$$f_y(x,y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{x(y+h)^2 - x^3(y+h) - [xy^2 - x^3y]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{xy^2} + 2xyh + xh^2 - \cancel{x^3y} - x^3h - \cancel{xy^2} + \cancel{x^3y}}{h}$$

$$= \lim_{h\to 0} 2xy + xh - x^3$$

$$\therefore f_y(x,y) = 2xy - x^3 \;\#$$

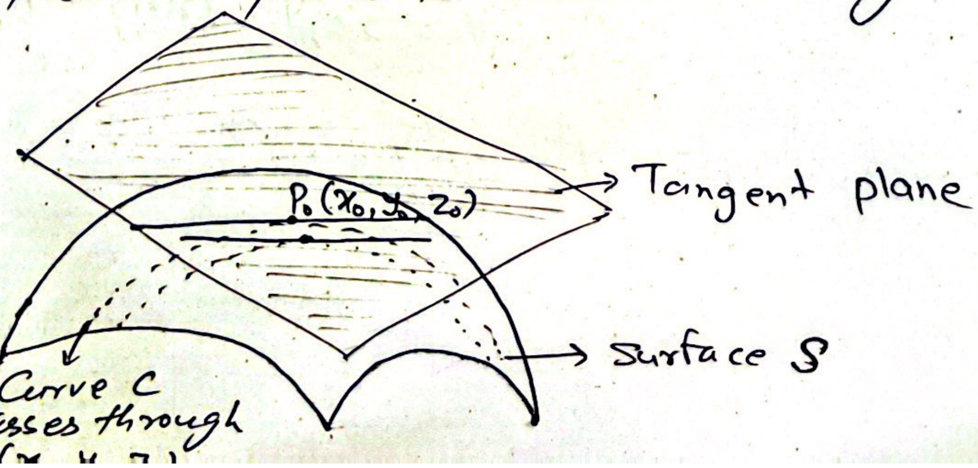
Equation of tangent line:

Let $y=f(x)$ be a curve in 2D and 'm' be the slope of tangent to the curve $y=f(x)$ at $x=a$: $f'(a)=m$ then, the eq^n of tangent line at the point $x=a$ is;

$$\boxed{y = f(a) + f'(a)(x-a)}$$

Tangent Plane:

Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 & lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the tangent plane to S at P_0.



$P_0 = (x_0, y_0)$ a point in the domain of f. Suppose that f has continuous partial derivatives. Then the tangent plane to S at P_0 is given by the equation.

$$\boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1,1,3)$.

Sol^n:

$z = 2x^2 + y^2$ at the point $(1,1,3) = (x_0, y_0, z_0)$

$$f_x = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(2x^2 + y^2) = 4x$$

$$f_y = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(2x^2 + y^2) = 2y$$

$\therefore$ The eq^n of tangent plane is;

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$\Rightarrow z = 2x_0^2 + y_0^2 + 4x_0(x - x_0) + 2y_0(y - y_0)$$

$$\Rightarrow z = 2\times 1^2 + 1^2 + 4\times 1(x - 1) + 2\times 1(y - 1)$$

$$\Rightarrow z = 3 + 4x - 4 + 2y - 2$$

$$\therefore z = 4x + 2y - 3 \ \#$$

Linear Approximation;

In the above example we found that an equation of the tangent plane to the graph of the function $f(x,y) = 2x^2 + y^2$ at the point $(1,1,3)$ is $z = 4x + 2y - 3$

Set,

$$L(x,y) = 4x + 2y - 3$$

This is a linear function in two variables.

For instance at a point $(1.1, 0.95)$ near to $(1,1)$ we have,

$f(x,y) = f(1.1, 0.95) = 2\times(1.1)^2 + (0.95)^2 = 3.3225$ but

$L(x,y) = L(1.1, 0.95) = 4\times 1.1 + 2\times 0.95 - 3 = 3.3$

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Clearly $f(x,y) \approx L(x,y)$

$\Rightarrow$ The function $L(x,y)$ is a good approximation to the function $f(x,y)$ at a point near to (x_0, y_0).

$\therefore$ The function $L(x,y)$ is called a <u>Linear approximation</u> or <u>tangent plane approximation</u> of $f(x,y)$ at a point near to (x_0, y_0).

In general if f has continuous partial derivatives, then the ~~expres~~ eq^n of a tangent plane to the graph of a function $f(x,y)$ at a point $(x_0, y_0, z_0) = (a, b, f(a,b))$ is given by the expression $L(x,y)$ as;

$$\boxed{L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

This expression is called the Linear approximation or tangent plane approximation of f at a point near (a,b). Also, $\boxed{L(x,y) \approx f(a,b) + \underbrace{\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}_{\text{Total differential } dz}}$

Differentiable functions

Let $z = f(x,y)$. We say that f is differentiable at a point (a,b) if Δz can be expressed in the form:

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Ex: Show that the function $f(x,y) = x^2 + 3y$ is differentiable at every point in the plane.

$\rightarrow$ $f(x,y) = x^2 + 3y$

$f_x = 2x$ & $f_y = 3$

Suppose Δx & Δy be increment in x & y respectively for the function $z = f(x,y)$ then Δz be the increment in z given as,

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$\Rightarrow \Delta z = (x+\Delta x)^2 + 3(y+\Delta y) - [x^2+3y]$$

$$\Rightarrow \Delta z = \cancel{x^2} + 2x\Delta x + \Delta x^2 + \cancel{3y} + 3\Delta y - \cancel{x^2} - \cancel{3y}$$

$$\Rightarrow \Delta z = f_x \Delta x + f_y \Delta y + \Delta x \, \Delta x + 0 \times \Delta y$$

where $\varepsilon_1 = \Delta x$ & $\varepsilon_2 = 0$ & $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

This satisfies the condition, hence, $f(x,y) = x^2 + 3y$ is a differentiable function at every point in the plane.

Show that $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$ & find ~~to~~ its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Sol^n:

Let $z = f(x,y) = xe^{xy}$ be a function. Let Δx & Δy be ~~increpre~~ increment in x & y respectively for the function $z = f(x,y)$ then Δz be the increment in z given by,

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$\Rightarrow \Delta z = (x+\Delta x)e^{(x+\Delta x)(y+\Delta y)} - xe^{xy}$$

$$\Rightarrow \Delta z = (x+\Delta x)e^{xy + x\Delta y + y\Delta x + \Delta x\Delta y} - xe^{xy}$$

$$\Rightarrow \Delta z = xe^{xy + x\Delta y + y\Delta y + \Delta x\Delta y} + \Delta x e^{xy + x\Delta y + y\Delta x + \Delta x\Delta y} - xe^{xy}$$

$$\Rightarrow \Delta z =$$

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Show that $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$ & find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

<u>Soln:</u>

$$f(x,y) = xe^{xy}$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xe^{xy}) = xe^{xy}.y + e^{xy}.1$$

$$\therefore f_x(1,0) = 1e^{1\times 0} \times 0 + e^{1\times 0}.1 = 1$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xe^{xy}) = xe^{xy}.x$$

$$\therefore f_y(1,0) = 1\times e^{1\times 0} \times 1 = 1$$

Both f_x & f_y are continuous functions so f is differentiable.

Now, the linearization at $(a,b) = (1,0)$ is;

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-b)$$

$$= 1 + 1(x-1) + 1(y-0)$$

$$= 1 + x - 1 + y$$

$\therefore L(x,y) = x+y$ is the linear approximation of $f(x,y) = xe^{xy}$

So, $f(1.1, -0.1) = (1.1)e^{(1.1)\times(-0.1)} = 0.9854$

& $L(2.1, -0.1) = 1.1 + (-0.1) = 1$

where, $L(1.1, -0.1) \approx f(1.1, -0.1)$ hence linear approximation approximates the function $f(x,y)$.

Theorem: Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} & f_{yx} are both continuous on D, then

$$\boxed{f_{xy}(a,b) = f_{yx}(a,b)}$$

Proof:

For small values of $h \neq 0$, consider the difference

$$\Delta(h) = [f(a+h, b+h) - f(a+h, b)] - [f(a, b+h) - f(a,b)]$$

let $g(x) = f(x, b+h) - f(x, b)$ then

$$\Delta(h) = g(a+h) - g(a) \quad \text{— (i)}$$

By mean value theorem, there is a number c between a & $a+h$ such that

$$g(a+h) - g(a) = g'(c)h = h[f_x(c, b+h) - f_x(c,b)] \quad \text{— (ii)}$$

Applying the MVT again to f_x, we get a number d between $b+h$ such that,

$$f_x(c, b+h) - f_x(c,b) = f_{xy}(c,d)h \quad \text{— (iii)}$$

From (i), (ii) & (iii);

$$\Delta(h) = h^2 f_{xy}(c,d)$$

If $h \to 0$ then $(c,d) \to (a,b)$, so the continuty of f_{xy} at (a,b) gives.

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = \lim_{(c,d) \to (a,b)} f_{xy}(c,d) = f_{xy}(a,b)$$

Similarly, by writing

$$\Delta(h) = [f(a+h, b+h) - f(a, b+h)] - [f(a+h, b) - f(a,b)]$$

& using the MVT twice & the continuity of f_{yx} at (a,b) we obtain

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a,b)$$

$$\Rightarrow f_{xy}(a,b) = f_{yx}(a,b) \quad \#$$

Theorem: If a function $z = f(x,y)$ is ... at a point, then it is continuous at the point.

Proof:

Assume that f is differentiable at a point (a,b). To prove that f is continuous at (a,b) we must show that,

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Let $\Delta x = x - a$, $\Delta y = y - b$ then,

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. However, by definition, we know that

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

setting $x = a + \Delta x$ & $y = b + \Delta y$ we get;

$$\begin{aligned}
\Delta z &= f(x,y) - f(a,b) \\
&= f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \\
&= f_x(a,b)(x-a) + f_y(a,b)(y-b) + \varepsilon_1(x-a) + \varepsilon_2(y-b)
\end{aligned}$$

Taking limit as $(x,y) \to (a,b)$, we have

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

f is continuous at (a,b).

...tal differential

$$\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$

If $z = f(x,y) = x^2 + 3xy - y^2$, find dz. If x changes from 2 to 2.05 & y changes from 3 to 2.96, compute the values of Δz & dz.

Soln.

$$z = f(x,y) = x^2 + 3xy - y^2$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$= (2x + 3y)dx + (3x - 2y)dy$$

$$= (2\times 2 + 3\times 3)(2.05 - 2) + (3\times 2 - 2\times 3)(2.96 - 3)$$

$$= 0.65$$

$$\boxed{\begin{aligned} x &= 2 \\ x + \Delta x &= 2.05 \\ \Delta x &= 0.05 \\ y &= 3 \\ y + \Delta y &= 2.96 \\ \Delta y &= -0.04 \end{aligned}}$$

$$\therefore \Delta z = f(x + \Delta x, y + \Delta y) - f(x,y)$$

$$= f(2.05, 2.96) - f(2,3)$$

$$= (2.05)^2 + 3\times 2.05\times 2.96 - (2.96)^2 - \left[2^2 + 3\times 2\times 3 - 3^2\right]$$

$$= 0.6449$$

$$\therefore \Delta z \approx dz$$

The base radius & height of a right circular cone are measured as 10 cm & 25 cm, respectively with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

Soln.

$$V = \frac{1}{3}\pi r^2 h$$

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh$$

$$dV = \frac{2}{3}\pi r h\,dr + \frac{\pi}{3}r^2\,dh$$

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$\Rightarrow dV = \frac{2\pi}{3} rh\, dr + \frac{\pi}{3} r^2 dh$$

Here $r = 10$ cm, $h = 25$ cm & since error is at most 0.1 cm so,

$|\Delta r| \leq 0.1$ & $|\Delta h| \leq 0.1$

To find the largest error in the measurement of r and h, take $dr = 0.1$ & $dh = 0.1$.

$$\therefore dV = \frac{2\pi}{3} \times 10 \times 25 \times 0.1 + \frac{\pi}{3} \times 10^2 \times 0.1$$

$$= \frac{500\pi}{3} \times 0.1 + \frac{100\pi}{3} \times 0.1$$

$$\approx 63 \text{ cm}^3 \#$$

Linear Approximation:

$$f(x,y,z) \approx L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

Differentiability:

Let $u = f(x,y,z)$

$$\Delta u = f_x(a,b,c)\Delta x + f_y(a,b,c)\Delta y + f_z(a,b,c)\Delta z + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z$$

Total differentials:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

The dimensions of a rectangular box are measured to be 75 cm, 60 cm & 40 cm. Each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

Soln:

length $(x) = 75$ cm
breadth $(y) = 60$ cm
height $(z) = 40$ cm
Error $= 0.2$ cm
$\therefore dx = dy = dz = 0.2$ cm

$\therefore V = x \times y \times z$

$$\therefore \Delta V = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$= yz\,dx + xz\,dy + xy\,dz$$

$$= 60 \times 40 \times 0.2 + 75 \times 40 \times 0.2 + 75 \times 60 \times 0.2$$

$$= 1980 \text{ cm}^3$$

Thus an error of only 0.2 cm in measuring each dimension could lead to an error of as much as 1980 cm^3 in the calculated volume. This may seem like a large error, but it's only 1.1% of the volume of the box.

Chain rule:- Let $x = x(t)$, $y = y(t)$ & $z = f(x, y)$ be differentiable functions. Then $z = f(x(t), y(t))$ is a differentiable function of t &

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

where the ordinary derivative are evaluated at t & the partial derivatives are evaluated at (x, y).

Proof: A change of Δt in t produces changes of Δx in x & Δy in y. These, in turn produce a change of Δz in z. Since $z = f(x, y)$ is differentiable, we have

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \quad \text{---} \ (i)$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Dividing both sides of this equation by Δt, we have

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t} \quad \text{---} \ (ii)$$

If $\Delta t \to 0$ then, $\Delta x = x(t + \Delta t) - x(t) \to 0$

Since, x is differentiable so, it is continuous also.

Similarly,

If $\Delta t \to 0$ then $\Delta y = y(t + \Delta t) - y(t) \to 0$

So, from (i) at $\Delta t \to 0$, $\Delta x \to 0$ & $\Delta y \to 0$ which gives $\varepsilon_1, \varepsilon_2 \to 0$

Now taking $\lim_{\Delta t \to 0}$ on (ii);

$$\lim_{\Delta t \to 0}\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\lim_{\Delta t \to 0}\frac{\Delta y}{\Delta t} + 0 + 0$$

$$\therefore \frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ & $y = \cos t$, find dz/dt when $t=0$.

Soln:

$$z = x^2y + 3xy^4, \quad x = \sin 2t, \; y = \cos t$$

$$\therefore \left.\frac{dz}{dt}\right|_{t=0} = \left.\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right|_{t=0}$$

$$= \left.(2xy + 3y^4)\times(2\cos 2t) + (x^2+12xy^3)(-\sin t)\right|_{t=0}$$

$$= (2xy + 3y^4)(2\cos 2\times 0) - (x^2 + 12xy^3)\times(\sin 0)$$

$$= 4xy + 6y^4 - 0$$

$$= 4xy + 6y^4 \;\#$$

Chain rule - II : Let $x = x(s,t)$, $y = y(s,t)$ & $z = f(x,y)$ be differentiable functions. Then $z = f(x(s,t), y(s,t))$ be a differentiable function of s, t.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

If $z = e^x \sin y$, where $x = st^2$, $y = s^2t$, find $\partial z/\partial s$, $\partial z/\partial t$

Soln:

$$z = e^x \sin y, \quad x = st^2, \; y = s^2t$$

$$\therefore \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)t^2 + (e^x\cos y)\,2st$$

$$\therefore \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)\,2st + (e^x\cos y)s^2$$

Write out the chain Rule for the case, where $w = f(x, y, z, t)$ & $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ & $t = t(u,v)$.

$\rightarrow$ $\cancel{\frac{\partial w}{\partial}}$ $\cancel{\frac{dw}{dx} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial x}\frac{\partial x}{\partial x}}$

$$\rightarrow \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\rightarrow \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

If $g(s,t) = f(s^2-t^2, t^2-s^2)$ & f is differentiable, show that g satisfies the equation:

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

Soln:

$$g(s,t) = f(s^2-t^2, t^2-s^2) = f(u,v)$$

where, $u = s^2 - t^2$ & $v = t^2 - s^2$

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial s}$$

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial u}(2s) + \frac{\partial f}{\partial v}(-2s)$$

$$\Rightarrow \frac{\partial g}{\partial s} = 2s\frac{\partial f}{\partial u} - 2s\frac{\partial f}{\partial v} \quad \text{—— (i)}$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} = \frac{\partial f}{\partial u} \cdot (-2t) + \frac{\partial f}{\partial v} 2t$$

$$\Rightarrow \frac{\partial g}{\partial t} = -2t\frac{\partial f}{\partial u} + 2t\frac{\partial f}{\partial v} \quad \text{—— (ii)}$$

To prove,

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

$$L.H.S = t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t}$$

$$= t\left[2s\frac{\partial f}{\partial u} - 2s\frac{\partial f}{\partial v}\right] + s\left[-2t\frac{\partial f}{\partial u} + 2t\frac{\partial f}{\partial v}\right]$$

$$= 2st\frac{\partial f}{\partial u} - 2st\frac{\partial f}{\partial v} - 2st\frac{\partial f}{\partial u} + 2st\frac{\partial f}{\partial v}$$

$$= 0$$

$= R.H.S$ proved //

\# If $z = f(x,y)$ & f has continuous 2nd order partial derivatives & $x = r^2 + s^2$, $y = 2rs$, find $\frac{\partial z}{\partial r}$, $\frac{\partial^2 z}{\partial r^2}$.

Soln:

$$z = f(x,y), \quad x = r^2 + s^2, \quad y = 2rs$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial z}{\partial x}\cdot 2r + \frac{\partial z}{\partial x}\cdot 2s$$

$$\therefore \frac{\partial z}{\partial r} = 2\frac{\partial z}{\partial x}(r+s)$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left[2\frac{\partial z}{\partial x}(r+s)\right] =$$

$$\frac{\partial^2 z}{\partial r^2} = 2\frac{\partial}{\partial r}\left[\left(\frac{\partial z}{\partial x}\cdot r\right) + \left(\frac{\partial z}{\partial x}\cdot s\right)\right] = 2\left[\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cdot r\right) + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cdot s\right)\right]$$

$$\frac{\partial^2 z}{\partial r^2} = 2\left[\left[\frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial r}(r) + r\cdot\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)\right] + \frac{\partial^2 z}{\partial r\partial}\right.$$

If $z = f(x,y)$ & f has continuous second order partial derivatives & $x = r^2 + s^2$, $y = 2rs$. Find $\frac{\partial z}{\partial r}$ & $\frac{\partial^2 z}{\partial r^2}$

Soln:

$z = f(x, y)$, $x = r^2 + s^2$, $y = 2rs$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot 2r + \frac{\partial z}{\partial y}\,2s \; \#$$

Now,

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)$$

$$= \frac{\partial}{\partial r}\left[\frac{\partial z}{\partial x}\cdot 2r + \frac{\partial z}{\partial y}\,2s\right]$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cdot 2r\right) + \frac{\partial}{\partial r}\left[\frac{\partial z}{\partial y}\cdot 2s\right]$$

$$= \left[\frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial r}(2r) + 2r\cdot\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)\right] + \frac{\partial^2 z}{\partial r\,\partial y}\cdot 2s$$

$$= \frac{\partial z}{\partial x}\cdot 2 + 2r\,\frac{\partial^2 z}{\partial r\,\partial x} + \frac{\partial^2 z}{\partial r\,\partial y}\cdot 2s$$

$$= 2\left[\frac{\partial z}{\partial x} + r\,\frac{\partial^2 z}{\partial r\,\partial x} + s\,\frac{\partial^2 z}{\partial r\,\partial y}\right] \; \#$$

Implicit function - I

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

for $f(x,y)$ where $y = f(x)$

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Find $\frac{dy}{dx}$ if $x^3+y^3 = 6xy$.

Soln:

$x^3+y^3 = 6xy$

diff. w.r.to x,

$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy)$

$3x^2+3y^2\frac{dy}{dx} = 6\left(x\cdot\frac{dy}{dx}+y\cdot 1\right)$

$\Rightarrow 3x^2+3y^2\frac{dy}{dx} = 6x\frac{dy}{dx}+6y$

$\Rightarrow 3x^2-6y = (6x-3y^2)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{3(x^2-2y)}{3(2x-y^2)}$

$\Rightarrow \frac{dy}{dx} = \frac{x^2-2y}{2x-y^2}$ #

$x^3+y^3-6xy=0$

$F_x = 3x^2-6y$

$F_y = 3y^2-6x$

$\therefore \frac{dy}{dx} = \frac{-F_x}{F_y}$

$= \frac{-(3x^2-6y)}{3y^2-6x}$

$= \frac{3(2y-x^2)}{3(y^2-2x)}$

$= \frac{2y-x^2}{y^2-2x}$

$\frac{dy}{dx} = \frac{x^2-2y}{2x-y^2}$ #

Implicit function - II :

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz = 1$.

Soln: $x^3+y^3+z^3+6xyz = 1$

~~diff. w.r.to x;~~ ~~$3x^2+3y^2\frac{dy}{d}$~~

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-(3x^2+6yz)}{3z^2+6xy}$

$\frac{\partial z}{\partial x} = \frac{-(x^2+2yz)}{z^2+2xy}$ #

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{-(3y^2+6xz)}{3z^2+6xy} = \frac{-(y^2+2xz)}{(z^2+2xy)}$ #

Directional Derivative :

The directional derivative of f at (x_0, y_0) in the direction of a unit vector $u = (a, b)$ is

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if the limit exists.

Theorem : If f is a differentiable function of x & y, then f has a directional derivative at (x_0, y_0) in the direction of a unit vector $u = (a, b)$ &

$$\boxed{D_u f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b}$$

Proof:

For a fixed point (x_0, y_0) let,

$$\left.\begin{matrix} x = x_0 + ha, & y = y_0 + hb \end{matrix}\right\} \quad (i)$$

Diff both x & y w.r. to h, we get;

$$x'(h) = a, \quad y'(h) = b$$

Let $g(h) = f(x_0 + ha, y_0 + hb) = f(x, y)$

$\because f$ is differentiable so we can apply chain rule as;

$$g'(h) = f_x(x, y)\, x'(h) + f_y(x, y)\, y'(h)$$

$$g'(h) = f_x(x, y)\, a + f_y(x, y)\, b$$

If $h = 0$ then, from (i); $x = x_0$ & $y = y_0$ so,

$$g'(0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$

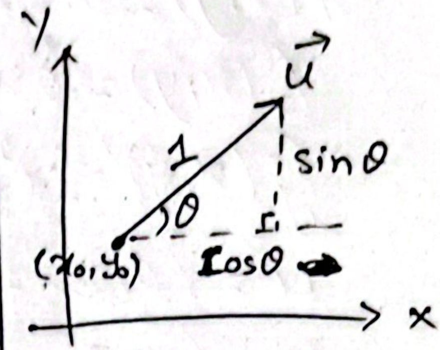
$$\therefore g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

$$= D_u f(x_0, y_0)$$

$$\therefore D_u f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b \quad \#$$

Find the directional derivative $D_u f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$ and u is the unit vector given by angle $\theta = \pi/6$. What is $D_u f(1,2)$?



