{z_i, t_n} + \frac{(\Delta z)^4}{12} \frac{\partial^4 E}{\partial x^4}\bigg|_{\xi_3, t_n}$$

Rearranging terms, we get

$$\left.\frac{\partial^2 E_x}{\partial z^2}\right|_{z_i,\, t_n} = \left[\frac{E_x(z_i + \Delta z) - 2E(z_i) + E(z - \Delta z)}{(\Delta z)^2}\right]_{t_n} + \mathcal{O}\left[(\Delta z)^2\right]$$

- error goes as $(\Delta z)^2$

"second-order accurate, central-difference approximation to the second partial space derivative of E"

Let's now use i as subscript for space position
n for time observation.
u for either E or H.
let's also have prop. in x for compat. with book.

$$\cancel{\frac{\partial^2 u}{\partial z^2}} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + \mathcal{O}\left((\Delta x)^2\right)$$

$$x_i = i\Delta x$$
$$t_n = n\Delta t$$

- We can repeat for time derivative:

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} + \mathcal{O}\left((\Delta t)^2\right)$$

- Can you do this for fourth order?
- what does accuracy mean for very short wavelengths?
  very long wavelengths?

3 Finite differences for solving 1D PDEs

we have two approximations so far: one for $\frac{\partial^2 u}{\partial x^2}$ one for $\frac{\partial^2 u}{\partial t^2}$

let's plug them into the wave equation to see if we can come up with a simulation ~~technique~~ method!

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} + \mathcal{O}((\Delta t)^2) = c^2 \left\{ \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + \mathcal{O}(\Delta x^2) \right\}$$

let's see if we can use the present and past values of the field to predict the future:

$$\boxed{(9) \quad u_i^{n+1} = \frac{(c\,\Delta t)^2}{(\Delta x)^2} \left[u_{i+1}^n - 2u_i^n + u_{i-1}^n \right] - 2u_i^n - u_i^{n-1}}$$

3 Finite Differences (Magic Time step)

if $\frac{c(\Delta t)}{(\Delta x)} = 1$

- exact solution of wave equation b/c
- Taylor Series expansion is the wave eq.
- $U_i^{n+1} = U_i^n + U_{i+1}^n - U_i^{n-1}$

  ↓

  no approx

- only works in 1d

$\tilde{k} = \tilde{k}_{real} + j\tilde{k}_{imag}$.

4 Numerical Dispersion in 1D

- put in a sinusoidal wave only

~~look at~~

$\cdot\ U_i^n = e^{j(\omega n(\Delta t) - \tilde{k} i \Delta x)}$

put this in (9)

$$(10)\quad e^{j[\omega(n+1)\Delta t - \tilde{k} i \Delta x]} = \left[\frac{c(\Delta t)}{\Delta x}\right]^2 \left\{ e^{j[\omega n\Delta t - \tilde{k}(i+1)\Delta x]} - 2e^{j[\omega n(\Delta t) - \tilde{k} i \Delta x]} + e^{j[\omega n\Delta t - \tilde{k}(i-1)\Delta x]} \right\} + 2e^{j[\omega n\Delta t - \tilde{k} i \Delta x]} - e^{j[\omega(n-1)\Delta t - \tilde{k} i \Delta x]}$$

divide by $e^{j(\omega n\Delta t - \tilde{k} i \Delta x)}$

$$e^{j\omega\Delta t} = \left[\frac{c(\Delta t)}{\Delta x}\right]^2 \cdot \left(e^{-j\tilde{k}\Delta x} - 2 + e^{j\tilde{k}\Delta x}\right) + 2 - e^{-j\omega\Delta t}$$

$\Downarrow$

$$\frac{e^{j\omega\Delta t} + e^{-j\omega\Delta t}}{2} = \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(\frac{e^{j\tilde{k}\Delta x} + e^{-j\tilde{k}\Delta x}}{2} - 1\right) + 1$$

Eq 10a $$\boxed{\cos(\omega \Delta t) = \left(\frac{c(\Delta t)}{(\Delta x)}\right)^2 \left[\cos(\tilde{k}\Delta x) - 1\right] + 1}$$

Eq (10) b $$\Rightarrow \boxed{\tilde{k} = \frac{1}{\Delta x} \cos^{-1}\left\{1 + \left(\frac{\Delta x}{c\Delta t}\right)^2 \left[\cos(\omega \Delta t) - 1\right]\right\}}$$

not $k = \frac{\omega}{c}$

a) in limit of $\omega \Delta t \to 0$ i.e. very fine sampling

$k\Delta x \to 0$

$$\tilde{k} = \frac{1}{\Delta x} \cos^{-1}\left[1 - \frac{1}{2}\left[k(\Delta x)\right]^2\right]$$

expansion of $\cos^{-1}$ to two terms gives

$$\tilde{k} = \frac{1}{\Delta x}(k\Delta x) = k \quad \text{ie (no dispersion!)}$$

main idea: very ~~small so~~ fine sampling reduces numerical dispersion

b) $\frac{c\Delta x}{\Delta t} = 1$ $$\tilde{k} = \frac{1}{c\Delta t}\cos^{-1}\left[1 + \cos(\omega\Delta t) - 1\right]$$

$$\tilde{k} = \frac{1}{c\Delta t}\cos^{-1}\left[\cos(\omega\Delta t)\right] = \frac{\omega\Delta t}{c\Delta t} = k$$

c) in between: $c(\Delta t) = \frac{\Delta x}{2}$ and $\Delta x = \lambda_0/10$ λ_0 freespace....

