Tribhuvan University
School of Mathematical Sciences, Kirtipur,Kathmandu, Nepal
Problem Set For Master in Data Sciences I Year (II Sem)-2080
Course: Multivariable Calculus For Data Science -II (MDS 554)
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Unit 1: Vectors and Geometry of Spaces

The problems are taken from the book "***Multivariable Calculus***", ***By James Stewart***, 7th Edition From Chapter-12, Vectors and Geometry of Spaces. Only few answers are given.

Note: Students are advised to give their answers in their own words. The final date of submission is………. …….. The marks of assignment will be added in the internal marks.

Assignment Problem Set –I(Exercise 12.1,12.2, and 12.3 (3D Co-ordinate and Vector, Dot Product)):

3D Co-ordinate System:

1. (i) Find an equation of a sphere with centre (-3,2,5) and radius 4. What is the intersection of this sphere with the *yz* plane? **Ans: $(y - 2)^2 + (z - 5)^2 = 7, x = 0$(a circle)**

(ii) If $\vec{r} = (x, y, z)$ and $\vec{r_0} = (x_0, y_0, z_0)$, describe the set of all points (x, y, z) such that $|\vec{r} - \vec{r_0}| = 1$.
Ans: Sphere with centre radius 1 and centre (x_0, y_0, z_0).

(iii) Find equations of the spheres with center (2, - 3, 6) that touch the -*xy*-plane
Ans: (i) $(x-2)^2 + (y + 3)^2 + (z - 6)^2 = 6^2$

Vectors and Vector Geometry:

1. (i) Find a unit vector that has be same direction as the vector $\vec{a} = 8\vec{i} - \vec{j} + 4\vec{k}$. Also find a a vector that has the same direction as $\vec{a}$ but has length 6.

(ii) What is the angle between the vector $\vec{i} + \sqrt{3}\,\vec{j}$ and the positive direction of the *x*-axis?

(iii) If $\vec{v}$ lies in the first quadrant and makes an angle π/3 with the positive *x*-axis and $|\vec{v}| = 4$, find $\vec{v}$ in component form. **Ans: $(2, 3\sqrt{3})$**

(iv) A quarterback throws a football with angle of elevation θ = 40° and speed $\vec{v}$ = 60 ft/s. Find the horizontal and vertical components of the velocity vector. **Ans: $\vec{v}\cos\theta$ and $\vec{v}\sin\theta$**

2. **(i) The position vectors of *P*, *Q*, *R*, *S* are (2, 0, 4), $(5, 3\sqrt{3}, 4)$, $(0, -2\sqrt{3}, 1)$ and (2, 0, 1) respectively. Prove that *RS* is parallel to *PQ* and *PQ* : *RS* = 3 : 2.**

(ii) Show that the following three points are collinear $\vec{i} + 2\vec{j} + 4\vec{k}$, $2\vec{i} + 5\vec{j} - \vec{k}$ and $3\vec{i} + 8\vec{j} - 6\vec{k}$.

(iii) Show that the three points whose position vectors are $7\vec{j} + 10\vec{k}$, $-\vec{i} + 6\vec{j} + 6\vec{k}$ and $-4\vec{i} + 9\vec{j} + 6\vec{k}$ forms an isosceles right-angled triangle.

3. (i) If *A, B* and *C* are vertices of a triangle , find $\vec{AB} + \vec{BC} + \vec{CA}$. **Ans: $\vec{O}$**

(ii) Let *C* be the point on the line segment *AB* that is twice as far from *B* as it is from *A*. If $\vec{a} = \vec{OA}$, $\vec{b} = \vec{OB}$, and $\vec{c} = \vec{OC}$, show that $\vec{c} = \frac{2}{3}\vec{a} + \frac{1}{3}\vec{b}$.

(iii) If *D* be the middle point of *BC* of the triangle *ABC*, show that: $\vec{AB} + \vec{AC} = 2\,\vec{AD}$.

4. (i) Find the direction cosines and direction angles of the vector $\vec{i} - 2\vec{j} - 3\vec{k}$

(ii) If $\vec{OP} = \vec{i} + 3\vec{j} - 7\vec{k}$ and $\vec{OQ} = 5\vec{i} - 2\vec{j} + 4\vec{k}$, find $\vec{PQ}$ and determine its direction cosines.

(iii) If a vector has direction angles α = π/4 and β = π/3, find the direction angle γ.

5. **(i) Find the linear combination between the following system of vectors:**

$\vec{a} - \vec{b} + \vec{c}$, $\vec{b} + \vec{c} - \vec{a}$, $\vec{c} + \vec{a} + \vec{b}$, $2\vec{a} - 3\vec{b} + 4\vec{c}$, where, $\vec{a}$, $\vec{b}$, $\vec{c}$ being any three non coplanar vectors.

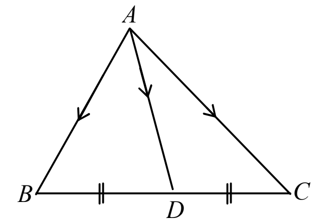
(ii) Are the vectors $-\vec{a} + 4\vec{b} + 3\vec{c}$, $2\vec{a} - 3\vec{b} - 5\vec{c}$, $2\vec{a} + 7\vec{b} - 3\vec{c}$ coplanar? where $\vec{a}$, $\vec{b}$, $\vec{c}$ are any vectors.

Hints of Selected Problems

1. **If *D* be the middle point of *BC* of the triangle *ABC*, show that: $\vec{AB} + \vec{AC} = 2\,\vec{AD}$.**

Hint:

Since D be the middle point of BC of the triangle ABC, then $BD = DC$.



$\therefore\ \overrightarrow{AB} + \overrightarrow{AC} = (\overrightarrow{AD} + \overrightarrow{DB}) + (\overrightarrow{AD} + \overrightarrow{DC})$

$= \overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{AD} + (-\overrightarrow{DB}) = 2\,\overrightarrow{AD}.$

$\therefore\ \overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$.

2. The position vectors of P, Q, R, S are (2, 0, 4), (5, 3 $\sqrt{3}$, 4), (0, –2 $\sqrt{3}$, 1) and (2, 0, 1) respectively. Prove that RS is parallel to PQ and $PQ : RS = 3 : 2$.

Hint:

Let O be the fixed origin, so that $\overrightarrow{OP} = (2, 0, 4)$, $\overrightarrow{OQ} = (5, 3\sqrt{3}, 4)$ and $\overrightarrow{OR} = (0, -2\sqrt{3}, 1)$, $\overrightarrow{OS} = (2, 0, 1)$.

Now, $\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = (2, 2\sqrt{3}, 0)$, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (3, 3\sqrt{3}, 0)$

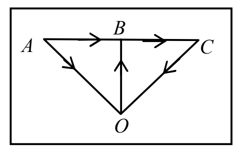
$\therefore\quad \overrightarrow{PQ} = 3\,(1, \sqrt{3}\,, 0) = \frac{3}{2}(2, 2\sqrt{3}, 0) = \frac{3}{2}\,\overrightarrow{RS}.$

Therefore, $\overrightarrow{PQ} = \frac{3}{2}\,\overrightarrow{RS}$. Hence, $\overrightarrow{PQ}$ is parallel to $\overrightarrow{RS}$.

Again, $RS = |\overrightarrow{RS}| = 4$, $PQ = |\overrightarrow{PQ}| = 6$. $\therefore\ PQ : RS = 6 : 4 = 3 : 2$. Thus $\overrightarrow{PQ} \parallel \overrightarrow{RS}$ and $PQ : RS = 3 : 2$.

3. Show that the following three points are collinear $\vec{i} + 2\vec{j} + 4\vec{k}$, $2\vec{i} + 5\vec{j} - \vec{k}$ and $3\vec{i} + 8\vec{j} - 6\vec{k}$.

Hint:



Let O be the origin and A, B, C are given points such that

$\overrightarrow{OA} = \vec{i} + 2\vec{j} + 4\vec{k}$, $\overrightarrow{OB} = 2\vec{i} + 5\vec{j} - \vec{k}$, $\overrightarrow{OC} = 3\vec{i} + 8\vec{j} - 6\vec{k}$.

Then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \ = \vec{i} + 3\vec{j} - 5\vec{k}$.

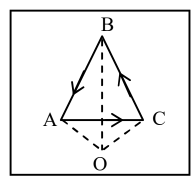
and $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \vec{i} + 3\vec{j} - 5\vec{k} = \overrightarrow{AB}$.

$\therefore\quad \overrightarrow{BC} = \overrightarrow{AB}$ and B is the common point to both $\overrightarrow{BC}$ and $\overrightarrow{AB}$.

Consequently, the points A, B, C are collinear.

4. Show that the three points whose position vectors are $7\vec{j} + 10\vec{k}$, $-\vec{i} + 6\vec{j} + 6\vec{k}$ and $-4\vec{i} + 9\vec{j} + 6\vec{k}$ forms an isosceles right-angled triangle.

Hint:



Let O be the origin such that

$\overrightarrow{OA} = 7\vec{j} + 10\vec{k} = (0, 7, 10)$, $\overrightarrow{OB} = -\vec{i} + 6\vec{j} + 6\vec{k} = (-1, 6, 6)$

and $\overrightarrow{OC} = -4\vec{i} + 9\vec{j} + 6\vec{k} = (-4, 9, 6)$

$\therefore\quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-1, -1, -4)$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (-3, 3, 0),$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = (4, -2, 4)$

$\therefore\quad AB = |\overrightarrow{AB}| = \sqrt{(-1)^2 + (-1)^2 + (-4)^2} = \sqrt{18} = 3\sqrt{2}$

$BC = |\overrightarrow{BC}| = \sqrt{(-3)^2 + 3^2 + 0^2} = \sqrt{18} = 3\sqrt{2}$.

$CA = |\overrightarrow{CA}| = \sqrt{4^2 + (-2)^2 + 4^2} = \sqrt{36} = 6$

Since AB = BC i.e. two sides are equal and $(AB)^2 + (BC)^2 = (CA)^2$ i.e. $\angle ABC = 90°$. Hence the three points A,B,C are the vertices of isosceles right angled triangle.

5. If $\overrightarrow{OP} = \vec{i} + 3\vec{j} - 7\vec{k}$ and $\overrightarrow{OQ} = 5\vec{i} - 2\vec{j} + 4\vec{k}$, find $\overrightarrow{PQ}$ and determine its direction cosines.

Hint:

Given that $\overrightarrow{OP} = \vec{i} + 3\vec{j} - 7\vec{k}$ and $\overrightarrow{OQ} = 5\vec{i} - 2\vec{j} + 4\vec{k}$

$\therefore\quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 4\vec{i} - 5\vec{j} + 11\vec{k}$ and $PQ = |\overrightarrow{PQ}| = \sqrt{4^2 + (-5)^2 + (11)^2} = \sqrt{162} = 9\sqrt{2}$.

Direction cosines of $\overrightarrow{PQ} = \frac{4}{PQ}, \frac{-5}{PQ}, \frac{11}{PQ}$ i.e. $\frac{4}{9\sqrt{2}}, \frac{-5}{9\sqrt{2}}, \frac{11}{9\sqrt{2}}$

6. Find the linear combination between the following system of vectors:

$\vec{a} - \vec{b} + \vec{c}$, $\vec{b} + \vec{c} - \vec{a}$, $\vec{c} + \vec{a} + \vec{b}$, $2\vec{a} - 3\vec{b} + 4\vec{c}$, where, $\vec{a}$, $\vec{b}$, $\vec{c}$ being any three non coplanar vectors.

Hint:

Let $\vec{r_1} = \vec{a} - \vec{b} + \vec{c}$, $\vec{r_2} = \vec{b} + \vec{c} - \vec{a}$, $\vec{r_3} = \vec{c} + \vec{a} + \vec{b}$, $\vec{r_4} = 2\vec{a} - 3\vec{b} + 4\vec{c}$.

Suppose, the linear relationship between four vectors is

$\vec{r_4} = x\vec{r_1} + y\vec{r_2} + z\vec{r_3}$,...... (1) where x, y, z are scalar quantities, to be determined.

From (1) , $2\vec{a} - 3\vec{b} + 4\vec{c} = x(\vec{a} - \vec{b} + \vec{c}) + y(\vec{b} + \vec{c} - \vec{a}) + z(\vec{c} + \vec{a} + \vec{b})$

or, $2\vec{a} - 3\vec{b} + 4\vec{c} = (x - y + z)\vec{a} + (-x + y + z)\vec{b} + (x + y + z)\vec{c}$

Equating the coefficients of like vectors, we get

$x - y + z = 2$.... (2) , $-x + y + z = -3$..... (3) and $x + y + z = 4$...... (4)

Solving (2), (3) & (4), we get $x = \frac{7}{2}$, $y = 1$ and $z = -\frac{1}{2}$.

Substituting the value of x, y and z in (1), we get $\vec{r_4} = \frac{7}{2}\vec{r_1} + \vec{r_2} - \frac{1}{2}\vec{r_3}$.

7. Are the vectors $-\vec{a} + 4\vec{b} + 3\vec{c}$, $2\vec{a} - 3\vec{b} - 5\vec{c}$, $2\vec{a} + 7\vec{b} - 3\vec{c}$ coplanar? where $\vec{a}$, $\vec{b}$, $\vec{c}$ are any vectors.

Hint:

Let $\vec{r_1} = -\vec{a} + 4\vec{b} + 3\vec{c}$, $\vec{r_2} = 2\vec{a} - 3\vec{b} - 5\vec{c}$ and $\vec{r_3} = 2\vec{a} + 7\vec{b} - 3\vec{c}$.

If the given three vectors are coplanar, then any one of them can be expressed as the linear combination of others.

So let $\vec{r_3} = x\vec{r_1} + y\vec{r_2}$, where x and y are scalars.

Now, $\vec{r_3} = x\vec{r_1} + y\vec{r_2}$ gives

$2\vec{a} + 7\vec{b} - 3\vec{c} = x(-\vec{a} + 4\vec{b} + 3\vec{c}) + y(2\vec{a} - 3\vec{b} - 5\vec{c})$

or, $2\vec{a} + 7\vec{b} - 3\vec{c} = (-x + 2y)\vec{a} + (4x - 3y)\vec{b} + (3x - 5y)\vec{c}$.

Equating the coefficients of like vectors, we get

$-x + 2y = 2$ ...... (1) , $4x - 3y = 7$ ..... (2) and $3x - 5y = -3$ ..... (3)

Solving (1) and (2), we get $x = 4$ and $y = 3$.

Substituting the value of x and y in the remaining equation $3x - 5y = -3$, we get

$3.4 - 5.3 = -3$ i.e ., $-3 = -3$, which is true.

Hence the given vectors are coplanar.

Dot Product of Vectors:

1. Determine whether the given vectors are orthogonal, parallel, or neither.

(i) $\vec{a} = (-5, 3, 7)$ and $\vec{b} = (6, -8, 2)$ **Ans: Neither**

(ii) $\vec{a} = -\vec{i} + 2\vec{j} + 5\vec{k}$ and $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$ **Ans: Orthogonal**

(iii) $\vec{a} = 2\vec{i} + 6\vec{j} - 4\vec{k}$ and $\vec{b} = -3\vec{i} - 9\vec{j} + 6\vec{k}$ **Ans: Parallel**

2. (i) Find $\vec{a}\cdot\vec{b}$ if $|\vec{a}| = 6$, $|\vec{b}| = 5$ and the angle between $\vec{a}$ and $\vec{b}$ is $2\pi/3$. **Ans: -15**

(ii) Find the cosine of the angle between following pair of vectors: $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j} + 2\vec{k}$.

(iii) Find the values of x such that the angle between the vectors (2,1, - 1), an (1, x, 0) is 45°.

(iv) Show that the line AB is perpendicular to CD if A, B, C, D are the points (2, 3,4), (5, 4,– 1), (3, 6,2) and (1, 2, 0).

(v) Find the angles of a triangle whose vertices are $7\vec{j} + 10\vec{k}$, $-\vec{i} + 6\vec{j} + 6\vec{k}$ and $-4\vec{i} + 9\vec{j} + 6\vec{k}$ respectively.

3. (i) Give the geometrical meaning of scalar product. Find the scalar projection and vector projection of $\vec{a} = \vec{i} - \vec{j} + \vec{k}$ onto $\vec{b} = \vec{i} + \vec{j} + \vec{k}$

(ii) If $\vec{a} = (3, 0, -1)$, find a vector $\vec{b}$ such that the scalar projection of $\vec{a}$ on $\vec{b}$ is 2. **Ans: $(0,0,-2\sqrt{10})$**

4. Find the angle between a diagonal of a cube and one of its edges. **Ans: $\cos^{-1}(1/\sqrt{3})$**

5. Find the work done by a force $\vec{F} = 8\vec{i} - 6\vec{j} + 9\vec{k}$ that moves an object from the point (0, 10, 8) to the point (6, 12, 20) along a straight line. The distance is measured in meters and the force in newtons. **Ans: 144J**

6. (i) Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonal are perpendicular.

(ii) In a right angled triangle ABC, right angle at A, use vector methods to show that $(AB)^2 + (AC)^2 = (BC)^2$.

(iii) Use vector methods to Show that a parallelogram whose diagonals are equal is a rectangle.

7. (i) Use formula $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ to prove Cauchy-Schwarz Inequality $|\vec{a}.\vec{b}| \le |\vec{a}||\vec{b}|$.

(ii) Give a geometric interpretation of the Triangle Inequality $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$ for vectors. Use Cauchy-Schwarz Inequality to prove the Triangle Inequality.

[***Hint***: Use $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b})$ and use Property of the dot product.]

(iii) Prove the Parallelogram Law $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2|\vec{a}|^2 + 2|\vec{b}|^2$ for vectors. Give a geometric interpretation of the Parallelogram Law.

(iv) If $\vec{a}$ and $\vec{b}$ are two vectors of unit length and θ is the angle between them, show that $\frac{1}{2}|\vec{a}-\vec{b}| = \sin\frac{\theta}{2}$

8. (i) If $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are orthogonal, show that the vectors $\vec{u}$ and $\vec{v}$ must have the same length.

