MDS 504 Mathematics for Data Science

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1	Introduction, Motivation, and Overview	9 hrs.
2	Introduction to Matrices and Vectors	15 hrs.
3	Spetral Theorems	14 hrs.
4.	System of Linear Equations	10 hrs.

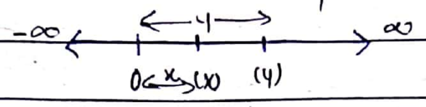
Book 2: Matrix algebra: Shayle R. & Andre.

Arpan Sapkota
Roll No - 07

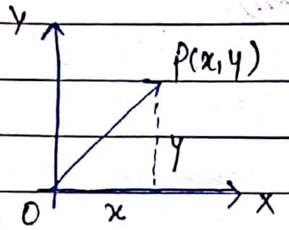
2 Introduction to Matrices and Vector.

Euclidean Space R^n

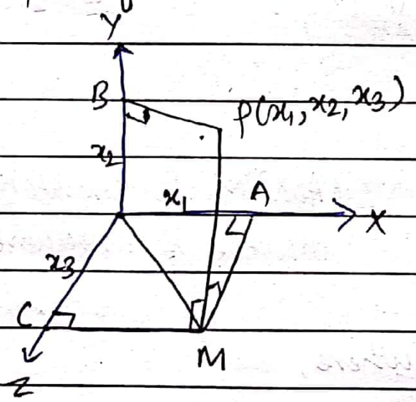
R^1 = Real line = R^1 - Space (x)



R^2 = Set of all ordered pair = plane i.e. R^2 - Space (x_1, x_2)



R^3 = Space of all order triple $= (x_1, x_2, x_3)$



Euclidean space R^n

Let n be a natural number. The set of all ordered n tuples $(x_1, x_2, x_3, \overset{x_k}{\dots}, x_n)$ is called n-dimensional vector space with n-components. & is denoted by R^n.

Here, x_k is called kth components.

Basic Properties:

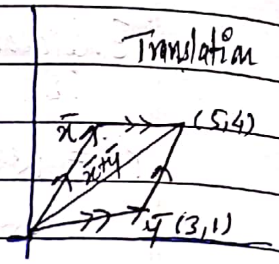
① If $\bar{x} = (x_1, x_2, \dots x_n)$, $\bar{y} = (y_1, y_2, \dots; y_n) \in R^n$ then,
$\bar{x} = \bar{y}$ if $x_1 = y_1$, $x_2 = y_2$, ..., $x_n = y_n$

$\exists \Rightarrow$ There exist $\quad$ iff $\Rightarrow$ if and only if

$\forall \Rightarrow$ for all

(ii) Addition.

$\bar{x}+\bar{y} = (x_1+y_1, x_2+y_2, \dots, x_n+y_n)$



In particular,

$\bar{x} = (2,3)$, $\bar{y} = (3,1)$

$\therefore \bar{x}+\bar{y} = (5,4)$

(iii) Multiplication by Scalar

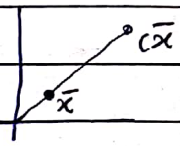
If c be a scalar & $\bar{x} = (x_1, x_2, \dots x_n)$ then,

$c\bar{x} = (cx_1, cx_2, \dots, cx_n)$.

In particular,

If, $\bar{x} = (1,2)$, $c = 2$

$c\bar{x} = (2,4)$



↗ extension
$c>0$

$c<0$
↳ shrink

Basic Algebraic Properties of Vectors in R^n

Addition Properties.

If $\bar{x}, \bar{y}, \bar{z} \in R^n$

A_1: $\bar{x}+\bar{y} = \bar{y}+\bar{x}$ (Addition is commutative)

A_2: $\bar{x}+(\bar{y}+\bar{z}) = (\bar{x}+\bar{y})+\bar{z}$ (Addition is associative)

A_3: $\exists -x \in R^n$ such that

$(-\bar{x})+\bar{x} = \bar{0}$, where,

$\bar{0} = \underbrace{(0,0,\dots,0)}_{n\text{-zero's}}$

A_4: $\bar{x}+\bar{0} = \bar{x} \quad \forall \bar{x} \in R^n$

Multiplication Property.

(i). $c(\bar{x}+\bar{y}) = c\bar{x}+c\bar{y}$ (Distributive law of vector)

(ii). $(c+d)\bar{x} = c\bar{x}+d\bar{x}$ (Distributive law of scalar)

(iii). $\exists\ 1 \in R$ such that

$1\bar{x} = \bar{x}$

Scalar product or Dot product

If $\bar{x} = (x_1, x_2, \ldots x_n)_{1\times n}$ & $\bar{y} = (y_1, y_2, \ldots y_n)_{1\times n}$

Then,

$$\bar{x}\cdot\bar{y} = x_1y_1 + x_2y_2 + \ldots + x_ny_n \quad {}_{1\times 1}$$

In term of Matrix,

$$\bar{x}\cdot\bar{y} = (\bar{x})_{1\times n}(\bar{y})_{n\times 1}$$

$$= (x_1, x_2, \ldots x_n) \times \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$\therefore \bar{x}\cdot\bar{y} = (x_1y_1 + x_2y_2 + \ldots + x_ny_n)_{1\times 1}$$

Norm of a vector,

→ Norm = Distance,

Let $\bar{x} = (x_1, x_2, \ldots x_n) \in R^n$

Then the norm of $\bar{x}$ is denoted by $\|\bar{x}\|$ and is defined by.

$$\|\bar{x}\| = \sqrt{\bar{x}\cdot\bar{x}}$$

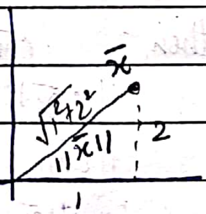
$$\therefore \|\bar{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \qquad \because (\text{from} \uparrow \bar{x}\cdot\bar{x})$$

which is non-negative real numbers and represents the distance from origin to the point $\bar{x}$.

Note *: R^1 = use modulus

R^2 or R^3 ...

i.e. $R^n, n>1$ = Norm.



$$\therefore \|\bar{x}\| = \sqrt{1^2+2^2} = \sqrt{5}$$

Some Facts on Norm,

① $\|\bar{x}\| \geq 0$ and $\|\bar{x}\| = 0$ iff $\bar{x} = \bar{0}$

$\|\bar{x}\| = 0$ iff $\bar{x} = \bar{0}$

proof:

Let $\bar{x} = (x_1, x_2, \ldots x_n)$ then,

$$\|\bar{x}\| = \sqrt{x_1^2+x_2^2+\dots+x_n^2}$$

i.e.

$$\|\bar{x}\|^2 = x_1^2+x_2^2+\dots+x_n^2 \qquad \text{(Sq. in bs)}$$

$\|\bar{x}\|^2 > 0$ if $x_i \neq 0$

But if all $x_i = 0$, Then $\|\bar{x}\|^2 = 0$

$\|\bar{x}\| = 0$

$\Rightarrow \bar{x} = \bar{0}$ //

(ii). $\|-\bar{x}\| = \|\bar{x}\|$

proof:

Let $\bar{x} = (x_1, x_2, \dots x_n)$ then,

$$\|\bar{x}\| = \sqrt{x_1^2+x_2^2+\dots+x_n^2}$$

$$\|-\bar{x}\| = \sqrt{(-x_1)^2+(-x_2)^2+\dots+(-x_n)^2}$$

$$= \sqrt{x_1^2+x_2^2+\dots+x_n^2}$$

$\therefore \|-\bar{x}\| = \|\bar{x}\|$ //

(iii). $\|c\bar{x}\| = |c|\,\|\bar{x}\|$

proof:

Let $\bar{x} = (x_1, x_2, \dots x_n)$ and scalar $c \in F$

then, $c\bar{x} = (cx_1, cx_2, \dots cx_n)$

$$\|c\bar{x}\| = \sqrt{(cx_1)^2+(cx_2)^2+\dots (cx_n)^2}$$

$$\|c\bar{x}\| = |c|\sqrt{x_1^2+x_2^2+\dots+x_n^2}$$

$\therefore \|c\bar{x}\| = |c|\,\|\bar{x}\|$ //

(iv) $\|\bar{x}-\bar{y}\| = \|\bar{y}-\bar{x}\|$ (Symmetry)

proof:

Let $\bar{x} = (x_1, x_2, \dots x_n)$, $\bar{y} = (y_1, y_2 \dots y_n)$

$\bar{x}-\bar{y} = (x_1-y_1, x_2-y_2, \dots, x_n-y_n)$

$$\|\bar{x}-\bar{y}\| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\dots+(x_n-y_n)^2}$$

$$= \sqrt{(y_1-x_1)^2+\dots (y_n-x_n)^2}$$

$\therefore \|\bar{x}-\bar{y}\| = \|\bar{y}-\bar{x}\|$ //

Some Norms on R^n

Let $\bar{x} = (x_1 x_2, \dots x_n)$ be a points in R^n then the more used norms on R^n are:

(i). L_1 Norm

L_1 Norm of $\bar{x} = \|\bar{x}\|_1 = |x_1| + |x_2| + \dots + |x_n|$

In term of Summation $= \sum_{k=1}^{n} |x_k|$

✓ (ii) L_2 Norm (Euclidean Norm)

L_2 Norm of $\bar{x} = \|\bar{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

(iii) L_∞ Norm

L_∞ Norm of $\bar{x} = \|\bar{x}\|_\infty = \max\{|x_1|, |x_2|, \dots |x_n|\}$

This shows that:

$\|\bar{x}\|_1 \geq 0$, $\|\bar{x}\|_2 \geq 0$, $\|\bar{x}\|_\infty \geq 0$

Example:- let $\bar{x} = (1,2) \in R^2$, find L_1, L_2, L_∞ Norm. How they are connected.

Soln,

L_1 Norm of $\bar{x} = \|\bar{x}\|_1 = |1| + |2| = 3$

L_2 Norm of $\bar{x} = \|\bar{x}\|_2 = \sqrt{1^2 + 2^2} = \sqrt{5} = 2.24$.

L_∞ Norm of $\bar{x} = \|\bar{x}\|_\infty = \max\{|1|, |2|\} = 2$

From this, we get

L_1 Norm $>$ L_2 Norm $>$ L_∞ Norm.

Example: let $\bar{x} = (1,2,3) \in R^3$; find L_1, L_2, L_∞ Norm. & their connection?

Soln,

L_1 Norm of $\bar{x} = \|\bar{x}\|_1 = |1| + |2| + |3| = 6$

L_2 Norm of $\bar{x} = \|\bar{x}\|_2 = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} = 3.74$

L_∞ Norm of $\bar{x} = \|\bar{x}\|_\infty = \max\{|1|, |2|, |3|\} = 3$

$\therefore$ L_1 Norm $>$ L_2 Norm $>$ L_∞ Norm //

* Note: • Unless and otherwise stated, we understand the norm on $\mathbb{R}^n$ as the Euclidean Norm (L_2 Norm)
• We use $\|\bar{x}\|_2$ as $\|\bar{x}\|$

Some other properties of Euclidean Norm.

$\|\bar{x}\| = \sqrt{\bar{x}\cdot\bar{x}}$

$\|\bar{x}\|^2 = \bar{x}\cdot\bar{x}$

(i). $\|\bar{x}+\bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2 + 2\bar{x}\cdot\bar{y}$

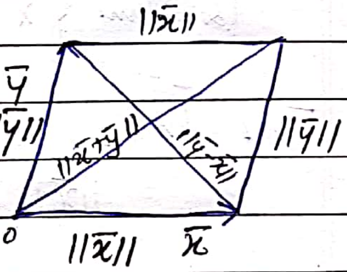
proof:

$$\|\bar{x}+\bar{y}\|^2 = (\bar{x}+\bar{y})\cdot(\bar{x}+\bar{y})$$
$$= \bar{x}\cdot\bar{x} + \bar{x}\bar{y} + \bar{y}\cdot\bar{x} + \bar{y}\cdot\bar{y}$$
$$= \|\bar{x}\|^2 + \|\bar{y}\|^2 + 2\bar{x}\cdot\bar{y} \quad //$$

(ii) Parallelogram Law

If, $\bar{x}, \bar{y} \in \mathbb{R}^n$, then,

$$\|\bar{x}+\bar{y}\|^2 + \|\bar{x}-\bar{y}\|^2 = 2(\|\bar{x}\|^2 + \|\bar{y}\|^2)$$

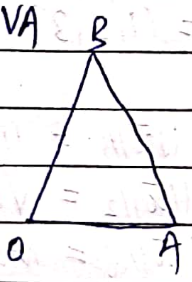


i.e. Sum of Squares of all sides
= Sum of squares of Diagonal.

Proof:

$$LHS = \|\bar{x}+\bar{y}\|^2 + \|\bar{x}-\bar{y}\|^2 = (\bar{x}+\bar{y})\cdot(\bar{x}+\bar{y}) + (\bar{x}-\bar{y})\cdot(\bar{x}-\bar{y})$$
$$= \bar{x}\cdot\bar{x} + \bar{x}\cdot\bar{y} + \bar{y}\cdot\bar{x} + \bar{y}\cdot\bar{y} + \bar{x}\cdot\bar{x} - \bar{x}\cdot\bar{y} - \bar{y}\cdot\bar{x} + \bar{y}\cdot\bar{y}$$
$$= \|\bar{x}\|^2 + 2\bar{x}\bar{y} + \|\bar{y}\|^2 + \|\bar{x}\|^2 - 2\bar{x}\bar{y} + \|\bar{y}\|^2$$
$$= 2(\|\bar{x}\|^2 + \|\bar{y}\|^2) \quad //$$

* Triangle Law of VA



$\vec{AB} = \vec{OB} - \vec{OA}$

$\vec{BA} = \vec{OA} - \vec{OB}$

* $|xy| = |x||y|$

$|\bar{x}\bar{y}| = |\bar{x}||\bar{y}|$
$||c\bar{x}|| = |c|\,||\bar{x}||$

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Cauchy-Schwarz Inequality.

If $\bar{x}$ & $\bar{y}$ be two vectors in R^n, then,
$|\bar{x}.\bar{y}| \le ||\bar{x}||.||\bar{y}||$.
The equality holds if they are co-linear (same line).
i.e,
$\exists\ c \in R$, such that $\bar{x} = c\bar{y}$

when x or $y = 0$
when $x = (1,2)$
$y = (2,4) = 2(1,2)$
$= 2(x)$
colinear

Proof:
Case I: If $\bar{x}$ & $\bar{y}$ are colinear, then $\bar{x} = c\bar{y}$ for some real number c.
Now,

$$|\bar{x}.\bar{y}| = |c\bar{y}.\bar{y}| = |c(\bar{y}.\bar{y})| = |c|\,||\bar{y}||^2 \quad (\to |c|.|\bar{y}.\bar{y}|)$$

$$\& \quad ||\bar{x}||\,||\bar{y}|| = ||c\bar{y}||\,||\bar{y}||$$
$$= |c|\,||\bar{y}||.||\bar{y}||$$
$$= |c|\,||\bar{y}||^2$$

Therefore, LHS = RHS $\Rightarrow |\bar{x}.\bar{y}| = ||\bar{x}||\,||\bar{y}||$ — (*)

Case II: If $\bar{x}$ & $\bar{y}$ are not collinear i.e. they are inclined to each other. then,

$\bar{x} = c\bar{y}$ for no real number c.
i.e. $\bar{x} - c\bar{y} = 0$ for no real number c.
or, $||\bar{x} - c\bar{y}|| = 0$ for no real number c.
or, $||\bar{x} - c\bar{y}||^2 = 0$ for no real number c.
or, $(\bar{x} - c\bar{y})(\bar{x} - c\bar{y}) = 0$ for no real number c.
or, $\bar{x}.\bar{x} - c(\bar{x}\bar{y}) - c(\bar{x}.\bar{y}) + c^2(\bar{y}.\bar{y}) = 0$ for no real number c.
or, $||\bar{x}||^2 - 2c(\bar{x}.\bar{y}) + c^2\,||\bar{y}||^2 = 0$ for no real number c.
or, $||\bar{y}||^2 c^2 - 2(\bar{x}.\bar{y})c + ||\bar{x}||^2 = 0$ — (1) for no real number c.
This is quadratic in c.

This is possible if the roots of (1) is complex.
So, its discriminants is less than 0.
i.e. $B^2 - 4AC < 0$
or, $[-2(\bar{x}.\bar{y})]^2 - 4\,||\bar{y}||^2\,||\bar{x}||^2 < 0$
or, $(\bar{x}.\bar{y})^2 < ||\bar{x}||^2\,||\bar{y}||^2$

*Note: $ax^2+bx+c=0$ for no real value of x.
means $\Rightarrow$ it has complex roots.
then,
$\underbrace{b^2-4ac}_{\text{discriminants}} < 0$

$x^2 = y^2$

$x = \pm y$

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taking the square roots, we get,

$|\bar{x}\,\bar{y}| < ||\bar{x}||\ ||\bar{y}||$ —— (**)

Combining (*) and (**)

$|\bar{x}\,\bar{y}| \leq ||\bar{x}||\ ||\bar{y}||$

HW Verify Cauchy Schwarz Inequality for the vectors:

(i) $\bar{x} = (1,2,3)$, $\bar{y} = (2,4,6)$

Here,

$|\bar{x}.\bar{y}| = |(1,2,3).(2,4,6)| = |2+8+18| = |28| = 28$

and,

$||\bar{x}|| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

$||\bar{y}|| = \sqrt{2^2+4^2+6^2} = \sqrt{56}$

so,

$||\bar{x}||\ ||\bar{y}|| = \sqrt{14}.\sqrt{56} = 28$.

$\therefore$ $|\bar{x}.\bar{y}| = ||\bar{x}||.||\bar{y}||$ // and $\bar{x}$ & $\bar{y}$ are colinear.

(ii) $\bar{x} = (1,2,3)$, $\bar{y} = (2,-4,-6)$

Here,

$|\bar{x}.\bar{y}| = |(1,2,3).(2,-4,-6)| = |2+(-8)+(-18)| = |-24| = 24$

and,

$||\bar{x}|| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

$||\bar{y}|| = \sqrt{2^2+(-4)^2+(-6)^2} = \sqrt{56}$.

so,

$||\bar{x}||\ ||\bar{y}|| = \sqrt{14}\sqrt{56} = 28$.

$\therefore$ $|\bar{x}.\bar{y}| < ||\bar{x}||\ ||\bar{y}||$ and $\bar{x}$ & $\bar{y}$ are not-colinear

Triangle Inequality.

If $x, y \in \mathbb{R}^n$, then: equlidian norm,

$\|\bar{x}+\bar{y}\|_2 \leq \|x\|_2 + \|y\|_2$

i.e,

$$\|\bar{x}+\bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|$$

Proof!

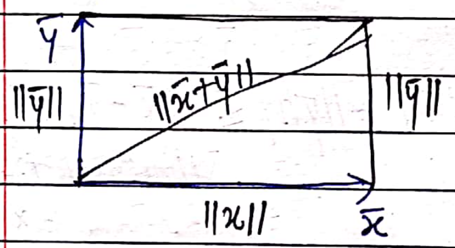
$$\|\bar{x}+\bar{y}\|^2 = (\bar{x}+\bar{y}).(\bar{x}+\bar{y})$$
$$= (\bar{x}.\bar{x} + \bar{x}.\bar{y} + \bar{y}.\bar{x} + \bar{y}.\bar{y})$$
$$= \|\bar{x}\|^2 + 2\bar{x}\bar{y} + \|\bar{y}\|^2$$
$$\leq \|\bar{x}\|^2 + 2|\bar{x}\bar{y}| + \|\bar{y}\|^2 \quad \because \bar{x}.\bar{y} \leq |\bar{x}.\bar{y}|$$
$$\leq \|\bar{x}\|^2 + 2\|\bar{x}\|\|\bar{y}\| + \|\bar{y}\|^2 \quad \because \text{from Cauchy Schwaz}$$

$$\therefore \|\bar{x}+\bar{y}\|^2 \leq (\|\bar{x}\|^2 + \|\bar{y}\|)^2$$

taking +ve square root.

$$\|\bar{x}+\bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|$$

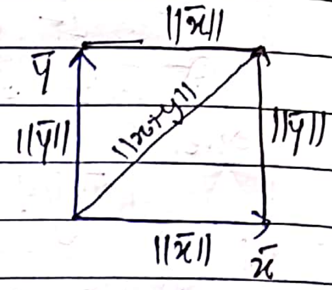
Use of Triangle Inequality.



In Δ 2 sides > than third sides,

\# If $\bar{x}, \bar{y} \in R^n$ and $\bar{x} \cdot \bar{y} = 0$ (Orthogonal / $\perp r$)

$$\|\bar{x}+\bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2$$



proof:

$$\|\bar{x}+\bar{y}\|^2 = (x+y)\cdot(x+y)$$
$$= \|\bar{x}\|^2 + 2\bar{x}\cdot\bar{y} + \|\bar{y}\|^2$$
$$\therefore \|\bar{x}+\bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2 \,//$$

\# If $\bar{x}, \bar{y} \in R^n$ then,

$\|\bar{x}-\bar{y}\| = \|\bar{x}+\bar{y}\|$, then $\bar{x}$ is perpendicular to $\bar{y}$. Prove it.

Given,

$$\|\bar{x}-\bar{y}\| = \|\bar{x}+\bar{y}\|$$
$$\|\bar{x}-\bar{y}\|^2 = \|\bar{x}+\bar{y}\|^2 \quad (\because \text{sq. b.s})$$
$$(\bar{x}-\bar{y})(\bar{x}-\bar{y}) = (\bar{x}+\bar{y})(\bar{x}+\bar{y})$$
$$\cancel{\|\bar{x}\|^2} - 2\bar{x}\cdot\bar{y} + \cancel{\|\bar{y}\|^2} = \cancel{\|\bar{x}\|^2} + 2\bar{x}\cdot\bar{y} + \cancel{\|\bar{y}\|^2}$$
$$4\bar{x}\cdot\bar{y} = 0.$$
$$\therefore \bar{x}\cdot\bar{y} = 0.$$

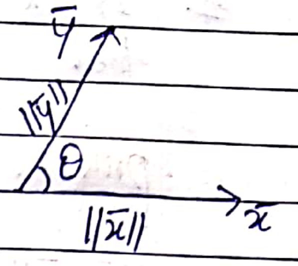
Hence, $\bar{x}$ $\perp r$ to $\bar{y}$ //

HW \# Let $\bar{x}, \bar{y} \in R^n$, prove that $\big|\|\bar{x}\| - \|\bar{y}\|\big| \leq \|\bar{x}-\bar{y}\|$.

Hint: $|\bar{x}| = +x$ if $x > 0$
$-x$ if $x < 0$
i.e. $|x| = \pm x$.

Angle between two Vectors.

Let $\bar{x}$ & $\bar{y}$ be two vectors in $\mathbb{R}^n$. Then the cosine of the angle between $\bar{x}$ & $\bar{y}$ is,



$$\cos\theta = \frac{\bar{x}\cdot\bar{y}}{\|\bar{x}\|\,\|\bar{y}\|}$$

where, θ is the angle between $\bar{x}$ & $\bar{y}$

If $\bar{x} = (x_1, x_2, \dots x_n)$;
$\bar{y} = (y_1, y_2, \dots y_n)$,
Then,

$$\cos\theta = \frac{x_1y_1 + x_2y_2 + \dots + x_ny_n}{\sqrt{x_1^2+x_2^2+\dots+x_n^2}\ \sqrt{y_1^2+y_2^2+\dots+y_n^2}} \qquad \cos\theta = [-1, 1]$$

\# Can we show $-1 \le \cos\theta \le +1$ by Cauchy schwarz Inequality. ?

Soln,

By Cauchy schwarz Inequality.

$$|\bar{x}\cdot\bar{y}| \le \|\bar{x}\|\,\|\bar{y}\|$$

or $\pm\, \bar{x}\,\bar{y} \le \|\bar{x}\|\,\|\bar{y}\|$ $\qquad$ ($\because |x| = \pm x$ by defn of mod)

or $\pm \dfrac{\bar{x}\cdot\bar{y}}{\|\bar{x}\|\,\|\bar{y}\|} \le 1$

or, $\pm \cos\theta \le 1$

Taking +ve : $\cos\theta \le 1$ —(i)

Taking -ve : $-\cos\theta \le 1$

or $\cos\theta \ge -1$ $\qquad$ (Ineq. sign change $- \to +$)

$\therefore -1 \le \cos\theta$ —(ii)

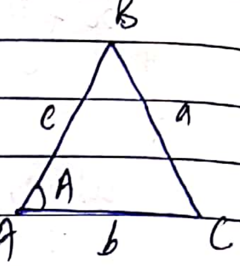
Combining (i) & (ii)

$$-1 \le \cos\theta \le 1 \quad //$$

If $\bar{x}, \bar{y} \in R^n$ and θ be the angle between them, then prove the cosine law

$$\|\bar{x}-\bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2 - 2\|\bar{x}\|\,\|\bar{y}\| \cos\theta$$

proof:



$$\|\bar{x}-\bar{y}\|^2 = (\bar{x}-\bar{y})\cdot(\bar{x}-\bar{y})$$
$$= \|\bar{x}\|^2 - 2\bar{x}\bar{y} + \|\bar{y}\|^2$$

$$\therefore \|\bar{x}-\bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2 - 2\|\bar{x}\|\,\|\bar{y}\| \cos\theta \;//$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

Cosine law

Vector Space and sub space:

A Vector space is denoted by V consist of elements $v_1, v_2, v_3 \ldots$ over the field F with operations.

(i) Addition of two vectors $(+)$

(ii) Multiplication of vector by scalar $(\cdot)$

Satisfying the following properties.