$\Downarrow$ plugging in, we get

$$\tilde{k} = \frac{1}{\Delta x}\cos^{-1}(0.8042) = \frac{0.63642}{\Delta x}$$

$\frac{1}{2}$ space step $\Rightarrow \frac{1}{4}$ phase error

$$\Rightarrow V_p = \frac{\omega}{\tilde{k}} = \tilde{V}_p = \frac{2\pi f}{(0.63642/\Delta x)} = \frac{2\pi(c/\lambda_0)\Delta x}{0.63642} = \frac{2\pi \times (.1)}{.63642}c = 0.987c$$

(8)

slower than speed of light!

We can get a general solution:

$$S = c\frac{\Delta t}{\Delta x}$$

$$N_\lambda = \frac{\lambda_0}{\Delta x}$$

$$\tilde{K} = \frac{1}{\Delta x}\cos^{-1}\left\{1 + \left(\frac{1}{S}\right)^2\left[\cos\left(\frac{2\pi S}{N_\lambda}\right) - 1\right]\right\}$$

~~example~~ show graph

$$\boxed{\begin{array}{l} S = 0.5 \\ 1 \le N_\lambda \le 10 \end{array}}$$

then

Draw Dispersed square pulses

Simulation Demo

1) Dispersion

   1) look at wave — $dt = \frac{dx}{c}$ "magic time step"

      sinusoid

~~[illegible]~~

- comment on un-physical source.
- comment on radiation boundary condition

Play with:

5) Numerical Stability

- An explicit numerical solution is stable if it produces a bounded result given a bounded input. The numerical solution is unstable if it produces an unbounded result given a bounded input.

von Neumann analysed stability for numerical sol. PDEs.

idea: Fourier domain ⇒ fourier series describes the error

if each term of fourier series representing error has a unity-or-less growth factor over a time step growth of error remains bounded.

we will do simpler approach: look at numerical estimate of ω (angular frequency)

$$\tilde{\omega} = \tilde{\omega}_{real} + j\tilde{\omega}_{imag}$$

$$U_i^n = e^{j(\tilde{\omega} n \Delta t - \tilde{k} i \Delta x)}$$

$$= e^{-\tilde{\omega}_{imag} n \Delta t} e^{j(\tilde{\omega}_{real} n \Delta t - \tilde{k} i \Delta x)}$$

$\tilde{k}$ estimated wavevector

leads to exp growth, exp decay, or $\tilde{\omega}_{imag} = 0$ (neither)

(2.22 from last time)

$$\cos(\tilde{\omega}\Delta t) = \left(\frac{c\Delta t}{\Delta x}\right)^2 \left[\cos(\tilde{k}\Delta x) - 1\right] + 1$$

$$\tilde{\omega} = \frac{1}{\Delta t}\cos^{-1}\left\{S^2\left[\cos(\tilde{k}\Delta x) - 1\right] + 1\right\}$$

$$= \frac{1}{\Delta t}\cos^{-1}(\xi) = \frac{1}{\Delta t}\left[\frac{\pi}{2} - \sin^{-1}(\xi)\right]$$