$\vec{u} = (a,b) = (\cos\theta, \sin\theta)$

$\cos\theta = \frac{b}{h} = \frac{b}{1} = b$

$\sin\theta = \frac{p}{h} = \frac{p}{1} = p$

Sol^n: $f(x,y) = x^3 - 3xy + 4y^2$

$f_x(x,y) = 3x^2 - 3y$

$f_y(x,y) = 8y - 3x$

$\theta = \pi/6$ so,

$\vec{u} = (a,b) = (\cos\theta, \sin\theta)$

$\Rightarrow (a,b) = (\cos\pi/6, \sin\pi/6)$

$\Rightarrow (a,b) = (\sqrt{3}/2, 1/2)$

$$\therefore D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$$

$$= (3x^2 - 3y) \times \frac{\sqrt{3}}{2} + (8y - 3x)\frac{1}{2}$$

$$= \frac{3\sqrt{3}x^2 - 3\sqrt{3}y + 8y - 3x}{2}$$

$$= \frac{3x(\sqrt{3}x - 1) + y(8 - 3\sqrt{3})}{2}$$

Now,

$$D_u f(1,2) = \frac{3\times1(\sqrt{3}\times1 - 1) + 2(8 - 3\sqrt{3})}{2}$$

$$= \frac{3\sqrt{3} - 3 + 16 - 6\sqrt{3}}{2}$$

$$= \frac{13 - 3\sqrt{3}}{2} \; \#$$

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Find the directional derivative of $f(x, y) = x^2 \sin 2y$ at $(1, \pi/2)$ in the direction of $u = 3\vec{i} - 4\vec{j}$.

Soln:

$$f(x,y) = x^2 \sin 2y$$

$$f_x(x,y) = 2x \sin 2y$$

$$f_y(x,y) = 2x^2 \cos 2y$$

$$u = (a, b) = \cancel{(\cos\theta, \sin\theta)} = (1, \pi/2)$$

$$\cancel{u = (\cos\tfrac{\pi}{2}, \sin\tfrac{\pi}{2}) = (0, 1)}$$

$$\therefore \cancel{D_u f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b}$$

$$= \cancel{2x \sin 2y \times 0 + 2x^2 \cos 2y \times 1}$$

$$\therefore \cancel{D_u f(x,y) = 2x^2 \cos 2y}$$

$$\cancel{D_u f(3, -4) = 2 \times 3^2 \cos(2 \times (-4)) = 17.824} \#$$

$$\therefore \cancel{D_u f(x,y) = f_x(x,y)\, a} + f_y(x,y)\, b$$

$$= \cancel{2x \sin 2y \times}$$

Find the directional derivative of $f(x,y) = x^2 \sin 2y$ at $(1, \pi/2)$ in the direction of $v = 3\vec{i} - 4\vec{j}$.

Sol^n: $f(x,y) = x^2 \sin 2y$

$f_x(1, \pi/2) = 2x \sin 2y = 2\times 1 \sin 2\times\frac{\pi}{2} = 2\times 0 = 0$

$f_y(1, \pi/2) = 2x^2 \cos 2y = 2\times 1^2 \times \cos 2\times\frac{\pi}{2} = 2\times(-1) = -2$

Here, $v = 3\vec{i} - 4\vec{j}$. Let, $\vec{u}$ be a unit vector along the direction of v, $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{3\vec{i} - 4\vec{j}}{\sqrt{3^2 + (-4)^2}} = \left(\frac{3}{5}, -\frac{4}{5}\right)$

$\Rightarrow (a,b) = \left(\frac{3}{5}, \frac{-4}{5}\right)$

$\therefore D_u f(x,y) = f_x(x,y)\times a + f_y(x,y)\times b$

$\Rightarrow D_u f(1, \frac{\pi}{2}) = f_x(1, \frac{\pi}{2})\times\frac{3}{5} + f_y(1, \pi/2)\times\left(\frac{-4}{5}\right)$

$= 0\times 3 + (-2)\times\left(\frac{-4}{5}\right)$

$= \frac{8}{5}$ #

Gradient:

The gradient is a vector function f denoted by ∇f or grad(f) & defined by,

$$\boxed{\nabla f(x,y) = \left(f_x(x,y), f_y(x,y)\right) = f_x\vec{i} + f_y\vec{j}}$$

Directional Derivative expressed in terms of gradient.

$D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$

$= \left(f_x(x,y), f_y(x,y)\right)\cdot(a,b)$

$$\therefore \boxed{D_u f(x,y) = \nabla f \cdot \vec{u}}$$

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Find ∇f if $f(x,y) = \sin x + e^{xy}$. What is $\nabla f(0,1)$?

Sol^n:

$$f(x,y) = \sin x + e^{xy}$$

$$\nabla f(x,y) = f_x \vec{i} + f_y \vec{j}$$

$$\cancel{\nabla f(x,y) = \cos x \vec{i} + (\cos x + ye^{xy}) \vec{i} + e^{y}}$$

$$\nabla f(x,y) = (\cos x + ye^{xy})\vec{i} + (xe^{xy})\vec{j}$$

$$\therefore \nabla f(0,1) = (\cos 0 + 1e^{0\times 1})\vec{i} + (0e^{0\times 1})\vec{j}$$

$$= 2\vec{i} + 1\vec{j}$$

$$= (2,1) \#$$

(Using gradient vector to find the directional derivative). Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ in the direction of the vector $V = 2\vec{i} + 5\vec{j}$.

Sol^n:

$$f(x,y) = x^2y^2 - 4y$$

$$f_x(x,y) = 2xy^2 \; ; \; f_x(2,-1) = 2\times 2\times(-1)^2 = 4$$

$$f_y(x,y) = 2x^2y - 4 \; ; \; f_y(2,-1) = 2\times(2)^2\times(-1) - 4 = -12$$

$$\therefore \nabla f(x,y) = (f_x(x,y), f_y(x,y))$$

$$\nabla f(2,-1) = (4, -12)$$

$$\vec{v} = 2\vec{i} + 5\vec{j}$$

The unit vector $\vec{u}$ in the direction of $\vec{v}$ is;

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2\vec{i} + 5\vec{j}}{\sqrt{2^2+5^2}} = \frac{2}{\sqrt{29}}\vec{i} + \frac{5}{\sqrt{29}}\vec{j} = \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$$

∴ The directional derivative of $f(x,y)$ at point $(2,-1)$ in the direction of unit vector $\vec{u}$ is;

$$D_u f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

$$D_u f(2,-1) = \nabla f(2,-1) \cdot \vec{u}$$

$$= (4,-12) \cdot \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$$

$$= 4 \times \frac{2}{\sqrt{29}} + (-12) \times \frac{5}{\sqrt{29}}$$

$$= \frac{-52}{\sqrt{29}} \text{ //}$$

Directional derivative of $f(x,y,z)$ in the direction of unit vector $\vec{u} = (a,b,c)$.

$$D_u f(x,y,z) = f_x(x,y,z) \times a + f_y(x,y,z) \times b + f_z(x,y,z) \times c$$

Gradient of $f(x,y,z)$

$$\nabla f(x,y,z) = \left(f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z)\right)$$

$$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

Directional derivative of $f(x,y,z)$ in the direction of unit vector $\vec{u} = (a,b,c)$ using gradient.

$$D_u f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}$$

Find ∇f and $D_u f(x,y,z)$ at $(1,3,0)$ in the direction of $\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$, where $f(x,y,z) = x\sin yz$

Solⁿ:

$$f(x,y,z) = x \sin yz$$

$$\nabla f(x,y,z) = (f_x, f_y, f_z) = (\sin yz, zx\cos yz, xy\cos yz)$$

$$\nabla f(1,3,0) = (\sin(3\times 0), 0\times 1 \cos(3\times 0), 1\times 3\times \cos(3\times 0))$$

$$= (0, 0, 3)$$

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Let $\vec{u}$ be the unit vector in the direction of

$\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$

$$\therefore \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{i}+2\vec{j}-\vec{k}}{\sqrt{1^2+2^2+(-1)^2}} = \frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} - \frac{1}{\sqrt{6}}\vec{k} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$$

$$\therefore D_u f(1,3,0) = \nabla f(1,3,0) \cdot \vec{u}$$

$$= (0,0,3) \cdot \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$$

$$= 0 \times \frac{1}{\sqrt{6}} + 0 \times \frac{2}{\sqrt{6}} + 3 \times \left(-\frac{1}{\sqrt{6}}\right)$$

$$= \frac{-3}{\sqrt{6}} \#$$

Let $f(x,y) = xe^y$

a) Find the directional derivative of f at $P(2,0)$ in the direction from P to $Q(1/2, 2)$.

b) In what direction does f have the maximum rate of change? What is the maximum rate of change?

Soln:

$f(x,y) = xe^y$

$f_x = e^y$ & $f_y = xe^y$

$\nabla f(x,y) = (f_x, f_y)$

$\nabla f(2,0) = (e^0, 2 \times e^0) = (1, 2)$

Now, the vector $\vec{PQ} = \vec{OQ} - \vec{OP} = (1/2, 2) - (2, 0) = (-3/4, 2)$

Let $\vec{u}$ be the unit vector along $\vec{PQ}$ then,

$$\vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{(-3/4, 2)}{\sqrt{(-3/4)^2 + (2)^2}} = \left(\frac{-3}{\sqrt{73}}, \frac{8}{\sqrt{73}}\right)$$

$\therefore$ The directional derivative of f at $P(2,0)$ in the direction from P to Q i.e along unit vector $\vec{u}$ is,

$$D_u f(2,0) = \nabla f(2,0) \cdot \vec{u}$$

$$= \left(-\tfrac{3}{4}, 2\right) \cdot \left(\frac{-3}{\sqrt{73}}, \frac{8}{\sqrt{73}}\right)$$

$$\cancel{= \left(\frac{9}{4\sqrt{73}}, \frac{16}{\sqrt{73}}\right)}$$

$$= \frac{9}{4\sqrt{73}} + \frac{16}{\sqrt{73}}$$

$$= \frac{9 + 64}{4\sqrt{73}} = \frac{73}{4\sqrt{73}} = \frac{\sqrt{73}}{4} = 2.136$$

(b) The function f always have the maximum rate of change in the direction of ∇f ~~i.e. from P(2,0) to Q~~

$\therefore D_u f(x,y) = \nabla f \cdot \vec{u}$

where $\vec{u}$ is the unit vector along the direction of ∇f ~~$\vec{PQ} = \vec{OQ} - \vec{OP} = (1,2) - (2,0) = (1,$~~

$\nabla f = (1,2)$ so,

i.e. $\vec{u} = \dfrac{(+1, 2)}{\sqrt{(+1)^2 + 2^2}} = \left(\dfrac{+1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$

$$\therefore D_u f(x,y) = (1,2) \cdot \left(\frac{+1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$= \frac{+1}{\sqrt{5}} + \frac{4}{\sqrt{5}}$$

$$= \frac{5}{\sqrt{5}}$$

$$= \sqrt{5} = 2.236 \text{ (Max value)}$$

Suppose that the temperature in degrees Celsius on the surface of a metal plate is given by $T(x,y) = 20-4x^2-y^2$, where x & y are measured in centimeters. In what direction from $(2,-3)$ does the temperature increase most rapidly? What is the rate of increase?

Soln:

$T(x,y) = 20-4x^2-y^2$

$f_x = -8x$ & $f_y = -2y$

$\therefore \nabla T(x,y) = (f_x, f_y) = (-8x, -2y)$.

At point $(2,-3)$,

$\nabla T(2,-3) = (-8\times 2, -2\times(-3)) = (-16, 6)$

Now, the unit vector in the direction of $(-16,6)$ is given as,

$$\frac{(-16,6)}{\sqrt{(-16)^2+6^2}} = \frac{1}{\sqrt{292}}(-16,6)$$

This is the direction from $(2,-3)$ in which the temperature increases most rapidly.

$\therefore$ Rate of increase $= |\nabla T(2,-3)| = \sqrt{(-16)^2+6^2} = \sqrt{292}$

$= 17.08$ °c/cm

Find the equations of tangent plane & normal line at point $(-2,1,-3)$ to the ellipsoid $\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$

Soln:

The eqn of ellipsoid : $\frac{x^2}{4}+y^2+\frac{z^2}{9}-3=0$ at point $(-2,1,-3)$

$f_x(-2,1,-3) = \frac{2x}{4} = \frac{2\times(-2)}{4} = -1$

$f_y(-2,1,-3) = 2y = 2\times 1 = 2$

$f_z(-2,1,-3) = \frac{2z}{9} = \frac{2\times(-3)}{9} = -2/3$

: The eq

$$\boxed{f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0}$$

$$-1(x-(-2)) + 2(y-1) + \left(-\frac{2}{3}\right)(z-(-3)) = 0$$

$$-x - 2 + 2y - 2 - \frac{2z}{3} - 6 = 0$$

$$\Rightarrow -3x + 6y - 2z - 18 = 0 \ \#$$

Also,

The eq^n of normal line is;

$$\boxed{\frac{x-x_0}{f_x} = \frac{y-y_0}{f_y} = \frac{z-z_0}{f_z}}$$

$$\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{(-2/3)} \ \#$$

Critical Point : A critical point (a,b) of a function $f = f(x,y)$ is a point in the domain of f at which $f_x(a,b) = f_y(a,b) = 0$, or $f_x(a,b)$ or $f_y(a,b)$ fails to exist.

Stationary point : A stationary point (a,b) of a function $f = f(x,y)$ is a specific type of critical point in the domain of f where $f_x(a,b) = f_y(a,b) = 0$.

Saddle point : Given the function $z = f(x,y)$, the point $(a,b,f(a,b))$ is a saddle point if there are two distinct vertical planes through this point such that the intersection of the surface with one of the planes has a relative maximum at (a,b) & the intersection with the other has a relative minimum at (a,b).

Second Derivative Test :-

Suppose the 2^{nd} partial derivatives of f are continuos on a disk with center (a,b) & suppose that $f_x(a,b)=0$ & $f_y(a,b)=0$. Let D be the quantity defined by Hessian matrix's determinant

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 \qquad [\because f_{xy} = f_{yx}]$$

$$\boxed{D = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2}$$

1. If $D>0$ & $f_{xx}(a,b)<0 \rightarrow f$ has local max at (a,b).
2. If $D>0$ & $f_{xx}(a,b)>0 \rightarrow f$ has local min at (a,b).
3. If $D<0 \rightarrow f$ has saddle point at (a,b).
4. If $D=0 \longrightarrow$ No information about f with (a,b).

Ex: Find the local maximum & minimum values & saddle point of $f(x,y)=x^4+y^4-4xy+1$

Soln:

$$f(x,y) = x^4+y^4-4xy+1$$
$$f_x(x,y) = 4x^3-4y \quad \& \quad f_y(x,y) = 4y^3-4x$$

For critical points,

$$f_x(x,y)=0 \quad \& \quad f_y(x,y)=0$$
$$4x^3-4y=0 \quad \& \quad 4y^3-4x=0$$
$$x^3=y \text{ --- (i)} \quad \& \quad y^3=x \text{ --- (ii)}$$

From (i) & (ii);

$$y^9=y$$
$$\Rightarrow y^9-y=0$$

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$\Rightarrow y(y^4+1)(y^4-1)=0$

$\Rightarrow y(y^4+1)(y^2+1)(y^2-1)=0$

$\Rightarrow y(y^4+1)(y^2+1)(y+1)(y-1)=0$

$\therefore y = 0, +1, -1$ & other are complex roots.

from (ii);

$x = 0, +1, -1$

$\therefore$ The critical points are $(0,0), (1,1)$ & $(-1,-1)$.

$f_{xx}(x,y) = 12x^2$, $f_{yy}(x,y) = 12y^2$, $f_{xy}(x,y) = -4$

~~fxx~~

$\therefore \cancel{D = f_{xx}(a,b) f_{yy}(a,b}$

$\therefore D = f_{xx} f_{yy} - f_{xy}^2$

$= 12x^2 \, 12y^2 - (-4)^2$

$D = 144x^2y^2 - 16$

At $(0,0)$, $D = -16$ ~~& $f_{xx}(0,0) = 0$~~

$\rightarrow$ f has saddle point at $(0,0)$.

At $(1,1)$,

$D = 128 > 0$, $f_{xx}(1,1) = 12 > 0$

$\rightarrow$ f has local minimum at $(1,1)$.

At $(-1,-1)$,

$D = 128 > 0$, $f_{xx}(-1,-1) = 12 > 0$

$\rightarrow$ f has local min^m at $(-1,-1)$.

$\therefore f(1,1) = f(-1,-1) = -1$ # (min^m value)

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Investigate critical points & find local min values if exists.

$f(x,y) = x^2 + y^2 - 2x - 6y + 14.$

Soln.

$f(x,y) = x^2 + y^2 - 2x - 6y + 14$

$f_x = 2x - 2, \quad f_y = 2y - 6$

For critical points;

$f_x = 0$ & $f_y = 0$

$2x - 2 = 0$ & $2y - 6 = 0$

$\Rightarrow x = 1$ & $y = 3$

$\therefore$ the critical point is $(1,3)$.

$f_{xx} = 2, \quad f_{yy} = 2$ & $f_{xy} = 0$

$\therefore D = f_{xx} f_{yy} - (f_{xy})^2 = 2 \times 2 - 0^2 = 4$

At $(1,3)$, $D = 4 > 0$ & $f_{xx}(1,3) = 2 > 0$

$\rightarrow f$ has local minm at $(1,3)$

$\therefore f(1,3) = 1^2 + 3^2 - 2\times1 - 6\times3 + 14 = 4$ (minm value)

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Unit-4

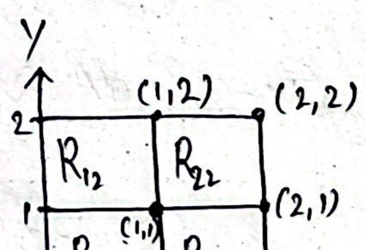
Multiple Integrals

 # Double Integrals: If $f(x,y) \geq 0$ then the volume V of the solid that lies above the rectangle R & below the surface $z = f(x,y)$ is

$$\boxed{\text{Volume} = \iint_R f(x,y)\, dA}$$

$$\boxed{\text{Double Riemann Sum} = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_i^*)\, \Delta A}$$

Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ & below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. [Divide R into 4 equal squares & choose the sample point to be the upper right corner of each square. Sketch the solid & the approximating rectangular boxes.]



Sol^n:

$f(x,y) = 16 - x^2 - 2y^2$

The area of each square $\Delta A = 1$

Approximating the volume by Riemann Sum with $m = n = 2$,

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_i)\, \Delta A$$

$$= f(1,1)\Delta A + f(1,2)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A$$

$$= 13 \times 1 + 7 \times 1 + 10 \times 1 + 4 \times 1$$

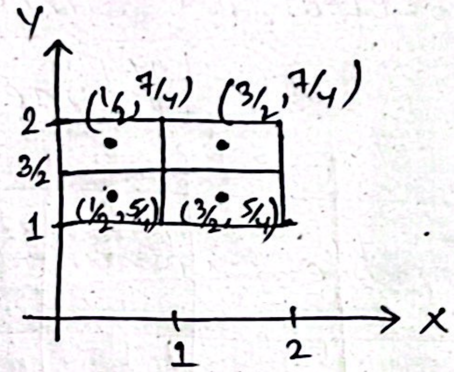
$$= 34 \#$$

Midpoint Rule for Double Integrals

$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\,\Delta A$ where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$

Use the Midpoint Rule with $m=n=2$ to estimate the value of the integral $\iint_R (x-3y)^2\,dA$, where $R=\{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$

Soln:



$\Delta x = 1-0 = 2-1 = 1$

$\Delta y = \frac{3}{2}-1 = 2-\frac{3}{2} = \frac{1}{2}$

$\therefore \Delta A = \Delta x \times \Delta y = 1 \times \frac{1}{2} = \frac{1}{2}$

$(\bar{x}_1, \bar{y}_1) = \left(\frac{1+0}{2}, \frac{3/2+1}{2}\right) = \left(\frac{1}{2}, \frac{5}{4}\right)$

$(\bar{x}_2, \bar{y}_1) = \left(\frac{2+1}{2}, \frac{3/2+1}{2}\right) = \left(\frac{3}{2}, \frac{5}{4}\right)$

$(\bar{x}_1, \bar{y}_2) = \left(\frac{1+0}{2}, \frac{2+3/2}{2}\right) = \left(\frac{1}{2}, \frac{7}{4}\right)$

$(\bar{x}_2, \bar{y}_2) = \left(\frac{2+1}{2}, \frac{2+3/2}{2}\right) = \left(\frac{3}{2}, \frac{7}{4}\right)$

$\therefore \iint_R (x-3y^2)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\,\Delta A$

$= f(\bar{x}_1,\bar{y}_1)\Delta A + f(\bar{x}_1,\bar{y}_2)\Delta A + f(\bar{x}_2,\bar{y}_1)\Delta A + f(\bar{x}_2,\bar{y}_2)\Delta A$

$= \Delta A\left[\left[\frac{1}{2} - 3\times\left(\frac{5}{4}\right)^2\right] + \left[\frac{1}{2} - 3\times\left(\frac{7}{4}\right)^2\right] + \left[\frac{3}{2} - 3\times\left(\frac{5}{4}\right)^2\right] + \left[\frac{3}{2} - 3\times\left(\frac{7}{4}\right)^2\right]\right]$

$= \frac{1}{2}\left[4 - 3\left(\frac{25}{16}\times 2 + \frac{49}{16}\times 2\right)\right]$

$= -\frac{99}{8} = -12.875$

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$$\iint_R [f(x,y) + g(x,y)]\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$$

$$\iint_R c\, f(x,y)\, dA = c \iint_R f(x,y)\, dA$$

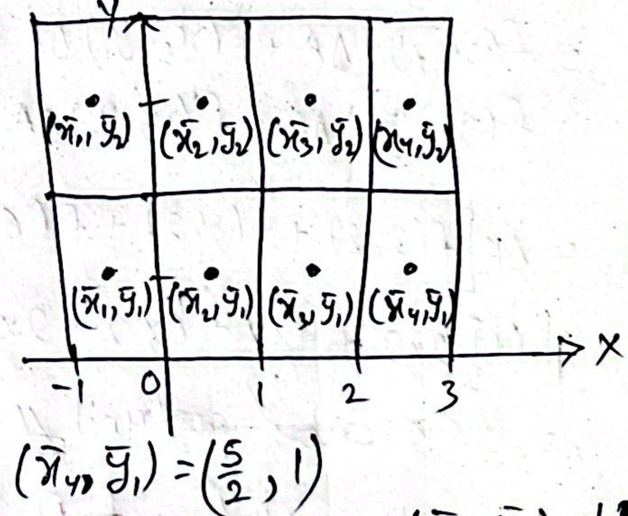
If $f(x,y) \geq g(x,y)\ \forall\ (x,y) \in R$, then

$$\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA$$

Assignment on Double Integrals

\# Use the Midpoint Rule to estimate the volume under $f(x,y) = x^2 + y$ & above the rectangle given by $-1 \leq x \leq 3,\ 0 \leq y \leq 4$ in the xy-plane. Use 4 subdivisions in the x-direction & 2 subdivisions in the y-direction.

