Unit 4: Double Integrals in Rectangles

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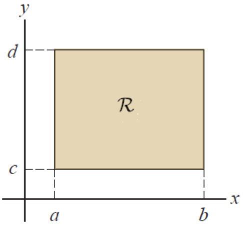
Summary

1. **Double integrals**
   - Subdivision
   - Geometric interpretation
   - The Midpoint Rule
   - Properties of Double Integrals
2. **Iterated integration**
   - Partial integration and iterated integrals
3. **Double integrals and iterated integrals**

Domain of integration: Rectangle

We consider a rectangle in $\mathbb{R}^2$ given by

$$\begin{aligned} R &= [a,b] \times [c,d] \\ &= \{(x,y) \in \mathbb{R}^2 : \ a \le x \le b, \ c \le y \le d\}. \end{aligned}$$



Let

$$A = \text{Area of } R.$$

Solid under a surface above a rectangle

Also, we will initially assume that

$$z = f(x, y) \geq 0.$$

The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x, y, z) \in \mathbb{R}^3 : \ 0 \leq z \leq f(x, y), \ (x, y) \in R\}.$$

Here is a sketch of this set up.

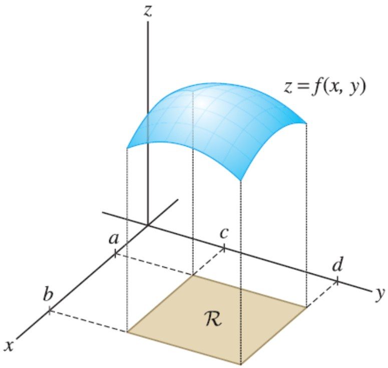


Figure 1: The solid above R and under f.

We now ask: What is the volume of the solid S?
Ans.: It is expressed by a *double integral*.

Three-step process of definition

Like integrals in one variable, double integrals are defined through a three-step process: **subdivision**, **summation**, and **passage to the limit**.

We first divide up $[a, b]$ into m subintervals of equal width $\Delta x = (b - a)/m$ and $[c, d]$ into n subintervals of equal width $\Delta y = (d - c)/n$ by choosing partitions:

$$a = x_0 < x_1 < \ldots < x_m = b, c = y_0 < y_1 < \ldots < y_n = d,$$

where m and n are positive integers to create an $n \times m$ grid of subrectangles R_{ij}.

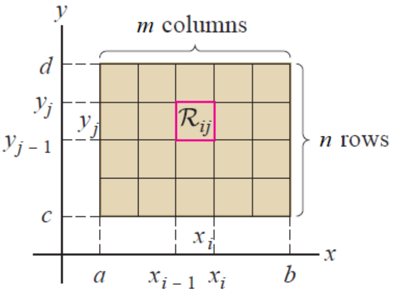


Figure 2: An $n \times m$ grid of R

The area of each subrectangle R_{ij} is given by

$$\Delta A = \Delta x \Delta y.$$

From each of these subrectangles we will choose a point (x_i^*, y_j^*), as shown in the figure given below.

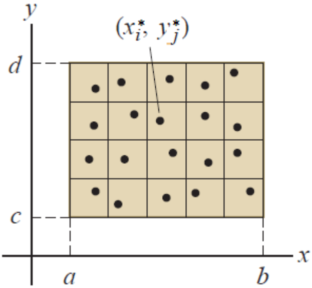


Figure 3: An $n \times m$ grid of R with sample points (x_i^*, y_j^*)

Now, over each of these smaller rectangles we will construct a box whose height is given by $f(x_i^*, y_j^*)$. See the figure given below.

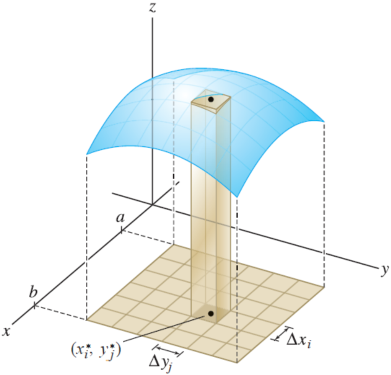


Figure 4: A box with volume $f(x_i^*, y_j^*)\Delta A$

Each of the rectangles has a base area of ΔA and a height of $f(x_i^*, y_j^*)$ so the volume of each of these boxes is

$$f(x_i^*, y_j^*)\Delta A.$$

Summation

The volume of the solid S is now approximated as follows:

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A.$$

Double Riemann sum

A **double Riemann sum** is defined as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A.$$

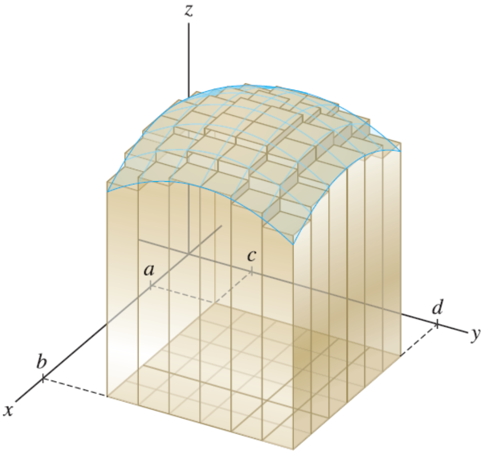


Figure 5: The solid S is approximated by the sum of the volumes of all boxes.

Passage to the limit

- We have a double sum since we will need to add up volumes in both the x and y directions.
- To get a better estimation of the volume we will take n and m larger and larger.
- And to get the exact volume we will need to take the limit as both n and m go to infinity.

In other words,

$$V = \lim_{n,m\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A.$$

This looks a lot like the definition of the integral of a function of single variable. In fact, this is also the definition of an integral of a function of two variables over a rectangle.

Here is the formal definition of a double integral of a function of two variables over a rectangle R as well as the notation that we'll use for it.

Double integral over a rectangle

$$\iint_R f(x,y)\,dA = \lim_{n,m\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A.$$

The sample point (x_i^*, y_j^*) in the definition can be chosen to be any point (x_i, y_j) in the subrectangle.

If f happens to be a positive function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 5, and is an approximation to the volume of the solid under the graph of f and above the rectangle R. Thus, we have the following difinition:

Geometric interpretation of a double integral

If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$\text{Volume } = \iint_R f(x,y)\, dA.$$

Example

Estimate the volume of the solid that lies above the square

$$R = [0, 2] \times [0, 2]$$

and below the elliptic paraboloid

$$z = 16 - x^2 - 2y^2.$$

Divide R into four equal squares and choose the sample point to be the upper right corner of each square . Sketch the solid and the approximating rectangular boxes.

Solution

The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$. The squares are shown in Figure 6.

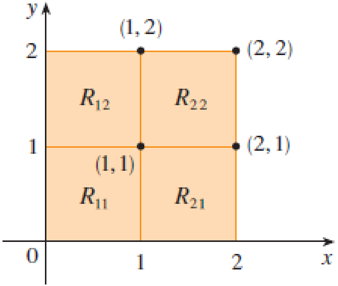


Figure 6:

Solution ...

Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$
\begin{aligned}
V &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_i)\Delta A \\
&= f(1,1)\Delta A + f(1,1)\Delta A + f(1,2)\Delta A + f(2,2)\Delta A \\
&= 13(1) + 7(1) + 10(1) + 4(1) = 34.
\end{aligned}
$$

Solution ...

This is the volume of the approximating rectangular boxes shown in Figure 7.

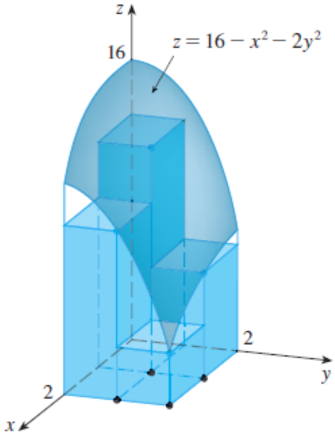


Figure 7:

Let $\overline{x}_i$ be the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ be the midpoint of $[y_{j-1}, y_j]$. Then we can choose the center $(\overline{x}_i, \overline{y}_j)$ of R_{ij} as the sample point (x_i^*, y_j^*).

Midpoint Rule for Double Integrals

$$\iint_A f(x,y)\ dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\Delta A,$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Example

Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y)^2 \, dA$, where

$$R = \{(x, y) |\ 0 \le x \le 2, 1 \le y \le 2\}.$$

Solution

In using the Midpoint Rule with $m = n = 2$, we evaluate $f(x, y) = (x - 3y^2)$ at the centers of the four subrectangles shown in Figure 8.

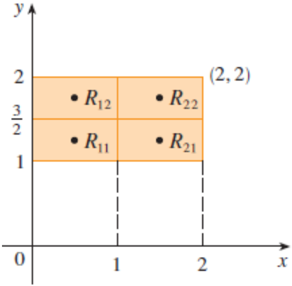


Figure 8:

Solution...

Since $m = n = 2$, we have

$\Delta x = 1 - 0 = 2 - 1 = 1,$

$\Delta y = \frac{3}{2} - 1 = 2 - \frac{3}{2} = \frac{1}{2}.$

The area of each subrectangle is

$\Delta A = \Delta x \Delta y = 1 \times \frac{1}{2} = \frac{1}{2}.$

We also have

$\overline{x}_1 = \frac{1}{2}, \overline{x}_2 = \frac{3}{2}, \overline{y}_1 = \frac{5}{4}$, and $\overline{y}_2 = \frac{7}{4}.$

Thus, the centers of the rectangles $R_{11}, R_{12}, R_{21}, R_{22}$ are respectively

$(\frac{1}{2}, \frac{5}{4}),\ (\frac{1}{2}, \frac{7}{4}),\ (\frac{3}{2}, \frac{5}{4}),\ (\frac{3}{2}, \frac{7}{4}).$

Solution...

