

Chapter 2

1. (a) Show every number field of degree 2 over \mathbb{Q} is one of the quadratic fields.

Let K be a number field of degree 2, and $f(x) = x^2 + px + q$ be its minimum polynomial over \mathbb{Q} . Since $p, q \in \mathbb{Q}$ we can multiply through to clear the denominators and give us a polynomial $g(x) = ax^2 + bx + c$ over \mathbb{Z} with the same roots as $f(x)$. Therefore $K = \mathbb{Q}[\sqrt{b^2 - 4ac}]$ is a quadratic field for $m = b^2 - 4ac$.

1. (b) Suppose $K = \mathbb{Q}[\sqrt{m}]$ contains \sqrt{n} for n a squarefree integer. Since K has the basis $\{1, \sqrt{m}\}$, so $\sqrt{n} = p + q\sqrt{m}$ for $p, q \in \mathbb{Q}$. Therefore $n = p^2 + 2pq\sqrt{m} + q^2m$, so either $p = 0$ or $q = 0$.

If $p = 0$, then $\sqrt{n} = q\sqrt{m}$ and so $\sqrt{n}/\sqrt{m} = q$. This can only happen if $q = 1$, meaning $m = n$.

If $q = 0$, then $\sqrt{n} = p$, which can only happen if p is also an integer, contradicting n squarefree.

Therefore the quadratic fields are each distinct.

2. Let I be the ideal generated by 2 and $1 + \sqrt{-3}$ in the ring $\mathbb{Z}[\sqrt{-3}]$.

We have $I \neq (2)$ because $1 + \sqrt{-3} (\in I)$ does not have the form $2a + b\sqrt{-3}$ for $a, b \in \mathbb{Z}$. The ideal I^2 is generated by $(4, 2 + 2\sqrt{-3}, -2 + 2\sqrt{-3})$. The number $-2 + 2\sqrt{-3} = 2 + 2\sqrt{-3} - 4$ and so is redundant as a generator; therefore $I^2 = (4, 2 + 2\sqrt{-3}) = 2I$.

Since $I^2 = 2I$, prime factorization of ideals in $\mathbb{Z}[\sqrt{-3}]$ must not hold; if we did then I would be invertible, meaning it could be cancelled from the right-hand-side of each equality, giving us $I = (2)$ which is not true (from above).

Suppose P is a prime ideal of $\mathbb{Z}[\sqrt{-3}]$ containing 2. Then $4 \in P$ also. Since $(1 + \sqrt{-3})(1 - \sqrt{-3}) = 4$ and P is a prime ideal, one of $1 + \sqrt{-3}$ and $1 - \sqrt{-3}$ are also in P . However, if $1 - \sqrt{-3} \in P$ then $1 + \sqrt{-3} \in P$ since $-1 \cdot (1 - \sqrt{-3}) + 2 = 1 + \sqrt{-3}$. Therefore any prime ideal containing (2) also contains I and I is the unique prime ideal that contains (2) . Since I cannot be expressed as a product of prime ideals, neither can (2) .

(We should expect this; $\mathbb{Z}[\sqrt{-3}]$ is an order of conductor 2 in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ and I is not prime to the conductor, meaning it is not invertible.)

3. Complete the proof of Corollary 2, Theorem 1.

The statement of the text leaves off with α being an algebraic integer if and only if $2r$ and $r^2 - ms^2$ are both integers, where $r, s \in \mathbb{Q}$.

$2r$ being an integer requires that $r = \frac{a}{2}$, where a is an integer. Substituting $r = \frac{a}{2}$ into the second equation, we see that $a^2 - 4ms^2$ is an integer divisible by 4. In order for the quantity to be an integer, $s = \frac{b}{2}$, where b is an

integer. Therefore α is an algebraic integer of the form $\frac{a+b\sqrt{m}}{2}$ if and only if $a^2 - mb^2 \equiv 0 \pmod{4}$.

We finish by considering $m \pmod{4}$ and seeing under which statements the given equation is solvable. The key is that integer squares are either equivalent to 0 or 1 modulo 4.

- $m \equiv 1 \pmod{4}$: Let a be even - then $a^2 \equiv 0 \pmod{4}$, and to satisfy the equality, $b^2 \equiv 0 \pmod{4}$ and so b must also be even. Similarly, if a is odd, then $a^2 \equiv 1 \pmod{4}$ - to satisfy the equality, b must also be odd. Therefore $\alpha = \frac{a+b\sqrt{m}}{2}$ for all $a \equiv b \pmod{2}$ as required.
- $m \equiv 2, 3 \pmod{4}$: For the equation to be solvable, both a and b must be equivalent to 0 or 2 modulo 4 (and so even), meaning $\alpha = c + d\sqrt{m}$ for $c, d \in \mathbb{Z}$ as required.

4. Suppose a_0, \dots, a_{n-1} are algebraic integers and α is a complex number satisfying $\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_1\alpha + a_0 = 0$. Show the ring $\mathbb{Z}[a_0, \dots, a_{n-1}, \alpha]$ has a finitely generated additive group.

For each a_i let k_i be the degree of the algebraic integer a_i over \mathbb{Q} : therefore for any power $k \geq k_i$, it can be written as a linear combination of powers of a_i less than k_i . Additionally any power of α^k where $k \geq n$ can be written as a linear combination of powers of α multiplied by each of the a_i . Therefore only a finite number of powers of $a_0^{m_0} \dots a_{n-1}^{m_{n-1}} \alpha^m$ are needed; the a_i terms are capped to be lower than k_i and the α term is capped to be lower than n .

Since α is a member of a subring of \mathbb{C} that is finitely generated, α is therefore an algebraic integer.

5. Let f be a polynomial over \mathbb{Z}_p where p is a prime. We prove $f(x^p) = (f(x))^p$ by induction on number of terms.

If $f(x) = kx^b$ where $k \in \mathbb{Z}_p$, then $f(x^p) = kx^{pb} = k^p x^{bp} = (kx^b)^p$ (since $k^p = k$ for all $k \in \mathbb{Z}_p$).

Next, let $f(x) = g(x) + h(x)$ where $g(x)$ and $h(x)$ have fewer terms than $f(x)$.

$$\begin{aligned} f(x)^p &= (g(x) + h(x))^p \\ &= g(x)^p + h(x)^p + \sum_{k=1}^{p-1} \binom{p}{k} g(x)^k h(x)^{p-k} \\ &= g(x)^p + h(x)^p \\ &= g(x^p) + h(x^p) \text{ (using the inductive hypothesis)} \\ &= f(x^p) \end{aligned}$$

This is the required result.

6. If f and g are polynomials over a field K and $f^2 \mid g$, then $g = f^2h$. Therefore $g' = f^2h' + 2fhf'$, so $f \mid g'$.

7. Complete the proof of Corollary 2, Theorem 3.

Let ϕ_k be the automorphism of $\mathbb{Q}[\omega]$ sending ω to ω^k . Then $(\phi_a \circ \phi_b)(\omega) = (\omega^a)^b = \omega^{ab} = \phi_{ab}$, giving the required result that composition of automorphisms corresponds to multiplication modulo m .

8. (a) Let $\omega = e^{2\pi i/p}$ where p is an odd prime. Then

$$\text{disc}(\omega) = \prod_{1 \leq r < s \leq n} (\alpha_r - \alpha_s)^2 = \pm p^{p-2}$$

Therefore

$$\left| \prod_{1 \leq r < s \leq n} (\alpha_r - \alpha_s) \right| = \sqrt{\pm p^{p-2}} = p^{(p-3)/2} \sqrt{\pm p}$$

Let $\zeta = e^{2\pi i/3}$. Using the above we have the identity $(\zeta - \zeta^2) = \sqrt{-3}$.

Let $\zeta = e^{2\pi i/5}$. Note $\zeta^4 = -(\zeta^3 + \zeta^2 + \zeta + 1)$.

We expand the product:

$$(\zeta - \zeta^2)(\zeta - \zeta^3)(\zeta - \zeta^4)(\zeta^2 - \zeta^3)(\zeta^2 - \zeta^4)(\zeta^3 - \zeta_4) = 10\zeta^3 + 10\zeta^2 + 1$$

Observing that this product is negative we flip the signs and divide by $5^{(5-3)/2} = 5$ to get the identity $\sqrt{5} = -2\zeta^3 - 2\zeta^2 - 1$.

8. (b) The 8th cyclotomic polynomial is x^4+1 , so the 8th cyclotomic field contains all the roots of this equation, which includes $\sqrt{i} = (1/\sqrt{2})(1+i)$ and its complex conjugate $(1/\sqrt{2})(1-i)$. Thus the 8th cyclotomic field also contains their sum $2/\sqrt{2} = \sqrt{2}$.

8. (c) Let m be a squarefree number. Then m can be written as $2^i q$ where $2 \nmid q$, and $i \in \{0, 1\}$. We proceed by case analysis, showing for each that \sqrt{m} is contained in the d th cyclotomic field, where $d = \text{disc}(\mathbb{A} \cap \mathbb{Q}[\sqrt{m}])$.

$m = -1$: $\sqrt{-1}$ is contained in the 4th cyclotomic field which contains the complex unit i ($d = -4$).

$m = 2$: $\sqrt{2}$ is contained in the 8th cyclotomic field by part (b) ($d = 4 \cdot 2 = 8$).

$m = -2$: The 8th cyclotomic field contains i (since it contains the 4th cyclotomic field as a subfield) so it contains $\sqrt{-2} = i\sqrt{2}$ ($d = 4 \cdot -2 = -8$).

$m = q$ where $q \equiv 1 \pmod{4}$: Because $q \equiv 1 \pmod{4}$, q has an even number of prime factors $\equiv 3 \pmod{4}$, meaning that \sqrt{q} must be contained in the q -th cyclotomic field ($d = q$ since $q \equiv 1 \pmod{4}$).

$m = q$ where $q \equiv 3 \pmod{4}$: The $4q$ -th cyclotomic field contains the q -th cyclotomic field (containing $\sqrt{-q}$) and the 4th cyclotomic field (containing $\sqrt{-1}$) ($d = 4q$ since $q \equiv 3 \pmod{4}$), and so contains \sqrt{q} .

$m = 2q$ where q is a product of odd primes: Here $d = 8q$. By the above, \sqrt{q} is contained in either the q -th or $4q$ -th cyclotomic field, depending on its residue mod 4. Thus $\sqrt{2q}$ is contained in the $8q$ -th cyclotomic field.

This shows every quadratic field $\mathbb{Q}[\sqrt{m}]$ is contained within the d -th cyclotomic field.

9. Let θ be a primitive k -th root of unity, i.e. $\theta = e^{2\pi i/k}$. Let $\gcd(k, m) = d$. Using Euclid's extended algorithm we can find u, v such that $uk + vm = d$. Then we have

$$\omega^u \theta^v = e^{(2\pi i u)/m} e^{(2\pi i v)/k} = e^{2\pi i (uk + vm)/km} = e^{2\pi i d/km} = e^{2\pi i/r}$$

where $r = \text{lcm}(k, m)$ ($\text{lcm}(k, m) = km/\gcd(k, m)$).

10. Show if m is even, $m \mid r$, and $\phi(r) \leq \phi(m)$ then $r = m$.

If $m \mid r$ there is some k such that $mk = r$. Let $d = \gcd(k, m)$, so $r = mdj$ with j satisfying $\gcd(j, m) = 1$. Therefore $\phi(r) = \phi(md)\phi(j)$. Since $d \mid m$, $\phi(md) = d \cdot \phi(m)$, so

$$\phi(r) = d \cdot \phi(m)\phi(j) \leq \phi(m)$$

The inequality forces $d = 1$ and $\phi(j) = 1$. Because $2 \mid m \mid r$, $\phi(j) = 1$ implies $j = 1$. Therefore $m = r$.

11. (a) Suppose all the roots to a monic polynomial f have absolute value 1. Show that the coefficient of x^r has absolute value $\leq \binom{n}{r}$, where n is the degree of f and $\binom{n}{r}$ is the binomial coefficient.

Factor f as $f = (x - \alpha_0) \cdots (x - \alpha_n)$. Re-expanding f we see that the coefficient of x^r is equal to $\sum_{S \subseteq \{0, \dots, n\}, |S|=r} x^r \prod_{i \in S} \alpha_i$. By assumption $|\alpha_i| = 1$ for all i , so $|\prod_{i \in S} \alpha_i| = 1$. There are $\binom{n}{r}$ of these subsets of S .