(i)

A_1 Closure

For each $v_1, v_2 \in V$
$$\Rightarrow v_1 + v_2 \in V$$

If $x+(-x)=0$, then x & $-x$ are additive inverse of each other

A_2 Commutative

For each $v_1, v_2 \in V$
$$\Rightarrow v_1 + v_2 = v_2 + v_1$$

If $xy = 1$ then, x & y are multiplicative inverse of each other,

A_3 Associative

For each $v_1, v_2, v_3 \in V$
$$\Rightarrow (v_1+v_2)+v_3 = v_1 + (v_2+v_3)$$

Field F = Real Number or Complex Number

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A4 Existance of additive identity.
There exist $0 \in V$ such that
$v+0 = v \ \forall \ v \in V$
Here 0 is called additive identity and is unique.

A5 Existance of additive inverse
For each, $v \in V$, then there exist
$-v \in V$, such that
$v+(-v) = 0$
Here $-v$ is called additive inverse of v.

(ii)

M1. Multiplication by scalar,
For each $v \in V$ and every scalar $c \in F$ such that
$cv \in V$

M2 Distributive of vector addition.
For each vectors $v_1, v_2 \in V$ and scalar $c \in F$
$c(v_1+v_2) = cv_1 + cv_2$

M3 Distributive of scalar addition.
For each vectors, $v \in V$ & scalars $c_1, c_2 \in F$
$(c_1+c_2)v = c_1v + c_2v$

M4 Associativity of product of scalars.
For each $v \in V$ & scalars, $c_1, c_2 \in F$
$c_1(c_2v) = (c_1c_2)v = c_2(c_1v)$

M5 Existance of unit element in the field
For each $v \in V$, $\exists \ 1 \in F$
Such that, $1v = v$.

Here if V satisfies $A_1 - A_5$ & $M_1 - M_5$ then we say that $(V, +, \cdot)$ is a vector space.

Example of Vector Space (VS)

(I) The set of all real number R forms a vector space with addition & scalar multiplication defined below,

for $\bar{v}_1 = (x_1)$ & $\bar{v}_2 = (x_2)$ then,

$$\bar{v}_1 + \bar{v}_2 = (x_1 + x_2)$$

and,

$\bar{v} = (x)$ & scalar $c \in F$, then

$$c\bar{v} = c(x) = (cx)$$

The zero element is $\bar{0} = (0)$

The unit element is $U = (1)$

HW# Verify R satisfies all the condition of Vector Space.

A_1 M_1

A_2 and, M_2

A_5 M_5

(II) The set of ordered pair (x_1, x_2) i.e. two dimentional plane form a vector Space (VS)

i.e.

$$R^2 = \{(x_1, x_2) : x_1, x_2 \in R\}$$

where, Addition & Scalar multiplication operation are defined below.

(i) For $\bar{v}_1 = (x, y_1)$, $V_2 = (x_2, y_2)$.

$$\bar{v}_1 + \bar{v}_2 = (x_1 + x_2, y_1 + y_2)$$

(ii) For $\bar{v} = (x, y)$, Scalar $c \in F$ then,

$$c\bar{v} = c(x, y) = (cx, cy)$$

The zero element $\bar{0} = (0, 0)$

The unit element $U = (1, 1)$

Verify R^2 satisfies all the condition of VS.

Verify R^2 satisfies all conditions of Vector Space (VS)

We show, R^2 is V.S. A_1 : Closure.

(I). A_1 : Let $\bar{v}_1 = (x_1, y_1)$, $\bar{v}_2 = (x_2, y_2)$ then

$\bar{v}_1 + \bar{v}_2 = (x_1 + x_2, y_1 + y_2) \in R^2$

$\therefore R^2$ is closed.

(II) A_2 : Commutative.

Let $\bar{v}_1 = (x_1, y_1)$, $\bar{v}_2 = (x_2, y_2)$ then,

$$
\begin{aligned}
\bar{v}_1 + \bar{v}_2 &= (x_1, y_1) + (x_2, y_2) \\
&= (x_1 + x_2, y_1 + y_2) \\
&= (x_2 + x_1, y_2 + y_1) \quad (\because R \text{ is commutative over addition}) \\
&= (x_2, y_2) + (x_1, y_1)
\end{aligned}
$$

$\therefore \bar{v}_1 + \bar{v}_2 = \bar{v}_2 + \bar{v}_1$

(III) A_3 : Associative over addition.

Let $\bar{v}_1 = (x_1, y_1)$, $\bar{v}_2 = (x_2, y_2)$, $\bar{v}_3 = (x_3, y_3)$ then,

$\bar{v}_1 + (\bar{v}_2 + \bar{v}_3) = (\bar{v}_1 + \bar{v}_2) + \bar{v}_3$

$$
\begin{aligned}
\text{LHS} &= \bar{v}_1 + (\bar{v}_2 + \bar{v}_3) \\
&= (x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) \\
&= (x_1, y_1) + (x_2 + x_3, y_2 + y_3) \\
&= (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3)) \\
&= ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3) \quad (\because R \text{ is associative over addition}) \\
&= (x_1 + x_2, y_1 + y_2) + (x_3, y_3) \\
&= ((x_1, y_1), (x_2, y_2)) + (x_3, y_3) \\
&= (\bar{v}_1 + \bar{v}_2) + \bar{v}_3 \\
&= \text{RHS}
\end{aligned}
$$

(IV) A_4 : $\exists\, \bar{0} = (0, 0)$ such that $\bar{v} + \bar{0} = (x_1, y_1) + (0, 0)$
$= (x_1, y_1) = \bar{v}$

$\therefore \bar{0} = (0, 0)$ acts as the additive identity.

(v) A_5: Existence of Inverse,

For each $\bar{v} = (x_1, y_1)$ there exists,

$$-\bar{v} = (-x, -y) \quad \text{such that}$$

$$\bar{v} + (-\bar{v}) = (0,0) = 0$$

So $(-x, -y)$ is the additive inverse of (x_1, y_1)

Ily, verify $M_1 - M_5$ (DIY)

So, R^2 is Vector Space.

Example 3: The Euclidean space R^n, $n > 1$ forms a vector space, where addition & scalar multiplication defined by.

(i) For $\bar{x} = (x_1, x_2 \dots x_n)$

$$\bar{y} = (y_1, y_2 \dots y_n)$$

$$\bar{x} + \bar{y} = (x_1 + y_1, x_2 + y_2 + \dots x_n + y_n)$$

(ii) For $\bar{x} = (x_1, x_2, \dots x_n)$, Scalar c.

$$c\bar{x} = (cx_1, cx_2, \dots cx_n)$$

The zero element is $\bar{0} = (0, 0, \dots 0)$ ⇒ n touple.

The additive inverse of $\bar{x} = (x_1, x_2 \dots x_n)$ is.

$$-\bar{x} = (-x_1, -x_2, \dots -x_n)$$

Example 4. Let V be the set of all 2×3 matrices

$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ with real entries.

Then, V is vector space, where addition & scalar multiplication are defined by.

Vector Subspace

A non-empty subset W of a vector space V is said to be vector subspace if W is closed with respect to vector addition and scalar multiplication.

(i) $w_1 + w_2 \in W$
(ii) $cw \in W$

In other words,

$W \subset V$ is said to be vector subspace of V if

(i) For each $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$

(ii) For each scalar $c \in F$ & vector $w \in W \Rightarrow cw \in W$

This defination equivalent to:

$c_1 w_1 + c_2 w_2 \in W$ for all $w_1, w_2 \in W$ & $c_1, c_2 \in F$

Note: In condition (ii), if $c = 0$ then $\bar{0} \in W$.
if $c = -1$, then $-\bar{w} \in W$

Example:

(i) Trivial Subspace

If $W = \{\bar{0}\}$ in a vector space V. Then it is clearly subspace of V. This is called trivial subspace of V.

(ii) $\overset{\to W}{V}$ itself is a subspace of V. (How?)
This is also subspace of V.

Example:

The set of all points in the plane $W = \{(x,y,0) : x, y \in R\}$ is a subspace of $R^3 = \{(x,y,z) : x, y, z \in R\}$. Verify.

Let $W = \{(x,y,z) : x+2y+z=0\}$. Then W is a subspace of R^2

proof:

Let $\bar{w}_1 = (x_1, y_1, z_1) \in W$ & $\bar{w}_2 = (x_2, y_2, z_2) \in W$ and c_1, c_2 scalars $\in F$

$\therefore$ By defination of W,

$$x_1 + 2y_1 + z_1 = 0$$

$$\& \quad x_2 + 2y_2 + z_2 = 0$$

Now,

$$c_1 w_1 + c_2 w_2 = c_1(x_1, y_1, z_1) + c_2(x_2, y_2, z_2)$$

$$= (c_1x_1, c_1y_1, c_1z_1) + (c_2x_2, c_2y_2, c_2z_2)$$

$$= (c_1x_1 + c_2x_2, c_1y_1 + c_2y_2, c_1z_1 + c_2z_2) \rightarrow (x, y, z)$$

Now, $x + 2y + z = 0$.

or, $(c_1x_1+c_2x_2)+2(c_1y_1+c_2y_2)+(c_1z_1+c_2z_2)=0$

or $c_1(x_1+2y_1+z_1)+c_2(x_2+2y_2+z_2)=0$

or $c_1 0+c_2\cdot 0=0$

$\therefore 0=0$ //

$\therefore c_1W_1+c_2W_2\in W$

W is a subspace of R^3

Example:

Let $W=\{(x,y,z): x,y,z\geq 0\}$. This is called orthant of $R^3=\{(x,y,z):(x,y,z)\in R\}$. Is W subspace of R^3?

Ans: Negative / No.

Since, $\overline{w}=(1,2,3)\in W$

& $c=-1\in F$

$\therefore cw=(-1,-2,-3)\notin W$

Example:

The line $y=mx+c$ (is or is not) subspace of R^2. $c\neq 0$

Here,

$W=\{(x,y): y=mx+c, x,y\in R \quad m \text{ \& } c \text{ fixed constant} \in F\}$

Let. $\overline{W}_1=(x_1,y_1)$, $\overline{W}_2=(x_2,y_2)\in W$

$y_1=mx_1+c.$

$y_2=mx_2+c.$

Now,

$c_1W_1+c_2W_2=c_1(x_1,y_1)+c_2(x_2,y_2)$

$=(c_1x_1+c_2x_2, c_1y_1+c_2y_2) \dashrightarrow (x,y)$

Now, $y=mx+c.$

or $c_1y_1+c_2y_2=m(c_1x_1+c_2x_2)+c$

or, $c_1(mx_1+c)+c_2(mx_2+c)=c_1(mx_1)+c_2(mx_2)+c$

or, $c_1(mx_1+c)+c_2(mx_2+c)=c_1(mx_1+c)+c_2(mx_2+c)+-c_1c-c_2c+c$

add. add.

or, $-c_1c-c_2c+c=0 \Rightarrow c_1+c_2=1.$

which is not true for some c_1, c_2

$\therefore$ W is not subspace of R^2

B (center, radius)

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HW # Is $W = \{(x,y) : y = mx ; x, y \in R, m \in F\}$ Subspace of R^2?

Ans: positive / Yes! Yes.

Example: ← closed unit circle with centre $\bar{0}$ & radius 1.

Let L_2 ball $= B(0,1) = \{(x,y) : x^2+y^2 \leq 1\}$.

OR → $\sqrt{x^2+y^2} = 1$.

$= \{\bar{x} = (x, x_2) : \|\bar{x}\| \leq 1\}$

Is l_2 ball $B(0,1)$ subspace of R^2?

This is not subspace of R^3,

since $w = (1,0) \in W$

& $c = 5 \in F$

$5w = (5,0) \notin W$

Example:

If the probabilities of happening the events $\overset{x}{P(A)}, \overset{y}{P(B)}, \overset{z}{P(C)}$.

Such that.

$P(A) + P(B) + P(C) = 1$.

i.e. $x + y + z = 1$.

If x, y, z are all positive, then the collection $W = \{(x,y,z) : x+y+z = 1\}$ is called positive simplex.

Is W subspace of R^3.

⇒ No,

Since, $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \in W$

& $6 \in F$

$6w = (2, 2, 2) \notin W$

∴ W is not the subspace of R^3.

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Linear Combination:

Let, $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ be the n vectors in the vector Space V. Then their linear combination is of the form.

$c_1\bar{v}_1 + c_2\bar{v}_2 + \ldots + c_n\bar{v}_n$, $c_1, c_2, \ldots c_n \in F$

i.e.

$$\sum_{k=1}^{n} c_k \bar{v}_k$$ Linear Hull

The Set of all linear combination of $\bar{v}_1, \bar{v}_2, \ldots \bar{v}_n$ is called Linear Hull.

and denoted by, $L(\bar{v}_1, \bar{v}_2, \ldots \bar{v}_n)$

Example:

Is the Linear hull $L(v_1, v_2 \ldots v_n)$ is a vector subspace of V.

Ans: Yes.

Proof:

Let, w_1 & $w_2 \in L(v_1, v_2, \ldots v_n)$

Then, w_1, w_2 can be expressed as the linear combination of $v_1, v_2, \ldots v_n$.

$\therefore$ $w_1 = c_1v_1 + c_2v_2 + \ldots + c_nv_n$, where, $c_1, c_2 \ldots c_n \in F$

and, $w_2 = d_1v_1 + d_2v_2 + \ldots + d_nv_n$, where $d_1, d_2, \ldots d_n \in F$

Let α, β be the scalars, then,

$$\alpha w_1 + \beta w_2 = \alpha(c_1v_1 + c_2v_2 + \ldots + c_nv_n) + \beta(d_1v_1 + d_2v_2 + \ldots + d_nv_n)$$
$$= (\alpha c_1 + \beta d_1)v_1 + (\alpha c_2 + \beta d_2)v_2 + \ldots + (\alpha c_n + \beta d_n)v_n$$
$$= a_1v_1 + a_2v_2 + \ldots + a_nv_n$$ $\because$ say, $\alpha c_1 + \beta d_1 = a_1$

$\vdots$ a_2

$\alpha c_n + \beta d_n = a_n$.

$\therefore$ $\alpha w_1 + \beta w_2 \in L(v_1, v_2, \ldots v_n)$.

$\therefore$ $L(v_1, v_2, \ldots v_n)$ is a vector subspace.

Linearly independent & dependent vectors.

Let $V_1, V_2, \dots V_n$ be n vectors in the vector space V. over the field F. Then they are said to be linearly dependent if there exist $c_1, c_2, \dots c_n \in F$ such that,

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0 \quad \text{(Linearly dependent)}$$

$\Rightarrow$ at least one c_i different from zero.

Otherwise,

they are said to be linearly independent.

i.e. $v_1, v_2, \dots v_n$ are linearly independent if

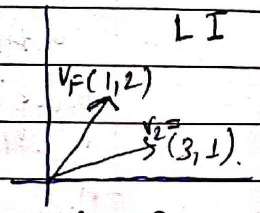
$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0 \quad \text{(Linearly independent)} \rightarrow \text{all } c_i = 0$$

$\Rightarrow$ all c_i are zero. $, c_1 = 0, c_2 = 0, \dots c_n = 0$

$R^n \Rightarrow n = \text{no. of constants } c_1 \dots c_n$

Example: In R^2, let $v_1 = (1,2)$, $v_2 = (3,1)$.

LI



If $c_1 v_1 + c_2 v_2 = 0$

$\Rightarrow c_1 = 0, \ c_2 = 0$.

Linearly independent

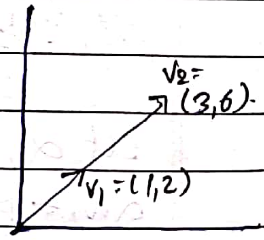
If $c_1 v_1 + c_2 v_2 = 0$

$v_1 = (1,2)$

$v_2 = (3,6) = 3(1,2) = 3v_1$

$3v_1 - v_2 = 0$

$\therefore 3v_1 + (-1)v_1 = 0$



or $c_1 v_1 + c_2 v_2 = 0$, $c_1 \neq 0$

$c_2 \neq 0$.

$\therefore \{v_1, v_2\}$ are linearly dependent.

lot, No. of vector = No of constants.

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Example:

Show that the vectors, $v_1 = (1,2)$ & $v_2 = (3,5)$ are linearly independent in R^2.

proof:

Let c_1, c_2 be the scalars (real number) such that,

$c_1 v_1 + c_2 v_2 = 0$

or, $c_1(1,2) + c_2(3,5) = 0$

or, $(c_1, 2c_1) + (3c_2, 5c_2) = 0$

or, $(c_1 + 3c_2,\ 2c_1 + 5c_2) = 0$

or, $\left.\begin{array}{l} c_1 + 3c_2 = 0, \\ 2c_1 + 5c_2 = 0 \end{array}\right\}$

$\therefore$ $c_1 = 0$, $c_2 = 0$

$\therefore$ $\{v_1, v_2\}$ are linearly independent.

#10 Show that the set of vectors $e_1 = (1,0)$ & $e_2 = (0,1)$ are linearly independent.

(11) Show that the matrices (2x3 matrices)

$M_1 = \begin{pmatrix} 1 & -2 & 4 \\ 3 & 0 & -1 \end{pmatrix}$ & $M_2 = \begin{pmatrix} 2 & -4 & 8 \\ 6 & 0 & -2 \end{pmatrix}$ are linearly dependent

Hint: $c_1 M_1 + c_2 M_2 = 0$

or, $c_1(\quad) + c_2(\quad) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

or, $c_1 . 1 + 2c_2 = 0$

Span or Generate

Let $v_1, v_2, \ldots, v_n$ be n vectors in a vector space V over the field F. Then we say that a vector $v \in V$ generated by $\{v_1, v_2, \ldots, v_n\}$

If v can be expressed as the linear combination of $\{v_1, v_2, \ldots v_n\}$ i.e. there exists $c_1, c_2, \ldots c_n \in F$ such that

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n.$$

Example: Show that:

(I) The vectors $\{(1,0), (0,1)\}$ generates every elements of R^2.

proof:

Let $v \in R^2$ and $v = (x_1, x_2)$.

If $v = c_1 v_1 + c_2 v_2$

or $(x_1, x_2) = c_1(1,0) + c_2(0,1)$ — ①.

or $(x_1, x_2) = (c_1, 0) + (0, c_2)$

$\therefore (x_1, x_2) = (c_1, c_2)$

$\therefore c_1 = x_1$

$c_2 = x_2$

Eqn ① becomes,

$(x_1, x_2) = x_1(1,0) + x_2(0,1)$

Thus, every vectors $v = (x_1, x_2)$ can be written as a linear comb. of v_1, v_2

$\therefore \{(1,0), (0,1)\}$ generates R^2.

(II) The vectors $\{(1,2), (-1,1)\}$ generates every elements of R^2

proof

Let $v \in R^2$ and $v = (x_1, x_2)$.

If $v = c_1 v_1 + c_2 v_2$.

or, $(x_1, x_2) = c_1(1,2) + c_2(-1,1)$ — ①.

or $(x_1, x_2) = (c_1, 2c_1) + (-c_2, c_2)$

$\therefore (x_1, x_2) = (c_1 - c_2, 2c_1 + c_2)$

$x_1 = c_1 - c_2 \Rightarrow c_1 = x_1 + c_2$

$x_2 = 2c_1 + c_2 \Rightarrow c_2 = x_2 - 2c_1 =$

Adding: $3c_1 = x_1 + x_2$

$c_1 = \frac{x_1 + x_2}{3}$

Also,

$c_2 = c_1 - x_1 = \frac{x_1 + x_2}{3} - x_1$

$c_2 = \frac{x_2 - 2x_1}{3}$

Eqn ① becomes,

$(x_1, x_2) = \left(\frac{x_1 + x_2}{3}\right)(1,2) + \left(\frac{x_2 - 2x_1}{3}\right)(-1,1)$

Thus, every vector $v = (x_1, x_2)$ can be written as Linear combination of v_1, v_2 $\therefore \{(1,2), (-1,1)\}$ generates R^2

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Basis of Vector Space.

A set of vectors $\{v_1, v_2, \dots, v_n\}$ in a vector space V over the field F is called basis of V. if.

① $\{v_1, v_2, \dots, v_n\}$ is linearly independent.

(ii) $\{v_1, v_2, \dots, v_n\}$ generate or span V.

Here, the finite number n is called dimension of the vector space V. In other words, the number of element in basis in a vector space V is called dimension of V. It is denoted by. dim V

Example:

I. let, $e_1 = (1,0)$ & $e_2 = (0,1)$ in $V = R^2$. Then,

(i). $\{e_1, e_2\}$ is linearly independent

(ii) $\{e_1, e_2\}$ generates R^2

Since for each $(x_1, x_2) \in R^2$

$V = (x_1, x_2) = (x_1, 0) + (0, x_2) = x_1(1,0) + x_2(0,1)$.

$= x_1 e_1 + x_2 e_2$.

$\therefore \{e_1, e_2\}$ form a basis of R^2.

II let, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. in $V = R^3$ Then, $\{e_1, e_2, e_3\}$ form a basis of R^3. Verify.

III let $v_1 = (1,2)$, $v_2 = (-1,1)$, Then $\{v_1, v_2\}$ form a basis of R^2. Verify.

Soln, II (i) L.I: let c_1, c_2, c_3 be scalars such that

$c_1 e_1 + c_2 e_2 + c_3 e_3 = 0$.

$c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$

$(c_1, c_2, c_3) = (0,0,0)$

$\therefore c_1 = c_2 = c_3 = 0$

$\therefore \{e_1, e_2, e_3\}$ are L.I.

(ii) G/S: $\{e_1, e_2, e_3\}$ generates R^3

Since for each $(x_1, x_2, x_3) \in R^3$.

$V = (x_1, x_2, x_3) = (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3)$

$= x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$

$\therefore V = x_1 e_1 + x_2 e_2 + x_3 e_3$.

$\therefore \{e_1, e_2, e_3\}$ form a basis of R^3.

IV Let V = vector space of all 2×2 matrices $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}\right\}$

Let, $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

We show that $\{E_1, E_2, E_3, E_4\}$ form a basis of V.

(i). $\{E_1, E_2, E_3, E_4\}$ is linearly independent

Let, c_1, c_2, c_3, c_4 be scalars such that,

$c_1E_1 + c_2E_2 + c_3E_3 + c_4E_4 = 0$

or, $c_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore$ $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$

(ii). $\{E_1, E_2, E_3, E_4\}$ generates V.

Let, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$.

Then one can write,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= aE_1 + bE_2 + cE_3 + dE_4.$$

$\therefore$ $\{E_1, E_2, E_3, E_4\}$ generates all the elements of V.

Theorem :

Let V be the vector space over F with $\dim V = n$. Then, the representation of any vector in terms of linear combination basis vectors is unique.

proof.

Let $\{v_1, v_2, \ldots v_n\}$ be a basis of V.

Let $v \in V$, then $v \in V$ can be written as the linear combination of $\{v_1, v_2, \ldots v_n\}$.

So $\exists$ constants $c_1, c_2, \ldots c_n \in F$.

Such that,

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \quad \text{--- (i)}$$

If possible, Suppose there is another representation of v i.e.

$$v = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n \text{ --- (ii)}, \text{ where } b_1, b_2, \ldots b_n \in F.$$

Then from (i) & (ii)

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n$$

or, $$0 = (b_1 - c_1) v_1 + (b_2 - c_2) v_2 + \cdots + (b_n - c_n) v_n$$

Since, $\{v_1, v_2 \ldots v_n\}$ form a basis of V. So, they are linearly independent.

$$\therefore (b_1 - c_1) v_1 + (b_2 - c_2) v_2 + \cdots + (b_n - c_n) v_n = 0$$

$$\Rightarrow b_1 - c_1 = 0, \quad b_2 - c_2 = 0, \ldots, b_n - c_n = 0$$

$$\therefore b_1 = c_1, \quad b_2 = c_2, \quad \ldots, \quad b_n = c_2$$

So, the representation (i) is unique.

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Coordinate of a vector $v \in V$ in terms of basis vectors.

Let V be a vector space over the field F with $\dim V = n$. Let $\{v_1, v_2, \dots v_n\}$ be a basis of V. Then we know, each $v \in V$ can be represented uniquely as,

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

Then, we say that the n-triple $(c_1, c_2, \dots c_n)$ is called coordinates of the vector $v \in V$ with respect to the basis $\{v_1, v_2, \dots v_n\}$.

Example:

Let $V = R^2$ and $\{v_1, v_2\}$ is a basis of R^2, where, $v_1 = (1,2)$ & $v_2 = (2,1)$

Let $v = (3,2) \in R^2$, find the coordinate of v w.r.t the basis $\{v_1, v_2\}$

Soln,

Let $v = c_1 v_1 + c_2 v_2$

or $(3,2) = c_1(1,2) + c_2(2,1)$

or $(3,2) = (c_1, 2c_1) + (2c_2, c_2)$

or $(3,2) = (c_1 + 2c_2, 2c_1 + c_2)$

$c_1 + 2c_2 = 3$ — (i)

$2c_1 + c_2 = 2$ — (ii)

Solving (i) & (ii)

$c_1 = \frac{1}{3}$ and $c_2 = \frac{4}{3}$

$\therefore (c_1, c_2) = \left(\frac{1}{3}, \frac{4}{3}\right)$ which is the co-ordinate of $v = (3,2)$ w.r.t the basis $\{v_1, v_2\}$.

#(1) Let $V = \mathbb{R}^2$ and $v = (5,4)$, let $\{(1,2), (3,1)\}$ be a basis of $\mathbb{R}^2$. Find the co-ordinates of v with respect to the basis $\{(1,2), (3,1)\}$.

Ans: $\left(\frac{17}{4}, -\frac{14}{5}\right)$

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#2. Let $V = \mathbb{R}^3$

(i). Show that $B = \{(1,1,1), (1,3,2), \{-1,0,1\}\}$ form a basis of $\mathbb{R}^3$.

(ii). Find the coordinate of $(2,1,1) \in \mathbb{R}^3$ w.r.t. the basis B.