Thus

$$\begin{aligned}\iint_R (x-3y^2)\,dA &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i,\overline{y}_j)\Delta A\\ &= f(\overline{x}_1,\overline{y}_1)\Delta A + f(\overline{x}_1,\overline{y}_2)\Delta A + f(\overline{x}_2,\overline{y}_1)\Delta A\\ &\quad + f(\overline{x}_2,\overline{y}_2)\Delta A\\ &= f(\tfrac{1}{2},\tfrac{5}{4})\Delta A + f(\tfrac{1}{2},\tfrac{7}{4})\Delta A + f(\tfrac{3}{2},\tfrac{5}{4})\Delta A\\ &\quad + f(\tfrac{3}{2},\tfrac{7}{4})\Delta A\\ &= (-\tfrac{67}{16})\tfrac{1}{2} + (-\tfrac{130}{16})\tfrac{1}{2} + (-\tfrac{51}{16})\tfrac{1}{2} + (-\tfrac{123}{16})\tfrac{1}{2}\\ &= -\tfrac{95}{8} = -11.875.\end{aligned}$$

Average Value

Let f be a function of two variables defined on a rectangle R. We define the average value of f to be

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\, dA,$$

where $A(R)$ is the area of R.
If $f(x, y \geq 0$, the equation

$$A(R) \times f_{ave} = \iint_R f(x,y)\, dA$$

says that the box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f.

Here are some properties of the double integral. Note that all three of these properties are really just extensions of properties of single integrals that have been extended to double integrals.

Some properties of the double integral

1. $$\iint_R [f(x,y) + g(x,y)]\, dA$$
$$= \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$$

2. If c is a constant, then
$$\iint_R cf(x,y)\, dA = c \iint_R f(x,y)\, dA$$

3. If $f(x,y) \geq g(x,y)$ for all in $(x,y) \in R$, then
$$\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA$$

Iterated Integrals

Just like with the definition of a single integral, it is usually difficult to evaluate double integrals from first principles. So we need to start looking into how we actually compute double integrals.

In the previous unit we found that it was useful to differentiate functions of several variables with respect to one variable, while treating all the other variables as constants or coefficients. We can integrate functions of several variables in a similar way.

For instance, suppose that $f_x(x, y) = 2xy$. We can treat y as staying constant and integrate to obtain $f(x, y)$:

$$\begin{aligned} f(x,y) &= \int f_x(x,y)\ dx \\ &= \int 2xydx \\ &= \int x^2y + C. \end{aligned}$$

Make a careful note about the constant of integration, C. This "constant" is something with a derivative of 0 with respect to x, so it could be any expression that contains only constants and functions of y.

For instance, if

$$f(x, y) = x^2y + \sin y + y^3 + 17,$$

then $f_x(x, y) = 2xy$. To signify that C is actually a function of y, we write:

$$f(x, y) = \int f_x(x, y)\, dx = x^2y + C(y).$$

Using this process we can evaluate definite integrals.

Example

Evaluate the integral $\int_1^4 2xy\ dx$.

Solution. We consider y as a constant and integrate with respect to x:

$$\begin{aligned}\int_1^4 2xy\,dx &= x^2y\Big|_1^4\\ &= 4^2y - 1^2y\\ &= 15y.\end{aligned}$$

We have considered y to be a constant. So, the limits of the integral may be functions of y as in the above example.

Example

Evaluate the integral $\displaystyle\int_1^{2y} 2xy\ dx$.

Solution. We consider y as a constant and integrate with respect to x:

$$\begin{aligned}\int_1^{2y} 2xy\,dx &= x^2y\Big|_1^{2y}\\ &= (2y)^2y - 1^2y\\ &= 4y^3 - y.\end{aligned}$$

Remark

Note how the limits of the integral are from $x = 1$ to $x = 2y$ and that the final answer is a function of y.

Example

Evaluate the integral $\int_1^x (5x^3y^{-3} + 6y^2)\ dy$.

Solution. Here, we consider x to be a constant and integrate with respect to y:

$$\int_1^x (5x^3y^{-3} + 6y^2)\, dy = \left(\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3}\right)\Bigg|_1^x$$
$$= \left(-\frac{5}{2}x^3x^{-2} + 2x^3\right) - \left(-\frac{5}{2}x^3\right.$$
$$= \frac{9}{2}x^3 - \frac{5}{2}x - 2.$$

Remark

Note how the limits of the integral are from $y = 1$ to $y = x$ and that the final answer is a function of x.

We can integrate the result obtained in the previous example with respect to x as well. This process is known as **iterated integration**, or **multiple integration**.

Example

Evaluate the integral

$$\int_1^2 \left(\int_1^x (5x^3 y^{-3} + 6y^2)\ dy \right) dx.$$

Solution. We follow a standard "order of operations" and perform the operations inside parentheses first (which is the integral evaluated in the previous example.

$$\begin{aligned}
& \int_1^2 \left(\int_1^x (5x^3y^{-3} + 6y^2)\ dy \right) dx \\
= & \int_1^2 \left(\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3} \right) \Bigg|_1^x dx \\
= & \int_1^2 \left(\frac{9}{2}x^3 - \frac{5}{2}x - 2 \right)\ dx \\
= & \left(\frac{9}{8}x^4 - \frac{5}{4}x^2 - 2x \right) \Bigg|_1^2 \\
= & \frac{89}{8}.
\end{aligned}$$

Remark

Note how the limits of the integral are from $x = 1$ to $x = 2$ and that the final result was a number.

The previous example showed how we could perform something called an *iterated integral*; we do not yet know why we would be interested in doing so nor what the result, such as the number 89/8, means. Before we investigate these questions, we offer some definitions.

We will continue to assume that we are integrating $f(x, y)$ over the rectangle

$$R = [a, b] \times [c, d].$$

If $x = x_0$ is kept fixed, we obtain a cross-section bounded by vertical lines $y = c$ and $y = d$, the horizontal line $z = 0$, and by the curve $z = f(x_0, y)$. The area of the cross-section is therefore given by

$$A(x_0) = \int_c^d f(x_0, y)\, dy.$$

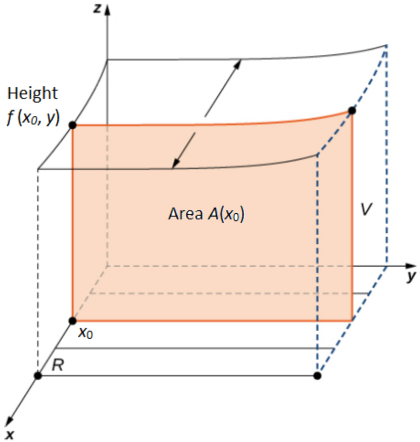


Figure 9: The cross-section $A(x_0)$.

Partial integration

We use the notation

$$\int_c^d f(x,y)\,dy$$

to mean that x is held fixed and $f(x,y)$ is integrated with respect to y from c to d. This procedure is called **partial integration** with respect to y.

We see that the cross-section area $\int_c^d f(x,y)\,dy$ is a number that depends on the value of x, so it defines a function of x:

$$\boxed{A(x) = \int_c^d f(x,y)\,dy.}$$

We now integrate the function A with respect to x from a to b. We then get

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx.$$

The integral on the right side is called an **iterated integral**. Usually the brackets are omitted.

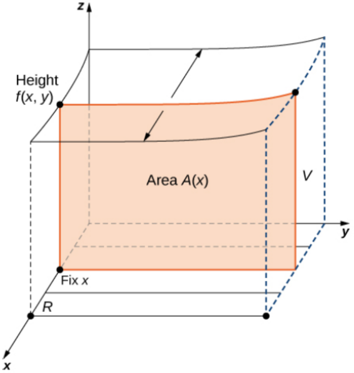
Thus

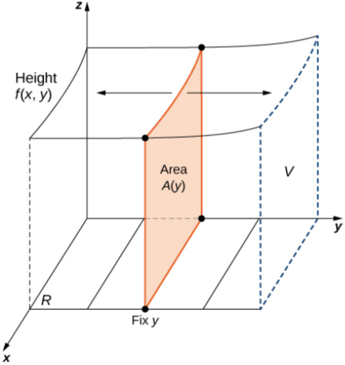
$$\int_a^b \int_c^d f(x,y)\ dydx = \int_a^b \left[\int_c^d f(x,y)\ dy\right] dx.$$

This means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly, we define the iterated integral:

$$\int_c^d \int_a^b f(x,y)\ dxdy = \int_c^d \left[\int_a^b f(x,y)\ dx\right] dy.$$





Integrating first w.r.t. y and then w.r.t. x to find the area $A(x)$ and then the volume V.

Integrating first w.r.t. x and then w.r.t. y to find the area $A(y)$ and then the volume V.

Thus,

Iterated integrals

- $$\int_a^b \int_c^d f(x,y)\ dydx = \int_a^b \left[\int_c^d f(x,y)\ dy\right] dx.$$
- $$\int_c^d \int_a^b f(x,y)\ dxdy = \int_c^d \left[\int_a^b f(x,y)\ dx\right] dy.$$

Example

Evaluate the iterated integrals:
(a) $\int_0^1 \int_1^2 x^2 y\ dydx$ (b) $\int_1^2 \int_0^1 x^2 y\ dxdy$.

Double integrals
and
Iterated integrals

The previous example illustrates a general fact: The value of an iterated integral does not depend on the order in which the integration is performed. This is a part of Fubini's Theorem. Even more important, Fubini's Theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral.

Fubini's theorem

Suppose that $f(x, y)$ is is continuous over a rectangle

$$R = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b,\ c \leq y \leq d\}.$$

Then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dydx = \int_c^d \int_a^b f(x, y)\, dxdy.$$

More generally, this is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves.