Using the identity $|a + b| \leq |a| + |b|$ we have:

$$\begin{aligned} \left| \sum_{S \subseteq \{0, \dots, n\}, |S|=r} \prod_{i \in S} \alpha_i \right| &\leq \sum_{S \subseteq \{0, \dots, n\}, |S|=r} \left| \prod_{i \in S} \alpha_i \right| \\ &\leq \sum_{S \subseteq \{0, \dots, n\}, |S|=r} 1 \\ &\leq \binom{n}{r} \end{aligned}$$

11. (b) We will consider all monic polynomials f of degree n and show that only a finite number of them can have a root α all of whose conjugates have absolute value 1.

By Theorem 1, if α is an algebraic integer, then the coefficients of f are integers. By (b), the absolute value of the coefficients of f are bounded above $\binom{n}{r}$, therefore there are at most $2\binom{n}{r}$ choices for each coefficient beyond the x^n th term. The constant term of the polynomial must be 1 (since α has absolute value 1) and the first term of the polynomial must also be 1 (since f is monic). This gives an upper bound of $\sum_{r=1}^{n-1} 2\binom{n}{r} = 2(2^n - 2) = 4(2^{n-1} - 1)$ on the number of algebraic integers satisfying the given condition.

11. (c) (TODO)

12. (a) Let u be a unit in $\mathbb{Z}[\omega]$, where $\omega = e^{2\pi i/p}$. Show u/\bar{u} is a root of 1.

The field $\mathbb{Q}[\omega]$ has Galois group $\simeq \mathbb{Z}_p^\times$, which has cardinality $p-1$ and so has an element of order 2 (complex conjugation). Therefore u has $p-1$ conjugates, which consist of $(p-1)/2$ elements along with their complex conjugates. Enumerate the conjugates of u as $a_1, \dots, a_n, \bar{a}_1, \dots, \bar{a}_n$.

Therefore, the conjugates of u/\bar{u} have the form a_i/\bar{a}_i or \bar{a}_i/a_i . Multiplying over all conjugates of u/\bar{u} , we have $\prod_{i=1}^n a_i/\bar{a}_i \cdot \prod_{i=1}^n \bar{a}_i/a_i = 1$, and so u/\bar{u} and all its conjugates have absolute value 1. By 11 (c), u/\bar{u} is then a root of 1, and so has form $\pm\omega^k$.

12. (b) Suppose $u/\bar{u} = -\omega^k$. We derive a contradiction. Raising both sides to the p -th power we have $u^p/\bar{u}^p = -(\omega^k)^p = -(\omega^p)^k = -1$, and so $u^p = -\bar{u}^p$. By exercise 1.25, $u^p \equiv a \pmod{p}$ for some $a \in \mathbb{Z}$. Applying exercise 1.23, we see $\bar{u}^p \equiv \bar{a} \pmod{p}$, and so $a \equiv -\bar{a} \pmod{p}$. There a must be 0, and $u^p \equiv 0 \pmod{p}$, so p divides u^p . This contradicts u^p being a unit, since if p divided u^p , p would also divide the absolute value of u^p , which is 1. Therefore $u/\bar{u} = \omega^k$.

13. Show that 1 and -1 are the only units in the ring $A \cap \mathbb{Q}[\sqrt{m}]$, m squarefree and $m < 0, m \neq -1, -3$. What if $m = -1, -3$?

Let u be a unit in $A \cap \mathbb{Q}[\sqrt{m}]$. Then $u = a + b\sqrt{m}$ where $p, q \in A \cap \mathbb{Q}[\sqrt{m}]$. Since $N(u) = 1$, then $(a + b\sqrt{m})(a - b\sqrt{m}) = a^2 - b^2m = 1$. We proceed by cases on whether $m \equiv 1 \pmod{4}$.

If $m \not\equiv 1 \pmod{4}$, then a and b must be integers and so $a^2 - b^2m = 1$ can only be satisfied if one of the terms is 1 and the other is 0. If $a^2 = 1$, then $b^2m = 0$. This corresponds to the units 1 and -1 in $A \cap \mathbb{Q}[\sqrt{m}]$. If $-b^2m = 1$, then $b^2m = -1$ and so $m = -1$. This corresponds to the units i and $-i$ in $A \cap \mathbb{Q}[\sqrt{-1}]$.

If $m \equiv 1 \pmod{4}$ then let $a = r/2$ and $b = s/2$. Therefore $r^2 - s^2m = 4$. Since m is negative, both r^2 and $-s^2m$ must be positive. r^2 must be either 0, 1, or 4.

If r^2 is 0 then $-s^2m = 4$, so $s^2m = -4$, forcing $m = -1$ which is not $\equiv 1 \pmod{4}$. (We have considered this case already.)

If r^2 is 1 then $-s^2m = 3$ so $s^2m = -3$ and $m = -3, s = \pm 1$. This corresponds to the unit $\pm \frac{1}{2} \pm \frac{\sqrt{-3}}{2}$ in the ring $A \cap \mathbb{Q}[\sqrt{-3}]$.

If r^2 is 4 then $-s^2m = 0$, which corresponds to the unit ± 1 in the ring $A \cap \mathbb{Q}[\sqrt{m}]$.

14. Show that $1 + \sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}]$, but not a root of 1.

$1 + \sqrt{2}$ is a unit, as $-(1 - \sqrt{2})$ is its inverse:

$$-(1 + \sqrt{2})(1 - \sqrt{2}) = -1 + (\sqrt{2})^2 = 1$$

If $1 + \sqrt{2}$ were a root of 1, we would have $(1 + \sqrt{2})^k = 1$ for some k . However by the Binomial Theorem, $(1 + \sqrt{2})^k = \sum_{i=0}^k \binom{k}{i} (\sqrt{2})^i$, which will always

contains a term $\sqrt{2}$ multiplied by a positive number. Therefore $1 + \sqrt{2}$ is not a root of 1.

Let $(1 + \sqrt{2})^k = a + b\sqrt{2}$. The inverse of this term is

$$((1 + \sqrt{2})^k)^{-1} = ((1 + \sqrt{2})^{-1})^k = (-1)^k (1 - \sqrt{2})^k = (-1)^k (a - b\sqrt{2})^k$$

Therefore, $(a + b\sqrt{2})^k \cdot (a - b\sqrt{2})^k = \pm 1$ and so the powers of $1 + \sqrt{2}$ give an infinite number of a, b such that $a^2 - 2b^2 = \pm 1$.

15. (a) Let $a + b\sqrt{-5}$ be an element of $\mathbb{Z}[\sqrt{-5}]$. Then the norm of $a + b\sqrt{-5}$ is $(a + b\sqrt{-5})(a - b\sqrt{-5}) = a^2 + 5b^2$, where $a, b \in \mathbb{Z}$. Since there are no integer solutions a, b such that $a^2 + 5b^2 = 2$ or $a^2 + 5b^2 = 3$, there can be no element of $\mathbb{Z}[\sqrt{-5}]$ with a norm of 2 or 3.
15. (b) In $\mathbb{Z}[\sqrt{-5}]$, $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. If unique factorization held in $\mathbb{Z}[\sqrt{-5}]$, there would be elements $a, b, c, d \in \mathbb{Z}[\sqrt{-5}]$ such that $a \cdot b = 2$, $c \cdot d = 3$, $a \cdot d = 1 + \sqrt{-5}$, $b \cdot c = 1 - \sqrt{-5}$. However by (a), 2 and 3 are irreducible in $\mathbb{Z}[\sqrt{-5}]$, meaning they are irreducible elements, and so no a, b, c, d can exist.
16. We argue in the style of K. Conrad: Trace and Norm, Section 4. Suppose $\sqrt[4]{3} \in \mathbb{Q}[\alpha]$ where $\alpha = \sqrt[4]{2}$; therefore $\sqrt{3} = a + b\alpha + c\alpha^2 + d\alpha^3$. We have the following traces:

$$\begin{aligned} \text{Tr}(\sqrt{3}) &= \sqrt{3} - \sqrt{3} = 0 \\ \text{Tr}(\alpha) &= \alpha - \alpha + i\alpha - i\alpha = 0 \\ \text{Tr}(\alpha^2) &= \alpha^2 - \alpha^2 + i\alpha^2 - i\alpha^2 = 0 \\ \text{Tr}(\alpha^3) &= \alpha^3 - \alpha^3 + i\alpha^3 - i\alpha^3 = 0 \end{aligned}$$

Since $\sqrt{3} = a + b\alpha + c\alpha^2 + d\alpha^3$,

$$\begin{aligned} \text{Tr}(\sqrt{3}) &= \text{Tr}(a + b\alpha + c\alpha^2 + d\alpha^3) \\ 0 &= a\text{Tr}(1) + b\text{Tr}(\alpha) + c\text{Tr}(\alpha^2) + d\text{Tr}(\alpha^3) \\ 0 &= 4a \end{aligned}$$

Therefore $a = 0$, and we have $\sqrt{3} = b\alpha + c\alpha^2 + d\alpha^3$. We have $\text{Tr}(\sqrt{3}\alpha) = \text{Tr}(\sqrt[4]{9/2}) = \sqrt[4]{9/2} - \sqrt[4]{9/2} + i\sqrt[4]{9/2} - i\sqrt[4]{9/2} = 0$, so $0 = b\text{Tr}(1) + c\text{Tr}(\alpha) + d\text{Tr}(\alpha)^2 = 4b$ and so $b = 0$.

Similarly $\text{Tr}(\sqrt{3}/\alpha^2) = \text{Tr}(\sqrt{3/2}) = 0$, and so $c = 0$.

From eliminating the coefficients a, b, c , we have $d\sqrt[4]{8} = \sqrt{3}$ and so $3 = d^2\sqrt{8} = 2d^2\sqrt{2}$. Therefore $\sqrt{2}$ is expressible as a rational number $3/d^2$, a contradiction. Therefore $\sqrt{3} \notin \mathbb{Q}[\alpha]$.

(Where would this argument break down for $\sqrt{2}$? $\sqrt{2} = \alpha^2$ so $\sqrt{2}/\alpha^2 = 1$ and so we would conclude that $c = 1$ rather than $c = 0$.)

17 - TODO

18 - TODO

19 - TODO

20. Write $f(x) = (x - \alpha)g(x)$. By the chain rule $f'(x) = (x - \alpha)g'(x) + g(x)$, so $f'(\alpha) = g(\alpha) = \prod_{\beta \neq \alpha} (\alpha - \beta)$.

21. Let $f(x) = g(x)h(x)$, where $g(x)$ is the minimum polynomial of α over \mathbb{Z} . Then $f'(x) = g'(x)h(x) + g(x)h'(x)$ and $f'(\alpha) = g'(\alpha)h(\alpha)$. We have

$$N(f'(\alpha)) = N(g'(\alpha))N(h(\alpha))$$

. By Theorem 8, $N(g'(\alpha)) = \pm \text{disc}(\alpha)$, so

$$N(f'(\alpha)) = \pm \text{disc}(\alpha)N(h(\alpha))$$

Therefore $\text{disc}(\alpha)$ divides $N(f'(\alpha))$ as required.

23. (c) Let $\{\alpha_1, \dots, \alpha_n\}$ be an integral basis for K ($n = [K : \mathbb{Q}]$) and let $\{\beta_1, \dots, \beta_m\}$ be an integral basis for L ($m = [L : \mathbb{Q}]$). Therefore

$$\{\alpha_i \beta_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$$

is an integral basis for KL .

We have the tower of field extensions $KL : K : \mathbb{Q}$ where $[KL : K] = m$, $[K : \mathbb{Q}] = n$. By the formula established in (b),

$$\text{disc}(\alpha_i \beta_j) = (\text{disc}(\alpha_i))^m N_{\mathbb{Q}}^K \text{disc}(\beta_j) = (\text{disc } R)^m (\text{disc } S)^n$$

Because $\text{disc } S$ is an integer, its norm is the degree of K over \mathbb{Q} .

- 24 Let G be a free abelian group of rank n and let H be a subgroup. Take $G = \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$. We show by induction that H is a free abelian group of rank $\leq n$.

First prove the result for $n = 1$.

If G is a free abelian group of rank 1, $G = \mathbb{Z}$. If H is a subgroup of G then H must have a least non-negative element, call it m . Then H is generated by m (all subgroups of \mathbb{Z} are generated by a single element).

Next, we assume the result holds for $n - 1$, and define $\pi : G \rightarrow \mathbb{Z}$ the projection of G onto the first factor. Let K denote the kernel of π .