#2 (i) Soln,

Let $c_1, c_2 \& c_3$ be scalars $\in F$, such that,

$c_1v_1 + c_2v_2 + c_3v_3 = 0$

$c_1(1,1,1) + c_2(1,3,2) + c_3(-1,0,1) = (0,0,0)$

$\therefore$ $c_1 + c_2 - c_3 = 0$ —(i)

$c_1 + 3c_2 = 0$ —(ii)

$c_1 + 2c_2 + c_3 = 0$ —(iii)

On, solving we get, $c_1 = c_2 = c_3 = 0$, which shows that B is linearly independent

And,

Since for each $(x_1, x_2, x_3) \in \mathbb{R}^3$

$v = (x_1, x_2, x_3) = x_1(1,1,1) + x_2(1,3,2) + x_3(-1,0,1) = x_1v_1 + x_2v_2 + x_3v_3$

$\therefore$ $\{v_1, v_2, v_3\}$ generates all the elements of V. and hence forms basis of $\mathbb{R}^3$.

(ii). Let $a_1, a_2, \& a_3$ be co-ordinates of $(2,1,1)$ w.r.t the basis B. , then,

$(2,1,1) = a_1(1,1,1) + a_2(1,3,2) + a_3(-1,0,1)$

or $(2,1,1) = (a_1 + a_2 - a_3, a_1 + 3a_2, a_1 + 2a_2 + a_3)$

$\therefore$ $a_1 + a_2 - a_3 = 2$ —(i)

$a_1 + 3a_2 = 1$ —(ii)

$a_1 + 2a_2 + a_3 = 1$ —(iii)

On solving, $a_1 = 2$, $a_2 = -1/3$, $a_3 = -1/3$

$\therefore$ $(2, -1/3, -1/3)$ is the coordinates of $(2,1,1) \in \mathbb{R}^3$ w.r.t basis B.

#1 Soln,

Let $c_1 \& c_2$ be coordinates of $v = (5,4)$ w.r.t basis $\{(1,2), (3,1)\}$ then,

$v = c_1v_1 + c_2v_2$

$(5,4) = (c_1 + 3c_2, 2c_1 + c_2)$

$\therefore$ $c_1 + 3c_2 = 5$ —(i)

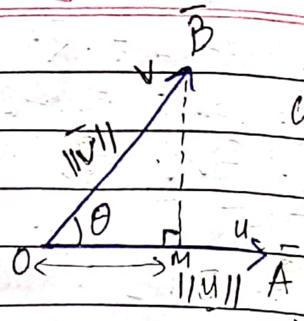
$2c_1 + c_2 = 4$ —(ii).

On, solving (i) & (ii)

$2\times$(i) − (ii), we get,

$c_1 = \frac{7}{5}$ and $c_2 = \frac{6}{5}$.

Projection.



$\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

Scalar and Vector Projection.

$M \Rightarrow$ Pr of $\vec{OB}$ on $\vec{OA}$
$\Rightarrow$ Pr. of $\vec{v}$ on $\vec{u}$

Let $u \neq 0$ & $v \neq 0$ be two vectors in vector space V.

Let $\vec{OA} = \vec{u}$

$\vec{OB} = \vec{v}$

Draw $\perp^r$ BM from B on OA.

Then OM is called scalar projection of the vector $\vec{v}$ on the vector $\vec{u}$.

From,

$$\triangle OBM, \quad \cos\theta = \frac{OM}{OB} = \frac{OM}{\|\vec{v}\|}.$$

$$\because \cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}, \quad u \neq 0,\ v \neq 0$$

$$\therefore OM = \|\vec{v}\|\cos\theta.$$

$$= \frac{\cancel{\|\vec{v}\|}\,(\vec{u}\cdot\vec{v})}{\|\vec{u}\|\,\cancel{\|\vec{v}\|}}$$

$$\boxed{\underset{(\vec{v}\text{ on }\vec{u})}{OM} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}} \qquad SP(\vec{v}\text{ on }\vec{u})$$

which is scalar projection (SP) of $\vec{v}$ on $\vec{u}$

Also, the vector projection of $\vec{v}$ on $\vec{u}$ $= \vec{OM}$

$= \|\vec{OM}\|$

$= (OM).(\text{Unit vector along } \vec{u})$

$$\boxed{VP(\vec{v}\text{ on }\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}\left(\frac{\vec{u}}{\|\vec{u}\|}\right)}$$

OR

$$\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\cdot\vec{u}.$$

Example:

let $\vec{u} = (1,2)$ & $v = (3,1)$ on R^2. find scalar projection & vector projection of $\vec{u}$ on $\vec{v}$ & $\vec{v}$ on $\vec{u}$.

Soln,

$\vec{u} = (1,2)$, $\bar{v} = (3,1)$

$\|\vec{u}\| = \sqrt{1^2+2^2} = \sqrt{5}$

$\|\vec{v}\| = \sqrt{3^2+1^2} = \sqrt{10}$

$\vec{u}.\vec{v} = (1,2).(3,1)$

$= 1\times3 + 2\times1$

$= 5.$

Now

$$\text{SP. of } \bar{v} \text{ on } \bar{u} = \frac{\bar{u}.\bar{v}}{\|\bar{u}\|} = \frac{5}{\sqrt{5}} = \sqrt{5}.$$

$$\text{V.P of } \bar{v} \text{ on } \bar{u} = \frac{\bar{u}.\bar{v}}{\|\bar{u}\|}\left(\frac{\bar{u}}{\|\bar{u}\|}\right) = \sqrt{5}.\left(\frac{(1,2)}{\sqrt{5}}\right) = (1,2).$$

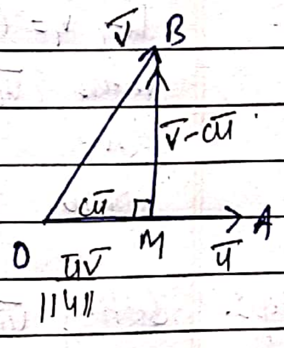
And,

$$\text{SP of } \bar{u} \text{ on } \bar{v} = \frac{\bar{u}.\bar{v}}{\|\bar{v}\|} = \frac{5}{\sqrt{10}}.$$

$$\text{VP of } \bar{u} \text{ on } \bar{v} = \frac{\bar{u}.\bar{v}}{\|\bar{v}\|}.\left(\frac{\bar{v}}{\|\bar{v}\|}\right) = \frac{5}{\sqrt{10}}\left(\frac{(3,1)}{\sqrt{10}}\right) = \left(\frac{3}{2}, \frac{1}{2}\right)$$

Note:

From the figure,



$$\vec{OM} = \frac{\bar{u}.\bar{v}}{\|\bar{u}\|}.\frac{\bar{u}}{\|\bar{u}\|}.$$

$$= \frac{\bar{u}.\bar{v}}{\|\bar{u}\|}.\bar{u}$$

$= c\bar{u}$

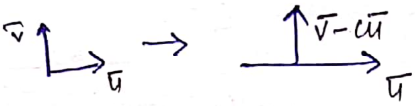
$\vec{OB} = \bar{v}$

$\bar{MB} = \bar{OB} - \bar{OM}$

$= \bar{v} - c\bar{u}$

Here we see that, $\bar{MB} = \bar{v} - c\bar{u}$ is always $\perp r$ to $\bar{u}$.

Here c is called fourier coefficient of $\bar{v}$ on $\bar{u}$.



Step I
every vector $\bar{u}_1 = \bar{v}_1$ containing more $0 \Rightarrow u_1$
consider $\bar{u}_1$

Gram Schmidt Orthogonalization Process.

Here from the two non-perpendicular vectors $\{u, v\}$.

We find a set of perpendicular vectors. $\{\bar{u}, \bar{v} - c\bar{u}\}$

Rules of finding Orthogonal Vectors by using G.S. Orthogonalization Process.

\# Find the orthogonal vectors corresponding to the vectors $\{V_1, V_2, \ldots V_n\}$.

Soln,

Let $\{u_1, u_2, u_3, \ldots, u_n\}$ be the required orthogonal vectors.

Step I: Fixing $\bar{u}_1 = \bar{v}_1$

Step II: Setting $\bar{u}_2 = \bar{V}_2 - (\underbrace{Proj_{\bar{u}_1} \bar{V}_2}_{\text{Projn of } \bar{V}_2 \text{ on } \bar{u}_1}) \vec{u}_1$ where, $Proj_{\bar{u}} \bar{V} = \left(\frac{\bar{V}.\bar{u}}{\|\bar{u}\|^2} \right) \bar{u}$ (VP)

Step III: Setting, $\bar{u}_3 = \bar{V}_3 - Proj_{\bar{u}_1} \bar{V}_3 - Proj_{\bar{u}_2} \bar{V}_3$

.....

$$\bar{U}_n = \bar{V}_n - Proj_{\bar{u}_1} \bar{V}_n - Proj_{\bar{u}_2} \bar{V}_n - \ldots - Proj_{\bar{u}_{n-1}} \bar{V}_n$$

Example:

Let, $v_1 = (1,0,0)$, $v_2 = (1,1,0)$, $v_3 = (1,1,1)$. find the orthogonal vectors $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ corresponding to $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ by using GS Or. Process.

Soln,

Here, $V_1 = (1,0,0)$, $V_2 = (1,1,0)$, $V_3 = (1,1,1)$.

StepI. Setting $\bar{u}_1 = \bar{V}_1 = (1,0,0)$

Now, $Proj_{\bar{u}_1} \bar{V}_2 = \frac{\bar{V}_2 . \bar{u}_1}{\|u_1\|^2} . \bar{u}_1 = \left(\frac{(1,1,0) . (1,0,0)}{(\sqrt{1^2+0^2+0^2})^2} \right) (1,0,0)$

$= \frac{1+0+0}{1} (1,0,0)$

$= (1,0,0)$

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Step II. Setting $\overline{u_2} = \overline{v_2} - (\text{proj}_{\overline{u_1}} \overline{v_2})\, \overline{u_1}$

$= (1,1,0) - (1,0,0).\cancel{(1,0,0)}$

$\therefore \overline{u_2} = (0,1,0)$

Clearly, $\overline{u_1} \cdot \overline{u_2} = 0$ $\therefore \{u_1, u_2\}$ are perpendicular.

Step III: Setting, $\overline{u_3} = \overline{v_3} - \text{proj}_{u_1} \overline{v_3} - \text{Proj}_{\overline{u_2}} \overline{v_3}$ — (*)

Now,

$$\text{Proj}_{\overline{u_1}} \overline{v_3} = \left(\frac{\overline{v_2} \cdot \overline{u_1}}{\|\overline{u_1}\|^2}\right) \overline{u_1} = \left(\frac{(1,1,1) \cdot (1,0,0)}{1}\right)(1,0,0)$$

$$= (1,0,0)$$

$$\text{Proj}_{\overline{u_2}} \overline{v_3} = \frac{\overline{v_3} \cdot \overline{u_2}}{\|\overline{u_2}\|^2} \cdot \overline{u_2} = (0,1,0).$$

From (*)

$\overline{u_3} = (1,1,1) - (1,0,0) - (0,1,0)$

$\therefore \overline{u_3} = (1-1-0, 1-0-0, 1-0-0) = (0,0,1)$

Clearly, $\overline{u_3} \cdot \overline{u_1} = (0,0,1) \cdot (1,0,0) = 0$ $\therefore \{u_1, u_3\}$ are $\perp r$

$\overline{u_3} \cdot \overline{u_2} = (0,0,1) \cdot (0,1,0) = 0$ $\therefore \{u_1, u_2\}$ are $\perp r$.

Therefore, $\{\overline{u_1}, \overline{u_2}, \overline{u_3}\}$ is mutually orthogonal vectors in R^3.

Note:-

If $\{u_1, u_2, u_3\}$ be the set of orthogonal vectors. Then,

$w_1 = \frac{u_1}{\|u_1\|}$, $w_2 = \frac{u_2}{\|u_2\|}$, $w_3 = \frac{u_3}{\|u_3\|}$ } are orthonormal vectors.

Orthonormal = Orthogonal + With each norm 1.

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Example:

Find the orthonormal vectors corresponding to the vectors $\{v_1, v_2, v_3\}$ where, $v_1 = (1,0,0)$, $v_2 = (1,1,0)$, $v_3 = (1,1,1)$

Soln,

As before, orthogonal vectors are,

$\{u_1, u_2, u_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$.

The orthonormal vectors are $\{w_1, w_2, w_3\}$.

where,

$$w_1 = \frac{u_1}{\|u_1\|} = \frac{(1,0,0)}{\|(1,0,0)\|} = \frac{(1,0,0)}{\sqrt{1^2+0^2+0^2}} = \frac{(1,0,0)}{1} = (1,0,0)$$

$$w_2 = \frac{u_2}{\|u_2\|} = \frac{(0,1,0)}{\|(0,1,0)\|} = \frac{(0,1,0)}{\sqrt{0^2+1^2+0^2}} = \frac{(0,1,0)}{1} = (0,1,0)$$

$$w_3 = \frac{u_3}{\|u_3\|} = (0,0,1)$$

#1 Show that the set $S = \{\overset{v_1}{(1,0,1)}, \overset{v_2}{(0,1,0)}, \overset{v_3}{(-1,0,1)}\}$ form a orthogonal bases of R^3. Hence find corresponding orthonormal basis of R^3.

- LI, G (Basis)
- Orthogonal
- On

Soln,

Let c_1, c_2, c_3 be scalar F, such that,

$c_1v_1 + c_2v_2 + c_3v_3 = 0$

$c_1(1,0,1) + c_2(0,1,0) + c_3(-1,0,1) = (0,0,0)$

$$\left.\begin{aligned} c_1 - c_3 &= 0 \\ c_2 &= 0 \\ c_1 + c_3 &= 0 \end{aligned}\right\} \text{On solving, } c_1 = c_2 = c_3 = 0$$

which such that S is Linearly Independent.

And,

Since for each $v = (x_1, x_2, x_3) \in R^3$ then,

If, $v = c_1v_1 + c_2v_2 + c_3v_3$

$(x_1, x_2, x_3) = c_1(1,0,1) + c_2(0,1,0) + c_3(-1,0,1)$ —(i)

$(x_1, x_2, x_3) = (c_1 - c_3, c_2, c_1 + c_3)$

$x_1 = c_1 - c_3$
$x_2 = c_2$
$x_3 = c_1 + c_3$

On solving

$c_2 = x_2$

$c_1 = \dfrac{x_1 + x_3}{2}$

$c_3 = \dfrac{3x_2 - x_1}{2}$

putting in (i)

$$(x_1, x_2, x_3) = \left(\frac{x_1 + x_3}{2}\right)(1,0,1) + x_2(0,1,0) + \frac{3x_3 - x_1}{2}(-1,0,1)$$

Thus, every vector $V = (x_1, x_2, x_3)$ can be written as linear combination of V_1, V_2, V_3.

$\therefore$ S generates R^3 & Hence, S form a orthogonal basis of R^3.

Again,

$V_1 \cdot V_2 = (1,0,1) \cdot (1,1,0) = 0$
$V_2 \cdot V_3 = (0,1,0) \cdot (-1,0,1) = 0$
$V_1 \cdot V_3 = (1,0,1) \cdot (-1,0,1) = 0$

Here, vector V_1, V_2, V_3 are mutually perpendicular. So they are orthogonal.

let, $\{w_1, w_2, w_3\}$ be orthonormal vectors of $\{v_1, v_2, v_3\}$ then.

$$w_1 = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{(1,0,1)}{\sqrt{1^2+0^2+1^2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$w_2 = \frac{\vec{v_2}}{\|\vec{v_2}\|} = \frac{(0,1,0)}{\sqrt{0^2+1^2+0^2}} = (0,1,0)$$

$$w_3 = \frac{v_3}{\|\vec{v_3}\|} = \frac{(-1,0,1)}{\sqrt{(-1)^2+0^2+1^2}} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$\therefore \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $(0,1,0)$ & $\left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ are the required orthonormal vectors.

$||\bar{w}|| = 1$ (can check)

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#2. Find the orthogonal vectors & hence orthonormal vectors by using G.S. orthogonalization process from the vectors. $\{(1,0,1), (1,1,0), (1,1,1)\}$.

Hint: G.S. → u_1, u_2, u_3
Get → w_1, w_2, w_3.

Here,

$V_1 = (1,0,1)$, $V_2 = (1,1,0)$, $V_3 = (1,1,1)$

By G.S Orthogonalization process, we get u_1, u_2, u_3.

Now,

I. setting $\bar{u}_1 = \bar{v}_1 = (1,0,1)$.

$$\text{proj}_{u_1} \bar{v}_2 = \left(\frac{\bar{v}_2 \cdot \bar{u}_1}{||\bar{u}_1||^2}\right) u_1 = \frac{(1,1,0).(1,0,1)}{1^2+0^2+1^2}.(1,0,1) = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$$

II. $$u_2 = \bar{v}_2 - \text{proj}_{\bar{u}_1} \bar{v}_2 = (1,1,0) - \left(\frac{1}{2}, 0, \frac{1}{2}\right) = \left(\frac{1}{2}, 1, \frac{-1}{2}\right)$$

Clearly, $\bar{u}_1 \cdot \bar{u}_2 = 0$

Now,

III. $$\text{proj}_{\bar{u}_1} \bar{v}_3 = \left(\frac{\bar{v}_3 \cdot \bar{u}_1}{||\bar{u}_1||^2}\right) . \bar{u}_1 = \left(\frac{(1,1,1).(1,0,1)}{1^2+0^2+1^2}\right)(1,0,1) = (1,0,1)$$

$$\text{proj}_{\bar{u}_2} \bar{v}_3 = \left(\frac{\bar{v}_3 \cdot \bar{u}_2}{||\bar{u}_2||^2}\right) \bar{u}_2 = \left(\frac{(1,1,1).(1/2,1,-1/2)}{(1/2)^2+1^2+(-1/2)^2}\right) . \left(\frac{1}{2}, 1, \frac{-1}{2}\right)$$

$$= (1/3, 2/3, -1/3)$$

$$\bar{u}_3 = \bar{v}_3 - \text{proj}_{u_1} \bar{v}_3 - \text{proj}_{\bar{u}_2} \bar{v}_3$$

$$= (1,1,1) - (1,0,1) - (1/3, 2/3, -1/3)$$

$$\therefore \bar{u}_3 = (-1/3, 1/3, 1/3)$$

Here

$\bar{u}_2 \cdot \bar{u}_3 = (1/2, 1, -1/2) \cdot (-1/3, 1/3, 1/3) = 0$

and,

$\bar{u}_1 \cdot \bar{u}_3 = (1,0,1) \cdot (-1/3, 1/3, 1/3) = (-1/3 + 1/3) = 0$.

$\therefore (1,0,1), (1/2, 1, -1/2)$ & $(-1/3, 1/3, 1/3)$ are mutually orthogonal vectors in $\mathbb{R}^3$.

Now,

let w_1, w_2 & w_3 be orthonormal vector of u_1, u_2, u_3 respectively, then,

$$w_1 = \frac{\bar{u}_1}{\|\bar{u}_1\|^2} = \frac{(1,0,1)}{\sqrt{1^2+0^2+1^2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$w_2 = \frac{\bar{u}_2}{\|\bar{u}_2\|^2} = \frac{(1/2, 1, -1/2)}{\sqrt{(1/2)^2+(1)^2+(-1/2)^2}} = \left(\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{-1}{\sqrt{6}}\right)$$

$$w_3 = \frac{\bar{u}_3}{\|u_3\|} = \frac{(-1/3, 1/3, 1/3)}{\sqrt{(-1/3)^2+(1/3)^2+(1/3)^2}} = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$\therefore$ $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{-1}{\sqrt{6}}\right)$ and $\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

are the required orthonormal vectors.

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Theorem:

A set of non-zero orthogonal vectors are linearly independent.

proof:

let V be a vector space over the field F.

let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of n vectors in V.

Also, $v_i \cdot v_j = 0 \quad \forall \; i \neq j$. ($\because$ orthogonal vectors)

Let $c_1, c_2, \ldots, c_n$ be scalars, such that,

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$$

Taking dot product by v_1 on both sides.

$$(c_1 v_1 + c_2 v_2 + \ldots + c_n v_n) \cdot v_1 = 0 \cdot v_1$$

or, $c_1 (v_1 \cdot v_1) + c_2 (v_2 \cdot v_1) + \ldots + c_n \cdot (v_n \cdot v_1) = 0$

or, $c_1 \|v_1\|^2 + c_2 \cdot 0 + \ldots + c_n \cdot 0 = 0$

or, $c_1 \|v_1\|^2 = 0$

Since, $v_1 \neq 0$, $\therefore \|v_1\| \neq 0$

$\therefore \|v_1\|^2 \neq 0$

$\Rightarrow c_1 = 0$

Similarly,

$c_2 = 0, \ldots \; c_n = 0$

$\therefore \{v_1, v_2, \ldots, v_n\}$ are linearly independent.

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Matrix

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Defination.

Order of Matrix / Shape / Dimension.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times3}$$ $n\times m \Rightarrow n$ rows followed by m column.

A $m\times n$ matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$$

$A = (a_{ij})$ $i = 1, 2, \ldots, m$
$j = 1, 2, \ldots, n.$

Row Matrix.

Column Matrix

Square matrix $\Rightarrow m = n.$

Diagonal matrix $\Rightarrow a_{ij} = 0 \;\forall i \neq j$ (Non-diagonal element $= 0$)

- A Square matrix $A = (a_{ij})$ is called diagonal matrix if $a_{ij} = 0 \;\forall\; i \neq j.$

Scalar matrix

- A square matrix $A = (a_{ij})$ is Scalar matrix if.

$$a_{ij} = \begin{cases} 0 & \text{if } i \neq j \text{ (Non-diagonal element)} \\ K & \text{if } i = j \text{ (Diagonal element)} \end{cases}$$

Unit Matrix.

A square matrix $A = (a_{ij})$ is called unit if

$$a_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

A unit matrix of order n is denoted by I_n.

Zero Matrix.

Upper Triangular Matrix (UTM)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \left.\begin{matrix} a_{21} \\ a_{31} \\ a_{32} \end{matrix}\right\} = 0$$

Ily, Lower Triangular Matrix (LTM)

$a_{12} = a_{13} = a_{23} = 0$.

$a_{ij} = 0 \ \forall \ i<j$

A sq. matrix $A = (a_{ij})$ is UTM if.

$a_{ij} = 0 \ \forall \ i>j$

Symmetric Matrix

A sq. matrix $A:(a_{ij})$ is Symmetric if

$a_{ij} = a_{ji} \ \forall \ i \& j$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

Skew-Symmetric Matrix.

if $a_{ij} = -a_{ji} \ \forall i \& j$

$$\begin{pmatrix} +0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{pmatrix}$$

All diagonal elements are zero in skew-Sym. Matrix.

$a_{11} = -a_{11}$

$\therefore a_{11} = 0$

$a_{22} = 0$

$a_{33} = 0$

4x4 Skew-Sym. Matrix.

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{bmatrix}$$

Addition of two Matrix.

If $A = (a_{ij})_{m \times n}$ & $B = (b_{ij})_{m \times n}$, Then,

$A+B = (a_{ij} + b_{ij})_{m \times n}$.

Law of Addition.

If A, B, C be the three matrices of order $m \times n$, Then.

(i). $A+B = B+A$ (Commutative).

(ii) $A+(B+C) = (A+B)+C$ (Associative)

(iii) $A+0 = A$. (0 is additive Identity)

(iv). $A+(-A) = 0$ (Existence of Additive Inv.)

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(11). $A+(B+C) = (A+B)+C$

proof:

let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$, $C = (c_{ij})_{m\times n}$.

we prove

$$\underbrace{A + \underbrace{(B+C)}_{m\times n}}_{m\times n} = \underbrace{\underbrace{(A+B)}_{m\times n}+C}_{m\times n}.$$

Clearly, both sides are of same order $m\times n$.

$$\therefore A+(B+C) = (a_{ij}) + (b_{ij}+c_{ij})$$
$$= (a_{ij} + (b_{ij}+c_{ij}))$$
$$= ((a_{ij}+b_{ij})+c_{ij}) \quad \text{(Associativity of Real Number)}$$
$$\therefore A+(B+C) = (A+B)+C \,//$$

OR

$(ij)^{th}$ element of $A+(B+C)$

$$= a_{ij} + (b_{ij}+c_{ij})$$
$$= (a_{ij}+b_{ij}) + c_{ij}$$
$$= (ij)^{th} \text{ element of } (A+B) + (ij)^{th} \text{ element of } C$$
$$= (ij)^{th} \text{ element of } (A+B+C) \,//$$

Transpose of a Matrix.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \; A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

The transpose of $A = (a_{ij})$ is new matrix $A^T = (a_{ji})$

Prove that:- $(A+B)^T = A^T + B^T$, where A, B are matrices of same order.

OR: The transpose of sum of 2 M = Transpose of sum of each matrices.

proof!

let $A = (a_{ij})_{m\times n}$ & $B = (b_{ij})_{m\times n}$

Then

$A+B$ is of order $m\times n$

$\therefore (A+B)^T$ is of order $n\times m$

Again, A^T is of order $n\times m$ and B^T is of order $n\times m$.

$\therefore A^T+B^T$ is of order $n\times m$.

Now,

$(i,j)^{th}$ element of $(A+B)^T = (j,i)^{th}$ element of $A+B$.

$= (j,i)^{th}$ element of $A + (j,i)^{th}$ element of B.

$= (i,j)^{th}$ element of $A^T + (i,j)^{th}$ element of B^T

$= (i,j)^{th}$ element of (A^T+B^T) $\forall$ i & j.

$\therefore (A+B)^T = A^T + B^T$

Similarly,

HW { (i) $(A^T)^T = A$.

(ii) $(KA)^T = KA^T$

Some Defn

1. Orthogonal Matrix.

. A square matrix $A = (a_{ij})$ is called orthogonal if $A^TA = AA^T = I$

Example:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ is a orthogonal matrix.

Now,

$$AA^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \times \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta.\cos\theta \\ \sin\theta.\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Similarly, $A^TA = I$

$\therefore$ A is orthogonal.

2. Idempotent matrix & Involutory Matrix.

A square matrix $A = (a_{ij})$ is called Idem

(i) Idempotent if $A^2 = A$

(ii) Involutory if $A^2 = I$

Show that the matrices A & B are idempotent.