What Fubini's Theorem says is that

- The value of an iterated integral does not depend on the order in which the integration is performed.

$$\int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy.$$

- A double integral can be calculated as an iterated integral.

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dydx$$

- The volume V can be calculated as the integral of the cross section perpendicular to the x or y-axis.

$$\iint_R f(x,y)\, dA = \int_a^b A(x)\, dx = \int_c^d A(y)\, dy.$$

Example

Compute the following double integral over the indicated rectangle.

$$\iint_R x\ dA,\ \ R = [0,2] \times [0,1].$$

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.

Example

Compute the following double integral over the indicated rectangle.

$$\iint_R (2x - 4y^3)\, dA, \;\; R = [-5, 4] \times [0, 3].$$

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.

Find the volume under the surface $z = \sqrt{1 - x^2}$ and above the triangle formed by $y = x, x = 1$, and the x-axis.

Solution

Let's consider the two possible ways to set this up:

$$\int_0^1 \int_0^x \sqrt{1-x^2}\, dy\, dx \quad \text{or} \quad \int_0^1 \int_y^1 \sqrt{1-x^2}\, dx\, dy.$$

Which appears easier? In the first, the inner integral is easy, because we need an anti-derivative with respect to y, and the entire integrand $\sqrt{1-x^2}$ is constant with respect to y.

Of course, the outer integral may be more difficult. In the second, the inner integral is mildly unpleasant – a trigonometric substitution.

Solution...

So let's try the first one, since the first step is easy, and see where that leaves us.

$$\int_0^1 \int_0^x \sqrt{1-x^2}\, dy\, dx = \int_0^1 y\sqrt{1-x^2}\Big|_0^x\, dx$$
$$= \int_0^1 x\sqrt{1-x^2}\, dx.$$

This is quite easy, since the substitution $u = 1 - x^2$ works:

$$\int x\sqrt{1-x^2}\, dx = -\frac{1}{2}\int \sqrt{u}\, du$$
$$= -\frac{1}{3}u^{2/3}$$
$$= -\frac{1}{3}(1-x^2)^{2/3}.$$

Therefore,

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{3}(1-x^2)^{2/3}\Big|_0^1 = \frac{1}{3}.$$

This is a good example of how the order of integration can affect the complexity of the problem. In this case it is possible to do the other order, but it is a bit messier. In some cases one order may lead to a very difficult or impossible integral; it's usually worth considering both possibilities before going very far.

Compute the following double integral over the indicated rectangle.

$$\iint_R y\sin(xy)\ dA,\ \ R = [1,2]\times[0,\pi].$$

It is easier to integrate first with respect to x and then with respect to y.

Find the volume of the solid S enclosed by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and three coordinate planes.

Solution

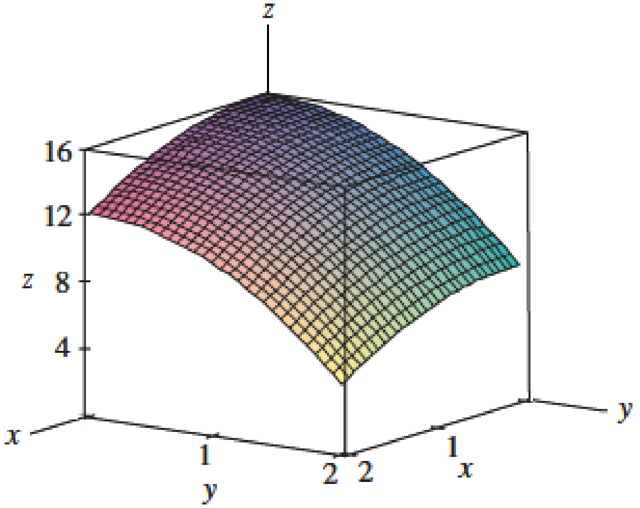


Figure 10:

Find the volume of the solid enclosed by the planes $4x + 2y + z = 10, y = 3x, z = 0, x = 0.$

Solution

Notice that the planes $4x + 2y + z = 10$ is the top of the volume and the planes $z = 0$ and $x = 0$ indicate that the plane $4x + 2y + z = 10$ does not go past the xy-plane and the yz-plane. So we are really looking for the volume under the plane

$$z = 10 - 4x - 2y$$

and above the region R in the xy-plane.

Solution...

The second plane, $y = 3x$, gives one of the sides of the volume as shown below.

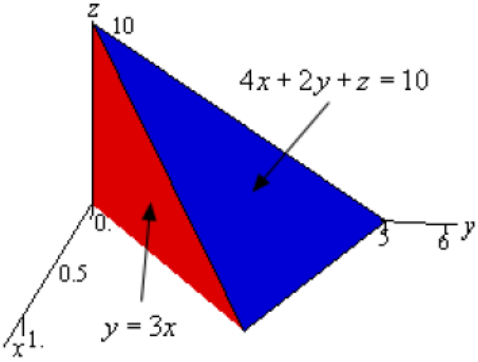


Figure 11:

The region R will be the region in the xy-plane (i.e. $z = 0$) that is bounded by $y = 3x, x = 0$, and the line where $4x + 2y + z = 10$ intersects the xy-plane. We can determine where $4x + 2y + z = 10$ intersects the xy-plane by plugging $z = 0$ into it.

$$\begin{aligned} & 4x + 2y + 0 = 10 \\ \Rightarrow \quad & 2x + y = 5 \\ \Rightarrow \quad & y = -2x + 5. \end{aligned}$$

Solution...

So, here is a sketch the region R.

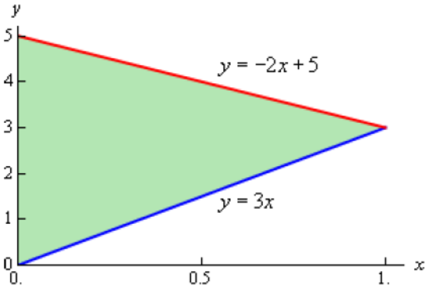


Figure 12:

Solution...

The region R is really where this solid will sit on the xy-plane and here are the inequalities that define the region.

$$0 \leq x \leq 1$$
$$3x \leq y \leq -2x + 5.$$

A special case:

$$f(x,y) = f(x)h(y) \text{ on } R = [a,b] \times [c,d].$$

In this case,

$$\begin{aligned}\iint_R f(x,y)\ dA &= \int_a^b \int_c^d g(x)h(y)\ dy\, dx \\ &= \int_a^b g(x)\ dx \int_c^d h(y)\ dy.\end{aligned}$$

Example

Compute the double integral of

$$f(x, y) = \frac{1 + x^2}{1 + y^2},$$

in the rectangular region $R = [0, 2] \times [0, 1]$.

Unit 4: Double Integrals in General Regions

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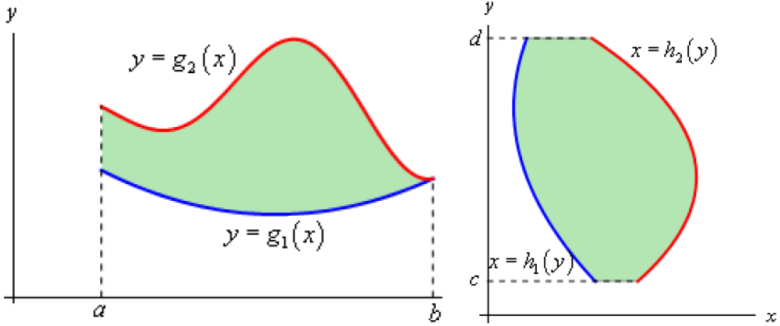
Summary

1. Integration Regions between Two Curves

To this point, we restricted our attention to rectangular domains (in some cases, triangular domains). Now we shall treat the more general case of domains.

When D is a region between two curves in the xy-plane, we can evaluate double integrals over D as iterated integrals.

There are two types of regions that we need to look at. Here is a sketch of both of them.



We will often use set builder notation to describe these regions.

Types of regions

Type I: $D = \{(x, y)|\ a \le x \le b, g_1(x) \le y \le g_2(x)\}$

Type II: $D = \{(x, y)|\ h_1(x) \le x \le h_2(x), c \le y \le d\}$

This notation indicates that we are going to use all the points, (x, y), in which both of the coordinates satisfy the two given inequalities.

Area of a plane region
Although we usually think of double integrals as representing volumes, it is worth noting that we can express the area of a domain D in the plane as the double integral of the constant function $f(x, y) = 1$:

$$\text{Area}(D) = \iint_D 1 \, dA. \tag{1}$$

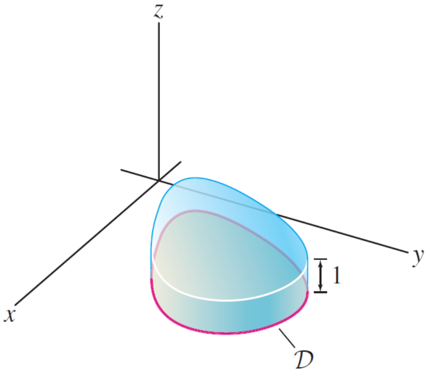


Figure 1:

Indeed, as we see in Figure 1, the the area of D is equal to the volume of the "cylinder" of height 1 with D as base. More generally, for any constant C,

$$\iint_D C \, dA = C \, \text{Area}(D).$$

Conceptual insight

Equation (1) tells us that we can approximate the area of a domain D by a Riemann sum for

$$\iint_D 1 \, dA.$$

In this case, $f(x, y) = 1$, and we obtain a Riemann sum by adding up the areas $\Delta x_i \Delta y_j$ of those rectangles in a grid that are contained in D or that intersects the boundary of D (See the figure given below).