$\sin^{-1}(\xi) = -j\ln\left[j\xi + \sqrt{1-\xi^2}\right]$

where

$$\xi = S^2\left[\cos(\tilde{k}\Delta x) - 1\right] + 1$$

because $\max(\cos) = 1$, $\min(\cos) = -1$

$\xi \le 1$

$\xi \ge 1 - 2S^2$

$$\boxed{1 - 2S^2 \le \xi \le 1}$$

If $-1 \le \xi \le 1$ then $0 \le S \le 1$,

then $\cos^{-1}(\xi)$ is real-valued

then $\tilde{w}$ is real valued

then $\tilde{w}_{imag} = 0$

then U_i^n is purely oscillatory

so output is bounded

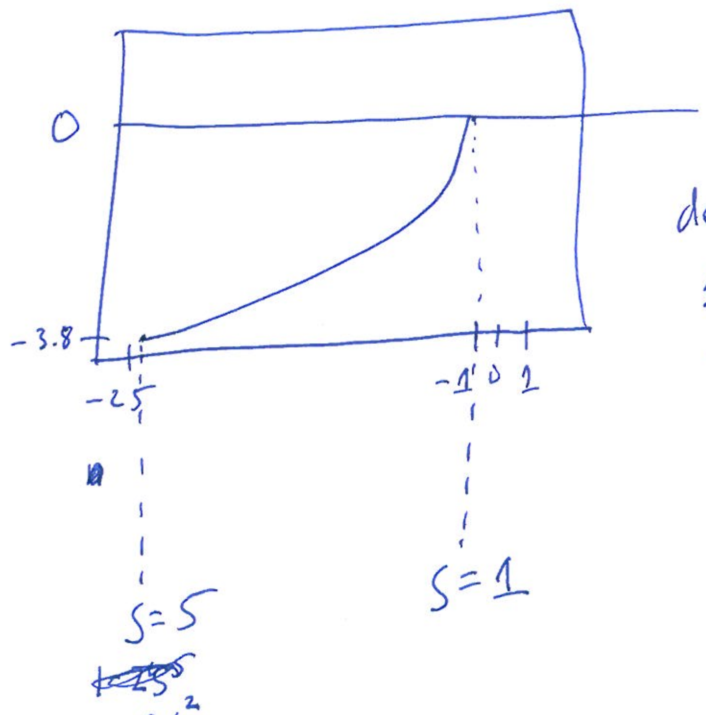
so the numerical simulation is stable

recall $S = \frac{c \Delta t}{\Delta x}$

so when $0 \le S \le 1$, $\boxed{\Delta t < \Delta x}$

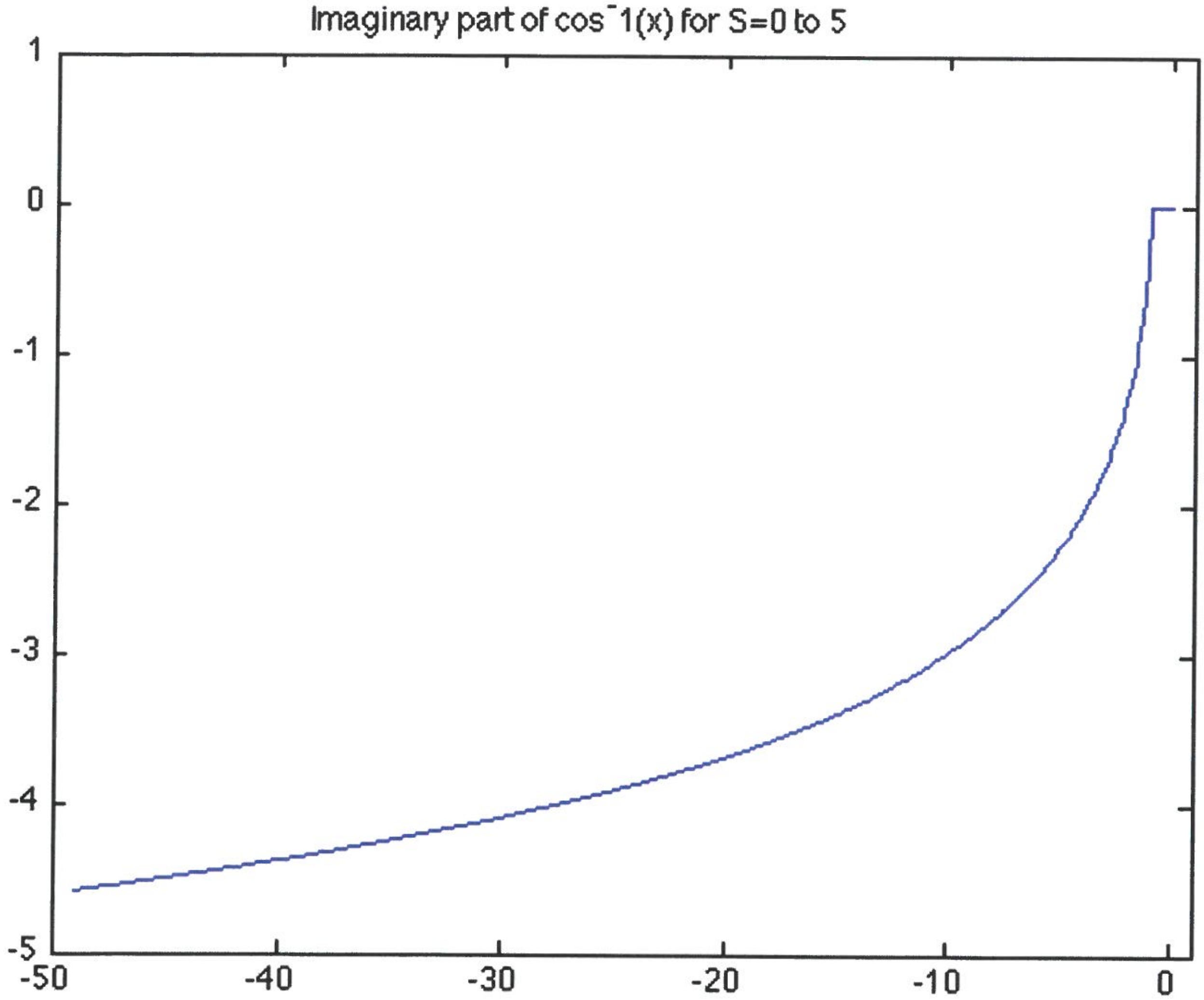
If $1 - 2S^2 \le \xi < -1$

then $\cos^{-1}(\xi)$ is complex-valued and its imaginary part is negative with increasing S



describe from printed

(12)


Imaginary part of cos¯1(x) for S=0 to 5
1
0
-1
-2
-3
-4
-5
-50
-40
-30
-20
-10
0

$$u_i^n = e^{\boxed{\text{real part}}\; -n \ln\left[-\xi - \sqrt{\xi^2-1}\right]} \; e^{j\left[\text{imag part}\right]\left[(\pi/\Delta t)(n\Delta t) - \tilde{k} i \Delta x\right]}$$

$$\Downarrow$$

$$\left(\frac{1}{-\xi - \sqrt{\xi^2-1}}\right)^n$$

now

$$= \frac{-\xi + \sqrt{\xi^2-1}}{(-\xi - \sqrt{\xi^2-1})(-\xi + \sqrt{\xi^2-1})} = \frac{-\xi + \sqrt{\xi^2-1}}{\xi^2 - (\xi^2-1)} = \frac{-\xi + \sqrt{\xi^2-1}}{}$$

$$g \text{ growth factor} = \left(\frac{1}{-\xi - \sqrt{\xi^2-1}}\right) = -\xi + \sqrt{\xi^2-1}$$

now for this case $\xi < -1$

so growth factor > 1

so

$$g_{grow} = -(1-2S^2) + \sqrt{(1-2S^2)^2 - 1}$$

$$\cancel{S + (S^2 - 1)}$$

$$= \left(S + \sqrt{S^2-1}\right)^2$$

$S = 1$ gives $g_{grow} = 1$, so it doesn't explode

$S > 1$ gives exponential growth in n

which frequencies explode the fastest?