Sol^n:



$\Delta x = 0-(-1) = 1-0 = 2-1 = 3-2 = 1$

$\Delta y = 4-2 = 2-0 = 2$

$\therefore \Delta A = \Delta x \times \Delta y = 1 \times 2 = 2$

$(\bar{x}_1, \bar{y}_1) = \left(\frac{-1+0}{2}, \frac{2+0}{2}\right) = \left(-\frac{1}{2}, 1\right)$

$(\bar{x}_2, \bar{y}_1) = \left(\frac{1}{2}, 1\right)$, $(\bar{x}_3, \bar{y}_1) = \left(\frac{3}{2}, 1\right)$, $(\bar{x}_4, \bar{y}_1) = \left(\frac{5}{2}, 1\right)$

$(\bar{x}_1, \bar{y}_2) = \left(-\frac{1}{2}, 3\right)$, $(\bar{x}_2, \bar{y}_2) = \left(\frac{1}{2}, 3\right)$, $(\bar{x}_3, \bar{y}_2) = \left(\frac{3}{2}, 3\right)$, $(\bar{x}_4, \bar{y}_2) = \left(\frac{5}{2}, 3\right)$

$$\therefore \iint_R (x^2+y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

$$= f(\bar{x}_1, \bar{y}_1)\Delta A + f(\bar{x}_1, \bar{y}_2)\Delta A + f(\bar{x}_2, \bar{y}_2)\Delta A + f(\bar{x}_2, \bar{y}_1)\Delta A + f(\bar{x}_3, \bar{y}_1)\Delta A + f(\bar{x}_3, \bar{y}_2)\Delta A + f(\bar{x}_4, \bar{y}_1)\Delta A + f(\bar{x}_4, \bar{y}_2)\Delta A$$

$$= \Delta A\left[\left(-\frac{1}{2}\right)^2 + 1 + \left(-\frac{1}{2}\right)^2 + 3 + \left(\frac{1}{2}\right)^2 + 3 + \left(\frac{1}{2}\right)^2 + 1 + \left(\frac{3}{2}\right)^2 + 1 + \left(\frac{3}{2}\right)^2 + 3 + \left(\frac{5}{2}\right)^2 + 1 + \left(\frac{5}{2}\right)^2 + 3\right] = 68 \text{ cubic units}$$

Properties of Double Integral :-

1. $\iint_R [f(x+y) + g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

2. $\iint_R c\,f(x,y)\,dA = c\iint_R f(x,y)\,dA$

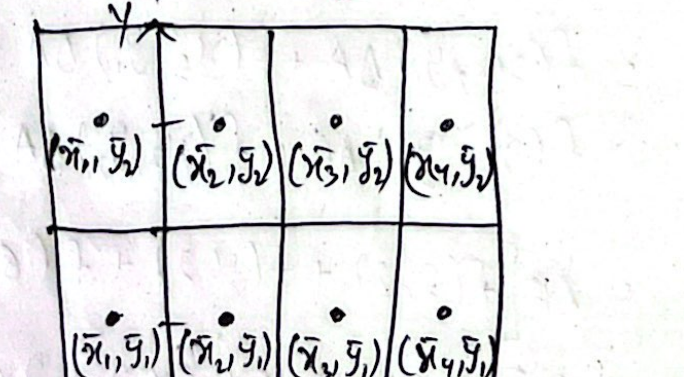
3. If $f(x,y) \geq g(x,y)\ \forall\ (x,y) \in R$. then,

   $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$

Assignment on Double Integrals

Use the Midpoint Rule to estimate the volume under $f(x,y) = x^2+y$ & above the rectangle given by $-1 \leq x \leq 3,\ 0 \leq y \leq 4$ in the xy-plane. Use 4 subdivisions in the x-direction & 2 subdivisions in the y-direction.

Solⁿ:



$\Delta x = 0-(-1) = 1-0 = 2-1 = 3-2 = 1$

$\Delta y = 4-2 = 2-0 = 2$

$\therefore \Delta A = \Delta x \times \Delta y = 1\times 2 = 2$

$(\bar{x}_1,\bar{y}_1) = \left(\frac{-1+0}{2}, \frac{2+0}{2}\right) = \left(-\frac{1}{2}, 1\right)$

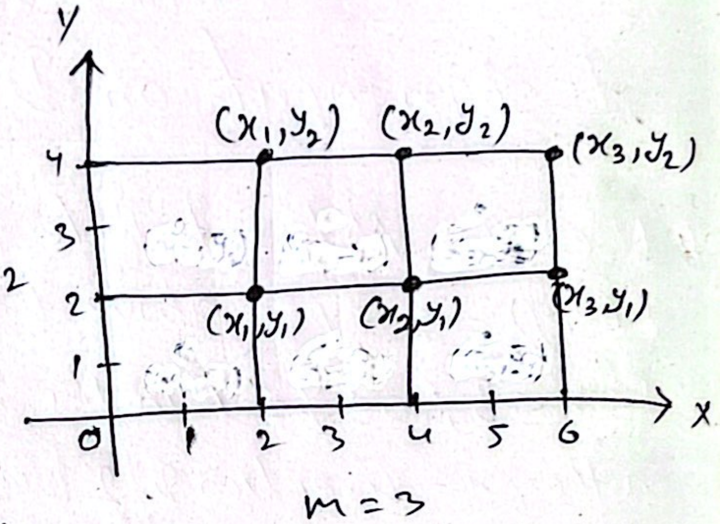
$(\bar{x}_2,\bar{y}_1) = \left(\frac{1}{2}, 1\right)$, $(\bar{x}_3,\bar{y}_1) = \left(\frac{3}{2}, 1\right)$, $(\bar{x}_4,\bar{y}_1) = \left(\frac{5}{2}, 1\right)$

$(\bar{x}_1,\bar{y}_2) = \left(-\frac{1}{2}, 3\right)$, $(\bar{x}_2,\bar{y}_2) = \left(\frac{1}{2}, 3\right)$, $(\bar{x}_3,\bar{y}_2) = \left(\frac{3}{2}, 3\right)$, $(\bar{x}_4,\bar{y}_2) = \left(\frac{5}{2}, 3\right)$

2.(a) Estimate the volume of the solid that lies below the surface $z = xy$ & above the rectangle,

$R = [0,6] \times [0,4]$

Use a Riemann sum with $m=3$, $n=2$ & take the sample point to be the upper right corner of each square.



Soln:

$z = xy$

$\Delta x = 2-0 = 4-2 = 6-4 = 2$

$\Delta y = 2-0 = 4-2 = 2$

$\therefore \Delta A = \Delta x \, \Delta y = 2 \times 2 = 4$

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\,\Delta A$$

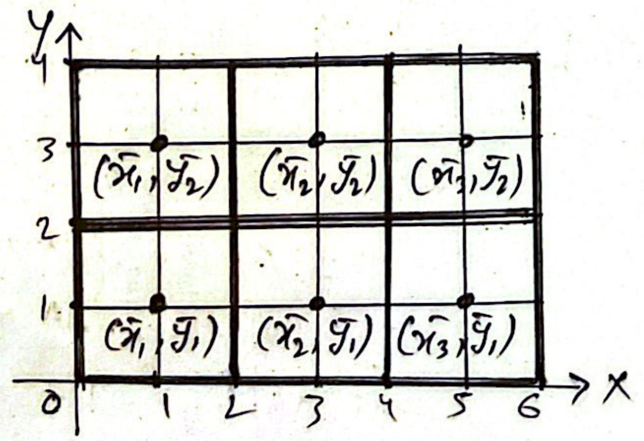
$$= f(x_1,y_1)\Delta A + f(x_2,y_1)\Delta A + f(x_3,y_1)\Delta A + f(x_1,y_2)\Delta A + f(x_2,y_2)\Delta A + f(x_3,y_2)\Delta A$$

$$= \Delta A\left[f(2,2) + f(4,2) + f(6,2) + f(2,4) + f(4,4) + f(6,4)\right]$$

$$= 4\left[2\times2 + 4\times2 + 6\times2 + 2\times4 + 4\times4 + 6\times4\right]$$

$= 288$ cubic unit #

(b) Use the Midpoint Rule to estimate the volume of the solid in part (a).



$\Delta A = 2 \times 2 = 4$

$z = xy$

$(\bar{x}_1, \bar{y}_1) = \left(\frac{2+0}{2}, \frac{2+0}{2}\right)$, $(\bar{x}_2, \bar{y}_1) = \left(\frac{4+2}{3}, \frac{2+0}{2}\right)$

$(\bar{x}_3, \bar{y}_1) = (5,1)$, $(\bar{x}_1, \bar{y}_2) = (1,3)$, $(\bar{x}_2, \bar{y}_2) = (3,3)$

$(\bar{x}_3, \bar{y}_2) = (5,3)$

$$\therefore V = \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_i)\, \Delta A$$

$$= f(\bar{x}_1, \bar{y}_1)\Delta A + f(\bar{x}_2, \bar{y}_1)\Delta A + f(\bar{x}_3, \bar{y}_1)\Delta A + f(\bar{x}_1, \bar{y}_2)\Delta A + f(\bar{x}_2, \bar{y}_2)\Delta A + f(\bar{x}_3, \bar{y}_2)\Delta A$$

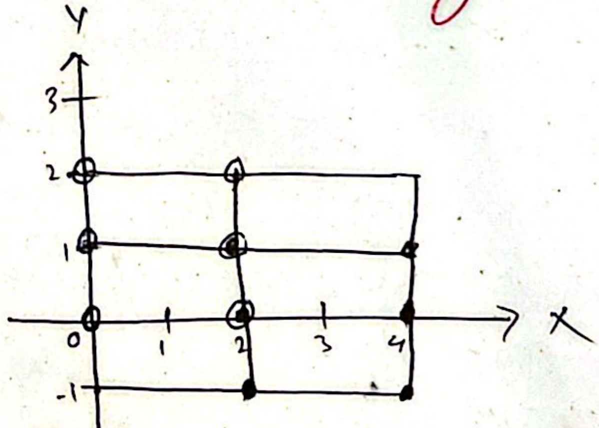
$$= \Delta A\left[1\times1 + 3\times1 + 5\times1 + 1\times3 + 3\times3 + 5\times3\right]$$

$$= 4\times36$$

$= 144$ cubic unit #

3. If $R = [0,4]\times[-1,2]$, use a Riemann sum with $m=2$, $n=3$ to estimate the value of $\iint_R (1-xy^2)\,dA$. Take the sample points to be (a) the lower right corners (b) the upper left corners of the rectangle

Sol^n:



(a) Using the lower right corners points (•)

$\Delta x = 2$, $\Delta y = 1$

$\therefore \Delta A = \Delta x\,\Delta y = 2\times1 = 2$

$(x_1, y_1) = (2,-1)$

$(x_2, y_1) = (4,-1)$

$(x_1, y_2) = (2,0)$

$(x_2, y_2) = (4,0)$

$(x_1, y_3) = (2,1)$

$(x_2, y_3) = (4,1)$

$$\therefore \iint_R (1-xy^2)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_i)\,\Delta A$$

$$= \Delta A\left[f(x_1,y_1) + f(x_2,y_1) + f(x_1,y_2) + f(x_2,y_2) + f(x_1,y_3) + f(x_2,y_3)\right]$$

$$= 2\left[1-2\times(-1)^2 + 1-4\times(-1)^2 + 1-2\times0^2 + 1-4\times0^2 + 1-2\times1^2 + 1-4\times1^2\right]$$

$$= -12$$

(b) Using the upper left corners of the rectangle as sample points (o).

$\Delta x = 2$

$\Delta y = 1$

$\Delta A = 2 \times 1 = 2$

$(x_1, y_1) = (0,0) \quad (x_2, y_1) = (2,0)$

$(x_1, y_2) = (0,1) \quad (x_2, y_2) = (2,1)$

$(x_1, y_3) = (0,2) \quad (x_2, y_3) = (2,2)$

$$\therefore \iint_R (1 - xy^2)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

$$= \Delta A \left[f(x_1,y_1) + f(x_2,y_1) + f(x_1,y_2) + f(x_2,y_2) + f(x_1,y_3) + f(x_2,y_3) \right]$$

$$= 2\left[1 - 0\times 0^2 + 1 - 2\times 0^2 + 1 - 0\times 1^2 + 1 - 2\times 1^2 + 1 - 0\times 2^2 + 1 - 2\times 2^2 \right]$$

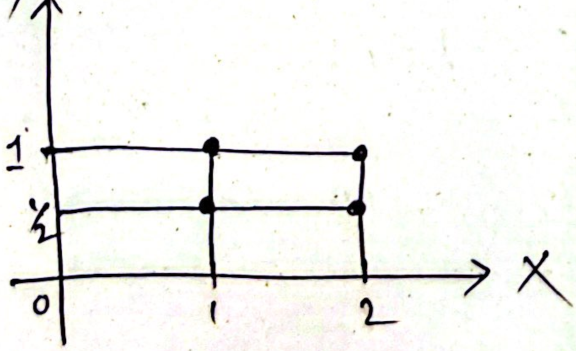
$$= -8 \#$$

4. (a) Use a Riemann sum with $m = n = 2$ to estimate the value of $\iint_R xe^{-xy}\, dA$, where $R = [0,2] \times [0,1]$. Take the sample points to be upper right corners.

(b) Use the Midpoint Rule to estimate the integral in part (a).

Soln:

(a) Using sample points ~~to~~ in the upper right corners.



$\Delta x = 1, \quad \Delta y = \frac{1}{2}$

$\therefore \Delta A = \Delta x \times \Delta y = 1 \times \frac{1}{2} = \frac{1}{2}$

$(x_1, y_1) = (1, \frac{1}{2}), \quad (x_2, y_1) = (2, \frac{1}{2})$

$(x_1, y_2) = (1, 1), \quad (x_2, y_2) = (2, 1)$

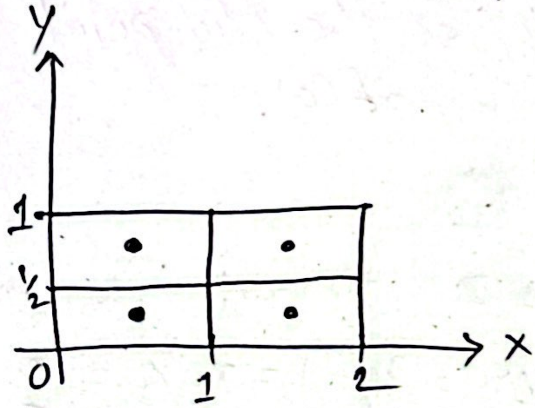
$$\iint_R (xe^{-xy})\,dA \approx \sum_{i=1}\sum_{j=1} f(x_i, y_j)\,\Delta A$$

$$= \Delta A\left[f(x_1,y_1) + f(x_2,y_1) + f(x_1,y_2) + f(x_2,y_2)\right]$$

$$= \frac{1}{2}\left[1\times e^{-1\times\frac{1}{2}} + 2\times e^{-2\times\frac{1}{2}} + 1\times e^{-1\times 1} + 2\times e^{-2\times 1}\right]$$

$$= 0.9904$$

b) Using mid-points as sample points



$$\Delta A = \Delta x\,\Delta y = 1\times\frac{1}{2} = \frac{1}{2}$$

$(\bar{x}_1, \bar{y}_1) = \left(\frac{1}{2}, \frac{1}{4}\right)$ $\quad (\bar{x}_1, \bar{y}_2) = \left(\frac{1}{2}, \frac{3}{4}\right)$

$(\bar{x}_2, \bar{y}_1) = \left(\frac{3}{2}, \frac{1}{4}\right)$ $\quad (\bar{x}_2, \bar{y}_2) = \left(\frac{3}{2}, \frac{3}{4}\right)$.

$$\therefore \iint_R (xe^{-xy})\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\,\Delta A$$

$$= \Delta A\left[f(\bar{x}_1,\bar{y}_1) + f(\bar{x}_1,\bar{y}_2) + f(\bar{x}_2,\bar{y}_1) + f(\bar{x}_2,\bar{y}_2)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}e^{-\frac{1}{2}\times\frac{1}{4}} + \frac{1}{2}e^{-\frac{1}{2}\times\frac{3}{4}} + \frac{3}{2}e^{-\frac{3}{2}\times\frac{1}{4}} + \frac{3}{2}e^{-\frac{3}{2}\times\frac{3}{4}}\right]$$

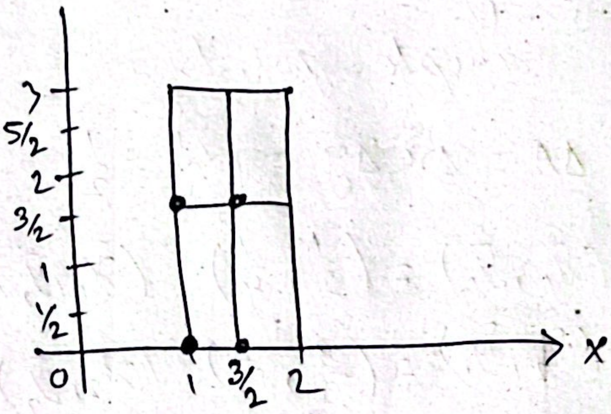
$$= \frac{1}{4}\left[e^{-\frac{1}{8}} + e^{-\frac{3}{8}} + e^{-\frac{3}{8}} + e^{-\frac{9}{8}}\right]$$

$$= 0.645$$

5(a) Estimate the volume of the solid that lies below the surface $z = 1+x^2+3y$ & above the rectangle $R = [1,2]\times[0,3]$. Use a Riemann sum with $m=n=2$ & choose the sample points to be lower left corners.

(b) Use the Mid point Rule to estimate the volume in part (a).

Soln:



$z = 1+x^2+3y$

$\Delta x = 3/2 - 1 = 2 - 3/2 = 1/2$

$\Delta y = 3/2 - 0 = 3 - 3/2 = 3/2$

$\therefore \Delta A = \Delta x \times \Delta y = \frac{1}{2}\times\frac{3}{2} = \frac{3}{4}$

Taking sample points to be lower left corners.

$(x_1, y_1) = (1, 0) \quad (x_2, y_1) = (3/2, 0)$

$(x_1, y_2) = (1, 3/2) \quad (x_2, y_2) = (3/2, 3/2)$

$$V = \iint_R (1+x^2 3y)\, dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\,\Delta A$$

$$= \Delta A\left[f(x_1,y_1) + f(x_1,y_2) + f(x_2,y_1) + f(x_2,y_2)\right]$$

$$= \frac{3}{4}\left[1+1^2+3\times 0 + 1+1^2+3\times\frac{3}{2} + 1+\left(\frac{3}{2}\right)^2+3\times 0 + 1+\left(\frac{3}{2}\right)^2+3\times\frac{3}{2}\right]$$

$= 14.625$ #

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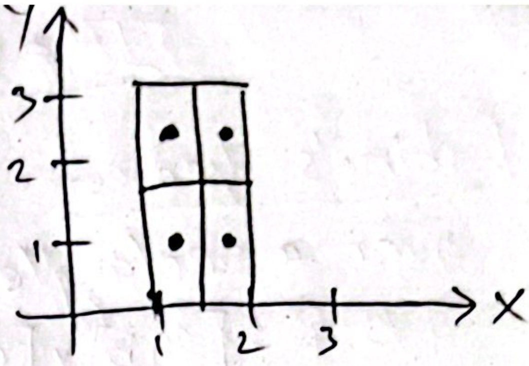
Taking sample points to be mid-points.



$$(\bar{x}_1, \bar{y}_1) = \left(\frac{1+1/2}{2}, \frac{3/2+0}{2}\right) = \left(3/4, 3/4\right)$$

$$(\bar{x}_2, \bar{y}_1) = \left(\frac{2+1/2}{2}, 3/4\right) = \left(5/4, 3/4\right)$$

$$(\bar{x}_1, \bar{y}_2) = \left(3/4, \frac{3+3/2}{2}\right) = \left(3/4, 9/4\right)$$

$$(\bar{x}_2, \bar{y}_2) = \left(5/4, 9/4\right)$$

$$\therefore V = \iint_R (1+x^2+3y)\,dA \simeq \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\,\Delta A$$

$$= \Delta A\left[f(\bar{x}_1, \bar{y}_1) + f(\bar{x}_2, \bar{y}_2) + f(\bar{x}_1, \bar{y}_2) + f(\bar{x}_2, \bar{y}_1)\right]$$

$$= \frac{3}{4} \times \left[1+\left(\frac{3}{4}\right)^2 + 3\times\frac{3}{4} + 1 + \left(\frac{5}{4}\right)^2 + 3\times\frac{9}{4} + 1+\left(\frac{3}{4}\right)^2 + 3\times\frac{9}{4} + 1+\left(\frac{5}{4}\right)^2 + 3\times 3/4\right]$$

$$= 19.6875 \;\#$$

Iterated Integrals

Fubini's theorem: Suppose that $f(x,y)$ is continuous over a rectangle $R=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ c\le y\le d\}$.
i.e $R=[a,b]\times[c,d]$.

Then
$$\boxed{\iint_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy}$$

Evaluate: $\int_1^4 2xy\,dx$

$$= 2y\int_1^4 x\,dx$$
$$= 2y\left[\frac{x^2}{2}\right]_1^4$$
$$= \cancel{4y^2-y}\ y(16-1)$$
$$= 15y \ \#$$

$\int_1^{2y} 2xy\,dx$

$$= 2y\int_1^{2y} x\,dx$$
$$= 2y\left[\frac{x^2}{2}\right]_1^{2y}$$
$$= y\left[(2y)^2-(1)^2\right]$$
$$= 4y^3-y \ \#$$

$\int_1^x (5x^3y^{-3}+6y^2)\,dy$

$$= 5x^3\left[\frac{y^{-2}}{-2}\right]_1^x+6\left[\frac{y^3}{3}\right]_1^x$$
$$= \frac{-5x^3}{2}\left[x^{-2}-(1)^{-2}\right]+2\left[x^3-1^3\right]$$
$$= \frac{-5x^3}{2}\times\frac{(1-x^2)}{x^2}+2(x^3-1)$$
$$= -\frac{5x}{2}+\frac{5x^3}{2}+2x^3-2$$
$$= \frac{1}{2}(9x^3-5x-4) \ \#$$

$\int_0^1\int_1^2 x^2y\,dy\,dx$

$$= \int_0^1 \frac{x^2}{2}\left[y^2\right]_1^2\,dx$$
$$= \int_0^1 \frac{3x^2}{2}\,dx$$
$$= \frac{3}{2}\times\frac{1}{3}\left[x^3\right]_0^1$$
$$= \frac{1}{2} \ \#$$

$\iint_R (2x-4y^3)\,dA$, $R=[-5,4]\times[0,3]$.

$\Rightarrow \int_{-5}^{4}\int_{0}^{3}(2x-4y^3)\,dy\,dx = \int_{-5}^{4}\left[2x[y]_0^3 - \frac{4}{4}[y^4]_0^3\right]dx$

$= \int_{-5}^{4}(6x-81)\,dx$

$= \frac{6}{2}[x^2]_{-5}^{4} - 81[x]_{-5}^{4}$

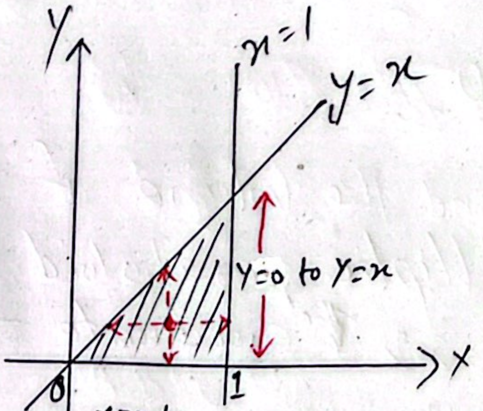
$= 3[4^2-(-5)^2] - 81(4-(-5))$

$= -27-729$

$= -756$ #

Find the volume under the surface $z=\sqrt{1-x^2}$ & above the triangle formed by $y=x$, $x=1$ & the x-axis

$\int_0^1\int_0^x \sqrt{1-x^2}\,dy\,dx$



$= \int_0^1 \sqrt{1-x^2}\,[y]_0^x\,dx$

$= \int_0^1 x\sqrt{1-x^2}\,dx$

Let $1-x^2 = u$

$-2x\,dx = du$

$\Rightarrow x\,dx = -\frac{1}{2}du$

$= -\frac{1}{2}\int_0^1 \sqrt{u}\,du$

$= -\frac{1}{2}\times\frac{2}{3}\left[u^{3/2}\right]_0^1$

$= -\frac{1}{3}\left[(1-x^2)\right]_0^1$

$= -\frac{1}{3}\left[(1-1^2)-(1-0)\right] = 1/3$ #

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$\iint_R y\sin(xy)\,dA$, $R=[1,2]\times[0,\pi]$.

$$\iint_R y\sin(xy)\,dA = \int_1^2\int_0^\pi y\sin(xy)\,dy\,dx = \int_0^\pi\int_1^2 y\sin(xy)\,dx\,dy$$

$\int_0^\pi\int_1^2 y\sin(xy)\,dx\,dy$ integral is easier to evaluate so,

$$= \int_0^\pi y\left[\frac{-\cos xy}{y}\right]_1^2 dy$$

$$= \int_0^\pi -[\cos 2y - \cos y]\,dy$$

$$= \int_0^\pi (-\cos 2y + \cos y)\,dy$$

$$= -\left[\frac{\sin 2y}{2}\right]_0^\pi + [\sin y]_0^\pi$$

$$= -\frac{\sin 2\pi}{2} + \sin\pi$$

$$= 0$$ #

Find the volume of the solid S enclosed by the elliptic paraboloid $x^2+2y^2+z=16$, the planes $x=2$, $y=2$ & the coordinate ~~axes~~ planes.

