Vectors and Geometry of spaces

Date
Page

1) i) Find an equation of sphere with centre (-3,2,5) and radius 4. What is the intersection of this sphere with yz plane.

2) Solution,

Given, Centre of sphere $=(-3,2,5)$

radius of sphere $=4$.

Now,

the equation of sphere is given by:

$$(x-f)^2+(y-g)^2+(z-h)^2=r^2$$

where, (f,g,h) is centre point.

So, equation of sphere is

$$(x+3)^2+(y-2)^2+(z-5)^2=4^2 \quad \text{--- (1)}$$

Now,

the equation of yz plane is found using,

$ax+by+cz=d$

where (a,b,c) ~~are~~ is vector normal to plane.

and the vector normal to yz plane is found as

$$\vec{n}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$=\vec{i}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}-\vec{j}\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}+\vec{k}\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$=\vec{i}$$

So, $(a,b,c)=(1,0,0)$

Hence equation of plane is $x=d$.

Vectors and Geometry of spaces.

Date
Page

1) i) Find an equation of sphere with centre $(-3,2,5)$ and radius 4. what is the intersection of this sphere with yz plane.

=> **Solution,**

Given, Centre of sphere $= (-3,2,5)$

radius of sphere $= 4$.

Now,

the equation of sphere is given by:

$$(x-f)^2 + (y-g)^2 + (z-h)^2 = r^2$$

where, (f,g,h) is centre point.

So, equation of sphere is

$$(x+3)^2 + (y-2)^2 + (z-5)^2 = 4^2 \quad \text{—(1)}.$$

Now,

the equation of yz plane is found using,

$$ax + by + cz = d$$

where (a,b,c) ~~are~~ is vector normal to plane.

and the vector normal to yz plane is found as

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= \vec{i}$$

So, $(a,b,c) = (1,0,0)$

Hence equation of plane is $x = d$.

Date
Page

Since the plane passes through origin,
$d = 0$.

Hence equation of plane is $x = 0$. — (ii)

From eqn (i) and (ii)

$$(x+3)^2 + (y-2)^2 + (z-5)^2 = 4^2$$
$$x = 0$$

So, place (ii) in (i) we get,

$$3^2 + (y-2)^2 + (z-5)^2 = 4^2$$
or, $(y-2)^2 + (z-5)^2 = 16 - 9 = 7$.

Hence, the equation of sphere with $x = 0$ plane is
$(y-2)^2 + (z-5)^2 = 7$ which is equation of circle.

ii) If $\vec{r} = (x, y, z)$ and $\vec{r_0} = (x_0, y_0, z_0)$, describe the set of all points (x, y, z) such that $|\vec{r} - \vec{r_0}| = 1$.

⇒ Solution,

Given $\vec{r} = (x, y, z)$
$\vec{r_0} = (x_0, y_0, z_0)$

Now,

$$|\vec{r} - \vec{r_0}| = |(x, y, z) - (x_0, y_0, z_0)|$$
$$= |(x - x_0, y - y_0, z - z_0)|$$
$$= \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

Since $|\vec{r} - \vec{r_0}| = 1$.

$$\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = 1.$$

Squaring both sides. we get,

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = 1$$

Hence, the set of points that satisfies the given condition is sphere with radius 1 and centre at (x_0, y_0, z_0)

iii) Find the equation of sphere with centre $(2,-3,6)$ that touch the xy plane.

⇒ Solution.

Given centre of sphere $(f,g,h) = (2,-3,6)$

So, equation of sphere with centre (f,g,h) is given by.

$$(x-2)^2 + (y+3)^2 + (z-6)^2 = r^2.$$

Also, given, the sphere touch xy plane so, we have, equation of the plane is $z = 0$.

Hence, the radius is $|6-0| = 6$.

thus, the equation of sphere is.

$$(x-2)^2 + (y+3)^2 + (z-6)^2 = 6^2$$

Vectors and Vector Geometry

Date ______
Page ______

1.) i) Find a unit vector that has same direction as vector $\vec{a} = 8\vec{i} - \vec{j} + 4\vec{k}$. Also find the vector that has same direction as $\vec{a}$ but length 6.

⇒ Solution:

Given vector $\vec{a} = 8\vec{i} - \vec{j} + 4\vec{k}$

Now,

unit vector is given by $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$= \dfrac{8\vec{i} - \vec{j} + 4\vec{k}}{\sqrt{8^2 + 1^2 + 4^2}}$

$= \dfrac{8\vec{i} - \vec{j} + 4\vec{k}}{9}$

To find the vector that has same direction as $\vec{a}$ but length 6 we have: $6 \times \hat{a}$

$= 6 \times \dfrac{8\vec{i} - \vec{j} + 4\vec{k}}{9}$

$= \frac{2}{3} \times (8\vec{i} - \vec{j} + 4\vec{k})$

ii) What is the angle between vector $\vec{i} + \sqrt{3}\,\vec{j}$ and positive direction of x axis.

⇒ Given.

vector is $\vec{r} = \vec{i} + \sqrt{3}\,\vec{j}$

So, its x component is $\vec{i}$

and y component is $\sqrt{3}\,\vec{j}$

So, the slope of vector w.r.t x axis is,

$\tan\theta = \dfrac{\text{y-component}}{\text{x-component}}$

Date
Page

or, $\tan\theta = \frac{\sqrt{3}}{1}$

or, $\theta = \tan^{-1}(\sqrt{3}) = 60°$

iii) If $\vec{v}$ lies in first quadrant and makes angle $\pi/3$ with positive x-axis and $|\vec{v}| = 4$, find $\vec{v}$ in component form.

=> Solution.

Given, ~~slot~~ angle that vector makes with x-axis $= \pi/3$.

and $|\vec{v}| = 4$.

Now,

slope of vector is $\tan\theta = \tan\pi/3$

also, $\tan\theta = \frac{y\text{-component}}{x\text{-component}}$

or, $\sqrt{3} = \frac{y\text{-component}}{x\text{-component}}$

if x is unit vector along x axis then,

y-component $= \sqrt{3}\vec{j}$ and x-component is $1.\vec{i}$

So, the desired vector is $\vec{r} = t(\vec{i} + \sqrt{3}j)$

also, we have

$|\vec{v}| = 4 = \sqrt{t^2 + (\sqrt{3}t)^2}$

or, $4 = \sqrt{t^2 + 3t^2}$

or, $4 = 2t$

or, $t = 2$.

thus, the required vector is $\vec{r} = 2\vec{i} + 2\sqrt{3}\vec{j}$

and y-component $= 2\sqrt{3}\vec{j}$ and x-component $= 2\vec{i}$

iv) A quarterback throws football with an angle of elevation $\theta = 40°$ and speed $\vec{v} = 60$ ft/s. Find the horizontal and vertical components of velocity vector.

=> Solution

Given, angle of elevation $\theta = 40°$

speed $(\vec{v}) = 60$ ft/s

Now,

the horizontal component is $= \cos\theta \, \vec{v}$

$= \cos 40° \times 60$

and vertical component is $= \sin\theta \, \vec{v}$

$= \sin 40° \times 60$

Q.No.2

i) The position vectors of P, Q, R, S are (2,0,4), $(5, 3\sqrt{3}, 4)$, $(0, -2\sqrt{3}, 1)$ and (2,0,1) respectively. prove RS is parallel to PQ and PQ : RS = 3 : 2

=> Solution,

Given, $P = (2,0,4)$

$Q = (5, 3\sqrt{3}, 4)$

$R = (0, -2\sqrt{3}, 1)$

$S = (2,0,1)$

Also, $PQ : RS = 3 : 2$.

Now, $\vec{PQ} = \vec{Q} - \vec{P} = (5, 3\sqrt{3}, 4) - (2,0,4) = (3, 3\sqrt{3}, 0)$

$\vec{RS} = \vec{S} - \vec{R} = (2,0,1) - (0,-2\sqrt{3},1) = (2, 2\sqrt{3}, 0)$

Here, clearly $\vec{PQ} = \frac{3}{2}\vec{RS}$ or, $\frac{PQ}{RS} = 3/2$.

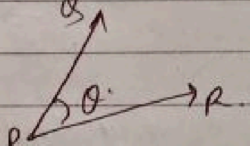
since, $\vec{PQ}$ can be represented in terms of $\vec{RS}$ with constant it is parallel to $\vec{RS}$

~~Again,~~

iii) Show that the following three points are collinear
$\vec{i}+2\vec{j}+4\vec{k}$, $2\vec{i}+5\vec{j}-\vec{k}$ and $3\vec{i}+8\vec{j}-6\vec{k}$

=> Given,
three points are $\vec{P} = \vec{i}+2\vec{j}+4\vec{k}$, $\vec{Q} = 2\vec{i}+5\vec{j}-\vec{k}$, $\vec{R} = 3\vec{i}+8\vec{j}-6\vec{k}$.

Now, lets find $\vec{PQ}$ and $\vec{PR}$ so,



$\vec{PQ} = \vec{Q} - \vec{P} = (2\vec{i}+5\vec{j}-\vec{k}) - (\vec{i}+2\vec{j}+4\vec{k})$
$= \vec{i}+3\vec{j}-5\vec{k}$

And, $\vec{PR} = \vec{R} - \vec{P} = (3\vec{i}+8\vec{j}-6\vec{k}) - (\vec{i}+2\vec{j}+4\vec{k}) = (2\vec{i}+6\vec{j}-10\vec{k})$

Now,

Angle between two vectors is $\cos\theta = \frac{\vec{PQ}\cdot\vec{PR}}{|\vec{PQ}|\,|\vec{PR}|}$

$= \frac{\cancel{1\times2+3\times8+(-5)\times(-6)}}{\sqrt{1^2+3^2+(-5)^2}\sqrt{3^2+8^2+(-6)^2}}$

$= \cancel{\frac{57}{\sqrt{}}}$

$= \frac{1\times2+3\times6+(-5)\times(-10)}{|\vec{PQ}|\,|\vec{PR}|}$

or, $\cos\theta = \dfrac{70}{\sqrt{1^2+3^2+(-5)^2}\ \sqrt{2^2+6^2+(-10)^2}}$

$= \dfrac{70}{\sqrt{35} \times \sqrt{140}}$

$= 1$

or, $\theta = \cos^{-1}(1) = 0^\circ$.

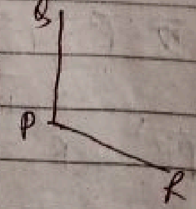
Here, since angle between vectors is 0° they are collinear.

iii) Show that three points whose position vector are $7\vec{j}+10\vec{k}$, $-\vec{i}+6\vec{j}+6\vec{k}$ and $-4\vec{i}+9\vec{j}+6\vec{k}$ forms an isosceles right-angled triangle.

=> Solution,

Given, vectors are $\vec{P} = 7\vec{j}+10\vec{k}$, $\vec{Q} = -\vec{i}+6\vec{j}+6\vec{k}$

$\vec{R} = -4\vec{i}+9\vec{j}+6\vec{k}$



so, lets find, $\vec{PQ} = \vec{Q} - \vec{P}$

$= (-\vec{i}+6\vec{j}+6\vec{k}) - (7\vec{j}+10\vec{k})$

$= -\vec{i} - \vec{j} - 4\vec{k}$

$\vec{PR} = \vec{R} - \vec{P}$

$= (-4\vec{i}+9\vec{j}+6\vec{k}) - (7\vec{j}+10\vec{k})$

$= -4\vec{i}+2\vec{j}-4\vec{k}$

And, $\vec{QR} = \vec{R} - \vec{Q} = (-4\vec{i}+9\vec{j}+6\vec{k}) - (-\vec{i}+6\vec{j}+6\vec{k})$

$= -3\vec{i}+3\vec{j}$

so, lets calculate magnitude.

$|\vec{PR}| = \sqrt{(-4)^2+(2)^2+(-4)^2} = \sqrt{36} = 6$

$$|\overrightarrow{PQ}| = \sqrt{(-1)^2+(-1)^2+(-4)^2} = \sqrt{18}$$
$$|\overrightarrow{QR}| = \sqrt{(-3)^2+(3)^2} = \sqrt{18}$$

Again, $$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{QR}}{|\overrightarrow{PQ}|\,|\overrightarrow{QR}|} = \frac{(-1)\times(-3)+(-1)\times 3+(-4)\times 0}{\sqrt{18}\times\sqrt{18}}$$
$$= \frac{3-3}{\sqrt{18}\times\sqrt{18}}$$
$$= 0$$

or, $\theta = \cos^{-1}(0) = 90°$.

Hence, the three points forms isosceles triangle with one angle $90°$.

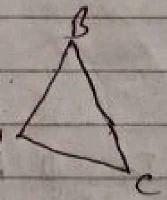
Q.No.3

i) If A, B and C are vertices of triangle, find $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}$

⇒ Solution:

let, A, B, C be vertices of triangle.

then, according to vector law of triangle



$$\overrightarrow{AC} = \overrightarrow{AB}+\overrightarrow{BC}$$

adding $\overrightarrow{CA}$ on both sides:

$$\overrightarrow{AC}+\overrightarrow{CA} = \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}$$

or, $\overrightarrow{AC}-\overrightarrow{AC} = \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}$ $[\because \overrightarrow{CA} = -\overrightarrow{AC}]$

or, $\vec{0} = \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}$

Hence, $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} = \vec{0}$

ii) let c be point on line segment AB that is twice as far from B as it is from A. If $\vec{a} = \vec{OA}$, $\vec{b} = \vec{OB}$, and $\vec{c} = \vec{OC}$, show that $\vec{c} = 2/3\,\vec{a} + 1/3\,\vec{b}$

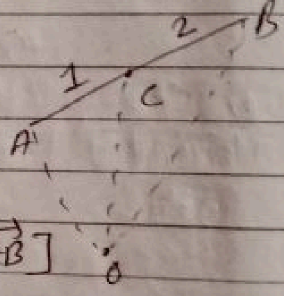
=> Solution.

Given, AB is line segment., AC : CB = 1:2

Also, $\vec{OA} = \vec{a}$

$\vec{OB} = \vec{b}$

and $\vec{OC} = \vec{c}$



Now,

$\vec{AB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a}$

Again,

$\vec{OA} + \vec{AC} = \vec{OC}$

or, $\vec{a} + \dfrac{\vec{AB}}{3} = \vec{OC}$ $[\because \vec{AC} = t\,\vec{AB}]$

or, $\vec{OC} = \vec{a} + \dfrac{(\vec{b} - \vec{a})}{3}$

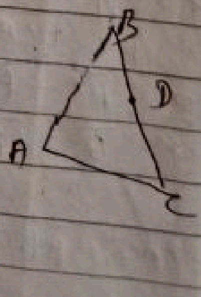
$= \dfrac{3\vec{a} + \vec{b} - \vec{a}}{3} = \dfrac{2\vec{a}}{3} + \dfrac{\vec{b}}{3}.$

iii) If D be the middle point of BC of triangle ABC, Show that: $\vec{AB} + \vec{AC} = 2\vec{AD}$

=> Given,

ABC is triangle,

D is middle point of BC



In $\triangle$ ABD

$\vec{AB} + \vec{BD} = \vec{AD}$ —(1)

Date
Page

In Δ ADC.

$\vec{AC} + \vec{CD} = \vec{AD}$ — (ii).

Adding (i) and (ii).

$$\begin{aligned} \vec{AB} + \vec{BD} &= \vec{AD} \\ \vec{AC} + \vec{CD} &= \vec{AD} \\ \hline \vec{AB} + \vec{AC} + \vec{BD} + \vec{CD} &= 2\vec{AD} \end{aligned}$$

or, $\vec{AB} + \vec{AC} + \vec{BD} - \vec{DC} = 2\vec{AD}$

or, $\vec{AB} + \vec{AC} + \vec{0} = 2\vec{AD}$ [$\because$ $\vec{BD} = \vec{DC}$ since D is mid]

or, $\vec{AB} + \vec{AC} = 2\vec{AD}$

4i) Find the direction cosines and directional angles of vector $\vec{i} - 2\vec{j} - 3\vec{k}$

=> Solution.

Given vector $\vec{v} = \vec{i} - 2\vec{j} - 3\vec{k}$

Now, $||\vec{v}|| = \sqrt{1^2 + (-2)^2 + (-3)^2}$
$= \sqrt{14}$.

So, direction cosines are,

$\cos\alpha = \dfrac{v_x}{||\vec{v}||} = \dfrac{1}{\sqrt{14}}$. or, $\alpha = 74.5°$

$\cos\beta = \dfrac{v_y}{||\vec{v}||} = \dfrac{-2}{\sqrt{14}}$ or, $\beta = 122.3°$

$\cos\gamma = \dfrac{v_z}{||\vec{v}||} = \dfrac{-3}{\sqrt{14}}$ or, $\gamma = 143.3°$

Date
Page

4 ii) If $\vec{OP} = \vec{i} + 3\vec{j} - 7\vec{k}$ and $\vec{OQ} = 5\vec{i} - 2\vec{j} + 4\vec{k}$ find $\vec{PQ}$ and determine its direction cosines.

=> Solution.

Given, $\vec{OP} = \vec{i} + 3\vec{j} - 7\vec{k}$

$\vec{OQ} = 5\vec{i} - 2\vec{j} + 4\vec{k}$

so,

$\vec{v} = \vec{PQ} = \vec{OQ} - \vec{OP} = (5\vec{i} - 2\vec{j} + 4\vec{k}) - (\vec{i} + 3\vec{j} - 7\vec{k})$

$= (4\vec{i} - 5\vec{j} + 11\vec{k})$

And, $|\vec{PQ}| = \sqrt{4^2 + (-5)^2 + (11)^2} = \sqrt{16 + 25 + 121}$

$= \sqrt{162}.$

So,

$\cos\alpha = \frac{v_x}{\|\vec{v}\|} = 4/\sqrt{162}$

$\cos\beta = \frac{v_y}{\|\vec{v}\|} = \frac{-5}{\sqrt{162}}$

$\cos\gamma = \frac{v_z}{\|\vec{v}\|} = \frac{11}{\sqrt{162}}$

4. iii) If a vector has direction angles $\alpha = \pi/4$ and $\beta = \pi/3$, find the direction angle γ

=> Solution,

Given, $\alpha = \pi/4$, $\beta = \pi/3$, $\gamma = ?$

Also, we have,

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

or, $\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{3} + \cos^2\gamma = 1$

Q.No.

5i)

=>

Date
Page

or, $\cos^2 \gamma = 1 - \frac{1}{2} - \frac{1}{4} = 0.25$

or, $\cos \gamma = 1/2$.

or, $\gamma = \cos^{-1}(1/2) = \pi/3$

Q.No.

5i) Find the linear combinations between the following system of vectors:
$\vec{a} - \vec{b} + \vec{c}$, $\vec{b} + \vec{c} - \vec{a}$, $\vec{c} + \vec{a} + \vec{b}$, $2\vec{a} - 3\vec{b} + 4\vec{c}$; where $\vec{a}, \vec{b}, \vec{c}$ being any three non coplanar vectors.

=> Solution,

let $\vec{D} = \vec{a} - \vec{b} + \vec{c}$
$\vec{E} = \vec{b} + \vec{c} - \vec{a}$
$\vec{F} = \vec{c} + \vec{a} + \vec{b}$
$\vec{G} = 2\vec{a} - 3\vec{b} + 4\vec{c}$

Now,

let, $\vec{G} = \alpha \vec{D} + \beta \vec{E} + \gamma \vec{F}$
$= \alpha(\vec{a} - \vec{b} + \vec{c}) + \beta(\vec{b} + \vec{c} - \vec{a}) + \gamma(\vec{a} + \vec{b} + \vec{c})$

or, $2\vec{a} - 3\vec{b} + 4\vec{c} = \vec{a}(\alpha - \beta + \gamma) + \vec{b}(\beta - \alpha + \gamma) + \vec{c}(\alpha + \beta + \gamma)$

comparing we get,

$\alpha - \beta + \gamma = 2$ —(i)
$\beta - \alpha + \gamma = -3$. —(ii)
$\alpha + \beta + \gamma = 4$. —(iii)

Solving the above equations we get.

$\alpha, \beta, \gamma = 7/2, 1, -1/2$.

thus, $\vec{G} = \frac{7}{2}\vec{D} + \vec{E} + -\frac{1}{2}\vec{F}$

5 iii.) Are vectors $-\vec{a}+4\vec{b}+3\vec{c}$, $2\vec{a}-3\vec{b}-5\vec{c}$, $2\vec{a}+7\vec{b}-3\vec{c}$ coplanar? where $\vec{a}, \vec{b}, \vec{c}$ are any vectors.

⇒ Solution.

