$$= \int_0^5 \int_0^{\frac{z}{2}} \left[ \frac{12y^2}{2} - 8x^2 y \right]_0^{10-2z} dx\, dz$$

$$= \int_0^5 \int_0^{z/2} 6(10-2z)^2 - 8x(10-2z)\, dx\, dz$$

$$= \int_0^5 \int_0^{z/2} 600 - 240z + 24z^2 - 80x + 16xz\, dx\, dz$$

$$= \int_0^5 \left[ 600x - 240zx + 24z^2 x - \frac{80x^2}{2} + \frac{16zx^2}{2} \right]_0^{z/2} dz$$

$$= \int_0^5 300z - 120z^2 + 12z^3 - 10z^2 + 2z^3\, dz$$

$$= \int_0^5 14z^3 - 130z^2 + 300z\, dz$$

$$= \left[ \frac{14z^4}{4} - \frac{130z^3}{3} + \frac{300z^2}{2} \right]_0^5$$

$$= \frac{7 \times 5^4}{4} - \frac{130 \times 5^3}{3} + 150 \times 5^2$$

$$= 520.83 \#$$

⑥ Evaluate $\iiint_E yz\, dV$ where $E$ is the region bounded by $x = 2y^2 + 2z^2 - 5$ & the plane $x = 1$.

Soln:

$$2y^2 + 2z^2 - 5 \le x \le 1$$

$$\therefore V = \iiint_E yz\, dV = \iint_D \int_{2y^2+2z^2-5}^{1} (yz)\, dx\, dy\, dz$$

$$= \iint_D yz(1 - 2y^2 - 2z^2 + 5)\, dy\, dz$$

$$= \iint_D yz(6 - 2(y^2 + z^2))\, dy\, dz$$

Now,

$$x = 2y^2 + 2z^2 - 5$$
$$\Rightarrow 1 = 2y^2 + 2z^2 - 5$$
$$\Rightarrow y^2 + z^2 = 3$$

In polar coordinates; if $z = r\cos\theta$ & $y = r\sin\theta$

$$\Rightarrow y^2 + z^2 = 3$$
$$\Rightarrow r^2 = 3$$
$$\therefore 0 \le r \le \sqrt{3} \quad \& \quad 0 \le \theta \le 2\pi$$

Also, $yz(6 - 2(y^2+z^2))\,dy\,dz = r^2 \sin\theta\cos\theta\,(6 - 2r^2)\,r\,dr\,d\theta$
$= r^3 \sin 2\theta\,(3 - r^2)\,dr\,d\theta$

$$\therefore V = \iiint_E yz\,dv = \iint_D yz(6 - 2(y^2+z^2))\,dy\,dz$$
$$= \int_0^{2\pi}\int_0^{\sqrt{3}} r^3 \sin 2\theta\,(3 - r^2)\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \sin 2\theta\,(3r^3 - r^5)\,dr\,d\theta$$
$$= \int_0^{2\pi} \sin 2\theta \left[\frac{3r^4}{4} - \frac{r^6}{6}\right]_0^{\sqrt{3}} d\theta$$
$$= \int_0^{2\pi} \sin 2\theta \left[\frac{27}{4} - \frac{27}{6}\right] d\theta$$
$$= \frac{9}{4}\left[\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$
$$= \frac{9}{8}\left[\cos 4\pi - \cos 0\right]$$
$$= \frac{9}{8}[1 - 1]$$
$$= 0 \,\#$$

(7) Evaluate $\iiint_E 15z\,dV$ where $E$ is the region bet- $2x+y+z=4$ & $4x+4y+2z=20$ that is in front of the region in the $yz$-plane bounded by $z=2y^2$ & $z=\sqrt{4y}$.

Sol$^n$:

$2x+y+z=4$ —(i) $\Rightarrow x = 2-\frac{y}{2}-\frac{z}{2}$

$4x+4y+2z=20$ —(ii) $\Rightarrow x = 5-y-\frac{z}{2}$

$\Rightarrow 2-\frac{y}{2}-\frac{z}{2} \le x \le 5-y-\frac{z}{2}$

Also,

$z=2y^2$ & $z=\sqrt{4y}$

$\Rightarrow 2y^2 \le z \le 2\sqrt{y}$

For $y$; $z=2y^2=\sqrt{4y}$

$\Rightarrow 2y^2 = 2\sqrt{y}$

$\Rightarrow y^2=\sqrt{y}$

$\Rightarrow y^4 = y$

$\Rightarrow y^4-y=0$

$\Rightarrow y(y^3-1)=0$

$\Rightarrow y(y-1)(y^2+y+1)=0$

$\Rightarrow y=0, y=1$ are real roots

$\Rightarrow 0\le y\le 1$

$$\therefore V = \iiint_E 15z\,dv = \int_0^1\int_{2y^2}^{2\sqrt{y}}\int_{2-\frac{y}{2}-\frac{z}{2}}^{5-y-\frac{z}{2}} (15z)\,dx\,dz\,dy$$

$$= \int_0 \int_{2y^2} 15z\left(5 - y - \frac{z}{2} - 2 + \frac{y}{2} + \frac{z}{2}\right) dz\, dy$$