Soln:

$$x^2+2y^2+z=16$$

$$\Rightarrow z=f(x,y)=16-x^2-2y^2$$

$$V=\iint_R f(x,y)\,dA$$

where R is the region in the xy-plane & $f(x,y)$ represents the height at each point (x,y) above the xy-plane.

$x^2+y^2=16.$

$\Rightarrow y^2=16-x^2$

$\Rightarrow y=\pm\sqrt{16-x^2}$

$\therefore$ the limits for y are: $y=-\sqrt{16-x^2}$ to $y=2.$

Similarly,

For x: $x^2+y^2=16$ $(y=0)$

$\Rightarrow x=16$

$\Rightarrow x=\pm 4$

$\therefore$ The limits for x are: $x=-4$ to $x=2.$

$$\therefore V=\int_{-4}^{2}\int_{-\sqrt{16-x^2}}^{2}(16-x^2-y^2)\,dy\,dx$$

$$=\int_{-4}^{2}\left[16y-x^2y-\frac{y^3}{3}\right]_{-\sqrt{16-x^2}}^{2}dx$$

$$=\int_{-4}^{2}\left(16\times 2-2x^2-\frac{2^3}{3}+16\sqrt{16-x^2}-x^2\sqrt{16-x^2}-\frac{(\sqrt{16-x^2})^3}{3}\right)dx$$

$$=\int_{-4}^{2}-\frac{8}{3}+(16-x^2)2+(16-x^2)\sqrt{16-x^2}-\frac{(\sqrt{16-x^2})^3}{3}\,dx$$

$$=\int_{-4}^{2}-\frac{8}{3}+2(16-x^2)+(16-x^2)\sqrt{16-x^2}-\frac{\sqrt{16-x^2}(16-x^2)}{3}\,dx$$

$$=\int_{-4}^{2}-\frac{8}{3}+2(16-x^2)+\frac{2(16-x^2)\sqrt{16-x^2}}{3}\,dx$$

$$=\int_{-4}^{2}-\frac{8}{3}+2(16-x^2)\left(1+\frac{\sqrt{16-x^2}}{3}\right)dx$$

$$=\frac{1}{3}\int_{-4}^{2}-8+2(16-x^2)(3+\sqrt{16-x^2})\,dx$$

$$=\frac{2}{3}\int_{-4}^{2}-4+3(16-x^2)+(16-x^2)^{3/2}\,dx\approx 303.61$$ #

Find the volume of the solid enclosed by the plane

$4x+2y+z=10$, $y=3x$, $z=0$, $x=0$.

Solⁿ:

$z = 10 - 4x - 2y$

For range of y;

$(z=0)$,

$0 = 10 - 4x - 2y$

$\Rightarrow y = \dfrac{10-4x}{2}$

$\Rightarrow y = 5 - 2x$

$\therefore$ Range of y is: $y = 3x$ to $y = 5-2x$

For range of x;

$(z=0,\ y=3x)$

$0 = 10 - 4x - 2\times 3x$

$\Rightarrow 10x = 10$

$\Rightarrow x = 1$

$\therefore$ Range of x: $x=0$ to $x=1$

$$\therefore V = \iint_R f(x,y)\,dA = \int_0^1 \int_{3x}^{5-2x} (10-4x-2y)\,dy\,dx$$

$$= \int_0^1 \left[10y - 4xy - \frac{2y^2}{2}\right]_{3x}^{5-2x} dx$$

$$= \int_0^1 \left[10(5-2x) - 4x(5-2x) - (5-2x)^2 - \left(10\times 3x - 4x\times 3x - (3x)^2\right)\right] dx$$

$$= \int_0^1 50 - 20x - \cancel{20x} + 8x^2 - 25 + \cancel{20x} - 4x^2 - 30x + 12x^2 + 9x^2\,dx$$

$$= \int_0^1 25x^2 - 50x + 25\,dx$$

$$= 25\int_0^1 x^2 - 2x + 1 \, dx$$

$$= 25\int_0^1 (x-1)^2 \, dx$$

$$= 25\left[\frac{(x-1)^3}{3\times 1}\right]_0^1$$

$$= 25/3 \ \#$$

Special Case :

$f(x,y) = g(x) \cdot h(y)$ on $R = [a,b] \times [c,d]$

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

$$= \int_a^b \int_c^d g(x)\, h(y)\, dy\, dx$$

$$\boxed{\iint_R f(x,y)\, dA = \int_a^b g(x)\, dx \cdot \int_c^d h(y)\, dy}$$

Compute the double integral of $f(x,y) = \dfrac{1+x^2}{1+y^2}$, in the region $R = [0,2] \times [0,1]$.

Sol^n:

$$\iint_R f(x,y)\, dA = \int_0^2 \int_0^1 \frac{1+x^2}{1+y^2}\, dy\, dx = \int_0^2 (1+x^2)\, dx \cdot \int_0^1 \frac{1}{1+y^2}\, dy$$

$$= \left[x + \frac{x^3}{3}\right]_0^2 \cdot \left[\frac{1}{1}\tan^{-1}\left(\frac{y}{1}\right)\right]_0^1 \quad \left[\because \int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\tan^{-1} + C\right.$$

$$= \left(2 + \frac{8}{3}\right) \cdot \left(\tan^{-1}(1) - \tan^{-1}(0)\right)$$

$$= \frac{14}{3} \times \left(\frac{\pi}{4} - 0\right)$$

$$= \frac{7\pi}{6} \ \#$$

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Iterated Integrals

Practice Problems (Assignment)

1. Compute the following double integral over the indicated rectangle
(a) by integrating w.r.to x first
(b) by integrating w.r.to y first

$\iint_R 12x - 18y \, dA \quad R = [-1, 4] \times [2, 3]$.

Soln.:

(a) $$\iint_R 12x - 18y \, dA = \int_2^3 \int_{-1}^4 (12x - 18y) \, dx \, dy$$

$$= \int_2^3 \left[\frac{12}{2} x^2 - 18yx\right]_{-1}^4 dy$$

$$= \int_2^3 \left[6 \times 4^2 - 18y \times 4 - 6 \times (-1)^2 + 18y(-1)\right] dy$$

$$= \int_2^3 90 - 90y \, dy$$

$$= 90 \int_2^3 (1 - y) \, dy$$

$$= 90 \left[y - \frac{y^2}{2}\right]_2^3$$

$$= 90 \left[3 - \frac{3^2}{2} - 2 + \frac{2^2}{2}\right]$$

$$= 90 \times \left(\frac{-3}{2}\right)$$

$$= -135 \; \#$$

(b) $\iint_R (12x-18y)\, dA = \int_{-1}^{4}\int_{2}^{3} (12x-18y)\, dy\, dx$

$= \int_{-1}^{4} \left[12xy - 18\frac{y^2}{2}\right]_2^3 dx$

$= \int_{-1}^{4} \left[12x\times 3 - \frac{18\times 3^2}{2} - 12x\times 2 + 18\times\frac{2^2}{2}\right] dx$

$= \int_{-1}^{4} (12x - 45)\, dx$

$= \left[\frac{12}{2}x^2 - 45x\right]_{-1}^{4}$

$= \left(6\times 4^2 - 45\times 4 - 6\times(-1)^2 + 45\times(-1)\right)$

$= -135$ //

(2) $\iint_R 6y\sqrt{x} - 2y^3\, dA \quad R=[1,4]\times[0,3]$

$= \iint_R 6y\sqrt{x} - 2y^3\, dA$

$= \int_1^4\int_0^3 (6y\sqrt{x} - 2y^3)\, dy\, dx$

$(\sqrt{4})^3$

$= \int_1^4 \left[\frac{6\sqrt{x}}{2}y^2 - \frac{2y^4}{4}\right]_0^3 dx$

$= \int_1^4 \left(3\sqrt{x}\times 3^2 - \frac{3^4}{2}\right) dx$

$= \int_1^4 \left(27\sqrt{x} - \frac{3^4}{2}\right) dx$

$= \left[\frac{27x^{3/2}}{3/2} - \frac{81}{2}x\right]_1^4 = 18(4)^{3/2} - \frac{81}{2}\times 4 - 18 + \frac{81}{2}$

$= 4.5$ //

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③ $\iint_R \frac{e^x}{2y} - \frac{4x-1}{y^2}\, dA \cdot \quad R = [-1,0] \times [1,2]$

$= \int_{-1}^{0}\int_{1}^{2}\left(\frac{e^x}{2y} - \frac{4x-1}{y^2}\right) dy\, dx$

$= \int_{-1}^{0}\left[\frac{e^x}{2}\log_e y - (4x-1)\frac{y^{-2+1}}{(-2+1)}\right]_1^2 dx$

$= \int_{-1}^{0}\left[\frac{e^x}{2}\log_e 2 + \frac{(4x-1)}{2} - \frac{e^x}{2}\log_e 1 - (4x-1)\right] dx$

$= \int_{-1}^{0} \log_e \sqrt{2}\, e^x - \frac{(4x-1)}{2}\, dx$

$= \left[\log_e \sqrt{2}\, e^x - \frac{2x^2}{2} + \frac{1}{2}x\right]_{-1}^{0}$

$= \log_e \sqrt{2}\, e^0 - \log_e \sqrt{2}\, e^{-1} + (-1)^2 + \frac{1}{2}$

$= \frac{2\log_e \sqrt{2} - 2\log_e \sqrt{2}\, e^{-1} + 3}{2}$

$= \cancel{0.98} \quad 1.71$ #

④ $\iint_R \sin(2x) - \frac{1}{1+6y}\, dA \quad R = [\frac{\pi}{4}, \frac{\pi}{2}] \times [0,1]$

$= \int_{\pi/4}^{\pi/2}\int_{0}^{1}\left[\sin(2x) - \frac{1}{1+6y}\right] dy\, dx$

$= \int_{\pi/4}^{\pi/2}\left[\sin(2x)y - \frac{\log_e(1+6y)}{6}\right]_0^1 dx$

$= \int_{\pi/4}^{\pi/2}\left(\sin 2x - \frac{\log_e 7}{6}\right) dx$

$= \left[-\frac{\cos 2x}{2} - \frac{\log_e 7}{6}x\right]_{\pi/4}^{\pi/2}$

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$$= -\frac{\cos\pi}{2} - \frac{\log_e 7}{6}\times\frac{\pi}{2} + \frac{\cos\pi/2}{2} + \frac{\log_e 7}{6}\,\frac{\pi}{4}$$

$$= \frac{1}{2} - \frac{\pi\log_e 7}{12} + 0 + \frac{\pi\log_e 7}{24}$$

$$= \frac{12 - 2\pi\log_e 7 + \pi\log_e 7}{24}$$

$$= \frac{12 - \pi\log_e 7}{24}$$

$$= 0.245 \,/\!/$$

(5) $\iint_R y e^{y^2-4x}\, dA$, $R = [0,2]\times[0,\sqrt{8}]$

$$= \int_0^2\int_0^{\sqrt{8}} y e^{y^2-4x}\, dy\, dx$$

$$= \int_0^{\sqrt{8}}\int_0^2 y e^{y^2-4x}\, dx\, dy$$

$$= \int_0^{\sqrt{8}} \left[y\,\frac{e^{y^2-4x}}{-4}\right]_0^2 dy$$

$$= \int_0^{\sqrt{8}} -\frac{1}{4} y e^{y^2-8} + \frac{1}{4} y e^{y^2}\, dy$$

$$= \frac{1}{4}\int_0^{\sqrt{8}} (1-e^{-8})\, y e^{y^2}\, dy$$

$$= \frac{1-e^{-8}}{4}\int_0^{\sqrt{8}} e^{y^2}\, y\, dy$$

Let $y^2 = u$

$\Rightarrow 2y\,dy = du$

$\Rightarrow y\,dy = \frac{du}{2}$

$$= \frac{1-e^{-8}}{8}\int_0^8 e^u\, du$$

$$= \frac{1-e^{-8}}{8}\left[e^u\right]_0^8$$

$$= \frac{1-e^{-8}}{8}\times\left[e^8 - 1\right]$$

$$= \frac{e^8 - 1 - 1 + e^{-8}}{8}$$

$= 372.36$ #

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⑥ $\iint_R xy^2\sqrt{x^2+y^3}\,dA$, $R=[0,3]\times[0,2]$

$= \int_0^2\int_0^3 xy^2\sqrt{x^2+y^3}\,dx\,dy$

Let, $x^2+y^3=u$
$2x\,dx = du$
$x\,dx = \frac{du}{2}$

If $x=0 \Rightarrow u=y^3$
& $x=3 \Rightarrow u=9+y^3$

$= \int_0^2\left[\int_{y^3}^{9+y^3} y^2\sqrt{u}\times\frac{1}{2}\,du\right]dy$

$= \int_0^2 \frac{y^2}{2}\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{y^3}^{9+y^3} dy$

$= \frac{1}{2}\int_0^2 y^2\times\frac{2}{3}\left[(9+y^3)^{3/2}-(y^3)^{3/2}\right]dy$

$= \cancel{\frac{1}{3}\int_0^2 y^2((9+y^3)^{3/2}-y^{9/2})\,dy}$

$= \cancel{\frac{1}{3}\int_0^2\left[(y^4(9+y^3)^3)^{\frac{1}{2}}-y^{9/2}\right]dy}$

$= \cancel{\frac{1}{3}\int_0^2 y^3(9+y^3)^3 y\,dy}$

$= \frac{1}{3}\int_0^2 y^2(9+y^3)^{3/2}\,dy - \frac{1}{3}\int_0^2 y^{\frac{9}{2}}\cdot y^2\,dy$

Let $y^3=p$
$3y^2\,dy = dp$
$y^2\,dy = \frac{1}{3}\,dp$

For $y=0 \rightarrow p=0$
& $y=2 \Rightarrow p=8$

$= \frac{1}{3}\int_0^8 (9+p)^{3/2}\cdot\frac{1}{3}\,dp - \frac{1}{3}\int_0^8 p^{3/2}\times\frac{1}{3}\,dp$ $\left[\because y^{\frac{9}{2}}y^2\,dy = p^{3/2}\cdot\frac{1}{3}\,dp\right]$

$= \frac{1}{9}\left[\frac{(9+p)^{3/2+1}}{(3/2+1)\times 1}\right]_0^8 - \frac{1}{9}\left[\frac{p^{3/2+1}}{3/2+1}\right]_0^8$

$$= \frac{1}{9} \times \frac{2}{5}\left[(9+8)^{\cdots} - (9)^{\cdots}\right] - \frac{1}{9} \times \frac{2}{5}\left[(8)^{\cdots}\right]$$

$$= \frac{2}{45}\left[17^{5/2} - 9^{5/2} - 8^{5/2}\right]$$

$$= 34.11 \#$$

(7) $\iint_R xy\cos(yx^2)\,dA \quad R = [-2,3]\times[-1,1].$

$$= \int_{-2}^{3}\int_{-1}^{1} xy\cos(yx^2)\,dy\,dx$$

$$= \int_{-2}^{3}\left[x\int_{-1}^{1} y\cos(yx^2)\,dy\right]dx$$

$$= \int_{-2}^{3} x\left[y\int\cos(yx^2)\,dy - \int\left[\frac{d}{dy}(y)\cdot\int\cos(yx^2)\,dy\right]dy\right]_{-1}^{1} dx$$

$$= \int_{-2}^{3} x\left[y\,\frac{\sin yx^2}{x^2} - \int 1\cdot\frac{\sin yx^2}{x^2}\,dy\right]_{-1}^{1} dx$$

$$= \int_{-2}^{3} \frac{1}{x}\left[y\sin yx^2 + \frac{1}{x^2}\left[\cos yx^2\right]\right]_{-1}^{1} dx$$

$$= \int_{-2}^{3}\frac{1}{x}\left[1\cdot\sin x^2 + \frac{1}{x^2}\cos x^2 - (-1)\sin(-x^2) - \frac{1}{x^2}\cos(-x^2)\right]dx$$

$$= \int_{-2}^{3}\left(\frac{1}{x}\sin x^2 + \frac{1}{x^3}\cos x^2 - \frac{1}{x}\sin x^2 - \frac{1}{x^3}\cos x^2\right)dx$$

$$= \int_{-2}^{3} 0\,dx$$

$$= 0 \#$$

⑧ $\iint_R xy\cos(y) - x^2 \, dA \; , \; R = [1,2] \times [\pi/2, \pi]$

$= \int_1^2 \int_{\pi/2}^{\pi} (xy\cos y - x^2)\, dy\, dx$

$= \int_1^2 \left[x \int_{\pi/2}^{\pi} y\cos y\, dy - \int_{\pi/2}^{\pi} x^2 dy \right] dx$

$= \int_1^2 \left[x \left[y \int \cos y\, dy - \int \left[\frac{d}{dy}(y) \cdot \int \cos y\, dy\right] dy \right]_{\pi/2}^{\pi} - x^2 [y]_{\pi/2}^{\pi} \right] dx$

$= \int_1^2 \left[x \left[y\sin y - \int 1 \cdot \sin y\, dy \right]_{\pi/2}^{\pi} - x^2 \left(\pi - \frac{\pi}{2}\right) \right] dx$

$= \int_1^2 \left[x \left[y\sin y + \cos y \right]_{\pi/2}^{\pi} - \frac{\pi}{2} x^2 \right] dx$

$= \int_1^2 \left[x \left[\pi \sin\pi + \cos\pi - \frac{\pi}{2}\sin\frac{\pi}{2} - \cos\frac{\pi}{2} \right] - \frac{\pi}{2} x^2 \right] dx$

$= \int_1^2 \left[x \left[\pi \times 0 - 1 - \frac{\pi}{2} \times 1 - 0 \right] - \frac{\pi}{2} x^2 \right] dx$

$= \int_1^2 \left[-\left(\frac{\pi}{2} + 1\right) x - \frac{\pi}{2} x^2 \right] dx$

$= -\left(\frac{\pi}{2} + 1\right) \times \frac{1}{2} \left[x^2\right]_1^2 - \frac{\pi}{2} \times \frac{1}{3} \left[x^3\right]_1^2$

$= -\frac{(\pi + 2)}{4} \times 3 - \frac{\pi}{6} \times 7$

$= \frac{-9\pi - 18 - 14\pi}{12}$

$= \frac{-23\pi - 18}{12}$

$= -7.52$ #

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(9) Det. under $f(x,y)$
$= 9x^2 + 4xy + y$ & above the rectangle given by $[-1,1]\times[0,2]$ in the xy-plane.

Sol^n:

$$f(x,y) = 9x^2 + 4xy + 4$$

$$R = [-1,1] \times [0,2]$$

$$V = \iint_R f(x,y)\, dA$$

$$= \int_{-1}^{1}\int_{0}^{2} (9x^2 + 4xy + 4)\, dy\, dx$$

$$= \int_{-1}^{1} \left[9x^2 y + \frac{4xy^2}{2} + 4y\right]_0^2 dx$$

$$= \int_{-1}^{1} (18x^2 + 8x + 8)\, dx$$

$$= \left[\frac{18x^3}{3} + \frac{8x^2}{2} + 8x\right]_{-1}^{1}$$

$$= 6 + 4 + 8 - (-6) - 4 - (-8)$$

$$= 28$$

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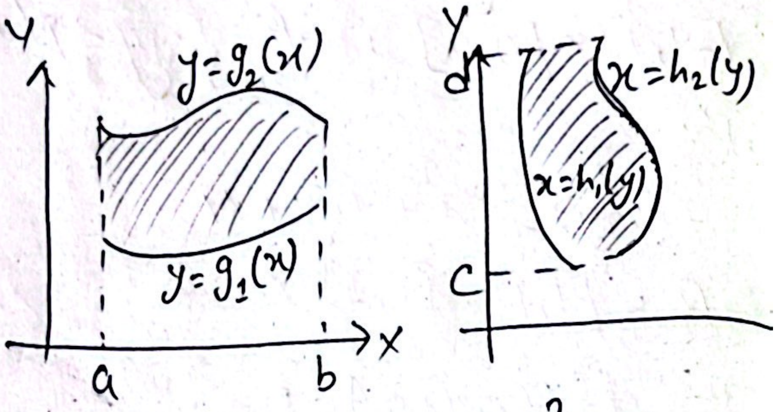
Double Integrals in general region :-

Let $f(x,y)$ be continuous,



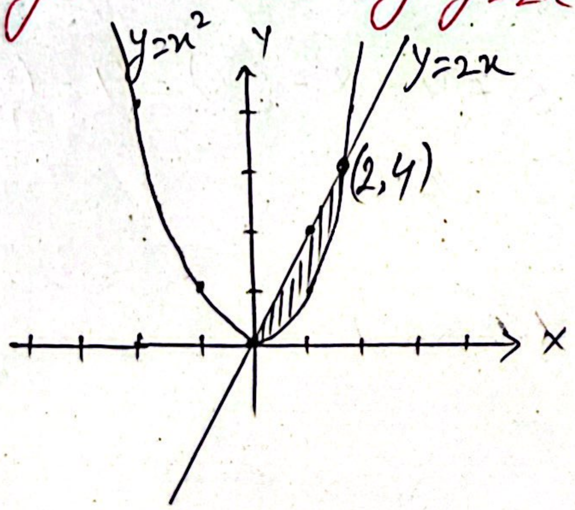
If $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ then

$$\boxed{\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

If $D = \{(x,y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$ then

$$\boxed{\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy}$$

Find the area of the region enclosed by $y = 2x$ & $y = x^2$



Solⁿ:

$y = 2x$ — (i)

$y = x^2$ — (ii)

Solving (i) & (ii);

$x^2 = 2x$

$\Rightarrow x^2 - 2x = 0$

$\Rightarrow x(x-2) = 0$

$\Rightarrow x = 0 \,/\, x = 2$

$\therefore\ 0 \le x \le 2$ & $x^2 \le y \le 2x$

$\therefore \cancel{\iint f(x,y)}$

For area $f(x,y) = 1$ [Unit height]

$$\therefore \iint_D f(x,y)\, dA = \int_0^2 \int_{x^2}^{2x} 1 \, dy\, dx$$

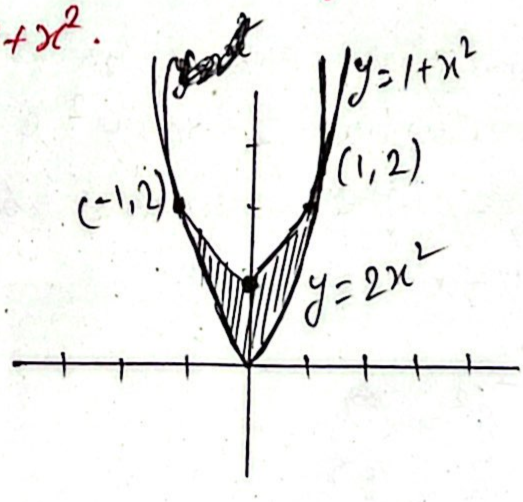
$$= \int_0^2 (2x - x^2)\, dx$$

$$= \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2$$

$$= 4 - \frac{8}{3}$$

$$= \frac{4}{3}$$ #

Evaluate $\iint_D (x+2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ & $y = 1+x^2$.