Given, let $\vec{A} = -\vec{a}+4\vec{b}+3\vec{c}$

$\vec{B} = 2\vec{a}-3\vec{b}-5\vec{c}$

$\vec{C} = 2\vec{a}+7\vec{b}-3\vec{c}$

If the given three vectors are coplanar, then any one of them can be expressed as the linear combination of others.

So, let, $\vec{C} = \alpha\vec{A}+\beta\vec{B}$

or, $2\vec{a}+7\vec{b}-3\vec{c} = \alpha(-\vec{a}+4\vec{b}+3\vec{c})+\beta(2\vec{a}-3\vec{b}-5\vec{c})$

$= \vec{a}(2\beta-\alpha)+\vec{b}(4\alpha-3\beta)+\vec{c}(3\alpha-5\beta)$

comparing we get.

$2\beta-\alpha = 2$

$4\alpha-3\beta = 7$

$3\alpha-5\beta = -3$

on solving we get, $\alpha = 4$, $\beta = 3$

Check. $3\times4-5\times3 = -3$

or, $12-15 = -3$

or, $-3 = -3$ (True)

Dot product of vectors

1) Determine whether the given vectors are orthogonal, parallel or neither.

i) $\vec{a} = (-5, 3, 7)$ and $\vec{b} = (6, -8, 2)$

Now,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{-5\times6+3\times(-8)+7\times2}{\sqrt{(-5)^2+(3)^2+7^2}\ \sqrt{6^2+(-8)^2+2^2}}$$

$$= \frac{-40}{\sqrt{83}\ \sqrt{104}}$$

or, $\cos\theta = \frac{-40}{\sqrt{83}\ \sqrt{104}}$ which is not 0 or 1.

Hence, the vectors are not orthogonal or parallel.

2

1

ii) $\vec{a} = -\vec{i} + 2\vec{j} + 5\vec{k}$ and $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$

Now,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{-1\times3+2\times4+5\times(-1)}{\sqrt{1^2+2^2+5^2}\ \sqrt{3^2+4^2+1^2}} = 0$$

Hence, ~~it is~~ they are orthogonal vectors.

iii) $\vec{a} = 2\vec{i} + 6\vec{j} - 4\vec{k}$ and $\vec{b} = -3\vec{i} - 9\vec{j} + 6\vec{k}$

Now,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{2\times(-3)+6\times(-9)+(-4)\times6}{\sqrt{2^2+6^2+4^2}\ \sqrt{(-3)^2+(-9)^2+6^2}}$$

$$= \frac{-84}{\sqrt{56}\ \sqrt{126}}$$

$$= -1.$$

Hence, they are parallel vectors.

Date
Page

2 i) Find $\vec{a}\cdot\vec{b}$ if $|\vec{a}| = 6$, $|\vec{b}| = 5$ and the angle between $\vec{a}$ and $\vec{b}$ is $2\pi/3$

⇒ Solution.

Given, $|\vec{a}| = 6$

$|\vec{b}| = 5$

and angle $(\theta) = 2\pi/3$

we have, $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$

or, $\cos\left(\dfrac{2\pi}{3}\right) = \dfrac{\vec{a}\cdot\vec{b}}{6\times 5}$

or, $\vec{a}\cdot\vec{b} = 30 \times -\dfrac{1}{2}$

$= -15$.

iii) Find the value of x such that angle between vectors $(2,1,-1)$ and $(1,x,0)$ is $45°$

⇒ Solution.

let, $\vec{a} = (2,1,-1)$

$\vec{b} = (1,x,0)$

$\theta = 45°$

we have,

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \dfrac{2\times 1 + 1\times x + (-1)\times 0}{\sqrt{2^2+1^2+(-1)^2}\ \sqrt{1^2+x^2}}$

or, $\cos 45 = \dfrac{2+x}{\sqrt{6\times(1+x^2)}}$

ii)

⇒

Date
Page

$$\text{or, } \frac{1}{\sqrt{2}} = \frac{2+x}{\sqrt{6+6x^2}}$$

Squaring both sides.

$$6+6x^2 = 2(2+x)^2$$

$$\text{or, } 6+6x^2 = 2(x^2+4x+4)$$

$$\text{or, } 6+6x^2 = 2x^2+8x+8$$

$$\text{or, } 4x^2+8x+2=0$$

$$\text{Solving } x = \frac{-8 \pm \sqrt{64-4(\cancel{4})(+2)}}{2\cdot\cancel{1}(-4)}$$

$$= \frac{-8 \pm \sqrt{\cancel{}\,96}}{2\cdot(-4)}$$

$$= \frac{-8 \pm \cancel{}\,4\sqrt{6}}{2\cdot(-4)} = \frac{-8 \pm 4\sqrt{6}}{-8}$$

$$= \frac{(4\sqrt{6}-8)}{-8} = -0.2247$$

ii) Find the cosine of angle between following pairs of vectors : $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j} + 2\vec{k}$

=> Solution.

Given $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$, $\vec{b} = \vec{i} + 3\vec{j} + 2\vec{k}$

Now, we have,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1\times1 - 2\times3 + 3\times2}{\sqrt{1^2+(-2)^2+3^2}\,\sqrt{1^2+3^2+2^2}}$$

$$= \frac{1}{\sqrt{14}\times\sqrt{14}}$$

$$= \frac{1}{14}.$$

Date
Page

2 iv) Show that line AB is perpendicular to CD if A, B, C, D are points (2,3,4), (5,4,−1), (3,6,2) and (1,2,0)

⇒ Solution:

Given points $A = (2,3,4)$, $B = (5,4,-1)$, $C = (3,6,2)$
$D = (1,2,0)$

Now,

$AB = B - A = (5,4,-1) - (2,3,4) = (3,1,-5)$

$CD = D - C = \cancel{(3,6,2)}\ (1,2,0) - (3,6,2) = (-2,-4,-2)$

Now,

$$\cos\theta = \frac{AB \cdot CD}{|AB|\,|CD|} = \frac{3\times(-2) + 1\times(-4) + (-5)\times(-2)}{\sqrt{3^2+1^2+(-5)^2}\ \sqrt{(-2)^2+(-4)^2+(-2)^2}}$$

$$= 0$$

or, $\theta = \cos^{-1}(0) = 90^\circ$

Hence, the line AB is perpendicular to CD.

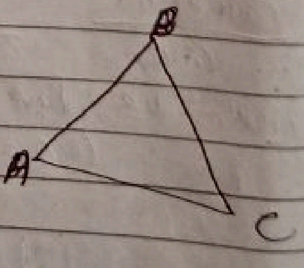
v) Find the angle of triangles whose vertices are $7\vec{j} + 10\vec{k}$, $-\vec{i} + 6\vec{j} + 6\vec{k}$ and $-4\vec{i} + 9\vec{j} + 6\vec{k}$ respectively.

⇒ Solution,

let, $\vec{OA} = 7\vec{j} + 10\vec{k}$

$\vec{OB} = -\vec{i} + 6\vec{j} + 6\vec{k}$

$\vec{OC} = -4\vec{i} + 9\vec{j} + 6\vec{k}$



Now, we have,

$\vec{AB} = \vec{OB} - \vec{OA} = (-\vec{i} + 6\vec{j} + 6\vec{k}) - (7\vec{j} + 10\vec{k})$

$= -\vec{i} - \vec{j} - 4\vec{j}$

Date
Page

$\vec{BC} = \vec{OC} - \vec{OB} = -4\vec{i} + 9\vec{j} + 6\vec{K} - (-\vec{i} + 6\vec{j} + 6\vec{K})$

$= -3\vec{i} + 3\vec{j}$

$\vec{CA} = \vec{OA} - \vec{OC} = 7\vec{j} + 10\vec{K} - (-4\vec{i} + 9\vec{j} + 6\vec{K})$

$= 4\vec{i} - 2\vec{j} + 4\vec{K}$

Now, we have, $\cos A = \dfrac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}|\,|\vec{AC}|}$

$= \dfrac{-1 \times (-4) + (-1) \times (2) + (-4) \times (-4)}{\sqrt{(-1)^2 + (-1)^2 + (-4)^2}\ \sqrt{(-4)^2 + 2^2 + (-4)^2}}$

$= \dfrac{18}{\sqrt{18} \times \sqrt{36}} = \dfrac{\cancel{18}}{\cancel{18}\sqrt{2}}$

or, $A = \cos^{-1}(1/\sqrt{2}) = 45°$.

Also,

$\cos B = \dfrac{\vec{BA} \cdot \vec{BC}}{|\vec{BA}|\,|\vec{BC}|} = \dfrac{1 \times (-3) + (1) \times 3 + 4 \times 0}{\sqrt{1^2 + 1^2 + 4^2}\ \sqrt{(-3)^2 + 3^2 + 0}}$

$= 0$

$B = \cos^{-1} 0$

$= 90°$

thus, $C = 180° - 90° - 45° = 45°$.

Date
Page

3.) i) Give the geometrical meaning of scalar product. Find the scalar projection and vector projection of $\vec{a} = \vec{i} - \vec{j} + \vec{k}$ onto $\vec{b} = \vec{i} + \vec{j} + \vec{k}$

→ Solution.

Given, $\vec{a} = \vec{i} - \vec{j} + \vec{k}$

$\vec{b} = \vec{i} + \vec{j} + \vec{k}$

The geometrical meaning of scalar product is the product of magnitude of vector A with magnitude of projection of vector B on A

Scalar projection of $\vec{a}$ into $\vec{b}$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{1\times1 + (-1)\times1 + 1\times1}{\sqrt{1^2+1^2+1^2}}$$

$$= \frac{1}{\sqrt{3}}.$$

Vector projection of $\vec{a}$ into $\vec{b}$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \cdot \frac{\vec{b}}{|\vec{b}|}$$

$$= \frac{1}{\sqrt{3}} \cdot \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{1^2+1^2+1^2}}$$

$$= \frac{1}{3}(\vec{i} + \vec{j} + \vec{k})$$

Date
Page

3. ii) If $\vec{a} = (3, 0, -1)$, find a vector $\vec{b}$ such that scalar projection of $\vec{a}$ on $\vec{b}$ is 2.

=> Solution,

Given, $\vec{a} = (3, 0, -1)$

and Scalar projection of $\vec{a}$ on $\vec{b} = 2$.

so,

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = 2$$

or, $\dfrac{3 \times a + 0 \times b + (-1) \times c}{\sqrt{a^2 + b^2 + c^2}} = 2$

or, $3a - c = 2\sqrt{a^2 + b^2 + c^2}$.

squaring both sides.

$(3a - c)^2 = 4(a^2 + b^2 + c^2)$

or, $9a^2 - 6ac + c^2 = 4(a^2 + b^2 + c^2)$

or, $5a^2 - 6ac - 4b^2 - 3c^2 = 0$.

If $a = 0$,

then, $4b^2 = -3c^2$ not possible.

If $b = 0$

then, $3a - c = 2\sqrt{a^2 + c^2}$

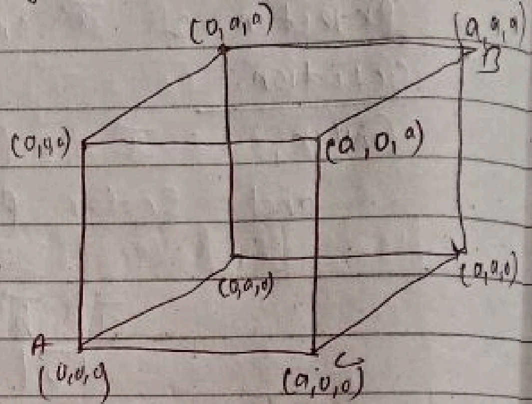
or, $\dfrac{3a - c}{2} = \sqrt{a^2 + c^2}$

squaring both $\dfrac{9a^2}{4} - 3ac + \dfrac{c^2}{4} = a^2 + c^2$

$9a^2 - 6ac + c^2 =$

4) Find the angle between diagonal of cube and one of its edges.

=> let, the side of cube be a.



then, O be origin and $B = (a,a,a)$ be the opposite end vertin of O

then we have,

$\vec{AB} = \vec{OB} - \vec{OA} = (a,a,a) - (0,0,0)$

$= (a,a,a)$

and we take edge AC so,

$\vec{AC} = \vec{OC} - \vec{OA} = (a,0,0) - (0,0,0)$

$= (a,0,0)$

Now,

let, θ be angle between diagonal and edge AC

then,

$$\cos\theta = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}|\,|\vec{AC}|} = \frac{(a,a,a)\cdot(a,0,0)}{\sqrt{a^2+a^2+a^2}\cdot\sqrt{a^2}}$$

$$= \frac{a \times a}{a\sqrt{3a^2}}$$

$$= \frac{\cancel{a}\cdot a}{\cancel{a}\cdot a\sqrt{3}}$$

$$= 1/\sqrt{3}.$$

so, $\theta = \cos^{-1}(1/\sqrt{3})$

$= 54.736°$

Date ______ Page ______

5) Find the work done by force $\vec{F} = 8\vec{i} - 6\vec{j} + 9\vec{k}$ that moves an object from point (0,10,8) to point (6,12,20) along a straight line. The distance is measured in meters and the force in newtons.

=> Solution.

Given, force vector $(\vec{F}) = 8\vec{i} - 6\vec{j} + 9\vec{k}$

Initial point A = (0,10,8), final point = B = (6,12,20).

Now,

$\vec{AB} = \vec{OB} - \vec{OA} = (6,12,20) - (0,10,8) = (6,2,12)$

Now,

We know work done is given by $W = f \cdot s$

$= \vec{F} \cdot \vec{AB}$

$= (8,-6,9) \cdot (6,2,12)$

$= 8 \times 6 - 6 \times 2 + 9 \times 12$

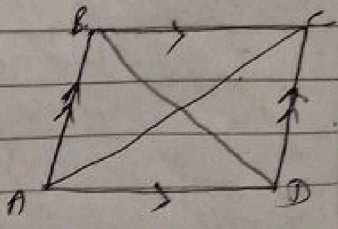
$= 144$ N-m. (J)

6i) Suppose that all sides of quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show diagonals are perpendicular

=> Solution.

Given, all sides of quadrilateral are equal.

opposite sides are parallel.



let, ~~A = (0,0,0) , B = (~~

$|\vec{AB}| = |\vec{DC}|$

we have in ΔABC

$\vec{AB} + \vec{BC} = \vec{AC}$

also, In $\triangle BCD$

$$\vec{DB} = \vec{DC} + \vec{CB}$$

we have angle between $\vec{AC}$ and $\vec{DB}$ is given by

$$\cos\theta = \frac{\vec{AC}\cdot\vec{DB}}{|\vec{AC}|\,|\vec{DB}|} = \frac{(\vec{AB}+\vec{BC})\cdot(\vec{DC}+\vec{CB})}{|\vec{AC}|\,|\vec{DB}|}$$

$$= \frac{\vec{AB}\cdot\vec{DC} + \vec{AB}\cdot\vec{CB} + \vec{BC}\cdot\vec{DC} + \vec{BC}\cdot\vec{CB}}{|\vec{AC}|\,|\vec{DB}|}$$

$$= \cancel{\vec{AB}\cdot\vec{DC}}\,(|AB||DC|\cdot\cos 0 + \vec{DC}\cdot\vec{DA} + \cancel{\vec{CB}\cdot\vec{CD}}\ \vec{CB}\cdot\vec{CD} + |\vec{BC}||\vec{CB}|\cos 180^\circ) / (|\vec{AC}||\vec{DB}|)$$

$$= \frac{a\cdot a + \vec{DB}\cdot\vec{CA} + \cancel{b\cdot b}\ a\cdot a(-1)}{|\vec{AC}|\,|\vec{DB}|}$$

$$= \frac{\vec{DC}+\vec{CB}+|\vec{CB}+\vec{BA}}{|\vec{AC}|\,|\vec{DB}|}$$

$$= \frac{\vec{AB}+2\vec{CB}+\vec{BA}}{|\vec{AC}|\,|\vec{DB}|}$$

$$= \frac{2\vec{CB}}{|(\vec{AB}+\vec{BC})\cdot(\vec{DC}+\vec{CB})|}$$

$$= \frac{2\vec{CB}}{|2\vec{CB}|}$$

$$= \frac{\vec{CB}}{|\vec{CB}|}$$

$$= a\cdot a + |\vec{DC}||\vec{DA}|\cdot\cos D + |\vec{CB}||\vec{CD}|\cdot\cos C - a\cdot a$$

$$= \frac{a^2\cdot\cos D + a^2\cos C}{|\vec{AC}|\,|\vec{DB}|}$$

$$= \frac{a^2(\cos D + \cos C)}{|\vec{AC}\cdot\vec{DB}|}$$

$$= \frac{a^2(\cos D + \cos C)}{|a^2\cos(D + \cos C)|}$$

$$= 1.$$

Here, $\vec{AC} \perp \vec{DB}$

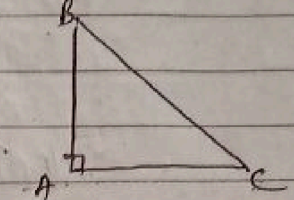
Date
Page

6 ii) In right angled triangle ABC, right angle at A, use vector methods to show $(AB)^2 + (AC)^2 = (BC)^2$

=> Solution:

Given, ΔABC is right angled triangle

$\angle A$ is $90°$.



Now,

from triangle law we have,

$$\vec{BA} + \vec{AC} = \vec{BC}$$

dot product ~~multiply~~ both sides by $\vec{BC}$.

$$(\vec{BA} + \vec{AC}) \cdot \vec{BC} = \vec{BC} \cdot \vec{BC}$$

or, $\vec{BA} \cdot \vec{BC} + \vec{AC} \cdot \vec{BC} = |\vec{BC}|\,|\vec{BC}| \cdot \cos 0$

or, $|\vec{BA}|\,|\vec{BC}| \cos B + \vec{CA} \cdot \vec{CB} = (BC)^2 \cdot 1.$

or, $|\vec{BA}|\,\cancel{|\vec{BC}|} \cdot \frac{|\vec{BA}|}{\cancel{|\vec{BC}|}} + |\vec{CA}| \cdot |\vec{CB}| \cdot \cos C = (BC)^2$

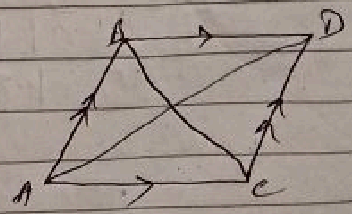
or, $(BA)^2 + |\vec{CA}| \cdot \cancel{|\vec{CB}|} \cdot \frac{|\vec{AC}|}{\cancel{|\vec{BC}|}} = (BC)^2$

or, $(AB)^2 + (AC)^2 = (BC)^2$

6 iii) Use vector method to show that parallelogram whose diagonal are equal is rectangle.

=> Given,

$|\vec{AD}| = |\vec{BC}|$ —— (i)



In Δ ABD.

$\vec{AD} = \vec{AB} + \vec{BD}$

$\vec{BC} = \vec{BA} + \vec{AC} = \vec{AC} - \vec{AB}$

squaring eqn (1)

$(\overrightarrow{AB})^2 = (\overrightarrow{BC})^2$

or, $(\overrightarrow{AB} + \overrightarrow{BD})^2 = (\overrightarrow{AC} - \overrightarrow{AB})^2$

or, $(\overrightarrow{AB} + \overrightarrow{AC})^2 = (\overrightarrow{AC} - \overrightarrow{AB})^2$

or, $\cancel{(\overrightarrow{AB})^2} + 2\overrightarrow{AB} \cdot \overrightarrow{AC} + \cancel{(\overrightarrow{AC})^2} = \cancel{(\overrightarrow{AC})^2} - 2\overrightarrow{AC} \cdot \overrightarrow{AB} + \cancel{(\overrightarrow{AB})^2}$

or, $4\overrightarrow{AB} \cdot \overrightarrow{AC} = 0$

or, $\overrightarrow{AB} \cdot \overrightarrow{AC}$ -

Hence, $AB \perp AC$.

thus, it is rectangle.

Q.No. 7 i) Use formula $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ to prove Cauchy-Schward enquality $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$

=> Solution:

we have dot product is given by $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|$. ~~cosθ~~

we have, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$.

taking modulus we have,

$|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}||\cos\theta|$

Here, $|\cos\theta|$ value ranges from 0 to 1.

so, we can write.

$|\vec{a}|\,|\vec{b}|\,|\cos\theta| \leq |\vec{a}|\,|\vec{b}|$

or, $|\vec{a} \cdot \vec{b}| \leq |\vec{a}|\,|\vec{b}|$

Date
Page

ii) Give a geometric interpretation of triangle inequality $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$ for vectors. Use cauchy-Schwaz inequality to prove the triangle inequality.

