Unit 3: Spectral theory III

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Summary

1. Spectral decompositions
2. A larger example

Spectral decompositions

Theorem (Spectral decomposing)

Let $v_1, \ldots, v_n$ be the eigenvectors associated with the eigenvalues $\lambda_1, \ldots, \lambda_n$ of an $n \times n$ symmetric matrix A respectively. If

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \qquad V = \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix},$$

then

$$A = V \Lambda V^T \tag{1}$$

$$A = \lambda_1 v_1 v_1^T + \cdots + \lambda_n v_n v_n^T. \tag{2}$$

Problem

Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Express A as a product of three matrices and as a sum of three outer products.

Solution. Note that this matrix is symmetric. We will show that for this matrix, we can compute 3 eigenvalues and 3 corresponding eigenvectors that form an orthonormal and that thus form an orthonormal basis for $\mathbb{R}^3$.

$$\begin{bmatrix} 1-\lambda & 2 & 1 \\ 2 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix}$$

In this case,

$$\begin{aligned}|A-\lambda I| &= (1-\lambda)^2(2-\lambda)+2+2\\ &\quad -2(1-\lambda)-4(2-\lambda)\\ &= (1-\lambda)^2(2-\lambda)-2(1-\lambda)-4(1-\lambda)\\ &= (1-\lambda)[(1-\lambda)(2-\lambda)-6]\\ &= (1-\lambda)[\lambda^2-3\lambda-4]\\ &= (1-\lambda)(1+\lambda)(\lambda-4).\end{aligned}$$

We see that if $\lambda = 4, 1, -1$, then

$$|A-\lambda I| = 0.$$

So, the eigenvalues are $\lambda = 4, 1, -1$.

To compute the eigenvectors for each of these eigenvalues, we can use Gauss-Jordan reduction method that we won't cover here. Instead, we'll simply state the eigenvectors and verify that they are eigenvectors.

For $\lambda_1 = 4$,

$$\begin{pmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

From this we get

$$\begin{cases} -3x_1 + 2x_2 + x_3 & = 0 \\ 2x_1 - 3x_2 + x_3 & = 0 \\ x_1 + x_2 - 2x_3 = 0 \end{cases}$$

By observation, the eigenvector associated with $\lambda_1 = 4$ is

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1$,

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

From this we get

$$\begin{cases} 2x_2 + x_3 & = 0 \\ 2x_1 + x_3 & = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases}$$

By observation, the eigenvector associated with $\lambda_2 = 1$ is

$$u_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

For $\lambda_3 = -1$,

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

From this we get

$$\begin{cases} 2x_1 + 2x_2 + x_3 & = 0 \\ 2x_1 + 2x_2 + x_3 & = 0 \\ x_1 + x_2 + 3x_3 & = 0 \end{cases}$$

By observation, the eigenvector associated with $\lambda_3 = -1$ is

$$u_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

To verify that u_1, u_2, u_3 are orthogonal, we have

$$u_1 \cdot u_2 = u_1^T u_2 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 0$$

$$u_2 \cdot u_3 = u_2^T u_3 = \begin{pmatrix} 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

$$u_1 \cdot u_3 = u_1^T u_3 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

So, these three vectors are eigenvectors, and they are orthogonal, and so they provide a basis for $\mathbb{R}^3$.

To normalize them, we divide each of them by its norm.

Here are the normalized eigenvectors.

$$\lambda_1 = 4: \qquad v_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\lambda_2 = 1: \qquad v_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$$\lambda_3 = -1: \qquad v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

Let's now work with these normalized eigenvectors. In this case, the 3×3 matrix of normalized eigenvectors (the columns of which form an orthonormal basis for $\mathbb{R}^3$) is

$$V = (v_1 \ v_2 \ v_3) = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \end{pmatrix}.$$

Finally, verify that

$$A = V \Lambda V^T,$$
$$A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \lambda_3 v_3 v_3^T.$$

Since V is orthogonal, we have $V^T V = I$.
Therefore,

$$\Lambda = V^T A V.$$

We say that V orthogonally **diagonalizes matrix** A, if

$$\Lambda = V^T A V,$$

where Λ is the diagonal eigenvalue matrix.

Problem

Let λ_1 and λ_2 be the eigenvalues of a symmetric matrix A, and u_1 and u_2 the corresponding eigenvectors. If

$$\lambda_1 = 4, \lambda_2 = -1, u_1 = (2, 1), u_2 = (1, -2),$$

find A.