Sol^n:

$y = 2x^2$ — (i)

$y = 1+x^2$ — (ii)

Solving (i) & (ii);

$2x^2 = 1 + x^2$

$\Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$

$\Rightarrow -1 \le x \le 1$ & $2x^2 \le y \le 1+x^2$

$$\therefore \iint_D (x+2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[xy + \frac{2y^2}{2}\right]_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^{1} \left[x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)^2\right] dx$$

$$= \int_{-1}^{1} x + x^3 + 1 + 2x^2 + x^4 - 2x^3 - 4x^4 \, dx$$

$$= \int_{-1}^{1} -3x^4 - x^3 + 2x^2 + x + 1 \, dx$$

$$= \left[-\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{x^2}{2} + x \right]_{-1}^{1}$$

$$= \left[-\frac{3}{5} - \cancel{\frac{1}{4}} + \frac{2}{3} + \cancel{\frac{1}{2}} + 1 - \frac{3}{5} + \cancel{\frac{1}{4}} + \frac{2}{3} - \cancel{\frac{1}{2}} + 1 \right]$$

$$= -\frac{6}{5} + \frac{4}{3} + 2$$

$$= \frac{-18 + 20 + 30}{15}$$

$$= \frac{32}{15}$$

$$= 2.13 \ \#$$

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ & above the region D in the xy-plane bounded by the line $y = 2x$ & the parabola $y = x^2$.

Solⁿ:

$y = x^2$ —(i)

$y = 2x$ —(ii)

Solving (i) & (ii);

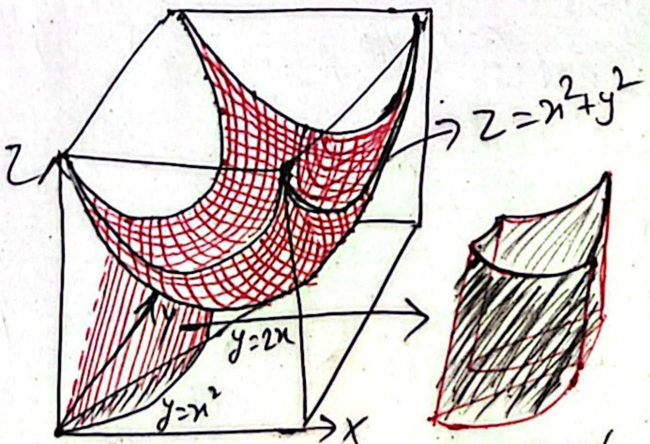
$x^2 = 2x$

$x^2 - 2x = 0$

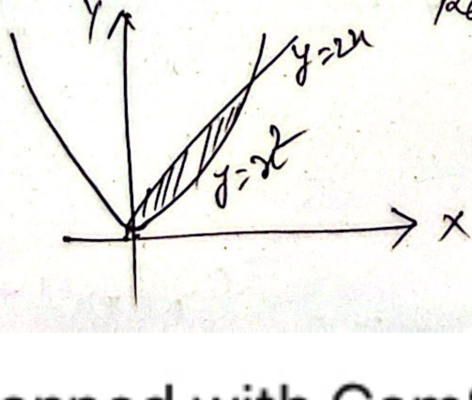
$x(x-2) = 0$

$\Rightarrow x = 0, \ x = 2$

$\therefore 0 \le x \le 2$ & $x^2 \le y \le 2x$



Required Volume



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$\therefore$ Volume $= \iint_D f(x,y)\, dA$

$$= \int_0^2 \int_{x^2}^{2x} (x^2+y^2)\, dy\, dx$$

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{2x} dx$$

$$= \int_0^2 x^2(2x) + \frac{(2x)^3}{3} - x^2(x^2) - \frac{(x^2)^3}{3}\, dx$$

$$= \int_0^2 \left(2x^3 + \frac{8}{3}x^3 - x^4 - \frac{x^6}{3}\right) dx$$

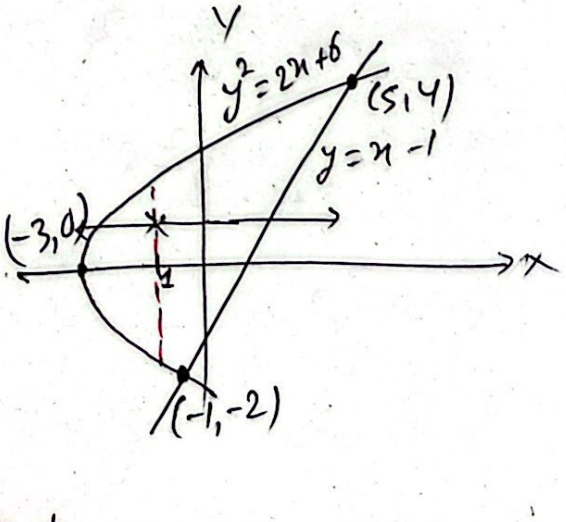
$$= \left[\frac{2}{4}x^4 + \frac{8}{3\times 4}x^4 - \frac{1}{5}x^5 - \frac{1}{3\times 7}x^7\right]_0^2$$

$$= \frac{1}{2}\times 2^4 + \frac{2}{3}\times 2^4 - \frac{1}{5}\times 2^5 - \frac{1}{21}\times 2^7$$

$$= 6.17 \ \#$$

\# Evaluate $\iint_D xy\, dA$ where D is the region bounded by the line $y = x-1$ & parabola $y^2 = 2x+6$.

Sol^n:



$y = x - 1$ —(i)

$y^2 = 2x + 6$ —(ii)

Solving (i) & (ii);

$(x-1)^2 = 2x + 6$

$x^2 - 4x - 5 = 0$

$x^2 - 5x + x - 5 = 0$

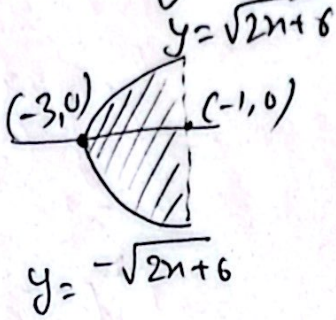
$x(x-5) + 1(x-5) = 0$

$(x-5)(x+1) = 0$

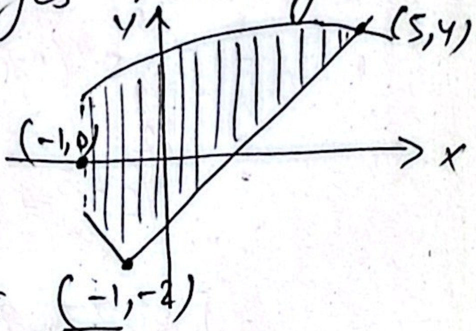
$\Rightarrow x = 5,\ x = -1$

This gives the range of x from -1 to 5 but it excludes the part from $x = -3$ to $x = -1$

where as in between $x=-3$ to $x=-1$ y also ranges from $-\sqrt{2x+6}$ to $\sqrt{2x+6}$



& from $x=-1$ to $x=5$, y ranges from $y=x-1$ to $y=\sqrt{2x+6}$



$$\therefore \iint_D xy\,dA = \int_{-3}^{1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

Instedd of evaluating this complex integral lets apply an alternative approach.

$y = x-1$ —— (i)

$y^2 = 2x+6$ —— (ii)

~~Can be~~ eq^{ns} (i) & (ii) can be written as,

$x = y+1$ —— (iii) & $x = \dfrac{y^2-6}{2}$ —— (iv)

Solving;

$$y+1 = \frac{y^2-6}{2}$$

$\Rightarrow y^2-2y-8=0$

$\Rightarrow y^2-4y+2y-8=0$

$\Rightarrow y(y-4)+2(y-4)=0$

$\Rightarrow (y-4)(y+2)=0$

$\Rightarrow y=4,-2$

$\Rightarrow$ y ranges from -2 to

$\Rightarrow -2 \le y \le 4$ &

~~$y+1 \le x \le \frac{y^2-6}{2}$~~

$$\frac{y^2-6}{2} \le x \le y+1$$

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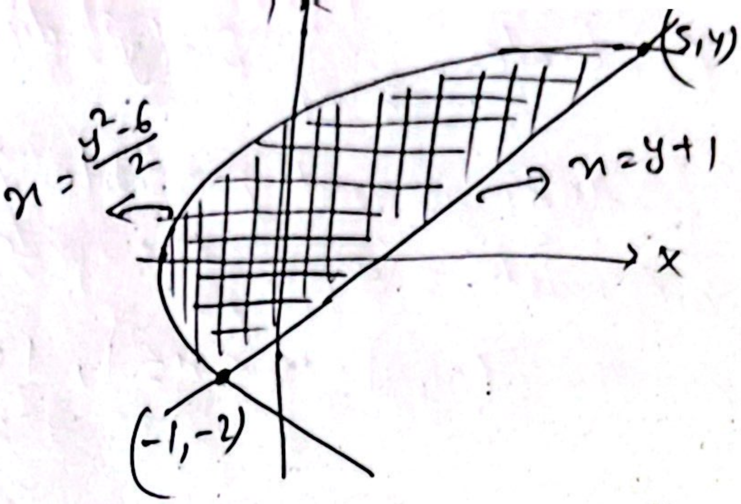
$\therefore \iint_D xy\,dA$



$= \int_{-2}^{5} \int_{\frac{y^2-6}{2}}^{y+1} xy \, dx \, dy$

$= \int_{-2}^{5} \left[\frac{x^2 y}{2}\right]_{\frac{y^2-6}{2}}^{y+1} dy$

$= \int_{-2}^{5} \frac{y(y+1)^2}{2} - \frac{y\left(\frac{y^2-6}{2}\right)^2}{2} \, dy$

$= \frac{1}{2}\int_{-2}^{5}\left(y^3 + 2y^2 + y - \frac{y^5}{2} + 6y^3 - 18y\right) dy$

$= \frac{1}{2}\int_{-2}^{5}\left(-\frac{y^5}{2} + 7y^3 + 2y^2 - 17y\right) dy$

$= \frac{1}{2}\left[-\frac{1}{12}y^6 + \frac{7}{4}y^4 + \frac{2}{3}y^3 - \frac{17}{2}y^2\right]_{-2}^{5}$

$= \frac{1}{2}\left[-\frac{1}{12}5^6 + \frac{7}{4}5^4 + \frac{2}{3}5^3 - \frac{17}{2}5^2 + \frac{1}{12}2^6 - \frac{7}{4}2^4 + \frac{2}{3}\times 2^3 + \frac{17}{2}\right.$

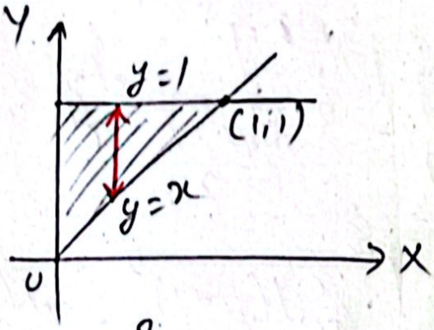
$= \frac{1}{2}\left[-5^6 + 21\times 5^4 + 8\times 5^3 - 102\times 5^2 + 2^6 - 21\times 2^4 + 64 + 408\right]\times \frac{1}{12}$

$= -17.354$

Evaluate $\int_0^1 \int_1^x \sin(y^2)\,dy\,dx$

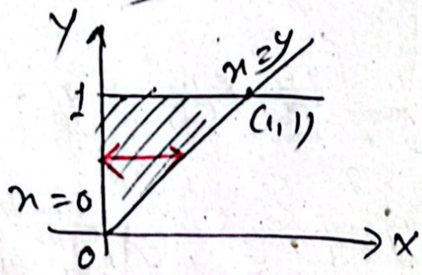
→ $\int \sin(y^2)dy$ is not possible in finite terms so we must change the order of integration.

Here, $D = \{(x,y) : 0 \le x \le 1,\ 1 \le y \le x\}$



Alternatively,

$$D = \{(x,y) : 0 \le x \le y,\ 0 \le y \le 1\}$$



$$= \int_0^1 \int_1^x \sin(y^2)\,dy\,dx$$

$$= \int_0^1 \int_0^y \sin(y^2)\,dx\,dy$$

$$= \int_0^1 y \sin y^2\,dy$$

Let $y^2 = p$, $2y\,dy = dp$, $y\,dy = \frac{1}{2}dp$

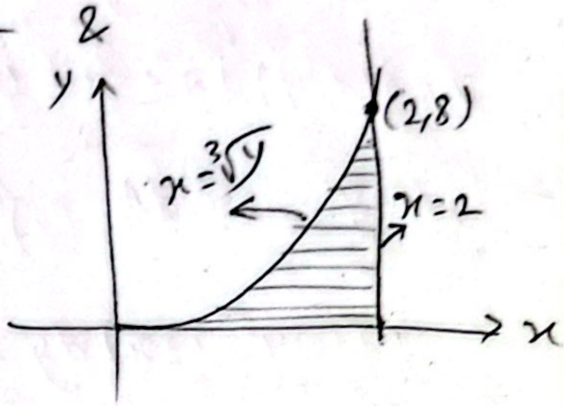
If $y = 0 \Rightarrow p = 0$
If $y = 1 \Rightarrow p = 1$

$$= \int_0^1 \frac{1}{2}\sin p\,dp = \frac{1}{2}\left[-\cos p\right]_0^1 = \frac{1}{2}\left[-\cos 1 + \cos 0\right]$$

$$= 7.61 \times 10^{-5}$$ #

Evaluate $\int_0 \int_{\sqrt[3]{y}} \sqrt{x^4+1}\; dx\, dy$

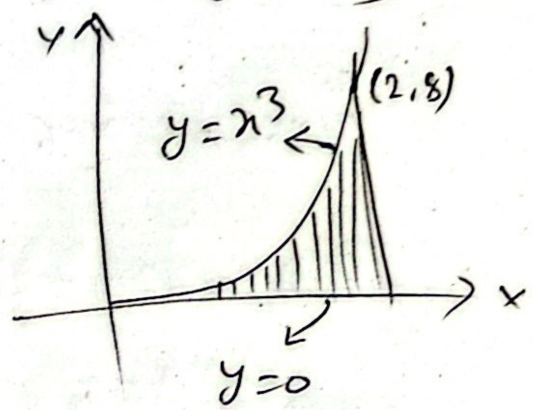
Here,
x ranges from $x = \sqrt[3]{y}$ to $x = 2$ &
y ranges from $y = 0$ to $y = 8$.



Here,
$D = \{(x,y) : \sqrt[3]{y} \le x \le 2,\ 0 \le y \le 8\}$

Alternatively,

$D = \{(x,y) : \cancel{0 \le y \le x^3},\ 0 \le x \le 2,\ 0 \le y \le x^3\}$



$$= \int_0^2 \int_0^{x^3} \sqrt{x^4+1}\; dy\, dx$$

$$= \int_0^2 \sqrt{x^4+1}\,(x^3)\, dx$$

put, $x^4 = p$
$4x^3\, dx = dp$
$x^3\, dx = \frac{1}{4} dp$

If $x = 0 \Rightarrow p = 0$
If $x = 2 \Rightarrow p = 16$

$$= \int_0^{16} \sqrt{p+1} \times \frac{1}{4}\, dp$$

$$= \left[\frac{(p+1)^{3/2}}{3/2} \times \frac{1}{4} \right]_0^{16} = \frac{1}{6}\left[(16+1)^{3/2} - 1^{3/2} \right]$$

$= 11.51$ #

Assignment (Practice Problems)

① Evaluate $\iint_D 42y^2 - 12x \, dA$ where D

$D = \{(x,y) \mid 0 \le x \le 4,\ (x-2)^2 \le y \le 6\}$

$\Rightarrow \iint_D 42y^2 - 12x \, dA$

$= \int_0^4 \int_{(x-2)^2}^{6} (42y^2 - 12x)\, dy\, dx$

$= \int_0^4 \left[\frac{42y^3}{3} - 12xy\right]_{(x-2)^2}^{6} dx$

$= \int_0^4 14 \times 6^3 - 12x \times 6 - 14(x-2)^6 + 12x(x-2)^2 \, dx$

$= \cancel{\int_0^4 3024 - 72x + 2(x-2)^2 (6x - 7(x-2)^4)\, dx}$

$= \cancel{3024[x]_0^4 - \frac{72}{2}[x^2]_0^4 + \int_0^4 2(x-2)^2}$

$= \int_0^4 3024 - 72x - 14(x-2)^3(x-2)^3 + 12x(x^2 - 4x + 4)\, dx$

$= 3024[x]_0^4 + \int_0^4 -72x - 14(x^3 - 6x^2 + 12x - 8)(x^3 - 6x^2 + 12x - 8) + 12x^3 - 48x^2 + 48x \, dx$

$3024 \times 4 + \int_0^4 \big[-24x - 14x^6 + 84x^5 - 168x^4 + 112x^3 + 84x^5 - 504x^4 + 1008x^3 - 588x^2 - 168x^4 + 1008x^3 - 2016x^2 - 1344x + 112x^3 - 672x^2 + 1344x - 896 + 12x^3 - 48x^2\big]\, dx$

$= 12096 + \int_0^4 (-14x^6 + 168x^5 - 840x^4 + 2252x^3 - 3324x^2 - 24x - 896)\, dx$

$= \cancel{12096 - \frac{14}{7}4^7 + \frac{168}{6}4^6 - \frac{840}{5}4^5 + \frac{\ldots}{4}4^4 - \frac{\ldots}{3} - \frac{\ldots}{2} - 896 \times 4}$

$= \cancel{-8576}$

$= \int_0^4 3024\, dx - 72\int_0^4 x\, dx - 14\int_0^4 (x-2)^6 dx + 12\int_0^4 (x-2)^2 dx$

$= 3024 \times 4 - \frac{72}{2}4^2 - 14\int_{-2}^{2} p^6 dp + 12\int_{-2}^{2} p^2 dp$

Let $x - 2 = p$
$\Rightarrow dx = dp$
for $x = 0 \Rightarrow p =$
for $x = 4 \rightarrow p =$

$= 12096 - 576 - \frac{14}{7}\left[p^7\right]_{-2}^{2} + \frac{12}{3}\left[p^3\right]_{-2}^{2}$

$= 11520 - 2\left[(2)^7 - (-2)^7\right] + 4\left[(2)^3 - (-2)^3\right]$

$= 11520 - 2^9 + 2^6$

$= 11072$ //

② $\iint_D 2yx^2 + 9y^3 \, dA$ where D is the region bounded by $y = \frac{2}{3}x$ & $y = 2\sqrt{x}$.

$= \iint_D 2yx^2 + 9y^3 \, dA$

Here, $y = \frac{2}{3}x$ — (i) & $y = 2\sqrt{x}$ — (ii)

Solving (i) & (ii);

$\frac{2}{3}x = 2\sqrt{x}$

$\Rightarrow x = 3\sqrt{x}$

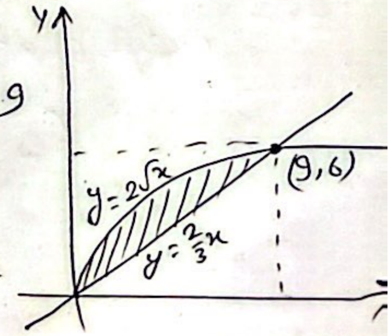
$\Rightarrow x^2 = 9x$

$\Rightarrow x^2 - 9x = 0$

$\Rightarrow x(x - 9) = 0$

$\Rightarrow x = 0 / x = 9$

Hence,
$0 \le x \le 9$ &
$\frac{2}{3}x \le y \le 2\sqrt{x}$



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$$\therefore \iint_D 2yx^2 + 9y^3\, dA = \int_0^9 \int_{\frac{2}{3}x}^{2\sqrt{x}} (2yx^2 + 9y^3)\, dy\, dx$$

$$= \int_0^9 \left[\frac{2x^2}{2} y^2 + \frac{9}{4} y^4\right]_{\frac{2}{3}x}^{2\sqrt{x}} dx$$

$$= \int_0^9 \left[x^2(2\sqrt{x})^2 + \frac{9}{4}(2\sqrt{x})^4 - x^2\left(\tfrac{2}{3}x\right)^2 - \frac{9}{4}\left(\tfrac{2}{3}x\right)^4\right] dx$$

$$= \int_0^9 \left(4x^3 + 36x^2 - \frac{4}{9}x^4 - \frac{4}{9}x^4\right) dx$$

$$= 4\int_0^9 \left[x^3 + 9x^2 - \frac{2}{9}x^4\right] dx$$

$$= 4\left[\left[\frac{x^4}{4}\right]_0^9 + \frac{9}{3}\left[x^3\right]_0^9 - \frac{2}{9\times 5}\left[x^5\right]_0^9\right]$$

$$= 4\left[\frac{9^4}{4} + 3\times 9^3 - \frac{2}{45}\times 9^5\right]$$

$$= 4811.4$$

③ $\iint_D (10x^2y^3 - 6)\, dA$, D is the region bounded by $x = -2y^2$ & $x = y^3$.

Soln:

$$\iint_D (10x^2y^3 - 6)\, dA,$$

$$x = -2y^2 \text{ — (i)}, \quad x = y^3 \text{ — (ii)}$$

$$y^3 = -2y^2$$

$$y^3 + 2y^2 = 0$$

$$y^2(y+2) = 0$$

$\Rightarrow y = 0 \ / \ y = -2$

$\therefore -2 \le y \le 0 \ \ \& \ \ -2y^2 \le x \le y^3$

$$\therefore \iint_D (10x^2y^3 - 6)\, dA = \int_{-2}^{0} \int_{-2y^2}^{y^3} (10x^2y^3 - 6)\, dx\, dy$$

$$= \int_{-2}^{0} \left[\frac{10y^3}{3} x^3 - 6x \right]_{-2y^2}^{y^3} dy$$

$$= \int_{-2}^{0} \left[\frac{10y^3}{3}(y)^9 - 6y^3 - \frac{10y^3}{3}(-2y^2)^3 + 6(-2y^2) \right] dy$$

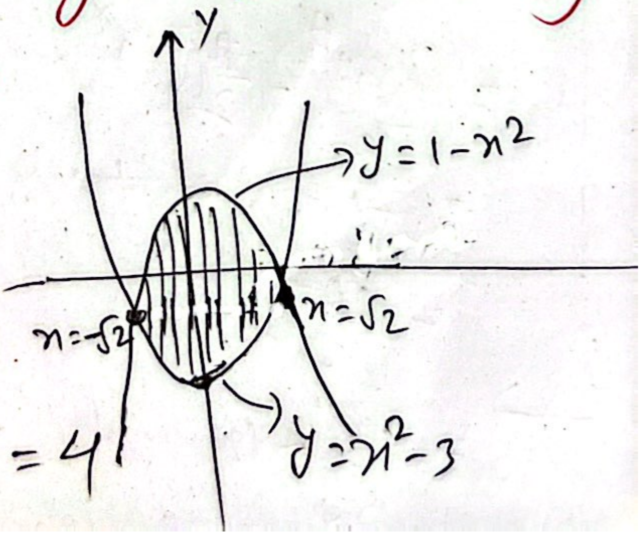
$$= \int_{-2}^{0} \left(\frac{10}{3}y^{12} - 6y^3 + \frac{80}{3}y^9 - 12y^2 \right) dy$$

$$= \left[\frac{10}{3 \times 13} y^{13} - \frac{6}{4}y^4 + \frac{80}{3 \times 10} y^{10} - \frac{12}{3} y^3 \right]_{-2}^{0}$$

$$= -\frac{10}{39}(-2)^{13} + \frac{3}{2}(-2)^4 - \frac{8}{3}(-2)^{10} + 4(-2)^3$$

$$= -638.15 \ \#$$

(4) $\iint_D x(y-1)\, dA$, D is the region bounded by $y = 1 - x^2$ & $y = x^2 - 3$.



