Exercise 11.1

(5) Find and sketch the domain of the function $f(x,y) = \ln(9 - x^2 - 9y^2)$. What is the range of f?

Soln:

$$f(x,y) = \ln(9 - x^2 - 9y^2)$$

The function $f(x,y)$ is defined for all (x,y) where,

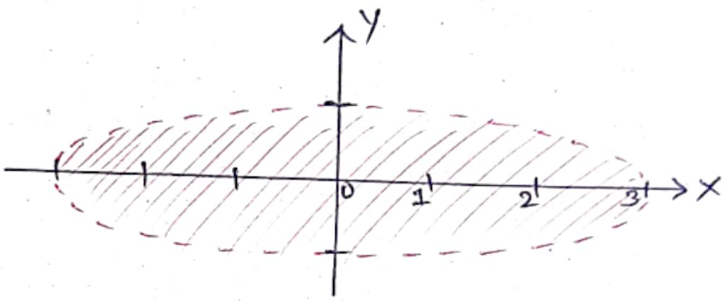
$$9 - x^2 - 9y^2 > 0$$

or, $9 - (x^2 + 9y^2) > 0$

or, $9 > x^2 + 9y^2$

or, $x^2 + 9y^2 < 9$

or, $\frac{x^2}{9} + \frac{y^2}{1} < 1$



So, the domain of the function $f(x,y)$ is the set of all points lying inside the ellipse $\frac{x^2}{3^2} + \frac{y^2}{1^2} = 1$.

Domain $= \{(x,y) \in \mathbb{R}^2 : x^2 + 9y^2 < 9\}$

→ Range of $f(x,y) = \ln(9 - x^2 - 9y^2) = z$

or, $z = \ln[9 - (x^2 + 9y^2)]$

$\because$ $x^2 + 9y^2 \geq 0$ and also $x^2 + 9y^2 < 9$

so, $0 \leq (x^2 + 9y^2) < 9$

$\Rightarrow z = \ln[9 - (x^2 + 9y^2)] \leq \log_e 9$

$\therefore$ Range of $\cancel{f(x,y) \in \mathbb{R}}$: $f(x,y) = z \in \mathbb{R} : z \leq \log_e 9$

(6.) Find and sketch the domain of the function $f(x,y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$

→ The function $f(x,y)$ is defined for all (x,y) where,

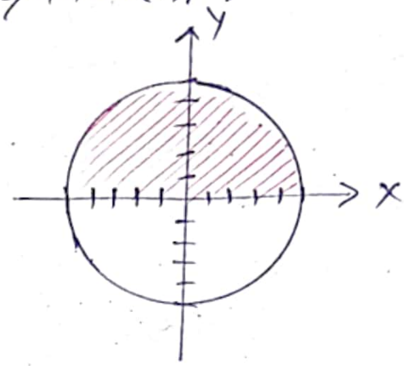
$y \geq 0$ & $25 - x^2 - y^2 \geq 0$

or, $25 \geq x^2 + y^2$

or, $x^2 + y^2 \leq 25$

→ The domain of the function $f(x,y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$ is the set of all points lying on or inside the circle $x^2+y^2=25$ except 3rd & 4th quardent.

Domain $= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 25,\ y \ge 0\}$



(8) Let $g(x,y,z) = x^3y^2z\sqrt{10-x-y-z}$

ⓐ Evaluate $g(1,2,3)$

ⓑ Find & describe the domain of g.

Soln:

$$g(x,y,z) = x^3y^2z\sqrt{10-x-y-z}$$

ⓐ $g(1,2,3) = 1^3 \times 2^2 \times 3\sqrt{10-1-2-3} = 24$ #

The function $g(x,y,z)$ exists if $\sqrt{10-x-y-z} \ge 0$

or, $10-(x+y+z) \ge 0$

or, $10 \ge x+y+z$

or, $x+y+z \le 10$

→ The domain of the function g are the set of points (x,y,z) that lies on or below the plane $x+y+z=10$.

Domain $= \{(x,y,z) \in \mathbb{R}^3 : x+y+z \le 10\}$

(19) Draw a contour map of the function showing several level curves.

$$f(x,y) = (y-2x)^2$$

Soln:

$$f(x,y) = (y-2x)^2,\quad c = 0,1,2,3.$$

For $c_0 = 0$, $(y-2x)^2 = 0$

or, $y - 2x = 0$ —— (i)

For $C_2 = 1$, $(y-2x) = 1$

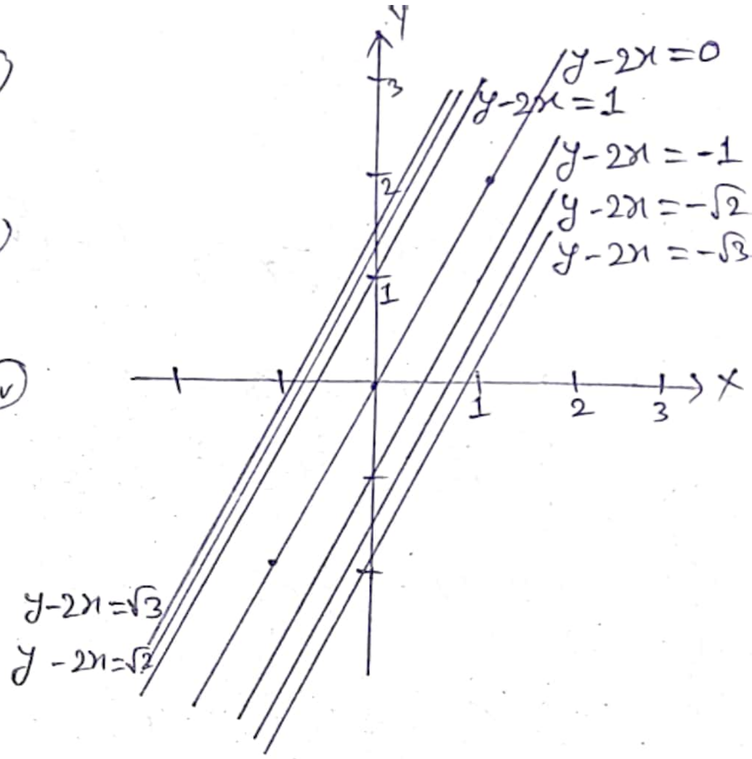
or, $y - 2x = \pm 1$ —— (ii)