ξ is smallest (most negative value)

when

$$\cos(\tilde{k}\Delta x) = -1$$

i.e. $\tilde{k}\Delta x = \pi$

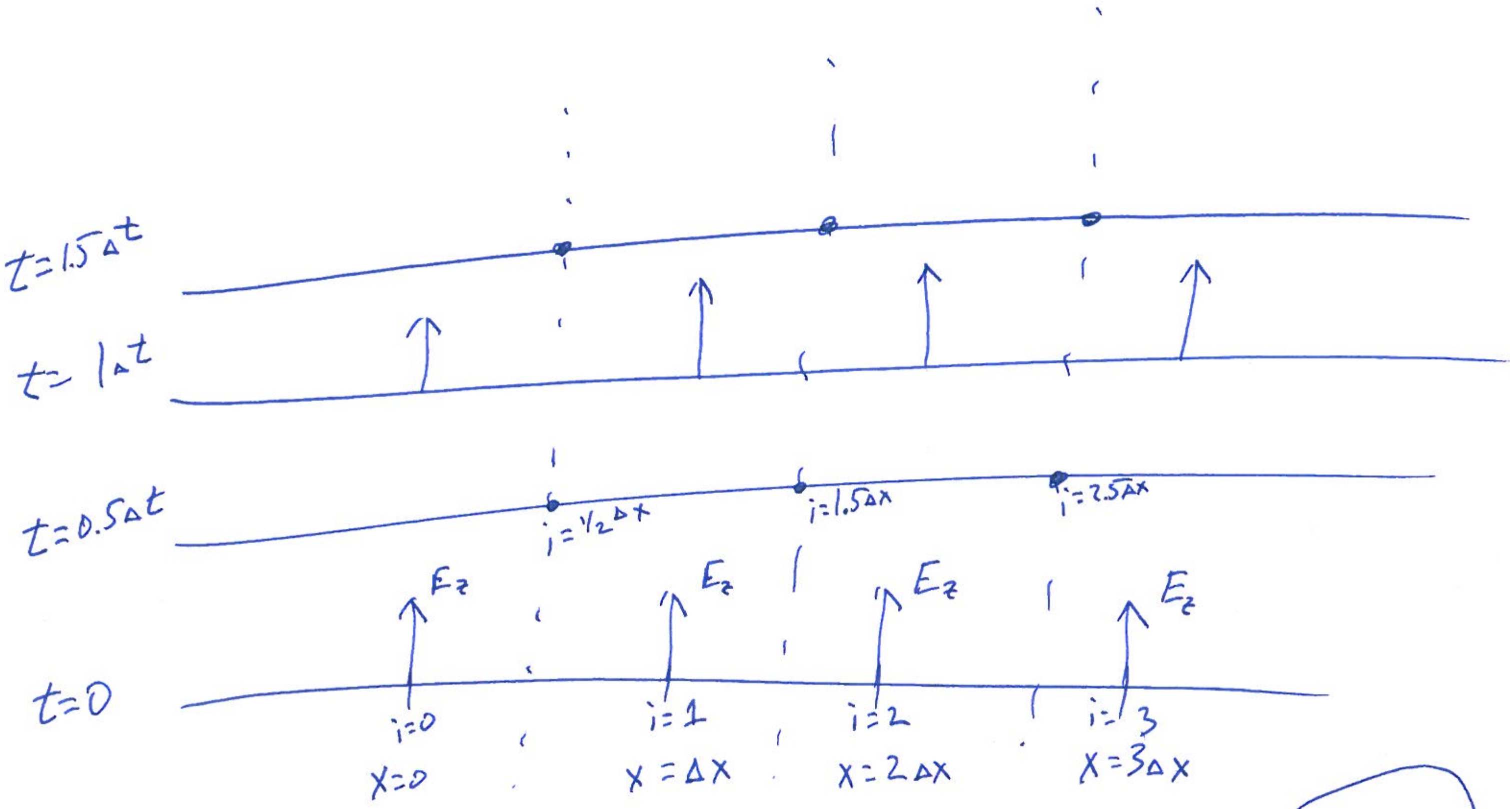
or $\tilde{\lambda} = 2\Delta x$ i.e. Nyquist freq. for grid.

i.e. highest possible freq.

In 1-D

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_z}{\partial x} - \rho' H_y\right)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon}\left(\frac{\partial H_y}{\partial x} - \sigma E_z\right)$$



loop

$$H^{n+1/2}_{i+1/2} = H^{n-1/2}_{i+1/2} + \frac{1}{\mu}\frac{\Delta t}{\Delta x}\left(E^n_{i+1} - E^n_i\right)$$

$$E^{n+1}_i = E^n_i + \frac{1}{\epsilon}\frac{\Delta t}{\Delta x}\left(H^{n+1/2}_{i+1/2} - H^{n+1/2}_{i-\frac{1}{2}}\right)$$