(ii) If $\vec{c} = |\vec{a}|\vec{b} + |\vec{b}|\vec{a}$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are all nonzero vectors, show that $\vec{c}$ bisects the angle between $\vec{a}$ and $\vec{b}$.

9. Using vector method, prove the projection law in any triangle that:

(i) b = c cosA + a cosC (ii) c = a cosB + b cosA (iii) a = b cos C + c cos B

10. Using vector method, prove the cosine laws of Trigonometry.

(i) $b^2 = c^2 + a^2 - 2ca\cos B$ (ii) $a^2 = b^2 + c^2 - 2bc\cos A$

<u>Hints of Selected Problems</u>

1. Find the cosine of the angle between following pair of vectors: $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j} + 2\vec{k}$.

Hint:

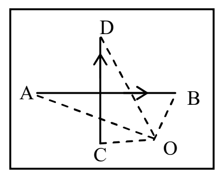
Since $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j} + 2\vec{k}$

$\therefore |\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$, $|\vec{b}| = \sqrt{14}$ and $\vec{a}.\vec{b} = 1 \times 1 - 2 \times 3 + 3 \times 2 = 1$.

If θ is the required angle between $\vec{a}$ and $\vec{b}$, then $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|} = \dfrac{1}{\sqrt{14}\times\sqrt{14}} = \dfrac{1}{14}$.

2. Show that the line AB is perpendicular to CD if A, B, C, D are the points (2, 3,4), (5, 4,– 1), (3, 6,2) and (1, 2, 0)

Hint:



Let O be the origin.

Then $\overrightarrow{OA} = (2, 3, 4)$, $\overrightarrow{OB} = (5, 4, -1)$

$\overrightarrow{OC} = (3, 6, 2)$, $\overrightarrow{OD} = (1, 2, 0)$

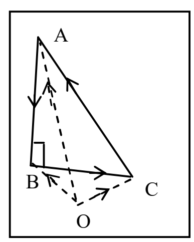
Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3, 1, -5)$

and $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = (-2, -4, -2)$

Now, $\overrightarrow{AB}.\overrightarrow{CD} = (3, 1, -5).(-2, -4, -6) = -6 - 4 + 10 = 0$. So $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CD}$

3. Find the angles of a triangle whose vertices are $7\vec{j} + 10\vec{k}$, $-\vec{i} + 6\vec{j} + 6\vec{k}$ and $-4\vec{i} + 9\vec{j} + 6\vec{k}$ respectively.

Hint:



Let O be the origin and A, B, C be the vertices of the triangle.

Then, $\overrightarrow{OA} = 7\vec{j} + 10\vec{k}$, $\overrightarrow{OB} = -\vec{i} + 6\vec{j} + 6\vec{k}$, $\overrightarrow{OC} = -4\vec{i} + 9\vec{j} + 6\vec{k}$.

$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{i} - \vec{j} - 4\vec{k}$.

$\therefore AB = |\overrightarrow{AB}| = \sqrt{(-1)^2 + (-1)^2 + (-4)^2} = \sqrt{18}$

Again, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -3\vec{i} + 3\vec{j}$.

$\therefore BC = |\overrightarrow{BC}| = \sqrt{(-3)^2 + 3^2} = \sqrt{18}$

Again, $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = 4\vec{i} - 2\vec{j} + 4\vec{k}$

$\therefore CA = |\overrightarrow{CA}| = \sqrt{4^2 + (-2)^2 + 4^2} = \sqrt{36} = 6$.

Now we see that $\overrightarrow{AB}.\overrightarrow{BC} = 0$, $\overrightarrow{BC}.\overrightarrow{CA} = -18$ and $\overrightarrow{CA}.\overrightarrow{AB} = -18$

For ∠B: Since $\overrightarrow{AB}.\overrightarrow{BC} = 0$ $\therefore \angle B = 90°$.

For ∠A: $\cos(\pi - A) = \dfrac{\overrightarrow{CA}.\overrightarrow{AB}}{|\overrightarrow{CA}|\,|\overrightarrow{AB}|} = \dfrac{-18}{6\times\sqrt{18}} = -\dfrac{1}{\sqrt{2}}$, so $-\cos A = -\dfrac{1}{\sqrt{2}} \Rightarrow \cos A = \dfrac{1}{\sqrt{2}}$ $\therefore A = 45°$.

For ∠C: $\cos(\pi - C) = \dfrac{\overrightarrow{BC}.\overrightarrow{CA}}{|\overrightarrow{BC}|\,|\overrightarrow{CA}|} = \dfrac{-18}{\sqrt{18}\times 6} = \dfrac{-1}{\sqrt{2}}$, so $-\cos C = -\dfrac{1}{\sqrt{2}} \Rightarrow \cos C = \dfrac{1}{\sqrt{2}}$ $\therefore C = 45°$

Thus A = 45°, B = 90°, C = 45° .

4. If $\vec{a}$ and $\vec{b}$ are two vectors of unit length and θ is the angle between them, show that $\frac{1}{2}|\vec{a}-\vec{b}| = \sin\frac{\theta}{2}$.

Hint:

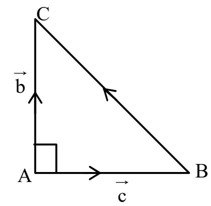
Since $\vec{a}$ and $\vec{b}$ are unit vectors , so $|\vec{a}| = 1$, $|\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = \cos\theta$ .... (i)

Now, L.H.S $= \frac{1}{2}|\vec{a} - \vec{b}| = \frac{1}{2}\sqrt{(\vec{a} - \vec{b})^2} = \frac{1}{2}\sqrt{a^2 - 2\vec{a}\cdot\vec{b} + b^2} = \frac{1}{2}\sqrt{(1)^2 - 2\cos\theta + (1)^2}$

$= \frac{1}{2}\sqrt{2(1 - \cos\theta)} = \frac{1}{2}\sqrt{2 \times 2\sin^2\theta/2} = \sin\frac{\theta}{2}$ = R.H.S.

5. In a right angled triangle ABC, right angle at A, show that $(AB)^2 + (AC)^2 = (BC)^2$.

Hint:



Let ABC be a triangle with A as origin.

Suppose $\overrightarrow{AB} = \vec{c}$ and $\overrightarrow{AC} = \vec{b}$.

Since $\angle A = 90°$. So $\vec{b} \cdot \vec{c} = bc\cos 90° = 0$......... (i)

Again by Triangle law of vector addition

$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\vec{c} + \vec{b}$ .... (ii)

Now, $BC^2 = (\overrightarrow{BC})^2 = (\vec{b} - \vec{c})^2 = b^2 - 2(\vec{b}\cdot\vec{c})^2 + c^2$ ($\because$ by formula)

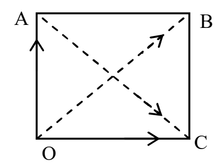
$= b^2 - 2.0 + c^2 = b^2 + c^2$ ($\because$ by (i))

$= (AC)^2 + (AB)^2$. Hence proved.

6. Use vector methods to Show that a parallelogram whose diagonals are equal is a rectangle.

Hint:

Let OABC is a parallelogram and OB and AC are two diagonal such that OB = OC.Now, by vector addition,



$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ ......(i) and $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$ ....(ii)

Now, According to question, OB = AC.

$\Rightarrow (OB)^2 = (AC)^2$ ....(iii)

Since OABC is a parallelogram $\therefore \overrightarrow{OC} = \overrightarrow{AB}$ ...(iv)

Also, we have the relation $(\overrightarrow{OB})^2 = (OB)^2$ etc. and $(\vec{a} + \vec{b})^2 = a^2 + 2\vec{a}\cdot\vec{b} + b^2$.

Hence using the relation (i) and (ii) on (iii), we get

$(\overrightarrow{OA} + \overrightarrow{AB})^2 = (-\overrightarrow{OA} + \overrightarrow{AB})^2$ [$\because$ By (iv)]

or, $(OA)^2 + 2\,\overrightarrow{OA}\cdot\overrightarrow{AB} + (AB)^2 = (OA)^2 - 2\,\overrightarrow{OA}\cdot\overrightarrow{AB} + (AB)^2$

or, $4\overrightarrow{OA}\cdot\overrightarrow{AB} = 0$

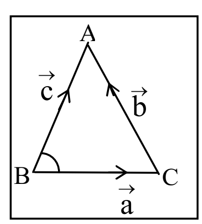
or, $\overrightarrow{OA}\cdot\overrightarrow{AB} = 0 \therefore \overrightarrow{OA} \perp \overrightarrow{AB}$ and hence $\angle OAB = 90°$. So OABC is a rectangle.

7.Using vector method, prove the projection law in any triangle that:

(i) b = c cosA + a cosC (ii) c = a cosB + b cosA (iii) a = b cos C + c cos B

Hint:

To prove b = c cosA + a cosC .



Let ABC be a triangle such that

$\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, $\overrightarrow{BA} = \vec{c}$

Then, $\vec{b} = \overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA} = -\vec{a} + \vec{c} = \vec{c} - \vec{a}$

Taking dot products both sides by $\vec{b}$, we get

$\vec{b}\cdot\vec{b} = (\vec{c}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})$

$b^2 = cb\cos(\pi - A) - ab\cos(\pi - C)$

or, $b^2 = b\,[c\cos A + a\cos C]$

or, $b = c\cos A + a\cos C$.

8. Using vector method, prove the cosine laws of Trigonometry.

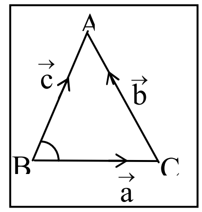
(i) $b^2 = c^2 + a^2 - 2ca\cos B$ (ii) $a^2 = b^2 + c^2 - 2bc\cos A$

Hint:

To prove $b^2 = c^2 + a^2 - 2ca\cos B$.

Let ABC be a triangle such that

$\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, $\overrightarrow{BA} = \vec{c}$



Then, $\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA} = -\vec{a} + \vec{c}$

or, $\vec{b} = \vec{c} - \vec{a}$

$\therefore b^2 = (\vec{b})^2 = (\vec{c} - \vec{a})^2$

$= c^2 - 2\vec{c}.\vec{a} + a^2 = c^2 - 2\,ca\cos B + a^2$

$\therefore$ $b^2 = c^2 + a^2 - 2ca\cos B$. Hence proved.

Assignment Problem Set – IIExercise 12.4,12.5 (Cross Product, Plane and Lines)):

Use Cross Product:

1. (i) Find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both $\vec{a}$ and $\vec{b}$ for the vectors $\vec{a} = t\,\vec{i} + \cos t\,\vec{j} + \sin t\,\vec{k}$ and $\vec{b} = \vec{i} - \sin t\,\vec{j} + \cos t\,\vec{k}$

(ii) Find two unit vectors orthogonal to both $\vec{a} = \vec{j} - \vec{k}$ and $\vec{b} = \vec{i} + \vec{j}$

(iii) Find the unit vector perpendicular to each of the pair of vectors: $\vec{a} = (3, 1, 2)$ and $\vec{b} = (2, -2, 4)$

2. **(i) Find the sine of the angle between the pair of vectors: $3\vec{i} + \vec{j} + 2\vec{k}$ and $2\vec{i} - 2\vec{j} + 4\vec{k}$**

(ii) If $\vec{a}.\vec{b} = \sqrt{3}$ and $\vec{a} \times \vec{b} = (1, 2, 2)$, find the angle between $\vec{a}$ and $\vec{b}$.

(iii) Find all vectors $\vec{v}$ such that $(1, 2, 1) \times \vec{v} = (3, 1, -5)$

(iv) Show that $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 . |\vec{b}|^2 - (\vec{a}.\vec{b})^2$

(v) If $\vec{a} + \vec{b} + \vec{c} = 0$, show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

3. (i) Find the area of the triangle determined by the vectors $3\vec{i} + 4\vec{j}$ and $-5\vec{i} + 7\vec{j}$.

(ii) Find the area of the parallelogram determined by the vector $\vec{i} + 2\vec{j} + 3\vec{k}$ and $-3\vec{i} - 2\vec{j} + \vec{k}$

(iii) Prove that the area of the parallelogram whose three of four vertices are (1, 1, 2), (2, – 1, 1) and (3, 2, – 1) is $5\sqrt{3}$ sq. units

(iv) Find the area of the parallelogram with vertices A(-2, 1), B(0, 4), C(4, 2) and D(2, -1) **Ans: 16**

(v Show that the area of the triangle PQR whose vertices are P(1, 2, 3), Q(3, 4, 5) and R(1, 3, 7) is $2\sqrt{6}$ sq. units.

(vi) Find a non zero vector orthogonal to the plane through the points P(1, 0, 1), Q(-2, 1, 3), R(4, 2, 5). Also find the area of the triangle PQR. **Ans: (0,18,-9) ; $\frac{9}{2}\sqrt{5}$**

4. **Prove the sine law of trigonometry by vector method that: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$**

Hints of Selected Problems

1. Find the unit vector perpendicular to each of the pair of vectors (3, 1, 2) and (2, –2, 4) .

Hint:

Let $\vec{a} = (3, 1, 2) = 3\vec{i} + \vec{j} + 2\vec{k}$, $\vec{b} = (2, -2, 4) = 2\vec{i} - 2\vec{j} + 4\vec{k}$

Since $\vec{a} \times \vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix}$$

$$= (4 + 4)\,\vec{i} - \vec{j}\,(12 - 4) + \vec{k}\,(-6 - 2) = 8\vec{i} - 8\vec{j} - 8\vec{k}$$

$\therefore$ (8, – 8, – 8) is a required vector perpendicular to both $\vec{a}$ and $\vec{b}$.

Also, unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

$$= \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{8\vec{i} - 8\vec{j} - 8\vec{k}}{|8\vec{i} - 8\vec{j} - 8\vec{k}|} = \frac{8\vec{i} - 8\vec{j} - 8\vec{k}}{\sqrt{8^2 + (-8)^2 + (-8)^2}} = \frac{\vec{i} - \vec{j} - \vec{k}}{\sqrt{3}}.$$

2. Find the sine of the angle between the pair of vectors: $3\vec{i} + \vec{j} + 2\vec{k}$ and $2\vec{i} - 2\vec{j} + 4\vec{k}$

Hint:

Since $\vec{a} = (3\vec{i} + \vec{j} + 2\vec{k}) = (3, 1, 2)$ and $\vec{b} = 2\vec{i} - 2\vec{j} + 4\vec{k} = (2, -2, 4)$

Now, $\vec{a} \times \vec{b} = (3, 1, 2) \times (2, -2, 4) = (8, -8, -8)$ (How??) and $|\vec{a} \times \vec{b}| = 8\sqrt{3}$

Again, $|\vec{a}| = \sqrt{14}$ and $|\vec{b}| = \sqrt{24} = 2\sqrt{6}$

If θ is the angle between $\vec{a}$ and $\vec{b}$,then, $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \dfrac{8\sqrt{3}}{\sqrt{14}\sqrt{24}} = \dfrac{2}{\sqrt{7}}$.

3. Find the area of the triangle determined by the vectors $3\vec{i} + 4\vec{j}$ and $-5\vec{i} + 7\vec{j}$.

Hint:

Suppose $\vec{a} = 3\vec{i} + 4\vec{j} = (3, 4, 0)$ and $\vec{b} = -5\vec{i} + 7\vec{j} = (-5, 7, 0)$

$\therefore \vec{a} \times \vec{b} = (3, 4, 0) \times (-5, 7, 0) = (0, 0, 41)$ and $|\vec{a} \times \vec{b}| = \sqrt{0^2 + 0^2 + (41)^2} = 41$.

$\therefore$ Area of the triangle $= \dfrac{1}{2}|\vec{a} \times \vec{b}| = \dfrac{41}{2} = 20.5$ sq. units.

4 Show that the area of the triangle PQR whose vertices are P(1, 2, 3), Q(3, 4, 5) and R(1, 3, 7) is $2\sqrt{6}$ sq. units.

Solution:

Let us take O as the origin. Since P, Q, R be the vertices of the triangle PQR.

$\therefore \overrightarrow{OP} = (1, 2, 3), \overrightarrow{OQ} = (3, 4, 5)$ and $\overrightarrow{OR} = (1, 4, 7)$

Now, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (2, 2, 2)$ and $\overrightarrow{OR} = OR - OQ = (-2, 0, 2)$

Now, $\overrightarrow{PQ} \times \overrightarrow{OR} = (2, 2, 2) \times (-2, 0, 2) = (4, -8, 4)$ and $|\overrightarrow{PQ} \times \overrightarrow{QR}| = \sqrt{96} = 4\sqrt{6}$.

Hence, required Area of the triangle PQR $= \dfrac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{QR}| = \dfrac{1}{2} \times 4\sqrt{6} = 2\sqrt{6}$ sq. units.

5. Find the area of the parallelogram determined by the vector $\vec{i} + 2\vec{j} + 3\vec{k}$ and $-3\vec{i} - 2\vec{j} + \vec{k}$.

Solution:

Suppose $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k} = (1, 2, 3)$

and $\vec{b} = -3\vec{i} - 2\vec{j} + \vec{k} = (-3, -2, 1)$

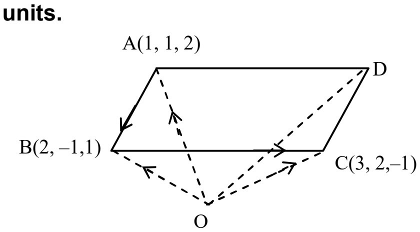
Now, $\vec{a} \times \vec{b} = (8, -10, 4)$

We know that $|\vec{a} \times \vec{b}|$ always represents the area of the parallelogram forming by the sides $\vec{a}$ and $\vec{b}$.

$\therefore$ Required area $= |\vec{a} \times \vec{b}| = |(8, -10, 4)| = \sqrt{8^2 + (-10)^2 + 4^2} = \sqrt{64 + 100 + 6} = 6\sqrt{5}$ sq. units.

6. Prove that the area of the parallelogram whose three of four vertices are (1, 1, 2), (2, – 1, 1) and (3, 2, – 1) is $5\sqrt{3}$ sq. units.

Solution:



Let O be the origin and ABCD be a parallelogram.