$$= \int_0^1 \int_{2y^2}^{2\sqrt{y}} 15z\left(3 - \frac{y}{2}\right) dz\, dy$$

$$\cancel{= \int_0^1 \left(3 - \frac{y}{2}\right)\left(2\sqrt{y} - 2y^2\right) dy} \qquad \cancel{= \int_0^1 \int_{2y^2}^{2\sqrt{y}} \left(45z - \frac{15}{2}yz\right) dz\, dy}$$

$$\cancel{= \int_0^1 \left(6\sqrt{y} - 6y^2 - y^{3/2} + y^3\right) dy} \qquad \cancel{= \int_0^1 \left[\frac{45z^2}{2} - \frac{15yz^2}{4}\right]_{2y^2}^{2\sqrt{y}} dy}$$

$$\cancel{= \left[\frac{6y^{3/2}}{\left(\frac{3}{2}\right)} - \frac{6y^3}{3} - \frac{y^{5/2}}{\left(\frac{5}{2}\right)} + \frac{y^4}{4}\right]_0^1} \qquad \cancel{= \int_0^1 90y - 15y^2 - 90y^4 +}$$

$$= \cancel{4 - 2 - \frac{2}{5} + \frac{1}{4}}$$

$$= \int_0^1 15\left(3 - \frac{y}{2}\right)\left[\frac{z^2}{2}\right]_{2y^2}^{2\sqrt{y}} dy$$

$$= \int_0^1 \frac{15}{2}\left(3 - \frac{y}{2}\right)\left(4y - 4y^4\right) dy$$

$$= 30\int_0^1 \left(3y - 3y^4 - \frac{y^2}{2} + \frac{y^5}{2}\right) dy$$

$$= 30\left[\frac{3y^2}{2} - \frac{3y^5}{5} - \frac{y^3}{6} + \frac{y^6}{12}\right]_0^1$$

$$= 30\left[\frac{3}{2} - \frac{3}{5} - \frac{1}{6} + \frac{1}{12}\right]$$

$$= 24.5 \#$$

⑧ Use a triple integral to determine the volume of the region below $z = 4 - xy$ & above the region in the xy-plane defined by $0 \le x \le 2,\ 0 \le y \le 1$.

Sol$^n$:

$$V = \iiint_E dV = \int_0^2 \int_0^1 \int_0^{4-xy} dz\, dy\, dx$$

$$= \int_0^2 \int_0^1 (4 - xy)\, dy\, dx$$

$$= \int_0^2 \left[4y - \frac{xy^2}{2}\right]_0^1 dx$$

$$= \int_0^2 \left(4 - \frac{x}{2}\right) dx$$

$$= \left[4x - \frac{x^2}{4}\right]_0^2$$

$$= 8 - 1$$

$$= 7 \#$$

⑨ Use a triple integral to determine the volume of the region that is below $z = 8 - x^2 - y^2$ & above of the region $z = -\sqrt{4x^2 + 4y^2}$ & inside $x^2 + y^2 = 4$.

Sol$^n$.

$$V = \iiint_E dV = \iint \int_{-\sqrt{4x^2+4y^2}}^{8-x^2-y^2} dz\, dy\, dx$$

$$= \iint \left(8 - x^2 - y^2 + \sqrt{4x^2 + 4y^2}\right) dy\, dx$$

$\therefore x^2+y^2=4 \Rightarrow r=4$ in polar coordinates,

$\Rightarrow 0 \le r \le 2$ & $0 \le \theta \le 2\pi$

$\therefore (8-x^2-y^2+\sqrt{4x^2+4y^2})\,dy\,dx = (8-r^2+2r)\,r\,dr\,d\theta$

$\therefore V=\iiint_E dV = \int_0^{2\pi}\int_0^{2}(8r-r^3+2r^2)\,dr\,d\theta$

$= \int_0^{2\pi}\left[\frac{8r^2}{2}-\frac{r^4}{4}+\frac{2r^3}{3}\right]_0^2 d\theta$

$= \int_0^{2\pi}\left(16-4+\frac{16}{3}\right)d\theta$

$= \frac{52}{3}\times 2\pi$

$= \frac{104\pi}{3}$ #

# Change of Variables - (Practice Problems)

For Problems 1-3 compute the Jacobian of each transformation.

1. $x = 4u-3v^2,\ y = u^2-6v$

Soln:

$x = 4u-3v^2,\ y=u^2-6v$

$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 4 & -6v \\ 2u & -6 \end{vmatrix} = -24+12uv$$ #

(2.) $x = u^2 v^3$, $y = 4 - 2\sqrt{u}$

Sol$^n$:

$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2uv^3 & 3u^2v^2 \\ -u^{-1/2} & 0 \end{vmatrix} = 3u^{3/2}v^2 \#$$

(3) $x = \frac{v}{u}$, $y = u^2 - 4v^2$

Sol$^n$:

$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ 2u & -8v \end{vmatrix} = \frac{8v^2}{u^2} - 2 \#$$

(4) If $R$ is the region inside $\frac{x^2}{4} + \frac{y^2}{36} = 1$. Determine the region we would get applying the transformation $x = 2u$, $y = 6v$ to $R$.