⇒ Solution:

let, $\vec{a}$ and $\vec{b}$ be two vecters then,

$$
\begin{aligned}
|\vec{a}+\vec{b}|^2 &= (\vec{a}+\vec{b})^2 \\
&= (\vec{a})^2 + 2\cdot\vec{a}\cdot\vec{b} + (\vec{b})^2 \\
&= |\vec{a}|^2 + 2\cdot|\vec{a}||\vec{b}|\cos\theta + |\vec{b}|^2 \\
&\leq |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2
\end{aligned}
$$

or, $|\vec{a}+\vec{b}|^2 \leq (|\vec{a}| + |\vec{b}|)^2$

or, $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$

iii) prove parallelogram law $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2|\vec{a}|^2 + 2|\vec{b}|^2$ for vecters. Give geometric interpretation of parallelogram law.

⇒ Solution.

let $\vec{a}$ and $\vec{b}$ be two vecters.

so, $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + \cancel{2|\vec{a}||\vec{b}|} + |\vec{b}|^2 + |\vec{a}|^2 - \cancel{2|\vec{a}||\vec{b}|} + |\vec{b}|^2$

$= 2|\vec{a}|^2 + 2|\vec{b}|^2$

Date
Page

7 iv) If $\vec{a}$ and $\vec{b}$ are two vectors of unit length and θ is the angle between them, show that $\frac{1}{2}|\vec{a}-\vec{b}| = \sin\theta/2$

⇒ Solution:

Given, $\vec{a}$ and $\vec{b}$ are unit vectors
θ is angle between $\vec{a}$ and $\vec{b}$

so,

$$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})^2$$
$$= |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$$
$$= |\vec{a}|^2 - 2|\vec{a}||\vec{b}|\cos\theta + |\vec{b}|^2$$
$$= 1^2 - 2\cdot 1\cdot 1\cdot\cos\theta + 1^2$$
$$= 2 - 2\cos\theta$$
$$= 2(1-\cos\theta)$$
$$= 2\cdot 2\sin^2\theta/2$$

or, $|\vec{a}-\vec{b}| = 2\sin\theta/2$

or, $\frac{1}{2}|\vec{a}-\vec{b}| = \sin\frac{\theta}{2}$

8 i) If $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are orthogonal show that vectors $\vec{u}$ and $\vec{v}$ must have same length

⇒ Solution:

Let $\vec{u}$ and $\vec{v}$ be two vectors then,
given, $(\vec{u}+\vec{v})$ and $(\vec{u}-\vec{v})$ are orthogonal.
that means

$$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = 0$$

or, $\vec{u}\cdot\vec{u} - \cancel{\vec{u}\cdot\vec{v}} + \cancel{\vec{v}\cdot\vec{u}} - \vec{v}\cdot\vec{v} = 0$

or, $|\vec{u}|^2 - |\vec{v}|^2 = 0$

or, $|\vec{u}|^2 = |\vec{v}|^2$.

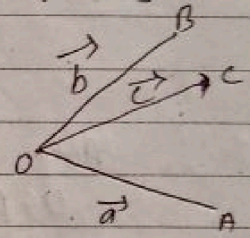
this shows that $\vec{u}$ and $\vec{v}$ must have same magnitude

ii) If $\vec{c} = |\vec{a}|\cdot\vec{b} + |\vec{b}|\cdot\vec{a}$, where $\vec{a}$ and $\vec{b}$ and $\vec{c}$ are non-zero vectors, show that $\vec{c}$ bisects the angle between $\vec{a}$ and $\vec{b}$

⇒ solution:

Given, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero vectors.

we have,



In Δ OAC,

$\vec{OC} = \vec{OA} + \vec{AC}$

$= \vec{a} + \vec{AC}$

or, $\vec{c} = \vec{a} + \vec{AC}$ —(i)

Also, in triangle OCB.

$\vec{OC} = \vec{OB} + \vec{BC}$

or, $\vec{c} = \vec{b} + \vec{BC}$ —(ii)

or, $|\vec{a}|\cdot\vec{b} + |\vec{b}|\cdot\vec{a} - \vec{b} = \vec{BC}$

or, $\vec{BC} = \vec{b}(|\vec{a}|-1) + |\vec{b}|\cdot\vec{a}$

Also, from (i).

$|\vec{a}|\cdot\vec{b} + |\vec{b}|\cdot\vec{a} = \vec{a} + \vec{AC}$

or, $\vec{AC} = \vec{a}(|\vec{b}|-1) + |\vec{a}|\cdot\vec{b}$

so, $|\vec{AC}| = \sqrt{(|\vec{b}|-1)^2 + |\vec{a}|^2} = \sqrt{|\vec{b}|^2 - 2|\vec{b}| + 1 + |\vec{a}|^2}$

and, $|\vec{BC}| = \sqrt{(|\vec{a}|-1)^2 + |\vec{b}|^2} = \sqrt{|\vec{a}|^2 - 2|\vec{a}| + 1 + |\vec{b}|^2}$

In Δ OAC

$$\cos\alpha = \frac{\vec{OA}\cdot\vec{OC}}{|\vec{OA}||\vec{OC}|} = \frac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{c}|}$$

$$= \frac{\vec{a}\cdot(|\vec{a}|\vec{b} + |\vec{b}|\vec{a})}{|\vec{a}||\vec{c}|}$$

$$= \frac{\vec{a}\cdot|\vec{a}|\vec{b} + \vec{a}\cdot\vec{a}\cdot|\vec{b}|}{|\vec{a}||\vec{c}|}$$

$$= \frac{|\vec{a}|(\vec{a}\cdot\vec{b}) + |\vec{b}||\vec{a}|^2}{|\vec{a}||\vec{c}|}$$

$$= \frac{\vec{a}\cdot\vec{b} + |\vec{b}||\vec{a}|}{|\vec{c}|}$$

Also,

$$\cos\beta = \frac{\vec{OC}\cdot\vec{OB}}{|\vec{OC}||\vec{OB}|} = \frac{\vec{c}\cdot\vec{b}}{|\vec{c}||\vec{b}|}$$

$$= \frac{(|\vec{a}|\vec{b} + |\vec{b}|\vec{a})\cdot\vec{b}}{|\vec{c}||\vec{b}|}$$

$$= \frac{|\vec{a}|(\vec{b}\cdot\vec{b}) + |\vec{b}|(\vec{a}\cdot\vec{b})}{|\vec{c}||\vec{b}|}$$

$$= \frac{|\vec{a}||\vec{b}|^2 + |\vec{b}|(\vec{a}\cdot\vec{b})}{|\vec{c}||\vec{b}|}$$

$$= \frac{|\vec{a}||\vec{b}| + \vec{a}\cdot\vec{b}}{|\vec{c}|}$$

Here,

$\cos\alpha = \cos\beta$

Hence, $\vec{c}$ bisects the angle between vectors $\vec{a}$ and $\vec{b}$

Date
Page

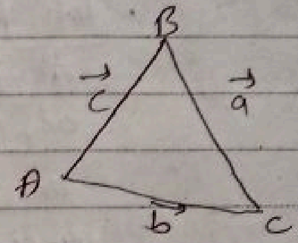
9) Using vector method prove the projection law in any triangle

i) $b = c\cos A + a\cos C$

⇒ Solution:

Given, in triangle ABC, $b = c\cos A + a\cos C$

Now, we have,



$\vec{b} = \vec{AB} + \vec{BC}$

or, $\vec{b} = \vec{a} + \vec{c}$

dot product on both sides.

$\vec{b}\cdot\vec{b} = \vec{a}\cdot\vec{b} + \vec{c}\cdot\vec{b}$

or, $(b)^2 = |\vec{a}||\vec{b}|\cos C + |\vec{c}||\vec{b}|\cos A$.

or, $b = a\cos C + c\cos A$.

Hence, $b = c\cos A + a\cos C$

ii) $c = a\cos B + b\cos A$

Solution:

We have,

$\vec{c} = \vec{AC} + \vec{CB} = \vec{b} + \vec{a}$

dot product with $\vec{c}$ on both side.

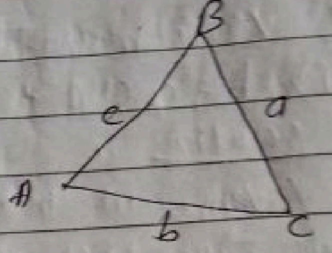
$\vec{c}\cdot\vec{c} = \vec{b}\cdot\vec{c} + \vec{a}\cdot\vec{c}$

or, $(c)^2 = |\vec{b}||\vec{c}|\cos A + |\vec{a}||\vec{c}|\cos B$

or, $c = b\cos A + a\cos B$.

Hence, $c = a\cos B + b\cos A$.

9) iii) $a = b\cos C + c\cos B$.



we have,

$\vec{a} = \vec{CA} + \vec{AB}$

or, $\vec{a} = \vec{b} + \vec{c}$

dot product with a on both sides.

$\vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}$

or, $(a)^2 = |\vec{b}||\vec{a}|\cos C + |\vec{c}||\vec{a}|\cos B$.

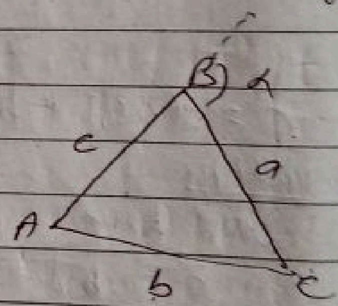
or, $a = b\cos C + c\cos B$.

Hence, $a = b\cos C + c\cos B$

10.) Using vector method prove the cosine of trigonometry

i) $b^2 = c^2 + a^2 - 2ac\cos B$.

⇒



we have,

$\vec{b} = \vec{AB} + \vec{BC}$

or, $\vec{B} = \vec{c} + \vec{a}$

Squaring we get.

$(\vec{B})^2 = (\vec{c} + \vec{a})^2$

or, $b^2 = c^2 + a^2 + 2\vec{c} \cdot \vec{a}$

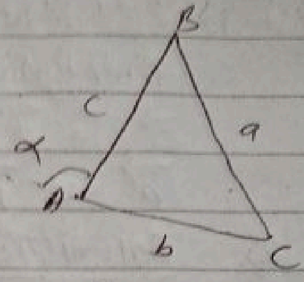
$= c^2 + a^2 + 2ac\cos\alpha$.

$= c^2 + a^2 + 2ac\cos(\cancel{B}90 - B)$

or, $b^2 = c^2 + a^2 - 2ac\cos B$.

Date
Page

10.) ii) $a^2 = b^2 + c^2 - 2bc\cos A$



⇒ solution.

we have,

$\vec{a} = \vec{CA} + \vec{AB}$

$= \vec{b} + \vec{c}$

Now,

squaring both sides we have.

$(\vec{a})^2 = (\vec{b} + \vec{c})^2$

or, $a^2 = b^2 + 2\vec{b}\cdot\vec{c} + c^2$

$= b^2 + 2b\cdot c\cos\alpha + c^2$

$= b^2 + c^2 + 2bc\cos(180 - A)$

$= b^2 + c^2 - 2bc\cos A.$

Hence, $a^2 = b^2 + c^2 - 2bc\cos A.$

cross product

1.) i) Find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both $\vec{a}$ and $\vec{b}$ for vectors
$\vec{a} = t\vec{i} + \cos t\vec{j} + \sin t\vec{k}$ and
$\vec{b} = \vec{i} - \sin t\vec{j} + \cos t\vec{k}$

⇒ Solution:

Given,

$$\vec{a} = t\vec{i} + \cos t\vec{j} + \sin t\vec{k}$$
$$\vec{b} = \vec{i} - \sin t\vec{j} + \cos t\vec{k}$$

Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & \cos t & \sin t \\ 1 & -\sin t & \cos t \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} - \vec{j}\begin{vmatrix} t & \sin t \\ 1 & \cos t \end{vmatrix} + \vec{k}\begin{vmatrix} t & \cos t \\ 1 & -\sin t \end{vmatrix}$$

$$= \vec{i}(\cos^2 t + \sin^2 t) - \vec{j}(t\cos t - \sin t) + \vec{k}(-t\sin t - \cos t)$$
$$= \vec{i} - \vec{j}(t\cos t - \sin t) - \vec{k}(t\sin t + \cos t)$$

Now,

$$\cos\theta = \frac{\vec{a}\cdot(\vec{a}\times\vec{b})}{|\vec{a}|\,|\vec{a}\times\vec{b}|} = \frac{t - \cos t(t\cos t - \sin t) - \sin t(t\sin t + \cos t)}{|\vec{a}|\,|\vec{a}\times\vec{b}|}$$

$$= \frac{t - t\cos^2 t + \cancel{\cos t\sin t} - t\sin^2 t - \cancel{\sin t\cos t}}{|\vec{a}|\,|\vec{a}\times\vec{b}|}$$

$$= \frac{t - t(\cos^2 t + \sin^2 t)}{|\vec{a}|\,|\vec{a}\times\vec{b}|}$$

$= 0$

so, $\theta = \cos^{-1}(0) = 90^\circ$

Hence, $\vec{a} \perp (\vec{a} \times \vec{b})$.

Also, for, $\vec{b} \cdot (\vec{a} \times \vec{b})$; $\dfrac{\vec{b} \cdot (\vec{a} \times \vec{b})}{|\vec{b}|\,|\vec{a} \times \vec{b}|}$

$$= \frac{1 + \sin t\,(t\cos t - \sin t) - \cos t\,(t\sin t + \cos t)}{|\vec{b}|\,|\vec{a} \times \vec{b}|}$$

$$= \frac{1 + \cancel{t\sin t\cos t} - \sin^2 t - \cancel{t\sin t\cos t} - \cos^2 t}{|\vec{b}|\,|\vec{a} \times \vec{b}|}$$

$$= \frac{1 - (\sin^2 t + \cos^2 t)}{|\vec{b}|\,|\vec{a} \times \vec{b}|}$$

$$= \frac{1 - 1}{|\vec{b}|\,|\vec{a} \times \vec{b}|} = 0$$

so, $\beta = \cos^{-1}(0) = 90^\circ$

Hence, $\vec{b} \perp (\vec{a} \times \vec{b})$

ii) Find two unit vector perpendicular to both $\vec{a} = \vec{j} - \vec{k}$ and $\vec{b} = \vec{i} + \vec{j}$

=> Solution,

we have, $\vec{a} = \vec{j} - \vec{k}$, $\vec{b} = \vec{i} + \vec{j}$.

Now, we have,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}$$

Date
Page

$= \vec{i}\begin{vmatrix}1 & -1\\1 & 0\end{vmatrix} - \vec{j}\begin{vmatrix}0 & -1\\1 & 0\end{vmatrix} + \vec{k}\begin{vmatrix}0 & 1\\1 & 1\end{vmatrix}$

$= \vec{i} - \vec{j} - \vec{k}$

and unit vector $= \frac{1}{\sqrt{3}}(\vec{i} - \vec{j} - \vec{k})$

Now $\vec{b} \times \vec{a} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\1 & 1 & 0\\0 & 1 & -1\end{vmatrix}$

$= \vec{i}\begin{vmatrix}1 & 0\\1 & -1\end{vmatrix} - \vec{j}\begin{vmatrix}1 & 0\\0 & -1\end{vmatrix} + \vec{k}\begin{vmatrix}1 & 1\\0 & 1\end{vmatrix}$

$= -\vec{i} + \vec{j} + \vec{k}$

and $|\vec{b} \times \vec{a}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.

so, unit vector $= \frac{1}{\sqrt{3}}(-\vec{i} + \vec{j} + \vec{k})$

iii) Find the unit vector perpendicular to each of the pair of vectors $\vec{a} = (3,1,2)$ and $\vec{b} = (2,-2,4)$

⇒ solution.

Given, $\vec{a} = (3,1,2)$, $\vec{b} = (2,-2,4)$

so, vector perpendicular to $\vec{a}$ and $\vec{b}$ is.

$\vec{a} \times \vec{b} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\3 & 1 & 2\\2 & -2 & 4\end{vmatrix}$

$= \vec{i}\begin{vmatrix}1 & 2\\-2 & 4\end{vmatrix} - \vec{j}\begin{vmatrix}3 & 2\\2 & 4\end{vmatrix} + \vec{k}\begin{vmatrix}3 & 1\\2 & -2\end{vmatrix}$

Date ________
Page ________

$= 8\vec{i} - 8\vec{j} - 8\vec{k}$

so, unit vector is, $\phi \cdot \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{8\vec{i} - 8\vec{j} - 8\vec{k}}{\sqrt{8^2 + (-8)^2 + (-8)^2}}$

$= \frac{1}{\sqrt{3}}(\vec{i} - \vec{j} - \vec{k})$

Q.No.2. i) Find the sine of angle between pair of vectors $3\vec{i} + \vec{j} + 2\vec{k}$ and $2\vec{i} - 2\vec{j} + 4\vec{k}$

$\Rightarrow$ Solution.

Let, $\vec{a} = 3\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{b} = 2\vec{i} - 2\vec{j} + 4\vec{k}$

Now, we have,

$\sin\theta = \frac{\vec{a} \times \vec{b}}{|\vec{a}||\vec{b}|}$

so, $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix}$

$= \vec{i}\begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}$

$= 8\vec{i} - 8\vec{j} - 8\vec{k}$

~~so, $\sin\theta = \frac{8\vec{i} - 8\vec{j} - 8\vec{k}}{\sqrt{3^2+1^2+2^2}\sqrt{4+(-2)^2+4^2}}$~~

$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{3 \times 2 + 1 \times (-2) + 2 \times 4}{\sqrt{3^2+1^2+2^2}\sqrt{2^2+(-2)^2+(4)^2}}$

$= \frac{12}{\sqrt{14}\sqrt{24}} = \cancel{\frac{\sqrt{21}}{7}}$

Date
Page

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{21}{49}} = \sqrt{\frac{4}{7}}$$

$$= 2/\sqrt{7}.$$

2) ii) If $\vec{a}\cdot\vec{b} = \sqrt{3}$ and $\vec{a}\times\vec{b} = (1,2,2)$, find the angle between $\vec{a}$ and $\vec{b}$

=> Solution:

Given, $\vec{a}\cdot\vec{b} = \sqrt{3}$.

and $\vec{a}\times\vec{b} = (1,2,2)$

We have, $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

or, $|\vec{a}||\vec{b}| = \dfrac{\vec{a}\cdot\vec{b}}{\cos\theta}$ — (i)

similarly. $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

or, $|\vec{a}||\vec{b}| = \dfrac{|\vec{a}\times\vec{b}|}{\sin\theta}$ — (ii)

from (i) and (ii). $\dfrac{\vec{a}\cdot\vec{b}}{\cos\theta} = \dfrac{|\vec{a}\times\vec{b}|}{\sin\theta}$

or, $\dfrac{\sqrt{3}}{\cos\theta}\times\sin\theta = \sqrt{1^2+2^2+2^2}$

or, $\tan\theta = \dfrac{1}{\sqrt{3}}\times\sqrt{9}$

or, $\tan\theta = \sqrt{3}$.

or, $\theta = \tan^{-1}\sqrt{3}$

$= 60^\circ$.

Date
Page

iii) Find all vectors $\vec{v}$ such that $(1,2,1) \times \vec{v} = (3,1,-5)$

$\Rightarrow$ Solution.

Given, $A = (1,2,1)$

$V = (x,y,z)$

$\vec{A} \times \vec{v} = (3,1,-5)$

Now, we have,

$$\vec{A} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ x & y & z \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 2 & 1 \\ y & z \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ x & z \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2 \\ x & y \end{vmatrix}$$

or, $(3,1,-5) = (2z-y)\vec{i} - (z-x)\vec{j} + (y-2x)\vec{k}$.

comparing we get.

$2z-y=3$, $x-z=1$, $y-2x=-5$.

~~solving three equations we get,~~

from (i) and (iii)

$y = 2z-3$	$2(x-1)-y=3$
$y = 2x-5$.	or, $2x-2-3=y$.
or, $2z-3=2x-5$.	or, $y = 2x-5$.
or, $2z = 2x-2$	
or, $z = x-1$.	

So, the vector $\vec{v} = (x, 2x-5, x-1)$

Date
Page

2) iv) Show that $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 \cdot |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$

$\Rightarrow$ Solution.

taking $R.H.S = |\vec{a}|^2 \cdot |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$

$= a^2 \cdot b^2 - a^2 b^2 \cos^2\theta$

$= a^2 b^2 (1 - \cos^2\theta)$

or, $|\vec{a} \times \vec{b}|^2 = a^2 b^2 \sin^2\theta$

or, $|\vec{a} \times \vec{b}|^2 = (ab)^2 \sin^2\theta$

taking L.H.S,

$|\vec{a} \times \vec{b}|^2 = (|\vec{a}||\vec{b}| \sin\theta)^2$

$= |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$

$= a^2 b^2 \sin^2\theta$

Hence, $= R.H.S$

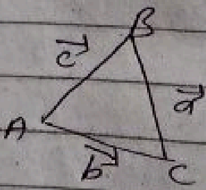
proved.