(a): Show that $H \cap K$ is a free abelian group of rank $\leq n - 1$.

Let ι be the map that drops the first factor from G ; as K is a subgroup of G , then $\iota(H \cap K)$ must be a subgroup of $\iota(G)$. $\iota(G)$ is a free abelian group of rank $n - 1$, and so applying the inductive hypothesis, we see $\iota(H \cap K) = 0 \oplus (H \cap K)$ is a free abelian group of order $n - 1$.

(b): The image $\pi(H) \subset \mathbb{Z}$ is either $\{0\}$ or infinite cyclic. If it is 0 , then $H = H \cap K$. Otherwise let $h \in \pi(H)$ be a generator of $\pi(H)$. Show H is the direct sum of its subgroups $\mathbb{Z}h$ and $K \cap H$.

Let h be as in the problem statement. Let $a \in H$. We will show a is a member of $\mathbb{Z}h \oplus (K \cap H)$. If $\pi(a) = 0$, then $a \in H \cap K$ and so a is a member of the required group. Otherwise $\pi(a) = m\pi(h)$ for some integer m and so $mh - a \in K \cap H$ (a free abelian group of rank $\leq n - 1$). Therefore a is the direct sum of $mh \in \mathbb{Z}h$ and the components of $mh - a$. Since a was chosen arbitrarily, $H = \mathbb{Z}h \oplus (K \cap H)$.

25. Let α be an algebraic number, so there is some $f \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$. We convert this polynomial into a (non-monic) $g \in \mathbb{Z}[x]$ by through multiplying by the GCD m for all of the denominators in the coefficients of f . Then $g = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ and $g(\alpha) = 0$. Multiplying through by a_n^{n-1} gives the relationship $(a_n \alpha)^n + a_{n-1} a_n^{n-1} \alpha^{n-1} + \dots + a_0 a_n^{n-1} = 0$. This is a monic polynomial with integer coefficients, so $ma_n^n \alpha$ is an algebraic integer.

Given any finite set of algebraic numbers, $\{\alpha_0, \dots, \alpha_n\}$ let m_i be such that $m_i \alpha_i$ is an algebraic integer. Therefore taking M to be the least common multiple of each m_i gives us a number M such that each $M \alpha_i$ is an algebraic integer.

26. The proof that two sets that generate the same subgroup have the same discriminant is the same as that of Theorem 11: as $\{\beta_1, \dots, \beta_n\}$ and $\gamma_1, \dots, \gamma_n$ generate the same additive subgroup, we can write the γ_i in terms of the β_i through an matrix M with entries in \mathbb{Z} , and vice versa. This shows that the translate matrices must have determinant 1, so the discriminants are equal.

27. Let G and H be two free abelian subgroups of rank n in K , with $H \subset G$.

27. (a) Show G/H is a finite group.

Since G and H are free abelian subgroups of rank n , $G \simeq \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$ and since H is a subgroup of G , then $H \simeq I_1 \oplus \dots \oplus I_n$, where each $I_i \subseteq \mathbb{Z}$ is an additive subgroup of \mathbb{Z} . Each \mathbb{Z}/I_i is finite, having cardinality equal to the generating element of I_i . Therefore G/H is finite, having cardinality $\prod_{i=1}^n |\mathbb{Z}/I_i|$.

27. (b) The well-known finite structure theorem for abelian groups says G/H is a direct sum of at most n cyclic groups. Use this to show that G has a generating set β_1, \dots, β_n such that for appropriate integers d_i , $d_1 \beta_1, \dots, d_n \beta_n$ is a generating set for H .

Let β_i be 1 projected to the i th-factor and 0 elsewhere. Then the set of $\{\beta_i\}$ generate G . Let d_i be the minimum element of I_i , an additive subgroup of \mathbb{Z} : we show $\{d_i \beta_i\}$ generates H . Take $a \in H$, and let $\iota_i(a)$ be the i th factor of a , so $\iota_i(a) \in I_i$. By choice of d_i , $\iota_i(a) = d_i m$ for some

integer m , and $a = \iota_1(a) \oplus \cdots \oplus \iota_n(a) = d_1\beta_1 + \cdots + d_n\beta_n$. Since a was chosen arbitrarily, the $\{d_i\beta_i\}$ generates H .

27. (c) $\text{disc}(H) = \text{disc}(d_1\beta_1, \dots, d_n\beta_n)$: by Exercise 3.18 (a),

$$\text{disc}(H) = (d_1 \cdots d_n)^2 \text{disc}(\beta_1, \dots, \beta_n) = |G/H|^2 \text{disc}(G)$$

27. (d) Show that if $\alpha_1, \dots, \alpha_n \in R = \mathbb{A} \cap K$, then they form an integral basis iff $\text{disc}(\alpha_1, \dots, \alpha_n) = \text{disc}(R)$.

Let H be the additive subgroup formed by $\alpha_1, \dots, \alpha_n$. By (c), we have $\text{disc}(H) = |R/H|^2 \text{disc}(R)$. Therefore $\text{disc}(R) = \text{disc}(G)$ iff $|G/H|^2 = 1$, which is the same as saying that there is $b \in G$ such that $b \notin H$. Therefore $\text{disc}(\alpha_1, \dots, \alpha_n) = \text{disc}(R)$ if and only if they form an integral basis for R .

27. (e) Show that if $\alpha_1, \dots, \alpha_n \in R = \mathbb{A} \cap K$ and $\text{disc}(\alpha_1, \dots, \alpha_n)$ is squarefree, then the α_i form an integral basis for R .

If $\text{disc}(H)$ is squarefree then $|R/H| = 1$ which implies that $\text{disc}(H) = \text{disc}(R)$. By (d) the α_i form an integral basis for R .

28. (a) Taking the derivative of the polynomial, we have $f'(x) = 3x^2 + a$. We then have:

$$\begin{aligned} f'(\alpha) &= 3\alpha^2 + a \\ \alpha f'(\alpha) &= 3\alpha^3 + a\alpha \\ \alpha f'(\alpha) &= -3(a\alpha + b) + a\alpha \\ \alpha f'(\alpha) &= -2a\alpha - 3b \\ f'(\alpha) &= -(2a\alpha + 3b)/\alpha \end{aligned}$$

28. (b) It is straightforward that $2a\alpha + 3b$ is a root of the polynomial $g(x) = (\frac{x-3b}{2a})^3 + a(\frac{x-3b}{2a}) + b$. To calculate the norm of $2a\alpha + 3b$ over $\mathbb{Q}[\alpha]$, we thus divide the zero coefficient of $g(x)$ by negative the initial coefficient of $g(x)$ (negative since $n = 3$ is odd):

$$-(2a)^3 \left(\frac{(-3b)^3}{(2a)^3} - \frac{3b}{2} + b \right)$$

Reducing terms gives us

$$N(2a\alpha + 3b) = (3b)^3 + (2^2)a^3b = 27b^3 + 4a^3b$$

28. (c) By Theorem 8, $\text{disc}(a) = -N(f'(\alpha))$ (the negative sign holds since $n = 3 \neq 0, 1, 4$).

Note that given the factoring of $f(x)$ into $(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$, $(-1)\alpha_1\alpha_2\alpha_3 = -N(\alpha) = b$, $N(\alpha) = -b$.

We now compute the discriminant of α :

$$\begin{aligned}
\text{disc}(\alpha) &= -N(f'(\alpha)) \\
&= -N(-(2a\alpha + 3b)/\alpha) \\
&= \frac{27b^3 + 4a^3b}{-b} \\
&= -(27b^2 + 4a^3)
\end{aligned}$$

This is the required result.

28. (d) If $\alpha^3 = \alpha + 1$, then $a = -1$ and $b = -1$. By (c), $\text{disc}(\alpha) = -27 - 4 = -31$, which is squarefree. By 27 (c) the powers of α thus form an integral basis for $\mathbb{A} \cap \mathbb{Q}[\alpha]$.

Similarly if $a = 1$ and $b = -1$, then $\text{disc}(\alpha) = -27 + 4 = -23$ (squarefree) and so again by 27 (c) the powers of α form an integral basis for $\mathbb{A} \cap \mathbb{Q}[\alpha]$.

29. Let $\mathbb{Q}[\sqrt{m}, \sqrt{n}]$, where $(m, n) = 1$. Find an integral basis and the discriminant of this basis for (a): the case where $m, n \equiv 1 \pmod{4}$ and (b) where $m \equiv 1 \pmod{4}$, $n \not\equiv 1 \pmod{4}$.

For both given scenarios, the ring of integers is a linear combination of the ring of integers of $\mathbb{Q}[\sqrt{m}]$ and $\mathbb{Q}[\sqrt{n}]$, and so Theorem 12, Corollary 1 applies, and an integral basis can be found as a combination of the bases of the individual rings.

29. (a) $m, n \equiv 1 \pmod{4}$: The corresponding rings of integers for $\mathbb{Q}[\sqrt{m}]$ and $\mathbb{Q}[\sqrt{n}]$ are $\mathbb{Z}[(1 + \sqrt{m})/2]$ and $\mathbb{Z}[(1 + \sqrt{n})/2]$ with discriminants m and n . By assumption, these discriminants are relatively prime, so Theorem 12, Corollary 1 applies. The field $\mathbb{Q}[\sqrt{m}, \sqrt{n}]$ thus has an integral basis $\{1, (\sqrt{m} + 1)/2, (\sqrt{n} + 1)/2, (1 + \sqrt{m} + \sqrt{n} + \sqrt{nm})/4\}$. By Exercise 23 (c), the discriminant for this basis is m^2n^2 .
29. (b) The rings of integers for $\mathbb{Q}[\sqrt{m}]$ and $\mathbb{Q}[\sqrt{n}]$ are $\mathbb{Z}[(1 + \sqrt{m})/2]$ and $\mathbb{Z}[\sqrt{n}]$, with discriminants m and $4n$. Since m was assumed to be square-free, $(m, 4n) = 1$, so Theorem 12, Corollary 1 applies again. The field $\mathbb{Q}[\sqrt{m}, \sqrt{n}]$ thus has an integral basis $\{1, (\sqrt{m} + 1)/2, \sqrt{n}, (\sqrt{mn} + \sqrt{n})/2\}$. By Exercise 23 (c), the discriminant for this basis is $m^2(4n)^2 = 16m^2n^2$.
30. Let f be the monic irreducible polynomial for α over \mathbb{Z} and for each $g \in \mathbb{Z}[x]$, let \bar{g} denote the polynomial in $\mathbb{Z}_3[x]$ obtained by reducing the coefficients mod 3.
30. (a) Show that $g(\alpha)$ is divisible by 3 in $\mathbb{Z}[\alpha]$ if and only if \bar{g} is divisible by \bar{f} in $\mathbb{Z}_3[x]$.
- Suppose $g(\alpha)$ is divisible by 3. Then $g(\alpha) = 3m$ for some m and so $(g - 3m)(\alpha) = 0$. Since this is a polynomial in α and f is the minimum polynomial, $f \mid g - 3m$. Therefore $\bar{f} \mid \overline{g - 3m} = \bar{g}$.

If \bar{g} is divisible by \bar{f} in $\mathbb{Z}_3[x]$, then $\bar{g} = \bar{f}h$ for some $h \in \mathbb{Z}[x]$, and so $g = (f + 3j)h$ in $\mathbb{Z}[x]$ for some polynomial $j(x) \in \mathbb{Z}[x]$. So $g(\alpha) = 3j(\alpha)h(\alpha)$ and $g(\alpha)$ is divisible by 3.

30. (b) Consider the four algebraic integers:

$$\begin{aligned}\alpha_1 &= (1 + \sqrt{7})(1 + \sqrt{10}) \\ \alpha_2 &= (1 + \sqrt{7})(1 - \sqrt{10}) \\ \alpha_3 &= (1 - \sqrt{7})(1 + \sqrt{10}) \\ \alpha_4 &= (1 - \sqrt{7})(1 - \sqrt{10})\end{aligned}$$

The conjugates of each α_i are the other α_j , and each product $\alpha_i\alpha_j$ is divisible by 3: $\alpha_1\alpha_3$, $\alpha_2\alpha_3$, $\alpha_1\alpha_4$, and $\alpha_2\alpha_4$ are divisible by -6 , and $\alpha_1\alpha_2$, $\alpha_1\alpha_4$, $\alpha_2\alpha_3$, and $\alpha_3\alpha_4$ are divisible by -9 .