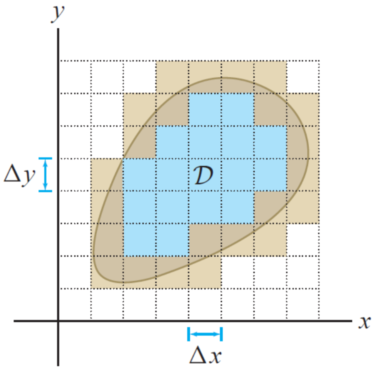


Figure 2: Approximation of D by small rectangles.

The finer the grid, the better the approximation. The exact area is the limit as the sides of the rectangles tend to zero.

Theorem (Area of a plane region of type I)

Let a plane region be given by

$$D = \{(x, y) |\ a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

where g_1 *and* g_2 *are continuous functions on* $[a, b]$*. Then the area* A *of* D *is given by*

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dydx.$$

Proof.
Consider the type I region
$D = \{(x, y)|\ a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$. We know that the area of D is given by

$$\int_a^b (g_2(x) - g_1(x))\ dx.$$

We can view the expression $g_2(x) - g_1(x)$ as

$$g_2(x) - g_1(x) = \int_{g_1(x)}^{g_2(x)} 1\ dy,$$

Proof...

That means, we can express the area of D as an iterated integral:

$$\begin{aligned}\text{Area of } D &= \int_a^b (g_2(x) - g_1(x))\, dx \\ &= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} 1\, dy \right) dx \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} 1\, dy dx.\end{aligned}$$

Using a process similar to that above, the area of a type II region D could also be obtained. We have

$$\text{Area of } D = \int_c^d \int_{h_1(y)}^{h_2(y)} 1\, dx dy. \quad \blacktriangleleft$$

Theorem (Area of a plane region of type II)

Let a plane region be given by

$$D = \{(x, y) |\ h_1(y) \le x \le h_2(y), c \le y \le d\},$$

where h_1 *and* h_2 *are continuous functions on* $[c, d]$. *Then the area* A *of* D *is given by*

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy.$$

Example

Find the area of the region enclosed by $y = 2x$ and $y = x^2$.

Solution.
We'll find the area of the region using both orders of integration.
For the type I region, we have

$$2x = x^2 \Rightarrow x = 0, 2.$$

Thus,

$$0 \le x \le 2, \quad x^2 \le y \le 2x.$$

Therefore, the required area is

$$\int_0^2 \int_{x^2}^{2x} 1 \; dy \; dx = \int_0^2 (2x - x^2) \; dx = \left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2 = \frac{4}{3}.$$

Solution...

For the type II region, we have

$$x = \frac{1}{2}y, \quad x = \sqrt{y}.$$

We then have

$$\frac{1}{2}y = \sqrt{y}.$$

This implies that

$$y^2 = 4y \Rightarrow y = 0,\ 4.$$

Therefore, the required area is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} 1\, dx\, dy = \int_0^4 (\sqrt{y} - y/2)\, dy = \left(\frac{2}{3}y^{3/2} - \frac{1}{4}y^2\right)\Big|_0^4 = \frac{4}{3}.$$

theorem

Let $f(x,y)$ be continuous.

1. If $D = \{(x,y) |\ a \le x \le b, g_1(x) \le y \le g_2(x)\}$, then
$$\iint_D f(x,y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\, dx.$$
2. If $D = \{(x,y) |\ h_1(x) \le x \le h_2(x), c \le y \le d\}$, then
$$\iint_D f(x,y)\ dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\, dy.$$

Setting up Limits of Integration

To apply the above theorem, it is helpful to start with a two-dimensional sketch of the region D. It is not necessary to graph $f(x, y)$. For a type I region, the limits of integration can be obtained as follows:

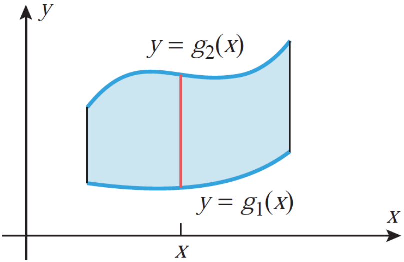


Fig. (a)

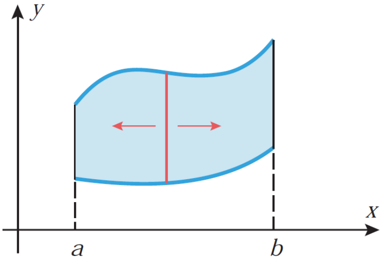


Fig. (b)

Determining Limits of Integration: Type I Region

1. x is held fixed for the first integration. We draw a vertical line through the region D at an arbitrary fixed value x (Figure (a)). This line crosses the boundary of D twice. The lower point of intersection is on the curve $y = g_1(x)$ and the higher point is on the curve $y = g_2(x)$. These two intersections determine the lower and upper y-limits of integration over the type I region.
2. Imagine moving the line drawn in Step 1 first to the left and then to the right (Figure (b)). The leftmost position where the line intersects the region D is $x = a$, and the rightmost position where the line intersects the region D is $x = b$. This yields the limits for the x-integration over the type I region.

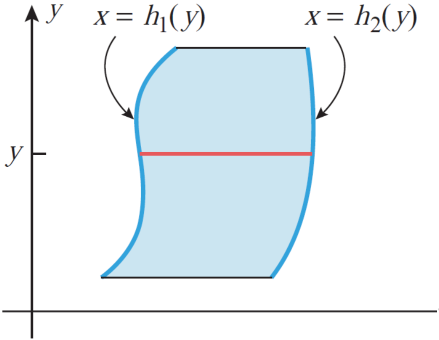


Fig. (c)

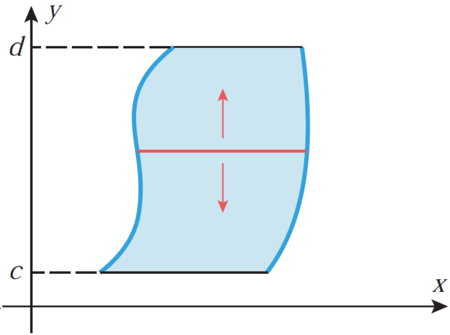


Fig. (d)

Determining Limits of Integration: Type II Region

1. y is held fixed for the first integration. We draw a horizontal line through the region D at a fixed value y (Figure (c)). This line crosses the boundary of D twice. The leftmost point of intersection is on the curve $x = h_1(y)$ and the rightmost point is on the curve $x = h_2(y)$. These intersections determine the x-limits of integration over the type II region.
2. Imagine moving the line drawn in Step 1 first down and then up (Figure (d)). The lowest position where the line intersects the region D is $y = c$, and the highest position where the line intersects the region D is $y = d$. This yields the y-limits of integration over the type II region.

Calculating a double integral over a type I region

Example

Example Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution.

We have

$$2x^2 = 1 + x^2 \Rightarrow x = \pm 1.$$

Then $y = 2$. Thus, the parabolas intersect at $(-1, 2)$ and $(1, 2)$. We see that the region D is given by

$$D = \{(x, y) : 1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\}$$

Solution...

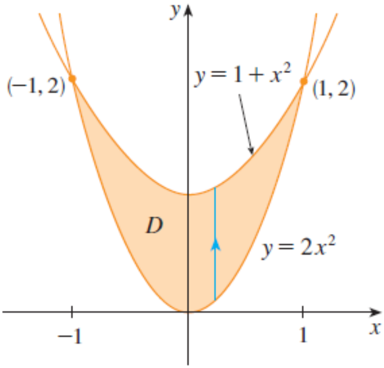


Figure 3: Type I region

We also see that the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$. Therefore, we have

$$\iint_D (x + 2y)\ dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\ dy\, dx$$
$$= \frac{12}{15}.$$

Calculating a double integral over both type I and type II region

Example

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution.

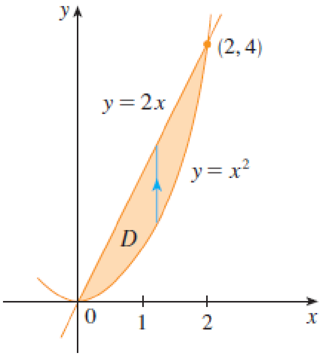


Figure 4: Type I region

From the figure we see that D can be viewed as a type I region:

$$D = \{(x, y) : \ 0 \leq x \leq 2, x^2 \leq y \leq 2x\}$$

Therefore the volume under $z = x^2 + y^2$ and above D is

$$\begin{aligned} V &= \iint_D (x^2 + y^2) \ dA \\ &= \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) \ dy \, dx \\ &= \frac{216}{35}. \end{aligned}$$

Solution.

Alternatively,

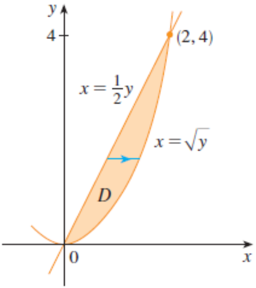


Figure 5: Type II region

From the figure we see that D can also be written as a type II region:

$$D = \{(x, y) : \ 0 \le y \le 4, \frac{1}{2}y \le x \le \sqrt{y}\}$$

Therefore another expression for V is

$$\begin{aligned} V &= \iint\limits_D (x^2 + y^2)\, dA \\ &= \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy \\ &= \frac{216}{35}. \end{aligned}$$

Choosing the better description of a region

Example

Example Evaluate $\iint\limits_{D} xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution...

The region D can be written as both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express D as a type II region:

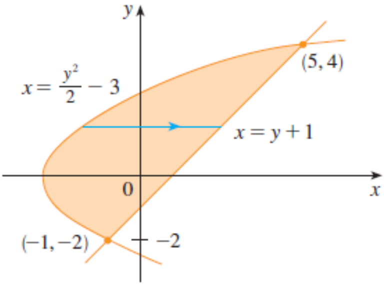


Figure 6: Type II region

Solution...