Let us take $\overrightarrow{OA} = (1, 1, 2), \overrightarrow{OB} = (2, -1, 1)$ and $\overrightarrow{OC} = (3, 2, -1)$.

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (1, -2, -1)$ and $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (1, 3, -2)$

$\therefore \overrightarrow{AB} \times \overrightarrow{BC} = (1, -2, -1) \times (1, 3, -2) = (7, 1, 5)$.(How??)

$\therefore$ Area of the parallelogram $= |\overrightarrow{AB} \times \overrightarrow{BC}| = |(7, 1, 5)| = \sqrt{75} = 5\sqrt{3}$ sq. units **Ans**.

7. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

Hence prove the sine law of trigonometry by vector method that: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Solution:

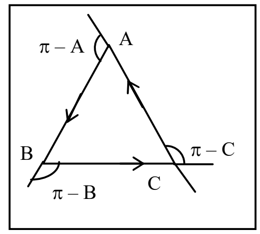
Multiplying both sides of $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ ....(i) vectorially by $\vec{a}$, we get

$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$

or, $\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$

or, $\vec{0} + \vec{a} \times \vec{b} - \vec{c} \times \vec{a} = \vec{0}$ $\quad [\because \vec{a} \times \vec{c} = -\vec{c} \times \vec{a}]$

$\therefore \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ $[\because \vec{a} \times \vec{a} = \vec{0}]$ ....(ii)



Similarly, multiplying both sides of (i) vectorially by $\vec{b}$, we get

$\vec{b} \times \vec{c} = \vec{a} \times \vec{b}$ ....(iii)

Combining (ii) and (iii), we get

$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$. Hence proved.

Finally for last part: Now $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

or, ab sin (π – C) = bc sin (π – A) = ca sin (π– B)

or, ab sin C = bc sin A = ca sin B

Dividing by abc, we get

$\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$ or, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin c}{c}$.

Use of Scalar Triple Product:

1. (i) Give the geometrical meaning of scalar triple product. Find the volume of the parallelepiped determined by the vectors $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = -\vec{i} + \vec{j} + 2\vec{k}$, and $\vec{c} = 2\vec{i} + \vec{j} + 4\vec{k}$ **Ans: 9**

(ii) Find the volume of the parallelepiped with adjacent edges *PQ, PR* and *PS*, where *P*(-2, 1, 0), *Q*(2, 3, 2), *R*(1, 4, -1), *S*(3, 6, 1) **Ans: 16**

2. Use the scalar triple product to

(i) Verify that the vectors $\vec{u} = \vec{i} + 5\vec{j} - 2\vec{k}$, $\vec{v} = 3\vec{j} - \vec{j}$, and $\vec{w} = 5\vec{i} + 9\vec{j} - 4\vec{k}$ are coplanar.

(ii) Show that the following four points (2, 3, –1), (1, –2, 3), (3, 4, –2) and (1,– 6, 4) **are coplanar.**

(iii) Show that the vectors $\vec{a} = \vec{i} + 3\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} - \vec{k}$ and $\vec{c} = 7\vec{j} + 3\vec{k}$ are parallel to the same plane.

(iv) Determine whether the points *A*(1, 3, 2), *B*(3, -1, 6), *C*(5, 2, 0), and *D*(3, 6, -4) lie in the same plane.

(v) The position vectors of the points *A*, *B*, *C* and *D* are $3\vec{i} - 2\vec{j} - \vec{k}$, $2\vec{i} + 3\vec{j} - 4\vec{k}$, $-\vec{i} + \vec{j} + 2\vec{k}$ and $4\vec{i} + 5\vec{j} + \lambda\vec{k}$ respectively, if the points *A*, *B*, *C*, *D* are coplanar, find the value of λ.

(vi) If the vectors $a\vec{i} + \vec{j} + \vec{k}$, $\vec{i} + b\vec{j} + \vec{k}$ and $\vec{i} + \vec{j} + c\vec{k}$ ($a \neq b$, $c \neq 1$) are coplanar, find the value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$

(vii) If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ **and the vectors $\vec{\alpha} = (1, a, a^2)$, $\vec{\beta} = (1, b, b^2)$ and $\vec{\gamma} = (1, c, c^2)$ are non coplanar.**

Find the value of *abc*.

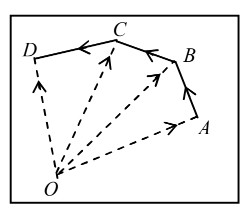
(viii) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly independen*t*, then show that $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$ are also linearly independent.

Hints of Selected Problems

1. Show that the following four points are coplanar:
(2, 3, –1), (1, –2, 3), (3, 4, –2) and (1,– 6, 4).

Hint:

Let *A*, *B*, *C*, *D* be the four points with position vectors with reference to the origin *O* be (2, 3, –1), (1, –2, 3), (3, 4, –2) and (1,– 6, 4) respectively.



Then we have,

$\overrightarrow{OA} = (2, 3, -1)$, $\overrightarrow{OB} = (1, -2, 3)$

$\overrightarrow{OC} = (3, 4, -2)$ and $\overrightarrow{OD} = (1, -6, 4)$.

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-1, -5, 4)$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (2, 6, -5)$

and $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = (-2, -10, 8)$.

If the given four points are coplanar, then the three vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ are also coplanar. For this, their scalar triple product must vanish i.e., $\overrightarrow{AB} \,.\, \overrightarrow{BC} \times \overrightarrow{CD} = 0$.

But $\overrightarrow{AB} \,.\, \overrightarrow{BC} \times \overrightarrow{CD} = \begin{vmatrix} -1 & -5 & 4 \\ 2 & 6 & -5 \\ -2 & -10 & 8 \end{vmatrix} = 0$. Hence the above four points are coplanar.

2. Show that the vectors $\vec{a} = \vec{i} + 3\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} - \vec{k}$ and $\vec{c} = 7\vec{j} + 3\vec{k}$ are parallel to the same plane.

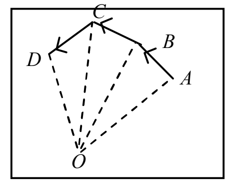
Hint:

The three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are parallel to the same plane if they are coplanar.

Here, $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \,.\, \vec{b} \times \vec{c} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ 0 & 7 & 3 \end{vmatrix} = 0$.

3. The position vectors of the points *A*, *B*, *C* and *D* are $3\vec{i} - 2\vec{j} - \vec{k}$, $2\vec{i} + 3\vec{j} - 4\vec{k}$, $-\vec{i} + \vec{j} + 2\vec{k}$ and $4\vec{i} + 5\vec{j} + \lambda\vec{k}$ respectively, if the points *A*, *B*, *C*, *D* are coplanar, find the value of λ.

Hint:



Here, $\overrightarrow{OA} = 3\vec{i} - 2\vec{j} - \vec{k}$, $\overrightarrow{OB} = 2\vec{i} + 3\vec{j} - 4\vec{k}$, $\overrightarrow{OC} = -\vec{i} + \vec{j} + 2\vec{k}$

and $\overrightarrow{OD} = 4\vec{i} + 5\vec{j} + \lambda\vec{k}$.

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{i} + 5\vec{j} - 3\vec{k}$

$\overrightarrow{BC} = -3\vec{i} - 2\vec{j} + 6\vec{k}$

$\overrightarrow{CD} = 5\vec{i} + 4\vec{j} + (\lambda - 2)\vec{k}$.

If the points *A*, *B*, *C*, *D* are coplanar, then the three vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ are also coplanar. For this, $[\overrightarrow{AB}\ \ \overrightarrow{BC}\ \ \overrightarrow{CD}] = 0$

or, $(-\vec{i} + 5\vec{j} - 3\vec{k}).[(-3\vec{i} - 2\vec{j} + 6\vec{k}) \times \{5\vec{i} + 4\vec{j} + (\lambda - 2)\vec{k}\}] = 0 \quad \therefore \quad \lambda = -\dfrac{146}{17}$.

4. If the vectors $a\vec{i} + \vec{j} + \vec{k}$, $\vec{i} + b\vec{j} + \vec{k}$ and $\vec{i} + \vec{j} + c\vec{k}$ ($a \neq b$, $c \neq 1$) are coplanar, find the value of $\dfrac{1}{1-a} + \dfrac{1}{1-b} + \dfrac{1}{1-c}$.

Hint:

Since the given three vectors are coplanar, so their scalar triple product is zero

i.e. $(a\vec{i} + \vec{j} + \vec{k}).(\vec{i} + b\vec{j} + \vec{k}) \times (\vec{i} + \vec{j} + c\vec{k}) = 0$.

or, $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

Which gives $a + b + c = abc + 2$. ...... (1)

$$\text{Now, } \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = \frac{3 + bc + ca + ab - 2(a+b+c)}{(1-a)(1-b)(1-c)}$$

$$= \frac{3 + bc + ca + ab - (a+b+c) - (a+b+c)}{(1-a)(1-b)(1-c)}$$

$$= \frac{3 + bc + ca + ab - (a+b+c) - (abc+2)}{(1-a)(1-b)(1-c)} \quad [\because \text{ by using (1)}]$$

$$= \frac{1 + bc + ca + ab - a - b - c - abc}{(1-a)(1-b)(1-c)} = \frac{(1-a)(1-b-c-bc)}{(1-a)(1-b)(1-c)} = \frac{(1-a)(1-b)(1-c)}{(1-a)(1-b)(1-c)} = 1.$$

5. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors $\vec{\alpha} = (1, a, a^2)$, $\vec{\beta} = (1, b, b^2)$ and $\vec{\gamma} = (1, c, c^2)$ are non coplanar. Find the value of *abc*.

Hint:

Since $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are non coplanar, so $[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] \neq 0$. $\therefore \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$ .... (1)

But given that

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0 \qquad \text{or, } \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

or, $\Delta + abc\,\Delta = 0$ or, $\Delta\ (1 + abc) = 0$

Since by (1), $\Delta \neq 0$, so $1 + abc = 0 \quad \therefore\ abc = -1$.

6. a) Prove that $[\vec{a} + \vec{b}\ \ \vec{b} + \vec{c}\ \ \vec{c} + \vec{a}] = 2[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]$, where $\vec{a}, \vec{b}, \vec{c}$ being any three vectors.

b) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar, then $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$ are also coplanar.

c) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly independent, then $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$ are also linearly independent.

Solutions:

(a) Now,

$$
\begin{aligned}
[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] &= (\vec{a}+\vec{b}).\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}\\
&= (\vec{a}+\vec{b}).[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}]\\
&= (\vec{a}+\vec{b}).[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]\ (\because \vec{c}\times\vec{c}=\vec{0})\\
&= \vec{a}.[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}] + \vec{b}.[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]\\
&= \vec{a}.\vec{b}\times\vec{c}+\vec{a}.\vec{b}\times\vec{a}+\vec{a}.\vec{c}\times\vec{a}+\vec{b}.\vec{b}\times\vec{c}+\vec{b}.\vec{b}\times\vec{a}+\vec{b}.\vec{c}\times\vec{a}\\
&= [\vec{a}.\vec{b}\times\vec{c}+0+0+0+0+\vec{b}.\vec{c}\times\vec{a}]\\
&= \vec{a}.\vec{b}\times\vec{c}+\vec{b}.\vec{c}\times\vec{a}.\\
&= [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{c}] = 2\,[\vec{a}\ \vec{b}\ \vec{c}].
\end{aligned}
$$

(b) Since $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar, $\therefore$ $[\vec{a}\ \vec{b}\ \vec{c}] = 0$. ....(1)

Then by part (a), $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2\,[\vec{a}\ \vec{b}\ \vec{c}] = 2.0 = 0$.

$\therefore$ $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ are also coplanar.

(c) Since $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly independent vectors $\therefore$ $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$...... (1)

Then by part (a), $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2\,[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$ [$\because$ by (1)]

This shows that $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are also linearly independent.

Vector Triple Product

1. **Verify that: $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c}$, where $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$.**
2. **If $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$, find**
   $\vec{a}\times(\vec{b}\times\vec{c})$, $(\vec{a}\times\vec{b})\times\vec{c}$ and $|\vec{a}\times(\vec{b}\times\vec{c})|$. Also, show that $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$.
3. **(i) Prove that: $b^2\vec{a} = (\vec{a}.\vec{b})\,\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$.**

   (ii) Show that : $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$.

   Hint: Use formula $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c}$.
4. **Prove that**

   $2\vec{a} = \vec{i}\times(\vec{a}\times\vec{i}) + \vec{j}\times(\vec{a}\times\vec{j}) + \vec{k}\times(\vec{a}\times\vec{k})$, where $\vec{i}$, $\vec{j}$, $\vec{k}$ are mutually unit vectors along the co–ordinate axes.

Hint:

Now, $\vec{i}\times(\vec{a}\times\vec{i}) + \vec{j}\times(\vec{a}\times\vec{j}) + \vec{k}\times(\vec{a}\times\vec{k})$

$$
\begin{aligned}
&= (\vec{i}.\vec{i})\,\vec{a} - (\vec{i}.\vec{a})\,\vec{i} + (\vec{j}.\vec{j})\,\vec{a} - (\vec{j}.\vec{a})\,\vec{j} + (\vec{k}.\vec{k})\,\vec{a} - (\vec{k}.\vec{a})\,\vec{k}\\
&= 1\vec{a} - (\vec{i}.\vec{a})\,\vec{i} + 1\vec{a} - (\vec{j}.\vec{a})\,\vec{j} + 1\vec{a} - (\vec{k}.\vec{a})\,\vec{k}\\
&= 3\vec{a} - [(\vec{i}.\vec{a})\,\vec{i} + (\vec{j}.\vec{a})\,\vec{j} + (\vec{k}.\vec{a})\,\vec{k}] \text{ .... (1)}
\end{aligned}
$$

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, then $\vec{i}.\vec{a} = \vec{i}.(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) = a_1$. etc

LHS $= 3\vec{a} - (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) = 3\vec{a} - \vec{a} = 2\vec{a}$.

5. **Show that: $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$ if, and only if, $(\vec{a}\times\vec{c})\times\vec{b} = \vec{0}$.**

Hint: Now, $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c}) \Leftrightarrow$ $-\vec{c}\times(\vec{a}\times\vec{b}) = \vec{a}\times(\vec{b}\times\vec{c})$

$$
\begin{aligned}
&\Leftrightarrow -\{(\vec{c}.\vec{b})\,\vec{a} - (\vec{c}.\vec{a})\,\vec{b}\} = (\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c}\\
&\Leftrightarrow -(\vec{c}.\vec{b})\,\vec{a} + (\vec{c}.\vec{a})\,\vec{b} = (\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c}\\
&\Leftrightarrow (\vec{a}.\vec{b})\,\vec{c} - (\vec{c}.\vec{b})\,\vec{a} = \vec{0}\\
&\Leftrightarrow (\vec{b}.\vec{c})\vec{a} - (\vec{b}.\vec{a})\vec{c} = \vec{0}\\
&\Leftrightarrow \vec{b}\times(\vec{a}\times\vec{c}) = \vec{0}\\
&\Leftrightarrow (\vec{a}\times\vec{c})\times\vec{b} = \vec{0}.
\end{aligned}
$$

6. Prove that the vector $\vec{a} \times (\vec{b} \times \vec{a})$ is coplanar with $\vec{a}$ and $\vec{b}$.

Hint:

Let $\vec{r} = \vec{a} \times (\vec{b} \times \vec{a})$, so that $\vec{r}$ is perpendicular to both $\vec{a}$ and $\vec{b} \times \vec{a}$. $\therefore \vec{r}.\vec{a} = 0$ and $\vec{r}.(\vec{b} \times \vec{a}) = 0$.

From the second relation, we have $[\vec{r}\ \vec{b}\ \vec{a}] = 0$.

This shows that the scalar triple product of $\vec{r}$, $\vec{a}$ and $\vec{b}$ is zero. Hence $\vec{r}$ is coplanar with $\vec{a}$ and $\vec{b}$ i.e. $\vec{a} \times (\vec{b} \times \vec{a})$ is coplanar with the vectors $\vec{a}$ and $\vec{b}$.

7. If $\vec{a}$, $\vec{b}$, $\vec{c}$ be three unit vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$, find the angle which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, $\vec{b}$ and $\vec{c}$ being non parallel.

Solution:

Since $|\vec{a}| = 1, |\vec{b}| = 1, |\vec{c}| = 1$. By given condition, we have

$$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$$

or, $(\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c} = \frac{1}{2}\,\vec{b}$

or, $\left(\vec{a}.\vec{c} - \frac{1}{2}\right)\vec{b} + [-(\vec{a}.\vec{b})]\,\vec{c} = \vec{0}$.

Since $\vec{b}$ and $\vec{c}$ are non parallel, so either $\vec{a}.\vec{c} - \frac{1}{2} = 0$ or $\vec{a}.\vec{b} = 0$.

Case I: **If $\vec{a}.\vec{c} - \frac{1}{2} = 0$**, then $\vec{a}.\vec{c} = \frac{1}{2}$.

Now, if θ is the angle between $\vec{a}$ and $\vec{c}$, then

$|\vec{a}|\,|\vec{c}|\cos\theta = \frac{1}{2}$ or, $\cos\theta = \frac{1}{2}$ $\therefore \theta = \frac{\pi}{3}$. Hence the angle between $\vec{a}$ and $\vec{c}$ is $\pi/3$.

Case II: If $\vec{a}.\vec{b} = 0$ then $|\vec{a}|\,|\vec{b}|\cos\phi = 0$, where ϕ is the angle between $\vec{a}$ and $\vec{b}$.

or, $1.1\cos\phi = 0$ $\therefore \phi = \frac{\pi}{2}$. Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\pi/2$.