Soln.:

$\iint_D x(y-1)\, dA$

$y = 1 - x^2$ — (i)

$y = x^2 - 3$ — (ii)

$\Rightarrow 1 - x^2 = x^2 - 3 \ \Rightarrow 2x^2 = 4$

$\Rightarrow x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}$

$\therefore -\sqrt{2} \le x \le \sqrt{2}$

~~For upper part ... 0~~

~~For lower part $x^2-3 \le y \le -2$~~

$$\therefore \iint_D x(y-1)\,dA = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2-3}^{1-x^2} x(y-1)\,dy\,dx \;\cancel{+\int_{-\sqrt{2}}^{\sqrt{2}}\int_{x^2-3}^{-2} x(y-1)\,dy\,dx}$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left[\frac{x}{2}y^2 - xy\right]_{x^2-3}^{1-x^2} dx \;\cancel{+\int_{-\sqrt{2}}^{\sqrt{2}}\left[\frac{x}{2}y^2 - xy\right]_{x^2-3}^{-2}}$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left[\frac{x}{2}(1-x^2)^2 - x(1-x^2) - \frac{x}{2}(x^2-3)^2 + x(x^2-3)\right] dx$$

$$\cancel{+\int \frac{x}{2}(-2)^2 - x(-2) - \frac{x}{2}(x^2-3)^2 + x(x^2 \ldots}$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{x}{2} - \cancel{x^3} + \cancel{\frac{x^5}{2}} - x + \cancel{x^3} - \cancel{\frac{x^5}{2}} + 3x^3 - \frac{9x}{2} + x^3 - 3x\right) dx$$

$$\cancel{+\int 2x + 2x - \frac{x^5}{2} + 3x^3 - \frac{9x}{2} + x^3 \ldots}$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} 4x^3 - 8x\,dx \quad \cancel{+\int -\frac{x^5}{2} + 4x^3 - \frac{7x}{2}\,dx}$$

$$= \left[\frac{4x^4}{4} - \frac{8x^2}{2}\right]_{-\sqrt{2}}^{\sqrt{2}} \cancel{+\left[-\frac{x^6}{12} + \frac{4x^4}{4} - \frac{7x^2}{4}\right]}$$

$$\cancel{= \frac{(\sqrt{2})^6}{12} - \frac{9}{2}(\sqrt{2})^2 - \frac{(-\sqrt{2})^6}{12} + \frac{9}{2}(-\sqrt{2})^2 - \frac{(\sqrt{2})^6}{12} + (\sqrt{2})^4 - \frac{7}{4}(\sqrt{2})^2}$$

$$\cancel{= 4 - 8 - 4 + 8} \qquad \cancel{+\frac{(\sqrt{2})^6}{12} - (\sqrt{2})^4 + \frac{7}{4}(\ldots}$$

$= \cancel{0}$

$$= \sqrt{2}^4 - \sqrt{2}^4 - 4(\sqrt{2})^2 + 4(\sqrt{2})^2$$

$= 0$ *

$-\frac{8x}{2}$ -4

$-\frac{16x}{2}$

$-5x$

$10x$

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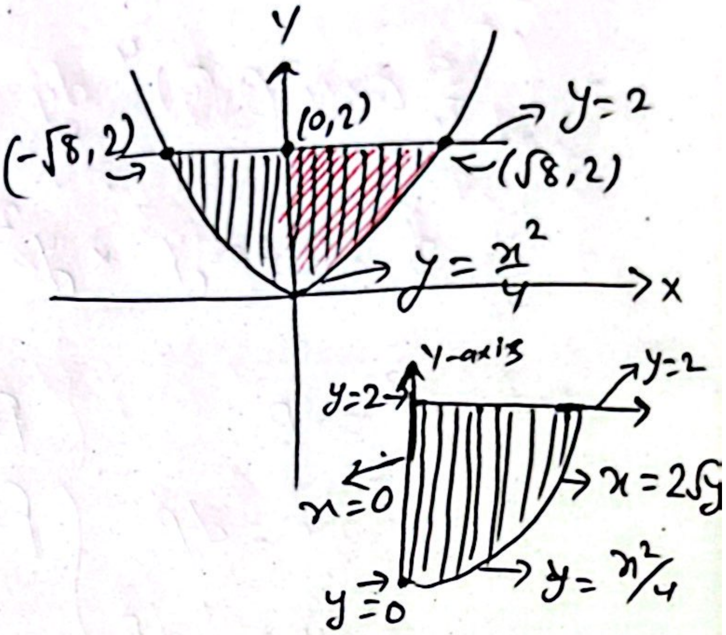
(5) $\iint_D 5x^3\cos(y^3)\,dA$, D is region bounded by $y=2$, $y=\frac{1}{4}x^2$ & y-axis.



Soln:

$y=2$ — (i)

$y=\frac{1}{4}x^2$ — (ii)

$2=\frac{1}{4}x^2$

$\Rightarrow x=\pm\sqrt{8}$

$\Rightarrow -\sqrt{8}\le x\le\sqrt{8}$

& $\frac{x^2}{4}\le y\le 2$

but region is bounded by y-axis so,

$0\le x\le\sqrt{8}$ & $\frac{x^2}{4}\le y\le 2$

$$\therefore \iint_D 5x^3\cos y^3\,dA = \int_0^{\sqrt{8}}\int_{x^2/4}^{2} 5x^3\cos(y^3)\,dy\,dx$$

$= \int_0^{\sqrt{8}} x^3$ This integral is difficult to evaluate so let's change the integral.

$y=\frac{1}{4}x^2$

$\Rightarrow x=2\sqrt{y}$

$\Rightarrow$ x changes from $x=0$ (y-axis) to $x=2\sqrt{y}$ $\Rightarrow 0\le x\le 2\sqrt{y}$

& y changes from 0 to 2. $\Rightarrow 0\le y\le 2$.

$$\therefore \iint_D 5x^3\cos y^3\,dA = \int_0^2\int_0^{2\sqrt{y}} 5x^3\cos(y)^3\,dx\,dy$$

$$= \int_0^2 \left[\frac{5\cos y^3 x^4}{4} \right]_0^{2\sqrt{y}} dy$$

$$= \int_0^2 \frac{5}{4} \cos y^3 (2\sqrt{y})^4 \, dy$$

$$= \int_0^2 20 \cos y^3 \times y^2 \, dy$$

put $y^3 = p$

$\Rightarrow 3y^2 \, dy = dp$

$\Rightarrow y^2 dy = \frac{1}{3} dp$

for $y = 0 \Rightarrow p = 0$

for $y = 2 \Rightarrow p = 8$

$$= 20 \int_0^8 \cos p \times \frac{1}{3} \, dp$$

$$= \frac{20}{3} [\sin p]_0^8$$

$$= \frac{20}{3} [\sin 8]$$

$$= 6.595 \#$$

⑥ Evaluate $\iint_D \frac{1}{\sqrt[3]{y}\,(x^3+1)} \, dA$ where D is the region bounde by $x = -y^{1/3}$, $x = 3$ & the x-axis.

Soln:

$x = -y^{1/3}$ —— (i)

$x = 3$ —— (ii)

Solving,

$3 = -y^{1/3}$

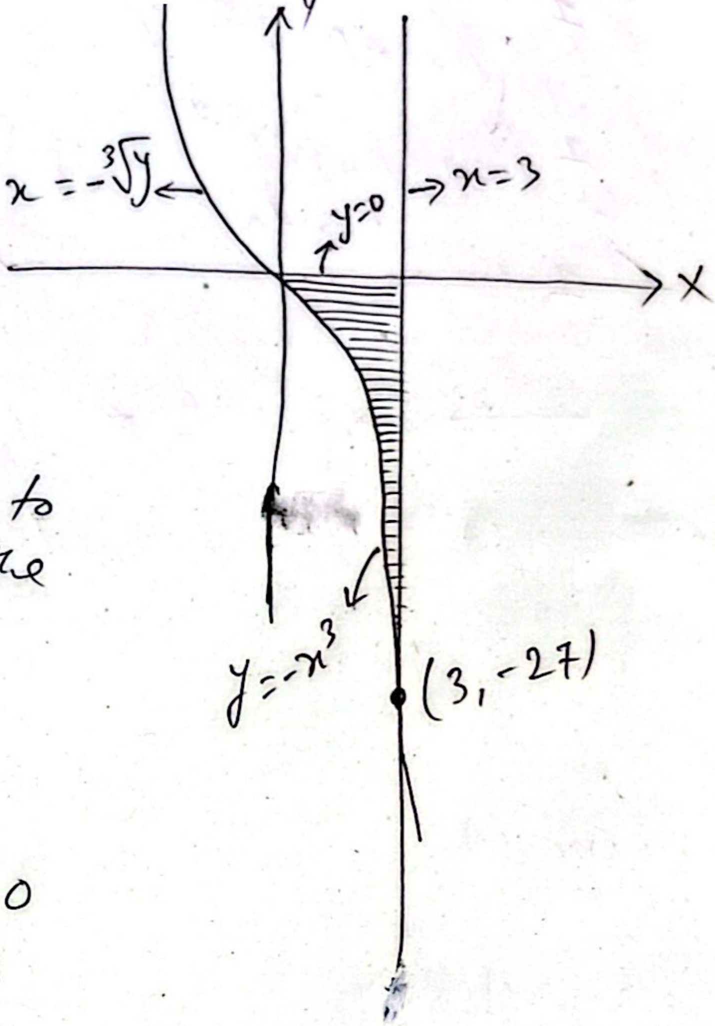
$3^3 = (-y^{1/3})^3$

$\Rightarrow 27 = -y$

$\Rightarrow y = -27$

$\therefore -27 \le y \le 0$ &

$-\sqrt[3]{y} \le x \le 3$



$$\int_{-27}^{0}\int_{-\sqrt[3]{y}}^{3}\left(\frac{1}{\sqrt[3]{y}(x^3+1)}\right) dx\, dy$$

This integral is difficult to evaluate so let's change the integral as;

$x = -y^{1/3}$

$\Rightarrow \cancel{y}\, x^3 = -y$

$\Rightarrow y = -x^3 \Rightarrow -x^3 \le y \le 0$

$\Rightarrow \cancel{0 \le y \le x}$ & $0 \le x \le 3$

$$\therefore \iint_D \frac{1}{\sqrt[3]{y}(x^3+1)} dA = \int_0^3\int_{-x^3}^{0} \frac{1}{\sqrt[3]{y}(x^3+1)} dy\, dx$$

$$= \int_0^3 \frac{1}{x^3+1}\int_{-x^3}^{0} y^{-1/3} dy\, dx$$

$$= \int_0^3 \frac{1}{x^3+1}\left[\frac{y^{-1/3+1}}{-1/3+1}\right]_{-x^3}^{0} dx$$

$$= \int_0^3 \frac{1}{x^3+1} \times \frac{3}{2}\left[(-x^3)^{2/3}\right] dx$$

$$= \frac{3}{2}\int_0^3 \frac{x^2}{x^3+1} dx$$

Let $x^3 = p$

$\Rightarrow 3x^2 dx = dp$

$\Rightarrow x^2 dx = \frac{1}{3} dp$

If $x = 0 \Rightarrow p = 0$

If $x = 3 \Rightarrow p = 27$

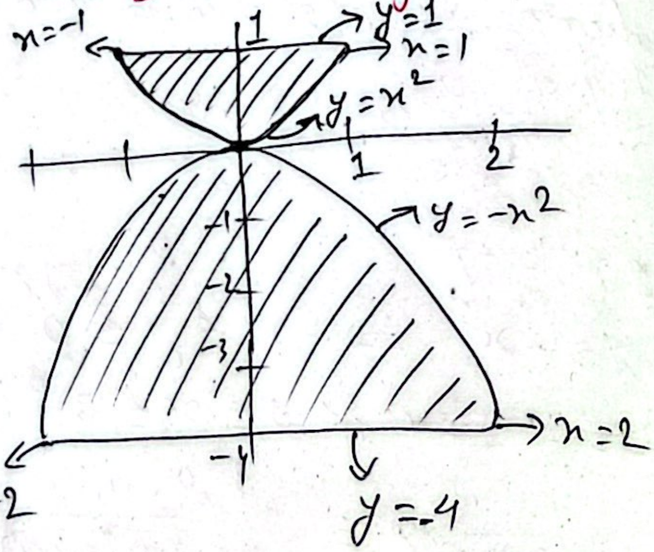
$$= \frac{3}{2}\int_0^{27} \frac{1}{p+1}\, dp$$

$$= \frac{3}{2}\left[\log_e (p+1)\right]_0^{27}$$

$$= \frac{3}{2} \log_e 28$$

$$= \cancel{4.998} \qquad 1.666$$ #

⑦ Evaluate $\iint_D 3-6xy\, dA$ where D is the region shown below.



Soln:

$$\iint_D 3-6xy\, dA$$

$$= \int_{-1}^{1}\int_{x^2}^{1} (3-6xy)\, dy\, dx + \int_{-2}^{2}\int_{-4}^{-x^2} (3-6xy)\, dy\, dx$$

$$= \int_{-1}^{1}\left[3y - \frac{6xy^2}{2}\right]_{x^2}^{1} dx + \int_{-2}^{2}\left[3y - \frac{6xy^2}{2}\right]_{-4}^{-x^2} dx$$

$$= \int_{-1}^{1} 3-3x-3x^2+3x(x^2)^2\, dx + \int_{-2}^{2} 3(-x^2) - 3x(-x^2)^2 - 3\times(-4) + 3x\times(-4)^2\, dx$$

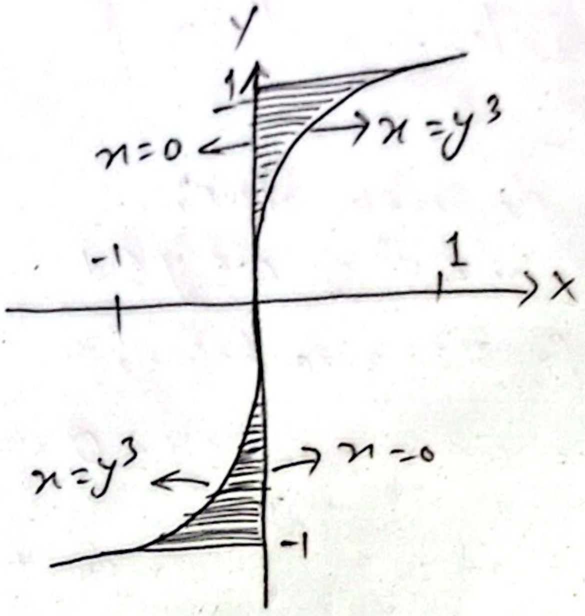
$$= 3\int_{-1}^{1} x^5 - x^2 - x + 1\, dx + 3\int_{-2}^{2} -x^5 - x^2 + 16x + 4\, dx$$

$$= 3\left[\frac{x^6}{6} - \frac{x^3}{3} - \frac{x^2}{2} + x\right]_{-1}^{1} + 3\left[-\frac{x^6}{6} - \frac{x^3}{3} + \frac{16x^2}{2} + 4x\right]_{-2}^{2}$$

$$= 3\left[\cancel{\frac{1}{6}} - \frac{1}{3} - \cancel{\frac{1}{2}} + 1 - \cancel{\frac{1}{6}} - \frac{1}{3} + \cancel{\frac{1}{2}} + 1\right] + 3\left[\cancel{-\frac{2^6}{6}} - \frac{2^3}{3} + \cancel{8\times 2^2} + 4\times 2 + \cancel{\frac{2^6}{6}} - \frac{2^3}{3} - \cancel{8\times 2^2} + \ldots\right]$$

$= 4 + 3\cancel{6}2$

$= \cancel{\quad}\ 36 \ //$



Here,

$0 \le x \le y^3$ & $0 \le y \le 1$

for the above part &

$y^3 \le x \le 0$ & $-1 \le y \le 0$

is for below part

$$\therefore \iint_D 3 - 6xy \, dA = \int_0^1 \int_0^{y^3} (3 - 6xy)\, dx\, dy + \int_{-1}^{0} \int_{y^3}^{0} (3 - 6xy)\, dx\, dy$$

$$= \int_0^1 \left[3x - \frac{6x^2y}{2}\right]_0^{y^3} dy + \int_{-1}^{0} \left[3x - \frac{6x^2y}{2}\right]_{y^3}^{0} dy$$

$$= \int_0^1 \left[3\times y^3 - 3\times(y^3)^2 y\right] dy + \int_{-1}^{0} \left[-3\times y^3 + 3(y^3)^2 y\right] dy$$

$$= \int_0^1 3y^3 - 3y^7 \, dy + \int_{-1}^{0} -3y^3 + 3y^7 \, dy$$

$$= 3\left[\frac{y^4}{4} - \frac{y^8}{8}\right]_0^1 + 3\left[\frac{-y^4}{4} + \frac{y^8}{8}\right]_{-1}^{0}$$

$$= \frac{3}{8}\left[2\times 1 - 1\right] + \frac{3}{8}\left[+2\times 1 - 1\right]$$

$$= \frac{3}{8} + \frac{3}{8}$$

$$= \frac{3}{4} \ //$$

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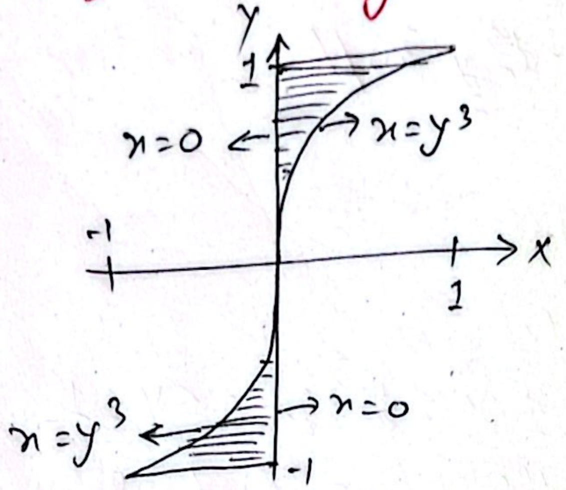
⑧ Evaluate $\iint_D e^{y^4}\,dA$ where D is the region shown below.



Soln:

For the area above;

$0 \le x \le y^3$ & $0 \le y \le 1$

& for the area below;

$y^3 \le x \le 0$ & $-1 \le y \le 0$

$$\therefore \iint_D e^{y^4}\,dA = \int_0^1\int_0^{y^3} e^{y^4}\,dx\,dy + \int_{-1}^{0}\int_{y^3}^{0} e^{y^4}\,dx\,dy$$

$$= \int_0^1 e^{y^4}\times y^3\,dy + \int_{-1}^{0} -e^{y^4} y^3\,dy$$

Let, $y^4 = p$
$\Rightarrow 4y^3 dy = dp$
$\Rightarrow y^3 dy = \frac{1}{4}dp$

If $y = 0 \Rightarrow p = 0$
If $y = 1 \Rightarrow p = 1$
If $y = -1 \Rightarrow p = 1$

$$= \int_0^1 e^p \times \frac{1}{4}\,dp + \int_1^0 -e^p \times \frac{1}{4}\,dp$$

$$= \frac{1}{4}\int_0^1 e^p\,dp + \frac{1}{4}\int_0^1 e^p\,dp$$

$$= \frac{1}{2}\int_0^1 e^p\,dp$$

$$= \frac{1}{2}\left[e^p\right]_0^1$$

$$= \frac{1}{2}[e-1]$$

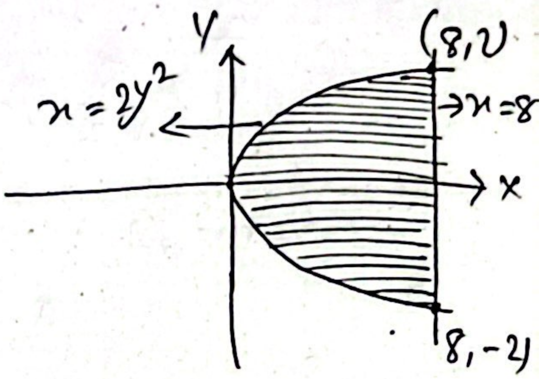
$$= 0.859$$ //

(4) Evaluate $\iint_D 7x^2+14y \, dA$ where D is the region bounded by $x=2y^2$ & $x=8$ in the order given below. Integrate w.r.to x first & then y. Integrate with respept to y first & then x.

Soln:

1st part:

$$x = 2y^2 \quad -(i)$$
$$x = 8 \quad -(ii)$$



Solving, $2y^2 = 8$

$$\Rightarrow y^2 = 4$$
$$\Rightarrow y = \pm 2$$

$\therefore 2y^2 \le x \le 8$ & $-2 \le y \le 2$

$$\therefore \iint_D 7x^2+14y \, dA = \int_{-2}^{2}\int_{2y^2}^{8} (7x^2+14y)\, dx\, dy$$

$$= \int_{-2}^{2}\left[\frac{7x^3}{3} + 14yx\right]_{2y^2}^{8} dy$$

$$= \int_{-2}^{2} \frac{7\times 8^3}{3} + 112y - \frac{7}{3}(2y^2)^3 - 14y(2y^2)\, dy$$

$$= \left[\frac{7\times 8^3}{3}y + \frac{112y^2}{2} - \frac{7\times 8}{3}\frac{y^7}{7} - \frac{28y^4}{4}\right]_{-2}^{2}$$

$$= \frac{7\times 8^3}{3}\times[2+2] + 56[4-4] - \frac{8}{3}[2^7-(-2)^7] - 7[2^4-(-2)^4]$$

$$= \frac{14336}{3} - \frac{2^{11}}{3}$$

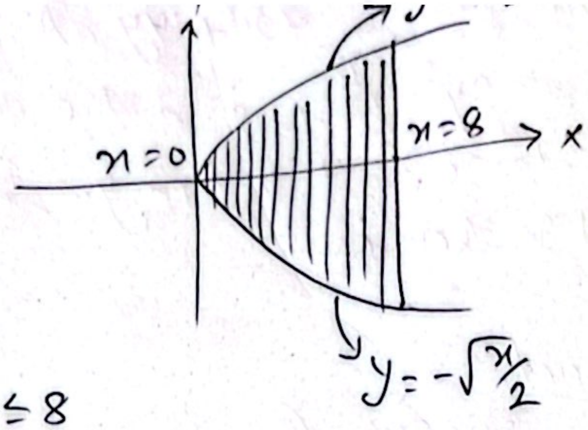
$$= 4096 \,\#$$

2nd part

$x = 2y^2$

$\Rightarrow \frac{x}{2} = y^2$

$\Rightarrow y = \pm\sqrt{\frac{x}{2}}$



$\therefore -\sqrt{\frac{x}{2}} \le y \le \sqrt{\frac{x}{2}}$ & $0 \le x \le 8$

$\therefore \iint_D 7x^2+14y\, dA = \int_0^8 \int_{-\sqrt{x/2}}^{\sqrt{x/2}} (7x^2+14y)\, dy\, dx$

$= \int_0^8 \left[7x^2y + 14\frac{y^2}{2}\right]_{-\sqrt{x/2}}^{\sqrt{x/2}} dx$

$= \int_0^8 14x^2\sqrt{x/2} + 7\left[\left(\sqrt{\frac{x}{2}}\right)^2 - \left(-\sqrt{x/2}\right)^2\right] dx$

$= \int_0^8 7\sqrt{2}\, x^{5/2}\, dx$

$= 7\sqrt{2}\left[\frac{x^{5/2+1}}{5/2+1}\right]_0^8$

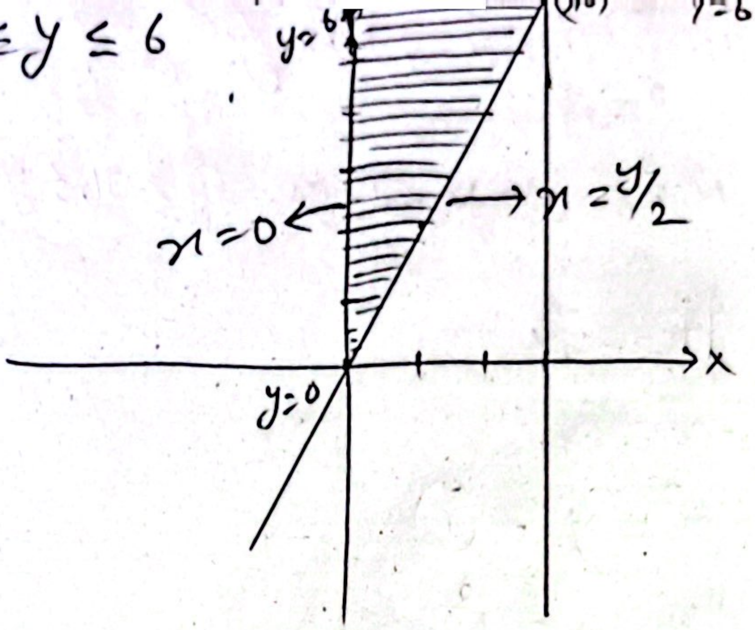
$= 7\sqrt{2} \times \frac{2}{7} \times 8^{7/2}$

$= 4096$ ##

(10) Evaluate the given integral by reversing the order of integration.