We show that $\alpha_i^n/3$ is not an algebraic integer by considering its trace: $\text{Tr}(\alpha_i^n/3) = \text{Tr}(\alpha_i^n)/3$, so we compute $\text{Tr}(\alpha_i^n)$. The conjugates of α_i^n are each of the other α_j^n , so $\text{Tr}(\alpha_i^n) = \alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n$. Modulo 3, $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^n \equiv \alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n$ because any of the monomials with any nonzero powers is divisible by 3 and so cancel out mod 3. However $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^n = 1^n = 1$. Since each α_i is conjugate to each of the α_j , their traces must be identical.

Therefore the trace of α_i^n is an integer $\equiv 1 \pmod{3}$, and so $\text{Tr}(\alpha_i^n/3)$ cannot be an integer, and so by Corollary 2 to Theorem 4, $\alpha_i^n/3$ must not be an algebraic integer.

30. (c) Let α_i from (b) be defined by $f_i(\alpha)$ (for any fixed α). Because $\alpha_i\alpha_j$ is divisible by 3, by (a), $\bar{f} \mid \overline{f_i f_j}$. However, $\bar{f} \nmid \overline{f_i}^n$ for any power of n (or else 3 would divide $\overline{f_i}^n$ which is not the case by (b)), so $\overline{f_i f_j} \neq \overline{f_i}^n$ for any n . Therefore, since $\mathbb{Z}_3[x]$ is a UFD, \bar{f} has an irreducible factor that does not divide $\overline{f_i}$ but does divide $\overline{f_j}$ for all $j \neq i$.
30. (d) The result of (c) is that \bar{f} has at least 4 irreducible factors in $\mathbb{Z}_3[x]$. However, \bar{f} is of degree at most 4, since $\alpha \in \mathbb{Q}[\sqrt{7}, \sqrt{10}]$. For there to be at least 4 irreducible factors of \bar{f} it would imply each are of degree 1, but there are only 3 monic polynomials of degree 1 in $\mathbb{Z}_3[x]$: x , $x - 1$, $x - 2$. Therefore $\mathbb{A} \cap \mathbb{Q}[\sqrt{7}, \sqrt{10}] \neq \mathbb{Z}[\alpha]$ for any α .

31. Show that $\frac{\sqrt{3} + \sqrt{7}}{2}$ is an algebraic integer.

$\frac{\sqrt{3} + \sqrt{7}}{2}$ is the root of the degree 4 polynomial $f(x) = x^4 - 5x^2 + 1$. This shows that the intersection of the ring of integers $\mathbb{Z}[\sqrt{3}]$ and $\mathbb{Z}[\sqrt{7}]$ is not $\mathbb{Z}[\sqrt{3}, \sqrt{7}]$; neither original ring contains fractional elements. (Their discriminants are 12 and 28 respectively, sharing a factor of 4.)

32. The fields $\mathbb{Q}[\sqrt[3]{2}]$ and $\mathbb{Q}[\omega\sqrt[3]{2}]$ where $\omega = e^{2\pi i/3}$ both have degree 3 over \mathbb{Q} , but their composition $\mathbb{Q}[\omega, \sqrt[3]{2}]$ has degree 6 over \mathbb{Q} .
33. Let $\omega = e^{2\pi i/m}$, where $m \geq 3$. We know that $N(\omega) = \pm 1$ because ω is a unit. Show the + sign holds.

Write $e^{2\pi i k/m}$ as ω_k . The conjugates of ω have the form ω_k where $(k, m) = 1$. There are $\phi(m)$ of these, which is even for all $m \geq 3$. If ω_k is a conjugate, then ω_{m-k} is also a conjugate, since $(k, m) = 1$ implies there exist integers a, b such that $ak + bm = 1$, so $-a(m-k) + (b+a)m = 1$, and so $(m-k, m) = 1$.

For each conjugate ω_k , $\omega_k \neq \omega_{m-k}$; if this were the case, $k = -k \pmod{m}$, so $2k = 0 \pmod{m}$ and so k would divide m , contradicting $(k, m) = 1$. Therefore all the conjugates are distinct.

Finally, for each conjugate ω_k , $\omega_k \cdot \omega_{m-k} = 1$, so in computing the norm of ω , all the conjugates cancel out and the norm of ω is seen to be 1.

34. (a) Show that $1 + \omega + \omega^2 + \dots + \omega^{k-1}$ is a unit in $\mathbb{Z}[\omega]$ if k is relatively prime to ω .

$$(1 + \omega + \omega^2 + \dots + \omega^{k-1}) \left(\frac{1 - \omega}{1 - \omega^k} \right) = \frac{1 - \omega^k}{1 - \omega^k} = 1$$

Therefore, if $\frac{1-\omega}{1-\omega^k} \in \mathbb{Z}[\omega]$ then $1 + \omega + \dots + \omega^{k-1}$ is a unit. Since $(k, m) = 1$, then there exist $a, b \in \mathbb{Z}$ such that $ak + bm = 1$, and so $\omega = \omega^{ak+bm} = \omega^{ak} \omega^{bm} = \omega^{ak}$. Since $\omega^{ak} = \omega^{(m-a)k}$ for negative a , a can be assumed to be positive. We then have

$$\frac{1 - \omega}{1 - \omega^k} = \frac{1 - \omega^{ak}}{1 - \omega^k} = 1 + \omega^k + \omega^{2k} + \dots + \omega^{(a-1)k} \in \mathbb{Z}[\omega]$$

This implies $1 + \omega + \omega^2 + \dots + \omega^{k-1}$ is a unit in $\mathbb{Z}[\omega]$.

34. (b) The conjugates of $1 - \omega$ are $\omega^{kp^{r-1}} - 1$ for $1 \leq k \leq p-1$. By (a), $1 - \omega^k = \frac{1 - \omega}{1 + \omega + \dots + \omega^k}$, so

$$N(1 - \omega) = (1 - \omega)^n \left(\prod_{(j, p^r)=1} \sum_{i=0}^j \omega^i \right)^{-1}$$

By (a) the sum of the ω^i factors is a unit in $\mathbb{Z}[\omega]$, so the inverse of the product of each of these is also a unit, call it u . Therefore

$$N(1 - \omega) = u(1 - \omega)^n$$

However as $f(x) = 1 + x^{p^{r-1}} + \dots + x^{(p-1)p^{r-1}}$ is the p^r th cyclotomic polynomial, the norm of $1 - \omega$ is the constant coefficient of the polynomial $1 + (1 - x)^{p^{r-1}} + \dots + (1 - x)^{(p-1)p^{r-1}} = p$, and so $N(1 - \omega) = p$. Setting both sides equal to one another gives $p = u(1 - \omega)^n$.

35. (a) Let $\omega = e^{2\pi i/m}$ and $\theta = \omega + \omega^{-1}$. Then $\omega^2 - (\omega + \omega^{-1})(\omega) + 1 = 0$ and so ω is a root of the polynomial $x^2 + \theta x + 1$. $\omega \notin \mathbb{Q}[\theta]$, therefore $\mathbb{Q}[\omega] : \mathbb{Q}[\theta]$ has degree 2.
35. (b) Since $\theta = \omega + \omega^{-1} \in \mathbb{R}$, clearly $\mathbb{Q}[\theta] \subseteq \mathbb{Q}[\omega] \cap \mathbb{R}$. We therefore have the tower of field extensions $\mathbb{Q}[\theta] \subseteq \mathbb{Q}[\omega] \cap \mathbb{R} \subsetneq \mathbb{Q}[\omega]$. By (a), $[\mathbb{Q}[\omega] : \mathbb{Q}[\theta]] = 2$. By the Tower Law, $[\mathbb{R} \cap \mathbb{Q}[\omega] : \mathbb{Q}[\theta]]$ must be a divisor of 2 by distinct from 2 (since $\omega \notin \mathbb{R}$). Therefore the degree must be 1 and so $\mathbb{R} \cap \mathbb{Q}[\omega] = \mathbb{Q}[\theta]$.
35. (c) Let σ be the automorphism defined by $\sigma(\omega) = \omega^{-1}$. Then $\sigma(\theta) = \theta$, and so $\mathbb{Q}[\theta]$ is in the fixed field of the automorphism σ . As the degree of $\mathbb{Q}[\omega]$ over $\mathbb{Q}[\theta]$ is 2, there can be no distinct intermediate field between $\mathbb{Q}[\omega]$ and $\mathbb{Q}[\theta]$. $\mathbb{Q}[\omega]$ is not in the fixed field of σ and so $\mathbb{Q}[\theta]$ must be the fixed field of this automorphism.
35. (d) Show that $\mathbb{A} \cap \mathbb{Q}[\theta] = \mathbb{R} \cap \mathbb{Z}[\theta]$.

$$\begin{aligned}
\mathbb{A} \cap \mathbb{Q}[\theta] &= \mathbb{A} \cap (\mathbb{R} \cap \mathbb{Q}[\omega]) && \text{By 35 (b).} \\
&= (\mathbb{A} \cap \mathbb{Q}[\omega]) \cap \mathbb{R} && \text{By associativity of intersection} \\
&= \mathbb{Z}[\omega] \cap \mathbb{R} && \text{By Theorem 12, Corollary 2}
\end{aligned}$$

This is the required result.

35. (e) Let $n = \phi(m)/2$. The set $\{1, \omega, \omega^2, \dots, \omega^{n-1}, \omega^n, \omega^{n+1}, \dots, \omega^{m-1}\}$ is an integral basis for $\mathbb{Z}[\omega]$.

Since $\omega^{n-k} = \omega^{-k}$, we can write this basis as $\{1, \omega, \omega^{-1}, \omega^2, \omega^{-2}, \dots, \omega^{-n}\}$ instead (note $\omega^n = \omega^{-n}$). We examine the set $\{1, \omega, \theta, \theta\omega, \theta^2, \theta^2\omega, \dots, \theta^n\}$.

Now we pair up the expressions $\theta^k\omega$ with ω^{k+1} and θ^k with ω^{-k} :

$$\{1, \omega, \omega^{-1}, \omega^2, \omega^{-2}, \omega^3, \dots, \omega^n\} \quad (1)$$

$$\{1, \omega, \theta, \theta\omega, \theta^2, \theta^2\omega, \dots, \theta^{n-1}\omega\} \quad (2)$$

We evaluate the expression θ^k using the Binomial Theorem:

$$\theta^k = (\omega + \omega^{-1})^k = \sum_{i=0}^k \binom{k}{i} \omega^i \omega^{-(k-i)} = \sum_{i=0}^k \binom{k}{i} \omega^{2i-k}$$

Therefore

$$\theta^k \omega = \sum_{i=0}^k \binom{k}{i} \omega^{2i-k+1}$$

For θ^k , the power of ω ranges between $-k$ and k for θ^k , and it uses 1 term of the power ω^{-k} and no power of ω with absolute value greater than k .

For $\theta^k\omega$, the power of ω ranges between $-k+1$ and $k+1$ for $\theta^k\omega$. It uses 1 power of ω^k and no other power of ω with absolute value of greater than or equal to k .

Therefore, there is a lower triangular translation matrix A between the basis (1) and (2). A has all 1s in the diagonal, and so A has determinant 1 and is invertible over \mathbb{Z} . Since (1) is an integral basis of $\mathbb{Z}[\omega]$, so is (2).

$$A = \begin{matrix} & 1 & \omega & \omega^{-1} & \omega^2 & \omega^{-2} & \dots \\ \begin{matrix} 1 \\ \omega \\ \theta \\ \theta\omega \\ \theta^2 \\ \vdots \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & \dots \\ 2 & 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{matrix}$$

35. (f) Show that $\{1, \theta, \theta^2, \dots, \theta^{n-1}\}$ is an integral basis for $\mathbb{A} \cap \mathbb{Q}[\theta]$.