(a) $A = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}$, (b) $B = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{pmatrix}$

Soln,

$$B = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{pmatrix}$$

Matrix B is Idempotent if $B^2 = B$.

So,

$$B^2 = B \times B = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{pmatrix} \times \begin{pmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times2+(-2)\times(-2)+2\times(-4) & (-1)\times(-1)+(-1)\times3+(-1)\times4 & 1-2-3 \\ -2\times2 & & \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{pmatrix} = B.$$

$\therefore B^2 = B\times B$ //

Show that the following matrices C & D are involutory.

(a) $C = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & -2 & -1 \end{pmatrix}$ (b) $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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3. Nilpotent Matrix

A square matrix $A = (a_{ij})$ is called nilpotent matrix of index or order r if

$A \neq 0,\ A^2 \neq 0, \ldots, A^{r-1} \neq 0$, but $A^r = 0$

Example :

Show that the following matrices A are nil potent of index 3.

(a)
$$A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$$

Theorem.

If A & B be two matrices, then $(AB)^T = B^T A^T$

OR: Transpose of product of two matrices is equal to transpose of their product taken in reverse order.

proof:

let $A = (a_{ij})_{m \times n}$ & $B = (b_{ij})_{n \times p}$ then,

AB is order $m \times p$.

$\therefore (AB)^T$ is order $p \times m$.

Again,

let $A^T = (C_{ij})_{n \times m}$ & $B^T = (d_{ij})_{p \times n}$

where, $C_{ij} = a_{ji}$ & $d_{ij} = b_{ji}$

$\therefore B^T A^T$ is order $p \times m$

$\therefore (AB)^T$ & $B^T A^T$ are of same order.

Now,

$(i,j)^{th}$ element of $(AB)^T$ = $(j,i)^{th}$ element of AB

$= a_{j1} b_{1i} + a_{j2} b_{2i} + \ldots + a_{jn} b_{ni}$

$= C_{1j} d_{i1} + C_{2j} d_{i2} + \ldots + C_{nj} d_{in}$

$= d_{i1} C_{1j} + d_{i2} C_{2j} + \ldots + d_{in} C_{nj}$

$= (i,j)^{th}$ element of $B^T A^T$

Viz:
$A_{2\times 2} \times B_{2\times 3} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$
$C_{12} = a_{11} b_{12} + a_{12} b_{22}$
$C_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j}$
$C_{ji} = a_{j1} b_{1i} + a_{j2} b_{2i}$

$\therefore (i,j)^{th}$ element of $(AB)^T = (i,j)^{th}$ element of $B^T A^T$, Hence $(AB)^T = B^T A^T$ //

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Inverse of a matrix.

The two square matrix A & B are said to be inverse of each, then if

$AB = BA = I$

∴ In this case, we write.

$A^{-1} = B$ & $B^{-1} = A$.

formula.

If $|A| \neq 0$, then $A^{-1} = \frac{Adj(A)}{|A|}$.

Example:-

Find adjoint & inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 3 & 2 \end{pmatrix}$

Soln,

Cofactor of :

$a_{11} = \begin{vmatrix} 5 & 6 \\ 3 & 2 \end{vmatrix} = -8$ $\quad a_{21} = -\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 5$ $\quad a_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3$

$a_{12} = -\begin{vmatrix} 4 & 6 \\ -1 & 2 \end{vmatrix} = 14$ $\quad a_{22} = \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} = 5$ $\quad a_{32} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = +6$

$a_{13} = \begin{vmatrix} 4 & 5 \\ -1 & 3 \end{vmatrix} = 17$ $\quad a_{23} = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = -5$ $\quad a_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3$

Now, cofactor of matrix $A = \begin{bmatrix} -8 & 14 & 17 \\ 5 & 5 & -5 \\ -3 & +6 & -3 \end{bmatrix}$

and,

Adj. of A = (cofactor of matrix $A)^T = \begin{bmatrix} -8 & 5 & -3 \\ 14 & 5 & 6 \\ 17 & -5 & -3 \end{bmatrix}$

$$|A| = \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

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Now,

$\because |A| = 1\begin{vmatrix} 5 & 6 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ -1 & 3 \end{vmatrix}$

$= -8 - 28 + 51$

$= 15$

Inverse of matrix $A = \frac{1}{|A|} Adj(A)$

$$= \frac{1}{15}\begin{bmatrix} -8 & 5 & -3 \\ 14 & 5 & 6 \\ 17 & -5 & -3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{15}\begin{bmatrix} -8 & 5 & -3 \\ 14 & 5 & 6 \\ 17 & -5 & -3 \end{bmatrix}$$

Some Facts on Inverse of a Matrix

Theorem:

The Inverse of a matrix if it exist is unique.

Proof:

If possible suppose there are two inverse B & C of the square matrix A.

$\therefore AB = BA = I$ —— (i)

$AC = CA = I$ —— (ii)

Our want,

$B = C$

Now,

$B = BI$ $\quad (\because A = AI)$

$= B(AC)$ $\quad$ (From (ii))

$= (BA)C$ $\quad$ (Associtivity)

$= IC$

$\therefore B = C$ $\quad (IA = AI = A)$

Hence, unique.

$|A| \neq 0$ $|AB| = |A||B|$
$|B| \neq 0$

Theorem:

If A & B be two square invertible matrics of the same order. then, $(AB)^{-1} = B^{-1}A^{-1}$

i.e. The inverse of product of two square matrices is equl to the product of inverse taken in reverse order.

Proof:

Since, A & B are invertible $\therefore |A| \neq 0$, $|B| \neq 0$

Also, $|AB| = |A||B| \neq 0$

$\therefore$ AB is non singular, so, $(AB)^{-1}$ exist.

Now,

$$(AB)(B^{-1}A^{-1}) = (ABB^{-1})A^{-1} \quad \text{(Associativity)}$$
$$= (AI)A^{-1} \quad (\because BB^{-1} = I)$$
$$= AA^{-1} \quad (AI = A)$$
$$= I \quad \text{---} (*)$$

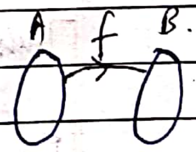
Again,

$$(B^{-1}A^{-1})AB = (B^{-1}A^{-1}A)B \quad \text{(Associativity)}$$
$$= (B^{-1}I)B \quad (\because AA^{-1} = I)$$
$$= B^{-1}B \quad (BI = B)$$
$$= I$$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$ Hence, proved.

Fact:

A function from a set A to the set B is a rule that assigns each element of A to the unique element of B.



Read Basic about:

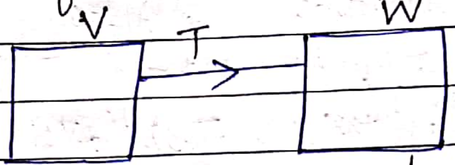
- function
- Injective function → One to One : 1 element → 1 image. $f(x) = x^2$
- Bijective function
- Surjective function → All the image should be mapped by the element
- Domain, Range.

Image of v under function/function :
↳ $TV = T(v)$

Linear Transformation.

Transformation.

let V & W be two vector spaces over the field F. The transformation $T.V \rightarrow W$ is a rule that assigns each element of V to the unique element of W.



Here, V is called Domain of T
W is called Co-domain of T

The set of all element in W, such that
$T(V) = \{T(v) : v \in V\} \subseteq W$ is called Image set

linear Transformation,

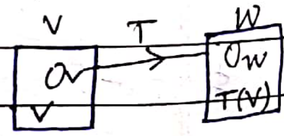
let V & W be two vector space over the field F. and, $T.V \rightarrow W$ is a transformation or mapping, then T is called linear transformation if

(i) For each $v_1, v_2 \in V$, $T(v_1+v_2) = TV_1 + TV_2 = T(V_1)+T(V_2)$

(ii) For each, $v \in V$, $\alpha \in F$, $T(\alpha v) = \alpha T(v)$

Special case :-

From (ii), if $\alpha = 0$, then, $T(0) = 0$
i.e. $T(0_V) = 0_W$



Thus T maps 0 element of V to 0 element of W.

Example: let $T: R^2 \rightarrow R^2$ be a mapping defined by $T(x,y) = (x+y, y)$. Is T linear?

Soln,

let $v_1, v_2 \in R^2$ (Domain) such that
$v_1 = (x_1, y_2)$, $v_2 = (x_2, y_2)$

Now,

(i) For each $V_1, V_2 \in R^2$

$T(V_1+V_2) = T((x_1,y_1) + (x_2,y_2))$

$= T(x_1+x_2, y_1+y_2)$

$= (x_1+x_2+y_1+y_2, y_1+y_2)$ $\because T(x,y) = (x+y, y)$

$= (x_1+y_1, y_1) + (x_2+y_2, y_2)$

$= T(x_1,y_1) + T(x_2,y_2)$

$\therefore T(V_1+V_2) = T(V_1) + T(V_2)$

(ii) Let $V \in R^2$, $\alpha \in R$ (field) where $V = (x,y)$ then,

$T(\alpha V) = T(\alpha(x,y))$

$= T(\alpha x, \alpha y)$

$= (\alpha x + \alpha y, \alpha y)$

$= \alpha(x+y, y)$

$= \alpha T(x,y)$

$\therefore T(\alpha V) = \alpha T(V)$

Therefore T is linear.

Example: Let $T: \overset{V}{R} \rightarrow \overset{W}{R}$ by, $T(x) = x+6$. Is T linear?

Soln,

Let $V_1, V_2 \in R$ (Domain) such that.

NOT

put, $x=0$

$T(0)=6$ which is

not true for linear function.

$(\because T(0)=0)$.

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Equivalent Defination of linear Transformation.

Theorem: Let V & W be two vector space over field F. & $T: V \to W$ be a mapping.
If T is linear, then.
$$T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2) \quad \forall \ c_1 c_2 \in F, \ v_1 v_2 \in V.$$

Proof:
Since T linear
$\therefore T(c_1v_1 + c_2v_2) = T(c_1v_1) + T(c_2v_2)$ [$\because$ First condition of LT]
$= c_1T(v_1) + c_2T(v_2)$ [$\because$ Second condition of LT]

proved

Note:
Above theorem can be extended to n-vectors, say,
$$T(c_1v_1 + c_2v_2 + \dots + c_nv_n) = c_1T(v_1) + c_2T(v_2) + \dots + c_nT(v_n)$$

Example:
Let: $R^2 \to R^2$ defined by $T(0,1) = (2,1)$, $T(1,4) = (0,-2)$
If T is linear, find $T(a,b)$. Also find $T(5,6)$.

Soln,
We express (a,b) as the linear combination of $(0,1)$ & $(1,4)$
Let $(a,b) = c_1(0,1) + c_2(1,4)$ — ①
or $(a,b) = (0,c_1) + (c_2, 4c_2)$
or $(a,b) = (c_2, c_1 + 4c_2)$
$\therefore c_2 = a$
$c_1 + 4c_2 = b.$
$\therefore c_1 = b - 4a$
Substituting c_1 & c_2 in ①
$(a,b) = (b-4a)(0,1) + a(1,4)$ — ②

Applying T on both sides.
$$T(a,b) = T(\{(b-4a)(0,1) + a(1,4)\})$$

$= (b-4a)T(0,1) + aT(1,4)$ $\{\because T \text{ is linear}\}$

$= (b-4a)(2,1) + a(0,-2)$

$= (2b-8a, b-4a) + (0,-2a)$

$\therefore T(a,b) = (2b-8a, b-6a)$

For, $T(5,6)$, put $a=5$, $b=6$.

$T(5,6) = (2\times6-8\times5, 6-6\times5)$

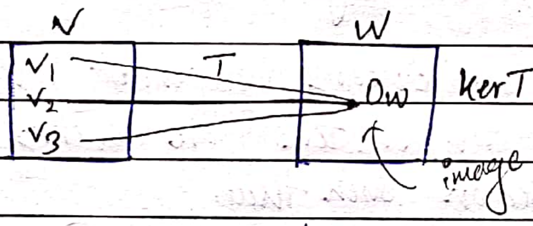
$\therefore T(5,6) = (-28, -24)$

Kernal and Image of a Linear Transformation.

Let V & W be two V.S. and $T: V \to W$ be a linear transformation. Then.

① The kernal of T is denoted by $\ker(T)$ and is defined by,

$\ker(T) = \{v \in V : T(v) = 0_W\}$



$\ker T = \{v_1, v_2, v_3\}$

$f(x) = \frac{x-1}{2}$: $f: \mathbb{R} \to \mathbb{R}$.

$f(1) = 0$

$\ker f = \{1\}$.

Note: Is $\ker(T)$ subspace of V? $\Rightarrow$ Yes.

proof:

Let $v_1, v_2 \in \ker T$.

So by defn $T(v_1) = 0_W$

& $T(v_2) = 0_W$

So for any scalars c_1, c_2,

$T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$

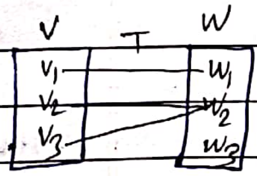
$= c_1 0 + c_2 0$

$= 0$

$\therefore$ By definition, $c_1v_1 + c_2v_2 \in \ker T$,

$\therefore \ker T$ is a subspace of V.

$Im(T) = \{w_1, w_2\}$



② The image of T is denoted by $Im(T)$ and is defined by,

$Im(T) = \{w \in W : T(v) = w \text{ for some } v\}$

Is $Im(T)$ subspace of W? $\rightarrow$ Yes.

Let, $w_1, w_2 \in Im\,T$,

Then $\exists\, v_1, v_2 \in V$ such that

$$T(v_1) = w_1 \;\&\; T(v_2) = w_2.$$

So, for scalar c_1, c_2.

$$T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2) \qquad \{\because T \text{ is linear}\}$$

$$= c_1w_1 + c_2w_2$$

$\therefore$ By definition, $c_1w_1 + c_2w_2 \in Im\,T$.

$\therefore$ $Im(T)$ is a subspace of W.

Theorem:

Let V & W be the V.S. over the field F &. $T: V \rightarrow W$ be a Linear Transformation. If $v_1, v_2, \dots v_n$ be n linear independent vectors in V and $\ker T = \{0\}$, Then, $\{T(v_1), T(v_2), \dots T(v_n)\}$ are linearly independent in W.

Since $\{v_1, v_2, \dots v_n\}$ are linearly independent in V.

We show that $\{T(v_1), T(v_2) \dots T(v_n)\}$ are linearly independent in W.

Let $c_1, c_2, \dots c_n$ be scalars such that,

$$c_1T(v_1) + c_2T(v_2) + \dots + c_nT(v_n) = 0_W \quad \text{--- (1)}$$

Using linearity of T,

$$T(c_1v_1 + c_2v_2 + \dots + c_nv_n) = 0_W$$

$\therefore$ $c_1v_1 + c_2v_2 + \dots + c_nv_n \in \ker T = \{0\}$.

$\therefore$ $c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$

But $\{v_1, v_2, \dots v_n\}$ are linearly independent.

$\therefore$ $c_1 = 0,\ c_2 = 0,\ \dots\ c_n = 0$.

Then,

$$c_1T(v_1) + c_2T(v_2) + \dots + c_nT(v_n) = 0$$

$$\Rightarrow c_1 = 0,\ c_2 = 0,\ \dots\ c_n = 0$$

$\therefore$ The set $\{T(v_1), T(v_2), \dots T(v_n)\}$ is linearly independent in W.

Algebra of Linear Transformation.

Let V & W be two vector spaces over the field F. and suppose, $T_1 : V \to W$ & $T_2 : V \to W$, be two linear transformations & α be the scalars Then,

(i) The sum is

$$(T_1+T_2)(v) = T_1(v) + T_2(v)$$

(ii) The scalar multiple

$$(\alpha T_1)(v) = \alpha T_1(v)$$

Example:

Let, $T_1 : R^2 \to R^2$ & $T_2 : R^2 \to R^2$ defined by

$T_1(x,y) = (x+y, y)$, $T_2(x,y) = (x,0)$. find $2T_1 + 3T_2$

Let $v = (x,y) \in R^2$ Then,

$$\begin{aligned}(2T_1+3T_2)(v) &= 2T_1(v) + 3T_2(v) \quad \text{(By dfn of sum)}\\ &= 2T_1(x,y) + 3T_2(x,y).\\ &= 2(x+y, y) + 3(x,0)\\ &= (2x+2y, 2y) + (3x, 0)\\ &= (2x+2y+3x, 2y+0)\end{aligned}$$

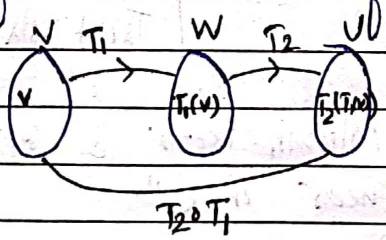
$\therefore (2T_1+3T_2)(v) = (5x+2y, 2y)$ ← formula/rule.

$(2T_1+3T_2)(x,y)$

if $(2T_1+3T_2)(5,6) = ?$ then, $(2T_1+3T_2)(5,6) = (5\times5+2\times6, 2\times6)$

$= (37, 12)$ //

Composite of two linear transformation.



Let $T_1 : V \to W$ & $T_2 : W \to U$ be two Linear Transformation. Then their compositions or composite transformation is defined by $(T_2 \circ T_1)(v) = T_2(T_1(v))$.

Example:

$T_1 : (x,y) = (x+y, y)$, $T_2(x,y) = (x,0)$. find $T_1 \circ T_2$ & $T_2 \circ T_1$

Is, $T_1 \circ T_2 = T_2 \circ T_1$?

Now,

$v = (x,y)$

$(T_1 \circ T_2)(v) = T_1(T_2(v))$ and, $(T_2 \circ T_1)(v) = T_2(T_1(v))$

$= T_1(T_2(x,y))$ $= T_2(T_1(x,y))$

$= T_1(x,0)$ $= T_2(x+y, y)$

$= (x,0)$ $= (x+y, 0)$

$\therefore (T_1 \circ T_2)(v) \neq (T_2 \circ T_1)(v)$

Linear map Associated with a matrix.

Let R^n and R^m be two vector spaces with order basis.

$B_{R^n} = \{v_1, v_2, \dots, v_n\}$

$B_{R^m} = \{w_1, w_2, \dots, w_n\}$

Let us consider a matrix $A = (a_{ij})_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$

Let $T : R^n \to R^m$ and let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in R^n$

T is defined by,

$T(X) = AX \in R^m$

→ Linear map associated with A.

$T_A(X) = AX$

Clearly AX is $m \times 1$ column vector so, $AX \in R^m$

Clearly T is linear on R^n.

Here, the Transformation T is called linear map associated with matrix A with respect to the basis B_{R^n} & B_{R^m},

No info in Qs, → standard Basis $(1,0)$ & $(0,1)$
$e_1 \quad e_2$

Example:

Let $A = \begin{pmatrix} 2 & 5 \\ 6 & 8 \end{pmatrix}_{2\times2}$ and $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$. Find the linear transformation $T_A : R^2 \to R^2$ corresponding to the matrix A.

Soln,

The transformation $T_A : R^2 \to R^2$ is,

$$T_A(X) = AX$$
$$= \begin{pmatrix} 2 & 5 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\therefore T_A(X) = \begin{pmatrix} 2x_1 + 5x_2 \\ 6x_1 + 8x_2 \end{pmatrix}$$

$A = \begin{pmatrix} 2 & 5 & 6 \\ 6 & 8 & 2 \end{pmatrix}_{2\times3}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in R^3$ $\quad T_A : R^3 \to R^2$?

Soln,

The transformation $T_A : R^3 \to R^2$ is

$$T_A(X) = AX$$
$$= \begin{pmatrix} 2 & 5 & 6 \\ 6 & 8 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
$$\therefore T_A(X) = \begin{pmatrix} 2x_1 + 5x_2 + 6x_3 \\ 6x_1 + 8x_2 + 2x_3 \end{pmatrix}$$

Example:

Find the matrix represented by linear map $T : R^2 \to R^2$ defined by $T(x,y) = (x, 2y)$ with respect to the standard basis.
$(1,0) \ (0,1)$

Soln,

Let $v_1 = (1,0)$ and $v_2 = (0,1)$

Now,
$$T(v_1) = T(1,0) = (1,0) = 1(1,0) + 0(0,1) = 1e_1 + 0e_2$$
$$T(v_2) = T(0,1) = (0,2) = 0(1,0) + 2(0,1) = 0e_1 + 2e_2$$

The matrix associated with the linear map

Date
Page

T = transpose of coefficient matrix

$$= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$ //

Verify:

$X = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad T(X) = AX = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2y \end{pmatrix}$

Example:
let, $T: R^2 \to R^2$ defined by, $T(x,y) = (x+y, 2y)$ relative to the basis of domain $(1,0)$ & $(1,-1)$

Soln,
let $v_1 = (1,0)$ & $v_2 = (1,-1)$ be the basis of Domain R^2.

$T(v_1) = T(1,2) = (1+2, 2\times 2) = (3,4)$
$= 3(1,0) + 4(0,1)$
$= 3e_1 + 4e_2$

Where e_1 & e_2 are basis vector of range R^2.

$T(v_2) = T(1,-1) = (1-1, 2\times(-1)) = (0,-2)$
$= 0(1,0) - 2(0,1)$
$= 0e_1 - 2e_2$.

$\therefore$ Coefficient matrix $= \begin{pmatrix} 3 & 4 \\ 0 & -2 \end{pmatrix}$

$\therefore$ Matrix associated with T = Transpose of Coefficient matrix

$$= \begin{pmatrix} 3 & 4 \\ 0 & -2 \end{pmatrix}^T$$

$$= \begin{pmatrix} 3 & 0 \\ 4 & -2 \end{pmatrix}$$

3

Spetral Theorems

Eigen Value and Eigen Vector

Motivation.

Consider a matrix $A = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$

let $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then,

$$AV = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda V$$

No such λ such that $AV = \lambda V$

Similarly,

If we take $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, Then

$$AV = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Again, we take $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$AV = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\therefore AV = \lambda V$, where $\lambda = 5$

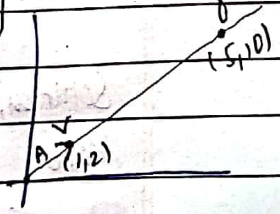
Thus for a square matrix A, we get

$AV = \lambda V$, $V \neq 0$,

Such vector V is called eigen vector of the matrix and λ is called eigen value.

Defination:

Let A is $n \times n$ square matrix. If there exist a non zero vector V such that $AV = \lambda V$, Then V is called eigen vector of A corresponding to the eigen value λ.



From defination of eigen vector V of matrix A,

$AV = \lambda V$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$$

$(a_{11}-\lambda)x_1 + a_{12}x_2 = 0$

$a_{21}x_1 + (a_{22}-\lambda)x_2 = 0$

$$\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Alt. Method to V's.

$AV = \lambda V$

or, $AV = \lambda IV$

or $AV - \lambda IV = 0$

or, $(A-\lambda I)V = 0$

or $$\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

or, $(A-\lambda I)V = 0$

$\therefore \boxed{(A-\lambda I)V = 0}$

Then,

(i) The matrix $A-\lambda I$ is called characteristic matrix of A.

(ii) The determinant $|A-\lambda I|$ is called characteristic polynomial.

(iii) The equation $|A-\lambda I|=0$ is called charactestic equation.

(iv) The solution of characteristic equation or root are called eigen value or characteristic roots.

(v) The set of all eigen value of the matrix A is called Spectrum.

Example:

Let $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$, Find : (0) Characteristic Matrix.

(i). Characteristic polynomial

(ii). Characteristic Equation

(iii). Characteristic roots or values or eigen values

(iv). Multiplicity of Characteristic value.

(v). Eigen vector (Characteristic vector)

Solution,

Here, $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(i). Characteristic Matrix.

$$A-\lambda I = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix}$$

(ii) Characteristic polynomial,

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) - (1)(-1)$$

$$= 3-3\lambda-\lambda+\lambda^2+1.$$

$\therefore |A-\lambda I| = \lambda^2-4\lambda+4.$

(iii) Characteristic roots

$|A-\lambda I| = 0$

or $(\lambda^2-4\lambda+4) = 0$

or, $(\lambda-2)^2 = 0$

$\therefore \lambda = 2, 2.$

(iv) Clearly, $\lambda = 2$ i.e. repeats two times.

$\therefore$ Multiplicity of λ is. 2.

Finally, to find eigen vectors,

(v). Let $V = \begin{pmatrix} x \\ y \end{pmatrix}$ be the eigen vector of the matrix A, Then,

$(A-\lambda I)V = 0$

$$\begin{pmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

Here, $\lambda = 2$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

$$\begin{pmatrix} x-y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or, $x-y = 0$

$\therefore y = x$

Let $x = k$,

Then $y = k$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k \\ k \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore$ Eigen vector $= k(1,1)$ where k is non-zero scalar.
i.e. $k \neq 0$.

Find eigen value & eigen vector of the matrix $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

Here,

$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A - \lambda I = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$

then,

$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$

$\therefore |A - \lambda I| = (3-\lambda)^2 - 1$

then, for eigen value,

$|A - \lambda I| = 0$

or, $(3-\lambda)^2 - 1^2 = 0$

or, $((3-\lambda)+1)((3-\lambda)-1) = 0$

or, $(2-\lambda)(4-\lambda) = 0$

$\therefore \lambda = 2, 4$

To find eigen vector let $V = \begin{pmatrix} x \\ y \end{pmatrix}$ be eigen vector of matrix A.

$(A - \lambda I)V = 0$

$\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

Here,

$\lambda = 2$

$\begin{pmatrix} 3-2 & 1 \\ 1 & 3-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\begin{pmatrix} x+y \\ x+y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore x + y = 0 \Rightarrow y = -x$

Let $x = k$, $\therefore y = -k$

$V = (x, y) = (k, -k) = k(1, -1)$

EV of A corresponding to EV $\lambda = 2$ is $k(1, -1)$, $k \neq 0$

For $\lambda = 4$.