$\int_0^3 \int_{2x}^{6} \sqrt{y^2+2}\, dy\, dx$

→ Here $0 \le x \le 3$ & $2x \le y \le 6$



~~From~~

$y = 2x$

$\Rightarrow x = y/2$

$\Rightarrow 0 \le x \le y/2$

& for $x = 0 \Rightarrow y = 0$

for $x = 3 \Rightarrow y = 6$

$\Rightarrow 0 \le y \le 6$

$$\therefore \int_0^3 \int_{2x}^6 \sqrt{y^2+2}\, dy\, dx = \int_0^6 \int_0^{y/2} \sqrt{y^2+2}\, dx\, dy$$

$$= \int_0^6 \left(\sqrt{y^2+2}\right) y/2 \, dy$$

Let, $y^2+2 = p$

$\Rightarrow 2y\, dy = dp$

$\Rightarrow y\, dy = \frac{1}{2} dp$

for $y=0 \Rightarrow p = 2$

for $y=6 \Rightarrow p = 38$

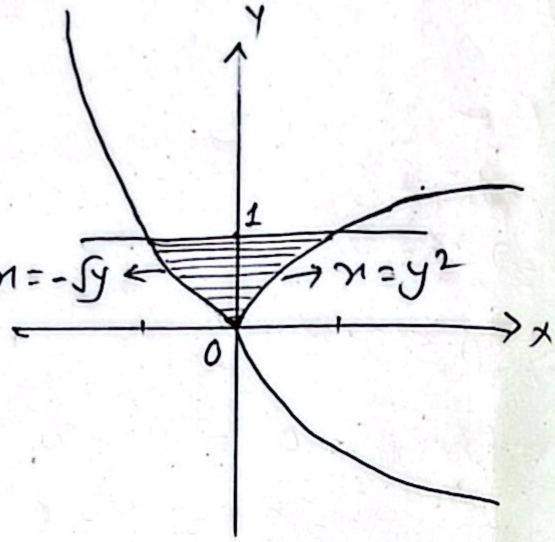
$$= \int_2^{38} \sqrt{p} \times \frac{1}{2} \times \frac{1}{2}\, dp$$

$$= \frac{1}{4} \int_2^{38} p^{1/2}\, dp$$

$$= \frac{1}{4} \left[\frac{p^{3/2}}{3/2} \right]_2^{38}$$

$$= \frac{1}{6} \left[38^{3/2} - 2^{3/2} \right] = 38.56$$

(11) $\int_0^1\int_{-\sqrt{y}}^{y^2}(6x-y)\,dx\,dy$



Here, $0 \le y \le 1$ & $-\sqrt{y} \le x \le y^2$

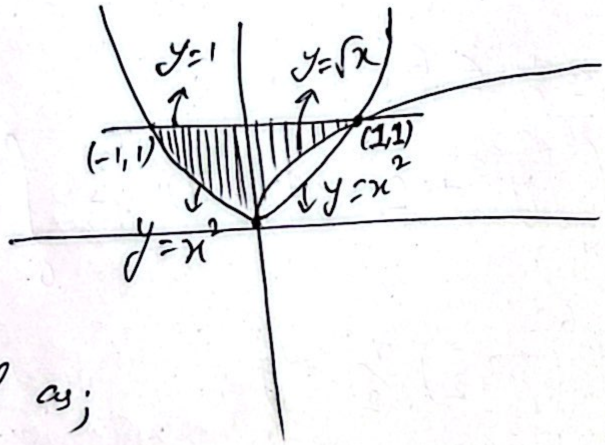
$x = -\sqrt{y}$

$-x = \sqrt{y}$

$x^2 = y$ — (i)

Also, $x = y^2$

$\Rightarrow y = \pm\sqrt{x}$ — (ii)



Now to determine

The $\int_0^1\int_{-\sqrt{y}}^{y^2}(6x-y)\,dx\,dy$.

we can split the integral as;

$$\int_{-1}^{0}\int_{x^2}^{1}(6x-y)\,dy\,dx + \int_{0}^{1}\int_{\sqrt{x}}^{1}(6x-y)\,dy\,dx$$

$$= \int_{-1}^{0}\left[6xy - \frac{y^2}{2}\right]_{x^2}^{1} dx + \int_0^1\left[6xy - y^2/2\right]_{\sqrt{x}}^{1} dx$$

$$= \int_{-1}^{0} 6x - 6x^3 - \frac{1}{2} + \frac{x^4}{2}\,dx + \int_0^1 6x - 6x^{3/2} - \frac{1}{2} + \frac{x}{2}\,dx$$

$$= \left[\frac{6x^2}{2} - \frac{6x^4}{4} - \frac{x}{2} + \frac{x^5}{10}\right]_{-1}^{0} + \left[\frac{6x^2}{2} - \frac{2\cdot 6x^{5/2}}{5} - \frac{x}{2} + \frac{x^2}{4}\right]_0^1$$

$$= -\cancel{3} + \frac{3}{2} - \frac{1}{2} + \frac{1}{10} + \cancel{3} - \frac{12}{5} - \frac{1}{2} + \frac{1}{4}$$

$$= -1.55$$ #

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17) Use a double integral to determine the area of the region bounded by $y = 1 - x^2$ & $y = x^2 - 3$.

<u>Sol^n:</u>

Here, $y = 1 - x^2$ — (i)

$y = x^2 - 3$ — (ii)

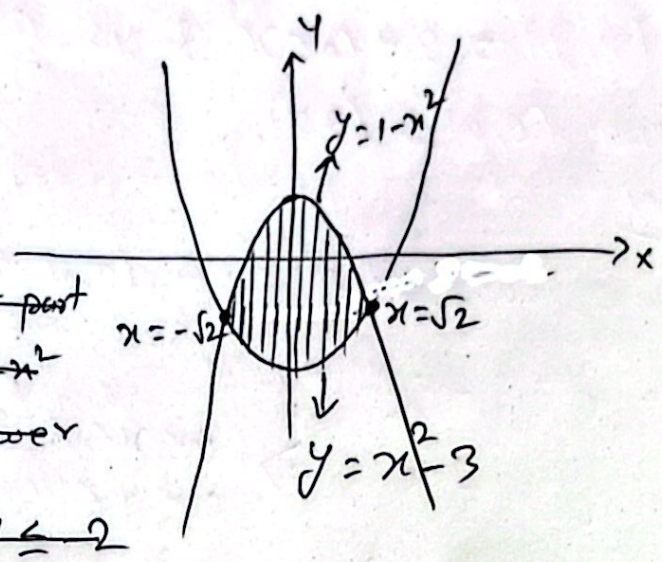
$\Rightarrow x^2 - 3 = 1 - x^2$

$\Rightarrow 2x^2 = 4$

$\Rightarrow x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}$

~~For upper part~~ $\cancel{-2 \le y \le 1 - x^2}$ ~~& for lower part~~ $\cancel{x^2 - 3 \le y \le 2}$



$\therefore \cancel{x^2 - 3 \le y \le 1 - x^2, \; -\sqrt{2} \le x \le \sqrt{2}}$

$\therefore$ Area of bounded region is $\iint_D f(x,y)\, dy\, dx$

Here, $f(x,y) = 1$ [Unit height for area]

$$\iint_D 1\, dy\, dx = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2-3}^{1-x^2} dy\, dx \cancel{+ \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2-3}^{2} dy\, dx}$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} (1 - x^2 - x^2 + 3)\, dx \cancel{+ \int_{-\sqrt{2}}^{\sqrt{2}} -2 - x^2 + 3\, dx}$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} 4 - 2x^2\, dx + \cancel{\int_{-\sqrt{2}}^{\sqrt{2}} 1 - x^2\, dx}$$

$$= \left[4x - \frac{2x^3}{3}\right]_{-\sqrt{2}}^{\sqrt{2}} \cancel{+ \left[x - \frac{x^3}{3}\right]_{-\sqrt{2}}^{\sqrt{2}}}$$

$$= 4\left[\sqrt{2} + \sqrt{2}\right] - \frac{2}{3}\left[\sqrt{2}^3 + \sqrt{2}^3\right] \cancel{\left[\sqrt{2} - \sqrt{2}\right] - \frac{2}{3}\left[(\sqrt{2})^3 + (\sqrt{2})^3\right]}$$

$$= 8\sqrt{2} - \frac{4(\sqrt{2})^3}{3} \cancel{3\sqrt{2} - \frac{(\sqrt{2})^3}{3} + 3\sqrt{2} - \frac{(\sqrt{2})^3}{3} - \sqrt{2} + \frac{(\sqrt{2})^3}{3}}$$

$$= \frac{24\sqrt{2} - 8\sqrt{2}}{3} = \frac{16\sqrt{2}}{3} \; //$$

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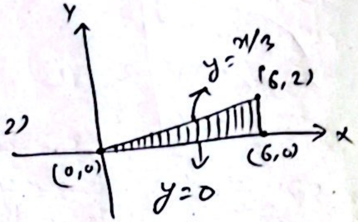
(13) Use a double integral to determine the volume of the region i.e. in between the xy-plane & $f(x,y) = 2+\cos x^2$ & is above the Δ with vertices (0,0), (6,0) & (6,2).



Soln:

The eqn of line from (0,0) to (6,2) is, $y-0 = \frac{2-0}{6-0}(x-0)$

$\Rightarrow y = \frac{1}{3}x \Rightarrow 0\le x\le 6$ & $0\le y\le x/3$

$\therefore V = \iint_D 2+\cos x^2\, dy\,dx = \int_0^6\int_0^{x/3}(2+\cos x^2)\,dy\,dx$

$= \int_0^6 (2+\cos x^2)\frac{x}{3}\,dx$

$= \frac{2}{3}\int_0^6 x\,dx + \frac13\int_0^6 \cos x^2 \cdot x\,dx$

$= \frac23\times\frac12[x^2]_0^6 + \frac13\int_0^{36}\cos p \times \frac12\, dp$

$= \frac13\times 36 + \frac16[\sin p]_0^{36}$

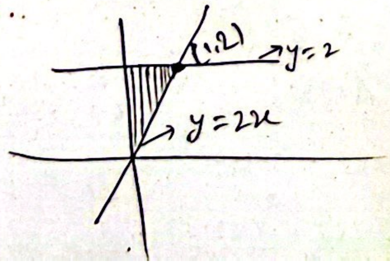
$= 12 + \frac{\sin 36}{6}$

$= 11.83$ #

let $x^2=p$
$2x\,dx = dp$
$x\,dx = \frac12 dp$
For $x=0\to p=0$
$x=6\to p=36$

(14) Use a double integral to determine the volume of the region bounded by $z=6-5x^2$ & the planes $y=2x$, $y=2$, $x=0$ & the xy-plane.

Soln:
Here, $0\le x\le 1$ & $2x\le y\le 2$



$\therefore V = \iint_D f(x,y)\,dy\,dx = \int_0^1\int_{2x}$ …

$= \int_0^1 (6-5x^2)(2-2x)$ …

$= \int_0^1 12-12x-10x^2+1$ …

$= \Big[12x - \frac{12x^2}{2} - 1$ …

$= 12 - 6 - \frac{10}{3} - \frac{5}{2}$

$= \frac{36-20-15}{6}$

$= \frac16$ #

(15) ~~Use a double i~~ … ~~region formed by~~ … ~~$x^2+y^2=4$ & x^2+z~~ …

Soln:

$x^2+y^2=4$ —(i)

$x^2+z^2=4$ —(ii)

$\Rightarrow z^2 = 4-x^2$

$\Rightarrow z = \pm\sqrt{4-x^2}$

The ~~post~~ height of cylinder is $\sqrt{4-x^2}$ …

$\therefore f(x,y) = 2\sqrt{4-x}$ …

$\therefore V = \iint_D f(x,y)\,d$ …

$\therefore V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2\sqrt{}$ …

$\therefore V = \iint_D f(x,y)\, dy\, dx = \int_0 \int_{2x} (6-5x^2)\, dy\, dx$

$= \int_0^1 (6-5x^2)(2-2x)\, dx$

$= \int_0^1 12 - 12x - 10x^2 + 10x^3\, dx$

$= \left[12x - \frac{12x^2}{2} - \frac{10x^3}{3} + \frac{10x^4}{4} \right]_0^1$

$= 12 - 6 - \frac{10}{3} - \frac{5}{2}$

$= \frac{36-20-15}{6}$

$= \frac{1}{6}$ #

(15) Use a double integral to determine the volume of the region formed by the intersections of the two cylinders $x^2+y^2=4$ & $x^2+z^2=4$.

Sol^n:

$x^2+y^2=4$ — (i)

$x^2+z^2=4$ — (ii)

$\Rightarrow z^2 = 4-x^2$

$\Rightarrow z = \pm\sqrt{4-x^2}$

The height of this cylinder is $\sqrt{4-x^2} - (-\sqrt{4-x^2})$

$\therefore f(x,y) = 2\sqrt{4-x^2}$

$\therefore V = \iint_D f(x,y)\, dA = \iint_D 2\sqrt{4-x^2}\, dA$

$\therefore V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2\sqrt{4-x^2}\, dy\, dx$

$= \int_{-2}^{2} 2\sqrt{4-x^2} \times 2\sqrt{4-x^2}\, dx$

$= 4\int_{-2}^{2} (4-x^2)\, dx$

$= 4\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$

$= 4\left[4(2+2) - \frac{1}{3}(2^3+2^3)\right]$

$= 64 - \frac{64}{3}$

$= \frac{128}{3}$ #

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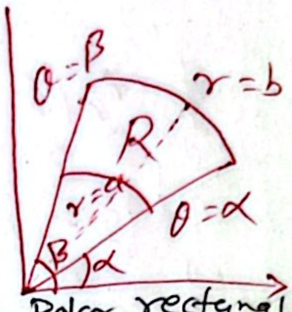
[Double Integral in polar coordinate]



Polar rectangle

$$\boxed{\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta}$$

$0 \le a \le r \le b$, $\alpha \le \theta \le \beta$ also, $0 \le \beta - \alpha$.

$$\boxed{\therefore \iint_D f(x,y)\,dA = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_\alpha^\beta \int_a^b f(r,\theta)\, r\, dr\, d\theta}$$

Example: Let $f(x,y) = e^{x^2+y^2}$ on the disk $D = \{(x,y) : x^2+y^2 \le 1\}$

Evaluate $\iint_D f(x,y)\, dA$

Soln:

Here, $x^2+y^2 \le 1$

$\Rightarrow r^2 \le 1$ $[\because x = r\cos\theta,\ y = r\sin\theta]$

Here, the limits of r is from $r=0$ to $r=1$

& $0 \le \theta \le 2\pi$

$$\therefore \iint_D f(x,y)\, dA = \int_0^{2\pi} \int_0^1 e^{r^2}\, r\, dr\, d\theta$$

Let $r^2 = p$

$\Rightarrow 2r\, dr = dp$

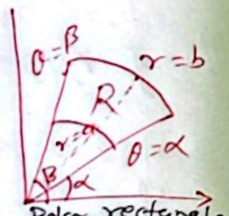
$r\, dr = \frac{1}{2} dp$

For $r=0 \rightarrow p=0$

For $r=1 \rightarrow p=1$

Double Integral in polar coordinate

$$\boxed{\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$



Polar rectangle

$0 \le a \le r \le b$, $\alpha \le \theta \le \beta$ also, $0 \le \beta - \alpha \le 2\pi$

$$\boxed{\therefore \iint_D f(x,y)\,dA = \iint_R f(r,\theta)\, r\,dr\,d\theta = \int_\alpha^\beta \int_a^b f(r,\theta)\, r\,dr\,d\theta}$$

Example: Let $f(x,y) = e^{x^2+y^2}$ on the disk $D = \{(x,y) : x^2+y^2 \le 1\}$

Evaluate $\iint_D f(x,y)\,dA$

Soln: Here, $x^2+y^2 \le 1$

$\Rightarrow r^2 \le 1 \quad [\because x = r\cos\theta,\ y = r\sin\theta]$

Here, the limits of r is from $r=0$ to $r=1$

& $0 \le \theta \le 2\pi$

$$\therefore \iint_D f(x,y)\,dA = \int_0^{2\pi}\int_0^1 e^{r^2}\, r\,dr\,d\theta$$

Let $r^2 = p$
$\Rightarrow 2r\,dr = dp$
$r\,dr = \frac{1}{2}dp$

For $r=0 \to p=0$
For $r=1 \to p=1$

$$= \int_0^{2\pi}\left\{\frac{1}{2}\int_0^1 e^p\,dp\right\}d\theta$$

$$= \int_0^{2\pi} \frac{(e-1)}{2}\,d\theta$$

$$= \left(\frac{e-1}{2}\right)\times 2\pi$$

$$= \pi(e-1)\ \#$$

Determine the volume of the region that lies under the sphere $x^2+y^2+z^2=9$ & above the plane $z=0$ & inside the cylinder $x^2+y^2=5$.

Soln:

$V = \iint_D f(x,y)\,dA$

$f(x,y) = z = \sqrt{9-x^2-y^2} = \sqrt{9-r^2}$

Also, the region D is given as: $x^2+y^2=5$

$\Rightarrow r^2 = 5$

$\Rightarrow$ The range of r is from $r=0$ to $r=\sqrt{5}$.

$\therefore D = \{(r,\theta) : 0 \le r \le \sqrt{5},\ 0 \le \theta \le 2\pi\}$

$$V = \iint_D \sqrt{9-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^{\sqrt{5}} \sqrt{9-r^2}\, r\,dr\,d\theta$$

Let $9-r^2 = p$
$-2r\,dr = dp$
$r\,dr = -\frac{1}{2}dp$

for $r=0 \to p=9$
for $r=\sqrt{5} \to p=4$

$$= \int_0^{2\pi} -\frac{1}{2}\int_9^4 p^{1/2}\,dp\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\left[\frac{2}{3}p^{3/2}\right]_9^4 d\theta$$

$$= \int_0 -\frac{1}{3}\left[4^{3/2} - 9^{3/2}\right] d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}\left[8 - 27\right] d\theta$$

$$= \int_0^{2\pi} -\frac{19}{3}\, d\theta$$

$$= \cancel{\frac{19\pi}{3}}\ \cancel{2\pi}$$

$$= \frac{38\pi}{3}$$ #

Find the volume of the region that lies inside $z = x^2+y^2$ & below the plane $z=16$.

Sol^n:

$z = x^2+y^2$ — (i)

$z = 16$

$\Rightarrow x^2+y^2 = 16$

$\Rightarrow r^2 = 16$

$\Rightarrow \therefore D = \{(r,\theta) : 0 \le r \le 4,\ 0 \le \theta \le 2\pi\}$

$$= \iint_D f(x,y)\, dA = \iint_D 16\, dA - \iint_D x^2+y^2\, dA$$

$$\int_0^{2\pi}\int_0^4 (16-r^2)\, r\, dr\, d\theta$$

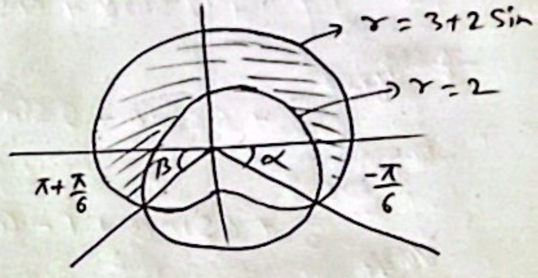
$$\int_0^{2\pi}\int_0^4 16r - r^3\, dr\, d\theta$$

$$\int_0^{2\pi}\left[\frac{16r^2}{2} - \frac{r^4}{4}\right]_0^4 d\theta$$

$$\int_0^{2\pi} 64\, d\theta = 128\pi$$

Use double integral to determine the area of the region that is inside $r = 3+2\sin\theta$ & outside $r=2$.