For $C_2 = 2$, $(y-2x)^2 = 2$

or, $y - 2x = \pm\sqrt{2}$ —— (iii)

For $C_3 = 3$, $(y-2x)^2 = 3$

or, $(y-2x) = \pm\sqrt{3}$ —— (iv)



(20) $f(x,y) = x^3 - y$

Sol^n:

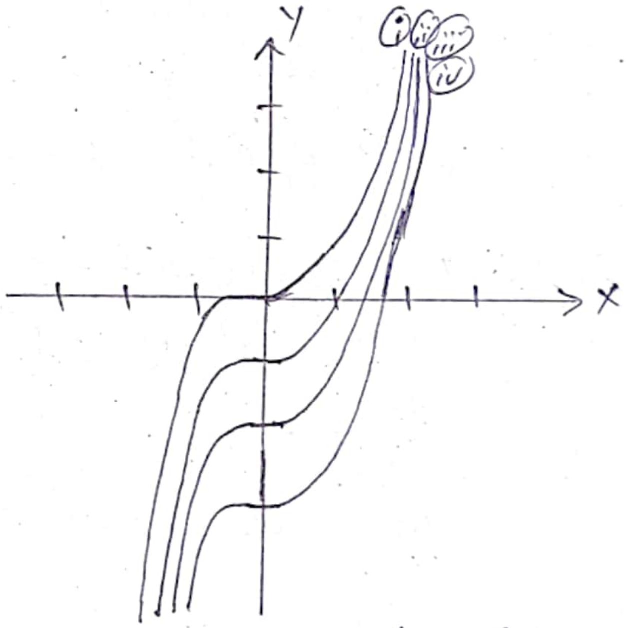
$f(x,y) = x^3 - y$, $C = 0, 1, 2, 3$

For $C = 0$, $x^3 - y = 0$ —— (i)

For $C = 1$, $x^3 - y = 1$ —— (ii)

For $C = 2$, $x^3 - y = 2$ —— (iii)

For $C = 3$, $x^3 - y = 3$ —— (iv)



(27) Sketch both a contour map and a graph of the function & compare them.

$f(x,y) = x^2 + 9y^2$

Sol^n:

$f(x,y) = x^2 + 9y^2$, $C = 0, 1, 2, 3$

For $C = 0$, $x^2 + 9y^2 = 0$ —— (i)

For $C = 1$, $x^2 + 9y^2 = 1$

or, $\dfrac{x^2}{1^2} + \dfrac{y^2}{\left(\frac{1}{3}\right)^2} = 1^2$ —— (ii)

For $C = 2$, $x^2 + 9y^2 = 2$

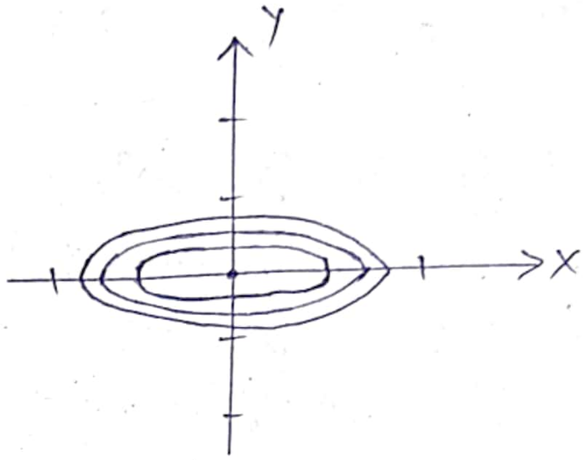
or, $\dfrac{x^2}{2} + \dfrac{9y^2}{2} = 1$

or, $\dfrac{x^2}{(\sqrt{2})^2} + \dfrac{y^2}{\left(\frac{\sqrt{2}}{3}\right)^2} = 1$ —— (iii)

For $c=3$; $x^2+9y^2=3$

or, $\frac{x^2}{3}+\frac{9y^2}{3}=1$

or, $\frac{x^2}{(\sqrt{3})^2}+\frac{y^2}{\left(\frac{1}{\sqrt{3}}\right)^2}=1$ —— (iv)



Graph of $f(x,y) = 9x^2+9y^2$

or $Z = x^2+9y^2$ which is an ellipsoid.