- leapfrog
-

if no magnetic source + no charges,

$$C_a\Big|_{i,j,k} = \frac{\left(1 - \frac{\sigma_{i,j,k}\Delta t}{2\epsilon_{i,j,k}}\right)}{\left(1 + \frac{\sigma_{i,j,k}\Delta t}{2\epsilon_{i,j,k}}\right)}$$

$$C_b\Big|_{i,j,k} = \frac{\left(\frac{\Delta t}{\epsilon_{i,j,k}\Delta}\right)}{\left(1 + \frac{\sigma_{i,j,k}\Delta t}{2\epsilon_{i,j,k}}\right)}$$

$$\cancel{C_{b_2}\Big|_{i,j,k} = \frac{\left(\frac{\Delta t}{\epsilon_{i,j,k}\Delta_2}\right)}{\left(1 + \frac{\sigma_{i,j,k}\Delta t}{2\epsilon_{i,j,k}}\right)}}$$

$$D_a\Big|_{i,j,k} = \cancel{\frac{1}{1+}} \quad 1$$

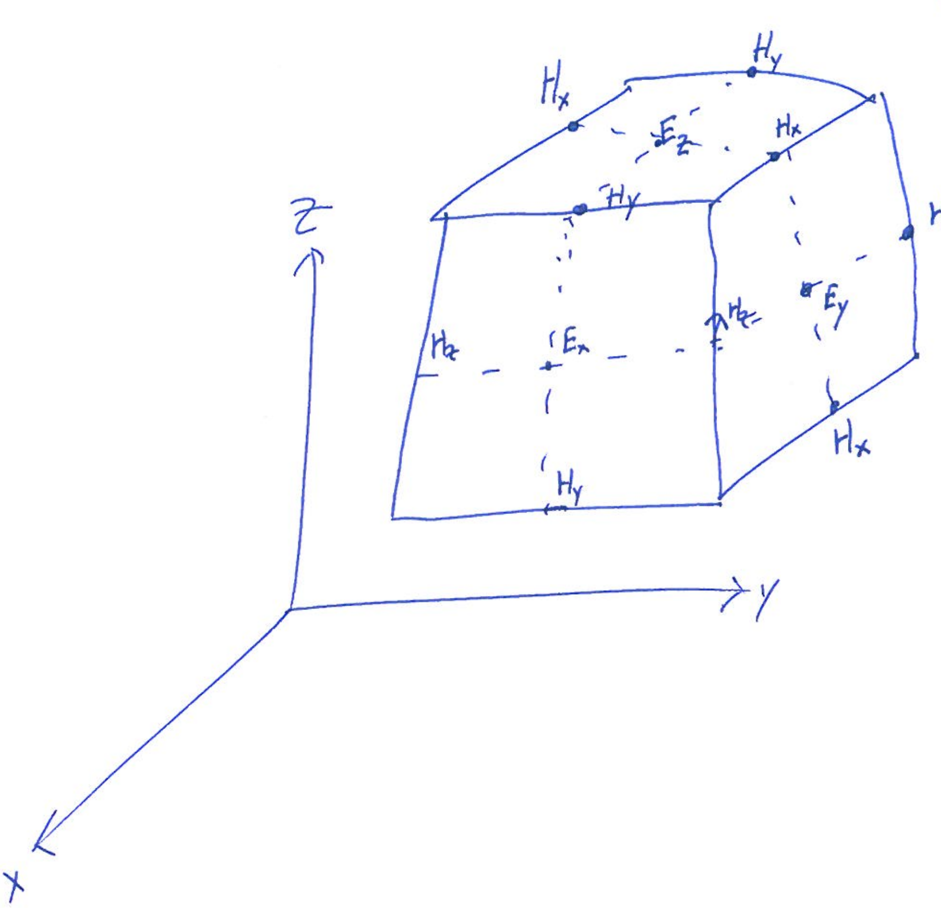
$$D_b\Big|_{i,j,k} = \frac{\Delta t}{\mu_{i,j,k}\Delta}$$

$\boxed{\text{is}}$

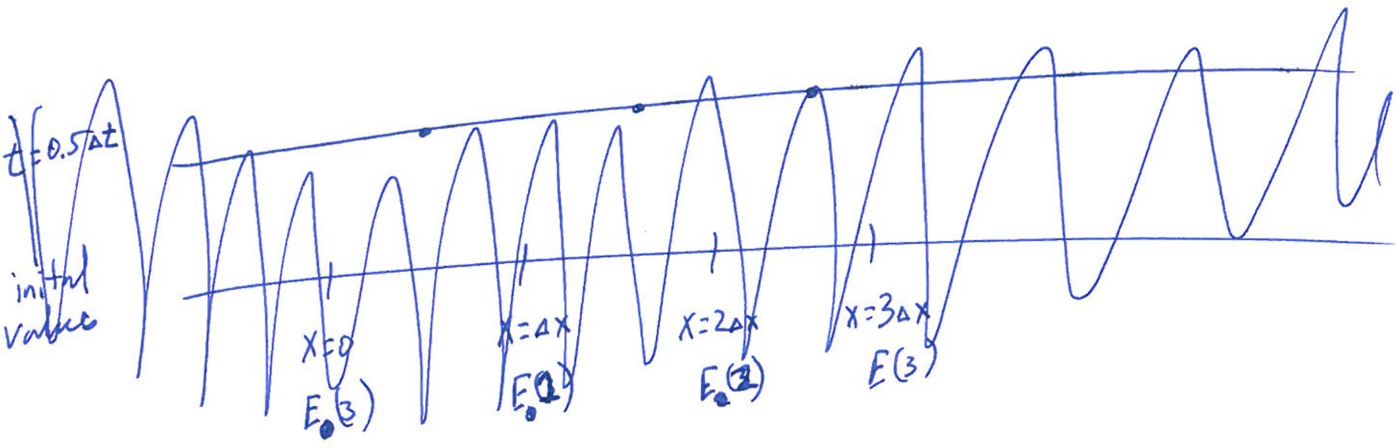
1) Divergence free

2) can make alt. grids

Yee Grid



do last



$(i,j,k) = (i\Delta x, j\Delta y, k\Delta z)$

$$\frac{\partial u}{\partial t}(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = \frac{U^{n+1/2}_{i,j,k} - U^{n-1/2}_{i,j,k}}{\Delta t} + O(\Delta t)^2$$

same for space
$$\frac{\partial u}{\partial x}(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = \left(\frac{U^{n}_{i+1/2,j,k} - U^{n}_{i-1/2,j,k}}{\Delta x}\right) + O[\Delta t]^2$$