Equation of the Line

1. Find a vector equation of a straight line through the
   (i) given point point $\vec{a}$ and parallel to the vector $\vec{b}$. **Ans: $\vec{r} = \vec{a} + t\,\vec{b}$**
   (ii) two points $\vec{a}$ and $\vec{b}$. **Ans: $\vec{r} = \vec{a} + t\,(\vec{b} - \vec{a})$**
2. Find a vector equation and parametric equation for the line through the point
   (i) (2, 2.4, 3.5) and parallel to the vector $3\vec{i} + 2\vec{j} - \vec{k}$ **Ans: $\vec{r} = (2\vec{i} + 2.4\vec{j} + 3.5\vec{k}) + t(3\vec{i} + 2\vec{j} - \vec{k})$**
   (ii) (1, 0, 6) and perpendicular to the plane $x + 3y + z = 5$. **Ans: $\vec{r} = (\vec{i} + 6\vec{k}) + t(\vec{i} + 3\vec{j} + \vec{k})$**
3. Find a vector equation for the line segment from (2, -1, 4) to (4, 6, 1).
   Ans: $\vec{r} = (2\vec{i} - \vec{j} + 4\vec{k}) + t(2\vec{i} + 7\vec{j} - 3\vec{k}),\ 0 \le t \le 1$
4. Find the parametric equations and symmetric equation for the lines through
   (i) the points (-8, 1,4)and (3, -2, 4) **Ans: $x = -8 + 11t$, $y = 1 - 3t$, $z = 4$; $\frac{x+8}{11} = \frac{y-1}{-3}$, $z = 4$**
   (ii) (2, 1,0) and perpendicular to both the vectors $\vec{i} + \vec{j}$ and $\vec{j} + \vec{k}$
   Hint: Find equation for the lines through $\vec{a} =$ (2, 1,0) and parallel to $\vec{b} = (\vec{i} + \vec{j}) \times (\vec{j} + \vec{k})$
   (iii) (1, -1, 1) and parallel to the line $x + 2 = \frac{1}{2}y = z - 3$ **Ans: $x = 1 + t$, $y = -1 + 2t$, $z = 1 + t$**
   (iv) the line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$
5. **(a)** Determine whether the lines L_1 and L_2 are parallel, skew or intersecting. If they intersect, find the point of intersection.
   (i). $L_1: x = 3 + 2t, y = 4 - t, z = 1 + 3t$ and $L_2: x = 1 + 4s, y = 3 - 2s, z = 4 + 5s$ **Ans: Skewed**
   (ii). $L_1: \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3}$ and $L_2: \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7}$ **Ans: (4,-1,-5)**

   (b) Prove that the lines $L_1: \frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$, $L_2: \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ intersect, find also their point of intersection .

Hints of Selected Problems

1.Find the symmetric equation for the lines of intersection of the planes $x + 2y + 3z - 6 = 0 = 3x+4y+5z-2$.

Hint:

The equation of the line is $x + 2y + 3z - 6 = 0 = 3x+4y+5z-2$ ....(1)
To transform the equation (1) in symmetrical form, we need
(i) direction ratios of the line. and (ii) the co-ordinates of any point on it.

The direction ratios of the line:

Let l, m, n be the direction ratios of the line (1). Since the line lies on both the planes. So it is perpendicular to the normal of these planes. So applying the conditions of perpendicularity, we get

$1l + 2m + 3n = 0$ and $3l + 4m + 5n = 0$.

Solving by rules of cross multiplication, we get

$$\frac{l}{10-12} = \frac{m}{9-5} = \frac{n}{4-6} \quad \text{or ,} \quad \frac{l}{-2} = \frac{m}{4} = \frac{n}{-2}$$

or, $\frac{l}{1} = \frac{m}{-2} = \frac{n}{1} = k$ (say)$\therefore$ $\quad l = k, m = -2k, n = k$.

Thus the direction ratios of the line are proportional to 1, –2, 1.

The co-ordinates of any point on it.

Suppose the line meets the plane $z = 0$ at $(x_1, y_1, 0)$. Then, $x_1 + 2y_1 - 6 = 0$ and $3x_1 + 4y_1 - 2 = 0$.

Solving, we get $\frac{x_1}{-4+24} = \frac{y_1}{-18+2} = \frac{1}{4-6} \quad \Rightarrow \quad \frac{x_1}{20} = \frac{y_1}{-16} = \frac{1}{-2} \quad \therefore x_1 = -10, y_1 = 8$.

Hence the line (1) intersect the plane $z = 0$ at point $(-10, 8, 0)$.

Hence the equation of straight line in symmetrical form is $\frac{x+10}{1} = \frac{y-8}{-2} = \frac{z-0}{1}$.

2.Find the equation of the line through the point (1, 2, 4) and parallel to the line $3x + 2y - z = 4$, $x - 2y - 2z = 5$.

Hint:

The required equation of the line through (1, 2, 4) is $\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-4}{n}$ ....(1)

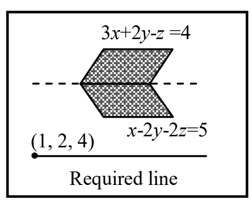
where l, m, n are the direction ratios of the line (1).
If the line (1) is parallel to the line $3x + 2y - z = 4$, $x - 2y - 2z = 5$
Then it is perpendicular to the normal to each of the plane.
Hence applying the condition of perpendicularity, we get
$3l + 2m - n = 0$ and $l - 2m - 2n = 0$
Solving by rules of cross multiplication, we get

$$\frac{l}{-4-2} = \frac{m}{-1+6} = \frac{n}{-6-2}$$

or, $\frac{l}{-6} = \frac{m}{5} = \frac{n}{-8}$

or, $\frac{l}{6} = \frac{m}{-5} = \frac{n}{8} = k$(say).

$\therefore$ $l = 6k, m = -5k, n = 8k$.



Substituting the value of l, m, n in (1), the required equation of the line is $\frac{x-1}{6} = \frac{y-2}{-5} = \frac{z-4}{8}$.

3. Prove that the lines $L_1 : \frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$, $L_2 : \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ intersect, find also their point of intersection .

Hint:

Let $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} = r$ and $\quad \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} = r'$.

Then the co-ordinates of any point on the given two lines are $(2r+1, -3r-1, 8r-10)$ and $(r'+4, -4r'-3, 7r'-1)$ respectively. The two lines will intersect if they meet at a single point. For the point of intersection, for suitable values of r and r', we have

$2r+1 = r'+4 \qquad \therefore\ 2r - r' = 3$ .... (1)
$-3r - 1 = -4r' - 3 \qquad -3r + 4r' = -2$ .... (2)
and $8r - 10 = 7r' - 1 \qquad 8r - 7r' = 9$ .... (3)

Solving (1) and (2), we get $r = 2$ and $r' = 1$.
As $r = 2$ and $r' = 1$ satisfy the equation (3). Hence two given lines intersect.
The co-ordinates of point of intersection is $(2r+1, -3r-1, 8r-10)$ or, $(5, -7, 6)$.

<u>Equation of the Planes</u>

1.Find an equation of the plane.

(i) through the point $\left(-1, \frac{1}{2}, 3\right)$ and with normal vector $\vec{i}+4\vec{j}+\vec{k}$ **Ans: $x+4y+z=4$.**

(ii) through the point (1, -1, -1) and parallel to the plane $5x-y-z=6$ **Ans: $5x-y-z=7$**

(iii) that contains the line $x=1+t$, $y=2-t$, $z=4-3t$ and is parallel to the plane $5x+2y+z=1$.

(iv) through the points (3, -1, 2), (8, 2, 4), and (-1, -2, -3) **Ans: $-13x+17y+7z=-42$**

2. Determine the equation of the plane

(i) that passes through the point (6, 0, -2) and contains the line $x=4-2t$, $y=3+5t$, $z=7+4t$ **Ans: $33x+10y+4z=190$**

(ii) that passes through the pint (1, -1, 1) and contains the line $x=2y=3z$.

(iii)that passes through the point (-1, 2, 1) and contains the line of intersection of the planes $x+y-z=2$ and $2x-y+3z=1$. **Ans: $x-2y+4z=-1$**

3.Find an equation of the plane.

(i)that passes through the points (0,-2, 5) and (-1, 3, 1) and is perpendicular to the plane $2z=5x+4y$

(ii)that passes through the point (– 1, 3, 2) and is perpendicular to the planes $x+2y+2z=5$ and $3x+3y+2z=8$. **Ans: $2x-4y+3z+8$**

(iii) through the point (2, 0, 1) and perpendicular to the line $x=3t$, $y=2-t$, $z=3+4t$

(iv)that passes through the line of intersection of the planes $x-z=1$ and $y+2z=3$ and is perpendicular to the plane $x+y-2z=1$.

Hints of Selected Problems

1. Find the equation of the plane through three points (1, 1, 0), (1, 2, 1) and (–2, 2, – 1).

Solution:

Any equation of the plane through the point (1, 1, 0) is

$a(x-1)+b(y-1)+c(z-0)=0$...... (1)

But (1) passes through the point (1, 2, 1) and (–2, 2, – 1).

So we have, $a(1-1)+b(2-1)+c(1-0)=0$

$\Rightarrow \quad a.0+b.1+c.1=0$ .... (2)

and $\quad a(-2-1)+b(2-1)+c(-1-0)=0$

$\Rightarrow \quad a.(-3)+b.1+c(-1)=0$ .... (3)

From (2) and (3) by cross multiplication, we have

$$\frac{a}{1.(-1)-1.1}=\frac{b}{1.(-3)-0.(-1)}=\frac{c}{0.1-(-3).1} \Rightarrow \frac{a}{-2}=\frac{b}{-3}=\frac{c}{3}=k \text{ (say)}$$

$\therefore \quad a=-2k,\ b=-3k,\ c=3k$.......... (4)

Substituting the values of a, b, c, in (1), we get

$-2k(x-1)-3k(y-1)+3kz=0$ or, $2x+3y-3z=5$.

2. Obtain the equation of the plane through the intersection of the plane $x+2y-3z=5$ and $5x+7y+3z=10$ and passing through the point (1, 2, 3) .

Solution:

The given planes are

$x+2y-3z-5=0,\ 5x+7y+3z-10=0$ .... (1)

The equation of the plane through the intersection of the planes (1) is

$(x+2y-3z-5)+\lambda(5x+7y+3z-10)=0$. .... (2)

If (2) passes through the point (1, 2, 3), then

$(1+4-9-5)+\lambda(5+14+9-10)=0 \quad \Rightarrow \quad \lambda=\frac{9}{18}=\frac{1}{2}$.

Substituting the value of λ in (2), the required equation of the plane is

$(x+2y-3z-5)+\frac{1}{2}(5x+7y+3z-10)=0$ or, $7x+11y-3z-20=0$.

3. Find the equation of the plane which passes through the point (– 1, 3, 2) and is perpendicular to each of the planes $x+2y+2z=5$, $3x+3y+2z=8$.

Solution:

Any equation of the plane through the point (– 1, 3, 2) is

$a(x+1)+b(y-3)+c(z-2)=0$ .... (1)

If this plane is perpendicular to each of the planes

$x+2y+2z=5$ and $3x+3y+2z=8$, then applying the condition of perpendicularity, we get

$a.1+b.2+c.2=0$ .... (2) and $a.3+b.3+c.2=0$ .... (3)

By cross multiplication on (2) and (3), we have

$$\frac{a}{4-6}=\frac{b}{6-2}=\frac{c}{3-6} \Rightarrow \frac{a}{-2}=\frac{b}{4}=\frac{c}{-3}=k \text{ (say)}$$

$\therefore \quad a = -2k, b = 4k, c = -3k.$

Substituting these values of a, b, c in (1), we get

$-2k(x + 1) + 4k(y - 3) - 3k(z - 2) = 0$ or, $2x - 4y + 3z + 8 = 0$

which is the required equation of the plane.

4. Find the equation of the plane through the points (2, 2, 1) and (9, 3, 6) and is perpendicular to the plane $2x + 3y + 6z = 9$.

Solution:

Any equation of the plane through the point (2, 2, 1) is

$a(x - 2) + b(y - 2) + c(z - 1) = 0$ .... (1)

If this plane passes through (9, 3, 6), then

$a(9 - 2) + b(3 - 2) + c(6 - 1) = 0$

or, $a.7 + b.1 + c.5 = 0$ .... (2)

Again, if the plane (1) is perpendicular to the plane $2x + 3y + 6z = 9$.

So applying the perpendicularity condition, we have

$a.2 + b.3 + c.6 = 0$ .... (3)

By cross multiplication on (2) and (3), we get

$$\frac{a}{6-15} = \frac{b}{10-42} = \frac{c}{21-2} \Rightarrow \frac{a}{-9} = \frac{b}{-32} = \frac{c}{19} = k \text{ (say)}$$

$\therefore \quad a = -9k, b = -32k, c = 19k.$

Substituting the values of a, b, c in equation (1), we get

$-9k(x - 2) - 32k(y - 2) + 19k(z - 1) = 0$

or, $9x + 32y - 19z - 63 = 0$, which is the required equation of the plane.

Point of Intersection of Line and Plane:

1. (i) Where does the line through (1, 0, 1) and (4, -2, 2) intersect the plane $x + y + z = 6$?
   (ii) Find the point at which the lines $\vec{r} = (1, 1, 0) + t(1, -1, 2)$ and $\vec{r} = (2, 0, 2) + s(-1, 1, 0)$ intersect.
2. Find direction numbers and direction cosines for the line of intersection of the planes $x + y + z = 1$ and $x + z = 0$.
   Ans:1,0,-1
3. Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.
   (i). $x + 4y - 3z = 1$, $-3x + 6y + 7z = 0$ **Ans: perpendicular**
   (ii). $x + y + z = 1$, $x - y + z = 1$ **Ans: Neither, 70.5°**
   (iii). $x = 4y - 2z$, $8y = 1 + 2x + 4z$ **Ans: Parallel**

4. Find the points in which the line $\frac{x+1}{-1} + \frac{y-12}{5} = \frac{z-7}{2}$ cuts the surface $11x^2 - 5y^2 + z^2 = 0$.

Ans: (2, –3, 1) and (1, 2, 3)

Hints of Selected Problem

1.Find the point where the line joining (2, –3, 1), (3, –4, –5) cuts the plane $2x + y + z = 7$.

Solution:

The equation of straight line joining two points (2, – 3, 1) and (3, – 4, – 5) is

$$\frac{x-2}{3-2} = \frac{y-(-3)}{-4-(-3)} = \frac{z-1}{-5-1} \quad \text{or,} \quad \frac{x-2}{1} = \frac{y+3}{-1} = \frac{z-1}{-6} = r\text{(say)}.$$

Then the co-ordinates of any point on this line are $(r + 2, -r - 3, -6r + 1)$ ....(1)

If this point lies on the plane $2x + y + z = 7$, then we must have

$2(r + 2) + (-r - 3) + (-6r + 1) = 7$ $\therefore r = -1$.

Substituting the value of r in (1), required point is $(-1 + 2, 1 - 3, 6 + 1)$ i.e. $(1, -2, 7)$.

Distance from a point to a plane:

1. Find equation for the plane consisting of all points that are equidistant from the points (1, 0, -2) and (3, 4, 0). **Ans: $x + 2y + z = 5$**

2.(i) Find the distance from the point (1, -2, 4) to the plane $3x + 2y + 6z = 5$ **Ans: 18/7**

(ii) **Formula:** The distance between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$. Using this formula, find the distance between the parallel planes

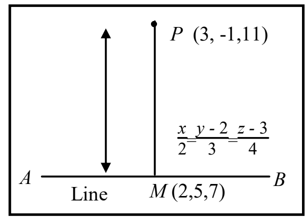
$2x - 3y + z = 4$ and $4x - 6y + 2z = 3$ **Ans:** $\frac{5}{2\sqrt{14}}$

(iii) Find the distance from the point (4, 1, -2) to the given line $x = 1 + t,\ y = 3 - 2t,\ z = 4 - 3t$ **Ans:** $\sqrt{61/14}$

Hints of Selected Problem

1. Find the length of the perpendicular from the point (3, – 1, 11) to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$.

Solution:



Given line is $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$..........(1)

Let $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r$(say).

Then the co-ordinate of any point on the line (1) is

$(2r, 3r+2, 4r+3)$ ....(2)

Let the foot of the perpendicular from $P(3, -1, 11)$ be M.

Then the co-ordinates of M is of the form $(2r, 3r+2, 4r+3)$.

The direction ratios of PM are $2r-3,\ 3r+2+1,\ 4r+3-11$ i.e. $2r-3,\ 3r+3,\ 4r-8$.

Also, the direction ratios of the line AB are 2, 3,4. But PM is perpendicular to AB. So applying the perpendicularity condition, we get

$2(2r-3) + 3(3r+3) + 4(4r-8) = 0$ or, $29r - 29 = 0 \therefore\ r = 1$.

Hence the co-ordinates of M are (from 2)

$(2.1, 3.1 + 2, 4.1 + 3)$ i.e. $(2, 5, 7)$.

Hence the required perpendicular distance $PM = \sqrt{(3-2)^2 + (-1-5)^2 + (11-7)^2} = \sqrt{53}$.

Distance Between Skew Lines:

1.(i) Show that the lines with symmetric equation $x = y = z$ and $x + 1 = y/2 = z/3$ are skew, and find the distance between these lines. **Ans:** $\frac{1}{\sqrt{6}}$

(ii) Let L_1 be the line through the origin and the point (2, 0, -1). Let L_2 be the line through the points (1, -1, 1) and (4, 1, 3). Find the distance between L_1 and L_2. **Ans:** $\frac{13}{\sqrt{69}}$

Hints of Similar Problem

1. Show that the shortest distance between the lines with symmetric equation

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5} \text{ is } \frac{1}{\sqrt{6}}.$$

Solution:

The given lines are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{.......... (1)} \quad \text{and} \quad \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5} \text{......... (2)}$$

Now, the equation of the plane containing the first line and parallel to the second line is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

or, $(x-1)(15-16) + (y-2)(12-10) + (z-3)(8-9) = 0$

or, $-1(x-1) + 2(y-2) - 1(z-3) = 0$

or, $-x + 2y - z + 1 - 4 + 3 = 0$

or, $x - 2y + z = 0$ ..... (3)

Also, (2, 4, 5) is a point on second line.