By (d), $\mathbb{A} \cap \mathbb{Q}[\theta] = \mathbb{R} \cap \mathbb{Z}[\theta]$, and by (e), any member α of $\mathbb{Z}[\theta]$ is expressible in terms of the basis vectors $\{1, \omega, \theta, \theta\omega, \theta^2, \dots\}$:

$$\beta = a_0 + a_1\omega + a_2\theta + a_3\theta\omega + \dots + a_{m-1}\theta^{n-1}$$

Since $\beta \in \mathbb{R}$, $\sigma(\beta) = \beta$ (where σ is complex conjugation). Therefore:

$$\begin{aligned} \beta &= \sigma(a_0 + a_1\omega + a_2\theta + a_3\theta\omega + \dots + a_{m-1}\theta^{n-1}) \\ &= \sigma(a_0) + \sigma(a_1\omega) + \sigma(a_2\theta) + \sigma(a_3\theta\omega) + \dots + \sigma(a_{m-1}\theta^{n-1}) \\ &= a_0 + a_1\sigma(\omega) + a_2\sigma(\theta) + a_3\theta\sigma(\omega) + \dots + a_{m-1}\theta^{n-1} \\ &= a_0 + a_1\omega^{-1} + a_2\theta + a_3\theta\sigma(\omega) + \dots + a_{m-1}\theta^{n-1} \end{aligned}$$

Since the elements of basis are linearly independent, each odd a_i must be equal to 0, and so β must be expressible as $a_0 + a_2\theta + \dots + a_{m-1}\theta^{n-1}$, and so $\mathbb{Q}[\theta]$ is an integral basis for $\mathbb{A} \cap \mathbb{Q}[\theta]$.

35. (g) Let p be an odd prime. Use exercise 23 to show that $\text{disc}(\theta) = \pm p^{(p-3)/2}$. Show the plus sign must hold.

By Exercise 23,

$$\begin{aligned} \text{disc}(1, \omega, \theta, \theta\omega, \dots, \theta^{n-1}) &= \text{disc}(\theta)^2 N_{\mathbb{Q}}^{\mathbb{Q}[\theta]} \text{disc}_{\mathbb{Q}[\theta]}^{\mathbb{Q}[\omega]}(\omega) \\ p^{p-2} &= \text{disc}(\theta)^2 N_{\mathbb{Q}}^{\mathbb{Q}[\theta]}(2\omega - \theta) \\ &= \text{disc}(\theta)^2 N_{\mathbb{Q}}^{\mathbb{Q}[\theta]}(\omega - \omega^{-1}) \\ &= \text{disc}(\theta)^2 N_{\mathbb{Q}}^{\mathbb{Q}[\theta]}(\omega^{-1}(\omega + 1)(\omega - 1)) \\ &= \text{disc}(\theta)^2 p \\ \pm p^{(p-3)/2} &= \text{disc}(\theta) \end{aligned}$$

As pointed out in the exercise, the square root of the discriminant is present in $\mathbb{Q}[\theta]$. Since $\mathbb{Q}[\theta] \subseteq \mathbb{R}$, $\text{disc}(\theta) = p^{(p-3)/2}$.

37. Let α be an algebraic integer of degree n over \mathbb{Q} and let f and g be polynomials over \mathbb{Q} , each of degree $< n$, such that $f(\alpha) = g(\alpha)$. Show $f = g$.

Let $h(x)$ be the minimal polynomial for α . If $f(\alpha) = g(\alpha)$, then $(f - g)(\alpha) = 0$. Since h is the minimum polynomial for α , $h \mid f - g$. However, $f - g$ has degree $< n$, and so $f - g = 0$. Therefore $f = g$.

40. (a) Show $\text{disc}(\alpha) = (d_1 d_2 \cdots d_{n-1})^2 \text{disc}(R)$.

We first show $\text{disc}(\alpha) = \text{disc}(1, f_1(\alpha), \dots, f_{n-1}(\alpha))$.

$$\text{disc}(\alpha) = \text{disc}(1, \alpha, \dots, \alpha^{n-1})$$

Since f_{n-1} is a monic polynomial with degree $n-1$ it is a linear combination of $\alpha, \dots, \alpha^{n-1}$, and so generate the same additive subgroup of R_k . By Exercise 26,

$$\text{disc}(1, \alpha, \dots, \alpha^{n-1}) = \text{disc}(1, \alpha, \dots, \alpha^{n-2}, f_{n-1}(\alpha))$$

Proceeding in this way we have

$$\text{disc}(\alpha) = \text{disc}(1, f_1(\alpha), \dots, f_{n-1}(\alpha))$$

Finally, we have

$$\begin{aligned} \text{disc}(R) &= \text{disc}(1, f_1(\alpha)/d_1, \dots, f_{n-1}(\alpha)/d_{n-1}) \\ &= \frac{1}{d_1^2 \cdots d_{n-1}^2} \text{disc}(1, f_1(\alpha)/d_1, \dots, f_{n-1}(\alpha)/d_{n-1}) \\ &= \frac{1}{(d_1 \cdots d_{n-1})^2} \text{disc}(\alpha) \end{aligned}$$

Multiplying both sides by $(d_1 \cdots d_{n-1})^2$ gives the required result.

40. (b) We show that $R_k/\mathbb{Z}[\alpha]$ has order d_1, \dots, d_k by induction on k . Since $R = R_{n-1}$ the result will follow by induction.

For the base case we see that $1/\mathbb{Z}[\alpha]$ has order 1. Next let $R_k = R_{k-1} \oplus \frac{1}{d_k} f_k(\alpha) \mathbb{Z}$, so

$$R_k/\mathbb{Z}[\alpha] = R_{k-1}/\mathbb{Z}[\alpha] \oplus \frac{1}{d_k} f_k(\alpha)/\mathbb{Z}[\alpha]$$

By induction $R_{k-1}/\mathbb{Z}[\alpha]$ has order $d_1 \cdots d_{k-1}$. f_k is a monic polynomial in α and so $f_k(\alpha) \in \mathbb{Z}[\alpha]$, therefore $\frac{1}{d_k} f_k(\alpha)/\mathbb{Z}[\alpha] = \frac{1}{d_k}$ which has order d_k , so the order of $R_k = d_1 \cdots d_k$.

40. (c) Show if $i + j < n$ then $d_i d_j \mid d_{i+j}$.

Since $f_i(\alpha)/d_i$ and $f_j(\alpha)/d_j$ are members of the ring R , $f_i(\alpha)f_j(\alpha)/d_i d_j$ must also be a member of the ring R . $f_i(\alpha)f_j(\alpha)$ has order $i + j$. Since

this is $< n$, this element by be generated by the basis elements of order $\leq i + j$. Let a_k be the integers that generate this element. Then

$$\begin{aligned}\frac{f_i(\alpha)f_j(\alpha)}{d_id_j} &= a_{i+j}\frac{f_{i+j}(\alpha)}{d_{i+j}} + \sum_{k=0}^{i+j-1} a_k \frac{f_k(\alpha)}{d_k} \\ f_i(\alpha)f_j(\alpha) &= a_{i+j}d_id_j\frac{f_{i+j}(\alpha)}{d_{i+j}} + \text{Lower terms}\end{aligned}$$

We know $a_{i+j} \neq 0$. Since f_i , f_j , and f_{i+j} are each monic, the denominator must cancel with no remainder, giving $d_{i+j} = a_{i+j}d_id_j$. Therefore $d_id_j \mid d_{i+j}$.

40. (d) Take $\frac{f_1(\alpha)}{d_1}$ as the basis element of order 1, and raise this element to the i -th power. Each $(\frac{f_1(\alpha)}{d_1})^i$ is a polynomial of order i and so generated by the basis element $\frac{f_i(\alpha)}{d_i}$. By a similar argument as in 40. (c) (each of these terms is a monic polynomial and so the denominators must cancel with no remainder), $d_1^i \mid d_i$.

Let j_i be the remainder left when dividing d_i by d_1^i ($j_1 = 1$). Then:

$$\begin{aligned}\text{disc}(\alpha) &= (d_1 \cdots d_{n-1})^2 \text{disc}(R) \\ &= (d_1 d_1^2 \cdots d_1^{n-1} \prod_{i=0}^{n-1} j_i)^2 \text{disc}(R) \\ &= (d_1^{n(n-1)/2})^2 (\prod_{i=0}^{n-1} j_i)^2 \text{disc}(R) \\ &= d_1^{n(n-1)} (\prod_{i=0}^{n-1} j_i)^2 \text{disc}(R)\end{aligned}$$

Therefore $d^{n(n-1)} \mid \text{disc}(\alpha)$.

41. (a) Let m be a cubefree integer, $\alpha = \sqrt[3]{m}$, and write m as hk^2 with h, k relatively prime. Let $R = \mathbb{A} \cap \mathbb{Q}[\alpha]$. (Therefore k^2 has any square factors of m). Show $\text{disc}(\alpha) = -27m^2$ (the 2018 edition has a typo).

Let $f(x) = x^3 - m$; $f(x)$ is the minimum polynomial of α over \mathbb{Q} and has degree 3 (not $\equiv 0, 1 \pmod{4}$), so $\text{disc}(\alpha) = -N(f'(\alpha))$. $f'(\alpha) = 3\alpha^2$ so $\alpha f'(\alpha) = 3m$ and $f'(\alpha) = 3m/\alpha$. Note $N(\alpha) = m$ so $N(\alpha^{-1}) = 1/m$. Therefore we have

$$\begin{aligned}N(3m/\alpha) &= 27m^3 N(\alpha^{-1}) = 27m^2 \\ \text{disc}(\alpha) &= -27m^2\end{aligned}$$

Using Exercise 40, we see $-27m^2 = (d_1 d_2)^2 \text{disc}(R)$ and $d_1^2 \mid d_2$, so writing $d_2 = d_1^2 j$, we have

$$-27m^2 = d_1^4 j^2 \text{disc}(R)$$

Since d_1 has a sextic factor on the righthand-side, the only possibilities for d_1 are 1 or 3. If $d_1 = 3$ then $9 \mid m$.

41. (b) Show $d_1 = 1$ even when $9 \mid m$.

Suppose $9 \mid m$ and $d_1 = 3$. Then R has an integral basis with 1 and $(\alpha + a)/3$ as two of the three basis vectors.

Let $\beta = (\alpha + a)/3$ for some integer a . As suggested in the exercise hint we consider the trace of β^3 . First, we determine the trace of α and α^2 as these will be important to evaluate $\text{Tr}(\beta)$.

$$\begin{aligned}\text{Tr}(\alpha) &= \alpha + \omega\alpha + \omega^2\alpha = \alpha(\omega^2 + \omega + 1) = 0 \\ \text{Tr}(\alpha^2) &= \alpha^2 + \omega^2\alpha^2 + \omega\alpha^2 = \alpha^2(\omega^2 + \omega + 1) = 0\end{aligned}$$

With these in hand we now have

$$\beta^3 = \frac{(\alpha + a)^3}{27} = \frac{m + 3\alpha^2a + 3a^2\alpha + a^3}{27}$$

By the additive linearity of trace, we have

$$\begin{aligned}\text{Tr}(\beta^3) &= \frac{m}{9} + \frac{3a}{27}\text{Tr}(\alpha^2) + \frac{3a^2}{27}\text{Tr}(\alpha) + \frac{3a^3}{27} \\ &= \frac{m}{9} + \frac{3a^3}{27} \\ &= \text{Integer} + \frac{3a^3}{27}\end{aligned}$$

Since β is an algebraic integer, β^3 is also an algebraic integer, and its trace must be a member of \mathbb{Z} . Therefore $\frac{3a^3}{27}$ must be an integer, and so 27 must divide $3a^3$, which means that 9 divides a^3 and so 3 divides a .

Since 3 divides a , $\frac{\alpha+a}{3} = \frac{\alpha}{3} + \text{Integer}$. Therefore $\alpha/3$ is a member of R , so $(\alpha/3)^3 = m/27 \in R$. However, m is cubefree and so $m/27 \notin \mathbb{Z}$. This contradicts Corollary 1 of Theorem 1 - the only members of \mathbb{Q} that are algebraic integers are members of \mathbb{Z} .

Therefore $d_1 = 1$ in all cases, and so R has a basis containing 1 and α . The third basis vector has yet to be determined.