Then we have

$$D = \{(x,y) : \ -2 \le y \le 4, \tfrac{1}{2}y^2 - 3 \le x \le y+1\}.$$

Thus,

$$\iint\limits_D xy \; dA = \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy \; dx \, dy$$

$$= 36.$$

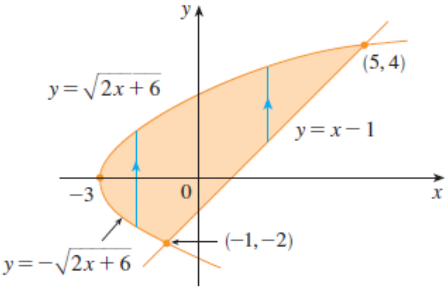


Figure 7: Type I region

If we had expressed D as a type I region using Figure 12(a), then we would have obtained

$$\iint_D xy \; dA = \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy \; dy \, dx + \int_{-1}^{5} \int_{x-1}^{\sqrt{2x+6}} xy \; dy \, dx$$

Reversing the order of integration

Example

Example Find the iterated integral

$$\int_0^1 \int_1^x \sin(y^2)\, dy\, dx.$$

Solution.

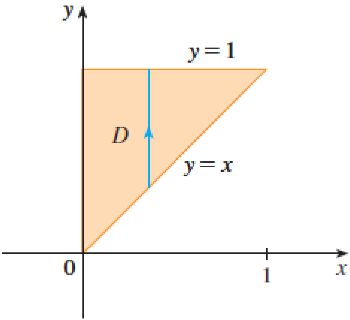
If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\, dy$. But it's impossible to do so in finite terms since $\int \sin(y^2)\, dy$ is not an elementary function. So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral.

Solution...

We have

$$\int_0^1 \int_x^1 \sin(y^2)\ dy\,dx = \iint_D \sin(y^2)\ dA,$$

where $D = \{(x, y):\ 0 \le x \le 1, x \le y \le 1\}$. The sketch of this region D is as follows:



Solution...

An alternative description of D is as follows:

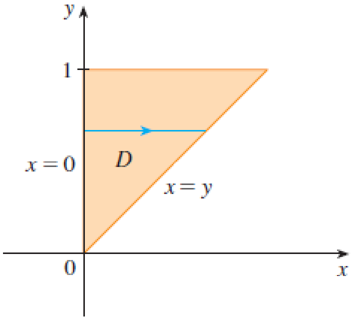


Figure 9: Type II region

Solution...

$$D = \{(x, y):\ 0 \leq y \leq 1, 0 \leq x \leq y\}$$

This enables us to express the double integral as an iterated integral in the reverse order:

$$\begin{aligned} \int_0^1 \int_x^1 \sin(y^2)\ dy\, dx &= \iint_D \sin(y^2)\ dA \\ &= \int_0^1 \int_0^y \sin(y^2)\ dx\, dy \\ &= \tfrac{1}{2}(1 - \cos 1). \end{aligned}$$

Problem

Evaluate the integral:

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\,dx\,dy.$$

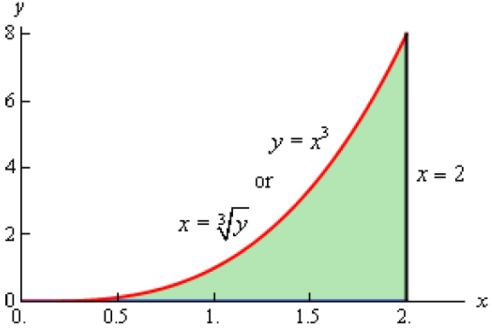


Figure 10: Domain of integration

Properties of Double Integrals

Note that all first three of these properties are really just generalizations of properties of double integrals over rectangles.

1. $\iint\limits_D [f(x,y)+g(x,y)]\ dA = \iint\limits_D f(x,y)\ dA + \iint\limits_D g(x,y)\ dA$

2. If c is a constant, then

$$\iint\limits_D cf(x,y)\ dA = c\iint\limits_D f(x,y)\ dA$$

3. If $f(x,y) \geq g(x,y)$ for all in $(x,y) \in D$, then

$$\iint\limits_D f(x,y)\ dA \geq \iint\limits_D g(x,y)\ dA$$

Assume that $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries. See the figure.

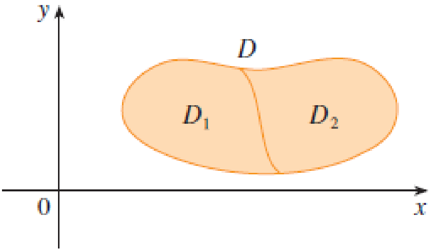


Figure 11:

Then

4. $\iint\limits_{D} [f(x,y)+g(x,y)]\, dA = \iint\limits_{D_1} f(x,y)\, dA + \iint\limits_{D_2} f(x,y)\, dA$

5.
$$\iint\limits_{D} 1\, dA = A(D),$$
where $A(D)$ is the area of D.

6. If $m \le f(x,y) \le M$ for all in $(x,y) \in D$, then
$$mA(D) \le \iint\limits_{D} f(x,y)\, dA \le MA(D)$$

Unit 4: Double Integrals in Polar Coordinates

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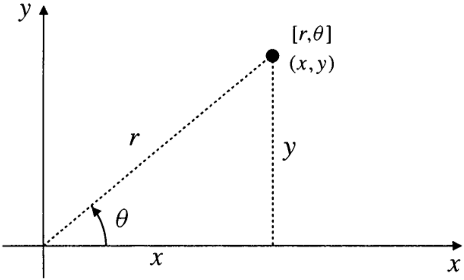
Summary

1. Polar coordinates

2. General Polar Regions of Integration

Rectangular and Polar Coordinates

Recall that the polar representation of a point P is an ordered pair (r, θ), where r is the distance from the origin to P and θ is the angle that the ray through the origin and P makes with the positive x-axis.



Relation between Rectangular and Polar Coordinates

The polar coordinates r and θ of a point (x, y) in rectangular coordinates satisfy the following relations:

- $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$.
- $x = r\cos\theta$ and $y = r\sin\theta$.

Polar coordinates are convenient when the domain of integration is an angular sector or a polar rectangle, as shown in the figure given below.

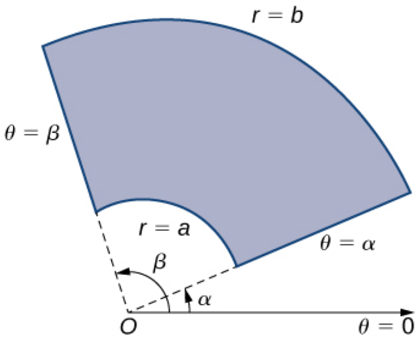


Figure 1: Polar rectangle

Moreover for many applications of double integrals, the integrand may be easier to integrate if it is in terms of polar coordinates than in terms of Cartesian coordinates.

Example

Consider the double integral

$$\iint\limits_D e^{x^2+y^2}\, dA,$$

where D is the unit disk.

Note that we cannot directly evaluate this integral in rectangular coordinates. However, a change to polar coordinates will convert it to one we can easily evaluate. First we establish the concept of a double integral in a polar rectangular region. Then we change rectangular coordinates to polar coordinates in double integrals.

Concept of a double integral in a polar rectangle

In polar coordinates, the shape we work with is a *polar rectangle.* See the figure given below.

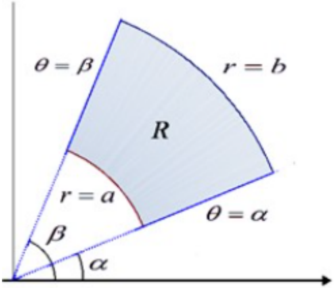


Figure 2: Polar rectangle

Polar rectangle

A polar rectangle is a region R given by

$$R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\},$$

where $0 \leq \beta - \alpha \leq 2\pi$.

Consider a function $f(r, \theta)$ over a polar rectangle R defined above.
Double integrals in polar coordinates are defined through a three-step process: **subdivision**, **summation**, and **passage to the limit**, as we did in the case of double integrals in rectangular coordinates.

Subdivision

We decompose R into an $n \times m$ grid of small polar subrectangles R_{ij} as follows:
We divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of length

$$\Delta r = (b - a)/m$$

and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{i-1}, \theta_i]$ of width

$$\Delta\theta = (\beta - \alpha)/n$$

by choosing partitions:

$$a = r_0 < r_1 < ... < r_m = b, \;\; \alpha = \theta_0 < \theta_1 < ... < \theta_n = \beta,$$

where m and n are positive integers.

This means that the circles of radii $r = r_i$ and rays with angles $\theta = \theta_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ divide the polar rectangle R into smaller polar subrectangles R_{ij} as in the figure given below.

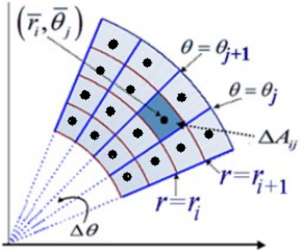


Figure 3: Polar grid

Choose the center $(\overline{r}_i, \overline{\theta}_l)$ of each polar subrectangle R_{ij} as a sample point. Then

$$\overline{r}_i = \frac{1}{2}(r_{i-1} + r_i), \quad \overline{\theta}_j = \frac{1}{2}(\theta_{j-1} + \theta_j).$$

The area of the polar subrectangle R_{ij} is given by

$$\begin{aligned}\Delta A_i &= \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta \\ &= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta \\ &= \overline{r}_i\Delta r\Delta\theta.\end{aligned}$$

So, the volume of each of the boxes with a base area of ΔA_i and a height of $f(\overline{r}_i, \overline{\theta}_j)$ is

$$f(\overline{r}_i, \overline{\theta}_j)\Delta A_i = f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i \Delta r \Delta\theta.$$

The volume of the solid with the vase R_{ij} is now approximated as follows:

$$\iint_{R_{ij}} f(r,\theta)\ dA \approx f(\overline{r}_i, \overline{\theta}_j)\Delta A_i = f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i \Delta r \Delta\theta.$$

Summation

The volume of the solid under the surface $z = f(r, \theta)$ with the vase R is now approximated as follows:

$$\begin{aligned}\iint_R f(r,\theta)\ dA &= \sum_{i=1}^{n}\sum_{j=1}^{m}\iint_{R_{ij}} f(r,\theta)\ dA \\ &\approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(\overline{r}_i,\overline{\theta}_j)\ \Delta A_i \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} f(\overline{r}_i,\overline{\theta}_j)\overline{r}_i\Delta r\Delta\theta.\end{aligned}$$

Riemann sum

The expression

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i \Delta r \Delta \theta$$

is called a **Riemann sum** for the double integral of $f(r, \theta)$ over the region

$$0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta,$$

where $0 \leq \beta - \alpha \leq 2\pi$.