(28) $f(x,y) = \sqrt{36-9x^2-4y^2}$

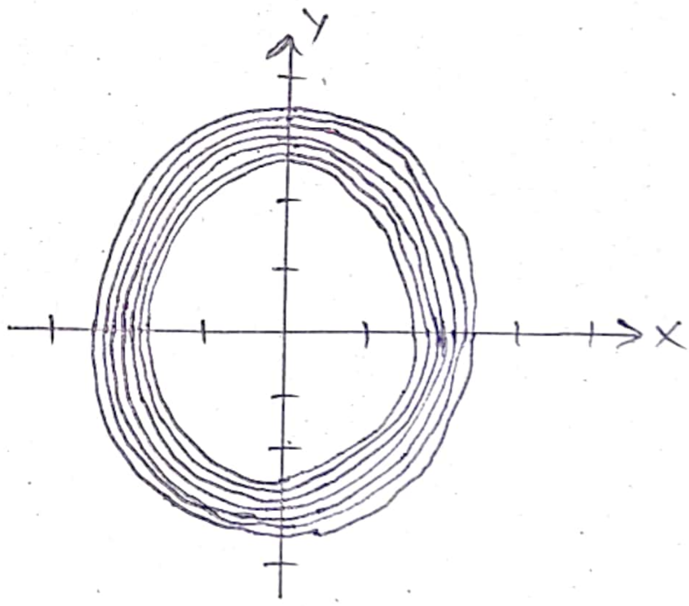
Soln:

$f(x,y) = \sqrt{36-9x^2-4y^2}$, $c=0,1,2,3$

For $c=0$, $\sqrt{36-9x^2-4y^2}=0$

or, $9x^2+4y^2=36$

or, $\frac{x^2}{4}+\frac{y^2}{9}=1$ —— (i)



For $c=1$, $\sqrt{36-9x^2-4y^2}=1$

or, $36-9x^2-4y^2=\pm 1$ —— (ii)

For $c=2$, $\sqrt{36-9x^2-4y^2}=2$

or, $36-9x^2-4y^2=\pm\sqrt{2}$ —— (iii)

For $c=3$, $\sqrt{36-9x^2-4y^2}=3$

or, $36-9x^2-4y^2=\pm\sqrt{3}$ —— (iv)

The graph of the function $f(x,y)=\sqrt{36-9x^2-4y^2}$ is,

$\pm Z = 36-9x^2-4y^2$ which is an ellipsoid.

where, $(9x^2+4y^2)\le 36$

③ Verify for the Cobb-Douglas production function

$$P(L,k) = 1.01 L^{0.75} k^{0.25}$$

that the production will be doubled if both the amount of labor (L) & the amount of capital (K) are doubled. Determine whether this is also true for the general production function

$$P(L,k) = bL^{\alpha} k^{1-\alpha}$$

Soln:

Cobb-Douglas production function:

$$P(L,k) = 1.01 L^{0.75} k^{0.25} \quad \text{--- (i)}$$

Let us double the labour (L) & capital (k).

$$\therefore P(2L,2k) = 1.01 (2L)^{0.75} \times (2k)^{0.25}$$

$$\text{or, } P(2L,2k) = 1.01 \times 2^{0.75} \times 2^{0.25} \times L^{0.75} \times k^{0.25}$$

$$\text{or, } P(2L,2k) = 2[1.01 L^{0.75} \times k^{0.25}]$$

$$\therefore P(2L,2k) = 2P(L,k)$$

Hence, the production will be doubled. #

General production function:

$$P(L,k) = bL^{\alpha} k^{1-\alpha} \quad \text{--- (ii)}$$

Let, $L = 2L$ & $k = 2k$ then,

$$P(2L,2k) = b(2L)^{\alpha} \cdot (2k)^{1-\alpha}$$

$$\text{or, } P(2L,2k) = b \times 2^{\alpha} \times 2^{1-\alpha} L^{\alpha} k^{1-\alpha}$$

$$\text{or, } P(2L,2k) = b \times 2^{\alpha+1-\alpha} L^{\alpha} k^{1-\alpha}$$

$$\text{or, } P(2L,2k) = 2[bL^{\alpha} k^{1-\alpha}]$$

$$\therefore P(2L,2k) = 2P(L,k)$$

Hence, general production function is also doubled.

) Describe the level surfaces of the function.

$f(x, y, z) = x + 3y + 5z$

Soln:

$f(x, y, z) = x + 3y + 5z$

let, $C = x + 3y + 5z$

here,

For $C = 0 \rightarrow x + 3y + 5z = 0$ — (i)

For $C = 1 \rightarrow x + 3y + 5z = 1$ — (ii)

For $C = 2 \rightarrow x + 3y + 5z = 2$ — (iii)

For $C = 3 \rightarrow x + 3y + 5z = 3$ — (iv)

Here, the level surfaces of the function f represents parallel plane surfaces.

(12) $f(x, y, z) = x^2 + 3y^2 + 5z^2$

Soln:

let $f(x, y, z) = x^2 + 3y^2 + 5z^2 = c$

For $c = 0 \longrightarrow x^2 + 3y^2 + 5z^2 = 0$ — (i) represents origin $(0,0,0)$

For $c = 1 \longrightarrow x^2 + 3y^2 + 5z^2 = 1$ — (ii)

For $c = 2 \longrightarrow x^2 + 3y^2 + 5z^2 = 2$ — (iii)

For $c = 3 \longrightarrow x^2 + 3y^2 + 5z^2 = 3$ — (iv)

Here, the level surfaces represents the faimly of ellipsoids.

I) Evaluating a Function. Find & Simplify the function values.