$\therefore$ Length of S.D. =Perpendicular distance from the point (2, 4, 5) to the plane (3)

$$= \left| \frac{2 - 2.4 + 5}{\sqrt{1^2 + (-2)^2 + 1^2}} \right| = \left| -\frac{1}{\sqrt{6}} \right| = \frac{1}{\sqrt{6}}.$$

Curvature of a curve at a point

1. Formula for Curvature and radius of Curvature of Cartesian Curves

Formula-1. For the Cartesian equation $y = f(x)$:

Radius of curvature at any point (x, y), $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$, where $y_1 = \dfrac{dy}{dx}$ and $y_2 = \dfrac{d^2y}{dx^2} \neq 0$.

But if the equation of the curve be $x = f(y)$, then

Radius of curvature at any point (x, y) , $\rho = \dfrac{(1+x_1^2)^{3/2}}{x_2}$, where $x_1 = \dfrac{dx}{dy}$ and $x_2 = \dfrac{d^2x}{dy^2} \neq 0$.

Formula-2. The curvature(κ) = $\dfrac{1}{\textbf{radius of curvature}} = \dfrac{1}{\rho}$.

Examples

1. Show that the circle $x^2 + y^2 = a^2$ is a curve of uniform curvature κ and its radius of curvature ρ at every point is constant.

Solution:

Consider a circle $x^2 + y^2 = a^2$ ...(i)

Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0 \qquad \therefore \frac{dy}{dx} = -\frac{x}{y}.$$

Again, differentiating w.r.t.x, we get

$$\frac{d^2y}{dx^2} = \frac{x\dfrac{dy}{dx} - y}{y^2} = \frac{x.\left(-\dfrac{x}{y}\right) - y}{y^2} = \frac{-(x^2+y^2)}{y^3} = \frac{-a^2}{y^3}.$$

If (x, y) be any point on the circle, then,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\dfrac{x^2}{y^2}\right)^{3/2}}{\dfrac{-a^2}{y^3}} = -\frac{(x^2+y^2)^{3/2}}{y^3} \times \frac{y^3}{a^2} = -\frac{(a^2)^{3/2}}{a^2} = -a. \quad \therefore \rho = a, \text{ numerically}.$$

Hence, curvature (κ) = $\dfrac{1}{\rho} = \dfrac{1}{a}$, which is constant.

2. Find the radius of curvature ρ at any point (x, y) on the parabola $y^2 = 4ax$

Solution:

Here, $y^2 = 4ax$ ....(i)

Differentiating both sides w.r.t. x, we get

$$2yy_1 = 4a \qquad \therefore y_1 = \frac{2a}{y} \qquad ......(ii)$$

Again differentiating w.r.t.x,

$$y_2 = \frac{-2a}{y^2}.y_1 = \frac{-2a}{y^2}.\frac{2a}{y} = -\frac{4a^2}{y^3} \quad [\because \text{ From (ii)}]$$

Applying the formula

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left[1+\left(\dfrac{2a}{y}\right)^2\right]^{3/2}}{\dfrac{-4a^2}{y^3}} = -\frac{(y^2+4a^2)^{3/2}}{4a^2}$$

But $y^2 = 4ax$, then

$$\therefore \rho = -\frac{(4ax+4a^2)^{3/2}}{4a^2} = -(4a)^{3/2}\frac{(x+a)^{3/2}}{4a^2} = \frac{-8a^{3/2}(x+a)^{3/2}}{4a^2} = -\frac{2(x+a)^{3/2}}{\sqrt{a}} = \frac{2(x+a)^{3/2}}{\sqrt{a}} \text{ (numerically)}.$$

3. Find the radius of curvature ρ and curvature (κ) at any point (x, y) on the catenary $y = c\cosh(x/c)$

Solution:

Here, $y = c \cosh (x/c)$ ……..(i)

Differentiating (i) w.r.t. x, we get

$\frac{dy}{dx} = c. \sinh \frac{x}{c} . \frac{1}{c}$ $\therefore$ $y_1 = \sinh \frac{x}{c}$

Again differentiating, we get $y_2 = \frac{1}{c} \cosh \frac{x}{c}$.

$$\text{Now, } \rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{\left(1 + \sinh^2 \frac{x}{c}\right)^{3/2}}{\frac{1}{c} \cosh \frac{x}{c}} = \frac{c\left(\cosh^2 \frac{x}{c}\right)^{3/2}}{\cosh \frac{x}{c}} = c \frac{\cosh^3 \frac{x}{c}}{\cosh \frac{x}{c}} = c \cosh^2 \frac{x}{c} = c . \frac{y^2}{c^2} = \frac{y^2}{c^2} \quad [\text{ By (i)}]$$

Hence, curvature $(\kappa) = \frac{1}{\rho} = \frac{c^2}{y^2}$, which is constant

4. Find the radius of curvature ρ and curvature (κ) at any point (x, y) on the rectangular hyperbola $xy = c^2$

Solution:

Here, $y = \frac{c^2}{x}$. ……..(i)

Differentiating (i) w.r.t. x, we get

$y_1 = \frac{-c^2}{x^2}$ and $y_2 = -c^2 . (-2x^{-3}) = \frac{2c^2}{x^3}$.

Applying the formula for radius of curvature in Cartesian form i.e.

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{\left(1 + \frac{c^4}{x^4}\right)^{3/2}}{\frac{2c^2}{x^3}} = \frac{(x^4 + c^4)^{3/2}}{2c^2x^3}$$

But $c^2 = xy$ and therefore, $c^4 = x^2y^2$

$$\therefore \quad \rho = \frac{(x^4 + x^2y^2)^{3/2}}{2c^2x^3} = \frac{x^3(x^2 + y^2)^{3/2}}{2c^2x^3} = \frac{(x^2 + y^2)^{3/2}}{2c^2}.$$

Hence, curvature $(\kappa) = \frac{1}{\rho} = \frac{2c^2}{(x^2 + y^2)^{3/2}}$

3. Show that the radius of curvature ρ at any point (x, y) on the curve $y = a \log \sec \frac{x}{a}$ is of constant length.

Solution:

Here, $y = a \log \sec \frac{x}{a}$ ………..(i)

Differentiating successively y (i) w.r.t x, we get

$y_1 = a. \frac{1}{\sec \frac{x}{a}} .\sec \frac{x}{a} .\tan \frac{x}{a} . \frac{1}{a} = \tan \frac{x}{a}$ and $y_2 = \frac{1}{a} \sec^2 \frac{x}{a}$.

Applying the formula for radius of curvature in Cartesian form i.e.

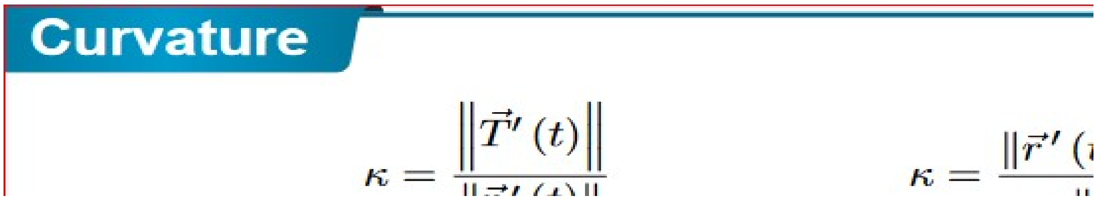
$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{\left(1 + \tan^2 \frac{x}{a}\right)^{3/2}}{\frac{1}{a}\sec^2 \frac{x}{a}} = \frac{\left(\sec^2 \frac{x}{a}\right)^{3/2}}{\frac{1}{a} \sec^2 \frac{x}{a}} = a \sec \frac{x}{a}.$$

4. Find the curvature of the curve $y = 2x^4$ at point $x = 2$.

5. Find the curvature of the curve $y = 3x^4$ at point $x = 1$.

Curvature in Parametric Curve for Vector Function:

1. Find the curvature of the vector function $\vec{r}(t) = p\cos t\,\vec{i} + p\sin t\,\vec{j}$, where p is constant.
Hint:



2. Determine the curvature for $\vec{r}(t) = (t, 3\sin(t), 3\cos(t))$.

3. Find the normal vector and binormal vector of the space curve $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ where $x = t^2, y = t^2, z = t^3$ at point (1, 1, 1).

4. Find the normal vector of the space curve

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ where $x = t^2, y = t^2, z = t^3$ at point (1, 0, 1).

2.Formula for Curvature and radius of Curvature of Parametric Curves

For the parametric equations $x = \phi(t), y = \Psi(t)$

The radius of curvature ρ is given by $\rho = \dfrac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$, where $x'y'' - y'x'' \neq 0$ and

Where, $x' = \dfrac{dx}{dt}$ and $y' = \dfrac{dy}{dt}$ etc.

Examples

1. Find the radius of curvature at any point θ for the parametric curve (Circle) : $x = a\cos\theta, y = a\sin\theta$

Solution:

Here, $x = a\cos\theta$ and $y = a\sin\theta$

Differentiating both sides w.r.t. θ, we get

$x' = -a\sin\theta$ and $y' = a\cos\theta$

Again differentiating, we get

$x'' = -a\cos\theta$ and $y'' = -a\sin\theta$.

Applying the formula of radius of curvature in parametric form i.e.

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{(a^2\sin^2\theta + a^2\cos^2\theta)^{3/2}}{(-a\sin\theta)\,.\,(-a\sin\theta) - (a\cos\theta)(-a\cos\theta)} = \frac{(a^2)^{3/2}\,.\,(1)^{3/2}}{a^2(\sin^2\theta + \cos^2\theta)} = \frac{a^3}{a^2} = a.$$

$\therefore \quad \rho = a$ Ans.

2. Find the radius of curvature at any point ϕ for the parametric curve (Ellipse): $x = a\cos\phi, y = b\sin\phi$.

Solution:

Here, $x = a\cos\phi$ and $y = b\sin\phi$

Differentiating both sides w.r.t. ϕ, we get

$x' = -a\sin\phi$ and $y' = b\cos\phi$

Again differentiating, we get

$x'' = -a\cos\phi$ and $y'' = -b\sin\phi$

Applying the formula for radius of curvature in parametric form

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x' y'' - x'' y'} = \frac{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}}{(-\, a \sin \phi)\,(-\, b \sin \phi) - (-\, a \cos \phi)\,.\, b \cos \phi}$$

$$= \frac{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}}{ab(\sin^2 \phi + \cos^2 \phi)} = \frac{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}}{ab}.$$

3. Prove that the radius of curvature at any point $\theta = 0$ for the parametric curve (Cycloid): $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ is $\rho = 4a$.

Solution:

Here, $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$

Differentiating w.r.t. θ, we get

$x' = a(1 + \cos\theta)$ and $y' = a \sin\theta$

Again, differentiating w.r.t. θ,

$x'' = -\, a \sin\theta$ and $y'' = a \cos\theta$

At $\theta = 0$: $x' = a(1 + 1) = 2a$ and $y' = a \sin 0 = 0$

$x'' = -\, a \sin 0 = 0$ and $y'' = a \cos 0 = a$.

Applying the formula

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} = \frac{[(2a)^2 + 0^2]^{3/2}}{2a\,.\,a - 0\,.\,0} = \frac{(4a^2)^{3/2}}{2a^2} = \frac{(2a)^3}{2a^2} = 4a. \text{ Hence proved.}$$

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Tribhuvan University
School of Mathematical Sciences, Kirtipur,Kathmandu, Nepal
<u>Problem Set</u> **For Master in Data Sciences I Year (II Sem)-2080**
Course: Multivariable Calculus For Data Science -II (MDS 554)
Prepared by Prof. Dr.Narayan Prasad Pahari

Unit II: Vector Functions

Unit 2: **Vector Functions**
Vector functions and space curves
Derivatives and integrals of vector functions
Arc length and curvature
Motion in space

Basic Formulae on Differentiation

If $\vec{r}$, $\vec{r}_1$ and $\vec{r}_2$ be three differentiable vector functions of scalar variable t and ϕ is a differentiable scalar function of t, then we have

1. $\frac{d}{dt}(\vec{r}_1 \pm \vec{r}_2) = \frac{d\vec{r}_1}{dt} \pm \frac{d\vec{r}_2}{dt}$.

2. If $\vec{r} = \vec{a}$, be a constant vector, then $\frac{d\vec{r}}{dt} = \vec{0}$.

3. $\frac{d}{dt}(\phi\,\vec{r}) = \frac{d\phi}{dt}\vec{r} + \phi\frac{d\vec{r}}{dt}$. In particular, if k is a constant, then $\frac{d}{dt}(k\vec{r}) = k\frac{d\vec{r}}{dt}$

4. $\frac{d}{dt}(\vec{r}_1.\vec{r}_2) = \vec{r}_1.\frac{d\vec{r}_2}{dt} + \frac{d\vec{r}_1}{dt}.\vec{r}_2$

5. $\frac{d}{dt}(\vec{r}_1 \times \vec{r}_2) = \vec{r}_1 \times \frac{d\vec{r}_2}{dt} + \frac{d\vec{r}_1}{dt} \times \vec{r}_2$.

6. **The Derivative of Scalar Triple Product:**

   The derivative of the scalar triple product $[\vec{r}_1\ \vec{r}_2\ \vec{r}_3]$ of three vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ is

$$\frac{d}{dt}[\vec{r}_1\ \vec{r}_2\ \vec{r}_3] = \left[\frac{d\vec{r}_1}{dt}\ \vec{r}_2\ \vec{r}_3\right] + \left[\vec{r}_1\ \frac{d\vec{r}_2}{dt}\ \vec{r}_3\right] + \left[\vec{r}_1\ \vec{r}_2\ \frac{d\vec{r}_3}{dt}\right].$$

7. **The Derivative of Vector Triple Product:**

   The derivative of the vector triple product $\vec{r}_1 \times (\vec{r}_2 \times \vec{r}_3)$ of three vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ is

$$\frac{d}{dt}[\vec{r}_1 \times (\vec{r}_2 \times \vec{r}_3)] = \frac{d\vec{r}_1}{dt} \times (\vec{r}_2 \times \vec{r}_3) + \vec{r}_1 \times \left(\frac{d\vec{r}_2}{dt} \times \vec{r}_3\right) + \vec{r}_1 \times \left(\vec{r}_2 \times \frac{d\vec{r}_3}{dt}\right).$$

Geometrical Interpretation of Derivative

Theorem:1. If $\vec{r} = \vec{f}(t)$ be a vector function of scalar variable t, then geometrically, the derivative $\frac{d\vec{r}}{dt}$ at a point P represents a vector along the tangent in the sense of t increasing.

Theorem: 2 The derivative $\frac{d\vec{r}}{dt}$ also represents the velocity of the particle at point P along the tangent PT and the second derivative $\frac{d^2\vec{r}}{dt^2}$ represents the acceleration of the particle at P along the tangent PT.

<u>Examples and Exercises</u>

<u>Example-1:</u> If $\vec{r} = \vec{a}\,e^{nt} + \vec{b}\,e^{nt}$, where $\vec{a}$ and $\vec{b}$ are constant vectors, show that: $\frac{d^2\vec{r}}{dt^2} - (m + n)\frac{d\vec{r}}{dt} + mn\vec{r} = \vec{0}$.

Proof:

Given that $\vec{r} = \vec{a}\, e^{mt} + \vec{b}\, e^{nt}$ .... (1)

$\therefore \quad \dfrac{d\vec{r}}{dt} = m\vec{a}\, e^{mt} + n\vec{b}\, e^{nt}$ .... (2), and $\dfrac{d^2\vec{r}}{dt^2} = m^2\vec{a}\, e^{mt} + n^2\, \vec{b}\, e^{nt}$ .... (3)

Using (1), (2) and (3), we get

$$\frac{d^2\vec{r}}{dt^2} - (m+n)\frac{d\vec{r}}{dt} + mn\vec{r} = (m^2\vec{a}\, e^{mt} + n^2\vec{b}\, e^{nt}) - (m+n)\,(m\vec{a}\, e^{mt} + n\vec{b}\, e^{nt}) + mn\,(\vec{a}\, e^{mt} + \vec{b}\, e^{nt}) = \vec{0}.$$

Example-2: **If $\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$,**

show that : $\vec{r} \times \dfrac{d\vec{r}}{dt} = \omega\,(\vec{a} \times \vec{b})$ and $\dfrac{d^2\vec{r}}{dt^2} = -\,\omega^2\vec{r}$, where $\vec{a}$ and $\vec{b}$ are constant vectors and ω is a constant.

Solution:

Here, $\vec{a}\cos\omega t + \vec{b}\sin\omega t$ .... (1)

$\therefore \quad \dfrac{d\vec{r}}{dt} = -\,\vec{a}\omega\sin\omega t + \vec{b}\omega\cos\omega t$

and $\dfrac{d^2\vec{r}}{dt^2} = -\,\vec{a}\,\omega^2\cos\omega t - b\omega^2\sin\omega t = -\,\omega^2\,(\vec{a}\cos\omega t + \vec{b}\sin\omega t) = -\,\omega^2\vec{r}.$ [$\because$ By using (1)]

To prove the first part:

Now, $\vec{r} \times \dfrac{d\vec{r}}{dt} = (\vec{a}\cos\omega t + \vec{b}\sin\omega t) \times (-\,\vec{a}\omega\sin\omega t + \vec{b}\omega\cos\omega t) = (\vec{a} \times \vec{b})\omega.$

$\therefore \quad \vec{r} \times \dfrac{d\vec{r}}{dt} = (\vec{a} \times \vec{b})\omega.$

Problem: **If $\vec{r} = \vec{a}\, e^{nt} + \vec{b}\, e^{-nt}$, where $\vec{a}$ and $\vec{b}$ are constant vectors. Show that : $\dfrac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0.$**

Example-3: **If $\vec{r}$ be a unit vector, prove that : $\left|\vec{r} \times \dfrac{d\vec{r}}{dt}\right| = \left|\dfrac{d\vec{r}}{dt}\right|$**

Solution:

Since $\vec{r}$ is a unit vector, so that $|\vec{r}| = 1$ i.e., $r = 1$ and $r^2 = 1$. $\therefore\ \vec{r}.\vec{r} = 1$ . ... (1)

Differentiating both sides of (1) w.r.t t, we get

$\dfrac{d\vec{r}}{dt}.\vec{r} + \vec{r}.\dfrac{d\vec{r}}{dt} = 0$ or, $2\vec{r}.\dfrac{d\vec{r}}{dt} = 0$ or, $\vec{r}.\dfrac{d\vec{r}}{dt} = 0.$

This shows that $\vec{r}$ and $\dfrac{d\vec{r}}{dt}$ are perpendicular to each other and therefore, the angle between $\vec{r}$ and $\dfrac{d\vec{r}}{dt}$ is 90°.