41. (c) Write m as hk^2 . Then $(\alpha^2/k)^3 = m^2/k^3 = (h^2k^4)(k^3) = h^2k$, and so α^2/k is the root of the polynomial $f(x) = x^3 - h^2k$, and so $\alpha^2/k \in R$.
41. (d) Suppose $m \equiv \pm 1 \pmod{9}$. Let $\beta = (\alpha \mp 1)^2/3$. Show that

$$\beta^3 - \beta^2 + \frac{1 \pm 2m}{3}\beta - \frac{(m \mp 1)^2}{27} = 0$$

As suggested we calculate $(\beta - 1/3)^3$ in two ways:

$$\begin{aligned}
(\beta - 1/3)^3 &= ((\alpha \mp 1)^2/3 - 1/3)^3 \\
\beta^3 - \frac{3\beta^2}{3} + \frac{3\beta}{9} - \frac{1}{27} &= \frac{(\alpha(\alpha \mp 2))^3}{27} \\
\beta^3 - \beta^2 + \frac{\beta}{3} - \frac{1}{27} &= m \left(\frac{m \mp 6\alpha^2 + 12\alpha \mp 8}{27} \right) \\
\beta^3 - \beta^2 + \frac{\beta}{3} - \frac{m^2 \mp 2m + 1}{27} &= m \left(\frac{\mp 6\alpha^2 + 12\alpha \mp 6}{27} \right) \\
\beta^3 - \beta^2 + \frac{\beta}{3} - \frac{(m \mp 1)^2}{27} &= \mp \frac{2m}{3} \left(\frac{\alpha^2 \pm 2\alpha + 1}{3} \right) = \mp \frac{2m}{3} \beta
\end{aligned}$$

Moving the terms around, we have the required result:

$$\beta^3 - \beta^2 + \frac{1 \pm 2m}{3} \beta - \frac{(m \mp 1)^2}{27} = 0$$

Since $m \equiv \pm 1 \pmod{9}$, $1 \pm 2m$ is divisible by 3, and $m \mp 1$ is divisible by 9, so $(m \mp 1)^2$ is divisible by 27. Therefore β is the root of a monic polynomial with integer coefficients and so $\beta \in R$.

41. (e) Using (c) and (d), show that if $m \equiv \pm 1 \pmod{9}$ then

$$\frac{\alpha^2 \pm k^2\alpha + k^2}{3k} \in R$$

Since $\alpha^2/k \in R$, we can add $k\alpha + k$ to the element to see that

$$\frac{\alpha^2 + k^2\alpha + k^2}{k} \in R$$

Next, observe that $k^2 \equiv 1 \pmod{3}$ - it cannot be 0 since $m \equiv \pm 1 \pmod{9}$. Therefore $(k^2 - 1)/3$ and $(k^2 + 2)/3$ are integers. Taking $(\alpha^2 \mp 2\alpha + 1)/3$, we add $(k^2 - 1)/3$ to see that

$$\frac{\alpha^2 \mp 2\alpha + k^2}{3} \in R$$

Next we have

$$\frac{\alpha^2 \mp 2\alpha + k^2}{3} \pm \frac{\alpha(k^2 - 2)}{3} = \frac{\alpha^2 \pm k^2\alpha + k^2}{3} \in R$$

Since $3 \nmid k$ and 3 is a prime, there exist integers a, b such that $3a + bk = 1$. Therefore

$$\begin{aligned}
b \left(\frac{\alpha^2 \pm k^2\alpha + k^2}{3} \right) + a \left(\frac{\alpha^2 \pm k^2\alpha + k^2}{k} \right) &= \frac{(kb + 3a)(\alpha^2 \pm k^2\alpha + k^2)}{3k} \\
&= \frac{\alpha^2 \pm k^2\alpha + k^2}{3k} \in R
\end{aligned}$$

This is the required result.

41. (f) We have $\text{disc}(\alpha) = -27m^2$. By Exercise 40(a), $d_2^2 \text{disc}(R) = \text{disc}(\alpha) = -27m^2 = -27h^2k^4$. We know $k \mid d_2$ so write $d_2 = jk$, thus $j^2k^2 \text{disc}(R) = -27h^2k^4$ and so $j^2 \text{disc}(R) = -27h^2k^2 = -27mh$. By assumption h is square-free, so $j^2 \mid -27m$, implying either $j \mid 3$ or $j \mid m$. Therefore $j \mid 3m$.
41. (g) Letting p be a prime such that $p \neq 3$, $p \mid m$, $p^2 \mid m$. Let $p \mid d_2$, and write $d_2 = pj$. Therefore if $(\alpha^2 + a\alpha + b)/d_2 \in R$, then

$$j(\alpha^2 + a\alpha + b)/d_2 = (\alpha^2 + a\alpha + b)/p \in R$$

Since $(\alpha^2 + a\alpha + b)/p \in R$, its trace must be an integer; however $\text{Tr}(\alpha^2) = \text{Tr}(\alpha) = 0$, and so $3b/p \in \mathbb{Z}$. $p \neq 3$, therefore $p \mid b$. Therefore $(\alpha^2 + a\alpha)/p \in R$.

$$\text{Tr}(((\alpha^2 + a\alpha)/p)^3) = \text{Tr}((m^2 + a^3m)/p^3)$$

Therefore $p^3 \mid 3(m^2 + a^3m)$. Since $p \neq 3$, $p^3 \mid m(m + a^3)$. m is cubefree and $p^2 \nmid m$, so $p^2 \mid m + a^3$. Therefore $a^3 \equiv 0 \pmod{p}$, meaning $a \equiv 0 \pmod{p}$. Considering the equation modulo p^2 we then have $m \equiv 0 \pmod{p^2}$, a contradiction. Therefore this case is impossible.

41. (h) Let $p \neq 3$ and $p^2 \mid m$. By the previous problem $(\alpha^2 + a\alpha)/p^2 \in R$ and so we consider the trace:

$$\text{Tr}(((\alpha^2 + a\alpha)/p^2)^3) = \text{Tr}((m^2 + a^3m)/p^6)$$

Therefore $p^6 \mid m(m + a^3)$. Since $p^2 \mid m$, $p^4 \mid m + a^3$. Considering the equation modulo p^2 , we must have $a^3 \equiv 0 \pmod{p^2}$, so $p^2 \mid a^3$. Therefore $p \mid a$ and so $p^3 \mid a^3$. Therefore $m + a^3 \equiv 0 \pmod{p^3}$ and so $m \equiv 0 \pmod{p^3}$ again contradicting m cubefree.

Together with (g) this shows that d_2 has no common prime factor with m that is not equal to 3.

41. (i) Take $(\alpha^2 + a\alpha + b)/d_2$.

$$\begin{aligned} \frac{(\alpha^2 + a\alpha + b)^2}{d_2^2} &= \frac{m\alpha + 2am + 2\alpha^2b + a^2\alpha^2 + 2ab\alpha + b^2}{d_2^2} \\ &= \frac{\alpha^2(a^2 + 2b) + \alpha(m + 2ab) + (2am + b^2)}{d_2^2} \end{aligned}$$

Since this is an element of the ring and the basis element of order 2 has denominator d_2 , d_2 must divide each of $a^2 + 2b$, $m + 2ab$, and $2am + b^2$.

41. (j) We now consider what power of 3 divides d_2 . We know $d_2 \mid 3m$. If $3 \nmid m$, then $9 \nmid d_2$. Therefore, if $m \equiv \pm 1 \pmod{9}$, $d_2 = 3k$; it cannot be any non-3 prime dividing m by (g) and (h), and 9 does not divide m .

We now consider the case where $m \not\equiv \pm 1 \pmod{9}$ and $3 \nmid m$. We assume $3 \mid d_2$ (to show a contradiction).

We evaluate the congruences obtained in (i) modulo 3. Since $a^2 + 2b \equiv 0 \pmod{3}$, $a^2 - b \equiv 0 \pmod{3}$, and so $b \equiv a^2 \pmod{3}$. Substituting b with a^2 in the equation $m + 2ab \equiv 0 \pmod{3}$, we have $m + 2a^3 \equiv 0 \pmod{3}$ and so $m - a^3 \equiv m - a \equiv 0 \pmod{3}$, so therefore $a \equiv m \pmod{3}$. Substituting m for a in the equivalence $b^2 + 2am \equiv 0 \pmod{3}$, we have $b^2 \equiv -2a^2 \equiv a^2 \pmod{3}$. Therefore since $a^2 + 2b \equiv 0 \pmod{3}$, we have $b(b + 2) \equiv b(b - 1) \equiv 0 \pmod{3}$. $b \not\equiv 0 \pmod{3}$ (as this would imply $m \equiv 0 \pmod{3}$) so we must have $b \equiv 1 \pmod{3}$.

Therefore we can write the basis element of order 2 as $\frac{\alpha^2 + (m+3l)\alpha + (3j+1)}{3i}$ for some i, l, j , and so by multiplying through by i and subtracting the term $3l\alpha + 3j$ from the resulting fraction, we have:

$$\frac{\alpha^2 + m\alpha + 1}{3} \in R$$

We now proceed by case on m congruence to 3. (Almost there!)

Suppose $m \equiv 1 \pmod{3}$. Then $\frac{\alpha^2 + \alpha + 1}{3} \in R$ and so by subtracting α , $\frac{\alpha^2 - 2\alpha + 1}{3} = \frac{(\alpha - 1)^2}{3} \in R$.

We raise this to the fourth power and take the trace. The only terms that contribute to the trace are those where α is raised to a power divisible by 3, so we have:

$$\begin{aligned} \text{Tr}\left(\frac{(\alpha - 1)^8}{3^4}\right) &= \frac{3}{3^4} \left(\binom{8}{6} \alpha^6 (-1)^2 + \binom{8}{3} \alpha^3 (-1)^5 + (-1)^8 \right) \\ &= \frac{1}{27} (28m^2 - 56m + 1) \end{aligned}$$

Therefore, 27 must divide $28m^2 - 56m + 1$. Congruent to 9, this equation reduces to $m^2 - 2m + 1 \equiv 0 \pmod{9}$ so $(m - 1)^2 \equiv 0 \pmod{9}$ and $m \equiv 1 \pmod{9}$. This contradicts $m \not\equiv \pm 1 \pmod{9}$. So m cannot be congruent to 1 mod 3.

Next, suppose $m \equiv 2 \pmod{3}$. Therefore $\frac{\alpha^2 + 2\alpha + 1}{3} = \frac{(\alpha + 1)^2}{3} \in R$. Again we raise this to the fourth power and take the trace. (The equation is the same except for the negative terms.)

$$\text{Tr}\left(\frac{(\alpha + 1)^8}{3^4}\right) = \frac{1}{27} (28m^2 + 56m + 1)$$

Modulo 9 we have $m^2 + 2m + 1 \equiv 0 \pmod{9}$ so $(m + 1)^2 \equiv 0 \pmod{9}$ and so $m \equiv -1 \pmod{9}$, again contradicting $m \not\equiv \pm 1 \pmod{9}$.

Therefore if $3 \nmid m$ and $m \not\equiv \pm 1 \pmod{9}$, $3 \nmid d_2$.

41. (k) Suppose $3 \mid m$ but $9 \nmid m$. We assume $3 \mid d_2$ to show a contradiction. By (i), $a^2 + 2b \equiv 0 \pmod{3}$, so $a^2 \equiv b \pmod{3}$ (*). Plugging this into $m + 2ab \equiv 0 \pmod{3}$ we have $m - a^3 \equiv 0 \pmod{3}$. Since $a^3 \equiv a \pmod{3}$, we thus have $m \equiv a \pmod{3}$ and so $a \equiv 0 \pmod{3}$, and also $b \equiv 0 \pmod{3}$ by (*).

Therefore we can write the basis element of order 2 as $\frac{\alpha^2+3i\alpha+3j}{3^l}$, and by multiplying through by l and subtracting $i\alpha + j$, we have $\frac{\alpha^2}{3} \in R$. Cubing this element and taking the trace we must have $m^2/9 \in \mathbb{Z}$, contradicting $9 \nmid m$. Therefore $3 \nmid d_2$.

41. (1) Suppose $9 \mid m$. We show $9 \nmid d_2$. Assume $9 \mid d_2$ (to show a contradiction). By (i), $9 \mid ab$ and $9 \mid b^2$ so either $9 \mid b$ or $3 \mid b$. Assume $3 \mid b$, therefore since $a^2 + 2b \equiv 0 \pmod{9}$, we must have $a^2 \equiv -6 \equiv 3 \pmod{9}$. However, 3 is not the square of any element mod 9, so this equation is unsatisfiable. We must have $9 \mid b$.

Therefore, $(a^2 + a\alpha)/9 \in R$. Taking this to the third power and considering the trace, we must have $9^3 \mid 3(m^2 + ma^3)$ and $9^2 \mid m(m + a^3)$. Since m is cube-free and $9 \mid m$, therefore $27 \mid m + a^3$. Considering $m + a^3$ modulo 9, we have $a^3 \equiv 0 \pmod{9}$; therefore a must be congruent to 0, 3, or 6 modulo 9. In all these cases we have $a^2 \equiv 0 \pmod{9}$. Since $9^2 \mid a^3$ and $9^2 \mid (m + a^3)$, $9^2 \mid m$, which contradicts m being cube-free. Therefore $9 \nmid d_2$.