LA2205wg

#1

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) = J_x + \frac{\partial D_x}{\partial t} \quad (4)$$
$$\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) = J_y + \frac{\partial D_y}{\partial t} \quad (5)$$
$$\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = J_z + \frac{\partial D_z}{\partial t} \quad (6)$$
$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = -\frac{\partial B_x}{\partial t} \quad (1)$$
$$\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) = -\frac{\partial B_y}{\partial t} \quad (2)$$
$$\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -\frac{\partial B_z}{\partial t} \quad (3)$$
$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$
$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho_e$$

$\frac{\partial}{\partial z} = 0$

$\vec{B} = \mu \vec{H}$
$\vec{D} = \epsilon \vec{E}$

#2

Maxwell's equations in the cylindrical coordinate can be expressed as:

$$\left(\frac{\partial H_z}{r\partial \theta} - \frac{\partial H_\theta}{\partial z}\right) = J_r + \frac{\partial D_r}{\partial t}$$
$$\left(\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right) = J_\theta + \frac{\partial D_\theta}{\partial t}$$
$$\left(\frac{\partial H_\theta}{\partial r} - \frac{\partial H_r}{r\partial \theta}\right) = J_z + \frac{\partial D_z}{\partial t}$$
$$\left(\frac{\partial E_z}{r\partial \theta} - \frac{\partial E_\theta}{\partial z}\right) = -\frac{\partial B_r}{\partial t}$$
$$\left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right) = -\frac{\partial B_\theta}{\partial t}$$
$$\left(\frac{\partial E_\theta}{\partial r} - \frac{\partial E_r}{r\partial \theta}\right) = -\frac{\partial B_z}{\partial t}$$
$$\frac{\partial}{r\partial r}(rH_r) + \frac{\partial H_\theta}{r\partial \theta} + \frac{\partial H_z}{\partial z} = 0$$
$$\frac{\partial}{r\partial r}(rE_r) + \frac{\partial E_\theta}{r\partial \theta} + \frac{\partial E_z}{\partial z} = \rho_e$$

Maxwell's equations in the spherical coordinate are:

ECE3025, Summer 2011, Class Project Help

Paul Voss

1 Advice

Advice for Project 2: The nice thing about this project is that it is two dimensional, not three dimensional. This means that we will either have perpendicular or parallel polarization and the equations will simplify quite a bit. In order to derive the following answer, you need to take the following steps:

1. We want to have the fields not vary as z changes, but have them do vary as x and y changes. If we write out Maxwell's Equations in Cartesian coordinates, but set all z derivatives equal to 0, we get

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu}\left(-\frac{\partial E_z}{\partial y}\right) \quad (1)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_z}{\partial x}\right) \quad (2)$$

other set (1), (2)

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right) \quad (3)$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon}\left(\frac{\partial H_z}{\partial y} - \sigma E_x\right) \quad (4)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\epsilon}\left(-\frac{\partial H_z}{\partial x} - \sigma E_y\right) \quad (5)$$

one set (3), (4), (5)

$$\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma E_z\right) \quad (6)$$

other set (6)

2. Please notice that we have included loss (σ) here, but in our simulations, $\sigma = 0$, so we have only dielectrics in our region of interest. We may as well set $\sigma = 0$ from now on. Please notice also that Equations 1, 2, and 6 are totally independent of Equations 3, 4, and 5. This means that for 2-D waves, the Perpendicular polarization (Equations 1, 2, and 6) and the Parallel polarization (Equations 3, 4, and 5) act totally independently. This means that in 2-D, perpendicularly polarized waves stay perpendicularly polarized and parallel polarization stays parallel. From a communications standpoint, you could code independent messages on the two polarizations and get double the data rate if the receiver could receive both polarizations separately. What we will do is to separate the two, so we will have two separate algorithms, one for perp. polarization (this is called TE or transverse electric field) and one for parallel polarization (this is called TM for transverse magnetic field).

3. We are next going to create an interlocking mesh of data points, the points where we will approximate the derivatives in Equations 1, 2, and 6. This gives TM updating algorithm. (You still need to derive this algorithm). If you are ok with it, you do not need to send TE waves through your imaginary space, only TM waves. This will simplify the project. The update algorithm you get is:

$$m = MEDIA_{H_x}|_{i,j} \quad (7)$$

$$H_x|_{i,j}^{n+1/2} = D_a(m)H_x|_{i,j}^{n-1/2} + D_b(m)\left(E_z|_{i,j-1/2}^{n} - E_z|_{i,j+1/2}^{n}\right) \quad (8)$$