Passage to the limit

Double integral in polar coordinates

Let f be continuous on a polar rectangle R given by

$$0 \leq a \leq r \leq b,\ \alpha \leq \theta \leq \beta,$$

where $0 \leq \beta - \alpha \leq 2\pi$. The double integral $\iint_R f(r,\theta)\, dA$ is defined as follows:

$$\iint_R f(r,\theta)\, dA = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i \Delta r \Delta\theta.$$

Just as in double integrals over rectangular regions, the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence

$$\iint_R f(r,\theta)\ dA = \iint_R f(r,\theta) r\ dr\, d\theta = \int_\alpha^\beta \int_a^b f(r,\theta) r\ dr\, d\theta.$$

Notice that the expression for dA is replaced by $r\,dr\,d\theta$ when working in polar coordinates.

We have the following theorem.

Theorem

If f is continuous on a polar rectangle R given by

$$0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta,$$

where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint\limits_D f(x, y)\ dA = \iint\limits_D f(r\cos\theta, r\sin\theta)\ r\,dr\,d\theta.$$

It is noteworthy that all the properties of the double integral in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

Change to Polar Coordinates in a Double Integral

If we are given a double integral

$$\iint\limits_D f(x,y)\;dA$$

in rectangular coordinates, we can write the corresponding iterated integral in polar coordinates by substitution.

Method for converting in polar coordinates

- Describe the domain of integration, R, and find bounds

$$a \leq r \leq b \text{ and } \alpha \leq \theta \leq \beta,$$

  where $0 \leq \beta - \alpha \leq 2\pi$.

- Convert the function $z = f(x, y)$ to a function with polar coordinates with the substitutions

$$x = r\cos\theta, y = r\sin\theta.$$

- Replace dA by $r\,dr\,d\theta$ to obtain

$$\iint_R f(x, y)\; dA = \iint_R f(r\cos\theta, r\sin\theta)\; r\,dr\,d\theta.$$

Example

Let $f(x,y) = e^{x^2+y^2}$ on the disk $D = \{(x,y) : \; x^2+y^2 \le 1\}$. Evaluate $\iint\limits_D f(x,y)\; dA$.

Solution.
We have the unit disk

$$D = \{(x, y) : \; x^2 + y^2 \leq 1\}.$$

We observe that

$$0 \leq r \leq 1, \; 0 \leq \theta \leq 2\pi.$$

Solution...

Using

$$x = r\cos\theta,\ y = r\sin\theta,\ dA = r\,dr\,d\theta,$$

we then have

$$\begin{aligned}\int_D e^{x^2+y^2}\ dA &= \int_0^{2\pi}\int_0^1 e^{r^2} r\ dr\,d\theta \\ &= \int_0^{2\pi} \frac{1}{2}e^{r^2}\Big|_0^1\ d\theta \\ &= \int_0^{2\pi} \frac{1}{2}(e-1)\ d\theta \\ &= \pi(e-1). \qquad \blacktriangleleft\end{aligned}$$

While there is no firm rule for when polar coordinates can or should be used, they are a natural alternative anytime the domain of integration may be expressed simply in polar form, and/or when the integrand involves expressions such as

$$\sqrt{x^2 + y^2}.$$

Example

Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$.

Solution.
We know that the formula for finding the volume of a region is

$$V = \iint_D f(x, y)\, dA.$$

We have

$$f(x, y) = z = \sqrt{9 - x^2 - y^2} = \sqrt{9 - r^2}.$$

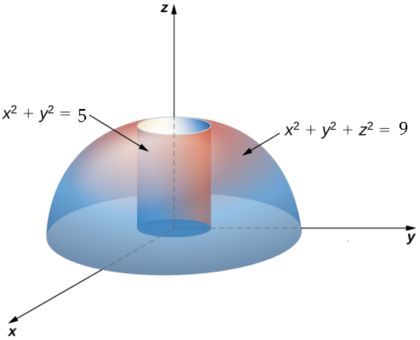
The region D is the bottom of the cylinder given by $x^2 + y^2 = 5$, that is, the disk

$$D = \{(x, y) |\ x^2 + y^2 \leq 5\}$$

in the xy-plane.

Solution...

So, the region that we want the volume for is really a cylinder with a cap that comes from the sphere.



Thus, the region D in polar coordinates is as follows:

$$D = \{(r, \theta) : \ 0 \leq r \leq \sqrt{5}, 0 \leq \theta \leq 2\pi\}$$

Solution...
Now, the volume is

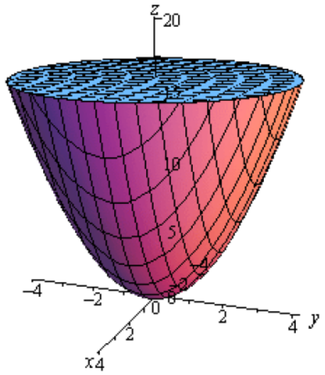
$$V = \iint\limits_D \sqrt{9 - x^2 - y^2}\, dA$$
$$= \int_0^{2\pi} \int_0^{\sqrt{5}} r\sqrt{9 - r^2} dr\ d\theta$$
$$= 38\pi/3. \qquad \blacktriangleleft$$

Example

Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$.

Solution.

Let's start this example off with a sketch of the region.



Now, we see that the top of the region, where the elliptic paraboloid intersects the plane $z = 16$, is the widest part of the region.

Solution...

So, setting $z = 16$ in the equation of the paraboloid gives

$$16 = x^2 + y^2,$$

which is the equation of a circle of radius 4 centered at the origin. Now, the domain of intgration, D, is given by

$$D = \{(r, \theta) : 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 4\}.$$

Solution...

Notice that the formula

$$\iint_D 16 \; dA.$$

will be the volume under plane $z = 16$ while the formula

$$\iint_D (x^2 + y^2) \; dA.$$

is the volume under the paraboloid $z = x^2 + y^2$, using the same D.

Solution...

Hence the required volume is

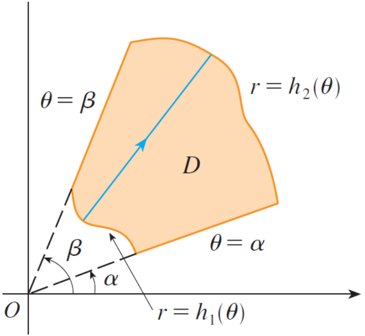
$$V = \iint\limits_D 16\ dA - \iint\limits_D (x^2 + y^2)\ dA$$
$$= \iint\limits_D (16 - x^2 - y^2)\ dA$$
$$= \int_0^{2\pi} \int_0^4 r(16 - r^2)\ dr\ d\theta$$
$$= 128\pi. \qquad \blacktriangleleft$$

General Polar Regions of Integration

To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular coordinates in Double Integrals over General Regions.

It is more common to write polar equations as $r = f(\theta)$ than $\theta = f(r)$, so we describe a general polar region as

$$D = \{(r, \theta) |\ \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$



Example

Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution.

Here is a sketch of the region, D, that we want to determine the shaded area (Figure (a)).

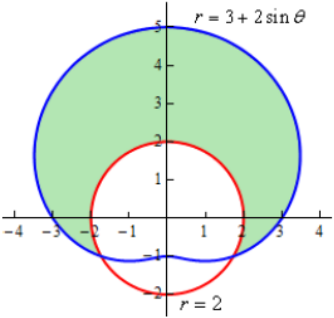


Figure 4: (a)

Solution...

To determine the range of θ, we solve the two equations. We have

$$3 + 2\sin\theta = 2 \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}.$$

Here is a sketch of the figure with these angles added.

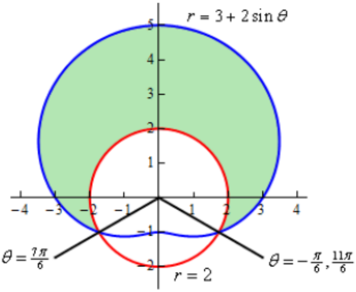


Figure 5: (b)

Solution...

Note as well that we've acknowledged that $-\frac{\pi}{6}$ is another representation for the angle

$$\frac{11\pi}{6} = 2\pi - \frac{\pi}{6}.$$

This is important since we need the range of θ to actually enclose the regions as we increase from the lower limit to the upper limit. If we'd chosen to use $\frac{11\pi}{6}$, then as we increase from $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we would be tracing out the lower portion of the circle and that is not the region that we are after.

Solution...
To get the ranges for r the function that is closest to the origin is the lower bound and the function that is farthest from the origin is the upper bound.
So, here are the ranges that will define the region.

$$-\frac{\pi}{6} \leq \theta \leq \frac{7\pi}{6}$$

$$2 \leq r \leq 3 + 2\sin\theta.$$

Solution...