Sol$^n$:

$$\frac{x^2}{4} + \frac{y^2}{36} = 1$$

This is an ellipse. Now applying the transformation

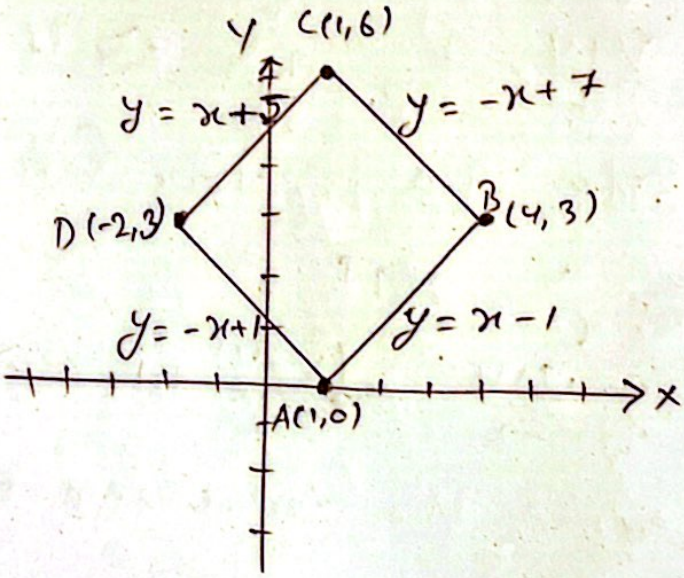
$x = 2u$, $y = 6v$

$$\Rightarrow \frac{(2u)^2}{4} + \frac{(6v)^2}{36} = 1$$

$\Rightarrow u^2 + v^2 = 1$

$\Rightarrow$ The transformed region is a unit circle.

(5) If $R$ is the parallelogram with vertices $(1,0)$, $(4,3)$, $(1,6)$ & $(-2,3)$ determine the region we would get applying the transformation $x = \frac{v-u}{2}$, $y = \frac{v+u}{2}$ to $R$.




Soln:

Let the vertices of the parallelogram be $A(1,0)$, $B(4,3)$, $C(1,6)$ & $D(-2,3)$.

The eqn of ~~line~~ side AB is,

$$y - 0 = \frac{3-0}{4-1}(x-1) \Rightarrow y = x-1$$

The eqn of side BC is,

$$y - 3 = \frac{6-3}{1-4}(x-4) \Rightarrow y = -x+7$$

The eqn of side CD is,

$$y - 6 = \frac{3-6}{-2-1}(x-1) \Rightarrow y = x+5$$

The eqn of side DA is,

$$y - 3 = \frac{0-3}{1-(-2)}(x-(-2)) \Rightarrow y = -x+1$$

Now applying transformations $x = \frac{v-u}{2}$, $y = \frac{v+u}{2}$

Side AB is transformed as,

$$y = x-1$$

$$\frac{v+u}{2} = \frac{v-u}{2} - 1 \Rightarrow u = -1$$

Side BC is transformed as,

$y = -x + 7$

$\frac{v+u}{2} = -\frac{(v-u)}{2} + 7 \Rightarrow v = 7$

Side CD is transformed as;

$y = x + 5$

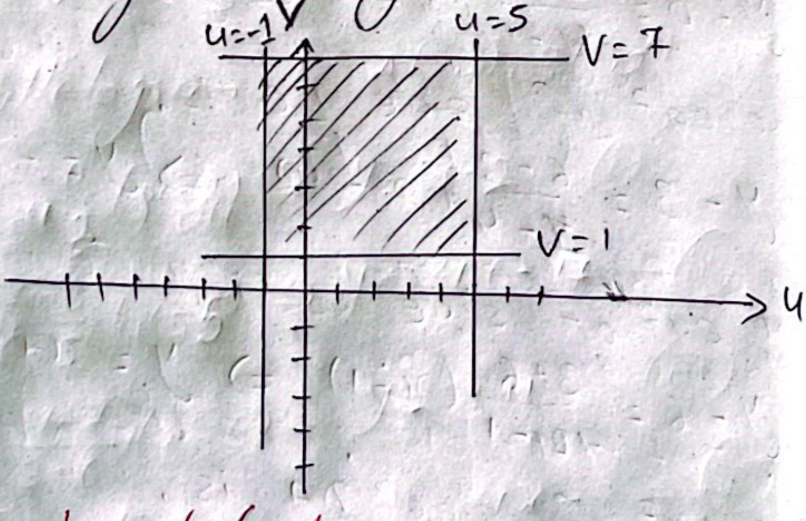
$\frac{v+u}{2} = \frac{v-u}{2} + 5 \Rightarrow u = 5$

Side DA is transformed as,

$y = -x + 1$

$\Rightarrow \frac{v+u}{2} = -\frac{(v-u)}{2} + 1 \Rightarrow v = 1$

Hence, the transformed figure is given as




(6) If R is the region bounded by $xy = 1$, $xy = 3$, $y = 2$ & $y = 6$ determine the region we would get applying the transformation $x = \frac{v}{6u}$, $y = 2u$ to R.