2) v) If $\vec{a} + \vec{b} + \vec{c} = 0$, Show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

$\Rightarrow$ Solution.

let $\vec{a}, \vec{b}, \vec{c}$ be three vectors

So, $\vec{AC} = \vec{b}$, $\vec{CB} = \vec{a}$, $\vec{BA} = \vec{c}$



Now, we have,

$\cancel{\vec{a} \times \vec{b}} = \vec{a} + \vec{b} + \vec{c} = 0$ —(1)

vector multiply on right part on both side by $\vec{a}$.

$(\vec{a} + \vec{b} + \vec{c}) \times \vec{a} = \vec{0} \times \vec{a}$

or, $\vec{a} \times \vec{a} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} = \vec{0}$

or, $\vec{b} \times \vec{a} + \vec{c} \times \vec{a} = 0$

Date
Page

or, $-\vec{a}\times\vec{b}+\vec{c}\times\vec{a}=0$

or, $\vec{a}\times\vec{b}=\vec{c}\times\vec{a}$ —(ii)

Again vecter multiply on left part on both side by $\vec{b}$

$\vec{b}\times(\vec{a}+\vec{b}+\vec{c})=\vec{b}\times\vec{0}$

or, $\vec{b}\times\vec{a}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c}=\vec{0}$

or, $-\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=0$.

or, $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}$. —(iii)

Hence from (ii) and (iii) we can write.

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$$

3 i) Find area of triangle determined by vectors $3\vec{i}+4\vec{j}$ and $-5\vec{i}+7\vec{j}$

Solution.

let, $\vec{a}=3\vec{i}+4\vec{j}$ and $\vec{b}=-5\vec{i}+7\vec{j}$

Now, lets find $\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 4 & 0\\ -5 & 7 & 0\end{vmatrix}$

$$=\vec{i}\begin{vmatrix}4 & 0\\ 7 & 0\end{vmatrix}-\vec{j}\begin{vmatrix}3 & 0\\ -5 & 0\end{vmatrix}+\vec{k}\begin{vmatrix}3 & 4\\ -5 & 7\end{vmatrix}$$

$=41\vec{k}$

So Area of triangle $=\frac{1}{2}|\vec{a}\times\vec{b}|=\frac{1}{2}\times 41$

$=20.5$.

ii) Find the area of parallelogram determined by vector $\vec{i} + 2\vec{j} + 3\vec{k}$ and $-3\vec{i} - 2\vec{j} + \vec{k}$

$\Rightarrow$ Solution:

Given, let $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$

$\vec{b} = -3\vec{i} - 2\vec{j} + \vec{k}$

Now,

area of parallelogram is given by;

$|\vec{a} \times \vec{b}|$. So,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 3 \\ -3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ -3 & -2 \end{vmatrix}$$

$$= \vec{i} \times 8 - 10\vec{j} + 4\vec{k}$$

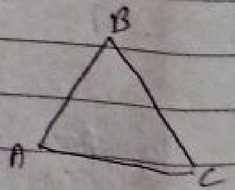
Now,

$$|\vec{a} \times \vec{b}| = \sqrt{8^2 + (-10)^2 + 4^2}$$

$$= 6\sqrt{5}$$

iii) prove that area of parallelogram whose three of four are $(1,1,2)$, $(2,-1,1)$ and $(3,2,-1)$ is $5\sqrt{3}$ sq units

$\Rightarrow$ Solution:



let, $A = (1,1,2)$, $B = (2,-1,1)$

$C = (3,2,-1)$

Now; ~~Area of~~ $\vec{AB} = \vec{OB} - \vec{OA} = (2,-1,1) - (1,1,2)$

$= (1,-2,-1)$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (3,2,-1) - (1,1,2) = (2,1,-3)$

Now,

Area of parallelogram is $= |\overrightarrow{AB} \times \overrightarrow{AC}|$

So, $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -1 \\ 2 & 1 & -3 \end{vmatrix}$

$$= \vec{i}\begin{vmatrix} -2 & -1 \\ 1 & -3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= \vec{i}(7) + \vec{j} + 5\vec{k}$$

So, Area of parallelogram $= |\overrightarrow{AB} \times \overrightarrow{AC}|$

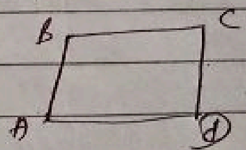
$$= \sqrt{7^2+1^2+5^2} = \sqrt{75}$$ $\Big]_1^2$

$$= 5\sqrt{3} \text{ sq. units.}$$

Hence, the area of parallelogram is $5\sqrt{3}$ sq. units.

3) iv) Find the area of parallelogram with vertices $A(-2,1)$, $B(0,4)$, $C(4,2)$ and $D(2,-1)$

⇒ Solution.



Given, vertices are $A(-2,1)$, $B(0,4)$, $C(4,2)$ and $D(2,-1)$

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (0,4) - (-2,1) = (2,3)$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (2,-1) - (-2,1) = (4,-2)$.

Now,

$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ 4 & -2 & 0 \end{vmatrix}$

Date
Page

$\vec{AC} = \vec{OC} - \vec{OA} = (3,2,-1) - (1,1,2) = (2,1,-3)$

Now,

Area of parallelogram is $= |\vec{AB} \times \vec{AC}|$

So, $\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -1 \\ 2 & 1 & -3 \end{vmatrix}$

$= \vec{i}\begin{vmatrix} -2 & -1 \\ 1 & -3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$

$= \vec{i}(7) + \vec{j} + 5\vec{k}$

So, Area of parallelogram $= |\vec{AB} \times \vec{AC}|$

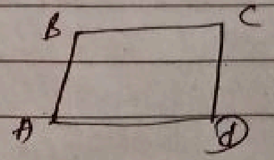
$= \sqrt{7^2 + 1^2 + 5^2} = \sqrt{75}$

$= 5\sqrt{3}$. Sq. units.

Hence, the area of parallelogram is $5\sqrt{3}$ Sq. units.

3) iv) Find the area of parallelogram with vertices $A(-2,1)$, $B(0,4)$, $C(4,2)$ and $D(2,-1)$

=> Solution:



Given, vertices are $A(-2,1)$, $B(0,4)$, $C(4,2)$ and $D(2,-1)$

Now, $\vec{AB} = \vec{OB} - \vec{OA} = (0,4) - (-2,1) = (2,3)$

$\vec{AD} = \vec{OD} - \vec{OA} = (2,-1) - (-2,1) = (4,-2)$.

Now,

$\vec{AB} \times \vec{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ 4 & -2 & 0 \end{vmatrix}$

Date ______ Page ______

$= \vec{k}(-4-12) = -16\vec{k}$

thus, Area of parallelogram is $|\vec{AB} \times \vec{AD}|$

$= \sqrt{(-16)^2}$

$= 16$ sq. unit.

v.) Show that area of triangle PQR whose vertices are $P(1,2,3)$, $Q(3,4,5)$ and $R(1,3,7)$ is $2\sqrt{6}$ sq. unit

$\Rightarrow$ solution,

Given $P(1,2,3)$, $Q(3,4,5)$, $R(1,3,7)$

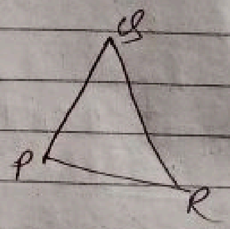
Now,

$\vec{PQ} = \vec{OQ} - \vec{OP} = (3,4,5) - (1,2,3)$

$= (2,2,2)$

$\vec{PR} = \vec{OR} - \vec{OP} = (1,3,7) - (1,2,3)$

$= (0,1,4)$



Now,

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 2 \\ 0 & 1 & 4 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 2 \\ 0 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix}$$

$= 6\vec{i} - 8\vec{j} + 2\vec{k}$

Now,

Area of triangle $= \frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{1}{2}\sqrt{6^2 + (-8)^2 + 2^2}$

$= \frac{1}{2}\sqrt{104} = \sqrt{26}$

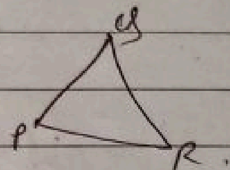
$= 2\sqrt{6.5}$

vi) Find a non-zero vector orthogonal to plane through points $P(1,0,1)$, $Q(-2,1,3)$, $R(4,2,5)$. Also, find the area of triangle PQR.

→ Solution,

Given points $P(1,0,1)$, $Q(-2,1,3)$, $R(4,2,5)$

Now,



$$\vec{PQ} = \vec{OQ} - \vec{OP} = (-2,1,3) - (1,0,1)$$
$$= (-3,1,2)$$
$$\vec{PR} = \vec{OR} - \vec{OP} = (4,2,5) - (1,0,1)$$
$$= (3,2,4)$$

Now,

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix}$$

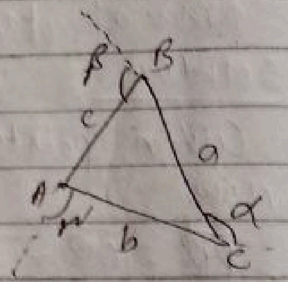
$$= \vec{i}\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} -3 & 2 \\ 3 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 18\vec{j} - 9\vec{k}$$

Now, Area of $\Delta PQR = \frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{1}{2}\sqrt{18^2 + (-9)^2}$

$$= \frac{9\sqrt{5}}{2} \text{ sq. unit}$$

Q.NO.4) prove the sine law of trigonometry by vector method that : $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$



→ Solution
we have,

$\vec{AC} + \vec{CB} + \vec{BA} = 0$

or, $\vec{b} + \vec{a} + \vec{c} = 0$ — (i)

multiply on both side vectorically by $\vec{a}$

$\vec{a} \times (\vec{b} + \vec{a} + \vec{c}) = \vec{a} \times \vec{0}$

or, $\vec{a} \times \vec{b} + \vec{a} \times \vec{a} + \vec{a} \times \vec{c} = 0$

or, $\vec{a} \times \vec{b} + 0 - \vec{c} \times \vec{a} = 0$

or, $\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ — (ii)

Similarly multiply on both side vectorically by $\vec{b}$

$\vec{b} \times (\vec{b} + \vec{a} + \vec{c}) = \vec{b} \times \vec{0}$

or, $\vec{b} \times \vec{b} + \vec{b} \times \vec{a} + \vec{b} \times \vec{c} = 0$

or, $0 - \vec{a} \times \vec{b} + \vec{b} \times \vec{c} = 0$

or, $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ — (iii)

from (ii) and (iii) we get,

$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Now, taking modulus we have,

$|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

or, $ab \sin\alpha = bc \sin\gamma = ca \sin\beta$

or, $ab \sin(180 - C) = bc \sin(180 - A) = ca \sin(180 - B)$

or, $ab \sin C = bc \sin A = ac \sin B$

Date
Page

dividing by abc

$$\frac{\cancel{ab}\sin C}{\cancel{ab}c} = \frac{\cancel{bc}\sin A}{a\cancel{bc}} = \frac{\cancel{ac}\sin B}{\cancel{a}b\cancel{c}}$$

or, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Use of Scalar Triple product.

1) i) Give the geometrical meaning of scalar triple product. Find the volume of parallelepiped determined by vectors $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = -\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{c} = 2\vec{i} + \vec{j} + 4\vec{k}$

=) Solution.

The geometrical interpretation of scalar triple product of three vectors is that it gives volume of parallelepiped and three vectors represent edges of parallelepiped.

Now,

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= 2\vec{i} + 8\vec{j} - 3\vec{k}$$

Now, $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{i} + 2\vec{j} + 3\vec{k}) \cdot (2\vec{i} + 8\vec{j} - 3\vec{k})$

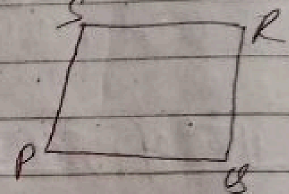
$= 2\times1 + 2\times8 - 3\times3 = 9$ cube unit

Date
Page

ii) Find the volume of parallelepiped with adjacent edges PQ, PR and PS, where P(-2,1,0), Q(2,3,2), R(1,4,-1), S(3,6,1)

=> Solution

Given, P(-2,1,0), Q(2,3,2)
R(1,4,-1), S(3,6,1)



Now, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (2,3,2) - (-2,1,0)$
$= (4,2,2)$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (1,4,-1) - (-2,1,0)$
$= (3,3,-1)$

$\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP} = (3,6,1) - (-2,1,0)$
$= (5,5,1)$

Now, $\overrightarrow{PR} \times \overrightarrow{PS} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3 & -1 \\ 5 & 5 & 1 \end{vmatrix}$

$= \vec{i}\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 3 \\ 5 & 5 \end{vmatrix}$

$= 8\vec{i} - 8\vec{j}$

And, $\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) = (4,2,2) \cdot (8,-8,0)$
$= 4 \times 8 + 2 \times (-8) + 2 \times 0$
$= 16$ sq. unit.

Date ______
Page ______

1) Use scalar triple product to

i) Verify that the vectors $\vec{u} = \vec{i} + 5\vec{j} - 2\vec{k}$, $\vec{v} = 3\vec{i} - \vec{j}$ and $\vec{w} = 5\vec{i} + 9\vec{j} - 4\vec{k}$ are coplanar.

⇒ Solution.

Given, $\vec{u} = \vec{i} + 5\vec{j} - 2\vec{k}$, $\vec{v} = 3\vec{i} - \vec{j}$

$\vec{w} = 5\vec{i} + 9\vec{j} - 4\vec{k}$

Now,

~~Now~~, $\vec{u} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & -2 \\ 5 & 9 & -4 \end{vmatrix}$

$= \vec{i}\begin{vmatrix} 5 & -2 \\ 9 & -4 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -2 \\ 5 & -4 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 5 \\ 5 & 9 \end{vmatrix}$

$= -2\vec{i} - 6\vec{j} - 16\vec{k}$

Now, $\vec{v} \cdot (\vec{u} \times \vec{w}) = \cancel{30 \times (-2) + 3 \times (-6) + (-1) \times (-16)}$

$= 3 \times (-2) + (-1) \times (-6) + 0 \times (-16)$

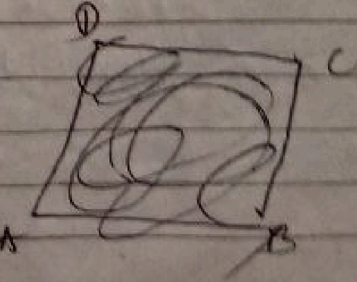
$= 0$

ii) Show the following four points ~~are~~ :(2, 3, −1), (1, −2, 3), (3, 4, −2) and (1, −6, 4) are coplanar.

⇒ Solution.

let $A(2,3,-1)$, $B(1,-2,3)$, $C(3,4,-2)$

$D(1,-6,4)$



Now

$\vec{AB} = \vec{OB} - \vec{OA} = (1,-2,3) - (2,3,-1)$

$= (-1, -5, 4)$

$\vec{BC} = \vec{OC} - \vec{OB} = (3,4,-2) - (1,-2,3) = (2,6,-5)$

$\vec{CD} = \vec{OD} - \vec{OC} = (1,-6,4) - (3,4,-2) = (-2,-10,6)$

Now,

$$\vec{AB} \cdot (\vec{BC} \times \vec{CD}) = \begin{vmatrix} -1 & -5 & 4 \\ 2 & 6 & -5 \\ -2 & -10 & 6 \end{vmatrix}$$

$$= -1\begin{vmatrix} 6 & -5 \\ -10 & 6 \end{vmatrix} + 5\begin{vmatrix} 2 & -5 \\ -2 & 6 \end{vmatrix} + 4\begin{vmatrix} 2 & 6 \\ -2 & -10 \end{vmatrix}$$

$$= 14 + 2 \times 5 - 32 = -8$$

$\vec{AC} = \vec{OC} - \vec{OA} = (3,4,-2) - (2,3,-1) = (1,1,-1)$

$\vec{AD} = \vec{OD} - \vec{OA} = (1,-6,4) - (2,3,-1) = (-1,-9,5)$

Now, $\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \begin{vmatrix} -1 & -5 & 4 \\ 1 & 1 & -1 \\ -1 & -9 & 5 \end{vmatrix}$

$$= -1\begin{vmatrix} 1 & -1 \\ -9 & 5 \end{vmatrix} + 5\begin{vmatrix} 1 & -1 \\ -1 & 5 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ -1 & -9 \end{vmatrix}$$

$$= 4 + 20 - 32 = -8$$

Here, the given four points are not coplanar

Date
Page

2.) iii) Show that vectors $\vec{a} = \vec{i} + 3\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} - \vec{k}$, $\vec{c} = 7\vec{j} + 3\vec{k}$ are parallel to same plane.

→ Solution

Given, vectors are $\vec{a} = \vec{i} + 3\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} - \vec{k}$, $\vec{c} = 7\vec{j} + 3\vec{k}$

Now,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ 0 & 7 & 3 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & -1 \\ 7 & 3 \end{vmatrix} - 3 \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 0 & 7 \end{vmatrix}$$

$$= 4 - 18 + 14 = 0$$

Here, scalar triple product

iv) Determine whether the points $A(1,3,2)$, $B(3,-1,6)$, $C(5,2,0)$ and $D(3,6,-4)$ lie in same plane.

→ Given, $A(1,3,2)$, $B(3,-1,6)$, $C(5,2,0)$, $D(3,6,-4)$

We have,

$\vec{AB} = \vec{OB} - \vec{OA} = (3,-1,6) - (1,3,2) = (2,-4,4)$

$\vec{AC} = \vec{OC} - \vec{OA} = (5,2,0) - (1,3,2) = (4,-1,-2)$

$\vec{AD} = (3,6,-4) - (1,3,2) = (2,3,-6)$

Now,

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \begin{vmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix}$$

Date
Page

$$= 2\begin{vmatrix} -1 & -2 \\ 3 & -8 \end{vmatrix} + 4\begin{vmatrix} 4 & -2 \\ 2 & -8 \end{vmatrix} + 4\begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix}$$

$= 24 - 80 + 56 = 0$

Here, scalar triple product is zero. Hence the points are collinear.

v.) The position vectors of points A, B, C and D are $3\vec{i} - 2\vec{j} - \vec{k}$, $2\vec{i} + 3\vec{j} - 4\vec{k}$, $-\vec{i} + \vec{j} + 2\vec{k}$ and $4\vec{i} + 5\vec{j} + \lambda\vec{k}$ respectively if A, B, C, D are coplanar find value of λ.

⇒ Solution.

Given, $\vec{OA} = 3\vec{i} - 2\vec{j} - \vec{k}$, $\vec{OB} = 2\vec{i} + 3\vec{j} - 4\vec{k}$

$\vec{OC} = -\vec{i} + \vec{j} + 2\vec{k}$, $\vec{OD} = 4\vec{i} + 5\vec{j} + \lambda\vec{k}$

Now,

$\vec{AB} = \vec{OB} - \vec{OA} = (2\vec{i} + 3\vec{j} - 4\vec{k}) - (3\vec{i} - 2\vec{j} - \vec{k})$

$= -\vec{i} + 5\vec{j} - 3\vec{k}$

$\vec{AC} = \vec{OC} - \vec{OA} = (-\vec{i} + \vec{j} + 2\vec{k}) - (3\vec{i} - 2\vec{j} - \vec{k})$

$= -4\vec{i} + 3\vec{j} + 3\vec{k}$

$\vec{AD} = \vec{OD} - \vec{OA} = (4\vec{i} + 5\vec{j} + \lambda\vec{k}) - (3\vec{i} - 2\vec{j} - \vec{k})$

$= \vec{i} + 7\vec{j} + (\lambda+1)\vec{k}$

Now,

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix}$$

$$\text{or, } 0 = -1\begin{vmatrix} 3 & 3 \\ 7 & \lambda+1 \end{vmatrix} - 5\begin{vmatrix} -4 & 3 \\ 1 & \lambda+1 \end{vmatrix} - 3\begin{vmatrix} -4 & 3 \\ 1 & 7 \end{vmatrix}$$

Date
Page

or, $0 = -1(3\lambda+3-21) - 5(-4\lambda-4-3) - 3(-4\times7-3)$

or, $0 = -3\lambda+18+20\lambda+35+93$

or, $0 = 146+17\lambda$

or, $\lambda = -146/17$.

vii) If vectors $a\vec{i}+\vec{j}+\vec{k}$, $\vec{i}+b\vec{j}+\vec{k}$ and $\vec{i}+\vec{j}+c\vec{k}$ $(a\neq b, c\neq 1)$ are coplanar, find value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$.