$\begin{pmatrix} 3-4 & 1 \\ 1 & 3-4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\begin{pmatrix} -x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore -x + y = 0$ $\qquad x - y = 0$

$y = x$ $\qquad x = y$

Let $x = k$, $\therefore y = k$.

$V = (x, y) = (k, k) = k(1, 1)$

EV of A corresponding to EV $\lambda = 4$ is $k(1, 1)$, $k \neq 0$.

Find characteristic values and vector of $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix}$

soln,

Characteristic Eqn is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 2+\lambda & 1 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$

Expanding along R_1

$(2-\lambda)\begin{vmatrix} -1-\lambda & 3 \\ 1 & 1-\lambda \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 \\ 1 & 1-\lambda \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 \\ -1-\lambda & 3 \end{vmatrix} = 0$

or, $(2-\lambda)\left[-(1+\lambda)(1-\lambda) - 3\right] = 0$

or, $(2-\lambda)\left[-(1-\lambda)^2 - 3\right] = 0$

Either $(2-\lambda) = 0$ or, $\left[-(1-\lambda)^2 - 3\right] = 0$

$\lambda = 2$ or $\lambda^2 - 4 = 0$

or, $\lambda = \pm 2$ $\therefore \lambda = +2, -2$.

$\therefore \lambda = 2, 2, -2$

Let, $V = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigen vector then,

$(A - \lambda I)V = 0$

$\begin{pmatrix} 2-\lambda & 1 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ — (X)

Case I: If $\lambda = 2$, then (X) give.

$\begin{pmatrix} 0 & 1 & 2 \\ 0 & -3 & 3 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\left.\begin{aligned} y + 2z &= 0 \\ -3y + 3z &= 0 \\ y - z &= 0 \end{aligned}\right\}$ $\begin{aligned} y + 2z &= 0 \quad \text{— (i)} \\ y - z &= 0 \quad \text{— (ii)} \end{aligned}$

Clearly, $y=z$ on ① gives,

$3z=0 \Rightarrow z=0$. Then, $y=0$.

Take, $x=k$, $y=0$, $z=0$

$\therefore (x,y,z) = (k,0,0) = k(1,0,0)$, $k \neq 0$

Case II: If $\lambda=-2$, then (*) gives,

$$\begin{pmatrix} 4 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 4x+y+2z &= 0 \\ y+3z &= 0 \\ y+3z &= 0 \end{aligned}\right\} \quad \begin{aligned} 4x+y+2z &= 0 \\ y+3z &= 0 \\ \Rightarrow y &= -3z. \end{aligned}$$

* Note: We can let any $x,y,z=k$

Then, $4x-3z+2z=0$

$\therefore 4x-z=0$

Let $x=k$, Then, $z=4k$

$y=-12k$.

$\therefore V=(x,y,z) = (k,-12k,4k) = k(1,-12,4)$, $k \neq 0$.

Cayley Hamilton Theorem.

Every square matrix A satisfies its characteristic equation. We shall verify by an example

Example: let $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$, verify Cayley Hamilton Theorem. Also find inverse by using this theorem.

Soln,

Now, $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then characteristic Eqn is

$|A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$

or $(3-\lambda)(1-\lambda) + 1 = 0$

or $3 - 3\lambda - \lambda + \lambda^2 + 1 = 0$

$\therefore \lambda^2 - 4\lambda + 4 = 0$

To verify Cayley Hamilton Theorem, we show, The square matrix A satisfies its characteristic equation, $|A - \lambda I| = 0$.

i.e. A satisfies $\lambda^2 - 4\lambda + 4 = 0$

we show, $A^2 - 4A + 4I = 0$

LHS $= A^2 - 4A + 4I$

$= \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} - 4\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3\times3 + 1\times(-1) & 3\times(-1) + (-1)\times1 \\ 1\times3 + 1\times1 & 1\times(-1) + (1)\times1 \end{pmatrix} - \begin{pmatrix} 12 & -4 \\ 4 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

$= \begin{pmatrix} 8 & -4 \\ 4 & 0 \end{pmatrix} - \begin{pmatrix} 12 & -4 \\ 4 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

$= \begin{pmatrix} 12 & -4 \\ 4 & 4 \end{pmatrix} - \begin{pmatrix} 12 & -4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Hence Cayley Hamilton theorem is verified.

To find A^{-1}: We know, $A^2 - 4A + 4I = 0$.

Here, $|A| = \begin{vmatrix} 3 & -1 \\ 1 & +1 \end{vmatrix} = 4 \neq 0$, $|A| \neq 0$. so, A^{-1} exist.

$$A^2 - 4A + 4I = 0$$

or, $A^2 - 4A = -4I$

or $A^2 - 4A = -4AA^{-1}$ $\quad (\because I = AA^{-1})$

or, $A(A-4) = -4AA^{-1}$

or $A(A - 4I + 4A^{-1}) = 0$

$A \neq 0$, so, $A - 4I + 4A^{-1} = 0$

$\therefore 4A^{-1} = 4I - A$.

$$A^{-1} = \frac{1}{4}(4I - A)$$

$$\therefore A^{-1} = \frac{1}{4}\left[\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\right] = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$$

Verify Cayley Hamilton theorem for $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ & find A^{-1}.

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

or $(1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 0\begin{vmatrix} 1 & -1 \\ 0 & 1-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1-\lambda & 0 \end{vmatrix} = 0$

or $(1-\lambda)(1-\lambda)^2 + 1 + (1-\lambda) = 0$

or $(1-\lambda)^3 + (1-\lambda) + 1 = 0$.

or, $(1-\lambda)(1 - 2\lambda + \lambda^2) + 2 - \lambda = 0$

or $1 - 2\lambda + \lambda^2 - \lambda + 2\lambda^2 - \lambda^3 + 2 - \lambda = 0$

or, $-\lambda^3 + 3\lambda^2 - 4\lambda + 2 = 0$

or, $\lambda^3 - 3\lambda^2 + 4\lambda - 2 = 0$

We prove, $A^3 - 3A^2 + 4A - 2I = 0$.

$$A^2 = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -2 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 0 & 2 & -2 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \Bigg($$

$$LHS = A^3 - 3A^2 + 4A - 2I$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Verified.}$$

Find the eigen value & eigen vector of $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

Quadratic form & Matrix.

Let A be a $n \times n$ symmetric matrix with real entries. A quadratic form is a real valued function on R^n.

i.e. $Q : R^n \longrightarrow R$ defined by,

$$Q(X) = X^T A X, \quad \text{where } X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in R^n.$$

$$= \underbrace{X^T_{1\times n} \boxed{A_{n\times n} X_{n\times 1}}_{n\times 1}}_{1\times 1}, \text{ which is real number.}$$

Example:

Let $Q : R^2 \longrightarrow R^1$ and $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Find $Q(x)$

Soln,

$$Q(x) = X^T I X = (x_1\ x_2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1\ x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + x_2^2$$

From this, $Q(x) = x_1^2 + x_2^2 = X \cdot X$.

So, Ordinary dot product is a special example of Quadratic form

Let $Q : R^3 \to R^1$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in R^3$ and $A = \begin{pmatrix} 1 & 5 & 6 \\ 5 & 2 & 8 \\ 6 & 8 & 3 \end{pmatrix}$ find $Q(x)$.

Soln,

$Q(x) = X^T AX.$

$$= (x_1 \ x_2 \ x_3)\begin{pmatrix} 1 & 5 & 6 \\ 5 & 2 & 8 \\ 6 & 8 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= (x_1 \ x_2 \ x_3)\begin{pmatrix} x_1+5x_2+6x_3 \\ 5x_1+2x_2+8x_3 \\ 6x_1+8x_2+3x_3 \end{pmatrix} = x_1(x_1+5x_2+6x_3)+x_2(5x_1+2x_2+8x_3) + x_3(6x_1+8x_2+3x_3)$$

$$Q(X) = x_1^2+5x_1x_2+6x_1x_3+5x_1x_2+2x_2^2+8x_2x_3+6x_1x_3+8x_2x_3+3x_3^2$$

$$= 1x_1^2+2x_2^2+3x_3^2+10x_1x_2+16x_2x_3+12x_3x_1.$$

* Relation between $Q(x)$ & Matrix A.

$$\begin{pmatrix} 1 & 5 & 6 \\ 5 & 2 & 8 \\ 6 & 8 & 3 \end{pmatrix}$$

2x 2x 2x

Example:

Let $Q: \mathbb{R}^2 \to \mathbb{R}^1$ defined by $Q(x,y) = 2x^2+3xy+y^2$, $x \in \mathbb{R}$. find the symmetric matrix w.r.t Quadratic form.

Soln,

Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$. The required matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

Then, $Q(X) = X^T AX$

or, $2x^2+3xy+y^2 = (x \ y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

or, $2x^2+3xy+y^2 = (x \ y)\begin{pmatrix} ax+by \\ bx+cy \end{pmatrix} = ax^2+bxy+bxy+cy^2$

or $2x^2+3xy+y^2 = ax^2+2bxy+cy^2$

Equating the coefficient of like terms.

$a=2, \ b=3/2, \ c=1.$

Now, the required matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 2 & 3/2 \\ 3/2 & 1 \end{pmatrix}$ //

Definiteness

let $Q: R^n \to R$ be a quadratic form associated with matrix A
i.e. $Q(X) = X^T A X$, $x \in R^n$

Then the value of $Q(X)$ will be >0, or ≥ 0 or <0 or ≤ 0 or no information.

Case I: If $Q(X) > 0$ $\forall$ $X \neq 0 \in R^n$, then Q is called positive definite.

Case II: If $Q(X) \geq 0$ $\forall$ $X \neq 0 \in R^n$, then Q is called positive semidefinite.

Case III: If $Q(X) < 0$ $\forall$ $X \neq 0 \in R^n$, then Q is called Negative definite.

Case IV: If $Q(X) \leq 0$ $\forall$ $X \neq 0 \in R^n$, then Q is called Negative semi-definite.

Case V: If $Q(X) \geq 0$ $\forall$ $X \in R^n$ & $Q(X) < 0$ $\forall$ $X \in R^n$ then Q is called indefinite.

Definiteness of 2 variable quadratic form.

let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$

Then. $Q(X) = ax_1^2 + cx_2^2 + 2bx_1x_2$

$$= a\left(x_1^2 + 2x_1 \cdot \frac{b}{a}x_2 + \frac{b^2}{a^2}x_2^2\right) + cx_2^2 - \frac{b^2 x_2^2}{a}$$

$$= a\left(x_1 + \frac{b}{a}x_2\right)^2 + \left(\frac{ac-b^2}{a}\right)x_2^2 \quad \text{---} (x)$$

The expression $ac-b^2$ is called discriminant of the quadratic form Q.
$= \begin{vmatrix} a & b \\ b & c \end{vmatrix}$

Thus, if $D_1 = a$ & $D_2 = ac-b^2$

Then (x) looks like

$$Q(X) = D_1\left(x_1 + \frac{b}{a}x_2\right)^2 + \frac{D_2}{D_1}x_2^2 \quad \text{---} \ (**)$$

Case I : If $D_1 > 0$, $D_2 > 0$, then $Q(X)$ is positive. So Q is positive definite.

Case II : If $D_1 < 0$, $D_2 > 0$, then $Q(X)$ looks like $-x^2 - y^2$, $Q(X)$ is negative so Q is negative definite.

Case III : If $D_1 > 0$, $D_2 < 0$, then $Q(X)$ looks like $x^2 - y^2$, then Q is indefinite.

If $D_1 < 0$, $D_2 < 0$, then $Q(X)$ looks like $-x^2 + y^2$, then Q is indefinite.

Above rule can be generalized to a quadratic form $Q: \mathbb{R}^3 \to \mathbb{R}^1$ as follows.

let $A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$

let the principal minor are,

$D_1 = |a| = a$

$D_2 = \begin{vmatrix} a & b \\ b & d \end{vmatrix}$, $D_3 = \begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}$

Then,

Case I : If $D_1 > 0$, $D_2 > 0$, $D_3 > 0$ i.e. all principal miners are positive, then Q is positive definite.

Case II : If $D_1 < 0$, $D_2 > 0$, $D_3 < 0$ (i.e. principal miners alternate its sign starting from negative). Then Q is negative definite.

\# let $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$, Check the definiteness of quadratic form

Definiteness of n-variable Form.

Theorem:-

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

Suppose $Q: \mathbb{R}^n \to \mathbb{R}$ be the quadratic form,

$$Q(X) = X^T A X$$

Consider the principal minors are:

$D_1 = |a_{11}|$, $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

Then, Q will be,

(i) Positive definite if

$D_1 > 0$, $D_2 > 0$, ... $D_n > 0$

(ii) Negative definite if,

$D_1 < 0$, $D_2 > 0$, $D_3 < 0$, $D_4 > 0$, ...

i.e. $D_1, D_2, \ldots$ changes alternatively its sign starting from negative.

Moreover we have,

(iii) Q will be positive semi-definite if

$D_1 \geq 0$, $D_2 \geq 0$ .... $D_n \geq 0$.

(iv) Q will be Negative semi-definite if.

$D_1 \leq 0$, $D_2 \geq 0$, $D_3 \leq 0$, $D_4 \geq 0$, ...

i.e. $D_1, D_2 \ldots$ changes alternatively its sign starting from negative.

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Theorem: $\to A = A^T$

Let A be a symmetric matrix, then any two eigen vector corresponding to different eigen values are orthogonal. $\lambda_1 \neq \lambda_2$.

Proof:

Let v_1 and v_2 be two eigen vectors corresponding to different eigen value λ_1 and λ_2.

$\therefore AV_1 = \lambda_1 V_1$ — (i)

& $AV_2 = \lambda_2 V_2$ — (ii)

We show, $V_1 \cdot V_2 = 0$

New,

Vector dot Prod. Matrix Mul.

$$\lambda_1 . V_1 \cdot V_2 = (\lambda_1 V_1)^T V_2 \qquad \because XY = X^T Y$$

$$= (AV_1)^T V_2 \qquad [\because \text{By (i)}]$$

$$= (V_1^T A^T) V_2 \qquad [\because (AB)^T = B^T A^T]$$

$$= V_1^T (AV_2) \qquad [\text{Associativity}]$$

$$= V_1^T (\lambda_2 V_2)$$

$$= \lambda_2 (V_1^T V_2)$$

$$= \lambda_2 (V_1 . V_2) \qquad [\because X^T Y = X . Y]$$

$\therefore \lambda_1 (V_1 . V_2) = \lambda_2 (V_1 . V_2)$

or $(\lambda_1 - \lambda_2)(V_1 . V_2) = 0$

This shows that either $\lambda_1 - \lambda_2 = 0$ or $V_1 . V_2 = 0$

But $\lambda_1 \neq \lambda_2$. So, $V_1 . V_2 = 0$

i.e. v_1 and v_2 are orthogonal.

Example:

Let (i) $A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix}$.

Find Eigen values & eigen vectors for both matrix. Verify that all eigen vectors in both matrices are orthogonal.

Hint: (i) Find $v_1\ v_2$, $v_1 . v_2 = 0$

(ii) Find $v_1 . v_2\ v_3$ & $v_1 v_2 = 0$, $v_2 v_3 = 0$, etc

(1) $A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$

Here,

$A - \lambda I = \begin{pmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{pmatrix}$

then,

$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) - 4$

$= 10 - 2\lambda - 5\lambda + \lambda^2 - 4$

$= \lambda^2 - 7\lambda + 10 - 4$

$= \lambda^2 - 7\lambda + 6$

Then for eigen value

$|A - \lambda I| = 0$

or $\lambda^2 - 7\lambda + 6 = 0$

or $\lambda^2 - 6\lambda - \lambda + 6 = 0$

or $\lambda(\lambda-6) - 1(\lambda-6) = 0$

or $(\lambda-1)(\lambda-6) = 0$

$\therefore \lambda = 1, 6$

To find eigen vector, let $v = \begin{pmatrix} x \\ y \end{pmatrix}$ be eigen vector of matrix A.

$(A - \lambda I)v = 0$

$\begin{pmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

Here, for $\lambda = 1$.

$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\begin{pmatrix} x - 2y \\ -2x + 4y \end{pmatrix} = 0$

$x - 2y = 0 \Rightarrow x = 2y$ — (i)

$x = 2y$ — (ii)

From (i): Let $x = k$; $y = k/2$

$\therefore v = (x, y) = (k, k/2) = k(1, 1/2)$

EV of A, when $\lambda = 1$ is $k.(1, 1/2)$; $k \neq 0$.

For $\lambda = 6$.

$\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$-4x - 2y = 0$ — (i)

$-2x - y = 0$ — (ii) $\Rightarrow x = -y/2$

From (i)

$x = -y/2$

let $x = k$, $y = -2k$ $k \in \mathbb{R}$.

$\therefore v = (x, y) = (k, -2k) = k(1, -2)$

$\therefore$ EV of A when $\lambda = 6$ is $k(1, -2)$.

Now, To verify orthogonal,

$v_1 = (1, 1/2)$

$v_2 = (1, -2)$

So, $v_1 \cdot v_2 = (1, 1/2) \cdot (1, -2) = 1 + 1/2 \times (-2)$
$= 0.$

$\therefore v_1 \cdot v_2 = 0$ Verified

(11) $B = \begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix}$

Here,

$B - \lambda I = \begin{pmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{pmatrix}$

then,

$|B - \lambda I| = \begin{vmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix} = (6-\lambda)\begin{vmatrix} 6-\lambda & -1 \\ -1 & 5-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & -1 \\ -1 & 5-\lambda \end{vmatrix} - \begin{vmatrix} -2 & 6-\lambda \\ -1 & -1 \end{vmatrix}$

$= (6-\lambda)\{(6-\lambda)(5-\lambda) - 1\} + 2\{(5-\lambda)(-2) - 1\} - \{2 + (6-\lambda)\}$

$= (6-\lambda)^2(5-\lambda) - 6 - \lambda + (-2)(5-\lambda) - 2 - 2 - 6 + \lambda$

$= (6-\lambda)(30 - 11\lambda + \lambda^2 - 1) + 2(-11 + \lambda.2) - (2 + 6 - \lambda)$

$= \lambda^3 - 17\lambda^2 + 90\lambda - 144$

$= (\lambda - 3)(\lambda - 6)(\lambda - 8)$

Then for eigen values.

$|B - \lambda I| = 0$

$(\lambda-3)(\lambda-6)(\lambda-8) = 0$

$\therefore \lambda = 3, 6, 8$

To find eigen vector, let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be eigen vector of matrix B.

$(B-\lambda I)v = 0$.

$$\begin{pmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

For $\lambda = 3$

$$\begin{pmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\left.\begin{array}{l} 3x - 2y - z = 0 \\ -2x + 3y - z = 0 \\ -x - y + 2z = 0 \end{array}\right\} \text{On solving, } x = y = z = k.$$

So, $(x, y, z) = k(1, 1, 1)$

$\therefore$ $v = (x, y, z) = (1, 1, 1)$ is the Eigen vector of matrix B, when $\lambda = 3$.

For, $\lambda = 6$.

$$\begin{pmatrix} 0 & -2 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\left.\begin{array}{l} -2y - z = 0 \\ -2x - z = 0 \\ -x - y - z = 0 \end{array}\right\} \text{on solving, } \begin{array}{l} y = x \\ z = -2x, \end{array}$$

$x = y = k$.

So, $(x, y, z) = k(1, 1, -2)$.

$\therefore$ $V = (x, y, z) = (1, 1, -2)$ is the Eigen vector of Matrix B, when $\lambda = 6$.

For $\lambda = 8$.

$$\begin{pmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ -2 & -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{array}{l} -2x - 2y - z = 0 \\ -2x - 2y - z = 0 \\ -x - y - 3z = 0 \end{array}$$

Here, $y = -x$, $z = 0$

Let $x = k$, $y = -k$, $z = 0$.

The eigen vector is $(x, y, z)' = (k, -k, 0) = k(1, -1, 0)$.

$\therefore V = k(1, -1, 0)$.

Now To show orthogonal.

λ	V	Let
3	$(1,1,1)$	$= V_1$
6	$(1,1,-2)$	$= V_2$
6	$(1,-1,0)$	$= V_3$

$V_1 . V_3 = 0$.

$V_1 . V_2 = (1,1,1).(1,1,-2)$ & $V_2 . V_3 = (1,1,-2).(1,-1,0)$

$= 1 + 1 - 2$ $\quad\quad = 1 - 1$

$\therefore V_1 . V_2 = 0$ $\quad\quad \therefore V_2 . V_3 = 0$.

Hence, V_1, V_2 and V_3 are orthogonal.

From above example, $A = (\ )$ symmetric. Then $\lambda_1, \lambda_2, \lambda_3$ all real.

Theorem,

If A is a real symmetric matrix. Then all the eigen values are real.

i.e. Solution of $|A - \lambda I| = 0$ gives real values of λ.

$A \stackrel{!}{=} A^{-1}$

$\lambda_1 \neq \lambda_2$

$V_1 \quad V_2$

$V_1 . V_2 = 0$

Spectral Decomposition of a Symmetric Matrix

Theorem: I

If A be a real symmetric matrix $(A = A^T)$ then there exist an orthogonal matrix P (which satisfy $P^{-1} = P^T$) such that,

$P^{-1}AP$ = Diagonal matrix.

i.e. $P^TAP = D$, where D = Diagonal matrix.

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Steps of Diagonalization.

Step. 1. Find eigen values by solving $|A-\lambda I|=0$, say λ_1 & λ_2. and corresponding eigen vectors v_1 & v_2.

step 2. find orthonormal vectors u_1 & u_2 corresponding to v_1 & v_2.

i.e, $u_1 = \dfrac{v_1}{\|v_1\|}$, $u_2 = \dfrac{v_2}{\|v_2\|}$.

step 3. P = matrix whose columns are u_1 & u_2.

$= \begin{pmatrix} u_1 & u_2 \\ \vdots & \vdots \end{pmatrix}$ & $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ = Diagonal matrix of eigen values.

Example:

let $A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$, a symmetric matrix. find the orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix.

Steps! soln,

1. Now,

$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0$

or, $(2-\lambda)(5-\lambda) - 4$

$\therefore \lambda_1 = 1$ & $\lambda_2 = 6$

For eigen vector,

If $v = \begin{pmatrix} x \\ y \end{pmatrix}$ be the eigen vector then. $AV = \lambda V \Rightarrow (A-\lambda I)v = 0$

or, $\begin{pmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Case I If $\lambda = 1$

$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x - 2y = 0$
$-2x + 4y = 0$ } same eqn

$x - 2y = 0 \Rightarrow y = 1/2\, x$. | let $y = k \Rightarrow x = 2k$.

let $x = 2$ $\therefore y = 1$. (chose any satisfying point)

$\therefore v_1 = (2, 1)$

Case II If $\lambda = 6$, then,

$$\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-4x - 2y = 0$
$-2x - y = 0$ } same eqn $2x + y = 0$

$x = k$
$y = -2x$.
$(k, -2k)$
$k(1, -2)$.

let $x = 1$, then $y = -2$.

$\therefore V_2 = (1, -2)$.

2. Corresponding orthonormal vectors are :

$$u_1 = \frac{V_1}{\|V_1\|} = \frac{(2, 1)}{\sqrt{5}} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

$$u_2 = \frac{V_2}{\|V_2\|} = \frac{(1, -2)}{\sqrt{5}} = \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$$

3. Take $P = \begin{pmatrix} u_1 & u_2 \\ \vdots & \vdots \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$ P is orthogonal, $u_1 \cdot u_2 = 1$ (Check $P^T = P^{-1}$)

We verify that

$P^{-1} A P = D$.

LHS $= P^{-1} A P$

$= P^T A P$ $(\because P^{-1} = P^T)$

$$= \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$ = Diagonal matrix of eigen values (D)

$= D$. //

Note :-

We had for symmetric matrix A, then there exist orthogonal matrix P such that $P^{-1} A P = D$.

Operating both sides by P (premultiply)

$P(P^{-1} A P) = P \cdot D$

or, $IAP = PD$ $\quad (\because PP^{-1} = I)$

or $AP = PD.$

Again, post operating by P^{-1}

$(AP)P^{-1} = PDP^{-1}$

or $AI = PDP^{-1}$

or $A = PDP^{-1}$

This shows that every symmetric matrix can be written on the product of three matrix,

where, P = Orthogonal matrix

D = Diagonal matrix.

HW. Diagonalize the symmetric matrix, $A = \begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix}$

Hint: $\lambda_1 = 8$, $v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 6$, $v_2 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

$\lambda_3 = 3$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Example:

Let $A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. Find eigen value λ_1, λ_2, eigen vectors v_1, v_2 & orthogonal matrix P. Such that,

$P^{-1}AP = D$ or $A = PDP^{-1}$

Hint: $\lambda_1 = 3, \lambda_2 = 1$

$u_1 = (1, 0)$ & $u_2 = (0, 1)$

Theorem II: Spectral Decomposition of a symmetric matrix.

Let $u_1, u_2, \ldots u_n$ be the column eigen vectors corresponding to eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$ of a $n \times n$ symmetric matrix A respectively. Then,

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$$

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Proof:

Here A is a symmetric matrix. So by spectral decomposition Theorem I,

A can be written as

$$A = PDP^{-1}$$

where, P = orthogonal matrix whose column vectors orthonormal.

D = Diagonal matrix

Now,

$$A = PDP^{-1}$$

$$= PDP^T \qquad (\because P^{-1} = P^T)$$

$$= (u_1\ u_2 \dots u_n) \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}_{n\times n} \begin{pmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{pmatrix}_{n\times 1}$$

$$= (u_1\ u_2 \dots u_n) \begin{pmatrix} \lambda_1 u_1^T \\ \lambda_2 u_2^T \\ \vdots \\ \lambda_n u_n^T \end{pmatrix}$$

$$= u_1\lambda_1 u_1^T + u_2\lambda_2 u_2^T + \dots + u_n\lambda_n u_n^T$$

$$\therefore A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \dots + \lambda_n u_n u_n^T$$

$$= \sum_{k=1}^{n} \lambda_k u_k u_k^T$$

Example: 1.