Sol^n:

$xy = 1$ — (i) $y = 2$ — (iii)

$xy = 3$ — (ii) $y = 6$ — (iv)

Now applying the transformations, $x = \frac{v}{6u}$, $y = 2u$

From (i);

$xy = 1$

$\Rightarrow \frac{v}{6u} \times 2u = 1$

$\Rightarrow v = 3$ — (a)

From (ii);

$xy = 3$

$\Rightarrow \frac{v}{6u} \times 2u = 3$

$\Rightarrow v = 9$ — (b)

From (iii);

$y = 2$

$2u = 2$

$\Rightarrow u = 1$ — (c)

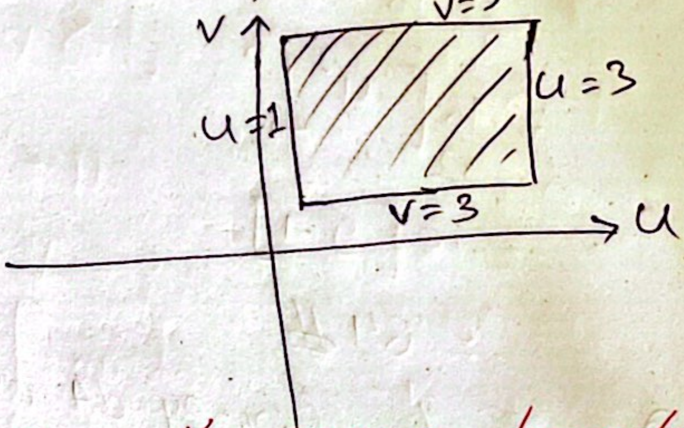
From (iv);

$y = 6$

$2u = 6$

$\Rightarrow u = 3$ — (d)

Mapping the transformations:




7) Evaluate $\iint_R xy^3\,dA$ where $R$ is the region bounded by $xy = 1$, $xy = 3$, $y = 2$ & $y = 6$ using the transformation $x = \frac{v}{6u}$, $y = 2u$.

Sol$^{n}$:

$xy = 1$ — (i), $xy = 3$ — (ii), $y = 2$ — (iii), $y = 6$ — (iv)

Applying transformations; $x = \frac{v}{6u}$ & $y = 2u$ we get;

$v = 3$, $v = 9$, $u = 1$, $u = 3$ $\Rightarrow 1 \le u \le 3$ & $3 \le v \le 9$

$$\text{Jacobian} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{v}{6u^2} & \frac{1}{6u} \\ 2 & 0 \end{vmatrix} = -\frac{1}{3u}$$

$$\therefore \iint_R xy^3\,dA = \int_{u=1}^{u=3}\int_{v=3}^{v=9} xy^3 \times |\text{Jacobian}|\,dv\,du$$

$$= \int_1^3\int_3^9 \frac{v}{6u} \times (2u)^3 \times \left|-\frac{1}{3u}\right| dv\,du$$

$\because$ The range of $u$ is from 1 to 3 which is +ve so $\left|-\frac{1}{3u}\right| = \frac{1}{3u}$

$$= \int_1^3\int_3^9 \frac{4}{9} uv\,dv\,du$$

$$= \int_1^3 \left[ \frac{4}{9} u \, \frac{v^2}{2} \right]_3^9 du$$

$$= \int_1^3 \frac{4u}{9} \left( \frac{9^2}{2} - \frac{3^2}{2} \right) du$$

$$= \int_1^3 16u \, du$$

$$= \frac{16}{2} \left[ u^2 \right]_1^3$$

$$= 8[9-1]$$

$$= 64 \ \#$$

(8) Evaluate $\iint_R 6x - 3y \, dA$ where $R$ is the parallelogram with vertices $(2,0)$, $(5,3)$, $(6,7)$ & $(3,4)$ using the transformation $x = \frac{v-u}{3}$, $y = \frac{4v-u}{3}$ to $R$.

Sol$^n$:

Let the vertices of parallelogram be $A(2,0)$, $B(5,3)$, $C(6,7)$ & $D(3,4)$.

Eq$^n$ of side AB is : $y - 0 = \frac{3-0}{5-2}(x-2) \Rightarrow y = x - 2$ —— (i)

Eq$^n$ of side BC is : $y - 3 = \frac{7-3}{6-5}(x-5) \Rightarrow y = 4x - 17$ —— (ii)

Eq$^n$ of side CD is : $y - 7 = \frac{4-7}{3-6}(x-6) \Rightarrow y = x + 1$ —— (iii)

Eq$^n$ of side DA is : $y - 4 = \frac{0-4}{2-3}(x-3) \Rightarrow y = 4x - 8$ —— (iv)

Applying transformations : $x = \frac{v-u}{3}$, $y = \frac{4v-u}{3}$ ;

From (i);

$y = x - 2$

$\frac{4v-u}{3} = \frac{v-u}{3} - 2$

$\Rightarrow v = -2$

From (ii);

$y = 4x - 17$

$\frac{4v-u}{3} = 4\left(\frac{v-u}{3}\right) - 17$

$\Rightarrow u = -17$

From (iii);

$y = x + 1$

$\frac{4v-u}{3} = \frac{v-u}{3} + 1$

$\Rightarrow v = 1$

From (iv);

$y = 4x - 8$

$\frac{4v-u}{3} = \frac{4(v-u)}{3} - 8$

$\Rightarrow u = -8$

$\Rightarrow -17 \le u \le -8$ & $-2 \le v \le 1$.