Hence, $\left|\vec{r} \times \dfrac{d\vec{r}}{dt}\right| = |\vec{r}|\left|\dfrac{d\vec{r}}{dt}\right|\sin 90° = \left|\dfrac{d\vec{r}}{dt}\right|.$

Example-4: **If $\vec{r} = t^2\vec{i} - t\vec{j} + (2t+1)\,\vec{k}$. Find (i) $\dfrac{d\vec{r}}{dt}.\dfrac{d^2\vec{r}}{dt^2}$ (ii) $\left|\dfrac{d\vec{r}}{dt}\right|$ (iii) $\left|\dfrac{d^2\vec{r}^2}{dt^2}\right|$ at $t = 0$.**

Solution:

Now, $\vec{r} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ $\quad\therefore \quad \dfrac{d\vec{r}}{dt} = 2t\,\vec{i} - \vec{j} + 2\,\vec{k}$ and $\dfrac{d^2\vec{r}}{dt^2} = 2\,\vec{i}$.

(i) $\dfrac{d\vec{r}}{dt}.\dfrac{d^2\vec{r}}{dt^2} = (2t\,\vec{i} - \vec{j} + 2\vec{k}).2\,\vec{i} = 4t\,(\vec{i}\,.\,\vec{i}) - 2(\vec{j}\,.\,\vec{i}) + 4\,(\vec{k}\,.\,\vec{i}) = 4t.$

$\therefore$ **At $t = 0$,** $\dfrac{d\vec{r}}{dt}.\dfrac{d^2\vec{r}}{dt^2} = 4.0 = 0$.

(ii) $\left|\dfrac{d\vec{r}}{dt}\right| = |2t\,\vec{i} - \vec{j} + 2\vec{k}| = \sqrt{(2t)^2 + (-1)^2 + (2)^2} = \sqrt{4t^2 + 5}.$ $\quad\therefore$ **At $t = 0$,** $\left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{5}$.

(iii) $\left|\frac{d^2\vec{r}}{dt^2}\right| = |2\vec{i}| = \sqrt{2^2} = 2.$ $\therefore$ **At $t = 0$,** $\left|\frac{d^2\vec{r}}{dt^2}\right| = 2.$

<u>Example-5:</u> If $\vec{r} = a\cos t\,\vec{i} + b\sin t\,\vec{j} + ct\vec{k}$, find $\dot{\vec{r}}$, $\ddot{\vec{r}}$, $|\dot{\vec{r}}|$ and $|\ddot{\vec{r}}|$. Also, find their values at $t = 0$.

Solution:

Given that $\vec{r} = a\cos t\,\vec{i} + b\sin t\,\vec{j} + ct\,\vec{k}$. .... (1)

Differentiating bothsides of (1) two times w.r.t. 't', we get

$$\dot{\vec{r}} = -a\sin t\,\vec{i} + b\cos t\,\vec{j} + c\,\vec{k} \quad \text{.......... (2)}$$

and $$\ddot{\vec{r}} = -a\cos t\,\vec{i} - b\sin t\,\vec{j}\,. \quad \text{.......... (3)}$$

From (2) and (3), we get

$$|\dot{\vec{r}}| = \sqrt{a^2\sin^2 t + b^2\cos^2 t + c^2} \quad \text{and} \quad |\ddot{\vec{r}}| = \sqrt{a^2\cos^2 t + b^2\sin^2 t}\,.$$

At $t = 0$

$$\dot{\vec{r}} = b\vec{j} + c\vec{k} \text{ and } \ddot{\vec{r}} = -a\,\vec{i} \qquad \therefore \ |\dot{\vec{r}}| = \sqrt{b^2 + c^2} \text{ and } |\ddot{\vec{r}}| = a.$$

Problem: **If $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + t\vec{k}$, find $\frac{d\vec{r}}{dt}$, $\frac{d^2\vec{r}}{dt^2}$ and $\left|\frac{d^2\vec{r}}{dt^2}\right|$.**

Ans: $-a\sin t\,\vec{i} + a\cos t\,\vec{j} + \vec{k}$; $-a\cos t\,\vec{i} - a\sin t\,\vec{j}$; a.

<u>Example-6:</u> If $\vec{r_1} = t^3\vec{i} + t^2\vec{j} + t\vec{k}$ and $\vec{r_2} = (t+1)\vec{i} + (t+2)\vec{j} - 3t\,\vec{k}$. Find (a) $\frac{d}{dt}(\vec{r_1}.\vec{r_2})$ (b) $\frac{d}{dt}(\vec{r_1}\times\vec{r_2})$ at $t = 2$.

Solution:

Now, $\vec{r_1} = t^3\vec{i} + t^2\vec{j} + t\vec{k}$ $\qquad \therefore \frac{d\vec{r_1}}{dt} = 3t^2\vec{i} + 2t\vec{j} + \vec{k}$

and $\vec{r_2} = (t+1)\vec{i} + (t+2)\vec{j} - 3t\,\vec{k}$ $\qquad \therefore \frac{d\vec{r_2}}{dt} = \vec{i} + \vec{j} - 3\vec{k}.$

(a) $$\frac{d}{dt}(\vec{r_1}.\vec{r_2}) = \vec{r_1}.\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}.\vec{r_2}$$
$$= (t^3\vec{i} + t^2\vec{j} + t\vec{k}).(\vec{i} + \vec{j} - 3\vec{k}) + (3t^2\vec{i} + 2t\vec{j} + \vec{k}).[(t+1)\vec{i} + (t+2)\vec{j} - 3t\vec{k}]$$
$$= 4t^3 + 6t^2 - 2t.$$

$\therefore$ **At t = 2,**

$$\frac{d}{dt}(\vec{r_1}.\vec{r_2}) = 4(2)^3 + 6(2)^2 - 2.2 = 32 + 24 - 4 = 52.$$

(b) $$\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \vec{r_1}\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}$$
$$= [(t^3\vec{i} + t^2\vec{j} + t\vec{k})\times(\vec{i} + \vec{j} - 3\vec{k})] + (3t^2\vec{i} + 2t\vec{j} + k)\times[(t+1)\vec{i} + (t+2)\vec{j} - 3t\vec{k}]$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t^3 & t^2 & t \\ 1 & 1 & -3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3t^2 & 2t & 1 \\ (t+1) & (t+2) & (-3t) \end{vmatrix}$$

$\therefore$ **At t = 2,**

$$\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 8 & 4 & 2 \\ 1 & 1 & -3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 12 & 4 & 1 \\ 3 & 4 & -6 \end{vmatrix}$$
$$= \vec{i}(-12-2) + \vec{j}(2+24) + \vec{k}(8-4) + \vec{i}(-2\,4-4) + \vec{j}(3+72) + \vec{k}(48-12)$$
$$= -42\vec{i} + 101\vec{j} + 40\vec{k}\,.$$

<u>Problem: 1.</u> If $\vec{u} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ and $\vec{v} = (2t-3)\,\vec{i} + \vec{j} - t\vec{k}$,

find (a) $\frac{d}{dt}(\vec{u}.\vec{v})$ (b) $\frac{d}{dt}(\vec{u}\times\vec{v})$,at t = 1.

Ans: At $t = 1$, $\frac{d}{dt}(\vec{u}.\vec{v}) = -6$ and $\frac{d}{dt}(\vec{u}\times\vec{v}) = 7\vec{j} + 3\vec{k}$.

Problem: 2. **If $\vec{r} = (a\cos t)\vec{i} + (a\sin t)\vec{j} + (at\tan\alpha)\vec{k}$, find $\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|$ and $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]$.**

Ans: $\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right| = a^2\sec\alpha$ and $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = \frac{d^3\vec{r}}{dt^3}.\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2} = a^3\tan\alpha$.

Example-8: **If $\vec{r_1} = (a\cos t, b\sin t, 0)$, $\vec{r_2} = (-a\sin t, b\cos t, t)$ and $\vec{r_3} = (1, 2, 3)$, find $\frac{d}{dt}[\vec{r_1}.(\vec{r_2}\times\vec{r_3})]$.**

Solution:

Here, $\vec{r_1} = (a\cos t, b\sin t, 0)$, $\vec{r_2} = (-a\sin t, b\cos t, t)$ and $\vec{r_3} = (1, 2, 3)$.

Now, $\vec{r_1}.\vec{r_2}\times\vec{r_3} = (a\cos t, b\sin t, 0).[(-a\sin t, b\cos t, t)\times(1, 2, 3)]$

$$= \begin{vmatrix} a\cos t & b\sin t & 0 \\ -a\sin t & b\cos t & t \\ 1 & 2 & 3 \end{vmatrix} = 3ab - 2at\cos t + bt\sin t .$$

$$\therefore\ \frac{d}{dt}[\vec{r_1}.\vec{r_2}\times\vec{r_3}] = -2a(\cos t - t\sin t) + b(\sin t + t\cos t).$$

$$\therefore\ \textbf{At } t = 0, \frac{d}{dt}[\vec{r_1}.\vec{r_2}\times\vec{r_3}] = -2a(1-0) + b(0+0) = -2a.$$

Problem: 1. **For the curve $x = 3t$, $y = 3t^2$, $z = 2t^3$, show that $[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = 216$.**

2. If $\vec{r_1} = (2t+1)\vec{i} - t^2\vec{j} + 3t^3\vec{k}$ and $\vec{r_2} = t^2\vec{i} + t\vec{j} - (t-1)\vec{k}$, verify that:

(a) $\frac{d}{dt}(\vec{r_1}.\vec{r_2}) = \vec{r_1}.\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}.\vec{r_2}$ (b) $\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \vec{r_1}\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}$.

Example-9: **Evaluate the derivatives of the following w.r.t t.**

(a) $\frac{\vec{r}}{r}$ (b) $\vec{r}^{\,2} + \frac{1}{\vec{r}^{\,2}}$ (c) $\frac{\vec{r}\times\vec{a}}{\vec{r}.\vec{a}}$ (d) $\frac{\vec{r}+\vec{a}}{\vec{r}\times\vec{a}}$, where $\vec{a}$ is a constant vector.

(e) $\vec{r}\times\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)$

Solutions:

(a) **Let $\vec{u} = \frac{\vec{r}}{r}$.**

Differentiating both sides w.r.t t yields

$$\frac{d\vec{u}}{dt} = \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \left[\mathrm{r}\frac{d\vec{r}}{dt} - \frac{dr}{dt}\vec{r}\right]\frac{1}{r^2} = \frac{1}{r}\frac{d\vec{r}}{dt} - \frac{\vec{r}}{r^2}\frac{dr}{dt}.$$

(b) Let $u = \vec{r}^{\,2} + \frac{1}{\vec{r}^{\,2}} = r^2 + \frac{1}{r^2}$ $(\because \vec{r}^{\,2} = r^2$ etc)

Differentiating w.r.t t, we get

$$\frac{du}{dt} = \frac{d}{dt}\left(r^2 + \frac{1}{r^2}\right) = 2r\frac{dr}{dt} + (-2r^{-3})\frac{dr}{dt} = 2\left(r - \frac{1}{r^3}\right)\frac{dr}{dt}.$$

(c) Let $\vec{u} = \frac{\vec{r}\times\vec{a}}{\vec{r}.\vec{a}}$.

Differentiating bothsides w.r.t t, we get

$$\frac{d\vec{u}}{dt} = \frac{(\vec{r}.\vec{a})\frac{d}{dt}(\vec{r}\times\vec{a}) - (\vec{r}\times\vec{a})\frac{d}{dt}(\vec{r}.\vec{a})}{(\vec{r}.\vec{a})^2} = \frac{1}{\vec{r}.\vec{a}}\left(\frac{d\vec{r}}{dt}\times\vec{a} + \vec{r}\times\frac{d\vec{a}}{dt}\right) - \frac{\vec{r}\times\vec{a}}{(\vec{r}.\vec{a})^2}\left(\vec{r}.\frac{d\vec{a}}{dt} + \frac{d\vec{r}}{dt}.\vec{a}\right)$$

Since $\vec{a}$ is a constant vector, therefore $\frac{d\vec{a}}{dt} = \vec{0}$.

$$\therefore \quad \frac{d\vec{u}}{dt} = \frac{1}{\vec{r}.\vec{a}}\left(\frac{d\vec{r}}{dt} \times \vec{a}\right) - \frac{(\vec{r} \times \vec{a})}{(\vec{r}.\vec{a})^2}\left(\frac{d\vec{r}}{dt}.\vec{a}\right).$$

(d) Let $\vec{u} = \dfrac{\vec{r} + \vec{a}}{\vec{r} \times \vec{a}}$.

Differentiating both sides w.r.t t, we get

$$\frac{d\vec{u}}{dt} = \frac{(\vec{r} \times \vec{a})\frac{d}{dt}(\vec{r} + \vec{a}) - (\vec{r} + \vec{a})\frac{d}{dt}(\vec{r} \times \vec{a})}{(\vec{r} \times \vec{a})^2} = \frac{1}{\vec{r} \times \vec{a}}\left(\frac{d\vec{r}}{dt} + \frac{d\vec{a}}{dt}\right) - \frac{\vec{r} + \vec{a}}{(\vec{r} + \vec{a})^2}\left[\vec{r} \times \frac{d\vec{a}}{dt} + \frac{d\vec{r}}{dt} \times \vec{a}\right]$$

Since $\vec{a}$ is a constant vector, therefore $\dfrac{d\vec{a}}{dt} = \vec{0}$.

$$\therefore \quad \frac{d\vec{u}}{dt} = \frac{1}{\vec{r} \times \vec{a}}\frac{d\vec{r}}{dt} - \frac{\vec{r} + \vec{a}}{(\vec{r} \times \vec{a})^2}\left(\frac{d\vec{r}}{dt} \times \vec{a}\right).$$

(e) Let $\vec{u} = \vec{r} \times \left(\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right)$

Differentiating both sides w.r.t t, we get

$$\frac{d\vec{u}}{dt} = \frac{d}{dt}\left[\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\right] = \vec{r} \times \frac{d}{dt}\left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right) + \frac{d\vec{r}}{dt} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)$$

$$= \vec{r} \times \left[\frac{d^2\vec{r}}{dt^2} \times \frac{d^2\vec{r}}{dt^2} + \frac{d\vec{r}}{dt} \times \frac{d^3\vec{r}}{dt^3}\right] + \frac{d\vec{r}}{dt} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)$$

Since $\dfrac{d^2\vec{r}}{dt^2} \times \dfrac{d^2\vec{r}}{dt^2} = \vec{0}$. So we have

$$\frac{d\vec{u}}{dt} = \vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^3\vec{r}}{dt^3}\right) + \frac{d\vec{r}}{dt} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right).$$

Problems: **Find the derivative of the following:**

(a) $\dfrac{\vec{r} + \vec{a}}{r^2 + a^2}$ **(b)** $\dfrac{\vec{r} + \vec{a}}{\vec{r}.\vec{a}}$ **(c)** $\vec{r}.\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}$

Ans: (a) $\dfrac{1}{r^2 + a^2}\dfrac{d}{dt}(\vec{r} + \vec{a}) - \dfrac{\vec{r} + \vec{a}}{(r^2 + a^2)^2}2\vec{r}.\dfrac{d\vec{r}}{dt}$ **(b)** $\dfrac{(\vec{r}.\vec{a})\dfrac{d\vec{r}}{dt} - (\vec{r} + \vec{a})\left(\dfrac{d\vec{r}}{dt}.\vec{a}\right)}{(\vec{r}.\vec{a})^2}$ **(c)** $\left[\vec{r}\ \dfrac{d\vec{r}}{dt}\ \dfrac{d^3\vec{r}}{dt^3}\right]$

Example-11: **If $\dfrac{d\vec{a}}{dt} = \vec{c} \times \vec{a}$ and $\dfrac{d\vec{b}}{dt} = \vec{c} \times \vec{b}$, show that: $\dfrac{d}{dt}(\vec{a} \times \vec{b}) = \vec{c} \times (\vec{a} \times \vec{b})$.**

Solution:

Now, $\dfrac{d}{dt}(\vec{a} \times \vec{b}) = \dfrac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \dfrac{d\vec{b}}{dt}$

$= (\vec{c} \times \vec{a}) \times \vec{b} + \vec{a} \times (\vec{c} \times \vec{b})$

$= -[\vec{b} \times (\vec{c} \times \vec{a})] + \vec{a} \times (\vec{c} \times \vec{b})$

$= -[(\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a}] + (\vec{a}.\vec{b})\vec{c} - (\vec{a}.\vec{c})\vec{b}$

$= -(\vec{a}.\vec{b})\vec{c} + (\vec{c}.\vec{b})\vec{a} + (\vec{a}.\vec{b})\vec{c} - (\vec{c}.\vec{a})\vec{b}$

$= (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}$

$= \vec{c} \times (\vec{a} \times \vec{b})$

Geometrical Problems:

Example-12:. Find the velocity and acceleration of a particle which moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ at time $t = \frac{\pi}{3}$. Also, find also their magnitudes.

Solution:

Let $\vec{r}$ be the position vector of a particle at any time t. So that

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ $\quad\therefore\quad \vec{r} = 2\sin 3t\,\vec{i} + 2\cos 3t\,\vec{j} + 8t\vec{k}$.

Since $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$ represent the velocity and acceleration of the moving particle at any time t.

Therefore, velocity $(\vec{v}) = \frac{d\vec{r}}{dt} = 6\cos 3t\,\vec{i} - 6\sin 3t\,\vec{j} + 8\vec{k}$

and acceleration $(\vec{a}) = \frac{d^2\vec{r}}{dt^2} = -18\sin 3t\,\vec{i} - 18\cos 3t\,\vec{j}$.

At $t = \frac{\pi}{3}$, *the velocity and acceleration are*

$\vec{v} = 6\cos\pi\,\vec{i} - 6\sin\pi\,\vec{j} + 8\vec{k} = -6\vec{i} + 8\vec{k}$ and $\vec{a} = -18\sin\pi\,\vec{i} - 18\cos\pi\,\vec{j} = 18\vec{j}$.