1. $f(x, y) = 2x + y^2$

(a) $\dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$

(b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

ⓐ Sol^n:

$$f(x,y) = 2x + y^2$$

$$\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \frac{2(x+\Delta x) + y^2 - (2x + y^2)}{\Delta x}$$

$$= \frac{2x + 2\Delta x + y^2 - 2x - y^2}{\Delta x}$$

$$= \frac{2\Delta x}{\Delta x}$$

$$= 2 \ \#$$

ⓑ Sol^n:

$$\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \frac{2x + (y+\Delta y)^2 - (2x + y^2)}{\Delta y}$$

$$= \frac{2x + y^2 + 2y\Delta y + \Delta y^2 - 2x - y^2}{\Delta y}$$

$$= \frac{2y\Delta y}{\Delta y} + \frac{\Delta y^2}{\Delta y} \left[\cancel{\because \Delta y^2 \text{ is very small}} \text{ so neglected}\right]$$

$$= 2y \cancel{\#} + \Delta y \ \#$$

② $f(x,y) = 3x^2 - 2y$

ⓐ $$\frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \frac{3(x+\Delta x)^2 - 2y - (3x^2 - 2y)}{\Delta x}$$

$$= \frac{\cancel{3x^2} + 6x\Delta x + 3\Delta x^2 - 3x^2 \cancel{+2y} \cancel{-2y} \; \cancel{-3x^2} + 2y}{\Delta x}$$

$$= 6x + 3\Delta x - \frac{3x^2}{\Delta x}$$

(b) $\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \frac{3x^2 - 2(y+\Delta y) - (3x^2 - 2y)}{\Delta y}$

$= \frac{3x^2 - 2y - 2\Delta y - 3x^2 + 2y}{\Delta y}$

$= -2$ #

(I) Find the domain & range of the function.

1. $f(x,y) = x^2 + y^2$

Domain $= \{(x,y) \in \mathbb{R}^2\}$ [The function $f(x,y)$ is defined for all real values of (x,y).]

Range of $f(x,y) = z$

or, $z = x^2 + y^2$ [Since, $x^2 + y^2 \geq 0$]

where, $z \geq 0$

$\therefore$ Range of $f(x,y) = [0, \infty)$

2. $f(x,y) = e^{xy}$

Domain $= \{(x,y) \in \mathbb{R}^2\}$ [The function $f(x,y)$ is defined for all real values of (x,y).]

Range of $f(x,y) = z \in \mathbb{R}$

where $z = (0, \infty)$ #

[Let, $e^{xy} = e^k$

If $k \geq 0$ then $e^k \to \infty$

& If $k < 0$ then $e^k \to 0$]

3. $g(x,y) = x\sqrt{y}$

$\rightarrow$ The function $g(x,y)$ exists for all real values of (x,y) ~~except~~ where $y \geq 0$

$\therefore$ Domain $= \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$

Range of $g(x,y) = x\sqrt{y} = z \in \mathbb{R}$.

where, $-\infty < z < \infty$

(4) $g(x,y) = \frac{y}{\sqrt{x}}$

→ The function $g(x,y)$ is defined for all real values of (x,y) where $x \geq 0$.

∴ Domain $= \{(x,y) \in \mathbb{R}^2 : x > 0\}$

→ Range of $g(x,y) = \frac{y}{\sqrt{x}} = z \in \mathbb{R}$.

where, $-\infty < z < \infty$.

(5) $z = \frac{x+y}{xy}$

→ The function $z = \frac{x+y}{xy}$ is defined for all real values of (x,y) where $xy \neq 0$

$\Rightarrow$ ~~either~~ $x \neq 0$, $y \neq 0$

∴ Domain $= \{(x,y) \in \mathbb{R}^2 : x \neq 0 \ \& \ y \neq 0\}$

∴ Range of function $z = \frac{x+y}{xy} \in \mathbb{R}$.

(6) $z = \frac{xy}{x-y}$

→ The function $z = \frac{xy}{x-y}$ is defined for all real values of (x,y) where $x \neq y$

∴ Domain $= \{(x,y) \in \mathbb{R}^2 : x \neq y\}$

∴ Range of function $z = \frac{xy}{x-y} \in \mathbb{R}$.

(7) $f(x,y) = \sqrt{4-x^2-y^2}$

→ The function f is defined for all real values of (x,y) where, $4-x^2-y^2 \geq 0$

or, $x^2+y^2 \leq 4$

∴ The domain of the function is the set of points lying on or inside the circle $x^2+y^2=4$.

(8).

Domain $= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 4\}$

$\because f(x) = \sqrt{4-x^2-y^2}$

or, $f(x) = \sqrt{4-(x^2+y^2)}$

$\because$ The value of x^2+y^2 lies in between $[0,4]$

So, The ~~one~~ range of the function $f(x) = \sqrt{4-x^2-y^2} = z$ where, $0 \leq z \leq 2$

$\therefore$ Range $= \{z \in \mathbb{R} : 0 \leq z \leq 2\}$

(8) $f(x,y) = \sqrt{4-x^2-4y^2}$

$\rightarrow$ The function f is defined for all real values of (x,y) where, $4-x^2-4y^2 \geq 0$

or, $x^2+4y^2 \leq 4$

or, $\frac{x^2}{2^2} + \frac{y^2}{1^2} \leq 1$

$\rightarrow$ The domain of the function f is the set of all points lying on or inside the ellipse $\frac{x^2}{2^2} + \frac{y^2}{1^2} \leq 1$.

$\therefore$ Domain $= \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{2^2} + \frac{y^2}{1^2} \leq 1\}$

$\rightarrow f(x,y) = \sqrt{4-x^2-4y^2}$

$= \sqrt{4-(x^2+4y^2)}$

Here, $x^2+4y^2 \leq 4$ & $x^2+4y^2 \geq 0$

$\therefore$ Range of $f(x,y) = \sqrt{4-x^2-4y^2} = z \in \mathbb{R}$.

where, $0 \leq z \leq 2$.

$\therefore$ Range $= \{z \in \mathbb{R} : 0 \leq z \leq 2\}$

(9) $f(x,y) = \ln(4-x-y)$

→ The function f is defined for all real values of (x,y) where, $4-x-y>0$

or, $x+y<4$

→ The domain of the function f is the set of all points that lies inside the line $x+y=4$ towards origin.