Also, $x = \frac{v-u}{3}$ & $y = \frac{4v-u}{3}$ so, Jacobian $= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$= \begin{vmatrix} -1/3 & 1/3 \\ -1/3 & 4/3 \end{vmatrix} = -\frac{1}{3}$

$\therefore \iint_R 6x - 3y \, dA = \int_{-17}^{-8} \int_{-2}^{1} 6\frac{(v-u)}{3} - 3\frac{(4v-u)}{3} \times \left| -\frac{1}{3} \right| du\, dv$

$= \int_{-17}^{-8} \int_{-2}^{1} -\frac{1}{3}(2v+u) \, dv \, du$

$= \int_{-17}^{-8} -\frac{1}{3}\left[ \frac{2v^2}{2} + uv \right]_{-2}^{1} du$

$= \int_{-17}^{-8} -\frac{1}{3}\left[ 1 + u - 4 + 2u \right] du$

$= \int_{-17}^{-8} 1 - u \, du$

$= \left[ u - \frac{u^2}{2} \right]_{-17}^{-8}$

$= -8 - \frac{64}{2} + 17 + \frac{17^2}{2}$

$= 121.5$ #

(9) Evaluate $\iint_R (x+2y)\, dA$ where $R$ is the triangle with vertices $(0,3)$, $(4,1)$ & $(2,6)$ using the transformations $x = \frac{1}{2}(u-v)$, $y = \frac{1}{4}(3u+v+12)$ to $R$.

Soln: Let the vertices of $\Delta$ be $A(0,3)$, $B(4,1)$ & $C(2,6)$

Eqn of side AB is : $y - 3 = \frac{1-3}{4-0}(x-0) \Rightarrow y = -\frac{x}{2} + 3$ — (i)

Eqn of side BC is : $y - 1 = \frac{6-1}{2-4}(x-4) \Rightarrow y = -\frac{5x}{2} + 11$ — (ii)

Eqn of side CA is : $y - 6 = \frac{3-6}{0-2}(x-2) \Rightarrow y = \frac{3x}{2} + 3$ — (iii)

Applying transformations: $x = \frac{u-v}{2}$ & $y = \frac{3u+v+12}{4}$

From eq$^n$ (i);

$y = -\frac{x}{2} + 3$

$\frac{3u+v+12}{4} = \frac{-(u-v)}{4} + 3$

$\Rightarrow u = 0$ — (iv)

from eq$^n$ (ii);

$y = -\frac{5x}{2} + 11$

$\frac{3u+v+12}{4} = \frac{-5(u-v)}{4} + 11$

$\Rightarrow v = 2u - 8$ — (v)

From eq$^n$ (iii);

$y = \frac{3x}{2} + 3$

$\frac{3u+v+12}{4} = \frac{3(u-v)}{4} + 3$

$\Rightarrow v = 0$ — (vi)

From (vi);

$v = 2u - 8$

$0 = 2u - 8$

$\Rightarrow u = 4$

$\therefore 0 \le u \le 4$,

$2u - 8 \le v \le 0$.

$$\text{Jacobian} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{2}$$

$$\therefore \iint_R (x+2y)\,dA = \int_0^4 \int_{2u-8}^0 \left(\frac{u-v}{2} + \frac{2(3u+v+12)}{4}\right) \left|\frac{1}{2}\right| \, dv\, du$$

$$= \int_0^4 \int_{2u-8}^0 (u+3)\, dv\, du$$

$$= \int_0^4 (u+3)(-2u+8)\, du$$

$$= \int_0^4 -2u^2 + 8u - 6u + 24\, du$$

$$= \int_0^4 -2u^2 + 2u + 24\, du$$

$$= \left[-\frac{2u^3}{3} + \frac{2u^2}{2} + 24u\right]_0^4$$

$$= 69.33 \,\#$$

(10) Derive the transformation used in problem 8.

Sol$^n$:

In problem (8) there was a parallelogram with vertices $A(2,0)$, $B(5,3)$, $C(6,7)$, $D(3,4)$

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Also, the eq$^{ns}$ of sides of parallelogram was;

$y = x-2 \Rightarrow y-x = -2$ — (i)

$y = 4x-17 \Rightarrow y-4x = -17$ — (ii)

$y = x+1 \Rightarrow y-x = 1$ — (iii)

$y = 4x-8 \Rightarrow y-4x = -8$ — (iv)