Let $A = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$, find (i) eigen values, eigen vectors of A.

(ii) Diagonalize A as $A = PDP^{-1}$

where, P = orthogonal matrix

D = Diagonal matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

(iii). Decompose A in the form:

$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$

(iv) Verify: $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$.

Hint:

$\lambda_1 = 8,\ \lambda_2 = 3$

$u_1 = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix},\ u_2 = \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$

Example 2: $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$; same

Example 3: $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

We had

In order to diagonalize the symmetric matrix A. We find a orthogonal matrix P ($P^T = P^{-1}$) such that,

$$P^{-1}AP = D \quad \text{---} (\otimes)$$

This process was called Diagonalization of matrix A.

Also, we had,

$$A = PDP^{-1}$$

In the relation $(\otimes)$, the two matrices A & D are called Similar Matrix.

Application of Orthogonal Diagonalization.
(Removing cross term in quadratic form).

Theorem: Principal Axes Theorem for quadratic form.

Let A be a $n \times n$ symmetric matrix and $X \in \mathbb{R}^n$. Then there is orthogonal change of variable. $X = PY$ that transform the quadratic form, $Q(X) = X^TAX$ to the form, Y^TDY, where $D = P^TAP$

where, D = Diagonal matrix

P = Orthogonal matrix (of column of orthonormal eigen vector).

Proof:

Here the quadratic form is,

$$Q(X) = X^TAX \quad \text{---} (1)$$

Writing $X = PY$ in (1), we get,

$$Q(PY) = (PY)^TA(PY)$$

$$= Y^TP^TAPY \qquad (\because (AB)^T = B^TA^T)$$

$$= Y^T(P^TAP)Y$$

$$= Y^TDY \qquad (\because D = P^TAP)$$

Which gives no cross term.

Example: let $A = \begin{pmatrix} 1 & -4 \\ -4 & -5 \end{pmatrix}$ (i). find the quadratic form associated with matrix A. Is cross term involved in the form?

(ii) Remove the cross term x_1x_2 by changing the variable.

(iii) Write the formula $X = PY$ — $(\otimes)$, (iv). What is value of Y if $X = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

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(i) Find the value of new quadratic form.

(i) Soln,

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$

So, the quadratic form is

$$Q(X) = X^T A X$$

$$= (x_1\ x_2) \begin{pmatrix} 1 & -4 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= 1.x_1^2 - 8x_1x_2 - 5x_2^2$$, Yes cross term involved.

(ii) In order to remove the cross term $x_1 x_2$ in $Q(X)$, let us change the variable.

$$X = PY$$

Now, for eigen value,

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 1-\lambda & -4 \\ -4 & -5-\lambda \end{vmatrix} = 0 \quad \Rightarrow (1-\lambda)(5-\lambda) - 16 = 0$$

$$5 - \lambda - 5\lambda + \lambda^2 - 16 = 0$$

or, $5 - 6\lambda + \lambda^2 + 16 = 0$

or, $\lambda^2 - 6\lambda + 11 = 0$

or, $(\lambda - 3)(\lambda + 7) = 0$

$\therefore \lambda = 3, -7$

Let $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigen vector, then,

$$AV = \lambda V$$

or, $(A - \lambda I)V = 0$

or, $\begin{pmatrix} 1-\lambda & -4 \\ -4 & -5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

If $\lambda_1 = 3$ then,

$$\begin{pmatrix} -2 & -4 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left.\begin{array}{l} -2x_1 - 4x_2 = 0 \Rightarrow x_1 = -2x_2 \\ -4x_1 + 8x_2 = 0 \Rightarrow x_2 = -2x_1 \end{array}\right\} x_1 + 2x_2 = 0.$$

Let $x_2 = k$, $x_1 = -2k$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2k \\ k \end{pmatrix} = k\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\therefore V_1 = k\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Take $v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

If $\lambda_2 = -7$

$$\begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

or $\left.\begin{matrix} 8x_1 - 4x_2 = 0 \\ -4x_1 + 2x_2 = 0 \end{matrix}\right\} x_2 = 2x_1$

let $x_1 = k$, $x_2 = 2k$.

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k \\ 2k \end{pmatrix} = k\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore V_2 = k\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Take $V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\therefore u_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \qquad u_2 = \frac{v_2}{\|v_2\|} = \begin{pmatrix} +1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$$\therefore P = \begin{pmatrix} u_1 & u_2 \\ \vdots & \vdots \end{pmatrix} = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

$$\& \ D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -7 \end{pmatrix}$$

Now,

To remove the cross term,

$$Q(X) = X^T P X = Y^T D Y$$

or $$x_1^2 - 8x_1x_2 - 5x_2^2 = (y_1 \ y_2)\begin{pmatrix} 3 & 0 \\ 0 & -7 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= 3y_1^2 - 7y_2^2 \ //$$

(iii) To find formula $X = PY$ — ①

$\therefore X = P^{-1}X$ (P is orthogonal, $P^{-1} = P^T$)

or, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix} (x_1 \ x_2)$ //

(iv) Also from ①, $Y = P^{-1}X$

or $$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Put, $x_1 = 2$, $x_2 = -2$.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -6/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix} \quad \therefore \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -6/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$$

$\therefore Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -6/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$

Finally,

(v) If $y_1 = -6/\sqrt{5}$ & $y_2 = -2/\sqrt{5}$, then,
The value of new quadratic form,
$3y_1^2 - 7y_2^2$

$= 3 \times \left(\frac{-6}{\sqrt{5}}\right)^2 - 7\left(\frac{-2}{\sqrt{5}}\right)^2$

$= \frac{3 \times 36}{5} - \frac{7 \times 4}{5}$

$= 16$

Theorem : Quadratic form with eigen value

Let A be a 2×2 matrix & $Q(X) = X^TAX$ be the quadratic form.

If λ_1 & λ_2 be two eigen values of A. Then the quadratic form Q is,

(i). Positive definite if both λ_1 & λ_2 are positive.
(ii). Negative definite if both λ_1 & λ_2 are negative.
(iii). Indefinite if one λ is +ve & other λ is -ve.
(iv) Semi positive definite if one λ is zero & other λ is +ve.
(v) Semi Negative definite if one λ is zero & other λ is -ve.

For Example:

1) The quadratic form determined by the matrix $A = \begin{pmatrix} 1 & -4 \\ -4 & -5 \end{pmatrix}$ is indefinite.

$\lambda_1 = 3$ +ve
$\lambda_2 = -7$ -ve

Example : for what value of k, the quadratic form $Q(X) = kx_1^2 - 6x_1x_2 + kx_2^2$ positive semi-definite?

Here,

$Q(X) = kx_1^2 - 6x_1x_2 + kx_2^2$ $\quad\rightarrow$ If $-\lambda x_1x_2$ $\quad \begin{pmatrix} k & -\lambda/2 \\ -\lambda/2 & k \end{pmatrix}$

and,

$$Q(X) = [x_1 \;\; x_2]\begin{pmatrix} k & -3 \\ -3 & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A = \begin{pmatrix} k & -3 \\ -3 & k \end{pmatrix}$$

Now, for eigen values.

$|A - \lambda I| = 0$ $\quad \begin{vmatrix} k-\lambda & -3 \\ -3 & k-\lambda \end{vmatrix} = 0 \Rightarrow (k-\lambda)^2 - 9 = 0.$

$\therefore k - \lambda = \pm 3.$

If $k - \lambda = 3$

$\therefore \lambda = k - 3$

If $k - 3 = -3$

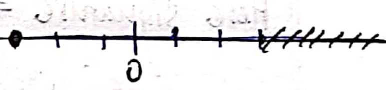
$\lambda = k + 3.$

For positive semi-definite,

If $\lambda_1 > 0$, $\lambda_2 = 0$ then,

$k - 3 > 0$, $k + 3 = 0$

$k > 3$, $k = -3$



$\therefore$ value of $k = \{-3\} \cup (3, \infty)$.

Same for λ_2.

Quadratic form expressed as the difference/sum of two squares.

For any quadratic form, $Q(X) = X^TAX$, it can be expressed as the sum or difference of squares.

$$\underbrace{[\alpha_1(x_1,x_2)]^2 + [\alpha_2(x_1,x_2)]^2 + \dots + [\alpha_k(x_1,x_2)]^2}_{k\text{-positive sign}} - \underbrace{[\alpha_{k+1}(x_1,x_2)]^2 - [\alpha_{k+2}(x_1,x_2)]^2 - \dots - [\alpha_{k+l}(x_1,x_2)]^2}_{l\text{-negative sign}}$$

Here, the order pair (k, l) is called signature of quadratic form.

This representation is not unique.

Example:- let $Q(x) = x_1^2 + x_1x_2$ be the quadratic form. Express $Q(X)$ as the difference of two squares and express it in matrix form.

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Soln, Here, $Q(X) = x_1^2 + x_1 x_2$

$$= x_1^2 + 2x_1 . \frac{x_2}{2} + \left(\frac{x_2}{2}\right)^2 - \left(\frac{x_2}{2}\right)^2$$

$$= \left(x_1 + \frac{x_2}{2}\right)^2 - \left(\frac{x_2}{2}\right)^2$$

$$= \alpha_1^2 - \alpha_2^2 \quad \text{where, } \alpha_1 = \left(x_1 + \frac{x_2}{2}\right), \alpha_2 = \frac{x_2}{2}$$

which is expressed as the difference of squares.
Writing in matrix form,

$$\alpha_1 = \left(x_1 + \frac{x_2}{2}\right)$$

$$\alpha_2 = \frac{x_2}{2}$$

$$\therefore \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2/2 \\ \frac{x_2}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\therefore \bar{\alpha} = AX \quad \text{where } A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here signature $= (1,1)$ $\begin{Bmatrix} +\text{ve sign} = 1 \\ -\text{ve sign} = 1 \end{Bmatrix}$

\# let $Q(X) = x_1^2 + x_1 x_2 - x_2^2$, Express $Q(X)$ as the sum/difference of two squares. Find signature.

Soln, first computing squares on x_1.

$$Q(X) = x_1^2 + 2x_1 . \frac{x_2}{2} + \left(\frac{x_2}{2}\right)^2 - \frac{x_2^2}{4} - x_2^2$$

$$= \left(x_1 + \frac{x_2}{2}\right)^2 - \frac{5x_2^2}{4} \Rightarrow \left(x_1 + \frac{x_2}{2}\right)^2 - \left(\frac{\sqrt{5}\, x_2}{2}\right)^2$$

$$= \alpha_1^2 - \alpha_2^2$$

$$\bar{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} x_1 + \frac{x_2}{2} \\ \frac{\sqrt{5}\, x_2}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{5}/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Next computing squares on x_2.

$$Q(X) = -\left[x_2^2 - x_1 x_2\right] + x_1^2$$

$$= -\left[x_2^2 - 2x_2 . \frac{x_1}{2} + \left(\frac{x_1}{2}\right)^2\right] + \left(\frac{x_1}{2}\right)^2 + x_1^2$$

$$= -\left(\frac{x_1}{2} - x_2\right)^2 + \left(\frac{\sqrt{5}}{2} x_1\right)^2 \Rightarrow \left(\frac{\sqrt{5}}{2} x_1\right)^2 - \left(\frac{x_1}{2} - x_2\right)^2$$

$\therefore\ Q(X) = \alpha_1^2 - \alpha_2^2$

$$\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \begin{pmatrix}\sqrt{5}/2\, x_1\\ \frac{x_1}{2} - x_2\end{pmatrix} = \begin{pmatrix}\sqrt{5}/2 & 0\\ 1/2 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \Rightarrow \bar{\alpha} = AX$$

From this example the representation of difference or sum of square is not unique.

Connection of quadratic form with Conic Sections.

Let the equation of conic section be,

$$Ax_1^2 + 2Bx_1x_2 + Cx_2^2 + 2Dx_1 + 2Ex_2 + F = 0,$$

This can be written as,

$$(x_1\ x_2)\begin{pmatrix}A & B\\ B & C\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + (2D\ \ 2E)\begin{pmatrix}x_1\\ x_2\end{pmatrix} + F = 0 \quad \text{---}(1)$$

The equation (1) represents different conic sections.

Case I: If $A = C$, & $B = 0$, then this give circle.

Case II: If $AC - B^2 > 0$, then this gives Ellipse

Case III: If $AC - B^2 < 0$, then this gives Hyperbola

Case IV: If $AC - B^2 = 0$, then this gives Parabola.

Identify the conic section and Trace the conic section

(a) $9x_1^2 - 4x_2^2 - 72x_1 + 8x_2 + 176 = 0$ (Hyperbola)

(b). $\frac{(x_1-4)^2}{4} + \frac{(x_2-1)^2}{9} = 0$ (Ellipse)

* Note:-

For any Square matrix A,

$$A = \underbrace{\left(\frac{A+A^T}{2}\right)}_{\text{symmetric matrix}} + \underbrace{\left(\frac{A-A^T}{2}\right)}_{\text{skew symmetric matrix.}}$$

i.e. A can be expressed as the sum of symmetric and skew symmetric matrix.

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Quadratic form and Symmetric Matrix.

Theorem; Let $Q(X) = X^TAX$ be a quadratic form determined by a matrix A. Then there exist a symmetric matrix B, such that,

$$X^TAX = X^TBX$$

proof: Here we deal with 3×3 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$$

$$Q(X) = X^TAX$$
$$= (x_1\ x_2\ x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
$$= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12}+a_{21})x_1x_2 + (a_{13}+a_{31})x_1x_3 + (a_{23}+a_{32})x_2x_3 \quad \text{--- (1)}$$

Letting $B = (b_{ij}) = \left(\frac{a_{ij}+a_{ji}}{2}\right)$, Then clearly, B is symmetric matrix

$b_{11} = a_{11}$, $b_{22} = a_{22}$, $b_{33} = a_{33}$.

Here, $b_{12} = \dfrac{a_{12}+a_{21}}{2}$, $b_{23} = \dfrac{a_{23}+a_{32}}{2}$

$b_{13} = \dfrac{a_{13}+a_{31}}{2}$,

Also, B is symmetric $\Rightarrow b_{12} = b_{21}$, $b_{23} = b_{32}$, $b_{13} = b_{31}$

Hence, eq (1) becomes,

$$X^TAX = b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + 2b_{12}x_1x_2 + 2b_{13}x_1x_3 + 2b_{23}x_2x_3$$
$$= b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + (b_{12}+b_{21})x_1x_2 + (b_{13}+b_{31})x_1x_3 + (b_{23}+b_{32})x_2x_3$$
$$= (x_1\ x_2\ x_3)\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\because 2b_{12} = b_{12} + b_{21}$, $\because b_{12} = b_{21}$

$\therefore X^TAX = X^TBX$ where B is symmetric matrix.

Here, this statement is true for 3×3 matrix. In general this statement is true for every square matrix.

Example:-

let $A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 6 & 8 \\ 0 & 0 & 9 \end{pmatrix}$, find symmetric matrix B, such that $X^T A X = X^T B X$.

Ans: $B = \frac{A + A^T}{2}$

Optimization using quadratic form.

Sometimes a quadratic form is maximized or minimized under certain constraint (condition).

Here we deal with the optimization of $Q(X) = X^T A X$, with the condition $\|X\| = 1$ or, $X^T X = 1$

Theorem: (Constrained Extreme Theorem)

Let $Q(X) = X^T A X$ be a quadratic form determined by a non symmetric matrix A, whose eigen values in descending order are.

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$. then,

(a) The quadratic form attains a maximum value and a minimum value on the set of point $\|X\| = 1$.

(b). The maximum attains at a vector corresponding to the eigen value λ_1 & minimum attains at a vector corresponding to the eigen value λ_n

Fact: The constraint.

$\|X\| = 1$ means $\sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = 1$ or $x_1^2 + x_2^2 + \dots x_n^2 = 1$

or,

$X^T X = 1$ or $x_1^2 + x_2^2 + \dots + x_n^2 = 1$.

Example:-

Find the maxima & minimum values of the quadratic form,

$z = 5x_1^2 + 5x_2^2 + 4x_1 . x_2$.

Here, $z = 5x_1^2 + 5x_2^2 + 4x_1 . x_2$

$= (x_1 \ \ x_2) \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = X^T A X$

$\lambda_1 = \lambda_2 \to$ Extreme Point
$\to$ No max, min determined.

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Here, $A = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$

To find eigen values
$$|A - \lambda I| = 0$$

or, $\begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0$

or, $(5-\lambda)^2 - 2^2 = 0$

or, $(5-\lambda+2) = 0$, $(5-\lambda-2) = 0$

$\therefore \lambda = 7$, $\lambda = 3$

$\therefore \lambda_1 = 7$, $\lambda_2 = 3$ $(\because \lambda_1 > \lambda_2)$

For eigen vector,

$AV = \lambda V$

$(A - \lambda I)V = 0$

or, $\begin{pmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

If $\lambda = \lambda_1 = 7$ then,

$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\left.\begin{array}{l} -2x_1 + 2x_2 = 0 \\ 2x_1 - 2x_2 = 0 \end{array}\right\} x_1 = x_2$

Let $x_1 = 1$, $x_2 = 1$

$\therefore V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

If $\lambda = \lambda_2 = 3$

$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\left.\begin{array}{l} 2x_1 + 2x_2 = 0 \\ 2x_1 + 2x_2 = 0 \end{array}\right\} x_1 = -x_2$

Let $x_1 = 1$, $x_2 = -1$

$\therefore V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Unit vector along V_2 is-

$u_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$\therefore$ Unit vector along v_1 is, $[\because \|x\| = 1]$

$u_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $\quad \because u_1 = \frac{v_1}{\|v_2\|}$

Thus eigen values & corresponding unit eigen vectors are,

$\lambda_1 = 7$, $u_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, $\lambda_2 = 3$, $u_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$\therefore$ The maximum value $= 7$ at point $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$
&
The minimum value $= 3$ at point $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

Check:- Put value in eqn:- $5x_1^2 + 5x_2^2 + 4x_1x_2$

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Example :- Find the maximum & minimum values of $Q(X) = 7x_1^2+4x_2^2+2x_3^2$ subject to the constraint $||X||=1$.

Hint: → No cross term. i.e. $x_1^2+x_2^2+x_3^2=1$.

Method I $Q(X) = X^T I X$

= Do as before.

Method II Here the condition. $x_1^2+x_2^2+x_3^2=1$. → Difficult if cross term.

$$Q(X) = 7x_1^2+4x_2^2+2x_3^2$$

$$\le 7x_1^2+7x_2^2+7x_3^2 \quad \text{(Making all highest term)}$$

$$\le 7(x_1^2+x_2^2+x_3^2)$$

$$= 7.1 = 7.$$

$\therefore Q(X) \le 7$

So, the maximum value of $Q(X) = 7$.

And, $Q(X) = 7x_1^2+4x_2^2+2x_3^2$.

$$\ge 2x_1^2+2x_2^2+2x_3^2.$$

$$= 2(x_1^2+x_2^2+x_3^2) \quad \text{(Making all lowest term)}$$

$$= 2.1 = 2.$$

$\therefore Q(X) \ge 2.$

Find the max & min value of $Q(X) = 8x_1^2 - 4x_1x_2 + 5x_2^2$. where $||X||=1$,

Soln

$$Q(X) = 8x_1^2 - 4x_1x_2 + 5x_2^2$$

$$= 8x_1^2 + 5x_2^2 - 4x_1x_2$$

$$= (x_1 \; x_2)\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = X^TAX.$$

Here, $A = \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$

To find eigen values,

$|A-\lambda I| = 0$

or $\begin{vmatrix} 8-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0.$

$(8-\lambda)(5-\lambda) - 4 = 0.$

: - - - - - - .

Bilinear form. $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

Note: The set $V \times W = \{(v, w) : v \in V \ \& \ w \in W\}$.

Find $v = \{1, 2, 3\}$, $w = \{5, 6, 7\}$

$V \times W = \{(1,5), (1,6), (1,7), (2,5), (2,6), (2,7) \ldots\}$

Let V be a vector space over $\mathbb{R}$. A bilinear form on V is a function

$B : V \times V \to \mathbb{R}$ such that

(i) $B(\alpha x_1 + \beta x_2, y) = \alpha B(x_1, y) + \beta B(x_2, y)$ (Linear in first Component)

(ii) $B(x, \alpha y_1 + \beta y_2) = \alpha B(x, y_1) + \beta B(x, y_2)$ (Linear in second Component)

where, $x, x_1, x_2 \in V$

$x, y_1, y_2 \in V$

A bilinear form $B : V \times V \to \mathbb{R}$ is called Symmetric if.

$B(x, y) = B(y, x) \quad \forall x, y \in V$

More formally, we have the following defn of symmetric bilinear form.

Dfn: Let V be V.S. over $\mathbb{R}$. A symmetric bilinear form $B : V \times V \to \mathbb{R}$ such that

$BF_1 : B(u, v) = B(v, u) \ \forall u, v \in V$

$BF_2 : B(\alpha u_1 + \beta u_2, v) = \alpha B(u_1, v) + \beta B(u_2, v) \ \forall u_1, u_2, v \in V$, α, β scalar.

Note: BF_1 is called Symmetric

BF_2 is called Linearity in first argument.

We can also prove the linearity in second argument with the help of BF_1 & BF_2.

Since,

$B(u, \alpha v_1 + \beta v_2) = B(\alpha v_1 + \beta v_2, u)$ [$\because$ By BF_1 Symmetric]

$= \alpha B(v_1, u) + \beta B(v_2, u)$

$= \alpha B(u, v_1) + \beta B(u, v_2)$

Example :-

Let A be a 2x2 matrix let $u, v \in \mathbb{R}^2$ such that $u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $v = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Define $B_A : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ by,

$B_A(u, v) = u^T A V$ Is it Bilinear fun? -> Yes.

The answer is Yes, and can be seen in the following Theorem.

Theorem:

Let A be a real symmetric nxn matrix, Then the mapping $B_A(u, v) = u^T A V$ is always symmetric Bilinear form. Where $u, v \in \mathbb{R}^n$ column vector

Proof:

Clearly the expression $u^T A V$ is 1x1 matrix i.e. number i.e. $u^T A V \in \mathbb{R}$.

BF$_1$: (i). B_A is symmetric

Now, $B_A(u, v) = u^T A V$ $\quad$ $u^T A V$ is 1x1 matrix

$= (u^T A V)^T$ $\quad$ $\therefore u^T A V = (u^T A V)^T$

$= (AV)^T (u^T)^T$ $\quad$ $\because (AB)^T = B^T A^T$

$= V^T A^T u$

$= V^T A u$ $\quad$ [A is symmetric $A = A^T$]

$= B_A(v, u)$

$\therefore B_A$ is symmetric.

BF$_2$: (ii) B_A is linear in first argument.

Let $\alpha, \beta \in \mathbb{R}$, Then

$B_A(\alpha u_1 + \beta u_2, v) = (\alpha u_1 + \beta u_2)^T A V$ $\quad$ [By dfn].

$= \left[(\alpha u_1)^T + (\beta u_2)^T\right] A V$

$= \alpha\, u_1^T A V + \beta\, u_2^T A V$

$= \alpha\, B_A(u_1, v) + \beta\, B_A(u_2, v)$

$\therefore B_A$ is linear in first argument

Combining BF$_1$ & BF$_2$. B_A is symmetric bilinear form.

Theorem : Let B_A be a symmetric bilinear form on $\mathbb{R}^n$ i.e.

$B_A : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by $B_A(u, v) = u^T A V$.

Then the matrix A is unique.

Singular Value Decomposition.

Fact ① If A is a $m \times n$ matrix, then A^TA is a square matrix

Fact ② For any matrix A, A^TA is a symmetric matrix. i.e. $(A^TA)^T = A^T(A^T)^T = A^TA$

$\therefore A^TA$ is a symmetric matrix.

Fact ③ If A is symmetric, then,
$A^TA = AA = A^2$ $\quad [\because A^T = A]$.

Fact ④ For any number m, if λ is the eigen value of A with unit eigen vector v. Then λ^m is the eigen value of λ^m.

Dfn: Singular Value.
Let A be a $m \times n$ matrix. Then A^TA is a symmetric $n \times n$ matrix. Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigen values of the matrix A^TA. Then the singular values of A are the numbers.
$\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}, \ldots,$ $\sigma_n = \sqrt{\lambda_n}$ (positive root only)

Example:
Let $A = \begin{pmatrix} 2 & 3 \\ 5 & 6 \\ 0 & 2 \end{pmatrix}$, find singular values of A.

Soln,

$$A^TA = \begin{pmatrix} 2 & 5 & 0 \\ 3 & 6 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 5 & 6 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2\times2+5\times5+0 & 2\times3+5\times6+0\times2 \\ 3\times2+6\times5+0 & 3\times3+6\times6+2\times2 \end{pmatrix}$$

$$= \begin{pmatrix} 29 & 36 \\ 36 & 49 \end{pmatrix}$$

$$\therefore A^TA = \begin{pmatrix} 29 & 36 \\ 36 & 49 \end{pmatrix}$$

Now, $|A^TA - \lambda I| = 0$

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$$\begin{vmatrix} 29-\lambda & 36 \\ 36 & 44-\lambda \end{vmatrix} = 0$$

or) $(29-\lambda)(49-\lambda) - 36\times36 = 0$

or) $1421 - 29\lambda - 49\lambda + \lambda^2 - 1296 = 0$

or) $\lambda^2 - 78\lambda + 125 = 0$

To Be Continued...

Unit 4

System of Linear Equation.

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An equation of the form $ax+by+cz=d$ is called Linear equation in variable x, y and z, where a,b,c,d are constants not all simultaneously zero.

If $d=0$ then, the above equation is homogeneous equation in x,y,z (of first degree).

If $d\neq 0$, then the above equation is non-homogeneous in x,y,z.