Domain $= \{(x,y) \in \mathbb{R}^2 : x+y<4\}$

$\because f(x,y) = \ln(4-x-y)$ is a logarithmic function

so, Range $= \{z \in \mathbb{R}\}$ ~~$0<z<\infty\}$~~

(10) $f(x,y) = \ln(xy-6)$

→ The function f is defined for all real values of (x,y) where $xy-6>0$

or, $xy>6$

Domain $= \{(x,y) \in \mathbb{R}^2 : xy>6\}$

Range of $f(x,y) = \ln(xy-6) = z \in \mathbb{R}$.

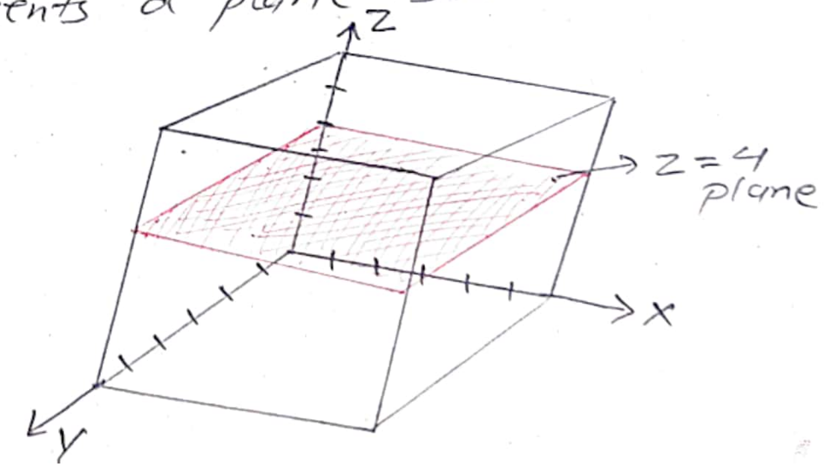
(II.) Sketch the surface given by the function.

(1). $f(x,y) = 4$.

Soln:

$f(x,y) = z = 4$

or, $z=4$ represents a plane surface in 3d.



② $f(x,y) = 6 - 2x - 3y$

Soln:

$f(x,y) = 6 - 2x - 3y$

or $6 - 2x - 3y = z$

For $z = 0, 1, 2, 3,$

$z=0$, $6-2x-3y=0$

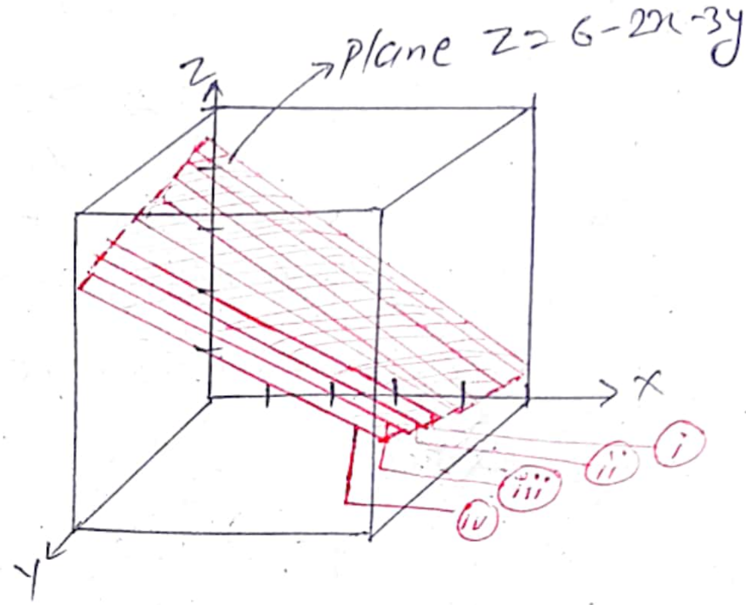
or, $2x+3y=6$ — (i)

$z=1$, $6-2x-3y=1$

or $2x+3y=5$ — (ii)

$z=2$, $2x+3y=4$ — (iii)

$z=3$, $2x+3y=3$ — (iv)



③ $f(x,y) = y^2$

Soln:

$f(x,y) = y^2$

or, $z = y^2$

For $z=0$, $y^2=0$ — (i)

For $z=1$, $y^2=1$

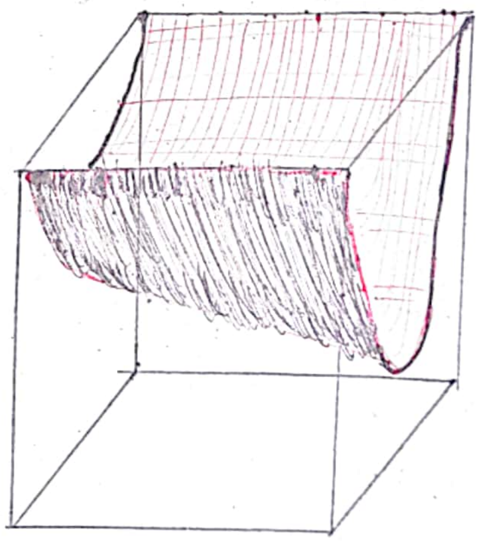
or, $y=\pm 1$ — (ii)

For $z=2$, $y^2=2$

or, $y=\pm\sqrt{2}$ — (iii)

For $z=3$, $y^2=3$

or, $y=\pm\sqrt{3}$ — (iv)



④ $g(x,y) = \frac{1}{2}y$

Solⁿ:

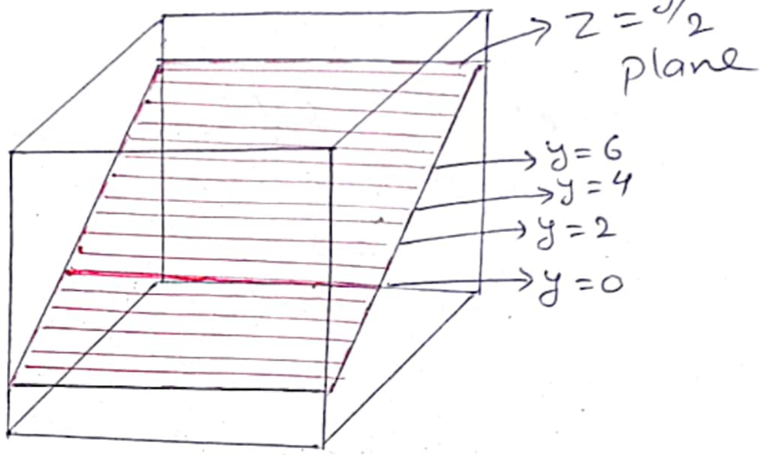
$g(x,y) = z = y/2$

For $z=0$, $y=0$ — (i)

For $z=1$, $y=2$ — (ii)

For $z=2$, $y=4$ — (iii)

for $z=3$, $y=6$ — (iv)



⑤ $z = -x^2 - y^2$

Solⁿ:

$z = -x^2 - y^2$

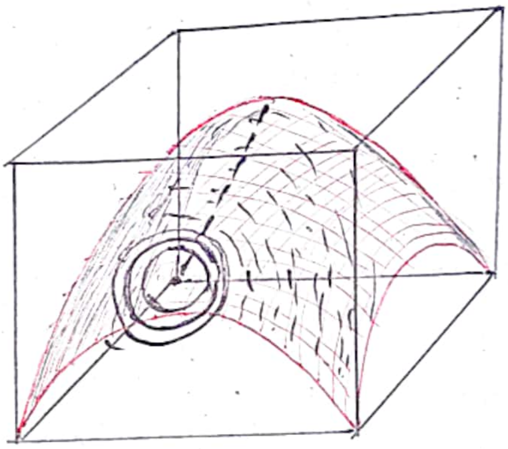
or, $z = -(x^2+y^2)$

For $z=0$, $x^2+y^2=0$ — (i)

for $z=-1$, $x^2+y^2=1$ — (ii)

for $z=-2$, $x^2+y^2=2$ — (iii)

For $z=-3$, $x^2+y^2=3$ — (iv)



⑥ $z = \dfrac{\sqrt{x^2+y^2}}{2}$

Solⁿ:

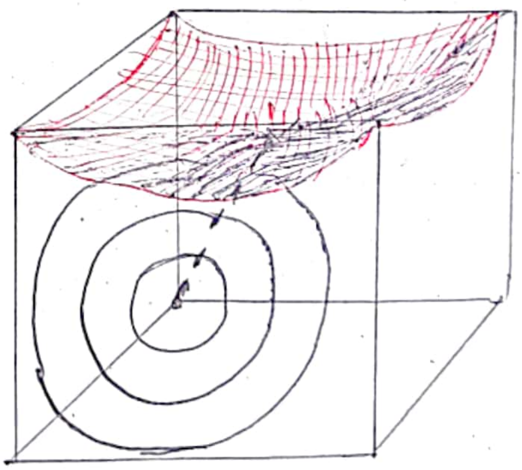
$z = \dfrac{\sqrt{x^2+y^2}}{2}$

for $z=0$, $x^2+y^2=0$ — (i)

For $z=1$, $x^2+y^2=4$ — (ii)

For $z=2$, $x^2+y^2=16$ — (iii)

For $z=3$, $x^2+y^2=36$ — (iv)



⑦ ~~Ex~~ $f(x,y) = e^{-x}$

Soln:

$f(x,y) = z = e^{-x}$

For $z = 1$, $e^{-x} = 1$

or, $e^{-x} = e^{0}$

or, $-x = 0$

or, $x = 0$ — (i)

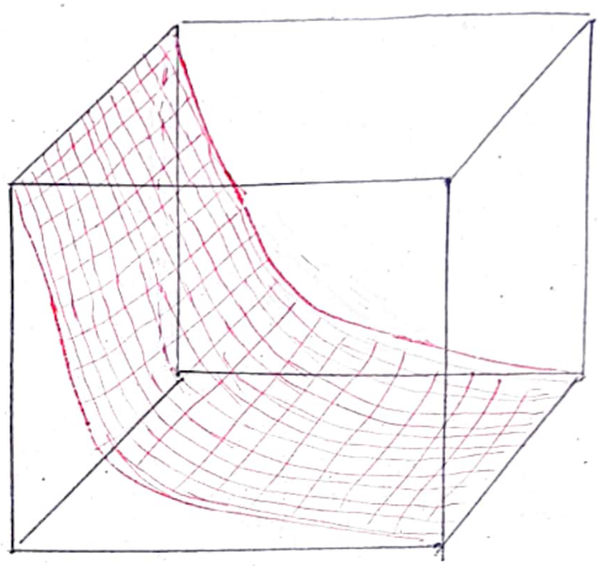
For $z = 2$, $e^{-x} = 2$

$-x = \ln(2)$

or, $x = \ln(2)$ — (ii)

For $z = 3$, $x = \ln(3)$ — (iii)

For $z = 4$, $x = \ln(4)$ — (iv)



Contour map.