Let, $y-x = v$ & $y-4x = u$

from eq$^{ns}$ (i) to (iv) we get; $v=-2$, $v=1$, $u=-17$, $u=-8$

Solving, $y-x = v$ & $y-4x = u$ as,

$y = v+x \Rightarrow v+x-4x = u$

$\Rightarrow v-3x = u$

$\Rightarrow x = \frac{v-u}{3}$

Also,

$y - \left(\frac{v-u}{3}\right) = v$

$\Rightarrow y = v + \frac{v-u}{3} = \frac{4v-u}{3}$

$\therefore$ The transformation is, $x = \frac{v-u}{3}$ & $y = \frac{4v-u}{3}$

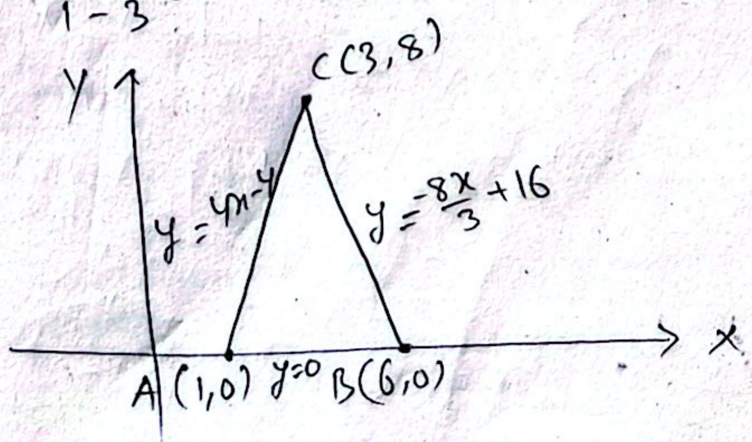
(ii) Derive a transformation that will convert the $\Delta$ with vertices $(1,0)$, $(6,0)$ & $(3,8)$ into a right triangle with the right angle occurring at the origin of the uv system.

Sol$^n$: Let the vertices of $\Delta$ be $A(1,0)$, $B(6,0)$, $C(3,8)$.

Eq$^n$ of side AB is: $y-0 = \frac{0-0}{6-1}(x-1) \Rightarrow y = 0$ — (i)

Eq$^n$ of side BC is; $y-0 = \frac{8-0}{3-6}(x-6) \Rightarrow y = -\frac{8x}{3}+16$ — (ii)

Eq$^n$ of side CA is: $y-8 = \frac{0-8}{1-3}(x-3) \Rightarrow y = 4x-4$ — (iii)

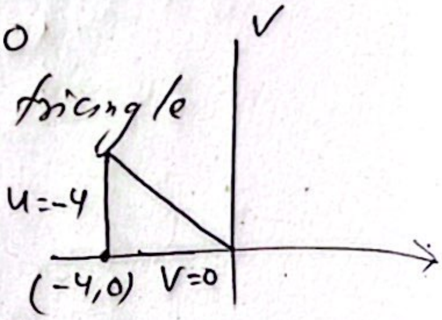


Let, $u = y - 4x$ & $v = y$

This will transform, eq$^n$ (iii) as,

$u = -4$ & eq$^n$ (ii) as $v = 0$

This will create a right angled triangle but at $(-4,0)$. Our goal is to get this $\Delta$ at $(0,0)$. So, let's fix the values of $u$ & $v$ as,




$$u = y - 4x + 4 \quad \& \quad v = y$$

Now, ~~from eq$^n$ (iii)~~;

$\cancel{y = 4x - 4}$

$\Rightarrow \cancel{y = u - 4 + 8}$

$\Rightarrow \cancel{y = u}$

$\Rightarrow u = v - 4x + 4$

$\Rightarrow 4x = 4 + v - u$

$\Rightarrow x = \dfrac{4 + v - u}{4}$

Now, Using this transformation to eq$^n$ (i), (ii) & (iii) we get;

From (i);

$y = 0$

$\Rightarrow v = 0$

From (ii);

$y = -\dfrac{8x}{3} + 16$

$\Rightarrow v = -\dfrac{8}{3} \times \dfrac{(4 + v - u)}{4} + 16$

$\Rightarrow v = \dfrac{2u}{5} + 8$

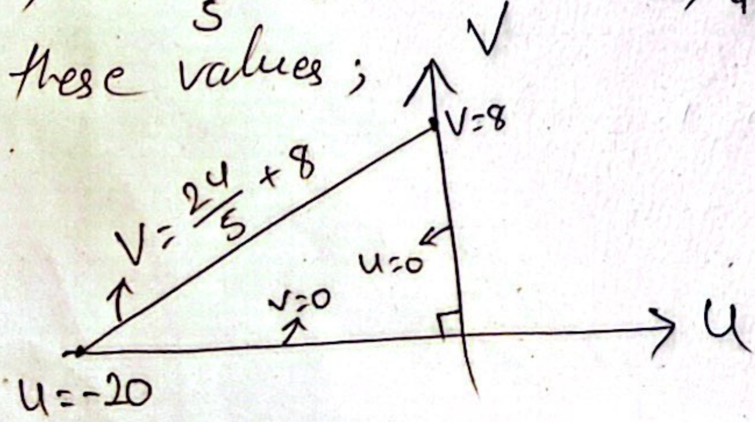
From (iii);

$y = 4x - 4$

$v = 4\dfrac{(4 + v - u)}{4} -$

$\Rightarrow u = 0.$

Mapping these values;