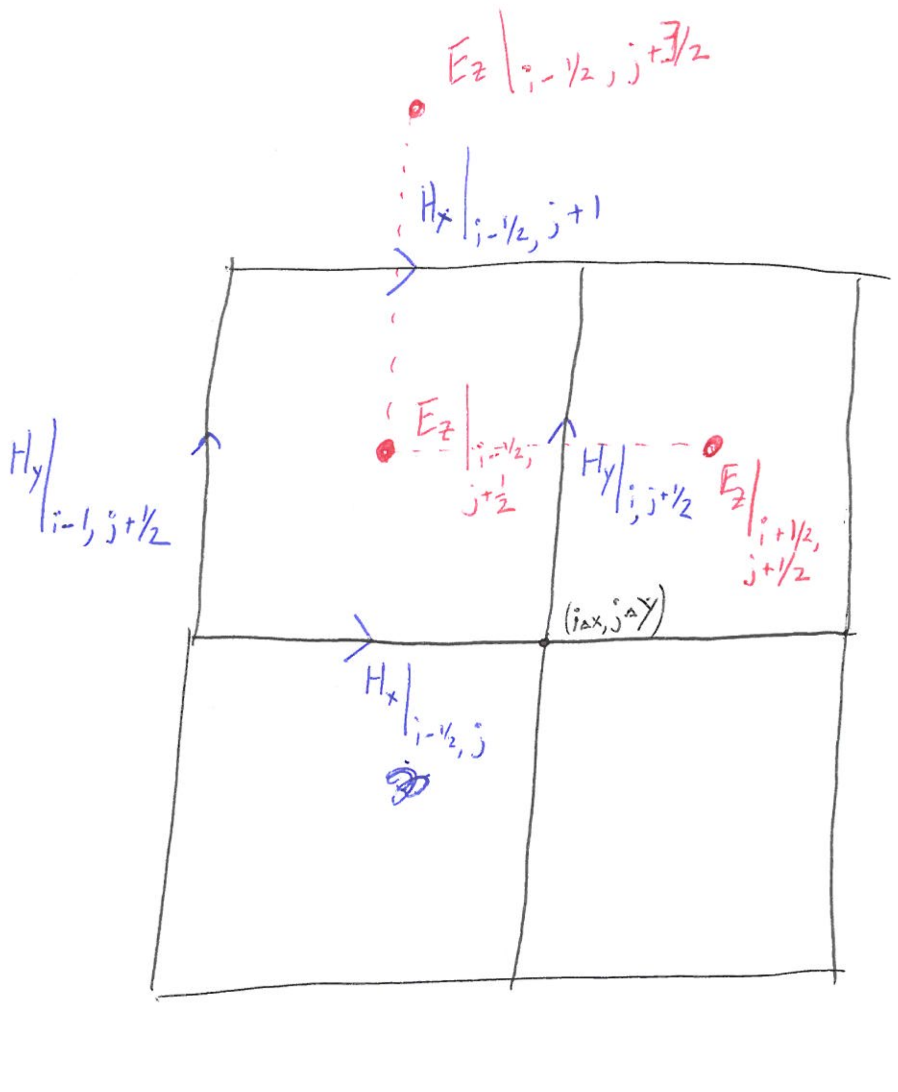
$$m = MEDIA_{H_y}|_{i,j} \quad (9)$$

$$H_y|_{i,j}^{n+1/2} = D_a(m)H_y|_{i,j}^{n-1/2} + D_b(m)\left(E_z|_{i+1/2,j}^{n} - E_z|_{i-1/2,j}^{n}\right) \quad (10)$$

$$m = MEDIA_{E_z}|_{i,j} \quad (11)$$

$$E_z|_{i,j}^{n+1} = C_a(m)E_z|_{i,j}^{n} + C_b(m)\left(H_y|_{i+1/2,j}^{n+1/2} - H_y|_{i-1/2,j}^{n+1/2} + H_x|_{i,j-1/2}^{n+1/2} - H_x|_{i,j+1/2}^{n+1/2}\right) \quad (12)$$

TM_z grid


$E_z|_{i-1/2,\, j+3/2}$
$H_x|_{i-1/2,\, j+1}$
$E_z|_{i-1/2,\, j+1/2}$
$H_y|_{i-1,\, j+1/2}$
$H_y|_{i,\, j+1/2}$
$E_z|_{i+1/2,\, j+1/2}$
$(i\Delta x, j\Delta y)$
$H_x|_{i-1/2,\, j}$

Yee Scheme for a 2D system -TM_z

2nd

1st

$$m = MEDIA_{H_x}|_{\cancel{i,j}\; i-1/2,j}$$

$$H_x|^{n+1\cancel{/2}}_{\cancel{i,j}\; i-1/2,j+1} = D_a(m)H_x|^{n\cancel{-1/2}}_{\cancel{i,j}\;(i-1/2,j+1)} + D_b(m)\left(E_z|^{n+1/2}_{\cancel{i,j-1/2}\; i-1/2,j+1/2} - E_z|^{n+1/2}_{\cancel{i,j+1/2}\; i-1/2,j+3/2}\right) - M_{source}\Big|^{n+1/2}_{i-1/2,j+1}\Delta$$

$$m = MEDIA_{H_y}|_{i,\cancel{j}\; j+1/2}$$

$$H_y|^{n+1\cancel{/2}}_{\cancel{i,j}\; i,j+1/2} = D_a(m)H_y|^{n\cancel{-1/2}}_{\cancel{i,j}\; i,j+1/2} + D_b(m)\left(E_z|^{n+1/2}_{\cancel{i+1/2,j}\; i+1/2,j+1/2} - E_z|^{n+1/2}_{\cancel{i-1/2,j}\; i-1/2,j+1/2}\right) - M_{source}\Big|^{n+1/2}_{i,j+1/2}\Delta$$