The magnitude of velocity and acceleration are

$|\vec{v}| = \sqrt{(-6)^2 + (8)^2} = 10$ and $|\vec{a}| = \sqrt{(18)^2} = 18$.

Example-13: A particle P is moving on a circle of radius r with constant angular velocity $\omega = \frac{d\theta}{dt}$. Show that its acceleration is $-\omega^2\vec{r}$.

Hint:

Since the particle P is moving on a circle of radius r with constant angular velocity ω. So that $x = r\cos\omega t$ and $y = r\sin\omega t$.

Therefore, $\vec{r} = x\vec{i} + y\vec{j} = r\cos\omega t\,\vec{i} + r\sin\omega t\,\vec{j}$.

Now, $\frac{d\vec{r}}{dt} = -r\omega\sin\omega t\,\vec{i} + r\omega\cos\omega t\,\vec{j}$

and $\frac{d^2\vec{r}}{dt^2} = -r\omega^2\cos\omega t\,\vec{i} - r\omega^2\sin\omega t\,\vec{j} = -\omega^2(r\cos\omega t\,\vec{i} + r\sin\omega t\,\vec{j}) = -\omega^2\vec{r}$.

Therefore, $\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$.

Example-14: A particle moves such that it's position vector is given by $\vec{r} = \cos\omega t\,\vec{i} + \sin\omega t\,\vec{j}$, where ω is constant. Then show that (a) Velocity $\vec{v}$ is perpendicular to $\vec{r}$. (b) $\vec{r} \times \vec{v}$ is a constant vector.

Hint:

(a) Since $\vec{r} = \cos\omega t\,\vec{i} + \sin\omega t\,\vec{j}$ $\quad\therefore\quad \vec{v} = \frac{d\vec{r}}{dt} = (-\sin\omega t)\,\omega\vec{i} + (\cos\omega t)\omega\vec{j}$

Now, $\vec{v}.\vec{r} = [(-\omega\sin\omega t)\vec{i} + (\omega\cos\omega t)\vec{j}].(\cos\omega t\,\vec{i} + \sin\omega t\,\vec{j}) = 0$.

Therefore, the velocity $\vec{v}$ is perpendicular to $\vec{r}$.

(b) Now, $\vec{r} \times \vec{v} = (\cos\omega t\,\vec{i} + \sin\omega t\,\vec{j}) \times (-\omega\sin\omega t\,\vec{i} + \omega\cos\omega t\,\vec{j}) = \omega\vec{k}$. (Prove it)

Therefore, $\vec{r} \times \vec{v} = \omega\vec{k}$, which is constant vector as ω is constant.

EXERCISE

1. (a) A particle moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$. Find the magnitude of velocity and acceleration at time $t = \frac{\pi}{3}$. **Ans:** Magnitude of velocity = 10 and acceleration = 18.

   (b) A particle moves along the curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the magnitude of velocity and acceleration at time $t = 0$ and $t = \pi$. **Ans:** $4\vec{i} + 6\vec{k}, -4\vec{j} + 6\vec{k}, -4\vec{i}, 4\vec{i}$

Integration of Vector Functions

Examples/ Exercises

Example-1: Evaluate: $\int \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$.

Solution:

Since $\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \vec{r} \times \frac{d^2\vec{r}}{dt^2} + \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = \vec{r} \times \frac{d^2\vec{r}}{dt^2}$.

Therefore, we have

$$\int \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \vec{r} \times \frac{d\vec{r}}{dt} + \vec{c}.$$

Example-2: Let $\vec{r}_1 = 3t^2\,\vec{i} + 4(t-2)\,\vec{j} + 5t^2\vec{k}$ and $\vec{r}_2 = 2(t-2)\,\vec{i} + t^2\vec{j} + (t-3)\vec{k}$. Find the value of $\int_0^1 (\vec{r}_1 \cdot \vec{r}_2)\, dt$.

Solution:

Now, $\vec{r}_1.\vec{r}_2 = [3t^2\vec{i} + 4(t-2)\vec{j} + 5t^2\vec{k}].[2(t-2)\,\vec{i} + t^2\vec{j} + (t-3)\vec{k}] = 15t^3 - 35t^2$.

$$\therefore \quad \int_0^1 (\vec{r}_1.\vec{r}_2)\, dt = \int_0^1 (15t^3 - 35t^2)\, dt = \left[15\frac{t^4}{4} - 35\frac{t^3}{3}\right]_0^1 = \frac{15}{4} - \frac{35}{3} = -\frac{95}{12}.$$

Example-2: If $\vec{r} = t^2\vec{i} + t\vec{j} + t^2\vec{k}$, then evaluate: $\int_0^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$.

Solution:

Here, $\vec{r} = t^2\vec{i} + t\vec{j} + t^2\vec{k}$. $\quad\therefore \frac{d\vec{r}}{dt^2} = 2t\vec{i} + \vec{j} + 2t\vec{k}$ and $\frac{d^2\vec{r}}{dt^2} = 2\vec{i} + 2\vec{k}$.

Now, $\vec{r} \times \frac{d^2\vec{r}}{dt^2} = (t^2\vec{i} + t\vec{j} + t^2\vec{k}) \times (2\vec{i} + 2\vec{k}) = 2t\vec{i} - 2t\vec{k}$.

Now, $\int_0^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \int_0^2 (2t\vec{i} - 2t\vec{k})\, dt = [t^2\vec{i} - t^2\vec{k}]_0^2 = 4\vec{i} - 4\vec{k}$.

Example-3: Solve the equation: $\frac{d^2\vec{r}}{dt^2} = \vec{a}t$, for any constant vector $\vec{a}$.

Solution:

Here, $\frac{d^2\vec{r}}{dt^2} = \vec{a}t$. Integrating twice w.r.t. t, we get

$$\frac{d\vec{r}}{dt} = \frac{\vec{a}t^2}{2} + \vec{b}.$$

or, $\vec{r} = \vec{a}\frac{t^3}{6} + \vec{b}t + \vec{c}$, where $\vec{b}$ and $\vec{c}$ are the constants of integration.

Example-4: If the acceleration of a particle at any time t is given by the function $\vec{a}(t) = e^{2t}\,\vec{i} + e^t\vec{j} + 2\vec{k}$, find the velocity $\vec{v}$ and displacement $\vec{r}$ at time t, given that $\vec{v} = \vec{i} + \vec{j}$ and $\vec{r} = \vec{0}$ at $t = 0$.

Solution:

Here, acceleration function $\vec{a}(t) = e^{2t}\,\vec{i} + e^t\vec{j} + 2\vec{k}$. Integrating w.r.t. t, the velocity function is

$$\vec{v}(t) = \int a(t)dt = \int (e^{2t}\,\vec{i} + e^t\,\vec{j} + 2\vec{k})\, dt$$

$\therefore \quad \vec{v}(t) = \frac{e^{2t}}{2}\,\vec{i} + e^t\vec{j} + 2t\vec{k} + \vec{c}.$ ………(1)

At $t = 0$, $\vec{v}(0) = \vec{i} + \vec{j}$, $\vec{v}(0) = \frac{1}{2}\vec{i} + \vec{j} + \vec{c}_1$ or, $\vec{i} + \vec{j} = \frac{1}{2}\vec{i} + \vec{j} + \vec{c}_1$ $\quad \therefore \quad \vec{c}_1 = \frac{1}{2}\vec{i}$.

Hence (1) becomes

$$\vec{v}(t) = \frac{e^{2t}}{2}\,\vec{i} + e^t\vec{j} + 2t\vec{k} + \frac{1}{2}\,\vec{i}$$

Again integrating, we get

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \left[\frac{e^{2t}}{2}\,\vec{i} + e^t\vec{j} + 2t\vec{k} + \frac{1}{2}\,\vec{i}\right] dt = \frac{e^{2t}}{4}\,\vec{i} + e^t\vec{j} + t^2\vec{k} + \frac{1}{2}\,\vec{i} + \vec{c}_2$$

But $\vec{r}(t) = \vec{0}$, when $t = 0$.

$\therefore \quad \vec{0} = \frac{\vec{i}}{4} + \vec{j} + \frac{1}{2}\vec{i} + \vec{c}_2 \qquad \therefore \quad \vec{c}_2 = -\frac{3\vec{i}}{4} - \vec{j}.$

Hence, $\vec{r}(t) = \frac{e^{2t}}{4}\vec{i} + e^t\vec{j} + t^2\vec{k} + \frac{1}{2}\vec{i} - \frac{3}{4}\vec{i} - \vec{j} = \frac{e^{2t}}{4}\vec{i} + e^t\vec{j} + t^2\vec{k} - \frac{1}{4}\vec{i} - \vec{j}.$

Example-5: If $\vec{r}.d\vec{r} = 0$, show that $|\vec{r}|$ is constant.

Solution:

Here, $\vec{r}.d\vec{r} = 0$ or, $2\vec{r}.\,d\vec{r} = 0 \quad \therefore \quad d(\vec{r}.\vec{r}) = 0$.

Integrating, we get

$\vec{r}.\vec{r} = c$ or, $\vec{r}^{\,2} = c$ i.e. $|\vec{r}|^2 = c$. i.e., $|\vec{r}|$ is constant.

Exercise (Vector Integration)

1. If $\vec{r} = 3t^2\vec{i} - 2\vec{j} + 3(t^2 - 1)\,\vec{k}$, find a) $\int \vec{r}\,dt$ b) $\int_0^1 \vec{r}\,dt$. ***Ans:*** a) $t^3\vec{i} - 2t\vec{j} + (t^3 - 3t)\,\vec{k}$, b) $\vec{i} - 2\vec{j} - 2\vec{k}$.
2. Given $\vec{r} = (t - t^2)\,\vec{i} + 2t^3\vec{j} - 3\vec{k}$, evaluate: $\int_1^2 \vec{r}\,dt$. ***Ans:*** $\frac{1}{6}(-5\vec{i} + 45\vec{j} - 18\vec{k})$
3. If $\vec{r}_1 = 2\vec{i} + t\vec{j} - \vec{k}$, $\vec{r}_2 = t\vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{r}_3 = 2\vec{i} - 3\vec{j} + 4\vec{k}$, find: a) $\int_0^2 (\vec{r}_1 \times \vec{r}_2)\,dt$ b) $\int_0^2 [\vec{r}_1\,\vec{r}_2\,\vec{r}_3]\,dt$.
Ans: a) $10\vec{i} - 14\vec{j} + \frac{16}{3}\vec{k}$ b) $\frac{253}{6}$
4. Evaluate: $\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$ for $\vec{r} = 2t^2\vec{i} + t\vec{j} - 3t^3\vec{k}$. ***Ans:*** $-42\vec{i} + 90\vec{j} - 6\vec{k}$
5. Integrate: $\frac{d^2\vec{r}}{dt^2} = -n^2\vec{r}$. ***Ans:*** $\left(\frac{d\vec{r}}{dt}\right)^2 = -n^2\vec{r}^{\,2} + c$
6. Solve: $\frac{d^2\vec{r}}{dt^2} = t\vec{a} + \vec{b}$, where $\vec{a}$ and $\vec{b}$ are constant vectors, given that $\vec{r} = \vec{0}$ and $\frac{d\vec{r}}{dt} = \vec{0}$, when $t = 0$.
Ans: $\vec{r}(t) = \frac{t^3}{6}\,\vec{a} + \frac{t^2}{2}\vec{b}$
7. a) The acceleration of a particle is given by $\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = e^{-t}\,\vec{i} - 6(t + 1)\vec{j} + 3sint\,\vec{k}$.
Find the velocity $\vec{v}$ and displacement $\vec{r}$, given that $\vec{v} = \vec{0}$, $\vec{r} = \vec{0}$ when $t = 0$.
Ans: $\vec{v} = (1 - e^{-t})\,\vec{i} - (3t^2 + 6t)\vec{j} + (3 - 3\cos t)\,\vec{k}$ and $\vec{r} = (t - 1 + e^{-t})\,\vec{i} - (t^3 + 3t^2)\vec{j} + (3t - 3\,sin\,t)\,\vec{k}$.
b) The acceleration of a moving particle at any time t is $\vec{a}\,(t) = \frac{d\vec{v}}{dt} = 12\,cos2t\,\vec{i} - 8\,sin2t\,\vec{j} + 16t\,\vec{k}$.
Find the velocity $\vec{v}$ and the displacement $\vec{r}$ at any time t, given that if $t = 0$, $\vec{v} = \vec{0}$ and $\vec{r} = \vec{0}$.
Ans: $\vec{v} = 6sin\,2t\,\vec{i} + 4(cos2t - 1)\,\vec{j} + 8t^2\vec{k}$; and $\vec{r} = 3(1 - cos\,2t)\,\vec{i} + 2(sin2t - 2t)\,\vec{j} + \frac{8}{3}\,t^3\,\vec{k}$.

□□

Unit 3: Home Work I – Partial Derivatives

Dr.P.M.Bajracharya

August 15, 2023

Textbook: **Multivariable Calculus: Concepts and Contexts**, Fourth Edition
by James Stewart

1 Functions of several variables

Exercises 11.1 (from the textbook)

3, 5, 6, 8, 19, 20, 27, 28, 41, 42

I Evaluating a Function. Find and simplify the function values.

1. $f(x, y) = 2x + y^2$
   (a) $\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$
   (b) $\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$
2. $f(x, y) = 3x^2 - 2y$
   (a) $\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$

(b) $\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$

I Find the domain and range of the function.

1. $f(x,y)=x^2+y^2$
2. $f(x,y)=e^{xy}$
3. $g(x,y)=x\sqrt{y}$
4. $g(x,y)=\frac{y}{\sqrt{x}}$
5. $z=\frac{x+y}{xy}$
6. $z=\frac{xy}{x-y}$
7. $f(x,y)=\sqrt{4-x^2-y^2}$
8. $f(x,y)=\sqrt{4-x^2-4y^2}$
9. $f(x,y)=\ln(4-x-y)$
10. $f(x,y)=\ln(xy-6)$

II Sketch the surface given by the function.

1. $f(x,y)=4$
2. $f(x,y)=6-2x-3y$
3. $f(x,y)=y^2$
4. $g(x,y)=\frac{1}{2}y$
5. $z=-x^2-y^2$
6. $z=\frac{1}{2}\sqrt{x^2+y^2}$

7. $f(x, y) = e^{-x}$

II Sketching a Contour Map Describe the level curves of the function. Sketch a contour map of the surface using level curves for the given -values.

1. $z = x + y, \quad c = -1, 0, 2, 4$
2. $z = 6 - 2x - 3y, \quad c = 0, 2, 4, 6, 8, 10$
3. $z = x^2 + 4y^2, \quad c = 0, 1, 2, 3, 4$
4. $f(x, y) = \sqrt{9 - x^2 - y^2}, \quad c = 0, 1, 2, 3$
5. $f(x, y) = xy, \quad c = \pm 1, \pm 2, ..., \pm 6$
6. $f(x, y) = e^{xy/2}, \quad c = 2, 3, 4, 1/2, 1/3, 1/4$

Textbook: **Multivariable Calculus:
Concepts and Contexts**, Fourth Edition
by James Stewart

2 Home Work: Limits and continuity

Due Date: **Bhadra 14, 2080**

Exercises 11.2

5-8, 23, 24,27, 29, 31

1. Use the definition of the limit of a function of two variables to verify the limit.

   (a) $\lim_{(x,y)\to(1,0)} x = 1$

   (b) $\lim_{(x,y)\to(1,-3)} y = -3$

   (c) $\lim_{(x,y)\to(a,b)} y = b$

2. Use limit laws to evaluate the limit.

   (a) $\lim_{(x,y)\to(0,0)} \dfrac{4x - y}{\sin y - 1}$

   (b) $\lim_{(x,y)\to(-1,2)} \dfrac{xy^3}{x + y}$

   (c) $\lim_{(x,y)\to(4,-2)} x\sqrt{y^3 + 2x}$

3. Investigate whether the limit exists or not.

   (a) $\lim_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4 + y^2}$

(b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3 y}{2x^6 + y^2}$

(c) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x + y}{x^2 + y}$

(d) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x}{x^2 - y^2}$

4. Discuss the continuity of the function.

(a) $$f(x,y) = \begin{cases} \dfrac{x^2 + 2xy + y^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

(b) $$f(x,y) = \begin{cases} \dfrac{\sin xy}{xy}, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

Textbook: **Multivariable Calculus: Concepts and Contexts**, Fourth Edition
by James Stewart

3 Home Work III: Partial derivatives

Due Date: **Bhadra 20, 2080**

Exercises 11.3

11, 15, 17, 18, 39, 43, 45, 46, 49, 51, 53, 57, 59, 71, 72, 78

1. Show that the function satisfies Laplace's equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$.

   (a) $u = x^2 - y^2 + 2xy$

   (b) $u = e^x \sin y + e^y \cos x$

   (c) $u = \ln(x^2 + y^2) + 2\tan^{-1}(y/x)$

2. Show that the function satisfies Wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

   (a) $u = \cos(4x + 4ct)$

   (b) $u = \ln(x + ct)$

   (c) $u = \sin c\omega t \sin \omega x$ for all real values of ω.

3. Show that the function satisfies the Heat (diffusion) equation $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

   (a) $u = e^{-t}\sin(x/c)$

   (b) $u = e^{-t}\cos(x/c)$

(c) $u = e^{-t/2}e^{-x^2/(4t)}$

4. (a) Verify for the Cobb-Douglas production function $P(L, K) = 1.01L^{0.75}K^{0.25}$ that the production will be doubled if both the amount of labor and the amount of capital are doubled. Determine whether this is also true for the general production function $P(L, K) = cL^{\alpha}K^{\beta}$. If not, under what condition this will be true.

5. Let's consider a small printing business where N is the number of workers, V is the value of the equipment (in units of \$25,000), and P is the production, measured in thousands of pages per day. Suppose the production function for this company is given by

$$P = f(N, V) = 2N^{0.6}V^{0.4}.$$

(a) If this company has a labor force of 100 workers and 200 units' worth of equipment, what is the production output of the company?