# Surface Area

1. Determine the surface area of the portion of $2x+3y+6z=9$ that is in the 1st octant.

**Sol^n:**

$$2x+3y+6z=9 \quad \text{--- (i)}$$

$$\Rightarrow 6z = 9-2x-3y$$

$$\Rightarrow z = \frac{3}{2} - \frac{x}{3} - \frac{y}{2}$$

$$\Rightarrow f(x,y) = \frac{3}{2} - \frac{x}{3} - \frac{y}{2}$$

$$f_x(x,y) = -\frac{1}{3}, \quad f_y(x,y) = -\frac{1}{2}$$

$$\therefore \text{Surface Area} = \iint_D \sqrt{f_x^2(x,y) + f_y^2(x,y) + 1}\, dA$$

$$= \iint_D \sqrt{(-\tfrac{1}{3})^2 + (-\tfrac{1}{2})^2 + 1}\, dA$$

$$= \iint_D \frac{7}{6}\, dA$$

For the range of $y$;

put $(z=0)$, in (i)

$$2x+3y=9 \quad \text{--- (ii)}$$

$$\Rightarrow 3y = 9-2x$$

$$\Rightarrow y = 3 - \frac{2x}{3} \Rightarrow 0 \le y \le 3 - \frac{2x}{3}$$

For range of $x$;

put $(y=0)$ in (ii);

$$2x = 9$$

$$\Rightarrow x = 9/2 \quad \Rightarrow 0 \le x \le 9/2$$

$$\therefore \text{Surface Area} = \int_0^{9/2} \int_0^{3-\frac{2x}{3}} \frac{7}{6}\, dA$$

Surface Area $= \int_0^{9/2} \frac{7}{6}\left(3 - \frac{2x}{3}\right) dx$

$= \int_0^{9/2} \left(\frac{7}{2} - \frac{7x}{9}\right) dx$

$= \left[\frac{7x}{2} - \frac{7x^2}{18}\right]_0^{9/2}$

$= \frac{7\times 9}{4} - \frac{7\times 81}{18\times 4}$

$= 7.875$ #

② Determine the surface area of the portion of $z = 13 - 4x^2 - 4y^2$ that is above $z = 1$ with $x \le 0$ & $y \le 0$.

Sol$^n$: $z = 13 - 4x^2 - 4y^2 = f(x,y)$

$f_x(x,y) = -8x$ , $f_y(x,y) = -8y$

$\therefore$ Surface Area $= \iint_D \sqrt{f_x^2(x,y) + f_y^2(x,y) + 1}\ dA$

$= \iint_D \sqrt{(-8x)^2 + (-8y)^2 + 1}\ dA$

$= \iint_D \sqrt{64x^2 + 64y^2 + 1}\ dA$

$= \iint_D \sqrt{64(x^2 + y^2) + 1}\ dA$

Here, $z = 13 - 4x^2 - 4y^2$ & also $z = 1$, so,

$1 = 13 - 4x^2 - 4y^2$

$\Rightarrow x^2 + y^2 = 3$

Using polar cordinate system we get;

$r^2 = 3 \Rightarrow 0 \le r \le \sqrt{3}$

also, $x = 0$ & $y = 0$

$\Rightarrow r\cos\theta = 0$ $\Rightarrow r\sin\theta = 0$

$\Rightarrow \cos\theta = \cos\frac{\pi}{2}$ $\Rightarrow \sin\theta = \sin 0$

$\Rightarrow \theta = \frac{\pi}{2}$ $\Rightarrow \theta = 0$

$\therefore 0 \le \theta \le \frac{\pi}{2}$

and. $\sqrt{64(x^2+y^2)+1}\, dA = \left(\sqrt{64r^2+1}\right) r\, dr\, d\theta$

$\therefore$ Surface area $= \int_0^{\pi/2}\int_0^{\sqrt{3}} \sqrt{64r^2+1}\; r\, dr\, d\theta$

Let, $r^2 = p$ $\Rightarrow 2r\,dr = dp$ $\left.\begin{matrix} \text{For } r=0, p=0 \\ \text{For } r=\sqrt{3}, p=3 \end{matrix}\right\}$

$= \int_0^{\pi/2}\int_0^{3} \left(\sqrt{64p+1}\right)\frac{1}{2}\, dp\, d\theta$

$= \int_0^{\pi/2} \frac{1}{2}\left[\frac{(64p+1)^{3/2}}{\frac{3}{2}\times 64}\right]_0^3 d\theta$

$= \int_0^{\pi/2} \frac{1}{2}\times\frac{2}{3}\times\frac{1}{64}\left[(64\times 3+1)^{3/2} - 1^{3/2}\right] d\theta$

$= 13.95 \times \frac{\pi}{2}$

$= 21.92$ #

(3) Determine the surface area of the portion of $z = 3 + 2y + \frac{x^4}{4}$ that is above the region in the xy-plane bounded by $y = x^5$, $x = 1$ & the x-axis.

Sol$^n$:

$$z = 3 + 2y + \frac{x^4}{4} = f(x,y)$$

~~Surf~~ $\therefore f_x(x,y) = x^3, \quad f_y(x,y) = 2$

$$\therefore \text{Surface Area} = \iint_D \sqrt{f_x^2(x,y) + f_y^2(x,y) + 1}\, dA$$

$$= \iint_D \sqrt{(x^3)^2 + 2^2 + 1}\, dA = \iint_D \sqrt{x^6 + 5}\, dA$$

Here, x-axis $\Rightarrow y = 0$ so, $0 \leq y \leq x^5$

& for $y = 0 \Rightarrow 0 = x^5$

$\Rightarrow x = 0$ (is a root)

$\Rightarrow 0 \leq x \leq 1$

$$\therefore \text{Surface Area} = \int_0^1 \int_0^{x^5} \sqrt{x^6 + 5}\, dy\, dx$$

$$= \int_0^1 x^5 \sqrt{x^6 + 5}\, dx$$

Let, $x^6 = p$ $\Rightarrow 6x^5 dx = dp$ $\Rightarrow x^5 dx = \frac{1}{6} dp$ $\Bigg\}$ For $x = 0 \Rightarrow p = 0$; for $x = 1 \Rightarrow p = 1$

$$= \frac{1}{6} \int_0^1 \sqrt{p + 5}\, dp$$

$$= \frac{1}{6} \left[ \frac{(p+5)^{3/2}}{3/2} \right]_0^1 = \frac{1}{6} \times \frac{2}{3} \left[ 6^{3/2} - 5^{3/2} \right]$$

(1) Determine the surface area of the portion of $y = 2x^2 + 2z^2 - 7$ that is inside the cylinder $x^2 + z^2 = 4$.