The area of the region D is then

$$\begin{aligned} A &= \iint\limits_D dA \\ &= \int_{-\pi/6}^{\pi/6} \int_2^{3+2\sin\theta} dr\, d\theta \\ &= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3}. \end{aligned}$$

◀

Example

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

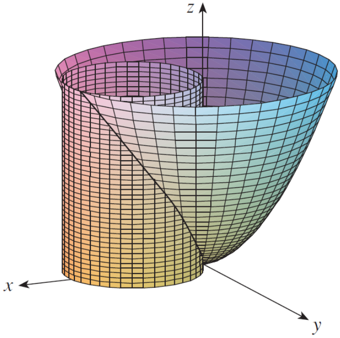
Solution.
The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square, we get

$$(x-1)^2 + y^2 = 1$$

(See the figures given below.) To find the volume of the required solid, we have to evaluate the integral:

$$V = \iint\limits_D (x^2 + y^2)\ dx\,dy.$$

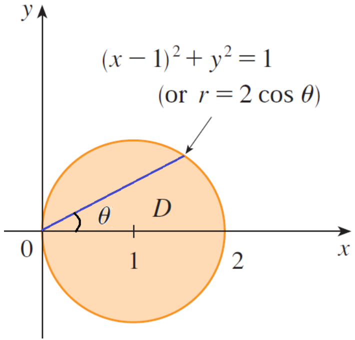
Solution...



Converting the equation of the circle in polar form, we get

$$\begin{aligned} x^2 + y^2 = 2x &\Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = 2r\cos\theta \\ &\Rightarrow r^2 = 2r\cos\theta \\ &\Rightarrow r = 0 \text{ or } 2\cos\theta. \end{aligned}$$

Solution...



Thus the disk D is given by

$$D = \{(r,\theta) : -\pi/2 \le \theta \le \pi/2,\ 0 \le r \le 2\cos\theta\}.$$

Solution.

Now, we have

$$V = \iint\limits_D (x^2+y^2)\ dx\,dy = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2 r\ dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\cos\theta}\ d\theta = 4\int_{-\pi/2}^{\pi/2}\cos^4\theta\ d\theta$$

$$= 8\int_0^{\pi/2}\cos^4\theta\ d\theta = 8\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2\ d\theta$$

$$= 2\int_0^{\pi/2}[1+2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)]\ d\theta$$

$$= 2\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = 2(3/2)(\pi/2)$$

$$= 3\pi/2. \qquad \blacktriangleleft$$

Unit 4: Applications of Double Integrals

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Summary

1. **Applications of double integrals**
   - Mass
   - Moments and Center of Mass
   - Moments of Inertia
   - Surface area

We saw before that the double integral over a region of the constant function 1 measures the area of the region. If the region has uniform density 1, then

$$\begin{aligned} \text{Mass} &= \text{Density} \times \text{Area} \\ &= 1 \times \text{Area} \\ &= \text{Area}. \end{aligned}$$

What if the density is not constant. Suppose that the density is given by the continuous function

$$\text{Density} \ = \rho(x, y)$$

In this case we can cut the region into tiny rectangles where the density is approximately constant. The area of mass rectangle is given by

$$\text{Mass} \ = \ \text{Density} \ \times \ \text{Area} \ = \rho(x, y)\Delta x \ \Delta y$$

You probably know where this is going. If we add all to masses together and take the limit as the rectangle size goes to zero, we get a double integral.

Mass

Let $\rho(x, y)$ be the density of a lamina (flat sheet) D at the point (x, y). Then the total mass of the lamina is the double integral

$$\text{Mass } = \iint\limits_{D} \rho(x, y)\; dy\; dx.$$

Finding the mass of a lamina with constant density

Find the mass of a square lamina, with side length 1, with a density of $\rho = 3\ \text{gm/cm}^2$.

Solution.

We represent the lamina with a square region in the plane as shown in the figure given below.

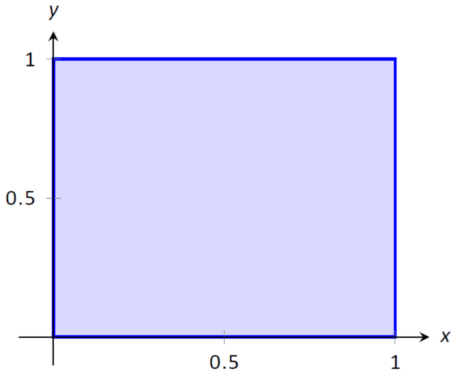


Figure 1: A region R representing a lamina.

As the density is constant, it does not matter where we place the square. Now, the mass M of the lamina is

$$M = \iint_R 3 \; dA = \int_0^1 \int_0^1 3 \; dx \, dy$$
$$= 3 \text{ gm.} \qquad \blacktriangleleft$$

This is all very straightforward.

Finding the mass of a lamina with variable density

Find the mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see the Figure), with variable density $\rho(x, y) = (x + y + 2)$ gm/cm^2.

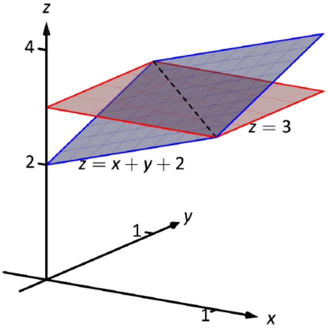


Figure 2: Graphing the density functions: $z = 3$ and $z = x + y + 2$

Solution.

The variable density ρ, in this example, is very uniform, giving a density of 3 in the center of the square and changing linearly. A graph of $\rho(x, y)$ can be seen in Figure 2; notice how "same amount" of density is above $z = 3$ as below.

The mass M is found by integrating $\rho(x, y)$ over R. The order of integration is not important; we choose $dx\,dy$ arbitrarily.

Solution...

Thus

$$\iint_R (x+y+2)\,dA = \int_0^1 \int_0^1 (x+y+2)\,dx\,dy$$
$$= \int_0^1 \left((1/2)x^2 + x(y+2)\right)\Big|_0^1 dy$$
$$= \int_0^1 \left(\frac{5}{2} + y\right) dy$$
$$= \left(\frac{5}{2}y + \frac{1}{2}y^2\right)\Big|_0^1$$
$$= 3 \text{ gm.}$$

◀

It turns out that since the density of the lamina is so uniformly distributed "above and below" $z = 3$ that the mass of the lamina is the same as if it had a constant density of 3. The density functions in the last two Examples are graphed in Figure 2, which illustrates this concept.

Moments

We know that the moments about an axis are defined by the product of the mass times the distance from the axis.

$$M_x = (\text{ Mass })(y), \qquad M_y = (\text{ Mass })(x).$$

If we have a region D with density function $\rho(x, y)$, then we do the usual thing. We cut the region into small rectangles for which the density is constant and add up the moments of each of these rectangles. Then take the limit as the rectangle size approaches zero. This will give us the total moment.

Moments of Mass and Center of Gravity

Suppose that $\rho(x, y)$ is a continuous density function on a lamina D. Then the **moments of mass** are

$$M_x = \iint_D \rho(x, y) y \; dy \; dx, \qquad M_y = \iint_D \rho(x, y) x \; dy \; dx.$$

and if m is the mass of the lamina, then the **center of mass** or **center of gravity** is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

Finding the center of mass of a lamina

Find the center mass of a square lamina, with side length 1, with a density of $\rho = 3$ gm/cm^2.

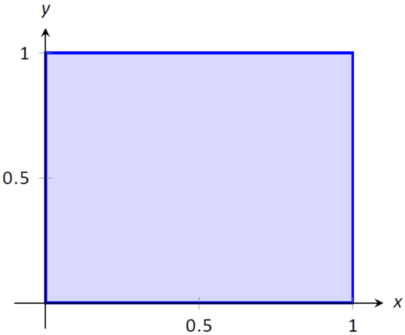


Figure 3: A region R representing a lamina.

Solution.

We represent the lamina with a square region in the plane as shown in the Figure. We have

$$M = \iint_R \rho(x, y)\ dA = \int_0^1 \int_0^1 3\ dx\, dy = 3 \text{ gm}.$$

$$M_x = \iint_R \rho(x, y)y\ dA = \int_0^1 \int_0^1 3y\ dx\, dy = 3/2 = 1.5.$$

$$M_y = \iint_R \rho(x, y)x\ dA = \int_0^1 \int_0^1 3x\ dx\, dy = 3/2 = 1.5.$$

Thus the center of mass is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = (1.5/3, 1.5/3) = (0.5, 0.5).$$

Finding the mass of a lamina with variable density

Find the center of the mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see the Figure), with variable density
$\rho(x, y) = (x + y + 2)$ gm/cm^2.

Solution.

We represent the lamina with a square region in the plane as before. We have

$$M = \iint\limits_R \rho(x, y)\ dA = \int_0^1 \int_0^1 (x + y + 2)\ dx\, dy = 3 \text{ gm.}$$

$$M_x = \iint\limits_R \rho(x, y)y\ dA = \int_0^1 \int_0^1 (x + y + 2)y\ dx\, dy = 19/12.$$

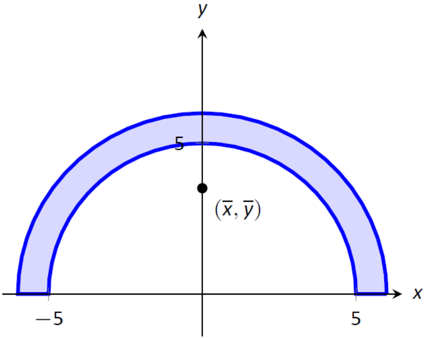
$$M_y = \iint\limits_R \rho(x, y)x\ dA = \int_0^1 \int_0^1 (x + y + 2)x\ dx\, dy = 19/12.$$

Thus the center of mass is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = (19/36, 19/36) = (0.528, 0.528).$$

Example

Find the center of mass of the lamina represented by the region R which is half an annulus with outer radius 6 and inner radius 5, with constant density 2 lb/ft^2.