(b) Find $f_N(100, 200)$ and $f_V(100, 200)$. Interpret your answers in terms of production.

6. A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.1091w^{0.425}h^{0.725}$$

where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet.

(i) Find and interpret it.

(ii) What is your own surface area?

Double Integrals - Practice Problems

Dr.P.M.Bajracharya

November 25, 2023

1. Use the Midpoint Rule to estimate the volume under $f(x,y) = x^2 + y$ and above the rectangle given by $-1 \le x \le 3, 0 \le y \le 4$ in the xy-plane. Use 4 subdivisions in the x direction and 2 subdivisions in the y direction.

2. (a) Estimate the volume of the solid that lies below the surface $z - xy$ and above the rectangle

$$R = [0,6] \times [0,4].$$

Use a Riemann sum with $m = 3, n = 2$ and take the sample point to be the upper right corner of each square.
(b) Use the Midpoint Rule to estimate the volume of the solid in part (a).

3. If $R = [0,4] \times [-1,2]$, use a Riemann sum with $m = 2, n = 3$ to estimate the value of $\iint_R (1 - xy^2)\, dA$. Take the sample points to be (a) the lower right corners and (b) the upper left corners of the rectangles.

4. (a) Use a Riemann sum with $m = n = 2$ to estimate the value of $\iint_R xe^{-xy}\, dA$, where $R = [0,2] \times [0,1]$. Take the sample points to be upper right corners.
(b) Use the Midpoint Rule to estimate the integral in part (a).

5. (a) Estimate the volume of the solid that lies below the surface $z = 1 + x^2 + 3y$ and above the rectangle $R = [1,2] \times [0,3]$. Use a Riemann sum with $m = n = 2$ and choose the sample points to be lower left corners. (b) Use the Midpoint Rule to estimate the volume in part (a).

Iterated Integrals - Practice Problems

Dr.P.M.Bajracharya

November 26, 2023

1. Compute the following double integral over the indicated rectangle (a) by integrating with respect to x first and (b) by integrating with respect to y first. $\iint\limits_R 12x - 18y\, dA \qquad R = [-1,4] \times [2,3]$.

For problems 2 – 8 compute the given double integral over the indicated rectangle.

2. $\displaystyle\iint\limits_R 6y\sqrt{x} - 2y^3\, dA \qquad R = [1,4] \times [0,3]$

3. $\displaystyle\iint\limits_R \frac{\mathbf{e}^x}{2y} - \frac{4x-1}{y^2}\, dA \qquad R = [-1,0] \times [1,2]$

4. $\displaystyle\iint\limits_R \sin(2x) - \frac{1}{1+6y}\, dA \qquad R = \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \times [0,1]$

5. $\displaystyle\iint\limits_R y\mathbf{e}^{y^2-4x}\, dA \qquad R = [0,2] \times \left[0, \sqrt{8}\right]$

6. $\displaystyle\iint\limits_R xy^2\sqrt{x^2+y^3}\, dA \qquad R = [0,3] \times [0,2]$

7. $\displaystyle\iint\limits_R xy\cos\left(y\,x^2\right)\, dA \qquad R = [-2,3] \times [-1,1]$

8. $\displaystyle\iint\limits_R xy\cos(y) - x^2\, dA \qquad R = [1,2] \times \left[\frac{\pi}{2}, \pi\right]$

9. Determine the volume that lies under $f(x,y) = 9x^2 + 4xy + 4$ and above the rectangle given by $[-1,1] \times [0,2]$ in the xy-plane.

Double Integrals over General Regions Practice Problems

Dr.P.M.Bajracharya

November 25, 2023

1. Evaluate $\iint\limits_D 42y^2 - 12x\, dA$ where $D = \left\{(x, y)\,|0 \le x \le 4, (x-2)^2 \le y \le 6\right\}$

2. Evaluate $\iint\limits_D 2yx^2 + 9y^3\, dA$ where D is the region bounded by $y = \frac{2}{3}x$ and $y = 2\sqrt{x}$.

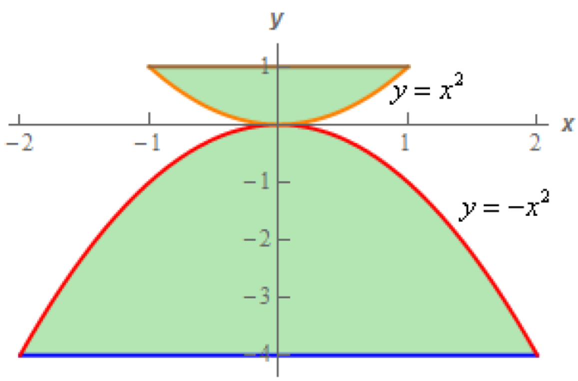
3. Evaluate $\iint\limits_D (10x^2y^3 - 6)\, dA$ where D is the region bounded by $x = -2y^2$ and $x = y^3$.

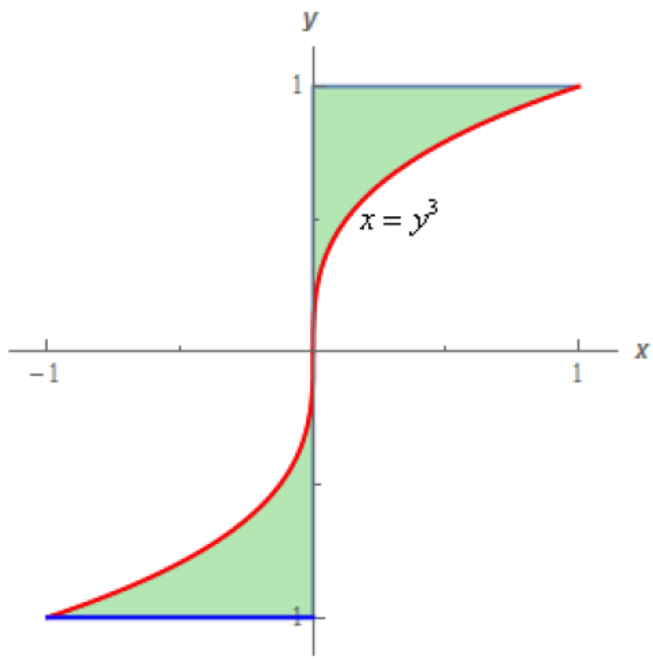
4. Evaluate $\iint\limits_D x(y-1)\, dA$ where D is the region bounded by $y = 1-x^2$ and $y = x^2-3$.

5. Evaluate $\iint\limits_D 5x^3 \cos\left(y^3\right)\, dA$ where D is the region bounded by $y = 2, y = \frac{1}{4}x^2$ and the y-axis.

6. Evaluate $\iint\limits_D \frac{1}{y^{\frac{1}{3}}\left(x^3+1\right)}\, dA$ where D is the region bounded by $x = -y^{\frac{1}{3}}, x = 3$ and the x-axis.

7. Evaluate $\iint\limits_D 3 - 6xy\, dA$ where D is the region shown below.





8. Evaluate $\iint\limits_D e^{y^4}\, dA$ where D is the region shown below.

9. Evaluate $\displaystyle\iint\limits_D 7x^2 + 14y\, dA$ where D is the region bounded by $x = 2y^2$ and $x = 8$ in the order given below. Integrate with respect to x first and then y. Integrate with respect to y first and then x.

10. For problems 10 & 11 evaluate the given integral by first reversing the order of integration.

$$\int_0^3 \int_{2x}^6 \sqrt{y^2 + 2}\, dy\, dx$$

11. $\displaystyle\int_0^1 \int_{-\sqrt{y}}^{y^2} 6x - y\, dx\, dy$

12. Use a double integral to determine the area of the region bounded by $y = 1 - x^2$ and $y = x^2 - 3$.

13. Use a double integral to determine the volume of the region that is between the xy-plane and $f(x, y = 2 + \cos(x^2)$ and is above the triangle with vertices $(0, 0)$, $(6, 0)$ and $(6, 2)$.

14. Use a double integral to determine the volume of the region bounded by $z = 6 - 5x^2$ and the planes $y = 2x$, $y = 2$, $x = 0$ and the xy-plane.

15. Use a double integral to determine the volume of the region formed by the intersection of the two cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$.

Double Integrals In Polar Coordinates Practice Problems

Dr.P.M.Bajracharya

November 25, 2023

1. Evaluate $\displaystyle\iint_D y^2 + 3x\, dA$ where D is the region in the 3rd quadrant between $x^2+y^2 = 1$ and $x^2 + y^2 = 9$.

2. Evaluate $\displaystyle\iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA$ where D is the bottom half of $x^2 + y^2 = 16$.

3. Evaluate $\displaystyle\iint_D 4xy - 7\, dA$ where D is the portion of $x^2 + y^2 = 2$ in the 1st quadrant.

4. Use a double integral to determine the area of the region that is inside $r = 4 + 2\sin\theta$ and outside $r = 3 - \sin\theta$.

5. Evaluate the following integral by first converting to an integral in polar coordinates.

$$\int_0^3 \int_{-\sqrt{9-x^2}}^0 \mathbf{e}^{x^2+y^2}\, dy\, dx$$

6. Use a double integral to determine the volume of the solid that is inside the cylinder $x^2 + y^2 = 16$, below $z = 2x^2 + 2y^2$ and above the xy-plane.

7. Use a double integral to determine the volume of the solid that is bounded by $z = 8 - x^2 - y^2$ and $z = 3x^2 + 3y^2 - 4$.

Triple Integrals - Practice Problems

Dr.P.M.Bajracharya

November 25, 2023

1. Evaluate $\int_{2}^{3}\int_{-1}^{4}\int_{1}^{0} 4x^2y - z^3\,dz\,dy\,dx$

2. Evaluate $\int_{0}^{1}\int_{0}^{z^2}\int_{0}^{3} y\cos\left(z^5\right)\,dx\,dy\,dz$

3. Evaluate $\iiint\limits_E 6z^2\,dV$ where E is the region below $4x+y+2z=10$ in the first octant.

4. Evaluate $\iiint\limits_E 3-4x\,dV$ where E is the region below $z=4-xy$ and above the region in the xy-plane defined by $0 \le x \le 2, 0 \le y \le 1$.

5. Evaluate $\iiint\limits_E 12y-8x\,dV$ where E is the region behind $y=10-2z$ and in front of the region in the xz-plane bounded by $z=2x, z=5$ and $x=0$.

6. Evaluate $\iiint\limits_E yz\,dV$ where E is the region bounded by $x=2y^2+2z^2-5$ and the plane $x=1$.

7. Evaluate $\iiint\limits_E 15z\,dV$ where E is the region between $2x+y+z=4$ and $4x+4y+2z=20$ that is in front of the region in the yz-plane bounded by $z=2y^2$ and $z=\sqrt{4y}$.

8. Use a triple integral to determine the volume of the region below $z=4-xy$ and above the region in the xy-plane defined by $0 \le x \le 2, 0 \le y \le 1$.

9. Use a triple integral to determine the volume of the region that is below $z=8-x^2-y^2$ above $z=-\sqrt{4x^2+4y^2}$ and inside $x^2+y^2=4$.

Change Of Variables - Practice Problems

Dr.P.M.Bajracharya

July 3, 2022

For problems 1 - 3 compute the Jacobian of each transformation.

1. $x = 4u - 3v^2 \qquad y = u^2 - 6v$

2. $x = u^2v^3 \qquad y = 4 - 2\sqrt{u}$

3. $x = \dfrac{v}{u} \qquad y = u^2 - 4v^2$

4. If R is the region inside $\dfrac{x^2}{4} + \dfrac{y^2}{36} = 1$ determine the region we would get applying the transformation $x = 2u, y = 6v$ to R.

5. If R is the parallelogram with vertices $(1,0), (4,3), (1,6)$ and $(-2,3)$ determine the region we would get applying the transformation $x = \dfrac{1}{2}(v-u)$, $y = \dfrac{1}{2}(v+u)$ to R.

6. If R is the region bounded by $xy = 1, xy = 3, y = 2$ and $y = 6$ determine the region we would get applying the transformation $x = \dfrac{v}{6u}, y = 2u$ to R.

7. Evaluate $\displaystyle\iint_R xy^3\, dA$ where R is the region bounded by $xy = 1, xy = 3, y = 2$ and $y = 6$ using the transformation $x = \dfrac{v}{6u}, y = 2u$.

8. Evaluate $\displaystyle\iint_R (6x - 3y)\, dA$ where R is the parallelogram with vertices $(2,0), (5,3), (6,7)$ and $(3,4)$ using the transformation $x = \dfrac{1}{3}(v-u), y = \dfrac{1}{3}(4v-u)$ to R.

9. Evaluate $\displaystyle\iint_R (x + 2y)\, dA$ where R is the triangle with vertices $(0,3), (4,1)$ and $(2,6)$ using the transformation $x = \dfrac{1}{2}(u-v), y = \dfrac{1}{4}(3u+v+12)$ to R.

10. Derive the transformation used in problem 8.

11. Derive a transformation that will convert the triangle with vertices $(1,0), (6,0)$ and $(3,8)$ into a right triangle with the right angle occurring at the origin of the uv system.

Math for Data Science: Problems for presentations

Dr.P.M.Bajracharya

November 25, 2023

1. Determine the surface area of the portion of $2x + 3y + 6z = 9$ that is in the 1st octant.

2. Determine the surface area of the portion of $z = 13 - 4x^2 - 4y^2$ that is above $z = 1$ with $x \leq 0$ and $y \leq 0$.

3. Determine the surface area of the portion of $3 + 2y + \frac{1}{4}x^4$ that is above the region in the xy-plane bounded by $y = x^5$, $x = 1$ and the x-axis.

4. Determine the surface area of the portion of $y = 2x^2 + 2z^2 - 7$ that is inside the cylinder $x^2 + z^2 = 4$.

5. Determine the surface area region formed by the intersection of the two cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$.

Math for Data Science: Problems for presentations

Dr.P.M.Bajracharya

December 9, 2023

Presentations on Wednesday, Mangsir 27, 2080.

roll no.	Name
20	Prabin Manee Shrestha
21	Pujan Bhusal
22	Rabin Khadka
23	Rachit Basnet
24	Rajendra Karki
25	Ram Krishna Pudasaini
26	Rohit Chaudhary
27	Rojen Bahadur Nhuchhe Pradhan
28	Sabin Mishra
29	Santosh Kumar Pandit
30	Sheryansh Lodha
31	Shishir Adhikari
32	Shubham Dhakal
33	Subash Khatiwada

roll no.	**Name**
34	Sudhanshu Neupane
35	Suraj Bhattarai
36	Sweta Neupane
37	Uma Dhakal

R. No.	**Problem**
20.	If $R = [0, 4] \times [-1, 2]$, use a Riemann sum with $m = 2, n = 3$ to estimate the value of $\iint_R (1 - xy^2)\, dA$. Take the sample points to be (a) the lower right corners and (b) the upper left corners of the rectangles.
21.	Use a double integral to determine the volume of the region that is between the xy-plane and $f(x, y) = 2 + \cos(x^2)$ and is above the triangle with vertices $(0, 0), (6, 0)$ and $(6, 2)$.
22	Find the volume of the ice-come bounded above by $z = \sqrt{1 - x^2 - y^2}$ and bounded below by the come $z = x^2 + y^2$.
23	(a) Use a Riemann sum with $m = n = 2$ to estimate the value of $\iint_R xe^{-xy}\, dA$, where $R = [0, 2] \times [0, 1]$. Take the sample points to be upper right corners. (b) Use the Midpoint Rule to estimate the integral in part (a).
24	Use a double integral to determine the volume of the region formed by the intersection of the two cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$.
25	Evaluate $\iiint\limits_E 6z^2\, dV$, where E is the region below $4x + y + 2z = 10$ in the first octant.
26	$\iint\limits_R \sin(2x) - \dfrac{1}{1 + 6y}\, dA \qquad R = \left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right] \times [0, 1]$
27	Evaluate $\iiint\limits_E e^{-x^2 - z^2}\, dV$, where E is the region between the two cylinders $x^2 + z^2 = 4$ and $x^2 + z^2 = 9$ with $1 \leq y \leq 5$ and $z \leq 0$.
28	Evaluate $\iiint\limits_E (x^2 + y^2)\, dV$, where E is the region bounded by $x^2 + y^2 + z^2 = 4$ with $y \leq 0$.
29	A lamina occupies the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant. Find its center of mass if the density at any point is proportional to its distance from the x-axis.

R. No.	Problem
30	Evaluate the following integral by first converting to an integral in polar coordinates. $$\int_0^3 \int_{-\sqrt{9-x^2}}^0 \mathbf{e}^{x^2+y^2}\, dy\, dx$$
31	$\displaystyle\iint\limits_R y\mathbf{e}^{y^2-4x}\, dA \qquad R = [0,2] \times \left[0, \sqrt{8}\right]$
32	Use a double integral to determine the volume of the solid that is inside the cylinder $x^2 + y^2 = 16$, below $z = 2x^2 + 2y^2$ and above the xy-plane.
33	Determine the surface area of the portion of $z = 13 - 4x^2 - 4y^2$ that is above $z = 1$ with $x \leq 0$ and $y \leq 0$.
34	If R is the parallelogram with vertices $(1,0), (4,3), (1,6)$ and $(-2,3)$ determine the region we would get applying the transformation $x = \frac{1}{2}(v-u), \frac{1}{2}(v+u)$ to R.
35	Evaluate $\displaystyle\iint\limits_R xy^3\, dA$ where R is the region bounded by $xy = 1, xy = 3, y = 2$ and $y = 6$ using the transformation $x = \dfrac{v}{6u}, y = 2u$.
36	Evaluate $\displaystyle\iint\limits_R (6x - 3y)\, dA$ where R is the parallelogram with vertices $(2,0), (5,3), (6,7)$ and $(3,4)$ using the transformation $x = \dfrac{1}{3}(v-u), y = \dfrac{1}{3}(4v-u)$ to R. Derive the transformation used in the problem.
37	Use a double integral to determine the volume of the region bounded by $z = 6 - 5x^2$ and the planes $y = 2x, y = 2, x = 0$ and the xy-plane.