Sol$^n$:

$$y = 2x^2 + 2z^2 - 7 = f(x, z)$$

$$f_x(x, z) = 4x, \quad f_z(x, z) = 4z$$

$\therefore$ surface Area $= \iint_D \sqrt{f_x^2(x,z) + f_z^2(x,z) + 1}\, dA$

$$= \iint_D \sqrt{(4x)^2 + (4z)^2 + 1}\, dA$$

$$= \iint_D \sqrt{16(x^2 + z^2) + 1}\, dA$$

$\because x^2 + z^2 = 4$

Using polar coordinate system as, $x = r\cos\theta$
$z = r\sin\theta$

$\Rightarrow r^2 = 4$

$\Rightarrow 0 \le r \le 2 \;\&\; 0 \le \theta \le 2\pi$

Also, $\sqrt{16(x^2+z^2)+1}\, dA = \left(\sqrt{16r^2+1}\right) r\,dr\,d\theta$

$\therefore$ Surface Area $= \int_0^{2\pi} \int_0^2 \left(\sqrt{16r^2+1}\right) r\,dr\,d\theta$

Let, $r^2 = p$
$\Rightarrow 2r\,dr = dp$
$\Rightarrow r\,dr = \frac{1}{2}dp$

For $r = 0 \Rightarrow p = 0$
For $r = 2 \Rightarrow p = 4$

$\therefore$ Surface Area $= \int_0^{2\pi} \int_0^4 \left(\sqrt{16p+1}\right) \frac{1}{2}\, dp\, d\theta$

$\therefore$ Surface Area $= \int_0^{2\pi} \frac{1}{2}\left[\frac{(16P+1)^{3/2}}{3/2 \times 16}\right]_0^4 d\theta$

$= \int_0^{2\pi} \frac{1}{2} \times \frac{2}{3} \times \frac{1}{16}\left[65^{3/2} - 1^{3/2}\right] d\theta$

$= \int_0^{2\pi} 10.89\, d\theta$

$= 10.89 \times 2\pi$

$=$ ~~36.17~~ $68.4$

(5) Determine the surface area of region formed by the intersection of two cylinders $x^2+y^2=4$ & $x^2+z^2=4$.

Sol$^n$:

$x^2+z^2=4$

$\Rightarrow z^2 = 4-x^2$

$\Rightarrow z = \pm\sqrt{4-x^2}$

$f_x(x,y) = \frac{1}{2}(4-x^2)^{-1/2} \times (-2x) = \frac{\pm x}{\sqrt{4-x^2}}$

$f_y(x,y) = 0$

Surface Area $= \iint_D \sqrt{f_x^2(x,y) + f_y^2(x,y) + 1}\; dA$

$= \iint_D \sqrt{\frac{x^2}{4-x^2} + 0 + 1}\; dA$

$= \iint_D \frac{2}{\sqrt{4-x^2}}\, dA$

Here, $x^2+y^2=4$

$\Rightarrow y = \pm\sqrt{4-x^2} \Rightarrow -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

Also, $x^2+z^2=4$

For $(z=0)$, $x^2=4$

$\Rightarrow x = \pm 2 \Rightarrow -2 \le x \le 2$

$\therefore$ Surface Area $= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{2}{\sqrt{4-x^2}}\, dy\, dx$

$= \int_{-2}^{2} \frac{2}{\sqrt{4-x^2}} \times \left[\sqrt{4-x^2}+\sqrt{4-x^2}\right] dx$

$= \int_{-2}^{2} 4\, dx$

$= \left[4x\right]_{-2}^{2}$

$= 4\times 2 - 4\times(-2)$

$= 16$ #

$y = mx$

$\frac{1}{\infty} \to \infty$

$\frac{mx^2}{x^2+m^2x^2}$

$\frac{m}{1+m^2}$

$(1, 10)$

$\frac{4}{1+16\times 16}$

$\frac{1}{2}$

$\frac{1}{1+1}$

$+x^3y$

$\frac{3mx^3}{x^2+1}$

$\frac{x^3y}{x^2+y^4}$

$\frac{m^2x^3}{x^2+m^4x^4}$

$\frac{m^2}{1+m^4x^2}$

$\frac{1.x}{2x}$

$\frac{xy}{x^2+y^2}$

$\frac{mx^2}{x^2+m^2x^2}$

$\frac{m}{1+m^2}$

$\frac{m^2m}{m^2} \to \frac{1}{2}$

$\frac{x^2-m^2x^2}{x^2+m^2x^2}$

$\frac{x^2(1-m^2)}{x^2(1+m^2)}$