Problem

Let $A = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$. If

$$v_1 = (1, 2), v_2 = (2, -1)$$

are the eigenvectors of A,

Note that this matrix is symmetric. We will show that for this matrix, we can compute 3 eigenvalues and 3 corresponding eigenvectors that form an orthonormal and that thus form an orthonormal basis for $\mathbb{R}^3$.

$$\begin{bmatrix} 1-\lambda & 2 & 1 \\ 2 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix}$$

To do so, consider

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 4 & 3 \\ 4 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{pmatrix}$$

In this case,

$$\begin{aligned}|A-\lambda I| &= (1-\lambda)((1-\lambda)^2-0)-4(4(1-\lambda)-0)\\ &\quad +3(0-3(1-\lambda))\\ &= (1-\lambda)^3-25(1-\lambda)\\ &= (1-\lambda)((1-\lambda)^2-25)\\ &= (1-\lambda)(\lambda-6)(\lambda+4).\end{aligned}$$

We see that if $\lambda=-4,1,6$, then

$$|A-\lambda I|=0.$$

So, the eigenvalues are $\lambda=-4,1,6$.

To compute the eigenvectors for each of these eigenvalues, we can use Gauss-Jordan reduction method that we won't cover here. Instead, we'll simply state the eigenvectors and verify that they are eigenvectors.

Here they are: $\lambda_1 = 6,\ v_1 = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 30 \\ 24 \\ 18 \end{pmatrix} = 6 \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}.$$

Similarly, $\lambda_1 = 1,\ v_2 = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} = 1 \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}.$$

And $\lambda_1 = -4,\ v_3 = \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$:

$$\begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} = -4 \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}.$$

To verify that the eigenvectors v_1, v_2, v_3 are orthogonal let $V_O = (v_1\ v_2\ v_3)$ and let's multiply it by it's transpose:

$$V_O^T V_O = \begin{pmatrix} 5 & 4 & 3 \\ 0 & -3 & 4 \\ -5 & 4 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 & -5 \\ 4 & -3 & 4 \\ 3 & 4 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 50 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 50 \end{pmatrix}$$

So, these three vectors are eigenvectors, and they are orthogonal, and so they provide a basis for $\mathbb{R}^3$.

To normalize them, we divide by their norms, the square of which are the diagonal elements of $V_O^T V_O$.

Here are the normalized eigenvectors.

$$\lambda_1 = 6 : \qquad v_1 = \frac{1}{\sqrt{50}} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

$$\lambda_1 = 1 : \qquad v_2 = \frac{1}{\sqrt{25}} \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$$

$$\lambda_1 = -4 : \qquad v_1 = \frac{1}{\sqrt{50}} \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}.$$

Let's now work with these normalized eigenvectors. In this case, the 3×3 matrix of normalized eigenvectors (the columns of which form an orthonormal basis for $\mathbb{R}^3$) is

$$V = (v_1\ v_2\ v_3) = \begin{pmatrix} 5/\sqrt{50} & 0 & -5/\sqrt{50} \\ 4/\sqrt{50} & -3/\sqrt{25} & 4/\sqrt{50} \\ 3/\sqrt{50} & 4/\sqrt{25} & 3/\sqrt{50} \end{pmatrix}.$$

Clearly, $V^T V = I$. Moreover,

$$AV = V\Lambda.$$

Here is how do we express the matrix in two standard forms.

Expressing A as a sum of 3 outer products. We have

$$\begin{aligned}
\sum_{i=1}^{2} \lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \lambda_3 v_3 v_3^T \\
&= 6 \begin{pmatrix} 5/\sqrt{50} \\ 4/\sqrt{50} \\ 3/\sqrt{50} \end{pmatrix} \begin{pmatrix} 5/\sqrt{50} & 4/\sqrt{50} & 3/\sqrt{50} \end{pmatrix} \\
&+ 1 \begin{pmatrix} 0 \\ -3/\sqrt{25} \\ 4/\sqrt{25} \end{pmatrix} \begin{pmatrix} 0 & -3/\sqrt{25} & 4/\sqrt{25} \end{pmatrix} \\
&- 4 \begin{pmatrix} -5/\sqrt{50} \\ 4/\sqrt{50} \\ 3/\sqrt{50} \end{pmatrix} \begin{pmatrix} -5/\sqrt{50} & 4/\sqrt{50} & 3/\sqrt{50} \end{pmatrix} \\
&= A.
\end{aligned}$$

Expressing A as a product of 3 matrices.

We have

$$
\begin{aligned}
VAV^T &= (v_1\ v_2\ v_3) \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \end{pmatrix} \\
&\dots \\
&= A
\end{aligned}
$$