If two or more linear equations is written of the form

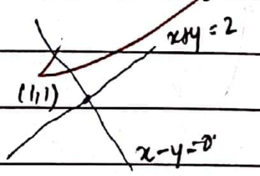
$$\left.\begin{array}{l} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{array}\right\}$$ is called system of linear eqn in 2 x & y.

Consistent & Inconsistent system of linear equation.

If the system has at least one solution, then it is called consistent, otherwise inconsistent.

A system of equation may be:

Consistent & Unique solution

- $\left.\begin{array}{l} x+y=2 \\ x-y=0 \end{array}\right\}$
- Solution $\Rightarrow x=1, y=1$
- 

Consistent & Infinite Solution

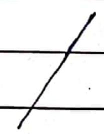
- $\left.\begin{array}{l} x+y=2 \\ 3x+3y=6 \end{array}\right\}$ $\quad$ $x+y=2$. $\quad$ $x+y=2$.
- solution $\Rightarrow$ $0=0$ (True)

or, $0x+0y=0$

which is true $\forall\, x,y \in R$.

So, it has infinitely solution.

Such that $x=k, y=2-k$

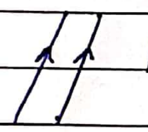


No-Solution

- $\left.\begin{array}{l} x+y=2 \\ 2x+2y=5 \end{array}\right\}$
- Solution $\Rightarrow$ FALSE

$2=5/2$

No solution.



No. of Variable > No. of Eqn

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Solution of Simultaneous Linear Equation.

There are various methods which can be found in following way.

(i) Gaussian Elimination Method
(ii) Gauss Jordan Method
(iii) Matrix Inversion Method

} Exact Method.

NR [(i). Gauss Jacobi Method.
(ii). Gauss Seidel Method

} Approximate Method.

(I) Gauss Elimination Method

Solve,

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\}$$

$\downarrow$

$$a_1x + by = c$$
$$0x + dy = e$$

find x & y

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\}$$

After successive elimination of y & z we find,

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ 0x + dy + ez &= f \\ 0x + 0y + gz &= h \end{aligned}\right\}$$

Then we find z, then y & hence x.

Solve : $\left.\begin{aligned} x + 2y &= 5 \\ 5x - 3y &= -1 \end{aligned}\right\}$ by Gaussian Elimination method.

Soln,

Here, $x + 2y = 5$ —(i)
$5x - 3y = -1$ —(ii).

(i) $\times$ 5 − (ii)

$$5x + 10y = 25$$
$$5x - 3y = -1$$
$$\underline{\quad - \quad + \quad\quad + \quad}$$
$$13y = 26 \quad \text{i.e } y = 2$$

Thus we have,

$x + 2y = 5$ —(iii)
$y = 2$ —(iv).

From (iv), $y = 2$.
From (iii), $x = 1$.

$\therefore (1, 2)$ is the solution.

$x \quad y \quad z$

$y \quad z$

z

Solve the system of Equation by Gaussian Elimination Method

$$\left.\begin{aligned} x+3y-2z&=0\\ 2x-3y+z&=1\\ 4x-3y+z&=3 \end{aligned}\right\}$$

Soln,

$x+3y-2z=0$ —— (I)

$2x-3y+z=1$ —— (II)

$4x-3y+z=3$ —— (III)

Writing in matrix form,

$$\begin{pmatrix} 1 & 3 & -2 \\ 2 & -3 & 1 \\ 4 & -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

Apply, $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -9 & 5 \\ 0 & -15 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

Applying, $R_3 \to 9R_3 - 15R_2$ OR LCM

$$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -9 & 5 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 12 \end{pmatrix}$$

Writing in equation form,

$x+3y-2z=0$ —— (IV)

$-9y+5z=1$ —— (V)

$6z=12$ —— (VI)

From (VI), $z=2$

From (V), $y=1$

From (IV), $x=1$

$\therefore (x,y,z)=(1,1,1)$

Solve by Gauss Elimination Method: $x_1 - x_2 + x_3 = 1$

$3x_1 + x_2 + 5x_3 = 11$

$4x_1 + 2x_2 + 7x_3 = 16$

Soln,

Writing in matrix form,

$$\begin{pmatrix} 1 & -1 & +1 \\ 3 & 1 & 5 \\ 4 & 2 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \\ 16 \end{pmatrix}$$

Apply, $R_2 \rightarrow R_2 - 3R_1$

$$\sim \begin{pmatrix} 1 & -1 & +1 \\ 0 & 4 & 6 \\ 4 & 2 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 16 \end{pmatrix}$$

Apply, $R_3 \rightarrow R_3 - 4R_1$

$$\sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 4 & 6 \\ 0 & 6 & 11 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 12 \end{pmatrix}$$

....

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 10 \end{pmatrix}$$

Infinite solution.

Writing in equation form,

$x_1 - x_2 + x_3 = 1$ —(i)

$4x_2 + 3x_3 = 8$ —(ii)

The equations are consistent & infinite solution.

from (ii)

$4x_2 = 8 - 2x_3$

$x_2 = 2 - \frac{1}{2}x_3$

from (i).

$x_1 = x_2 - x_3 + 1$

$= 2 - \frac{1}{2}k - k + 1$

letting, $x_3 = k$, Then $x_2 = 2 - \frac{1}{2}k$

$\therefore x_1 = \frac{3 - 3k}{2}$

Solution are,

$(x_1, x_2, x_3) = \left(\frac{3-3k}{2}, 2 - \frac{k}{2}, k\right)$ where $k \in R$.

#10 Solve by Gauss Elimination Method.

$$\left.\begin{aligned} x+3y+4z &= 8 \\ 2x+y+6z &= 5 \\ 5x+2z &= 7 \end{aligned}\right\}$$

Ans: $\left(\frac{7-2k}{5}, \frac{11-6k}{5}, k \right)$, $k \in R$

#11

$$\left.\begin{aligned} x_1+x_2+x_3 &= -3 \\ 3x_1+x_2-x_3 &= -2 \\ 2x_1+4x_2+Ax_3 &= 7 \end{aligned}\right\}$$

No Solution, FALSE

(II) Solution of system of Equation by Gauss Jordan Matrix Method.

Solve the system

$$\left.\begin{aligned} a_1x_1+b_1x_2+c_1x_3 &= d_1 \\ a_2x_1+b_2x_2+c_2x_3 &= d_2 \\ a_3x_1+b_3x_2+c_3x_3 &= d_3 \end{aligned}\right\}$$ by Gauss Jordan Matrix Method

Here, writing in matrix form,

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $C = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

$\therefore AX = C$ —— (1)

If $|A| \neq 0$, then A^{-1} exist,

From (1).

$$X = A^{-1}C$$

To find A^{-1},

Writing the matrix A with identity matrix I as,

$$[A:I] = \left[\begin{array}{ccc:ccc} a_1 & b_1 & c_1 & 1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 1 \end{array}\right]$$

14. 2×2 : $R_n \to R_n \pm R_m$

By elementary row by row operation, if we change the above matrix as,

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & a_1' & b_1' & c_1' \\ 0 & 1 & 0 & a_2' & b_2' & c_2' \\ 0 & 0 & 1 & a_3' & b_3' & c_3' \end{array}\right] = [I : B]$$

Then, the inverse of matrix A is matrix B; $B = \begin{bmatrix} a_1' & b_1' & c_1' \\ a_2' & b_2' & c_2' \\ a_3' & b_3' & c_3' \end{bmatrix}$

i.e. $A^{-1} = B$

from ①.

$X = A^{-1}C = BC$, we get x_1, x_2, x_3.

Example:

Find the inverse of the matrix, $M = \begin{pmatrix} 2 & -3 \\ -2 & 5 \end{pmatrix}$ by Gauss Jordan.

clearly, $|M| \neq 0$, M^{-1} exist

Soln,

Writing the augmented matrix of M with identity matrix I as,

$$[M : I] = \left[\begin{array}{cc:cc} 2 & -3 & 1 & 0 \\ -2 & 5 & 0 & 1 \end{array}\right]$$

Applying $R_1 \to \frac{1}{2} R_1$

$$\sim \left[\begin{array}{cc:cc} 1 & -3/2 & 1/2 & 0 \\ -2 & 5 & 0 & 1 \end{array}\right]$$

Applying $R_2 \to R_2 + 2R_1$

$$\sim \left[\begin{array}{cc:cc} 1 & -3/2 & 1/2 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right]$$

Applying $R_2 \to \frac{1}{2} R_2$

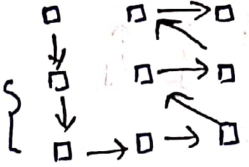
$$\sim \left[\begin{array}{cc:cc} 1 & -3/2 & 1/2 & 0 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right]$$

$R_1 \to R_1 + \frac{3}{2} R_2$.

$$\sim \left[\begin{array}{cc:cc} 1 & 0 & 5/4 & 3/4 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right]$$

$= [I : B]$, where $B = \begin{bmatrix} 5/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$

Here, $M^{-1} = B = \begin{bmatrix} 5/4 & 3/4 \\ 1/2 & 1/2 \end{bmatrix}$



Example : for the sys. of Equations,

$$\left.\begin{aligned} x+y+z&=0 \\ x-2y+z&=-3 \\ x+2y-z&=5 \end{aligned}\right\}$$

(i). Write in Matrix form $AX=C$

(ii) Does A^{-1} exist

(iii) Find A^{-1} by Gauss Jordan Matrix Method.

(iv) find the solution x, y, z.

Soln,

Writing in matrix form

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 5 \end{pmatrix}$$

$\therefore AX=C \quad \therefore X=A^{-1}C.$

Now,

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 6 \neq 0 \; ; |A| \neq 0, A^{-1} \text{ exist.}$$

To find A^{-1}, writing the augmented matrix A with identity matrix I, as

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

Applying, $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

Applying $R_3 \to R_3 + \frac{1}{3}R_2$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & -4/3 & 1/3 & 1 \end{array}\right]$$

Applying, $R_3 \to -\frac{1}{2}R_3$ and $R_2 \to -\frac{1}{3}R_2$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/3 & -1/3 & 0 \\ 0 & 0 & 1 & 2/3 & -1/6 & -1/2 \end{array}\right]$$

Applying $R_1 \to R_1 - R_2$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 1 & 2/3 & 1/3 & 0 \\ 0 & 1 & 0 & 1/3 & -1/3 & 0 \\ 0 & 0 & 1 & 2/3 & -1/6 & -1/2 \end{array}\right] \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1/3 & -1/3 & 0 \\ 0 & 0 & 1 & 2/3 & -1/6 & -1/2 \end{array}\right]$$

$$= [I : B]$$

$$A^{-1} = B = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & -1/3 & 0 \\ 2/3 & -1/6 & -1/2 \end{pmatrix}$$

(iv) To solve,

$$X = A^{-1}C$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & -1/3 & 0 \\ 2/3 & -1/6 & -1/2 \end{pmatrix} \begin{pmatrix} 0 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 - 3/2 + 5/2 \\ 0 + 1 + 0 \\ 0 + 1/2 - 5/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Solve by Gauss-Jordan Matrix Method,

$$x + 2y + z = 8$$
$$2x + 3y + 2z = 14$$
$$3x + 2y + 2z = 13$$

Soln,

Writing in matrix form,

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \\ 13 \end{pmatrix}$$

$$AX = C$$

$$\therefore X = A^{-1}C.$$

Now

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}$$

$$= (6-4) - 2(4-6) + (4-9)$$

$$= 6 - 4 - 8 + 12 \cancel{+4} - 9$$

$$= 1 \neq 0, \quad |A| \neq 0, \; A^{-1} \text{ exist.}$$

To find A^{-1}, writing the augmented matrix A with Identity matrix I as,

$$[A:I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

Applying, $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & -4 & -1 & -3 & 0 & 1 \end{array}\right]$$

Applying, $R_3 \rightarrow R_3 - 4R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 5 & -4 & 1 \end{array}\right]$$

Applying, $R_3 \rightarrow R_3 \times (-1)$.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & -1 \end{array}\right]$$

Applying $R_2 \rightarrow R_2 \times (-1)$.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -5 & 4 & -1 \end{array}\right]$$

$R_1 \rightarrow R_1 - 2R_2$.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -5 & 4 & -1 \end{array}\right]$$

$R_1 \rightarrow R_1 - R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -5 & 4 & -1 \end{array}\right]$$

$= [I : B]$.

$$A^{-1} = B = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ -5 & 4 & -1 \end{pmatrix}$$

To Solve,

$$X = A^{-1}C = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$|| \neq 0$ LI

$|| = 0$ LD

Rank of Matrix

Order of non-zero determinant.

The rank of a matrix is the number of linearly independent rows (or column).
In order to find the Rank of Matrix,
We first change into upper triangular matrix and count the no. of non-zero rows.

Example:
Reduce the matrix $A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 2 & 1 \end{pmatrix}$ into upper Triangular matrix and find rank.

Soln,
Apply $R_3 \leftrightarrow R_1$

$$\sim \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ 0 & 1 & 3 \end{pmatrix}$$

Apply, $R_2 \rightarrow R_2 - 2R_1$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

Apply $R_3 \rightarrow 3R_3 + R_2$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 11 \end{pmatrix} \Rightarrow \text{UTM. Canonical Form.}$$

The last matrix is the upper Triangular Matrix, with no. of non zero rows = 3.

$\therefore$ Rank $(A) = 3$.

Example: Let $AX = C$, where,

$$A = \begin{pmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 \\ 2 \\ 3 \\ 11 \end{pmatrix}$$

Find the rank of coefficient matrix A & Augmented Matrix $[A : C]$. Is Rank $(A) =$ Rank $([A : C])$?

Soln,

Here, $[A:C] = \left[\begin{array}{cccc:c} 2 & 4 & 3 & 2 & 2 \\ 3 & 6 & 5 & 2 & 2 \\ 2 & 5 & 2 & -3 & 3 \\ 4 & 5 & 14 & 14 & 11 \end{array}\right]$

Apply, $R_2 \to 2R_2 - 3R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - 2R_1$

$\sim \left[\begin{array}{cccc:c} 2 & 4 & 3 & 2 & 2 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 1 & -1 & -5 & 1 \\ 0 & -3 & 8 & 10 & 7 \end{array}\right]$

$R_2 \leftrightarrow R_3$.

$\sim \left[\begin{array}{cccc:c} 2 & 4 & 3 & 2 & 2 \\ 0 & 1 & -1 & -5 & 1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & -3 & 8 & 10 & 7 \end{array}\right]$

$R_4 \to R_4 + 3R_2$

$\sim \left[\begin{array}{cccc:c} 2 & 4 & 3 & 2 & 2 \\ 0 & 1 & -1 & -5 & 1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & -5 & 10 \end{array}\right]$

$R_4 \to R_4 - 5R_3$

$\sim \left[\begin{array}{cccc:c} 2 & 4 & 3 & 2 & 2 \\ 0 & 1 & -1 & -5 & 1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 5 & 20 \end{array}\right]$

Here, the no. of non-zero rows in reduced coefficient Matrix is 4
and,

$\therefore$ Rank$(A) = 4$.

The no. of non-zero rows in reduced Augmented Matrix is 4.

$\therefore$ Rank$([A:C]) = 4$

Solution of System of Equation (Test of Consistency & inconsistency by the help of Rank)

Let the system of equation be

$$\left.\begin{array}{l} a_1x_1 + b_1x_2 + c_1x_3 = d_1 \\ a_2x_1 + b_2x_2 + c_2x_3 = d_2 \\ a_3x_1 + b_3x_2 + c_3x_3 = d_3 \end{array}\right\}$$

be a system of equation.

Writing in matrix form.

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

or, $AX = C$

If $C = 0$, the system is called homogeneous equation.

The Augmented matrix is,

$$[A:C] = \begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{pmatrix}$$

- Find the rank of coefficient matrix A and augmented matrix $[A:C]$

Three cases may arise.

I. If $\text{Rank}(A) = \text{Rank}([A:C]) = $ No. of unknown variable $= 3$

Then the above sys. of Equation is consistent & give unique solution

i.e.
$$\begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & 0 & c_3' & : & d_3' \end{pmatrix}$$

II If $\text{Rank}(A) = \text{Rank}([A:C]) <$ No. of unknown variable $= 3$

Then the above sys. of Equation is Consistent & give infinite solution.

i.e.
$$\begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & 0 & 0 & : & 0 \end{pmatrix}$$

III If $\text{Rank}(A) \neq \text{Rank}([A:C])$

Then the above system of equation is inconsistent.

i.e.
$$\begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & 0 & 0 & : & d_3' \end{pmatrix}$$

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Is the following system consistent? $\left.\begin{array}{l} x+2y-z=3 \\ 3x-y+2z=1 \\ 2x-2y+3z=0 \end{array}\right\}$

If yes, solve.

Soln,

Now, the Augmented Matrix is

$$[A:C] = \left(\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \end{array}\right)$$

Applying $R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \end{array}\right]$$

Applying $R_3 \to 7R_3 - 6R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \end{array}\right]$$

The last matrix is in upper Triangular form and has 3 non-zero rows.

$\therefore$ Rank (A) = Rank $([A:C])$ = number of unknown variables = 3.

The above system of equation is consistent & unique solution.

Writing in equation form,

$$1x + 2y - 1z = 3$$
$$-7y + 5z = -8$$
$$5z = 20$$

On solving,

$z = 4$, $y = 4$, $x = -1$

Investigate the value of a and b, so that the system of equation, $\left.\begin{array}{l} x+y+z=6 \\ x+2y+3z=10 \\ x+2y+az=b. \end{array}\right\}$ has

(i) Unique solution
(ii). No solution
(iii) Infinite solution

Soln,

Now, the Augmented Matrix is,

$$[A:C] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & a & b \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & a-1 & b-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$.

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & a-3 & b-10 \end{array}\right]$$

Case I: The system of equation has unique solution if
Rank (A) = Rank ([A : C]) = no. of unknown variable = 3
$a-3 \neq 0$, $\therefore a \neq 3$ and b is any value

Case II: The system of equation has no solution if
Rank (A) $\neq$ Rank ([A : C])
If Rank (A) = 2 then $a-3 = 0$ $\therefore a = 3$
rank of ([A : C]) = 3, then, $b - 10 \neq 0$ $\therefore b \neq 10$.

Case III: The system of equation has infinite solution if
Rank (A) = Rank [A : C] < no. of unknown variable = 3
If Rank (A) = Rank [A : C] = 2, then
$a-3 = 0$, $b - 10 = 0$
$\therefore a = 3$, $b = 10$

In this case, the Augmented Matrix,

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then writing in eqn form

$$\left.\begin{array}{l} x + y + z = 6 \\ y + 2z = 4. \end{array}\right\}$$

$|A| = 0$, Rank = Order − 1

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If $z = k$, then $y = 4 - 2k$

Then, $x = 6 - y - z$

$= 6 - (4-2k) - k$.

$\therefore x = k+2$

$\therefore (x, y, z) = (k+2, 4-2k, k)$; $k \in R$

Example: For what value of p the system of equation $x + 2y + 3z = px$, $3x + y + 2z = py$, $2x + 3y + z = pz$ has a non zero solution & solve.

Soln,

$(1-p)x + 2y + 3z = 0$

$3x + (1-p)y + 2z = 0$

$2x + 3y + (1-p)z = 0$

Now, the determinant

$$|A| = \begin{vmatrix} 1-p & 2 & 3 \\ 3 & 1-p & 2 \\ 2 & 3 & (1-p) \end{vmatrix}$$

For the non-zero solution, i.e. infinite solution $|A| = 0$.

$|A| = 0$.

$$(1-p)\begin{vmatrix} 1-p & 2 \\ 3 & 1-p \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 1-p \end{vmatrix} + 3\begin{vmatrix} 3 & 1-p \\ 2 & 3 \end{vmatrix} = 0$$

or, $(1-p)\{(1-p)^2 - 6\} - 2\{3(1-p) - 4\} + 3\{9 - 2(1-p)\} = 0$.

or, $(1-p)^3 - 6(1-p) - 2\{3 - 3p - 4\} + 27 - 6 + 6p = 0$

or, $(1-p)^3 - 6 + 6p - 6 + 6p + 8 + 27 - 6 + 6p = 0$

or, $(1-p)^3 + 18p + 17 = 0$.

or, $(1-p)^2(1-p) + 18p + 17 = 0$

or, $(1-p)(1 - 2p + p^2) + 18p + 17 = 0$

or, $1 - p - 2p(1-p) + p^2(1-p) + 18p + 17 = 0$

or, $1 - p - 2p + 2p^2 + p^2 - p^3 + 18p + 17 = 0$

or, $-p^3 + 3p^2 + 15p + 18 = 0$

or, $p^3 - 3p^2 - 15p - 18 = 0$ —— (1)

Clearly $p = 6$ satisfies (1).

so $(p-6)$ is a factor of (1)

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$\therefore a^3 - 6a^2 + 3a^2 - 18a + 3a - 18 = 0$

or, $a^2(a-6) + 3a(a-6) + 3(a-6) = 0$

$\therefore (a-6)(a^2+3a+3) = 0$

Imaginary root $(b^2-4ac<0)$ i.e. complex root.

or, $a - 6 = 0$

$\therefore a = 6$ is only the solution.

Then above equation become

$-5x + 2y + 3z = 0$

$3x - 5y + 2z = 0$

$2x + 3y - 5z = 0.$

$$[A:C] = \left[\begin{array}{ccc|c} -5 & 2 & 3 & 0 \\ 3 & -5 & 2 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right]$$

Applying $R_2 \to 5R_2 + 3R_1$

$R_3 \to 5R_3 + 2R_1$

$$\sim \left[\begin{array}{ccc|c} -5 & 2 & 3 & 0 \\ 0 & -19 & 19 & 0 \\ 0 & 19 & -19 & 0 \end{array}\right]$$

$R_3 \to R_3 + R_2.$

$$\sim \left[\begin{array}{ccc|c} -5 & 2 & 3 & 0 \\ 0 & -19 & 19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Writing in equation form,

$$\left.\begin{array}{l} -5x + 2y + 3z = 0 \\ -19y + 19z = 0 \end{array}\right\}$$

If $z = k$, then $y = \frac{19k}{19} = k$, $x = k.$

$\therefore (x, y, z) = (k, k, k)$

For what value of p, the systems of equations

$$\left.\begin{array}{l} px + 2y - 2z = 1 \\ 4x + 2py - z = 2 \\ 6x + 6y + pz = 3 \end{array}\right\}$$

has a unique solution, then solve. Completely.

Soln, For unique solution,

$$\begin{vmatrix} p & 2 & -2 \\ 4 & 2p & -1 \\ 6 & 6 & p \end{vmatrix} \neq 0$$

or, $p\begin{vmatrix} 2p & -1 \\ 6 & p \end{vmatrix} - 2\begin{vmatrix} 4 & -1 \\ 6 & p \end{vmatrix} - 2\begin{vmatrix} 4 & 2p \\ 6 & 6 \end{vmatrix} \neq 0$

or, $p\{2p^2 + 6\} - 2\{4p + 6\} - 2\{24 - 12p\} \neq 0$

or, $2p^3 + 6p - 8p - 12 - 48 + 24p \neq 0$

or, $2p^3 + 22p - 60 \neq 0$

or, $p^3 + 11p - 30 \neq 0$

or, $p^3 - 2p^2 + 2p^2 - 4p + 15p - 30 \neq 0$

or, $p^2(p-2) + 2p(p-2) + 15(p-2) \neq 0$

$\therefore (p-2)(p^2 + 2p + 15) \neq 0$

Case I: If $p = 2$, the given eqn has infinite solution.
If $p \neq 2$, the given eqn has unique solution.

For infinite solution,

$2x + 2y - 2z = 1$

$4x + 4y - z = 2$

$6x + 6y + 2z = 3$

$$[A : C] = \left[\begin{array}{ccc|c} 2 & 2 & -2 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 8 & 0 \end{array}\right]$$

$2x + 2y - 2z = 1$

$\left.\begin{array}{l} 3z = 0 \\ 8z = 0 \end{array}\right\} z = 0$

$2x + 2y = 1$, Let $y = k$. Then $2x = 1 - 2k$

$x = \frac{1-2k}{2}$

$\therefore (x, y, z) = \left(\frac{1-2k}{2}, k, 0\right)$

Find the rank of the matrix?

Solve: $\left.\begin{aligned} x_1 + 9x_2 - 4x_3 &= 0 \\ x_1 + 9x_2 - 4x_3 &= 2 \end{aligned}\right\}$

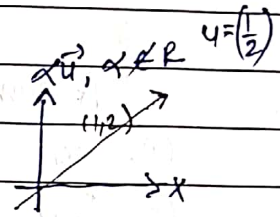
Solu,

Writing in Aug-matrix $\qquad R_2 \to R_2 - R_1$

$$[A:C] = \left[\begin{array}{ccc:c} 1 & 9 & -4 & 0 \\ 1 & 9 & -4 & 2 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$\therefore$ Rank(A) = 1

Rank([A:C]) = 2. $\qquad \therefore$ Eqns is inconsistent i.e. No solution.



Give a Geometrical meaning of

(i) Span(u) (ii) Span(u,v), where $u = \binom{1}{2}$ $v = \binom{1}{-1}$

(i) Geometrical span span(u) represents vectors $\{\vec{u} = \alpha \binom{1}{2}\}$ is a st. line passes through origin & point (1,2).

(ii) Span(u,v) = $\alpha\vec{u} + \beta\vec{v}$

It represents a plane containing $\vec{u}$ & $\vec{v}$

(iii) Span(u, v, w)

Cube

Without calculation find eigen value of two independent eigen vectors of

$$A = \begin{pmatrix} 4 & 4 & -4 \\ 4 & 4 & -4 \\ 4 & 4 & -4 \end{pmatrix} \qquad \therefore \lambda = 4,\ V_1 = \ldots$$

$$V_2 = \ldots$$

$$A = 4\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

Rank of a Matrix in terms of Minor

Defn

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a 3×3 matrix. Then,

(i) $a_{11}, a_{12}, a_{13}, \ldots$ are minor of order 1. (9)

(ii) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \ldots$ are minor of order 2 (9)

(iii) $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is a minor of order 3 (1)

A number r is said to be rank of a $m\times n$ matrix

(i) There exist at least one non-zero minor of order r, and

(ii) Every minor of order $r+1$ is zero.