→ solution:

Given, let, $\vec{m} = a\vec{i}+\vec{j}+\vec{k}$

$\vec{n} = \vec{i}+b\vec{j}+\vec{k}$

$\vec{p} = \vec{i}+\vec{j}+c\vec{k}$

Now,

$$\vec{m}\cdot(\vec{n}\times\vec{p}) = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

or, $a\begin{vmatrix} b & 1 \\ 1 & c \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & c \end{vmatrix} + 1\begin{vmatrix} 1 & b \\ 1 & 1 \end{vmatrix} = 0$

or, $a(bc-1) - 1(c-1) + 1(1-b) = 0$

or, $abc - a - c + 1 + 1 - b = 0$

or, $abc - (a+b+c) + 2 = 0$ ——(1)

~~If $c=1$ we have~~
~~$ab-(a+b)+1=0$~~

~~If $(a+b+c)=0$ then, we have,~~
~~$abc+2=0$~~
~~If $abc=0$ then, we have,~~
~~$a+b+c=2$.~~

Date
Page

2) vii) if $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors $\vec{\alpha} = (1, a, a^2)$, $\vec{\beta} = (1, b, b^2)$ and $\vec{\gamma} = (1, c, c^2)$ are non coplanar. if value of abc

⇒ <u>solution</u>

Given, $\vec{\alpha} = (1, a, a^2)$, $\vec{\beta} = (1, b, b^2)$, $\vec{\gamma} = (1, c, c^2)$ are non coplanar.

now, $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$

or, $\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$

or, $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

or, $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} (1 + abc) = 0$

since determinant can't be zero.

$(1 + abc) = 0$ or, $abc = -1$

Date
Page

viii) If $\vec{a}, \vec{b}, \vec{c}$ are linearly independent, then show that $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are also linearly independent.

⇒ Solution,

Given, ~~$\vec{a}+\vec{b}+c$~~ ~~$\vec{a}+\vec{b}$~~ $\vec{a}, \vec{b}, \vec{c}$ are linearly independent

Now,

$c_1(\vec{a}+\vec{b}) + c_2(\vec{b}+\vec{c}) + c_3(\vec{c}+\vec{a}) = 0$

or, $c_1\vec{a} + c_1\vec{b} + c_2\vec{b} + c_2\vec{c} + c_3\vec{c} + c_3\vec{a} = 0$

or, $\vec{a}(c_1+c_3) + \vec{b}(c_1+c_2) + \vec{c}(c_2+c_3) = 0$

Since, $\vec{a}, \vec{b}, \vec{c}$ are linearly independent.

$c_1 + c_3 = 0$ — (i)

$c_1 + c_2 = 0$ — (ii)

$c_2 + c_3 = 0$ — (iii)

from (i) and (iii)

$c_1 = -c_3$.

and $c_1 + c_2 = 0$

or, $-c_3 + c_2 = 0$

or, $c_3 = c_2$

from (iii)

$c_2 + c_3 = 0$

or, $c_2 + c_2 = 0$

or, $2c_2 = 0$

or, $c_2 = 0$.

So,

$c_1 + c_2 = 0$ becomes

$c_1 + 0 = 0$

or, $c_1 = 0$.

Hence, since, $c_1 = c_2 = c_3 = 0$.

the vectors $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are linearly independent

Vector Triple product

Date ____ Page ____

1) Verify that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b}) \cdot \vec{c}$, when $a = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$

⇒ Solution,

Given, $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$

$\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$

$\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$

Now, $\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$

$= \vec{i} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$

$= -3\vec{i} + 3\vec{j} + 3\vec{k}$

Now, $\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix}$

$= \vec{i} \begin{vmatrix} -2 & 1 \\ 3 & 3 \end{vmatrix} - j \begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix}$

$= -9\vec{i} - 6\vec{j} - 3\vec{k}$

Now, $(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b}) \cdot \vec{c}$

$= (1 \times 1 - 2 \times 2 - 1 \times 1)(2\vec{i} + \vec{j} + \vec{k}) - (1 \times 2 - 2 \times 1 + 1 \times 1) \cdot \vec{c}$

$= -4(2\vec{i} + \vec{j} + \vec{k}) - \vec{c}$

$= -8\vec{i} - 4\vec{j} - 4\vec{k} - (\vec{i} + 2\vec{j} - \vec{k})$

$= -9\vec{i} - 6\vec{j} - 3\vec{k}$

Hence, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b}) \cdot \vec{c}$

2) If $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$, $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$ find, $\vec{a} \times (\vec{b} \times \vec{c})$, $(\vec{a} \times \vec{b}) \times \vec{c}$ and $|\vec{a} \times (\vec{b} \times \vec{c})|$. Also, show that $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$.

→ Solution,

Given, $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$

$\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$

$\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$

Now,

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$= -3\vec{i} + 3\vec{j} + 3\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

Date ____ Page ____

$\vec{i}\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$

$= -3\vec{i} + \vec{j} + 5\vec{k}$

Now,

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -2 & 1 \\ 3 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix}$$

$$= -9\vec{i} - 6\vec{j} - 3\vec{k}$$

Again,

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & 5 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 5 \\ 2 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} -3 & 5 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= -11\vec{i} + 2\vec{j} - 7\vec{k}$$

And, $|\vec{a} \times (\vec{b} \times \vec{c})| = \sqrt{(-9)^2 + (-6)^2 + (-3)^2}$

$= 3\sqrt{14}.$

3) i)

$\Rightarrow$ Sol

$\vec{b}$

Date ______ Page ______

8) i) prove that $b^2 \vec{a} = (\vec{a}\cdot\vec{b})\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$

=> Solution.

Given $\vec{a}$ and $\vec{b}$

to prove $b^2\vec{a} = (\vec{a}\cdot\vec{b})\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$

~~let~~, $\vec{a} = m\vec{i} + n\vec{j} + p\vec{k}$

$\vec{b} = q\vec{i} + r\vec{j} + s\vec{k}$

Now, $(\vec{a}\cdot\vec{b})\cdot\vec{b} = (m\times q + n\times r + p\times s)\vec{b}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ m & n & p \\ q & r & s \end{vmatrix}$$

$$= \vec{i}(ns-pr) - \vec{j}(ms-qp) + \vec{k}(mr-nq)$$

$$\vec{b}\times(\vec{a}\times\vec{b}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ q & r & s \\ (ns-pr) & -(ms-qp) & (mr-nq) \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} r & s \\ (qp-ms) & (mr-nq) \end{vmatrix} - \vec{j}\begin{vmatrix} q & s \\ (ns-pr) & (mr-nq) \end{vmatrix}$$

$$+ \vec{k}\begin{vmatrix} q & r \\ (ns-pr) & (qp-ms) \end{vmatrix}$$

or, $(mr^2 - nqr - pqs + ms^2)\vec{i} - \vec{j}(qmr - nq^2 - ns^2 + prs) +$
$\vec{k}(q^2p - qms - nrs + pr^2)$

$]_1^2$

Date ______
Page ______

$(\vec{a}\cdot\vec{b})\cdot\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$

$= (mq^2+nqr+pqs)\vec{i} + (mqr+nr^2+prs)\vec{j} + (mqs+nsr+ps^2)\vec{k}$
$+ (mr^2-nqr-pqs+ms^2)\vec{i} - (qmr-nq^2-ns^2+prs)\vec{j} +$
$(q^2p-qms-nrs+pr^2)\vec{k}$

$= (mq^2+\cancel{nqr}+\cancel{pqs}+mr^2-\cancel{nqr}-\cancel{pqs}+ms^2)\vec{i}$
$+ (\cancel{mqr}+nr^2+\cancel{prs}-\cancel{qmr}+nq^2+ns^2-\cancel{prs})\vec{j}$
$+ (\cancel{mqs}+\cancel{nsr}+ps^2+q^2p-\cancel{qms}-\cancel{nrs}+pr^2)\vec{k}$

$b^2\vec{a} = (q^2+r^2+s^2)(m\vec{i}+n\vec{j}+p\vec{k})$

and, $(\vec{a}\cdot\vec{b})\cdot\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$
$= m(q^2+r^2+s^2)\vec{i} + n(r^2+q^2+s^2)\vec{j} + p(s^2+q^2+r^2)\vec{k}$
$= (q^2+r^2+s^2)(m\vec{i}+n\vec{j}+p\vec{k})$

Hence, $b^2\vec{a} = (\vec{a}\cdot\vec{b})\cdot\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$

3 ii) Show that $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$

⇒ Solution.

We have,

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$
$\vec{b}\times(\vec{c}\times\vec{a}) = (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}$
$\vec{c}\times(\vec{a}\times\vec{b}) = (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$

Now, $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$ =

Date ______ Page ______

$(\vec{a}\cdot\vec{b})\cdot\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$

$= (mq^2+nqr+pqs)\vec{i} + (mqr+nr^2+prs)\vec{j} + (mqs+nsr+ps^2)\vec{k}$
$+ (mr^2-nqr-pqs+ms^2)\vec{i} - (qmr-nq^2-ns^2+prs)\vec{j} + (q^2p-qms-nrs+pr^2)\vec{k}$

$= (mq^2+\cancel{nqr}+\cancel{pqs}+mr^2-\cancel{nqr}-\cancel{pqs}+ms^2)\vec{i}$
$+ (\cancel{mqr}+nr^2+\cancel{prs}-\cancel{qmr}+nq^2+ns^2-\cancel{prs})\vec{j}$
$+ (\cancel{mqs}+\cancel{nsr}+ps^2+q^2p-\cancel{qms}-\cancel{nrs}+pr^2)\vec{k}$

$b^2\vec{a} = (q^2+r^2+s^2)(m\vec{i}+n\vec{j}+p\vec{k})$

and, $(\vec{a}\cdot\vec{b})\cdot\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$
$= m(q^2+r^2+s^2)\vec{i} + n(r^2+q^2+s^2)\vec{j} + p(s^2+q^2+r^2)\vec{k}$
$= (q^2+r^2+s^2)(m\vec{i}+n\vec{j}+p\vec{k})$

Hence, $b^2\vec{a} = (\vec{a}\cdot\vec{b})\cdot\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$

3 ii) Show that $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$

$\Rightarrow$ Solution.

We have,

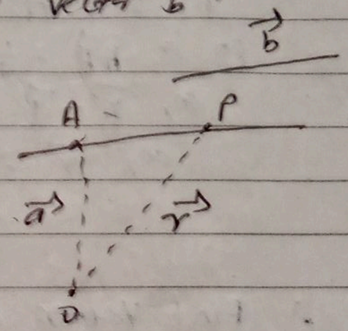
$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$
$\vec{b}\times(\vec{c}\times\vec{a}) = (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}$
$\vec{c}\times(\vec{a}\times\vec{b}) = (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$

Now, $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) =$
$\cancel{(\vec{a}\cdot\vec{c})\vec{b}} - \cancel{(\vec{a}\cdot\vec{b})\vec{c}} + \cancel{(\vec{b}\cdot\vec{a})\vec{c}} - \cancel{(\vec{b}\cdot\vec{c})\vec{a}}$
$+ \cancel{(\vec{c}\cdot\vec{b})\vec{a}} - \cancel{(\vec{c}\cdot\vec{a})\vec{b}}$
$= 0$

Equation of line

8) i) Find a vector equation of straight line through the given point $\vec{a}$ and parallel to vector $\vec{b}$.

→ Solution.



Here in Δ OAP

$\vec{AP} = \vec{AO} + \vec{OP}$

$= -\vec{OA} + \vec{r}$

$= \vec{r} - \vec{a}$

Also, $\vec{AP} = t\,\vec{b}$ [∵ since $\vec{AP} \parallel \vec{b}$

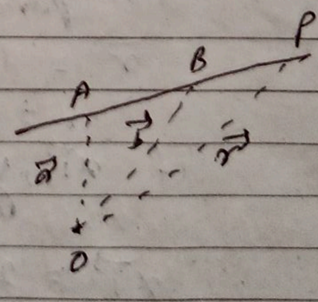
So, equation become.

$t\vec{b} = \vec{r} - \vec{a}$

or, $\vec{r} = \vec{a} + t\,\vec{b}$

ii) Find the vector equation of straight line through two points $\vec{a}$ and $\vec{b}$

→



Here, let, $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OP} = \vec{r}$.

Now,

In Δ OAB.

$\vec{AB} = \vec{AO} + \vec{OB}$

$= -\vec{OA} + \vec{OB}$

$= -\vec{a} + \vec{b} = \vec{b} - \vec{a}$.

Also, in Δ OAP.

$\vec{AP} = \vec{AO} + \vec{OP}$

$= -\vec{OA} + \vec{OP}$

$= \vec{r} - \vec{a}$

Date ______
Page ______

Since, A, B and P are collinear.

$\vec{AP} = t\,\vec{AB}$

So, or, $\vec{AP} = t(\vec{b} - \vec{a})$

or, $\vec{r} - \vec{a} = t(\vec{b} - \vec{a})$

or, $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

Q.) i) Find vector equation and parametric equation for line through point $(2, 2.4, 3.5)$ ~~throug~~ and parallel to vector $(3\vec{i} + 2\vec{j} - \vec{k})$

⇒ Solution.

Given, lines passes through $(2, 2.4, 3.5)$
line is parallel to $(3\vec{i} + 2\vec{j} - \vec{k})$

Now,

the vector equation of line that passes through a point and parallel to a vector can be written as.

$\vec{r} = (2, 2.4, 3.5) + t(3\vec{i} + 2\vec{j} - \vec{k})$

$= (2\vec{i} + 2.4\vec{j} + 3.5\vec{k}) + t(3\vec{i} + 2\vec{j} - \vec{k})$

If $x\vec{i}$, $y\vec{j}$ and $z\vec{k}$ are resulting vectors then we can write.

$x\vec{i} = 2\vec{i} + 3t\vec{i}$ ⇒ $x = 2 + 3t$

$y = 2.4 + 2t$

$z = 3.5 - t$

Date ______
Page ______

2ii) Find vector equation and parametric equation of line through point (1,0,6) and perpendicular to plane $x + 3y + z = 5$.

→ **Solution**

Given, point line is through is $= (1,0,6)$

Given, plane is $x + 3y + z = 5$.

Here, since, line is perpendicular to plane.

the line is parallel to normal of plane.

so, normal vector $\vec{n} = \hat{i} + 3\vec{j} + \vec{k}$

so, equation of line can be written as

$$\vec{r} = (\vec{i} + 0\vec{j} + 6\vec{k}) + t\,\vec{n}$$
$$= (\vec{i} + 6\vec{k}) + t(\vec{i} + 3\vec{j} + \vec{k})$$

If x, y, z are components of $\vec{r}$ then,

$x = 1 + t$

$y = 3t$

$z = 6 + t$

3) Find vector equation for line segment from $(2,-1,4)$ to $(4,6,1)$

→ **Solution**

Given points through which line segment pass are $(2,-1,4)$ and $(4,6,1)$

Now,

equation can be written as.

$$(x,y,z) = (2,-1,4) + t(4-2, 6+1, 4-1)$$
$$= (2,-1,4) + t(2,7,3)$$

Hence, $\vec{r} = (2\vec{i} - \vec{j} + 4\vec{k}) + t(2\vec{i} + 7\vec{j} + 3\vec{k})$ and $0 \le t \le 1$

Date
Page

4-i) Find the parametric equation and symmetric equations for lines through points $(-8,1,4)$ and $(3,-2,4)$

⇒ Solution

Given points through which line passes are $(-8,1,4)$, $(3,-2,4)$

Now,

the vector equation of line is

$\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

or, $(x,y,z) = (-8,1,4) + t(3+8, -2-1, 4-4)$

$= (-8,1,4) + t(11,-3,0)$

or, $(x,y,z) = (-8,1,4) + t(11,-3,0)$

So,

$x = -8 + 11t$ or, $t = (x+8)/11$

$y = 1 - 3t$ or, $t = (y-1)/(-3)$

$z = 4 + t\cdot 0$ or, $t = (z-4)/0$

Hence, symmetric equation can be written as.

$$\frac{x+8}{11} = \frac{y-1}{-3} = \frac{z-4}{0}$$

4-ii) Find the parametric equation and symmetric equation for lines through $(2,1,0)$ and perpendicular to both vectors $\vec{i}+\vec{j}$ and $\vec{j}+\vec{k}$

⇒ Solution.

Here, given point is $(2,1,0)$

and line is perpendicular to $\vec{i}+\vec{j}$ and $\vec{j}+\vec{k}$

Now, the vector perpendicular to these vectors is parallel to line.

Date
Page

4-i) Find the parametric equation and symmetric equations for lines through points $(-8, 1, 4)$ and $(3, -2, 4)$

⇒ Solution

Given points through which line passes are $(-8, 1, 4)$, $(3, -2, 4)$

Now,

the vector equation of line is

$\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

or, $(x, y, z) = (-8, 1, 4) + t(3+8, -2-1, 4-4)$

$= (-8, 1, 4) + t(11, -3, 0)$

or, $(x, y, z) = (-8, 1, 4) + t(11, -3, 0)$

So,

$x = -8 + 11t$ or, $t = (x+8)/11$

$y = 1 - 3t$ or, $t = (y-1)/(-3)$

$z = 4 + t \cdot 0$ or, $t = (z-4)/0$

Hence, symmetric equation can be written as.

$$\frac{x+8}{11} = \frac{y-1}{-3} = \frac{z-4}{0}$$

4-ii) Find the parametric equation and symmetric equation for lines through $(2, 1, 0)$ and perpendicular to both vectors $\vec{i} + \vec{j}$ and $\vec{j} + \vec{k}$

⇒ Solution.

Here, given point is $(2, 1, 0)$

and line is perpendicular to $\vec{i} + \vec{j}$ and $\vec{j} + \vec{k}$

Now, the vector perpendicular to these vectors is parallel to line.

Date
Page

to find vector perpendicular to ~~that~~ those vectors.

$$(\vec{i}+\vec{j}) \times (\vec{j}+\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= \vec{i} - \vec{j} + \vec{k}$$

so, ~~vector~~ equation is

$$\vec{r} = (2\vec{i}+\vec{j}) + t(\vec{i}-\vec{j}+\vec{k})$$

or, $(x,y,z) = (2,1,0) + t(1,-1,1)$

so, components are.

$x = 2+t$. or, $\frac{x-2}{1} = t$.

$y = 1-t$ or, $(y-1)/-1 = t$

$z = 0+t$ or, $\frac{z}{1} = t$.

Hence,

symmetric equation is

$$\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z-0}{1}.$$

Date ____ Page ____

4.iii) $(1,-1,1)$ and parallel to line $\frac{x+2}{1} = \frac{y}{2} = z-3$

⇒ Solution:

Given point through which line pass is $(1,-1,1)$ and parallel to line $\frac{x+2}{1} = \frac{y}{2} = \frac{z-3}{1}$

Now, the parametric form of the line parall to line is.

$x+2 = t$	$y = 2t$	$z-3 = t$
or, $x = -2+1t$	or, $y = 0+2t$	or, $z = 3+1t$

~~If $t = 1$ then,~~
~~$x = -1$, $y = 2$~~, $z = 4$

Hence, the equation of line through point $(1,-1,1)$ is.
$(x_1, y_1, z_1) = (1,-1,1) + t_1(1,2,1)$

So, we have,

$x_1 = 1 + 1t_1$ or, $t_1 = (x_1-1)/(+1)$
$y_1 = -1 + 2t_1$ or, $t_1 = (y_1+1)/2$
$z_1 = 1 + 1t_1$ or, $t_1 = (z_1-1)/1$

So, symmetric equation is,

$$\frac{x_1-1}{+1} = \frac{y_1+1}{2} = \frac{z_1-1}{1}$$

In general $$\frac{x-1}{+1} = \frac{y+1}{2} = \frac{z-1}{1}$$

Date
Page

4 (v) line through ~~int~~ line of intersection of planes $x+2y+3z=1$ and $x-y+z=1$

=> Solution.

Given planes are $x+2y+3z=1$, $x-y+z=1$

Now,

normal vectors of planes are $\vec{n_1} = (\vec{i}+2\vec{j}+3\vec{k})$

and $\vec{n_2} = \vec{i}-\vec{j}+\vec{k}$

Now

$$\vec{b} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 5\vec{i} + 2\vec{j} - 3\vec{k}$$

So, this $\vec{b}$ is parallel to line through line of intersection.

Now,

Solving the equations.

$x+2y+3z=1$

$x-y+z=1$.

put, $z=0$ then,

$x+2y=1$

$x-y=1$.

solving we get, $x=1, y=0, z=0$

Hence, the equation is found as.