1. $z = x + y$, $c = -1, 0, 2, 4$

Solr:

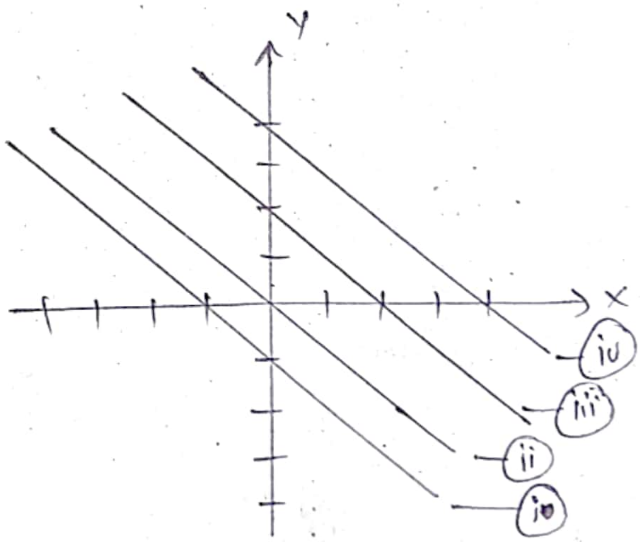
$z = x + y = c$

For $c = -1$, $x + y = -1$ — (i)

For $c = 0$, $x + y = 0$ — (ii)

For $c = 2$, $x + y = 2$ — (iii)

For $c = 4$, $x + y = 4$ — (iv)



② $z = 6 - 2x - 3y, \quad c = 0, 2, 4, 6, 8, 10$

Soln:

$z = 6 - 2x - 3y = c$

For $c=0$, $6 - 2x - 3y = 0$
or, $2x + 3y = 6$ —— (i)

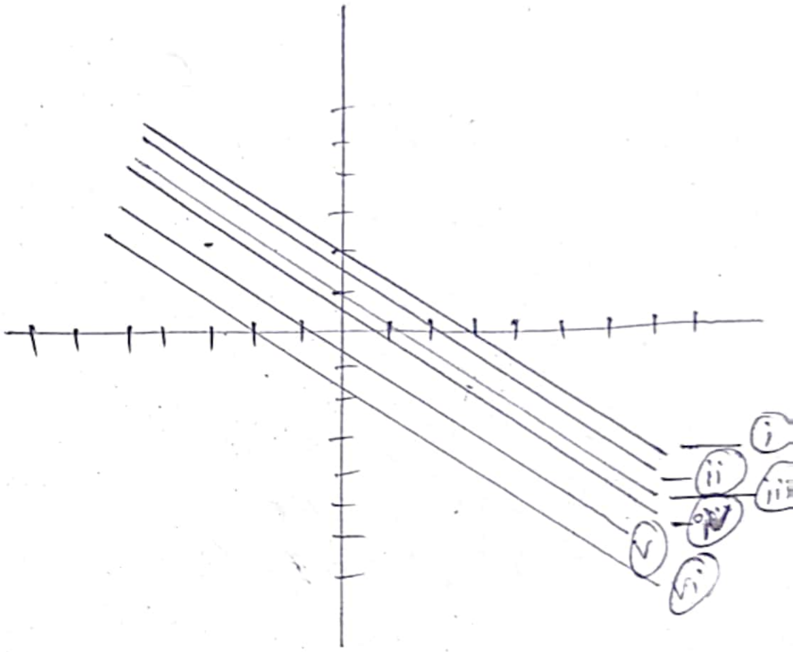
For $c=2$, $6 - 2x - 3y = 2$
or, $2x + 3y = 4$ —— (ii)

For $c=4$, $6 - 2x - 3y = 4$
or, $2x + 3y = 2$ —— (iii)

For $c=6$, $6 - 2x - 3y = 6$
or, $2x + 3y = 0$ —— (iv)

For $c=8$, $6 - 2x - 3y = 8$
or, $2x + 3y = -2$ —— (v)

For $c=10$, $6 - 2x - 3y = 10$
or, $2x + 3y = -4$ —— (vi)



③ $z = x^2 + 4y^2, \quad c = 0, 1, 2, 3, 4$

Soln: $z = x^2 + 4y^2 = c$

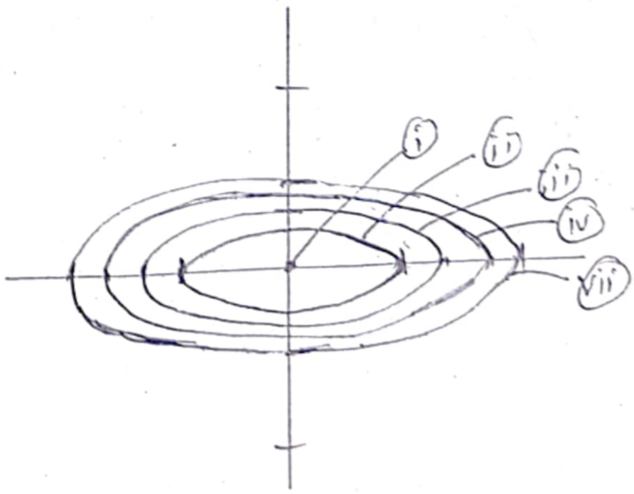
For $c=0$, $x^2 + 4y^2 = 0$ —— (i)

For $c=1$, $x^2 + 4y^2 = 1$ —— (ii)

For $c=2$, $x^2 + 4y^2 = 2$ —— (iii)

For $c=3$, $x^2 + 4y^2 = 3$ —— (iv)

For $c=4$, $x^2 + 4y^2 = 4$ —— (v)