$$m = MEDIA_{E_z}|_{\cancel{i,j}\; i-1/2,j+\frac{1}{2}}$$

$$E_z|^{n+1/2}_{\cancel{i,j}\; i-1/2,j+\frac{1}{2}} = C_a(m)E_z|^{n-1/2}_{\cancel{i,j}\; i-1/2,j+\frac{1}{2}} + C_b(m)\left(H_y|^{n\cancel{+1/2}}_{\cancel{i+1/2,j}\; i,j+1/2} - H_y|^{n\cancel{+1/2}}_{\cancel{i-1/2,j}\; i-1,j+1/2} + H_x|^{n\cancel{+1/2}}_{\cancel{i,j-1/2}\; i-1/2,j} - H_x|^{n\cancel{+1/2}}_{\cancel{i,j+1/2}\; i-1/2,j+1}\right) - J_{source_z}\Big|^{n}_{i-1/2,j+\frac{1}{2}}\Delta$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon}\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \left(J_{source_x} + \sigma E_x\right)\right]$$

if we do our normal FDTD scheme,
we get

E_x^{n} ~~on right side~~ on right side!

So we estimate

$$E_x^n \cong \frac{E_x^{n+1/2} + E_x^{n-1/2}}{2}$$

$\Downarrow$

plug into Yee cell
gives

$$\underline{\left(1 + \frac{\sigma_{i,j+1/2,k+1/2}\,\Delta t}{2\epsilon_{i,j+1/2,k+1/2}}\right)} E^{n+1/2}_{i,j+1/2,k+1/2} = \left(1 - \frac{\sigma_{i,j+1/2,k+1/2}\,\Delta t}{2\epsilon_{i,j+1/2,k+1/2}}\right) E_x\Big|^{n-1/2}_{i,j+1/2,k+1/2}$$

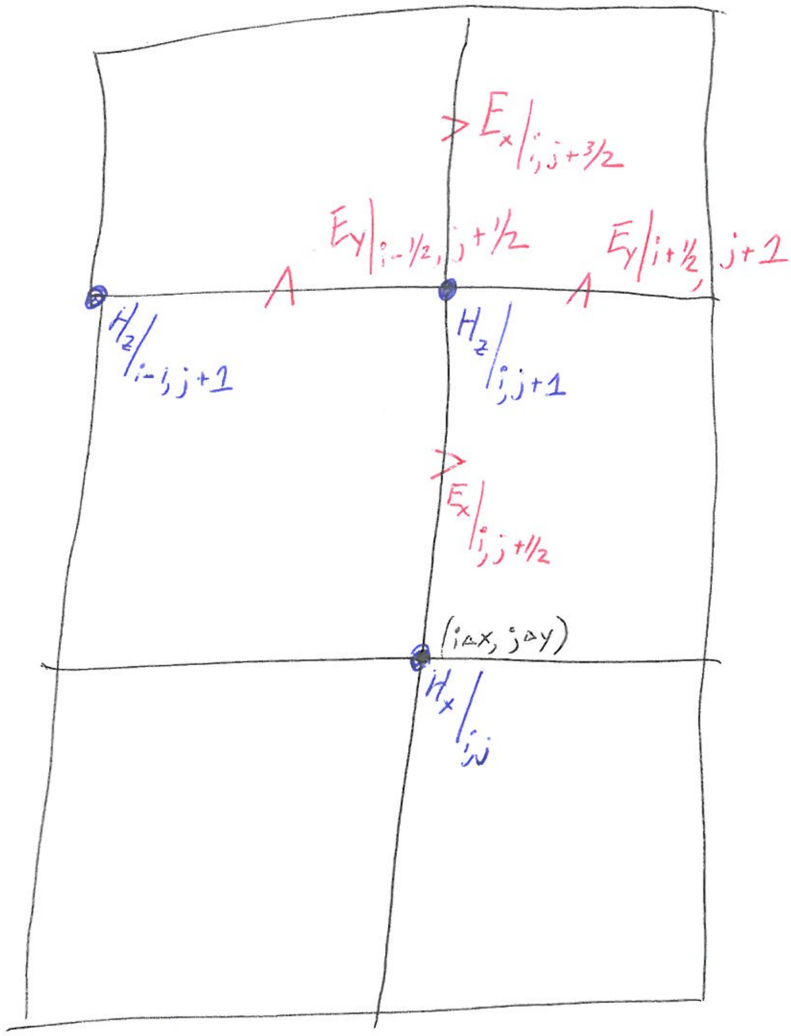
$$+ \frac{\Delta t}{\epsilon_{i,j+1/2,k+1/2}} \cdot \left[\frac{H_z\Big|^n_{i,j+1,k+1/2} - H_z\Big|^n_{i,j,k+1/2}}{\Delta y} - \frac{\left(H_y\Big|^n_{i,j+1/2,k+1} - H_y\Big|^n_{i,j+1/2,k}\right)}{} \right.$$

divide by

$$\left. - J_{source}\Big|^n_{i,j+1/2,k+1/2}\right]$$

gives formula for how to do time step

TE_z grid



$E_x|_{i,j+3/2}$
$E_y|_{i-1/2,j+1/2}$
$E_y|_{i+1/2,j+1}$
$H_z|_{i-1,j+1}$
$H_z|_{i,j+1}$
$E_x|_{i,j+1/2}$
$(i\Delta x, j\Delta y)$
$H_x|_{i,j}$