43. (a) Let $f(x) = x^5 + ax + b$ with $a, b \in \mathbb{Z}$ and f irreducible over \mathbb{Q} . Let α be a root of f . Show $\text{disc}(\alpha) = 4^4 a^5 + 5^5 b^4$.

We proceed in a similar fashion to Exercise 28: first, we determine $f'(\alpha)$, then we determine $N(f'(\alpha))$ by collecting the most and least significant the coefficients of its polynomial.

$f'(x) = 5x^4 + a$, so $\alpha f'(x) = 5\alpha^5 + a = -5(a\alpha + b) + a = -4a\alpha - 5b$ and $f'(\alpha) = (-4a\alpha - 5b)/\alpha$. The expression $4a\alpha + 5b$ is a root of the polynomial $(\frac{x-5b}{4a})^5 + a(\frac{x-5b}{4a}) + b$. The norm $N(4a\alpha + 5b)$ is the negative of the x^0 coefficient divided by the x^5 coefficient (again, negative because 5 is odd), so we calculate those values.

The x^0 coefficient is $(\frac{-5b}{4a})^5 + a(\frac{-5b}{4a}) + b = (\frac{-5b}{4a})^5 + \frac{-b}{4}$, and the x^5 coefficient is $(\frac{1}{4a})^5$, so $N(4a\alpha + 5b) = 5^5 b^5 + 4^4 a^5 b$.

Therefore,

$$\text{disc}(\alpha) = N(-(4a\alpha + 5b)/\alpha) = -\frac{5^5 b^5 + 4^4 a^5 b}{-b} = 5^5 b^4 + 4^4 a^5$$

This is the required result. (The plus sign for the discriminant holds because $5 \equiv 1 \pmod{4}$)

43. (b) Suppose $\alpha^5 = \alpha + 1$. We are given that this polynomial is irreducible because it is irreducible modulo 3. (The options are 0, 1, and 2: $0^5 \not\equiv 0 + 1 \pmod{3}$, $1^5 \not\equiv 1 + 1 \pmod{3}$, and $2^5 = 2 \not\equiv 1 + 2 = 0 \pmod{3}$.)

In this case $a = -1$ and $b = -1$ so the above formula gives $\text{disc}(\alpha) = 5^5 - 4^4 = 125 \cdot 25 - 16 \cdot 16 = 2869 = 19 \cdot 151$. Since the discriminant is squarefree, $\mathbb{A} \cap \mathbb{Q}[\alpha] = \mathbb{Z}[\alpha]$.

43. (c) Let a be squarefree and not equal to ± 1 . Let α be a root and d_1, d_2, d_3, d_4 be as in Theorem 13. Prove that if $4^4a + 5^5$ is squarefree then $d_1 = d_2 = 1$ and $d_3d_4 \mid a^2$.

By exercise 40,

$$\text{disc}(\alpha) = 5^5a^4 + 4^4a^5 = a^4(5^5 + 4^4a) = (d_1d_2d_3d_4)^2\text{disc}(R)$$

Here $d_1d_2 \mid d_3$, $d_1d_2 \mid d_4$, and $d_1d_3 \mid d_4$. Therefore d_1 and d_2 both have 6 factors represented in the $\text{disc}(\alpha)$ expression which is impossible unless they are both 1. Since $5^5 + 4^4a$ is squarefree, $(d_3d_4)^2$ must divide a^4 and so $d_3d_4 \mid a^2$.

Verify that $4^4a + 5^5$ is squarefree when $a = -2, -3, -6, -7, -10, -11, -13$, and -15 .

```
sage: [(factor(x), is_squarefree(x)) for x in
      map(lambda a: 5^5 + 4^4 * a,
          [-2, -3, -6, -7, -10, -11, -13, -15])]
```

```
[(3 * 13 * 67, True),
 (2357, True),
 (7 * 227, True),
 (31 * 43, True),
 (5 * 113, True),
 (3 * 103, True),
 (-1 * 7 * 29, True),
 (-1 * 5 * 11 * 13, True)]
```

Experimenting a bit more with Sage, we can quickly test integers using the following code:

```
sage: def test_poly_degree_5(a):
....:     return (is_squarefree(5^5 + 4^4 * a) and
....:             is_squarefree(a))
....:
sage: filter(lambda x: test_poly_degree_5(x),
....:         range(2, 30))
[2, 3, 5, 6, 7, 10, 11, 14, 15, 17, 19, 21, 23, 26, 29]
sage: filter(lambda x: test_poly_degree_5(x),
....:         range(-2, -30, -1))
[-2, -3, -6, -7, -10, -11, -13, -15, -17, -19, -21,
 -22, -26, -29]
```

43. (d) Let α be as in part (c) (α is the root of a polynomial $f(x) = x^5 + ax + a$). Show $\alpha + 1$ is a unit.

We have $\alpha^5 = -a(\alpha + 1)$, so we take the norm of both sides. $N(\alpha^5) = -a^5 = N(-a)N(\alpha + 1) = -a^5N(\alpha + 1)$, so $N(\alpha + 1) = 1$. Therefore $\alpha + 1$ is a unit in $\mathbb{A} \cap \mathbb{Q}[\alpha]$.

44. (a) Let $f(x) = x^5 + ax^4 + b$ where $a, b \in \mathbb{Z}$, and let α be a root of f . To determine the discriminant of α , we proceed as in exercise 28 and 43. The derivative of $f(x)$ is $f'(x) = 5x^4 + 4ax^3$, so

$$f'(\alpha) = \alpha^3(5\alpha + 4a)$$

$N(\alpha^3) = -b^3$ so determine the norm of $5\alpha + 4a$ by observing it is the root of the polynomial $(\frac{x-4a}{5})^5 + (\frac{x-4a}{5})^4 + b$. The x^0 term is $(\frac{-4a}{5})^5 + (\frac{-4a}{5})^4 + b$ while the x^5 term is $\frac{1}{5^5}$,

$$N(5\alpha + 4a) = (4a)^5 - 5a(4a)^4 - 5^5b = -(4a)^5 \cdot (-4 + 5) - 5^5b = -(4^5a^5 + 5^5b)$$

Therefore $\text{disc}(\alpha) = (4^5a^5 + 5^5b)b^3$ as required (the discriminant is positive since $5 \equiv 1 \pmod{4}$).

44. (b) TODO

45. Let α be the root of the polynomial $f(x) = x^n + ax + b$. Find a formula for $\text{disc}(\alpha)$.

We proceed in similar fashion to exercise 43. $f'(\alpha) = n\alpha^{n-1} + a$, so we have:

$$\begin{aligned} \alpha f'(\alpha) &= n\alpha + a\alpha \\ &= -n(a\alpha + b) + a\alpha \\ &= -((n-1)a\alpha + bn) \\ f'(\alpha) &= -((n-1)a\alpha + bn)/\alpha \end{aligned}$$

We now calculate $N((n-1)a\alpha + bn)$. This is the root of the polynomial

$$g(x) = \left(\frac{x - bn}{(n-1)a} \right)^n + a \left(\frac{x - bn}{(n-1)a} \right) + b$$

The norm is equal to $(-1)^n$ times the x_0 coordinate multiplied by the inverse of x_n coordinate. Therefore,

$$N((n-1)a\alpha + bn) = (bn)^n + (-1)^{n+1}a^n b(n-1)^{n-1}$$

The inverse of the x_n coordinate is seen to be $((n-1)a)^n$

The discriminant is then (with the \pm positive if $n \equiv 0, 1 \pmod{4}$, negative otherwise):

$$\begin{aligned} \text{disc}(\alpha) &= \frac{\pm(-1)^n N((n-1)a\alpha + bn)}{b(-1)^n} \\ &= \frac{\pm(bn)^n + (-1)^{n+1}a^n b(n-1)^{n-1}}{b} \\ &= \pm[b^{n-1}n^n + (-1)^{n+1}a^n(n-1)^{n-1}] \end{aligned}$$

Plugging values in gives:

$$\begin{aligned}n = 2 &= -(2^2b - a^2) = a^2 - 4b \\n = 3 &= -(27b^2) + a^32^2 = -27b^2 + 4a^3 \\n = 4 &= b^34^4 - a^43^3 = 256b^3 - 27a^4 \\n = 5 &= b^45^5 + a^54^4\end{aligned}$$

These agree with the known values of these polynomials.

Next, we calculate $\text{disc}(\alpha)$ if α is a root of $x^n + ax^{n-1} + b$. The derivative $f'(\alpha) = n\alpha^{n-1} + a(n-1)\alpha^{n-2} = \alpha^{n-2}(\alpha n + a(n-1))$, so

$$\text{disc}(\alpha) = \pm N(f'(\alpha)) = \pm N(\alpha^{n-2})N(n\alpha + (n-1)a)$$

The norm $N(\alpha^{n-2}) = N(\alpha)^{n-2} = (-1)^n b^{n-2}$, so we only need to calculate $N(n\alpha + (n-1)a)$. This is a root of the polynomial

$$\left(\frac{x - (n-1)a}{n}\right)^n + a\left(\frac{x - (n-1)a}{n}\right)^{n-1} + b$$

We now calculate the norm of this. The x_n coefficient is $\frac{1}{n^n}$, and the x_0 coefficient is

$$\left(-\frac{(n-1)a}{n}\right)^n + a\left(-\frac{(n-1)a}{n}\right)^{n-1} + b$$

Multiplying through by n^n gives us:

$$\begin{aligned}N(n\alpha + (n-1)a) &= (-1)^n[(-1)^n(n-1)^na^n + (-1)^{n-1}a^n(n-1)^{n-1}n + bn^n] \\&= (n-1)^na^n - a^n(n-1)^{n-1}n + (-1)^nbn^n \\&= a^n(n-1)^{n-1}(n-1-n) + (-1)^nbn^n \\&= -a^n(n-1)^{n-1} + (-1)^nbn^n\end{aligned}$$

Multiplying the norm by $(-1)^nb^{n-2}$ we have

$$\text{disc}(\alpha) = \pm[bn^n + (-1)^{n-1}a^n(n-1)^{n-1}]b^{n-2}$$

This agrees with the answer to Exercise 44 (a) ($n = 5$) and I confirmed via Sage that the formula holds for some examples where $n = 4$ and $n = 6$:

```
sage: a = 4; b = -7; n = 4
sage: K.<g> = QQ.extension(x^4 + a*x^3 + b)
sage: K.disc([1, g, g^2, g^3])
-426496
sage: (b*n^n - a^n * (n - 1)^(n - 1))*b^(n-2)
-426496
sage: a = 3; b = -5; n = 6
sage: K.<g> = QQ.extension(x^6 + a*x^5 + b)
sage: K.disc([1, g, g^2, g^3, g^4, g^5])
1569628125
sage: -(b*n^n - a^n * (n - 1)^(n - 1))*b^(n-2)
1569628125
```


Chapter 3

2. Prove that every finite integral domain D is a field.

For $\alpha \in D$, consider the set $\{1, \alpha, \alpha^2, \dots\}$. Since D is finite this set must also be finite, so there must be some i, j , $i \neq j$ such that $\alpha^i = \alpha^j$. Thus $\alpha^{j-i} = 1$, and $\alpha^{j-i-1}\alpha = \alpha^{j-i} = 1$, so every element in D has an inverse, and D is therefore a field.

3. Let G be a free abelian group of rank n , with additive notation. Show for any $m \in \mathbb{Z}$, G/mG is the direct sum of n cyclic group of order m .

Since G is a free abelian group of rank n ,

$$G \simeq \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{n \text{ copies}}$$

Therefore

$$G/mG \simeq \underbrace{\mathbb{Z}/m\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m\mathbb{Z}}_{n \text{ copies}}$$

Each $\mathbb{Z}/m\mathbb{Z}$ is a cyclic group of order m , so the order of G/mG is m^n .

4. Let K be any number field of degree n over \mathbb{Q} . Prove that every nonzero ideal I in $R = \mathbb{A} \cap K$ is a free abelian group of rank n .

As an additive subgroup of R , I must be a free abelian group of order $\leq n$. Let $\{\beta_1, \dots, \beta_n\}$ be a basis for R , and take $\alpha \in I$. $\{\alpha\beta_1, \dots, \alpha\beta_n\} \subseteq I \subseteq R$ is a free abelian group of order n . Since I contains αI , the rank of I must also be n .