$(x,y,z) = (1,0,0) + t(5,2,-3)$

and $\frac{x-1}{5} = \frac{y}{2} = \frac{z}{-3}$

14, 48

Date
Page

5. a) Determine whether lines L_1 and L_2 are parallel, skew or intersecting. If they intersect, find the point of intersection.

i) $L_1 : x = 3 + 2t, \ y = 4 - t, \ z = 1 + 3t$.
$L_2 : x = 1 + 4s, \ y = 3 - 2s, \ z = 4 + 5s$.

⇒ Solution.

Given, L_1 and L_2.

Now, from L_1 : $(l_1, m_1, n_1) = (2, -1, 3)$
from L_2 : $(l_2, m_2, n_2) = (4, -2, 5)$

Now, vector perpendicular to both direction ratio of L_1 and L_2

$$\vec{b} = (l_1, m_1, n_1) \times (l_2, m_2, n_2)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ 4 & -2 & 5 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix}$$

$$= \vec{i} + 2\vec{j} + 0\vec{k}$$

also, ~~$\vec{a} = (4-2, -2+1, 5-3) = (2, -1, 2)$~~

Now, $a = (3-1, 4-3, 1-4) = (2, 1, -3)$

Projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$$= \frac{2 \times 1 - 1 \times 2 + 2 \times 0}{\sqrt{1^2 + 2^2}} = 0.$$

Date ______ Page ______

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{2\times1 + 1\times2 + 3\times0}{\sqrt{1^2+2^2}} = \frac{4}{\sqrt{5}}.$$

Here, the vector projection of $\vec{a}$ on $\vec{b}$ is not zero.
Hence, the lines are skewed lines.

(ii) $L_1 : \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3}$ and $L_2 : \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7}$.

Solution.

Here, $\vec{n_1} = (1,-2,-3)$ and $\vec{n_2} = (1,3,-7)$ 2

$\vec{a} = (2,3,1)$, $\vec{b} = (3,-4,2)$ 1

Now, $\vec{c} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -3 \\ 1 & 3 & -7 \end{vmatrix}$

$= \vec{i}\begin{vmatrix} -2 & -3 \\ 3 & -7 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -3 \\ 1 & -7 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix}$

$= 23\vec{i} + 4\vec{j} + 5\vec{k}$

and, $\vec{d} = \vec{a} - \vec{b} = (2-3, 3+4, 1-2) = (-1, 7, -1)$

Now,
$\vec{c}\cdot\vec{d} = 23\times(-1) + 4\times7 + 5\times(-1) = 0.$

Here, since the dot product is zero the ~~distan~~ shortest distance between two lines is also zero. Hence, they intersect each other.

Date
Page

From, L_1 we get, $x = 2+t$, $y = 3-2t$, $z = 1-3t$.
Substitute these into L_2 we get.

$$\frac{2+t-3}{1} = \frac{3-2t+4}{3} = \frac{1-3t-2}{-7}$$

or, $$\frac{t-1}{1} = \frac{7-2t}{3} = \frac{1+3t}{7}$$

taking first two	taking 2nd and 3rd.
$\frac{t-1}{1} = \frac{7-2t}{3}$	$\frac{7-2t}{3} = \frac{1+3t}{7}$
or, $3t-3 = 7-2t$	or, $49-14t = 3+9t$
or, $5t = 10$	or, $46 = 23t$
or, $t = 2$	or, $t = 2$

Hence, we get.

$x = 2+t$	$y = 3-2t$.	$z = 1-3\times2$
$= 2+2$	$= 3-2\times2$	$= 1-6$
$= 4$	$= -1$	$= -5$.

Hence, the lines intersect at $(4,-1,-5)$

5b) Prove that lines $L_1: \frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$

$L_2: \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$

$\Rightarrow$ from, $L_1: x = 1+2t$, $y = -1-3t$, $z = -10+8t$.
Put these in L_2 we get

Date ______ Page ______

$\frac{1+2t-4}{1} = \frac{-1-3t+3}{-4} = \frac{-10+8t+1}{7}$

or, $\frac{2t-3}{1} = \frac{2-3t}{-4} = \frac{8t-9}{7}$

taking 1st and 2nd.	taking 2nd and 3rd.
$\frac{2t-3}{1} = \frac{3t-2}{4}$	$\frac{3t-2}{4} = \frac{8t-9}{7}$
or, $8t-12 = 3t-2$	or, $21t-14 = 32t-36$
or, $5t = 12-2$	or, $t = 2$.
or, $t = 10/5 = 2$	

so, the lines intersect

so, intersection points are $(x, y, z) = (1+2\times2, -1-3\times2, -10+8\times2)$

or, $(x, y, z) = (5, -7, 6)$

Point of intersection of line and plane.

1. i) where does the line through $(1, 0, 1)$ and $(4, -2, 2)$ intersect the plane $x + y + z = 6$?

$\Rightarrow$ Given, the points are $(1, 0, 1)$ and $(4, -2, 2)$

Now,

the symmetrical equation of lines through two points is

$$\frac{x-1}{4-1} = \frac{y-0}{-2-0} = \frac{z-1}{2-1}$$

Hence, coordinates are $(x, y, z) = (1+3t, -2t, 1+t)$.

Now,

Substitute these into equation of plane we get.

$(x+y+z) = 6$

or, $(1+3t) + (-2t) + (1+t) = 6$

or, $2 + 2t = 6$

or, $t = 2$.

Hence, coordinates are $(1+3\times2, -2\times2, 1+2)$

or, $(x, y, z) = (7, -4, 3)$

ii) Find the point at which the lines $\vec{r} = (1, 1, 0) + t(1, -1, 2)$ and $\vec{r} = (2, 0, 2) + s(-1, 1, 0)$ intersect

$\Rightarrow$ Given, $\vec{r} = (1, 1, 0) + t(1, -1, 2)$ — (i)

$\vec{r} = (2, 0, 2) + s(-1, 1, 0)$ — (ii)

$0 = (-1, 1, -2) + t(1, -1, 2) - s(-1, 1, 0)$

if lines intersect they intersect at s = t.

so, $0 = (-1, 1, -2) + t(2, -2, 2)$

or, $(1, -1, 2) = t(2, -2, 2)$

Q. N

28

Date ______
Page ______

or, $s(-1,1,0) = (-1,1,-2) + t(1,-1,2)$

or, $s = \dfrac{(-1,1,-2) + t(1,-1,2)}{(-1,1,0)}$

in symmetric form we get:

$$\frac{t-1}{-1} = \frac{1-t}{1} = \frac{-2+2t}{0}$$

taking (i) and (ii):

$t-1 = -1(1-t)$

or, $t-1 = t-1$

or, $-1 = -1$

(true)

taking (ii) and (iii)

$\dfrac{1-t}{1} = \dfrac{2t-2}{0}$

or, $2t-2 = 0$

or, $t = 1$.

taking (i) and (iii)

$\dfrac{t-1}{-1} = \dfrac{2t-2}{0}$

or, $2t-2 = 0$

or, $t = 1$.

So, we get,

$(x,y,z) = (1+t, 1-t, 0+2t)$

$= (1+1, 1-1, 2\times1)$

$= (2,0,2)$

Q.No.2. Find direction numbers and direction cosines for the line of intersection of planes $x+y+z=1$ and $x+z=0$

Solution.

Given planes, $x+y+z=1$, $x+z=0$

Now, $x+y+z=1$ —(i)

$x+z=0$ —(ii)

let, $x=0$ then, eq^n (i) become.

$y+z=1$, $\cancel{z=0}$

Solving (i) and (ii).

$z=0$, $y=1$, when, $x=0$.

normal vectors are $\vec{n_1} = (1,1,1)$ and $\vec{n_2} = (1,0,1)$

Now,

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}\,(1\times1 - 0\times1) - \vec{j}\,(1\times1 - 1\times1) + \vec{k}\,(1\times0 - 1\times1)$$

$$= \vec{i} - \vec{k}$$

Hence, direction ratio of point of intersection of lines are $(1,0,-1)$

3) Determine if planes are parallel, perpendicular or neither. If neither find the angle between them

i) $x+4y-3z=1$, $-3x+6y+7z=0$

⇒ Solution.

Given, Plane is $x+4y-3z=1$, $-3x+6y+7z=0$

now, normal vectors are, $\vec{n_1} = (1,4,-3)$

$\vec{n_2} = (-3,6,7)$

Now,

$$\vec{n_1} \cdot \vec{n_2} = 1\times(-3) + 4\times6 - 3\times7 = 0$$

Since, dot product is zero they are perpendicular

ii) $x+y+z=1$; $x-y+z=1$

⇒ Solution:

Given Plane is $x+y+z=1$ and $x-y+z=1$

So, normal vectors are $\vec{n_1} = (1,1,1)$

$\vec{n_2} = (1,-1,1)$

Date ____ Page ____

Now, $\vec{n_1} \cdot \vec{n_2} = (1\times1 + 1\times(-1) + 1\times1)$

So, if θ be angle then,

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|} = \frac{1}{\sqrt{1^2+1^2+1^2}\,\sqrt{1^2+(-1)^2+1^2}}$$

$$= \frac{1}{\sqrt{3}\cdot\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{3}\right)$$

$$= 70.52^\circ$$

iii) $x = 4y - 2z,\ 8y = 1 + 2x + 4z$

Solution,

Given plane is $x = 4y - 2z$, $8y = 1 + 2x + 4z$

Now,

normal vectors $\vec{n_1} = (1, -4, 2)$ and $\vec{n_2} = (2, -8, 4)$

Now,

Angle between them is

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|} = \frac{1\times2 + (-4)\times(-8) + 2\times4}{\sqrt{1^2+(-4)^2+2^2}\ \sqrt{2^2+(-8)^2+4^2}}$$

$$= \frac{42}{\sqrt{21}\ \sqrt{84}}$$

$$= 1$$

or, $\theta = \cos^{-1}(1) = 0^\circ$

Hence, they are parallel.

Date ____
Page ____

4.) Find the points in which line $\frac{x+1}{-1} + \frac{y-12}{5} = \frac{z-7}{2}$ cuts surface $11x^2 - 5y^2 + z^2 = 0$.

⇒ <u>Solution.</u>

Given, line is $\frac{x+1}{-1} = \frac{y-12}{5} = \frac{z-7}{2}$.

and surface is $11x^2 - 5y^2 + z^2 = 0$

we have from line, $x+1 = -1t$ | $y - 12 = 5t$ | $z-7 = 2t$

or, $x = -1 - 1t$ | ~~$y = 5+5t$~~ | $z = 7+2t$

$y = 12+5t$

Now, substituting in surface

$11(-1-1t)^2 - 5(12+5t)^2 + (7+2t)^2 = 0$

or, $11(1+2t+t^2) - 5(144+120t+25t^2) + (49+28t+4t^2) = 0$

or, $-110t^2 - 550t - 660 = 0$

or, $t^2 + 5t + 6 = 0$

or, $t = \frac{-5 \pm \sqrt{25 - 4 \cdot 6}}{2}$

$= \frac{-5 \pm 1}{2} = -3$ or -2.

So, point of intersection are, $(x, y, z) = (2, -3, 1)$ and $(x, y, z) = (1, 2, 3)$

Distance from point to plane

Date ____ Page ____

1) Find equation for plane consisting of all points that are equidistant from points $P(1,0,-2)$ and $Q(3,4,0)$

2) <u>solution.</u>

Given points are $P(1,0,-2)$ and $Q(3,4,0)$

<u>Now,</u> let, equation of plane be $ax+by+cz=d$. — (1)

<u>and,</u>

distance from P to plane is.

$$D_1 = \left|\frac{ax_1+by_1+cz-d_1}{\sqrt{a^2+b^2+c^2}}\right| = \left|\frac{a\times 1+b\times 0+c\times(-2)-d_1}{\sqrt{a^2+b^2+c^2}}\right|$$

similarly, distance from Q to plane is.

$$D_2 = \left|\frac{a\times 3+b\times 4+c\times 0-d_2}{\sqrt{a^2+b^2+c^2}}\right|$$

since $D_1 = D_2$ Given in question.

$$\frac{a-2c-d_1}{\cancel{\sqrt{a^2+b^2+c^2}}} = \frac{3a+4b-d_2}{\cancel{\sqrt{a^2+b^2+c^2}}}$$

or, $3a+4b+2c = d_1 - d_2 = 0$

Distance between points $= \sqrt{(1-3)^2+(0-4)^2+(-2-0)^2}$

$= \sqrt{24}$

so, $\cancel{P_1+P}$ $D_1 = D_2 = \frac{\sqrt{24}}{2} = \sqrt{6}$.

Date ____ Page ____

$$P_1 + P_2 = \left| \frac{a-2c-d_1 + 3a+4b-d_1}{\sqrt{a^2+b^2+c^2}} \right| = \sqrt{24}$$

or, $|a-2c-2d_1-2c| = \sqrt{a^2+b^2+c^2} \cdot \sqrt{24}$

or, $(a-4c-2d_1)^2 = (a^2+b^2+c^2) \cdot 24$

Also

$$\sqrt{(x-1)^2+(y-0)^2+(z+2)^2} = \sqrt{(x-3)^2+(y-4)^2+(z-0)^2}$$

or, $\cancel{x^2} - 2x + 1 + \cancel{y^2} + \cancel{z^2} + 4z + 4 = \cancel{x^2} - 6x + 9 + \cancel{y^2} - 8y + 16 + \cancel{z^2}$

or, $4x + 8y + 4z = 16 + 9 - 4 - 1$

or, $4x + 8y + 4z = 20$

or, $x + 2y + z = 5$

2.i) Find the distance from point $(1, -2, 4)$ to plane $3x + 2y + 6z = 5$.

→ Solution:

Given point $A = (1, -2, 4)$

Plane equation $3x + 2y + 6z = 5$

Now,

distance is given by $D = \left| \frac{ax_0 + by_0 + cz_0 - d}{\sqrt{a^2+b^2+c^2}} \right|$

$$= \left| \frac{3\times1 + 2\times(-2) + 6\times4 - 5}{\sqrt{3^2+2^2+6^2}} \right|$$

$$= \frac{18}{7}$$

Date ____ Page ____

2 ii) Find distance between parallel planes $2x-3y+z=4$ and $4x-6y+2z=3$.

→ Solution.

Given planes are $2x-3y+z=4$ and $4x-6y+2z=3$

Now,

multiply equation ① by 2 we get.

$4x-6y+2z=8$.

so, distance $D = \dfrac{|8-3|}{\sqrt{4^2+(-6)^2+2^2}}$

$= \dfrac{5\sqrt{14}}{28}$

iii) Find the distance from point $(4,1,-2)$ to the given line $x=1+t$, $y=3-2t$, $z=4-3t$

→ Solution.

Given point is $A(4,1,-2)$

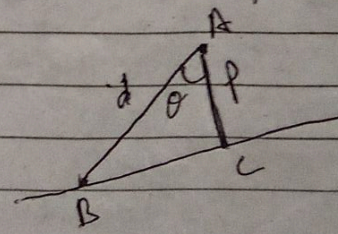
~~eq.~~ parametric eqn of line. $x=1+t$, $y=3-2t$, $z=4-3t$

direction ratio of line $(l,m,n) = (1,-2,-3)$.

$\vec{OA} = \vec{r}$ at $t=0$

Now, $\vec{AB} = \vec{OB}-\vec{OA} = (1,3,4)-(4,1,-2)$

$= (-3,2,6)$



Now,

$d = |\vec{AB}| = \sqrt{(-3)^2+2^2+6^2}$

$= \sqrt{49} = 7$.

Now, $\cos\theta = P/d$

or, $P = d\cos\theta = 7\cos\theta$

Now, $\vec{AB} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 2 & 6 \\ t & -2t & -3t \end{vmatrix}$

$= \vec{i}(-6t+12t) - \vec{j}(9t-\cancel{6t}) + \vec{k}(6t-2t)$

$= 6t\vec{i} - 3t\vec{j} + 4t\vec{k}$

And $|\vec{BC}| = \sqrt{t^2+(-2t)^2+(-3t)^2}$

$= t\sqrt{1+4+9}$

$= \sqrt{14}\, t$.

$|\vec{AB} \times \vec{BC}| = \sqrt{(6t)^2+(-3t)^2+(4t)^2}$

$= t\sqrt{36+9+16}$

$= \sqrt{61}\, t$

Now, Area of Parallelogram $= |\vec{AB} \times \vec{BC}| = 2 \times \frac{1}{2} \times |\vec{BC}| \times P$.

or, $\sqrt{61}\,\cancel{t} = \cancel{2} \times \frac{1}{\cancel{2}} \times \sqrt{14}\,\cancel{t} \times P$

or, $P = \sqrt{61/14}$.

Distance Between Skew Lines

Date
Page

1) i) Show that lines with symmetric equation $x = y = z$ and $x+1 = y/2 = z/3$ are skew and find the distance between these lines.

2) Solution.

Given, symmetric equation of lines. $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$ and

$$\frac{x+1}{1} = \frac{y}{2} = \frac{z}{3} \quad \text{(ii)}$$

from first equation we get following parametric form

$x = t, \; y = t, \; z = t$.

Now,

Substitute in eqn (ii) we get.

$$\frac{t+1}{1} = \frac{t}{2} = \frac{t}{3}$$

taking first two ratio

$$\frac{t+1}{1} = \frac{t}{2}$$

or, $2t + 2 = t$

or, $t = -2$

taking 2nd and 3rd ratio

$$\frac{t}{2} = \frac{t}{3}$$

or, $3t = 2t$

or, $3 = 2$ (not true)

$]^2_1$

Hence, the two lines are skew lines.

equation of plane containing first line and parallel to second line is

$$\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

$= x(3-2) - y(3-1) + z(2-1) = 0$

or, $x - 2y + z = 0$.

Date ______
Page ______

point on Second line is $(-1,0,0)$

So,

distance is $D = \left| \dfrac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2+b^2+c^2}} \right|$

$$= \left| \frac{1 \times (-1) + (-2)\times 0 + 1 \times 0 + 0}{\sqrt{1^2 + (-2)^2 + 1^2}} \right|$$

$$= 1/\sqrt{6}$$

1)ii) let L_1 be line through origin and point $(2,0,-1)$. let L_2 be line through points $(1,-1,1)$ and $(4,1,3)$. find the distance between L_1 and L_2.

=> Solution.

points on line L_1 are $(0,0,0)$ and $(2,0,-1)$

points on line L_2 are $(1,-1,1)$ and $(4,1,3)$

Now,

direction ratio for $L_1 = (2-0, 0-0, -1-0)$

$= (2, 0, -1)$

direction ratio for $L_2 = (4-1, 1+1, 3-1)$

$= (3, 2, 2)$

Now,

equation of plane ~~through~~ containing line L_1 and parallel to L_2 is
$$\begin{vmatrix} x-0 & y-0 & z-0 \\ 2 & 0 & -1 \\ 3 & 2 & 2 \end{vmatrix} = 0$$

Date ______
Page ______

or, $x(0+2) - y(4+3) + z(4-0) = 0$

or, $2x - 7y + 4z = 0$

Now,

distance $D = \left| \dfrac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2+b^2+c^2}} \right|$

$= \left| \dfrac{2\times 1 + (-7)\times(-1) + 4\times 1}{\sqrt{2^2+(-7)^2+4^2}} \right|$

$= \dfrac{13}{\sqrt{69}}.$

Unit II

Vector functions

Date ______ Page ______

Question If $\vec{r} = \vec{a}e^{nt} + \vec{b}e^{-nt}$, where $\vec{a}$ and $\vec{b}$ are constant vectors. Show that $\frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0$

⇒ Given

$\vec{r} = \vec{a}e^{nt} + \vec{b}e^{-nt}$, and $\vec{a}$ and $\vec{b}$ are constant vecters.

now,

$$\frac{d\vec{r}}{dt} = \frac{d(\vec{a}e^{nt} + \vec{b}e^{-nt})}{dt}$$

$$= \vec{a}e^{nt}.n + \vec{b}e^{-nt}.(-n)$$

$$= n\vec{a}e^{nt} - n\vec{b}e^{-nt}$$

Again,

$$\frac{d^2\vec{r}}{dt^2} = \frac{d(n\vec{a}e^{nt} - n\vec{b}e^{-nt})}{dt}$$

$$= n\vec{a}n.e^{nt} - n\vec{b}e^{-nt}.(-n)$$

$$= n^2\vec{a}e^{nt} + n^2\vec{b}e^{-nt}$$

Now, $\frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = n^2\vec{a}e^{nt} + n^2\vec{b}e^{-nt} - n^2(\vec{a}e^{nt} + \vec{b}e^{-nt})$

$= 0.$

Hence, $\frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0$

Date ______ Page ______

Question If $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + t\vec{k}$, find $\frac{d\vec{r}}{dt}$, $\frac{d^2\vec{r}}{dt^2}$ and $\left|\frac{d^2\vec{r}}{dt^2}\right|$.