Sol^n:



$r = 3+2\sin\theta$ — (i)

$r = 2$ — (ii)

Solving;

$3+2\sin\theta = 2$

$\sin\theta = -\frac{1}{2} = -\sin 30°$

$\Rightarrow \theta = -\frac{\pi}{6}$

Let $\alpha = -\frac{\pi}{6}$

$\Rightarrow \therefore$ ~~we know that~~, From figure;

$\beta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

$\Rightarrow \cancel{\beta = 2\pi - \alpha}$

$\Rightarrow \cancel{\beta = 2\pi - (0.523)}$

$\Rightarrow \cancel{\beta = 6.806}$

$\Rightarrow \cancel{-0.523 \le \theta \le 6.806}\ \ -\frac{\pi}{6} \le \theta \le \frac{7\pi}{6}$

& $2 \le r \le 3+2\sin\theta$

$\therefore$ Area $= \iint_D f(r,\theta)\, r\, dr\, d\theta$

Here, $f(r,\theta) = 1$ [Unit height for area]

$$\text{Area} = \int_{-\pi/6}^{7\pi/6}\int_2^{3+2\sin\theta} r\, dr\, d\theta$$

$$= \int_{-\pi/6}^{7\pi/6} \frac{1}{2}\left[r^2\right]_2^{3+2\sin\theta} d\theta$$

$$= \int_{-\pi/6}^{6} \frac{1}{2}\left[(3+2\sin\theta)^2 - 4\right] d\theta$$

$$= \int_{-\pi/6}^{7\pi/6} \frac{1}{2}\left(9 + 12\sin\theta + 4\sin^2\theta - 4\right) d\theta$$

$$= \int_{-\pi/6}^{7\pi/6} \frac{5}{2} + 6\sin\theta + 2\sin^2\theta \, d\theta$$

$$= \int_{-\pi/6}^{7\pi/6} \frac{5}{2} + 6\sin\theta + 1 - \cos 2\theta \, d\theta$$

$$= \int_{-\pi/6}^{7\pi/6} \frac{7}{2} + 6\sin\theta - \cos 2\theta \, d\theta$$

$$= \left[\frac{7\theta}{2} - 6\cos\theta - \frac{\sin 2\theta}{2}\right]_{-\pi/6}^{7\pi/6}$$

$$= \frac{7}{2}\left(\frac{7\pi}{6} + \frac{\pi}{6}\right) - 6\left(\cos\left(\frac{7\pi}{6}\right) - \cos\left(-\frac{\pi}{6}\right)\right) - \frac{1}{2}\left[\sin\left(2\times\frac{7\pi}{6}\right) - \sin\left(2\times\left(-\frac{\pi}{6}\right)\right)\right]$$

$= \cancel{24.78}$ // $\cancel{24.78}$

$$= \frac{14\pi}{3} - 6\left[\cos\left(\pi + \frac{\pi}{6}\right) - \frac{\sqrt{3}}{2}\right] - \frac{1}{2}\times 2\sin\frac{7\pi}{6}\cos\frac{7\pi}{6} + \frac{1}{2}\times 2\sin\left(-\frac{\pi}{6}\right)\cos\left(-\frac{\pi}{6}\right)$$

$$= \frac{14\pi}{3} - 6\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2}\right)\times\left(-\frac{\sqrt{3}}{2}\right) + \left(-\frac{\sqrt{3}}{4}\right)$$

$$= \frac{14\pi}{3} + 6\sqrt{3} - \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}$$

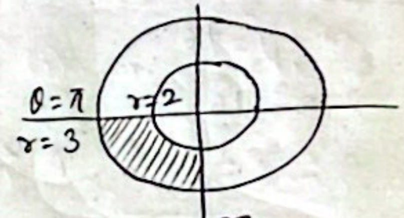
$$= \frac{56\pi + 66\sqrt{3}}{12} = \frac{1}{6}\left(33\sqrt{3} + 28\pi\right) \approx 24.187$$

$$= \cancel{\frac{14\pi}{3} + \frac{133\sqrt{3}}{12}}$$

Assignment (Practice Problems)

1. Evaluate $\iint_D y^2 + 3x \, dA$ where D is the region in 3^{rd} quadrant between $x^2+y^2=1$ & $x^2+y^2=9$.

Soln:



$x^2+y^2=1$ —(i) $\Rightarrow r^2 = 1$

$x^2+y^2=9$ —(ii) $\Rightarrow r^2 = 9$

$\Rightarrow 1 \le r \le 3$ & $\pi \le \theta \le \frac{3\pi}{2}$

$$\iint_D y^2 + 3x \, dA = \int_{\pi}^{3\pi/2}\int_1^3 \left(r^2\sin^2\theta + 3r\cos\theta\right) r \, dr \, d\theta$$

$$= \int_{\pi}^{3\pi/2}\int_1^3 \left(r^3\sin^2\theta + 3r^2\cos\theta\right) dr \, d\theta$$

$$= \int_{\pi}^{3\pi/2}\left[\frac{\sin^2\theta \, r^4}{4} + \frac{3r^3}{3}\cos\theta\right]_1^3 d\theta$$

$$= \int_{\pi}^{3\pi/2}\left(\frac{81}{4}\sin^2\theta + 27\cos\theta - \frac{\sin^2\theta}{4} - \cos\theta\right) d\theta$$

$$= \int_{\pi}^{3\pi/2}\left(20\sin^2\theta + 26\cos\theta\right) d\theta$$

$$= \int_{\pi}^{3\pi/2} 10(1-\cos 2\theta) + 26\cos\theta \, d\theta$$

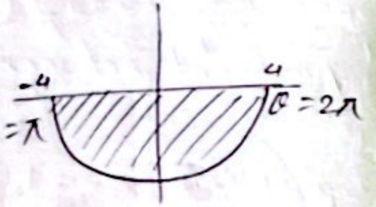
$$= \int_{\pi}^{3\pi/2} 10 - 10\cos 2\theta + 26\cos\theta \, d\theta$$

$$= \left[10\theta - \frac{10}{2}\sin 2\theta + 26\sin\theta\right]_{\pi}^{3\pi/2}$$

$$= 10\times\frac{3\pi}{2} - 5\sin\left(2\times\frac{3\pi}{2}\right) + 26\sin\frac{3\pi}{2} - 10\pi + 5\sin 2\pi - 26\sin\pi$$

(2.) Evaluate $\iint_D \sqrt{1+4x^2+4y^2}\,dA$ where D is ... half of $x^2+y^2=16$.

Soln:



$x^2+y^2=16$

$\Rightarrow r^2=16$

$\therefore 0 \le r \le 4$ & $\pi \le \theta \le 2\pi$

$\therefore \iint_D \sqrt{1+4x^2+4y^2}\,dA = \iint_D \sqrt{1+4(x^2+y^2)}\,dA$

$= \int_\pi^{2\pi}\int_0^4 \sqrt{1+4r^2}\; r\,dr\,d\theta$

Let, $1+4r^2 = p$

$\Rightarrow 8r\,dr = dp$

$\Rightarrow r\,dr = \frac{1}{8}dp$

For $r=0 \Rightarrow p=1$

For $r=4 \Rightarrow p=65$

$= \int_\pi^{2\pi}\int_1^{65} \sqrt{p}\cdot\frac{1}{8}\,dp\,d\theta$

$= \int_\pi^{2\pi} \frac{1}{8}\left[p^{3/2}\times\frac{2}{3}\right]_1^{65} dp\,d\theta$

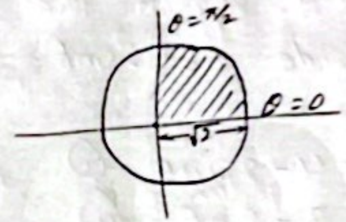
$= \frac{1}{12}\int_\pi^{2\pi}\left[p^{3/2}\right]_1^{65} d\theta$

$= \frac{1}{12}\int_\pi^{2\pi} (65)^{3/2} - 1\,d\theta$

$= \frac{\pi}{12}\left[65^{3/2}-1\right]$ #

3. Evaluate $\iint_D 4xy-7\,dA$ where D is ... in the 1st quadrant.

Soln:



$x^2+y^2=2$

$\Rightarrow r^2=2$

$\therefore 0 \le r \le \sqrt{2}$ & $0 \le \theta \le \pi/2$

$\therefore \iint_D 4xy-7\,dA = \int_0^{\pi/2}\int_0^{\sqrt{2}} (4r^2\sin\theta\cos\theta - 7)\,r\,dr\,d\theta$

$= \int_0^{\pi/2}\left(4\sin\theta\cos\theta\cdot\frac{r^4}{4} - \frac{7r^2}{2}\right)_0^{\sqrt{2}} d\theta$

$= \int_0^{\pi/2}\left(\sin\theta\cos\theta\,(\sqrt{2})^4 - \frac{7(\sqrt{2})^2}{2}\right) d\theta$

$= \int_0^{\pi/2} (\sin 2\theta - 7)\,d\theta$

$= \left[\frac{-2\cos 2\theta}{2} - 7\theta\right]_0^{\pi/2}$

$= -\cos\pi - \frac{7\pi}{2} + \cos 0$

$= -(-1) - \frac{7\pi}{2} + 1$

$= 2 - \frac{7\pi}{2}$

$= -\frac{3\pi}{2}$ #

$= \cancel{\frac{1}{2}\int_0^{\pi/2}(-13-\cos 4\theta)\,d\theta}$

$= \cancel{\frac{1}{2}\left[-13\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}}$

$= \cancel{\frac{1}{2}\left[-\frac{13\pi}{2} - \frac{\sin 2\pi}{4}\right]}$

$= -\frac{13\pi}{4}$

... area of the region that is inside $r=4+2\sin\theta$ & outside $r=3-\sin\theta$.

Sol^n:

$r=4+2\sin\theta$ —(i)

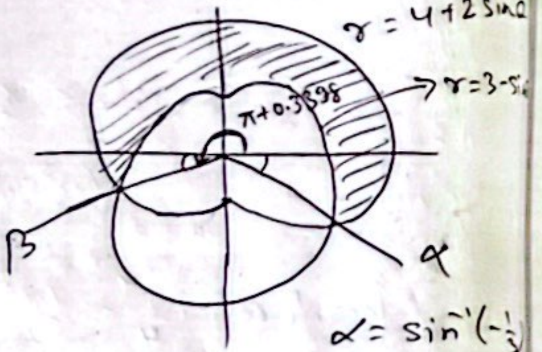
$r=3-\sin\theta$ —(ii)

$\Rightarrow 4+2\sin\theta = 3-\sin\theta$

$\Rightarrow 3\sin\theta=-1$

$\Rightarrow \sin\theta = -\frac{1}{3}$

$\Rightarrow \theta = \sin^{-1}(\frac{-1}{3})$



$\alpha = \sin^{-1}(-\frac{1}{3}) = -0.3398$

~~Let, $\alpha = \sin^{-1}(\frac{-1}{3})$~~

~~$\alpha+\beta=2\pi$~~

$\Rightarrow$ ~~$\beta = 2\pi - \alpha$~~

$\Rightarrow$ ~~$\beta = 2\pi - \sin^{-1}(\frac{-1}{3})$~~

Here,

$\beta = \pi + 0.3398 = 3.4814$

$\Rightarrow$ ~~$\beta = \pi - (-0.3398)$~~

$=$ ~~3.4814~~

$\therefore$ ~~$\sin^{-1}(\frac{-1}{3}) \le \theta \le (2\pi - \sin^{-1}(\frac{-1}{3}))$~~ $\Rightarrow -0.3398 \le \theta \le 3.4184$

$$\therefore \text{Area} = \int_{-0.3398}^{3.4184}\int_{3-\sin\theta}^{4+2\sin\theta} 1 \cdot r\,dr\,d\theta \quad [\text{For area } f(r,\theta)=1]$$

$$= \int_{-0.3398}^{3.4184}\left[\frac{r^2}{2}\right]_{3-\sin\theta}^{4+2\sin\theta} d\theta$$

$$= \int_{-0.3398}^{3.4184} \frac{1}{2}\left((4+2\sin\theta)^2 - (3-\sin\theta)^2\right) d\theta$$

$$= \frac{1}{2}\int_{-0.3398}^{3.4184} (16+16\sin\theta+4\sin^2\theta - 9 + 6\sin\theta - \sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_{-0.3398}^{3.4184}[7+22\sin\theta+3\sin^2\theta]\,d\theta$$

$$= \frac{1}{2}\int_{-0.3398}^{3.4184}\left[7+22\sin\theta+\frac{3}{2}(1-\cos2\theta)\right]d\theta$$

$$= \frac{1}{2}\left[7\theta - 22\cos\theta + \frac{3\theta}{2} - \frac{3\sin2\theta}{4}\right]_{-0.3398}^{3.4184}$$

$$= \frac{1}{2}\left[\frac{17\theta}{2} - 22\cos\theta - \frac{3\sin2\theta}{4}\right]_{-0.3398}^{3.4184}$$

$= \frac{1}{2}$ ~~$\left[\frac{17}{2}(2\pi - \sin^{-1}(-\frac{1}{3}) - \sin^{-1}(-\frac{1}{3})) - 22\cos[2\pi - \sin^{-1}(-\frac{1}{3})] + 22\cos\sin^{-1}(-\frac{1}{3}) - \frac{3}{4}[\sin(4\pi - 2\sin^{-1}(-\frac{1}{3})) - \sin(2\sin^{-1}\ldots\right.$~~

$= 36.5108$ #

(5) Evaluate the following integral by 1st converting to an integral in polar coordinates.

$$\int_0^3\int_{-\sqrt{9-x^2}}^{0} e^{x^2+y^2}\,dy\,dx$$

Sol^n:

In polar coordinates,

$x = r\cos\theta,\ y = r\sin\theta$

$\Rightarrow x^2+y^2=r^2$

For $x=0$

$\Rightarrow r\cos\theta = 0$

$\Rightarrow \theta = \cos^{-1}$

to an integral in polar coordinates.

$$\int_0^3 \int_{-\sqrt{9-x^2}}^0 e^{x^2+y^2}\,dy\,dx$$

Soln:

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = r^2$ — (i)

~~for diff. w.r.to x;~~ $\therefore$ ~~$x = r\cos\theta$~~

~~$2x\,dx = 2r\,dr$~~ For ~~$x=0 \Rightarrow r\cos\theta = 0$~~

~~$\Rightarrow$ for $x=0$~~ ~~$\Rightarrow \cos\theta = \cos\frac{\pi}{2}$~~

~~For $x=3$~~, Here: $-\sqrt{9-x^2} \le y \le 0$ ~~$\Rightarrow \theta = \frac{\pi}{2}$~~

~~$r\cos\theta = 3$~~ & $0 \le x \le 3$

~~$\Rightarrow \cos\theta$~~

$\Rightarrow y = -\sqrt{9-x^2}$

$\Rightarrow y^2 = 9-x^2$

$\Rightarrow r^2\sin^2\theta = 9 - r^2\cos^2\theta$

$\Rightarrow r^2\sin^2\theta + r^2\cos^2\theta = 9$

$\Rightarrow r^2 = 9$

$\Rightarrow 0 \le r \le 3$

For $x=0$

$\Rightarrow r\cos\theta = 0$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \cos\theta = \cos\frac{\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{2}$

For $x = 3$

$r\cos\theta = 3$

$3\cos\theta = 3$

$\cos\theta = 1$

$\cos\theta = \cos 0$

$\Rightarrow \theta = 0$

$\Rightarrow 0 \le \theta \le \frac{\pi}{2}$

$$\therefore \int_0^3 \int_{-\sqrt{9-x^2}}^0 e^{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^3 e^{r^2}\, r\,dr\,d\theta$$

put $r^2 = p$

$\Rightarrow 2r\,dr = dp$

$\Rightarrow r\,dr = \frac{1}{2}dp$

for $r = 0 \Rightarrow p = 0$

$r = 3 \Rightarrow p = 9$

$$= \int_0^{\pi/2} \left\{\frac{1}{2}\int_0^9 e^p\,dp\right\} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}[e^p]_0^9\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}[e^9 - e^0]\,d\theta$$

$$= \frac{1}{2}(e^9-1)\int_0^{\pi/2} d\theta$$

$$= \frac{1}{2}(e^9-1)\times\frac{\pi}{2}$$

$$= \frac{\pi}{4}(e^9-1)$$ #

⑥ Use a double integral to determine the volume of the solid that is inside the cylinder $x^2+y^2=16$, below $z = 2x^2+2y^2$ & above the xy-plane.

Soln:

$x^2+y^2 = 16$

$\Rightarrow r^2 = 16$

$\therefore 0 \le r \le 4$ & $0 \le \theta \le 2\pi$

$$V = \iint_D 2x^2+2y^2\,dA = \int_0^{2\pi}\int_0^4 2r^2\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{2}{4}[r^4]_0^4\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}[4^4]\,d\theta$$

$= 128 \times 2\pi$

$= 256\pi$ #

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① Use a double integral to de...
of the solid that is bounded by $Z = 8 - x^2 - y^2$ & $z = 3x^2 + 3y^2 - 4$.

Sol^n:

$$Z = 8 - x^2 - y^2 \quad \text{--- (i)}, \qquad Z = 3x^2 + 3y^2 - 4 \quad \text{--- (ii)}$$

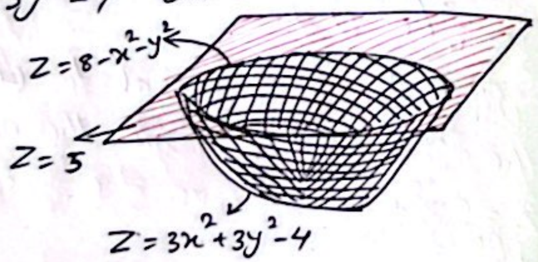
Solving (i) & (ii);

$$8 - x^2 - y^2 = 3x^2 + 3y^2 - 4$$
$$\Rightarrow 12 = 4(x^2 + y^2)$$
$$\Rightarrow x^2 + y^2 = 3 \quad \text{--- (iii)}$$
$$\Rightarrow r^2 = 3$$
$$\Rightarrow 0 \le r \le \sqrt{3}$$

Also, from (i);

$$Z = 8 - x^2 - y^2$$
$$\Rightarrow Z = 8 - (x^2 + y^2)$$
$$\Rightarrow Z = 8 - 3$$
$$\Rightarrow Z = 5$$

$\therefore Z = 8 - x^2 - y^2$ & $Z = 3x^2 + 3y^2 - 4$ intersects at $Z = 5$



The ~~area~~ volume of the region above z=3 is;

$$\iint_D (8 - x^2 - y^2)\, dA - \iint_D 5\, dA$$

$$\therefore V_{above} = \iint_D (8 - x^2 - y^2 - 5)\, dA$$

$$V_{above} = \int_0^{2\pi}\int_0^{\sqrt{3}} (3 - x^2 - y^2)\, dA = \int_0^{2\pi}\int_0^{\sqrt{3}} (3 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{3r^2}{2} - \frac{r^4}{4}\right]_0^{\sqrt{3}} d\theta$$

$$= \int_0^{2\pi} \left[\frac{3 \times (\sqrt{3})^2}{2} - \frac{(\sqrt{3})^4}{4}\right] d\theta$$

$$= \int_0^{2\pi} \frac{9}{4}\, d\theta = \frac{9}{4} \times 2\pi = \frac{9\pi}{2}$$

$$V_{below} = \iint_D 5\, dA - \iint_D (3x^2 + 3y^2 - 4)\, dA$$

$$= \iint_D (5 - 3x^2 - 3y^2 + 4)\, dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} (9 - 3r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{9r^2}{2} - \frac{3r^4}{4}\right]_0^{\sqrt{3}} d\theta$$

$$= \int_0^{2\pi} \left(\frac{27}{2} - \frac{27}{4}\right) d\theta$$

$$= \frac{27}{4} \times 2\pi$$

$$= \frac{27\pi}{2}$$

$$\therefore \text{Volume} = V_{above} + V_{below}$$
$$= \frac{9\pi}{2} + \frac{27\pi}{2}$$
$$= \frac{36\pi}{2}$$
$$= 18\pi$$

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Triple Integrals

1. Evaluate $\int_2^3\int_{-1}^4\int_1^0 (4x^2y - z^3)\,dz\,dy\,dx$

$= \int_2^3\int_{-1}^4 \left[4x^2yz - \frac{z^4}{4}\right]_1^0 dy\,dx$

$= \int_2^3\int_{-1}^4 \left[-4x^2y\times 1 + \frac{1^4}{4}\right] dy\,dx$

$= \int_2^3\int_{-1}^4 (-4x^2y + \frac{1}{4})\,dy\,dx$

$= \int_2^3 \left[-\frac{4x^2y^2}{2} + \frac{y}{4}\right]_{-1}^4 dx$

$= \int_2^3 \left[-2x^2 4^2 + \frac{4}{4} + 2x^2(-1)^2 - \frac{(-1)}{4}\right] dx$

$= \int_2^3 (-32x^2 + 1 + 2x^2 + \frac{1}{4})\,dx$

$= \int_2^3 (\frac{5}{4} - 30x^2)\,dx$

$= \left[\frac{5x}{4} - \frac{30x^3}{3}\right]_2^3$

$= \frac{15}{4} - 270 - \frac{5}{2} + 80$

$= -188.75$ #

2. $\int_0^1\int_0^{z^2}\int_0^3 y\cos(z^5)\,dx\,dy\,dz$

$= \int_0^1\int_0^{z^2} 3y\cos(z^5)\,dy\,dz$

$= \int_0^1 \frac{3\cos(z^5)}{2}\left[y^2\right]_0^{z^2} dz$

$= \int_0^1 \frac{3}{2} z^4\cos(z^5)\,dz$

put $z^5 = p$ For $z=0 \Rightarrow p=0$

$\Rightarrow 5z^4\,dz = dp$ For $z=1 \Rightarrow p=1$

$\Rightarrow z^4\,dz = \frac{1}{5}dp$

$= \frac{3}{2}\times\frac{1}{5}\int_0^1 \cos p\,dp$

$= \frac{3}{10}\left[\sin p\right]_0^1$

$= \frac{3\sin 1}{10}$ #

$= 0.2524$ #

3. Evaluate $\iiint_E 6z^2\,dV$ where E is the region below $4x+y+2z=10$ in the 1st octant.

Solⁿ:

$V = \iiint_E 6z^2\,dV$ where the region E is given as

$4x+y+2z=10$	For xy plane, $z=0$	Similarly, $y=0,\ z=0$
$\Rightarrow z = 5-2x-\frac{y}{2}$	$\Rightarrow 4x+y=10$	$\Rightarrow 4x=10$
$\Rightarrow 0\le z\le 5-2x-\frac{y}{2}$	$\Rightarrow y = 10-4x$	$\Rightarrow x=\frac{5}{2}$
	$\Rightarrow 0\le y\le 10-4x$	$\Rightarrow 0\le x\le 5/2$

$$\therefore V = \iiint_E 6z^2\, dV = \int_0 \int_0 \int_0 (6z^2)\, dz\, dy\, dx$$

$$= \int_0^{5/2}\int_0^{10-4x} \frac{6}{3}\left[z^3\right]_0^{5-2x-y/2} dy\, dx$$

$$= \int_0^{5/2}\int_0^{10-4x} 2\left(5-2x-\tfrac{y}{2}\right)^3 dy\, dx$$

$$= \int_0^{5/2} 2\left[\frac{(5-2x-\frac{y}{2})^4}{4\times(-\frac{1}{2})}\right]_0^{10-4x} dx$$

$$= -\int_0^{5/2} \left(5-2x-\frac{(10-4x)}{2}\right)^4 \cancel{dx} - (5-2x)^4\, dx$$

$$= \int_0^{5/2} (5-2x)^4\, dx$$

$$= \left[\frac{(5-2x)^5}{-5\times 2}\right]_0^{5/2}$$

$$= \frac{(5-2\times\frac{5}{2})^5}{-10} + \frac{5^5}{10}$$

$$= 0 + \frac{625}{2}$$

$$= 312.5 \ \#$$

(4) Evaluate $\iiint_E 3-4x\, dV$ where E is the region below $z=4-xy$ & above the region in the xy-plane defined by $0\le x\le 2$, $0\le y\le 1$.

Sol^n:

$$V = \iiint_E 3-4x\, dV = \int_0^2\int_0^1\int_0^{4-xy} (3-4x)\, dz\, dy\, dx$$

$$= \int_0^2\int_0^1 (3-4x)(4-xy)\, dy\, dx$$

$$= \int_0^2\int_0^1 (12-3xy-16x+4x^2y)\, dy\, dx$$

$$= \int_0^2 \left[12y - \frac{3xy^2}{2} - 16xy + \frac{4x^2y^2}{2}\right]_0^1 dx$$

$$= \int_0^2 \left(12 - \frac{3x}{2} - 16x + 2x^2\right) dx$$

$$= \left[12x - \frac{3x^2}{4} - \frac{16x^2}{2} + \frac{2x^3}{3}\right]_0^2$$

$$= 24 - 3 - 32 + \frac{16}{3}$$

$$= -\frac{17}{3} \ \#$$

(5) Evaluate $\iiint_E (12y-8x)\, dV$ where E is the region ~~below~~ behind $y=10-2z$ & in front of the ~~xy~~ region in the xz-plane bounded by $z=2x$, $z=5$, $x=0$.

Sol^n:

$y = 10-2z \Rightarrow 0 \le y \le 10-2z$

$z = 2x$
$z = 2\times 0 \Rightarrow 0 \le z \le 5$

$z = 2x$
$\Rightarrow 0 \le x \le z/2$

$z/2 = x$

$$\therefore \iiint_E (12y-8x)\, dV = \int_0^5\int_0^{z/2}\int_0^{10-2z} (12-8x)\, dy\, dx\, dz$$

$$= \int_0 \int_0^{z} \left[\frac{12y^2}{2} - 8xy\right]_0^{10-2z} dx\, dz$$

$$= \int_0^5 \int_0^{z/2} 6(10-2z)^2 - 8x(10-2z) \; dx\, dz$$

$$= \int_0^5 \int_0^{z/2} 600 - 240z + 24z^2 - 80x + 16xz \; dx\, dz$$

$$= \int_0^5 \left[600x - 240zx + 24z^2x - \frac{80x^2}{2} + \frac{16zx^2}{2}\right]_0^{z/2} dz$$

$$= \int_0^5 300z - 120z^2 + 12z^3 - 10z^2 + 2z^3 \; dz$$

$$= \int_0^5 14z^3 - 130z^2 + 300z \; dz$$

$$= \left[\frac{14z^4}{4} - \frac{130z^3}{3} + \frac{300z^2}{2}\right]_0^5$$

$$= \frac{7\times 5^4}{4} - \frac{130\times 5^3}{3} + 150\times 5^2$$

$$= 520.83 \#$$

⑥ Evaluate $\iiint_E yz \, dV$ where E is the region bounded by $x = 2y^2 + 2z^2 - 5$ & the plane $x = 1$.