Solution. Here, it is useful to represent R in polar coordinates. Using the description of R, we see that

$$R = \{(r, \theta) : \; 5 < r < 6, \, 0 < \theta < \pi\}.$$

As the lamina is symmetric about the y-axis, we should expect $M_y = 0$. We compute M, M_x and M_y.

Solution...

We have

$$M = \int_0^{\pi} \int_5^6 2r \; dr \, d\theta = 11\pi \text{ lb.}$$

$$M_x = \int_0^{\pi} \int_5^6 (r \sin\theta) 2r \; dr \, d\theta = \frac{364}{3} = 121.33.$$

$$M_y = \int_0^{\pi} \int_5^6 (r \cos\theta) 2r \; dr \, d\theta = 0.$$

Thus the center of mass is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right) = (0, 364/(33\pi)) = (0, 3.51).$$

Example

Set up the integrals that give the center of mass of the rectangle with vertices $(0,0), (1,0), (1,1)$, and $(0,1)$ and density function proportional to the square of the distance from the origin.

Solution.
Since the density function $\rho(x, y)$ is proportional to the square of the distance from the origin, $x^2 + y^2$, the mass is given by

$$m = \int_0^1 \int_0^1 k(x^2 + y^2)\ dy\ dx = \frac{2k}{3}.$$

The moments are given by

$$M_x = \int_0^1 \int_0^1 k(x^2 + y^2)y\ dy\ dx = 5k/12$$
$$M_y = \int_0^1 \int_0^1 k(x^2 + y^2)x\ dy\ dx = 5k/12$$

Solution...

It should not be a surprise that the moments are equal since there is complete symmetry with respect to x and y. Finally, we divide to get

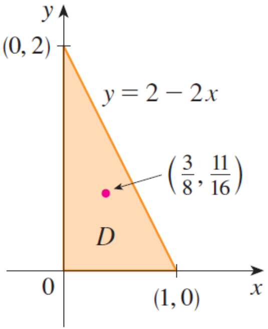
$$(\overline{x}, \overline{y}) = (5/8, 5/8)$$

This tells us that the metal plate will balance perfectly if we place a pin at (5/8 , 5/8).

Example

Find the mass and center of mass of a triangular lamina with vertices $(0,0), (1,0)$, and $(0,2)$ if the density function is

$$\rho(x,y) = 1 + 3x + y.$$



Moments of Inertia

We often call M_x and M_y the first moments. They have first powers of y and x in their definitions and help find the center of mass. We define the moments of inertia (or second moments) by introducing squares of y and x in their definitions. The moments of inertia help us find the kinetic energy in rotational motion. Below is the definition.

Moments of Inertia

Suppose that $\rho(x, y)$ is a continuous density function on a lamina D. Then the moments of inertia about the x-axis and the y-axis are

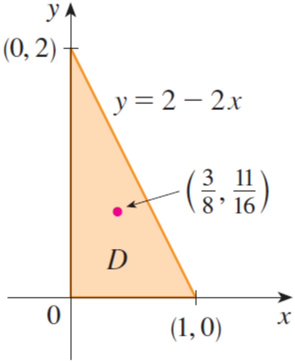
$$I_x = \iint\limits_D \rho(x, y)y^2 \, dy \, dx, \qquad I_y = \iint\limits_D \rho(x, y)x^2 \, dy \, dx.$$

It is also of interest to consider the moment of inertia about the origin, also called the **polar moment of inertia**:

$$I_x = \iint\limits_D \rho(x, y)(x^2 + y^2) \, dy \, dx.$$

Example

Find the moments of inertia for the square metal plate with vertices $(0, 0), (1, 0), (1, 1)$, and $(0, 1)$.



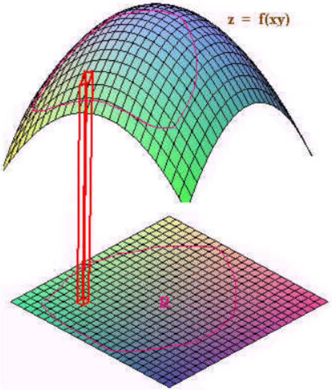
Problem

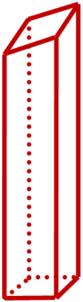
Find the moments of inertia I_x, I_y, and I_o of a homogeneous disk with density $\rho(x, y) = \rho$, center the origin, and radius a.

Solution.
The boundary of D is the circle $x^2 + y^2 = a$ and in polar coordinates D is described by $0 \leq \theta \leq 2\pi, 0 \leq r \leq a$. First compute I_o. Then use $I_x + I_y = I_o$ and $I_x = I_y$ (due to the symmetry of the problem) to find I_x and I_y. ◀

Surface area

Let $z = f(x, y)$ be a surface in $\mathbb{R}^3$ defined over a region D in the xy-plane. cut the xy-plane into rectangles. Each rectangle will project vertically to a piece of the surface as shown in the figure below.





Although the area of the rectangle in D is

$$\text{Area } = \Delta y \Delta x,$$

the area of the corresponding piece of the surface will not be $\Delta y \Delta x$ since it is not a rectangle. Even if we cut finely, we will still not produce a rectangle, but rather will approximately produce a parallelogram. With a little geometry we can see that the two adjacent sides of the parallelogram are (in vector form)

$$u = \Delta x \vec{i} + f_x(x, y) \Delta x \vec{k}$$

and

$$v = f_y(x, y) \Delta y \vec{i} + \Delta y \vec{k}$$

We can see this by realizing that the partial derivatives are the slopes in each direction. If we run Δx in the $\vec{i}$ direction, then we will rise $f_x(x, y)\Delta x$ in the $\vec{k}$ direction so that

$$\frac{\text{rise}}{\text{run}} = f_x(x, y),$$

which agrees with the slope idea of the partial derivative. A similar argument will confirm the equation for the vector v. Now that we know the adjacent vectors we recall that the area of a parallelogram is the magnitude of the cross product of the two adjacent vectors.

We have

$$
\begin{aligned}
|v \times w| &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & f_x(x,y)\Delta x \\ 0 & \Delta y & f_y(x,y)\Delta y \end{vmatrix} \\
&= |-(f_y(x,y)\Delta y\Delta x)\vec{i} - (f_x(x,y)\Delta y\Delta x)\vec{j} + (\Delta y\Delta x)\vec{k}| \\
&= \sqrt{f_y^2(x,y)(\Delta y\Delta x)^2 + f_x^2(x,y)(\Delta y\Delta x)^2 + (\Delta y\Delta x)^2} \\
&= \sqrt{f_y^2(x,y) + f_x^2(x,y) + 1}\ \Delta y\Delta x.
\end{aligned}
$$

This is the area of one of the patches of the quilt. To find the total area of the surface, we add up all the areas and take the limit as the rectangle size approaches zero. This results in a double Riemann sum, that is a double integral. We state the definition below.

Surface Area

Let $z = f(x, y)$ be a differentiable surface defined over a region D. Then its surface area is given by

$$\text{Surface Area } = \iint\limits_D \sqrt{1 + f_x^2(x, y) + f_y^2(x, y)}\, dy\, dx.$$

Problem

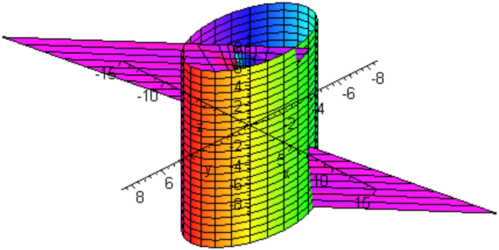
Find the surface area of the part of the plane

$$z = 8x + 4y$$

that lies inside the cylinder

$$x^2 + y^2 = 16.$$

Solution



We calculate partial derivatives

$$f_x(x, y) = 8, \qquad f_y(x, y) = 4$$

so that

$$1 + f_x^2(x, y) + f_y^2(x, y) = 1 + 64 + 16 = 81$$

Taking a square root and integrating, we get

$$\iint_D 9 \, dy \, dx.$$

We could work this integral out, but there is a much easier way. The integral of a constant is just the constant times the area of the region. Since the region is a circle, we get

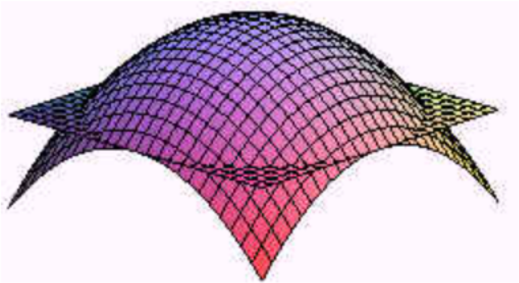
$$\text{Surface Area } = 9(16\pi) = 144\pi.$$

Example

Find the surface area of the part of the paraboloid

$$z = 25 - x^2 - y^2$$

that lies above the xy-plane.



Solution.

We calculate partial derivatives

$$f_x(x,y) = -2x \qquad f_y(x,y) = -2y$$

so that

$$1 + f_x^2(x,y) + f_y^2(x,y) = 1 + 4x^2 + 4y^2.$$

Using "Polar Coordinates", we realize that the region is just the circle

$$r = 5$$

Now convert the integrand to polar coordinates to get

$$\int_0^{2\pi} \int_0^5 \sqrt{(1+4r^2)}\ rdr\ d\theta$$

Solution...

Now let

$$u = 1 + 4r^2, \qquad du = 8rdr$$

and substitute

$$\frac{1}{8}\int_0^{2\pi}\int_1^{101} u^{1/2}\ du\ d\theta = \frac{1}{12}\int_0^{2\pi}\left[u^{3/2}\right]_1^{101}\ d\theta \approx 169.3\pi.$$