(4) $f(x,y) = \sqrt{9-x^2-y^2}$, $c = 0,1,2,3$

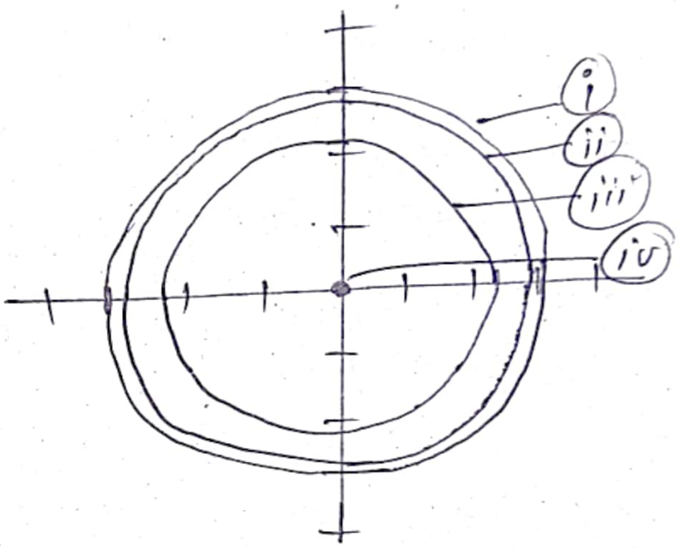
Soln:

$f(x,y) = \sqrt{9-x^2-y^2} = c.$

For $c=0$, $\sqrt{9-x^2-y^2} = 0$

or, $9-x^2-y^2 = 0$

or, $x^2+y^2 = 9$ — (i)



For $c=1$, $\sqrt{9-x^2-y^2} = 1$

$9-x^2-y^2 = 1$

or $x^2+y^2 = 8$ — (ii)

For $c=2$, $\sqrt{9-x^2-y^2} = 2$

or, $9-x^2-y^2 = 4$

or $x^2+y^2 = 5$ — (iii)

For $c=3$, $\sqrt{9-x^2-y^2} = 3$

or, $9-x^2-y^2 = 9$

or $x^2+y^2 = 0$ — (iv)

(5) $f(x,y) = xy$, $c = \pm 1, \pm 2, --- \pm 6$.

Soln: $f(x,y) = xy = c$

For $c = \pm 1$, $xy = 1$ — (i)

or, $xy = -1$ — (ii)

For $c = \pm 2$, $xy = 2$ — (iii)

or, $xy = -2$ — (iv)

For $c = \pm 3$, $xy = 3$ — (v)

or, $xy = -3$ — (vi)

For $c = \pm 4$, $xy = 4$ — (vii)

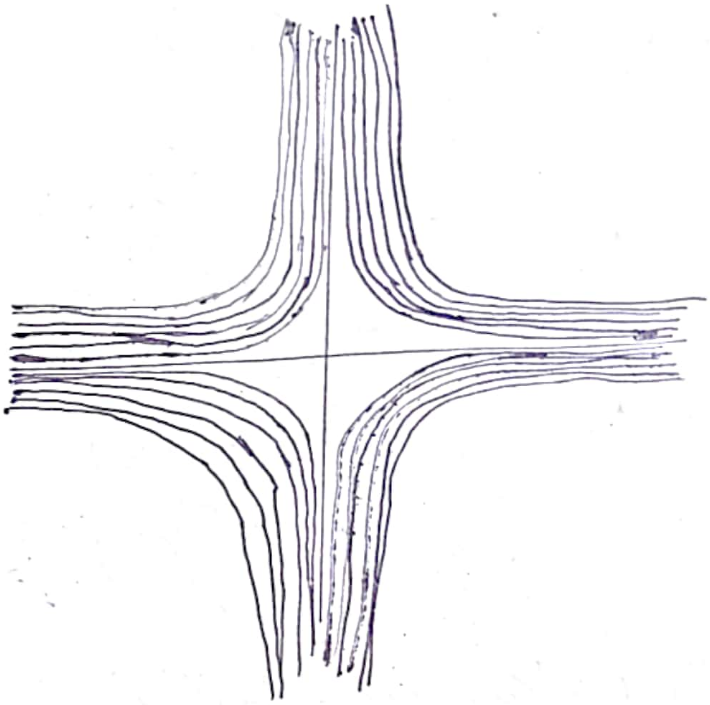
or, $xy = -4$ — (viii)

For $c = \pm 5$, $xy = 5$ — (ix)

or, $xy = -5$ — (x)

For $c = \pm 6$, $xy = 6$ — (xi)

or, $xy = -6$ — (xii)



(6) $f(x,y) = e^{xy/2}$, $c = 2, 3, 4, 1/2, 1/3, 1/4$

Soln,

$f(x,y) = e^{xy/2} = c$

For $c = 2$, $e^{xy/2} = 2$

or, $\frac{xy}{2} = \log_e 2$

or $xy = \log_e 4$ — (i)

For $c = 3$, $e^{xy/2} = 3$

or, $xy/2 = \log_e 3$

or $xy = \log_e 9$ — (ii)

For $c = 4$, $e^{xy/2} = 4$

or $xy = \log_e 16$ — (iii)

CamScanner

For $c = \frac{1}{2}$, $e^{xy/2} = \frac{1}{2}$

or $\frac{xy}{2} = \log_e(\frac{1}{2})$

or $xy = \log_e(\frac{1}{4})$

For $c = \frac{1}{3}$, $e^{xy/2} = \frac{1}{3}$

or $\frac{xy}{2} = \log_e(\frac{1}{3})$

or $xy = \log_e(\frac{1}{9})$

For $c = \frac{1}{4}$, $xy = \log_e(\frac{1}{16})$