$\Rightarrow$ Given, $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + t\vec{k}$

Now,

$$\frac{d\vec{r}}{dt} = a\vec{i}\,\frac{d\cos t}{dt} + a\vec{j}\,\frac{d\sin t}{dt} + \cancel{t\frac{d\vec{k}}{dt}}\;\vec{k}\,\frac{dt}{dt}$$

$$= -a\vec{i}\sin t + a\vec{j}\cos t + \vec{k}$$

Again, $\frac{d^2\vec{r}}{dt^2} = -a\vec{i}\cos t - a\vec{j}\sin t$

And, $\left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{(-a\cos t)^2 + (0 - a\sin t)^2}$

$$= a\sqrt{\cos^2 t + \sin^2 t}$$

$$= a.$$

Question If $\vec{u} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ and $\vec{v} = (2t-3)\vec{i} + \vec{j} - t\vec{k}$

Find a) $\frac{d}{dt}(\vec{u}\cdot\vec{v})$ b) $\frac{d}{dt}(\vec{u}\times\vec{v})$ at $t = 1$

$\Rightarrow$ ~~Now~~, Solution,

Given, $\vec{u} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$

$\vec{v} = (2t-3)\vec{i} + \vec{j} - t\vec{k}$

Now, $\vec{u}\cdot\vec{v} = t^2\cdot(2t-3) - t\cdot 1 + (2t+1)\cdot(-t)$

Date
Page

$= 2t^3 - 3t^2 - t - 2t^2 - t$

$= 2t^3 - 5t^2 - 2t$

And, $\frac{d}{dt}(\vec{u}\cdot\vec{v}) = \frac{d}{dt}(2t^3 - 5t^2 - 2t)$

$= 6t^2 - 10t - 2$

at, $t = 1$ $\frac{d}{dt}(\vec{u}\cdot\vec{v}) = 6 - 10 - 2 = -4 - 2 = -6$.

Again,

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t^2 & -t & (2t+1) \\ 2t-3 & 1 & -t \end{vmatrix}$$

$= \vec{i}(t^2 - 2t - 1) - \vec{j}(-t^3 - (4t^2 + 2t - 6t - 3)) + \vec{k}(t^2 + 2t^2 + 3t)$

$\frac{d\,\vec{u}\times\vec{v}}{dt} = \vec{i}(2t - 2) - \vec{j}(-3t^2 - (8t + 2 - 6)) + \vec{k}(2t + 4t + 3)$

at $t = 1$. $\frac{d\vec{u}\times\vec{v}}{dt} = 7\vec{j} + 3\vec{k}$

Question If $\vec{r} = (a\cos t)\vec{i} + (a\sin t)\vec{j} + (at\tan\alpha)\vec{k}$

Find $\left[\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right]$ and $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^2}\right]$

→ Soution,

Given, $\vec{r} = (a\cos t)\vec{i} + (a\sin t)\vec{j} + (at\tan\alpha)\vec{k}$

Now, $\frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k}$

Again, $\frac{d^2\vec{r}}{dt^2} = -a\cos t\,\vec{i} - a\sin t\,\vec{j}$

Also, $\frac{d^3\vec{r}}{dt^3} = a\sin t\,\vec{i} - a\cos t\,\vec{i}$

So, $\left[\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$

$= \vec{i}\,(a^2\sin t\tan\alpha) - \vec{j}\,(a^2\cos t\tan\alpha) + \vec{k}\,(a^2\sin^2 t + a^2\cos^2 t)$

$= \vec{i}\,(a^2\sin t\tan\alpha) - \vec{j}\,(a^2\cos t\tan\alpha) + a^2\,\vec{k}$

So, $\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{a^4\sin^2 t\tan^2\alpha + a^4\cos^2 t\tan^2\alpha + a^4}$

$= a^2\sqrt{\tan^2\alpha + 1}$

$= a^2\sec\alpha$

Now, $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = \frac{d\vec{r}}{dt} \cdot \left[\frac{d^2\vec{r}}{dt^2} \times \frac{d^3\vec{r}}{dt^3}\right]$

$= (-a\sin t\,\vec{i} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k}) \cdot \vec{k}(a^2\cos^2 t + a^2\sin^2 t)$

$= (-a\sin t \cdot 0 + a\cos t \cdot 0 + a\tan\alpha \cdot a^2$

$= a^3\tan\alpha.$

Date
Page

Question 1) For curve $x=3t$, $y=3t^2$, $z=2t^3$ show that $[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = 216$.

2) Solution.

Given, for curve, $x=3t$, $y=3t^2$, $z=2t^3$

So, $\vec{r} = 3t\vec{i} + 3t^2\vec{j} + 2t^3\vec{k}$

Now,

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = 3\vec{i} + 6t\vec{j} + 6t^2\vec{k}$$

Again, $$\ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2} = 6\vec{j} + 12t\vec{k}$$

Also, $$\dddot{\vec{r}} = \frac{d^3\vec{r}}{dt^2} = 12\vec{k}.$$

Now,

$$[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = \dot{\vec{r}}\cdot(\ddot{\vec{r}}\times\dddot{\vec{r}})$$
$$=(3\vec{i}+6t\vec{j}+6t^2\vec{k})\cdot(72\vec{i})$$
$$=72\times3+6t\times0+6t^2\times0$$
$$=216.$$

2) If $\vec{r_1} = (2t+1)\vec{i} - t^2\vec{j} + 3t^3\vec{k}$ and
$\vec{r_2} = t^2\vec{i} + t\vec{j} - (t-1)\vec{k}$, verify that

a) $\frac{d}{dt}(\vec{r_1}\cdot\vec{r_2}) = \vec{r_1}\cdot\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\cdot\vec{r_2})$

b) $\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \vec{r_1}\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}$

2) Solution,

Given, $\vec{r_1} = (2t+1)\vec{i} - t^2\vec{j} + 3t^3\vec{k}$

$\vec{r_2} = t^2\vec{i} + t\vec{j} - (t-1)\vec{k}$

Now, $\vec{r_1}\cdot\vec{r_2} = (2t+1)\cdot t^2 - t^2\cdot t + 3t^3\cdot(1-t)$

$= 2t^3 + t^2 - t^3 + 3t^3 - 3t^4$

$= -3t^4 + 4t^3 + t^2.$

$\dfrac{d\vec{r_1}}{dt} = 2\vec{i} - 2t\vec{j} + 9t^2\vec{k}$

$\dfrac{d\vec{r_2}}{dt} = 2t\vec{i} + \vec{j} - \vec{k}.$

Also, $\dfrac{d}{dt}(\vec{r_1}\cdot\vec{r_2}) = -3\cdot 4t^3 + 4\cdot 3\cdot t^2 + 2t$

$= -12t^3 + 12t^2 + 2t.$

And, $\vec{r_1}\cdot\dfrac{d\vec{r_2}}{dt} + \dfrac{d\vec{r_1}}{dt}\cdot\vec{r_2} = \left[(2t+1)\cdot 2t + (-t^2)\cdot 1 + 3t^3\cdot(-1)\right]$

$+ \left[2\cdot t^2 + (-2t)\cdot t + 9t^2\cdot(1-t)\right].$

$= 4t^2 + 2t - t^2 - 3t^3 + \cancel{2t^2} - \cancel{2t^2} + 9t^2 - 9t^3$

$= -12t^3 + 12t^2 + 2t.$

Hence, $\dfrac{d(\vec{r_1}\cdot\vec{r_2})}{dt} = \vec{r_1}\cdot\dfrac{d\vec{r_2}}{dt} + \dfrac{d\vec{r_1}}{dt}\cdot\vec{r_2}$

b) b) $\vec{r_1}\times\vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t+1 & -t^2 & 3t^3 \\ t^2 & t & 1-t \end{vmatrix}$

$= \vec{i}\,(-t^2 + t^3 \overset{-3t^4}{}) - \vec{j}\,(2t - 2t^2 + 1 - t - [illegible]) + \vec{k}\,\overset{(2t^2+t+t^4)}{\cancel{(2t^2+t+t^4)}}$

Also, $\vec{r_1}\times\dfrac{d\vec{r_2}}{dt} = \vec{i}\left((-t^2)\cdot(-1) - 1\cdot 3t^3\right) - \vec{j}\left((2t+1)\cdot(-1) - 2t\cdot 3t^3\right)$

$+ \vec{k}\left((2t+1)\cdot 1 - 2t\cdot(-t^2)\right)$

$$-2t-1-6t^4$$

$$= \vec{i}(t^2-3t^3) - \vec{j}(\cancel{-2t-1+2t}) + \vec{k}(2t+1+2t^3)$$

$$= \vec{i}(t^2-3t^3) + \vec{k}(2t+1+2t^3) + \vec{j}(6t^4+2t+1)$$

& also, $\dfrac{d\vec{r_1}}{dt} \times \vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2t & 9t^2 \\ t^2 & t & 1-t \end{vmatrix}$

$$= \vec{i}(-2t+2t^2-9t^3) - \vec{j}(2-2t-9t^4) + \vec{k}(2t+2t^3)$$

Now,

$$\vec{r_1} \times \frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt} \times \vec{r_2}$$

$$= \vec{i}(t^2-3t^3) + \vec{k}(2t+1+2t^3) + \vec{i}(-2t+2t^2-9t^3) + \vec{j}(-2+2t+9t^4) + \vec{k}(2t+2t^3) + \vec{j}(6t^3+2t+1)$$

$$= \vec{i}(t^2-3t^3-2t+2t^2-9t^3) + \vec{k}(2t+6+2t+2t^3) + \vec{j}(1-2+2t+9t^4)$$

$$= \vec{i}(t^2-3t^3) + \vec{j}(6t^4+2t+1) + \vec{k}(2t^3+2t+1)$$

$\cancel{+\vec{j}(6t^4+2t+1)}$ $$+ \vec{i}(-9t^3+2t^2-2t) + \vec{j}(9t^4+2t-2) + \vec{k}(2t+2t^3)$$

$$\Rightarrow \vec{i}(t^2-3t^3-9t^3+2t^2-2t) + \vec{j}(6t^4+2t+1+9t^4+2t-2)$$
$$+ \vec{k}(2t^3+2t+1+2t+2t^3) \cancel{+\vec{j}(6t^4+2t+1)}$$

$$= \vec{i}(-12t^3+3t^2-2t) + \vec{j}(9t^4+6t^4+4t-1)$$
$$+ \vec{k}(4t^3+4t+1)$$

Also, $\dfrac{d}{dt}(\vec{r_1} \times \vec{r_2}) = \vec{i}(-3\cdot4\cdot t^3+3\cdot t^2-2\cdot t) + \vec{j}(4t-1)$
$$+ \vec{k}(4t^3+4t+1)$$

$$= \vec{i}(-12t^3+3t^2-2t) + \vec{j}(4t-1) + \vec{k}(4t^3+4t+1)$$

Date
Page

Question Find the derivative of following.

a) $\dfrac{\vec{r}+\vec{a}}{\vec{r}^{\,2}+\vec{a}^{\,2}}$

Solution,

let, $u = \dfrac{\vec{r}+\vec{a}}{\vec{r}^{\,2}+\vec{a}^{\,2}}$

Here, $\vec{r}^{\,2} = r^2$ and $\vec{a}^{\,2} = a^2$.

And,

$$\frac{du}{dt} = \frac{d}{dt}\left(\frac{\vec{r}+\vec{a}}{r^2+a^2}\right)$$

$$= \frac{(r^2+a^2)\cdot(d\vec{r}/dt + d\vec{a}/dt) - (\vec{r}+\vec{a})\cdot\frac{d}{dt}(r^2+a^2)}{(r^2+a^2)^2}$$

$$= \frac{(r^2+a^2)\cdot d\vec{r}/dt - (\vec{r}+\vec{a})\cdot 2r\cdot dr/dt + (r^2+a^2)\,d\vec{a}/dt}{(r^2+a^2)^2}$$

$$= \frac{1}{r^2+a^2}\left(\frac{d\vec{r}}{dt}+\frac{d\vec{a}}{dt}\right) - \frac{2r\cdot(\vec{r}+\vec{a})}{(r^2+a^2)^2}\cdot\frac{dr}{dt}$$

b) $\dfrac{\vec{r}+\vec{a}}{\vec{r}\cdot\vec{a}}$

$\Rightarrow$ let, $u = \dfrac{\vec{r}+\vec{a}}{\vec{r}\cdot\vec{a}}$

then, $$\frac{du}{dt} = \frac{\vec{r}\cdot\vec{a}\cdot\frac{d}{dt}(\vec{r}+\vec{a}) - (\vec{r}+\vec{a})\cdot\frac{d\,\vec{r}\cdot\vec{a}}{dt}}{(\vec{r}\cdot\vec{a})^2}$$

$$= \frac{1}{\vec{r}\cdot\vec{a}}\cdot\frac{d\vec{r}}{dt} - \frac{(\vec{r}+\vec{a})}{(\vec{r}\cdot\vec{a})^2}\cdot\vec{a}\cdot\frac{d\vec{r}}{dt}$$

c) $\vec{r}\cdot\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)$

=> solution.

let, $\vec{u} = \vec{r}\cdot\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)$

Now, $\frac{d\vec{u}}{dt} = \frac{d}{dt}\left(\vec{r}\cdot\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)\right)$

$= \vec{r}\cdot\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right) + \left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)\cdot\frac{d\vec{r}}{dt}$

$= \vec{r}\cdot\frac{d^2\vec{r}}{dt^2}\times\frac{d^2\vec{r}}{dt^2} + \vec{r}\cdot\frac{d\vec{r}}{dt}\times\frac{d^3\vec{r}}{dt^3} + \frac{d\vec{r}}{dt}\cdot\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)$

$= \cancel{\left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\right]}$

Question 1) a) A particle moves along curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$. Find the magnitude of velocity and acceleration at time $t = \pi/3$.

=> Solution.

Given curve is $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$.

Now,

let, $\vec{r} = 2\sin 3t\,\vec{i} + 2\cos 3t\,\vec{j} + 8t\,\vec{k}$

Now,

$\vec{v} = \frac{d\vec{r}}{dt} = 2\cos 3t\cdot 3\,\vec{i} - 2\sin 3t\ 3\vec{j} + 8\vec{k}$

Also, $\vec{a} = \frac{d^2\vec{r}}{dt^2} = -6\sin 3t\cdot 3\,\vec{i} - 6\cos 3t\cdot 3\,\vec{j}$

$= -18\sin 3t\,\vec{i} - 18\cos 3t\,\vec{j}$

Date
Page

Now, $|\vec{v}| = \sqrt{(6\cos 3t)^2 + (-6\sin 3t)^2 + 8^2}$

At, $t = \pi/3$, $|\vec{v}| = \sqrt{(6\cos(\cancel{3}\times\frac{\pi}{\cancel{3}}))^2 + (-6\sin(\cancel{3}\times\frac{\pi}{\cancel{3}}))^2 + 8^2}$

$= 10$ units

Similarly, $|\vec{a}| = \sqrt{(-18\sin 3t)^2 + (-18\cos 3t)^2}$

at, $t = \pi/3$ $|\vec{a}| = \sqrt{(-18\sin 180)^2 + (-18\cos 180))^2}$

$= 18$ units.

1-b) A particle along curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the magnitude of velocity and acceleration at time $t = 0$ and $t = \pi$

2) Solution,

equation of curve is $x = 4\cos t$, $y = 4\sin t$, $z = 6t$.

let, $\vec{r} = 4\cos t\,\vec{i} + 4\sin t\,\vec{j} + 6t\,\vec{k}$

Now,

$\vec{v} = \frac{d\vec{r}}{dt}(4\cos t\,\vec{i} + 4\sin t\,\vec{j} + 6t\,\vec{k})$

$= -4\sin t\,\vec{i} + 4\cos t\,\vec{j} + 6\vec{k}$

and, $\vec{a} = \frac{d^2\vec{r}}{dt^2} = -4\cos t\,\vec{i} - 4\sin t\,\vec{j}$

at, $t = 0$ $|\vec{v}| = \sqrt{(-4\sin 0)^2 + (4\cos 0)^2 + 6^2}$

$=$ ~~4 units.~~ $\sqrt{52} = 2\sqrt{13}$ units

at, $t = \pi$ $|\vec{v}| = \sqrt{(-4\sin 180)^2 + (4\cos 180)^2 + 6^2}$

$=$ ~~4 units~~ $\sqrt{52} = 2\sqrt{13}$ units

Now, At $t=0$ $|\vec{a}| = \sqrt{(-4\cos 0)^2 + (-4\sin 0)^2}$
$= 4$ units.

at $t = \pi$ $|\vec{a}| = \sqrt{(-4\cos 180)^2 + (-4\sin 180)^2}$
$= 4$ units.

Vector integration:

1) If $\vec{r} = 3t^2\vec{i} - 2\vec{j} + 3(t^2-1)\vec{k}$ find.
   a) $\int \vec{r}\, dt$ b) $\int_0^1 \vec{r}\, dt$

=> Solution.
Given, $\vec{r} = 3t^2\vec{i} - 2\vec{j} + 3(t^2-1)\vec{k}$
Now,
$$\int \vec{r}\, dt = \int 3t^2\vec{i} - 2\vec{j} + (3t^2-3)\vec{k}\, dt$$
$$= t^3\vec{i} - 2t\vec{j} + (t^3-3t)\vec{k} + c$$

Now, $$\int_0^1 \vec{r}\cdot dt = [t^4\vec{i} - 2t\vec{j} + (t^3-3t)\vec{k} + c]_0^1$$
$$= \vec{i} - 2\vec{j} - 2\vec{k}$$

2) Given, $\vec{r} = (t-t^2)\vec{i} + 2t^3\vec{j} - 3\vec{k}$, evaluate
$\int_1^2 \vec{r}\, dt$

=> Given, $\vec{r} = (t-t^2)\vec{i} + 2t^3\vec{j} - 3\vec{k}$

Date
Page

Now, $\int \vec{r} \cdot dt = \int (t-t^2)\vec{i} + 2t^3\vec{j} - 3\vec{k}\, dt$

$= \left(\frac{t^2}{2} - \frac{t^3}{3}\right)\vec{i} + \frac{2t^4}{4}\vec{j} - 3t\vec{k} + c.$

Now, $\int_1^2 \vec{r}\, dt = \left[\left(\frac{t^2}{2} - \frac{t^3}{3}\right)\vec{i} + \frac{2t^4}{4}\vec{j} - 3t\vec{k} + c\right]_1^2$

$= \left(\frac{-2}{3}\vec{i} + 8\vec{j} - 6\vec{k} + \frac{1}{6}\vec{i} - \frac{1}{2}\vec{j} + 3\vec{k}\right)$

$= \cancel{\frac{-1}{2}\vec{i} + \frac{17}{2}\vec{j} - 3\vec{k}}$

$= -5/6\, \vec{i} + \frac{15}{2}\vec{j} + 3\vec{k}$

Q.No.3 If $\vec{r_1} = 2\vec{i} + t\vec{j} - \vec{k}$, $\vec{r_2} = t\vec{i} + 2\vec{j} + 3\vec{k}$
and, $\vec{r_3} = 2\vec{i} - 3\vec{j} + 4\vec{k}$

⇒ Solution.

Now,

$$\vec{r_1} \times \vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & t & -1 \\ t & 2 & 3 \end{vmatrix} = \vec{i}(3t+2) - \vec{j}(6+t) + \vec{k}(4-t^2)$$

$\int (\vec{r_1} \times \vec{r_2})dt = \int \vec{i}(3t+2) - \vec{j}(6+t) + k(4-t^2)\, dt.$

$= \vec{i}\left(\frac{3t^2}{2} + 2t\right) - \vec{j}\left(6t + \frac{t^2}{2}\right) + \vec{k}\left(4t - \frac{t^3}{3}\right) + c$

Now, $\int_0^2 (\vec{r_1} \times \vec{r_2})\, dt = \cancel{8\vec{i} - 14\vec{j} - \frac{16}{3}\vec{k}}\ 10\vec{i} - 14\vec{j} + \frac{16}{3}\vec{k}$

Date
Page

b) $[\vec{r_1}\ \vec{r_2}\ \vec{r_3}] = \begin{vmatrix} 2 & t & -1 \\ t & 2 & 3 \\ 2 & -3 & 4 \end{vmatrix}$

$= 2(2\times4 + 3\times3) - t(4t-6) - 1(-3t-4)$

$= 34 - 4t^2 + 6t + 3t + 4$

$= -4t^2 + 9t + 38$

Now, $\int [\vec{r_1}\ \vec{r_2}\ \vec{r_3}]\, dt = \int -4t^2 + 9t + 38\, dt$

$= -4\frac{t^3}{3} + \frac{9t^2}{2} + 38t + c$

And, $\int_0^2 [\vec{r_1}\ \vec{r_2}\ \vec{r_3}]\, dt = \left[-4\frac{t^3}{3} + \frac{9t^2}{2} + 38t\right]_0^2$

$= 250/3$

4.) Evaluate $\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$ for $\vec{r} = 2t^2\vec{i} + t\vec{j} - 3t^3\vec{k}$

=) Solution.