Example: $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 3 & 1 & 4 \end{bmatrix}_{3\times3}$

Here, $|A| = \begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 3 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{vmatrix} = 2\times0 = 0$

$r+1 = 0$

$2+1 = 0 \quad \therefore r = 2$

Consider 2x2 matrix: $\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} = 4-6 = -2 \neq 0$

Thus, we see that there is a minor of order 2 is $\neq 0$ and every minor of order $2+1$ i.e. 3 is 0

$\therefore$ Rank$(A) = 2$.

Find the rank of $A = \begin{pmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{pmatrix}$

Consider a minor of order 4.

$$|A| = \begin{vmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{vmatrix}$$

$R_3 \to R_3 - R_2 - R_1$

$R_4 \to R_4 - R_3 - R_1$

$$= \begin{vmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Also, we have every minor of order 3 is zero.
So the rank can not be 3.

Consider 2x2 submatrix. $\begin{pmatrix} 6 & 1 \\ 4 & 2 \end{pmatrix}$

$\therefore \begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 12 - 4 \neq 0$

Thus we see there is a minor of order 2 is not zero.
& Every minor of order 3 is zero.
$\therefore$ Rank of $A = 2$.

Fact:

(i) For a $m \times n$ matrix A, $\text{Rank}(A) \leq \min\{m, n\}$
(ii) Every non singular square matrix of order n has rank n.
(iii) The rank of zero matrix is 0.

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U2 # Express the quadratic form $Q(X) = x_1x_2 - x_1x_3 + x_2x_3$ as the sum/difference of square.

Let $x_1 = x_2 + x'$

$Q(x) = (x_2+x')x_2 - (x_2+x')x_3 + x_2x_3$

$= x_2^2 + x'x_2 - x_2x_3 - x'x_3 + x_2x_3$

$= x_2^2 + x'x_2 - x'x_3$

$= x_2^2 + 2.x_2.\frac{x'}{2} + \left(\frac{x'}{2}\right)^2 - x'x_3 - \left(\frac{x'}{2}\right)^2$

$= \left(x_2 + \frac{x'}{2}\right)^2 - \left[\left(\frac{x'}{2}\right)^2 + 2.\frac{x'}{2}.x_3 + x_3^2\right] + x_3^2$

$= \left(x_2 + \frac{x'}{2}\right)^2 - \left(\frac{x'}{2} + x_3\right)^2 + x_3^2$

where, $x' = x_1 - x_2$.

Four Fundamental Subspace of a Matrix.

Consider a system of linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= c_2 \\ \dots\dots\dots\dots\dots& \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= c_m \end{aligned}\right\}$$

or,

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

or, $AX = C$

where, A is $m\times n$ matrix

X is $n\times 1$ matrix $\in R^n$

C is $m\times 1$ matrix $\in R^m$

Here, the matrix A maps from R^n to R^m

We have the following four subspaces :-

(I) Column Space $C(A)$

The column space of A is the set

$C(A) = \{ c \in R^m : AX = C \text{ , where } X \in R^n \}$

In other words, the column space of A is composed of the vectors $c \in R^m$ such that c can be written as the linear combination of the column A.

Note: The column space is the subspace of R^m.

(II) The Null Space $N(A)$

The Null space of A is the set

$N(A) = \{ X \in R^n : AX = 0 \}$

Clearly $N(A)$ is the subspace of R^n

In other words, $N(A)$ is the solution of the homogeneous equation $AX = 0$

Example:

Consider a matrix

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}_{2\times2}$$

Then, if $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{2\times1}$

It's eqn form,

$$\left.\begin{array}{l} x_1 + 3x_2 = 0 \\ 3x_1 + 9x_2 = 0 \end{array}\right\} \quad x_1 + 3x_2 = 0$$

$\therefore$ It's solution is $(-3k, k)$; $k \in R$

i.e. $k(-3, 1)$; $k \in R$

$$\therefore X = k \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad ; \quad k \in R$$

Here, the set X is called null space of matrix A.

Further, if we transpose the matrix A as

$B = A^T = a_{ij}$, then

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$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & & b_{2m} \\ \vdots & \vdots & & \\ b_{n1} & b_{m2} & & b_{nm} \end{pmatrix}, \text{ where } b_{ij} = a_{ji}$$

If we consider the equation,

$$\underset{n\times m}{A^T} Y_{m\times 1} = D_{n\times 1}$$

where, A^T is $n\times m$ matrix

Y is $m\times 1$ vector $\in R^m$

D is $n\times 1$ vector $\in R^m$

Thus, we have the following definition,

(III) The Row Space

The Row Space of A is the set

$C(A^T) = \{D \in R^n : A^TY = D$ for some $Y \in R^m\}$

Evidently row space of A is composed of all vector $D \in R^n$ such that, D can be written as the linear combination of column of A^T (or rows of A)

Note: row space is the subspace of R^m.

(IV) The Left null Space of A

The left null space of A is the set

$N(A^T) = \{Y \in R^m : A^TY = 0,\ 0 \in R^n\}$

In other words, left null space is the solution of homogeneous equation,

$$A^TY = 0$$

It is subspace of R^m.

Example: Consider a matrix, $A = \begin{pmatrix} 3 & 12 \\ 1 & 4 \end{pmatrix}$, find the left null space of A.

Null space of A is the solution of

$A^TX = 0$

or, $\begin{pmatrix} 3 & 1 \\ 12 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$3y_1 + y_2 = 0$

$y = k \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \; k \in R$

Here, the left null space is the collection of all vectors. Spanned by the vector,

$$N(A^T) = \left\{ y = k \begin{pmatrix} 1 \\ -3 \end{pmatrix} : k \in R \right\}$$

$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ → Desc.

Spectral Value Decomposition.

If A be a symmetric matrix, then $\exists$ a orthogonal matrix P (satisfy $P^T = P^{-1}$) such that,

$A = PDP^{-1}$ where D = Diagonal matrix $= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$(\because \lambda_1 \geq \lambda_2)$

What about the theorem for the rectangular matrix of order $m \times n$. The answer can be seen in Singular Value Decomposition Theorem.

Singular Value Decomposition Theorem.

It states that a rectangular $m \times n$ matrix A can be decomposed (broken) as the product of 3 factors.

$$\underset{m\times n}{A} = \underbrace{\underset{m\times m}{U}\,\underset{m\times n}{S}\,\underset{n\times n}{V^T}}$$

Where, (i) U and V are orthogonal matrix satisfying $UU^T = I$ & $VV^T = I$

$\therefore UU^T = UU^{-1} = I$

(ii) The columns of U are orthonormal vectors of AA^T

(iii) The columns of V are orthonormal vectors of A^TA

(iv) If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are k non zero eigen values of the matrix A^TA corresponding to the column vectors of V, then the diagonal matrix S look like.

$$S = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \sqrt{\lambda_k} & \cdots & 0 \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

In particular if $m = 3$, $n = 4$, $k = 2$ then S looks like

$$S = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where, $\sqrt{\lambda_1} \geq \sqrt{\lambda_2}$

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spectral = Eigen value (λ)

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(v) The values,

$$\sigma_1 = \sqrt{\lambda_1}, \ \sigma_2 = \sqrt{\lambda_2}, \ldots, \sigma_k = \sqrt{\lambda_k}$$

are called singular value of matrix A.

(vi) The column vector of U i.e. $\{u_1, u_2, \ldots, u_m\}$ are called left singular vectors of A.

(vii) The column vector of V i.e. $\{v_1, v_2, \ldots v_n\}$ are called Right singular vectors of A.

Example:

Let $A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}$. Find the singular value decomposition of A.

Soln,

To find U,

Now, $A' = AA^T = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 1 \\ 1 & 11 \end{pmatrix}$

For eigen values.

$|A' - \lambda I| = 0$

or $\begin{vmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{vmatrix} = 0$

or $(11-\lambda)^2 - 1^2 = 0$

or $(11-\lambda+1)(11-\lambda-1) = 0$

or $(12-\lambda)(10-\lambda) = 0$

$\therefore \lambda_1 = 12, \quad \lambda_2 = 10$

For eigen vectors,

$(A' - \lambda I)V = 0$

or $\begin{pmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or, $\left.\begin{aligned} (11-\lambda)x_1 + x_2 &= 0 \\ x_1 + (11-\lambda_1)x_2 &= 0 \end{aligned}\right\} \text{ --- } (x)$

If $\lambda = 12$

$\left.\begin{aligned} -x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}\right\}$

or, $x_1 - x_2 = 0$

If $x_2 = 1$, then $x_1 = 1$

$\therefore V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ u_1 = \frac{V_1}{||V_1||} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

If $\lambda = 10$

$\left.\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}\right\} \ x_1 + x_2 = 0$

If $x_2 = -1$, then $x_1 = 1$

$\therefore V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$u_2 = \frac{V_2}{||V_2||} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$U =$ Columns are orthonormal vectors of AA^T

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Again for V,

Now,

$$A'' = A^TA = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

For Eigen value, of A''

$$|A'' - \lambda I| = 0$$

$$\begin{vmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{vmatrix} = 0$$

or, $(10-\lambda)\begin{vmatrix} 10-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} - 0 \begin{vmatrix} & \\ & \end{vmatrix} + 2\begin{vmatrix} 0 & 10-\lambda \\ 2 & 4 \end{vmatrix} = 0$

or, $(10-\lambda)\{(10-\lambda)(2-\lambda) - 16\} + 2\{-2.(10-\lambda)\} = 0$

or, $(10-\lambda)^2(2-\lambda) - 16(10-\lambda) - 4.(10-\lambda) = 0$

or, $(10-\lambda)\{(10-\lambda)(2-\lambda) - 16 - 4\} = 0$

or, $(10-\lambda)\{(10-\lambda)(2-\lambda) - 20\} = 0$

or, $(10-\lambda)\{\cancel{4} - 12\lambda + \lambda^2 - \cancel{4}\} = 0$

or, $\lambda(10-\lambda)\{-12+\lambda\} = 0$

or, $\lambda = 0, \quad \lambda = 10, \quad \lambda = 12$

$\therefore \lambda_1 = 12, \quad \lambda_2 = 10, \quad \lambda_3 = 0$

For Eigen vectors,

$$(A'' - \lambda I)V = 0$$

$$\begin{pmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or, $(10-\lambda)x_1 + 2x_3 = 0$
$(10-\lambda)x_2 + 4x_3 = 0$
$2x_1 + 4x_2 + (2-\lambda)x_3 = 0$ — ⓧ

If $\lambda = 12$ Get non zero EV by putting (Don't solve)

$-2x_1 + 2x_3 = 0$
$-2x_2 + 4x_3 = 0$
$2x_1 + 4x_2 - 10x_3 = 0$

let $x_1 = 1$, then $x_3 = 1$
Then, $-2x_2 + 4x_3 = 0$
$\Rightarrow -2x_2 + 4 \times 1 = 0 \quad \therefore x_2 = 2$

clearly, third eqn satisfy $(1, 2, 1)$

$\therefore V_1 = (1, 2, 1)$

$$u_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}$$

If $\lambda = 10$, $V_2 = (2, -1, 0)$, $u_2 = (-2/\sqrt{5}, 1/\sqrt{5}, 0)$
If $\lambda = 0$, $V_3 = (1, 2, -5)$, $u_3 =$

$$S = \begin{pmatrix}\sqrt{\lambda_1} & 0 & 0\\ 0 & \sqrt{\lambda_2} & 0\end{pmatrix} = \begin{pmatrix}\sqrt{12} & 0 & 0\\ 0 & \sqrt{10} & 0\end{pmatrix}$$

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Orthonormal basis of Column Space and Row Space.

Let A be a $m \times n$ matrix with Singular Value Decomposition.

$$A = USV^T$$

Let the matrix A has $k \le n$ +ve Singular values $\sigma_1, \sigma_2, \dots, \sigma_k$

Then we have,

(i) The set of vectors $\{u_1, u_2, \dots, u_k\}$ are called orthonormal basis of the column space of the matrix A.

(ii) The set of vectors $\{v_1, v_2, \dots, v_k\}$ are called orthonormal basis of the row space of the matrix A.

Example:

In above decomposition,

Orthonormal basis of Column Space $= \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\}$

Orthonormal basis of Row Space $= \left\{ \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix} \right\}$

\# Find the singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -2 & 1 \end{bmatrix}$ in the form USV^T.

Also, find the orthonormal basis of column space of A and row space of A.

Steps: (Short Rule)

1. Find A^TA and find Eigen values λ_1, λ_2 & eigen vectors v_1, v_2 of A^TA
2. Find the singular values $\sigma_1, \sigma_2, \dots$
3. To find u_1, u_2 we use the formula, $\sigma_1 u_1 = Av_1$ and $\sigma_2 u_2 = Av_2$ } find u_1, u_2

In some cases if u_3 is to be known << Viz with AA^T order:

use $u_1 \cdot u_3 = 0$ & $u_2 \cdot u_3 = 0$

4. Write $A = USV^T$

Eigen Vector ⇒ Non Zero.

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Soln,

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -2 & 1 \end{pmatrix}$, $A'' = A^T A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}$

For Eigen values,

$|A'' - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 6-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$

or, $(6-\lambda)(3-\lambda) = 0$

$\therefore \lambda = 6,\ \lambda = 3$

$\therefore \lambda_1 = 6,\ \lambda_2 = 3$

For Eigen vectors,

$(A'' - \lambda I)V = 0$

or, $\begin{pmatrix} 6-\lambda & 0 \\ 0 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

or, $\left.\begin{array}{l} (6-\lambda)x_1 + 0x_2 = 0 \\ 0x_1 + (3-\lambda)x_2 = 0 \end{array}\right\}$.

If $\lambda = 6$

$\left.\begin{array}{l} 0x_1 = 0 \\ -3x_2 = 0 \end{array}\right\}$ — (*) Take $x_1 = 1$ (say) (Any)

$\Rightarrow x_2 = 0$

$\therefore V_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

If $\lambda = 3$

$3x_1 = 0 \Rightarrow x_1 = 0$

$0x_2 = 0$ Take $x_2 = 1$ (say)

$\therefore V_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

To find u_1, u_2, we use the formula,

$\sigma_1 u_1 = AV_1 \Rightarrow \sqrt{\lambda_1}\, u_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\sqrt{6}\, u_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \Rightarrow u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$\sigma_1 u_1 = \underset{3\times2}{A} \underset{2\times2}{V_1}$ → If not x then get A^T inplace of A

And, for u_2

$\sigma_2 u_2 = A V_2 \Rightarrow \sqrt{3}\, u_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow u_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

* $\underset{3\times3}{AA^T}$ $\underset{3\times2}{S}$ $\underset{2\times2}{A^TA}$ so, u_3 needed *

To find $\{u_1, u_2, u_3\}$ orthonormal vectors

For u_3

$u_1 \cdot u_3 = 0$

$u_2 \cdot u_3 = 0$

Take $u_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

So,

$u_1 \cdot u_3 = 0$ $\qquad$ $u_2 \cdot u_3 = 0$ (Vector)

or, $\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ $\qquad$ or, $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

or, $x_1 + x_2 - 2x_3 = 0$ — (*) $\qquad$ or $x_1 + x_2 + x_3 = 0$ — (**)

(*) − (**)

$x_1 + x_2 - 2x_3 = 0$

$x_1 + x_2 + x_3 = 0$

$- x_3 = 0$

$\therefore x_3 = 0$

From (*),

$x_1 + x_2 = 0$

Take $x_1 = 1 \Rightarrow x_2 = -1$.

$\therefore u_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

The unit vector along $u_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$

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$$\therefore U = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \end{pmatrix}_{3\times 3}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2} = V^T, \quad S = \begin{pmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}_{3\times 2}$$

Finally,

$A = USV^T$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Orthonormal basis of column space of $A = \left\{ \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \right\}$

Orthonormal basis of Row space of $A = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

Singular Value Decomposition.

$$A_{m\times n} = U_{m\times m} \; S_{m\times n} \; V^T_{n\times n}$$

$$= [u_1 \; u_2 \; u_r] \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_r \end{bmatrix}_{r\times r} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_r^T \end{bmatrix}_{r\times 1}$$

$[u_1 \; u_2 \; u_r]$ ↓ Left singular Vectors

$\begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \end{bmatrix}$ Transpose of right singular vectors.

$$\therefore A_{m\times n} = [u_1 \; u_2 \dots u_r] \begin{bmatrix} \sigma_1 v_1^T \\ \sigma_2 v_2^T \\ \vdots \\ \sigma_r v_r^T \end{bmatrix}$$

$$= u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T + \dots + u_r \sigma_r v_r^T$$

$$\therefore A_{m\times n} = \sum_{i=1}^{r} u_i \sigma_i v_i^T$$

Defn:

1. The number r of the singular values $\{\sigma_1, \sigma_2, \dots \sigma_r\}$ is called rank of the matrix A.

Note: From previous explanation,

$$u_i = \frac{1}{\sigma_i} A v_i$$

$$A v_i = u_i \sigma_i$$

Taking norm on both sides,

$$\|A v_i\| = \|u_i \sigma_i\|$$

$$= |\sigma_i| \, \|u_i\|$$

$$= \sigma_i \cdot 1 \quad (\because u_i \text{ are orthonormal})$$

$$\therefore \|A v_i\| = \sigma_i$$

Therefore,

$$\|A v_1\| = \sigma_1$$

$$\|A v_2\| = \sigma_2$$

$$\dots$$

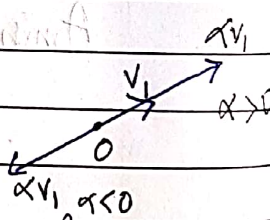
Thus, σ_i is the length of vector $A v_i$.

Recall: # Geometrical Meaning of Span

Let $v_1 v_2 \in \mathbb{R}^3$.

$\therefore$ Span $\{v_1\} = \{\alpha v_1 : \alpha \in \mathbb{R}\}$.

Span $\{v_1 v_2\} = \{\alpha_1 v_1 + \alpha_2 v_2 : \alpha_1, \alpha_2 \in \mathbb{R}\}$.



Then,

Span of single vector v_1 represents the st. line in $\mathbb{R}^3$ passes through origin.

The span of 2 vectors $\{v_1, v_2\} = \{\alpha_1 v_1 + \alpha_2 v_2 : \alpha_1, \alpha_2 \in \mathbb{R}\}$.

$\Rightarrow$ Parallelogram.

If v_1 & v_2 are not parallel then span $\{v_1 v_2\} = \{\alpha_1 v_1 + \alpha_2 v_2, \alpha_1, \alpha_2 \in \mathbb{R}\}$ represents the plane containing the vectors v_1 & v_2.

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Theorem: (Bilinear form)

If B is a symmetric bilinear form on $\mathbb{R}^n$, $B: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $B(u,v) = B_A(u,v) = u^T A v$, $u, v \in \mathbb{R}^n$ for some symmetric matrix A. Then A is unique.

proof:

Let $u, v \in \mathbb{R}^n$ & $\{e_1, e_2, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$,

where, $u = (u_1, u_2, \dots, u_n)$

$v = (v_1, v_2, \dots, v_n)$

Then u & v can be written as,

$$u = u_1 e_1 + u_2 e_2 + \dots + u_n e_n = \sum_{i=1}^{n} u_i e_i$$

$$\& \ v = v_1 e_1 + v_2 e_2 + \dots + v_n e_n = \sum_{j=1}^{n} v_j e_j$$

$B(\alpha u_1 + \beta u_2, v)$
$= \alpha B(u_1, v) + \beta B(u_2, v)$

Then using symmetric bilinear form,

$$B(u,v) = B(u_1 e_1 + u_2 e_2 + \dots + u_n e_n, v_1 e_1 + v_2 e_2 + \dots + v_n e_n)$$

Using linearity on the first component

$$= u_1 B(e_1, v_1 e_1 + v_2 e_2 + \dots + v_n e_n) + u_2 B(e_2, v_1 e_1 + v_2 e_2 + \dots + v_n e_n)$$
$$\dots\dots + u_n B(e_n, v_1 e_1 + v_2 e_2 + \dots + v_n e_n)$$

u, v = scalars
$u_i = c_i$
$v_i = d_i$

$$= \sum_{i=1}^{n} u_i B(e_i, v_1 e_1 + v_2 e_2 + \dots + v_n e_n)$$

Using linearity on the second component

$$= \sum_{i=1}^{n} u_i \left[v_1 B(e_i, e_1) + v_2 B(e_i, e_2) + \dots + v_n B(e_i, e_n) \right]$$

$$= \sum_{i=1}^{n} u_i \left[\sum_{j=1}^{n} v_j B(e_i, e_j) \right]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} u_i v_j B(e_i, e_j)$$

Since, $B(e_i, e_j)$ is a number, let $a_{ij} = B(e_i, e_j)$

$$\therefore B(u,v) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} u_i v_j$$

$$= (u_1 \ u_2 \dots u_n) \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$$\sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} u_i v_j$$
$$= \sum_{i=1}^{2} a_{i1} u_i v_1 + a_{i2} u_i v_2$$
$$= a_{11} u_1 v_1 + a_{12} u_1 v_2 + a_{21} u_2 v_1 + a_{22} u_2 v_2$$
$$= (u_1 \ u_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$= u_1 \ u_2 \begin{pmatrix} a_{11} v_1 + a_{12} v_2 \\ a_{21} v_1 + a_{22} v_2 \end{pmatrix}$$

Let, $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$

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Then,

$B(u,v) = u^TAV$

$= B_A(u,v)$

Thus we get the matrix A such that

$B(u,v) = B_A(u,v) = u^TAV.$ — (*)

Now we show that A is unique $= u^TBV$

If possible suppose there is another matrix B, Such that,

$B(u,v) = u^TBV$ — (**)

Then, from (*) & (**)

$u^TAV = u^TBV.$

or $u^T(A-B)V = 0 \quad \forall u,v \in R^n.$

or, $u^TCV = 0$, where, $C = A - B$ — (4)

Let $u = e_1, v = e_1$, (Generalize and let)

Then (4) becomes,

$$(1\ 0\ 0 \ldots 0)\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}_{n\times n}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}_{n\times 1}$$

$$\Rightarrow (1\ 0 \ldots 0)\begin{pmatrix}c_{11}\\c_{21}\\ \vdots \\ c_{n1}\end{pmatrix} = 0$$

$\Rightarrow c_{11} = 0.$

Similarly, $u = e_1, v = e_2$, then $c_{12} = 0$

$\ldots \quad c_{ij} = 0 \quad \forall i \neq j$

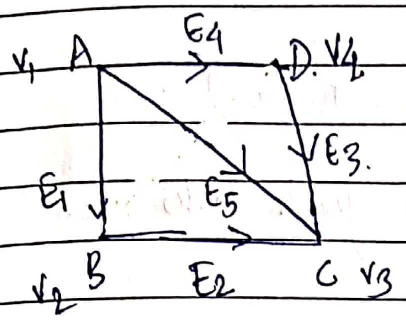
$\therefore C = 0.$

$A - B = 0$

$\therefore A = B$

A is unique.

Graph, Network & Incidence Matrix.



Edge = 5
Vertex = 4.
5x4

$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline E_1 & -1 & 1 & 0 & 0 \\ E_2 & 0 & -1 & 1 & 0 \\ E_3 & 0 & 0 & 1 & -1 \\ E_4 & -1 & 0 & 0 & 1 \\ E_5 & -1 & 0 & 1 & 0 \end{array}$$

Graph

A graph is the finite collections of nodes joined by edges. This graph consist of nodes & edges.

Example 1:

Consider the figure represented by graph, which can be described by the following electric circuit.

where,

(i) The nodes or vertex represents the point from which current flow.

(ii) The edges with the arrow represents the direction of flow.

Incidence Matrix.

The matrix represented by the directed graph is called incidence matrix

Note: In Incidence Matrix.

(i) One raw is associated for each edges.

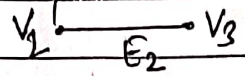
(ii) One column is associated for each node (or vertex)

In particular, if a graph consists of m edges & n nodes, then order of incidence is m×n.

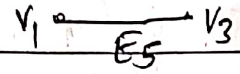
The above graph has incidence matrix of order 5×4

Note: If an edge runs from vertex v_1 to vertex v_3. Then the corresponding entry for v_1 is -1 & for v_3 is $+1$ & result is 0.

Example.



$$\begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{pmatrix} 0 & -1 & 1 & 0 \end{pmatrix} \end{array}$$

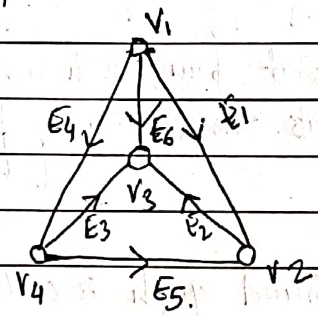


$$\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}$$

The incident matrix of above graph looks like

$$M(G) = \begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline E_1 & -1 & 1 & 0 & 0 \\ E_2 & 0 & -1 & 1 & 0 \\ E_3 & 0 & 0 & 1 & -1 \\ E_4 & -1 & 0 & 0 & 1 \\ E_5 & -1 & 0 & 1 & 0 \end{array}$$

Example: Find the incident matrix of the graph.



Here, number of edge $(m) = 6$.
number of vertices $(n) = 4$.

$\therefore$ The Order of Incidence matrix $M(G)$ of above graph is 6×4.

$$\begin{array}{c|cccc} & V_1 & V_2 & V_3 & V_4 \\ \hline E_1 & -1 & 1 & 0 & 0 \\ E_2 & 0 & -1 & 1 & 0 \\ E_3 & 0 & 0 & 1 & -1 \\ E_4 & -1 & 0 & 0 & 1 \\ E_5 & 0 & 1 & 0 & -1 \\ E_6 & -1 & 0 & 1 & 0 \end{array}$$