$\vec{r} = 2t^2\vec{i} + t\vec{j} - 3t^3\vec{k}$

Now, $\frac{d\vec{r}}{dt} = 4t\vec{i} + \vec{j} - 9t^2\vec{k}$

Again, $\frac{d^2\vec{r}}{dt^2} = 4\vec{i} - 18t\vec{k}$

~~step~~ And, $\vec{r} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t^2 & t & -3t^3 \\ 4 & 0 & -18t \end{vmatrix}$

Date
Page

$= \vec{i}(-18t^2) - \vec{j}(-18\times 2\cdot t\cdot t^2 + 4\cdot 3\cdot t^3) + \vec{k}(-4t)$

$= -18t^2\vec{i} + 24t^3\vec{j} - 4t\vec{k}$

Now,

$$\int \vec{r}\times\frac{d^2\vec{r}}{dt^2}\,dt = \int -18t^2\vec{i} + 24t^3\vec{j} - 4t\vec{k}\,dt$$

$$= -18\frac{t^3}{3}\vec{i} + 24\frac{t^4}{4}\vec{j} - 4\frac{t^2}{2}\vec{k} + c$$

So, $$\int_1^2\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = \left[-18\frac{t^3}{3}\vec{i} + 24\frac{t^4}{4}\vec{j} - 4\frac{t^2}{2}\vec{k}\right]_1^2$$

$$= -42\vec{i} + 90\vec{j} - 6\vec{k}$$

5) Integrate: $\frac{d^2\vec{r}}{dt^2} = -n^2\vec{r}$

=> Now,

$$\int\frac{d^2\vec{r}}{dt^2}\,dt = \int -n^2\vec{r}\,dt$$

$$= -n^2\frac{(\vec{r})^2}{2} + c$$

So, $$\int\frac{d^2\vec{r}}{dt^2}\,dt = \frac{d\vec{r}}{dt} = -n^2\cdot\frac{(\vec{r})^2}{2} + c$$

Date ______
Page ______

6.) Solve $\frac{d^2\vec{r}}{dt^2} = t\vec{a} + \vec{b}$, where $\vec{a}$ and $\vec{b}$ are constant vectors, given that $\vec{r} = \vec{0}$ and $\frac{d\vec{r}}{dt} = 0$ ~~and~~ $\frac{d\vec{r}}{dt}$, when $t=0$

7.a)

⇒ Solution:

⇒

Given, $\frac{d^2\vec{r}}{dt^2} = t\vec{a} + \vec{b}$

Now,

$$\int \frac{d^2\vec{r}}{dt^2}\, dt = \frac{d\vec{r}}{dt} = \int t\vec{a} + \vec{b}\ dt$$

$$= \vec{a}\frac{t^2}{2} + \vec{b}\cdot t + c.$$

and $\int \frac{d\vec{r}}{dt}\, dt = \int (\vec{a}\frac{t^2}{2} + \vec{b}t + c)\, dt$

or, $\vec{r}$. $= \frac{\vec{a}}{2}\cdot\frac{t^3}{3} + \vec{b}\frac{t^2}{2} + ct + d$

when, $t=0.$, $\vec{r} = \frac{\vec{a}}{2}\cdot 0 + \frac{\vec{b}}{2}\, 0 + c\cdot 0 + d = 0$

or, $d = 0$.

and, when, $t=0.$, $\frac{d\vec{r}}{dt} = \vec{a}\cdot\frac{0}{2} + \vec{b}\cdot 0 + c = 0$

or, $c = 0$.

Hence, $\vec{r} = \frac{\vec{a}}{6}t^3 + \frac{\vec{b}}{2}t^2$

Date ______ Page ______

a) The acceleration of particle is given by $\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt}$ $= e^{-t}\vec{i} - 6(t+1)\vec{j} + 3\sin t\,\vec{k}$. Find the velocity $\vec{v}$ and displacement $\vec{r}$, given $\vec{v} = 0$, $\vec{r} = 0$, when $t = 0$

⇒ Solution

Given, $\frac{d^2\vec{r}}{dt^2} = e^{-t}\vec{i} - 6(t+1)\vec{j} + 3\sin t\,\vec{k}$

Now, $\int \frac{d^2\vec{r}}{dt^2}\,dt = \vec{v} = \int e^{-t}\vec{i} - 6(t+1)\vec{j} + 3\sin t\,\vec{k}\;dt$

or, $\vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\vec{i} - 6\left(\frac{t^2}{2} + t\right)\vec{j} - 3\cos t\,\vec{k} + c$

Again, $\int \frac{d\vec{r}}{dt}\,dt = \int -e^{-t}\vec{i} - 6\left(\frac{t^2}{2} + t\right)\vec{j} - 3\cos t\,\vec{k} + c\;dt$

or, $\vec{r} = e^{-t}\vec{i} - 6\left(\frac{t^3}{6} + \frac{t^2}{2}\right)\vec{j} - 3\sin t\,\vec{k} + ct + d$

when, $t = 0$, $\vec{r} = e^{0}\vec{i} - 6(0+0)\vec{j} - 3\sin 0\,\vec{k} + c\cdot 0 + d = 0$

or, $d = -\vec{i}$

Also, when, $t = 0$, $\vec{v} = -e^{-0}\vec{i} - 6(0+0)\vec{j} - 3\cos 0\,\vec{k} + c = 0$

or, $-\vec{i} - 3\vec{k} + c = 0$

or, $c = \vec{i} + 3\vec{k}$

Hence,

$\vec{r} = e^{-t}\vec{i} - 6\left(\frac{t^3}{6} + \frac{t^2}{2}\right)\vec{j} - 3\sin t\,\vec{k} + (\vec{i} + 3\vec{k})t - \vec{i}$

$= \vec{i}(e^{-t} + t - 1) - \vec{j}\cdot 6\left(\frac{t^3}{6} + \frac{t^2}{2}\right) + \vec{k}(3t - 3\sin t)$

Date ______
Page ______

or, $\vec{r} = (t-1+e^{-t})\vec{i} - (t^3+3t^2)\vec{j} + (3t - 3\sin t)\vec{k}$

And, $\vec{v} = \vec{i}(1-e^{-t}) - (3t^2+6t)\vec{j} + \vec{k}(3-3\cos t)$

7.b) The acceleration of moving particle at any time t is $\vec{a}(t) = \frac{d\vec{v}}{dt} = 12\cos 2t\,\vec{i} - 8\sin 2t\,\vec{j} + 16t\,\vec{k}$

Find the velocity $\vec{v}$ and displacement $\vec{r}$ at any time t given, that if $t=0$, $\vec{v}=0$ and $\vec{r}=0$.

⇒ Solution,

Given, $\vec{a} = \frac{d\vec{v}}{dt} = 12\cos 2t\,\vec{i} - 8\sin 2t\,\vec{j} + 16t\,\vec{k}$

Now, $\int \vec{a}\,dt = \vec{v} = \int 12\cos 2t\,\vec{i} - 8\sin 2t\,\vec{j} + 16t\,\vec{k}\,dt$

$= \frac{12\sin 2t}{2}\vec{i} + \frac{8\cos 2t}{2}\vec{j} + \frac{16t^2}{2}\vec{k} + c$

Also, $\vec{r} = \int \vec{v}\,dt = \int 6\sin 2t\,\vec{i} + 4\cos 2t\,\vec{j} + 8t^2\vec{k} + c\,dt$

$= -\frac{6\cos 2t}{2}\vec{i} + \frac{4}{2}\sin 2t\,\vec{j} + \frac{8t^3}{3}\vec{k} + ct + d$

when, $t=0$, $\vec{r} = -\frac{6}{2}\cdot\cos 0\,\vec{i} + 2\sin 0\,\vec{j} + 0\,\vec{k} + 0 + d = 0$

or, $-3\vec{i} + d = 0$

or, $d = 3\vec{i}$

when, $t=0$, $\vec{v} = 6\sin 0\,\vec{i} + 4\cos 0\,\vec{j} + 8t^2\vec{k} + c = 0$

or, $4\vec{j} + c = 0$

Date
Page

or, $C = -4\vec{j}$

Hence, $\vec{r} = -3\cos 2t\,\vec{i} + 2\sin 2t\,\vec{j} + \frac{8t^3}{3}\vec{k} + (-4t)\vec{j} + 3\vec{i}$

$= \vec{i}(3 - 3\cos 2t) + \vec{j}(2\sin 2t - 4t) + \vec{k}\,\frac{8t^3}{3}$

$= 3(1-\cos 2t)\,\vec{i} + 2(\sin 2t - 2t)\,\vec{j} + \frac{8t^3}{3}\vec{k}$

Curvature of curve at a point

4.) Find the curvature of curve $y = 2x^4$ at point $x = 2$

⇒ Solution,

Given, curve equation $= 2x^4 = y$.

Now,

$y_1 = \frac{dy}{dx} = \frac{d\,2x^4}{dx} = 8x^3.$

and $y_2 = \frac{d^2y}{dx^2} = \frac{d\,8x^3}{dx}$

$= 24x^2.$

Now,

curvature is given by $K = \frac{y_2}{(1+y_1^2)^{3/2}}$

$= \frac{24x^2}{(1+(8x^3)^2)^{3/2}}$

$= \frac{24x^2}{(1+64x^6)^{3/2}}$

at, $x=2$, $K = \frac{24.2^2}{(1+64\times 2^6)^{3/2}} = 3.66\times 10^{-4}$

Date
Page

5) Find the curvature of curve $y = 3x^4$ at point $x = 1$.

⇒ Solution,

Given, equation of curve is $y = 3x^4$

Now,

$$y_1 = \frac{dy}{dx} = \frac{d\,3x^4}{dx} = 12x^3.$$

and, $$y_2 = \frac{d^2y}{dx^2} = \frac{d\,12x^3}{dx} = 36x^2.$$

Now,

curvature is given by $$K = \frac{y_2}{(1+y_1^2)^{3/2}}$$

$$= \frac{36x^2}{(1+144x^6)^{3/2}}.$$

at, $x = 1$.

$$K = \frac{36 \cdot 1^2}{(1+144 \cdot 16)^{3/2}}$$

$$= 0.0206.$$

Curvature in parametric curve for vector function.

1) Find the curvature of vector function $\vec{r}(t) = \rho \cos t \, \vec{i} + \rho \sin t \, \vec{j}$, where ρ is constant

⇒ Solution,

Given, vector function, $\vec{r}(t) = \rho \cos t \, \vec{i} + \rho \sin t \, \vec{j}$

Now,

$$(x_1, y_1) = \frac{d\vec{r}}{dt} = -\rho \sin t \, \vec{i} + \rho \cos t \, \vec{j}$$

$$(x_2, y_2) = \frac{d^2\vec{r}}{dt^2} = -\rho \cos t \, \vec{i} - \rho \sin t \, \vec{j}$$

Now, ~~the~~ curvature is
$$K = \frac{x_1 y_2 - x_2 y_1}{(x_1^2 + y_1^2)^{3/2}}$$

$$= \frac{-\rho \sin t \cdot (-\rho \sin t) - (-\rho \cos t) \cdot \rho \cos t}{(\rho^2 \sin^2 t + \rho^2 \cos^2 t)^{3/2}}$$

$$= \frac{\rho \sin^2 t + \rho \cos^2 t}{(\rho^2)^{3/2}}$$

$$= \frac{\rho}{\rho^{2 \times \frac{3}{2}}} = \rho^{-2}$$

2) Determine the curvature for $\vec{r}(t) = (t; 3\sin(t), 3\cos(t))$

⇒ solution,

Given, $\vec{r} = t\,\vec{i} + 3\sin(t)\,\vec{j} + 3\cos(t)\,\vec{k}$

Now,

$$(x_1, y_1, z_1) = \frac{d\vec{r}}{dt} = \vec{i} + 3\cos(t)\,\vec{j} - 3\sin(t)\,\vec{k}$$

Again, $$(x_2, y_2, z_2) = \frac{d^2\vec{r}}{dt^2} = 0 \cdot \vec{i} - 3\sin(t)\,\vec{j} - 3\cos(t)\,\vec{k}$$

Date ______ Page ______

Now, curvature $\kappa = \dfrac{|r_1 \times r_2|}{|r_1|^3}$

Now, $r_1 \times r_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3\cos t & -3\sin t \\ 0 & -3\sin t & -3\cos t \end{vmatrix}$

$= \vec{i}\,(-9\cos^2 t - 9\sin^2 t) - \vec{j}\,(-3\cos t) + \vec{k}\,(-3\sin t)$

$= -9\vec{i} + 3\cos t\,\vec{j} - 3\sin t\,\vec{k}.$

So, $|r_1 \times r_2| = \sqrt{(-9)^2 + (3\cos t)^2 + (-3\sin t)^2}$

$= \sqrt{81 + 9\cos^2 t + 9\sin^2 t}$

$= \sqrt{81 + 9}$

$= \sqrt{90}$

~~So~~, And, $|r_1| = \sqrt{1^2 + (3\cos t)^2 + (-3\sin t)^2}$

$= \sqrt{1 + 9\cos^2 t + 9\sin^2 t}$

$= \sqrt{1 + 9}$

$= \sqrt{10}$

Hence, $\kappa(t) = \dfrac{\sqrt{90}}{\sqrt{10}}$

$= 3.$

Date ____ Page ____

3.) Find the normal vector and binormal vector of the space curve $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ where, $x = t^2$, $y = t^2$, $z = t^3$ at point $(1,1,1)$ i.e $t = 1$.

=> Solution,

Given, vector is $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$= t^2\vec{i} + t^2\vec{j} + t^3\vec{k}$$

Now,

$$\vec{T} = \frac{d\vec{r}}{dt} = (2t\vec{i} + 2t\vec{j} + 3t^2\vec{k})$$

And, unit tangent vector $= \dfrac{(2t\vec{i} + 2t\vec{j} + 3t^2\vec{k})}{\sqrt{4t^2 + 4t^2 + 9t^4}}$

~~Again,~~
~~normal vector~~ $= \dfrac{1}{2t\sqrt{1+1+9/4t^2}} \cdot \vec{T}$

$$\vec{T} = \frac{1}{\sqrt{2+9/4t^2}} \cdot (\vec{i} + \vec{j} + 3/2t\vec{k})$$

Now,

$$\vec{N} = \frac{d\vec{T}}{dt} = \frac{\frac{1}{\sqrt{2+9/4t^2}} \cdot (\frac{3}{2}\vec{k}) - (\vec{i} + \vec{j} + 3/2t\vec{k}) \cdot \frac{1}{2\sqrt{2+9/4t^2}} \times \frac{9}{4} \times 2t}{(2+9/4t^2)}$$

$$= \frac{3\vec{k}}{2\sqrt{2+9/4t^2}} - \frac{9(t\vec{i} + t\vec{j} + 3/2t^2\vec{k})}{4\sqrt{2+9/4t^2}}$$

at, $t = 1$, $\vec{N} = \dfrac{3\vec{k}}{4\sqrt{17}} - \dfrac{9(\vec{i} + \vec{j} + 3/2\vec{k}) \times 2}{\sqrt{17} \cdot \times 2 \times 2}$

Date ______
Page ______

$= \cancel{\dfrac{-9\vec{i} - 9\vec{j} - 21\sqrt{2}\vec{k}}{2\sqrt{17}}}$

$= \dfrac{-18\vec{i} - 18\vec{j} - 24\vec{k}}{4\sqrt{17}}$

And, unit normal vector $= \dfrac{\vec{N}}{|\vec{N}|} = \dfrac{\vec{N}}{\sqrt{\dfrac{(-18)^2+(-18)^2+(-24)^2}{(4\sqrt{17})^2}}}$

$= \dfrac{4\sqrt{17}\,\vec{N}}{\cancel{\sqrt{17}}} \times \dfrac{1}{6\sqrt{34}}$

$= \dfrac{4\sqrt{17}}{6\sqrt{34}} \cdot \dfrac{(-18\vec{i} - 18\vec{j} - 24\vec{k})}{4\sqrt{17}}$

$= \dfrac{1}{\cancel{\sqrt{17}}}(-18\vec{i} - 18\vec{j} - 24\vec{k}) \times \dfrac{1}{6\sqrt{34}}$

And, unit tangent vector, at $t = 1$ is.

$\vec{T} = (2\vec{i} + 2\vec{j} + 3\vec{k}) \times \dfrac{1}{\sqrt{17}}.$

So, Binormal vector, $= \vec{T} \times \vec{N}$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 3 \\ -18 & -18 & -24 \end{vmatrix} \times \dfrac{1}{\sqrt{17}} \times \dfrac{1}{\cancel{\sqrt{17}}} \times \dfrac{1}{6\sqrt{34}}$

$= \dfrac{1}{3} \left[\vec{i}(-2\times 24 + 3\times 18) - \dfrac{1}{3}\vec{j}(-2\times 24 + 54) + \dfrac{1}{3}\vec{k}(-18\times 2 + 2\times 18) \right] \times \dfrac{1}{102\sqrt{2}}$

Date ____ Page ____

$$= \frac{24\vec{i}}{3\sqrt{1649}} - \frac{24\vec{j}}{3\sqrt{1649}}. \qquad = (6\vec{i} - 6\vec{j}) \times \frac{1}{102\sqrt{2}}.$$

$$= \frac{8\vec{i}}{3\sqrt{1649}} - \frac{8\vec{j}}{3\sqrt{1649}}$$

4) Find the normal vector of space curve $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ where $x = t^2$, $y = t^2$, $z = t^3$ at point $(1, 0, 1)$

=) Solution

Given, $\vec{r} = t^2\vec{i} + t^2\vec{j} + t^3\vec{k}$

Now,

$$\vec{r_1} = \frac{d\vec{r}}{dt} = 2t\vec{i} + 2t\vec{j} + 3t^2\vec{k}.$$

and,

$$|\vec{r_1}| = \sqrt{(2t)^2 + (2t)^2 + 9t^4}$$
$$= \sqrt{4t^2 + 4t^2 + 9t^4}$$
$$= t\sqrt{8 + 9t^2}$$

So,

$$\vec{T} = \frac{\vec{r_1}}{|\vec{r_1}|} = \frac{1}{t\sqrt{8+9t^2}}(2t\vec{i} + 2t\vec{j} + 3t^2\vec{k})$$
$$= \frac{1}{\sqrt{8+9t^2}}(2\vec{i} + 2\vec{j} + 3t\vec{k})$$

Again,

$$\vec{r_2} = \frac{d\vec{T}}{dt} = \sqrt{8+9t^2}\cdot 3\vec{k} - (2\vec{i} + 2\vec{j} + 3t\vec{k})\cdot\frac{1}{\cancel{2}\sqrt{8+9t^2}}\cdot 9\cdot\cancel{2}t$$
$$= \frac{3(8+9t^2)\vec{k} - 9t(2\vec{i} + 2\vec{j} + 3t\vec{k})}{\sqrt{8+9t^2}}$$

Date
Page

$$= \frac{\vec{i}(-18t) - \vec{j}\cdot 18t + \vec{k}(24 + \cancel{27t^2} - \cancel{27t^2})}{\sqrt{8+9t^2}}$$

$$= \frac{-18t\,\vec{i} - 18t\,\vec{j} + 24\vec{k}}{\sqrt{8+9t^2}}.$$

$$|r_2| = \frac{1}{\sqrt{8+9t^2}} \cdot \sqrt{(-18t)^2 + (-18t)^2 + 24^2}$$

$$= \frac{\sqrt{648t^2 + 576}}{\sqrt{8+9t^2}}.$$

Now, $\vec{N} = \frac{\vec{r_2}}{|\vec{r_2}|} = \frac{\cancel{\sqrt{8+9t^2}}}{\cancel{\sqrt{8+9t^2}} \cdot \sqrt{648t^2+576}} \times (-18t\,\vec{i} - 18t\,\vec{j} + 24\vec{k})$

$$= \frac{1}{\sqrt{648t^2+576}} \cdot (-18t\,\vec{i} - 18t\,\vec{j} + 24\vec{k})$$

At $t = 1$, $\vec{T} = (2\vec{i} + 2\vec{j} + 3\vec{k}) \times \frac{1}{\sqrt{17}}.$

At, $t = 1$, $\vec{N} = \frac{1}{\sqrt{648+576}} \times (-18\vec{i} - 18\vec{j} + 24\vec{k})$

$$= \frac{1}{6\sqrt{34}} (-18\vec{i} - 18\vec{j} + 24\vec{k})$$