7. If $I + J = 1$ then there exist $\alpha \in I, \beta \in J$ such that $\alpha + \beta = 1$. Raising both powers to the $m + n$ th power, we have $(\alpha + \beta)^{m+n} = 1^{m+n} = 1$. By the binomial theorem, $(\alpha + \beta)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n-k}{k} \alpha^{m+n-k} \beta^k$. If $k < n$, this element is a member of I^m (as $\alpha^{n+\text{positive}} \in I^m$); otherwise this element is a member of J^n . Therefore $(\alpha + \beta)^{m+n} \in I^m + J^n$.
8. (a) Suppose $I = (2, x)$ was generated by some $\alpha \in I$. Therefore there are $\beta, \gamma \in \mathbb{Z}[x]$ such that $\alpha\beta = 2$ and $\alpha\gamma = x$. Since $\alpha\beta = 2$, the rank of α must be 0; $\alpha \in \mathbb{Z}$. The only option is $\alpha = 2$ (since $1 \notin I$). However 2 is not a factor of x in $\mathbb{Z}[x]$. Therefore the ideal $(2, x)$ is not principal in $\mathbb{Z}[x]$.
8. (b) Let $f, g \in \mathbb{Z}[x]$ and let m, n be the gcd of the coefficients of f and g respectively. Prove mn is the gcd of the coefficients of fg .

Since m and n are the gcds of f and g we can write

$$f = m(a_0 + a_1x + \dots + a_jx^j) \tag{3}$$

$$g = n(b_0 + b_1x + \dots + b_kx^k) \tag{4}$$

where $(a_0, \dots, a_j) = 1$ and $(b_0, \dots, b_k) = 1$. Let d be the GCD of the coefficients of fg . As

$$fg = mn \left(\sum_{0 \leq l \leq j} \sum_{0 \leq m \leq k} a_l b_m \right)$$

we know that $mn \mid d$.

Suppose there is some prime p such that p divides $a_l b_m$ for all l, m . Since $(a_0, \dots, a_j) = 1$ and $(b_0, \dots, b_k) = 1$, there is some first a_l and first b_m such that $p \nmid a_l$ and $p \nmid b_m$; so $p \mid a_0, \dots, a_{l-1}$ but $p \nmid a_l$ and similarly $p \mid b_0, \dots, b_{m-1}$ but $p \nmid b_m$. The x^{l+m} term in fg has coefficient $a_l b_m + a_{l+1} b_{m-1} + \dots + a_{l-1} b_{m+1} + \dots$. Taken modulo p , $a_l b_m \not\equiv 0 \pmod{p}$ but p divides every other term in the expansion. This contradicts p being dividing the sum, and so there must be no other factor d beyond mn .

8. (c) Let $f \in \mathbb{Z}[x]$ be irreducible over \mathbb{Z} . Show f is irreducible over \mathbb{Q} .

Suppose f is irreducible over \mathbb{Z} but reducible over \mathbb{Q} , i.e. $f = gh$ for $g, h \in \mathbb{Q}[x]$. Then we can pull out the denominators from g, h , giving us $gh = \frac{g'h'}{d}$ where $g', h' \in \mathbb{Z}[x]$. Let a and b be the GCD of the coefficients of g' and h' respectively. We must have $ab \mid d$ because otherwise then f would be reducible into the product of two polynomials in $\mathbb{Z}[x]$. Therefore, reducing to lowest terms if necessary, we have $ab \nmid d$. However, multiplying both sides of the equation by d gives $df = g'h' = ab(g''h'')$ for some g'' and h'' and so by (b), $ab \mid d$; this is a contradiction. Therefore f must be also irreducible over $\mathbb{Q}[x]$.

9. Let K and L be number fields, $K \subset L$, $R = \mathbb{A} \cap K$, $S = \mathbb{A} \cap L$.

9. (a) - TODO Let I and J be ideals in R and suppose $IS \mid JS$. Show $I \mid J$.

10. Show e and f are multiplicative in terms of towers.

Let $K \subset L \subset M$ and $R \subset S \subset T$ be the associated number fields and $P \subset Q \subset U$ prime ideals.

f is multiplicative: By the third isomorphism theorem, there is the field series of field inclusions: $R/P \rightarrow S/Q \rightarrow T/U$. $[S/Q : R/P] = f(Q|P)$ and $[T/U : S/Q] = f(U|Q)$; therefore the composition map from $R/P \rightarrow T/U$ must have degree $f(U|P) = f(Q|P)f(U|Q)$ by the tower law for field extensions.

e is multiplicative: $P = Q^{e(Q|P)}I$ and $Q = U^{e(U|Q)}J$ for ideals I, J such that $I + Q$ and $J + U$ are relatively prime. Therefore $P = U^{e(U|Q)e(Q|P)}IJ$ with $U^{e(Q|P)e(U|Q)}$ and IJ relatively prime; the factor of U dividing P is $e(U|P)$ so $e(U|P) = e(U|Q)e(Q|P)$.

11. Since $\alpha \in I$, $I \mid (\alpha)$, and so $I \cdot J = (\alpha)$. Taking norms of both sides, $\|I\| \cdot \|J\| = \|(\alpha)\|$. By Theorem 22 (c), $\|(\alpha)\| = N_{\mathbb{Q}}^K(\alpha)$, so $\|I\| \mid N_{\mathbb{Q}}^K(\alpha)$, with equality holding if I is principal.

12. (a) Verify that $5S = (5, \alpha + 2)(5, \alpha^2 + 3\alpha - 1)$ in $S = \mathbb{Z}[\sqrt[3]{2}]$, $\alpha = \sqrt[3]{2}$.
Let $I = (5, \alpha + 2)(5, \alpha^2 + 3\alpha - 1)$. The generators of I are:

$$5^2 \quad (1)$$

$$5(\alpha^2 + 3\alpha - 1) \quad (2)$$

$$5(\alpha + 2) \quad (3)$$

$$(\alpha + 2)(\alpha^2 + 3\alpha - 1) = \alpha^3 + (3 + 2)\alpha^2 + (-1 + 6)\alpha - 2 = 5(\alpha^2 + \alpha) \quad (4)$$

All generators have a factor of 5 so $1 \notin I$; therefore $5 \subset I$. We have $\alpha \cdot (3) - (1) + 3 \cdot (2) = 45$, so $\gcd(45, 5^2) = 5 \in I$. Therefore $(3) - 10 = 5\alpha \in I$ and also $5\alpha^2 \in I$ by subtracting factors from (2); therefore $I = 5S$.

12. (b) Show there is an isomorphism between $\mathbb{Z}[x]/(5, x^2 + 3x - 1)$ and $\mathbb{Z}_5[x]/(x^2 + 3x - 1)$.

Let $a \in \mathbb{Z}[x]/(5, x^2 + 3x - 1)$. Then a can be associated with a coset representative $f(x) + 5g(x) + (x^2 + 3x - 1)h(x)$ where all of the coefficients of $f(x)$ and $h(x)$ are less than 5 (other terms can be placed in $g(x)$).

Let ρ be the mapping of $\mathbb{Z}[x] \rightarrow \mathbb{Z}_5[x]$ by reducing the coefficients mod 5. $\rho(a) = \rho(f(x)) + (x^2 + 3x - 1)\rho(h(x)) = f(x) + (x^2 + 3x - 1)h(x)$ and so ρ is an isomorphism from the quotient ring $\mathbb{Z}[x]/(5, x^2 + 3x - 1)$ to $\mathbb{Z}_5[x]/(x^2 + 3x - 1)$.

12. (c) Show there is a surjective homomorphism from $\mathbb{Z}[x]/(5, x^2 + 3x - 1)$ onto $S/(5, \alpha^2 + 3\alpha - 1)$.

The ring homomorphism ψ from $\mathbb{Z}[x] \rightarrow S$ defined by $\psi(x) = \alpha$ is a surjective. Let $\beta \in S$; $\beta = m_0 + m_1\alpha + m_2\alpha^2$ for integers m_0, m_1, m_2 , so $f(m_0 + m_1x + m_2x^2) = \beta$. Therefore the surjective ψ induces a surjection $\hat{\psi}$ on the quotient rings:

$$\mathbb{Z}[x]/(5, x^2 + 3x + 1) \rightarrow S/(5, \alpha^2 + 3\alpha - 1)$$

This utilizes the following lemma on ring homomorphisms:

Lemma 1. *Let R and R' be rings and ψ be a surjection $R \rightarrow R'$. Let I be an ideal of R . Then the mapping that ψ induces between the quotient groups $R/I \rightarrow R'/\psi(I)$ is also a surjection.*

Proof. Take $a \in R/\psi(I)$; then $a = r' + \psi(I)$ for $r' \in R'$. Since ψ is surjective there must be some $r \in R$ such that $\psi(r) = r'$; therefore the coset $r + I$ is mapped to $r' + \psi(I)$, and the mapping between the quotient groups is also surjective. \square

12. (d) In \mathbb{Z}_5 , the polynomial $f(x) = x^2 + 3x - 1 = x^2 + 3x + 4$ is irreducible. Any factor must be a root, and manual testing gives $f(0) = 4, f(1) = 3, f(2) =$

4, $f(3) = 2$, and $f(4) = 2$, so the polynomial has no root and is irreducible. Therefore $\mathbb{Z}_5/(x^2 + 3x - 1)$ is a field of order $5^2 = 25$.

Let $I = (5, \alpha^2 + 3\alpha - 1)$. By (b) and (c) there is a surjection $\hat{\psi}$ from $\mathbb{Z}_5/(x^2 + 3x - 1)$ onto S/I . As $\hat{\psi}$ is onto and the source ring has cardinality 25, S/I must have a cardinality dividing 25; the options are 1 ($R = S$), 5, and 25 ($S/I \simeq \mathbb{Z}_5/(x^2 + 3x - 1)$).

Assume $|S/I| = 5$; we derive a contradiction. Since $\alpha^3 = 2$, we must have $2 \notin \ker(\psi)$ (otherwise $\alpha^3 = 0$ and so $\alpha \in \ker(\psi)$, giving $S \subset \ker(\psi)$). The only cube root of 2 modulo 5 is 3, so $\psi(\alpha) = 3$. However then $\psi(\alpha^2) = 4 + I$, $\psi(3\alpha) = 4 + I$, and $\psi(-1) = 4 + I$; thus $\psi(\alpha^2 + 3\alpha - 1) = 2$. But we know $\psi(\alpha^2 + 3\alpha - 1) = 0$. This is a contradiction, so $|S/I| \neq 5$.

Therefore $I = (5, \alpha^2 + 3\alpha - 1)$ is either the whole ring or a prime ideal inducing S/I to be a field of order 25.

12. (e) Suppose $(5, \alpha^2 + 3\alpha - 1) = S$. Then by (a), $5S = (5, \alpha + 2)S$; however, $\alpha + 2 \notin 5$, so $S/(5, \alpha^2 + 3\alpha - 1)$ must be a field of order 25.

13. (a) Let $S = \mathbb{Z}[\alpha]$, $\alpha^3 = \alpha + 1$. Verify $23S = (23, \alpha - 10)^2(23, \alpha - 3)$.

Let $I = (23, \alpha - 10)^2(23, \alpha - 3)$. The generators of I are:

$$23^3 \tag{1}$$

$$23^2(\alpha - 3) \tag{2}$$

$$23^2(\alpha - 10) \tag{3}$$

$$(\alpha - 10)^2(\alpha - 3) = -23(\alpha^2 - 7\alpha + 13) \tag{4}$$

$$23(\alpha - 10)^2 = 23(\alpha^2 - 20\alpha + 100) \tag{5}$$

$$23(\alpha - 10)(\alpha - 3) = 23(\alpha^2 - 13\alpha + 30) \tag{6}$$

From the generators it is clear that 23 divides every member of I , and so $23S \subset I$. To show the required result we need to show $\{23, 23\alpha, 23\alpha^2\} \in I$.

$$(4) + (5) = 23(-13\alpha + 87) \tag{7}$$

$$2 \cdot (6) - (5) + (4) = 23\alpha + 53 \cdot 23 \tag{8}$$

$$13 \cdot (8) - (7) = 23 \cdot 602 \tag{9}$$

From (1), (2), and (3), we must have $23^2 \in I$ as this is the GCD of (1) with the sum of (2) and (3); since $23 \cdot 602 \in I$, therefore $23 \in I$ as it is the GCD of these two integers. Subtracting a multiple of $23 \in I$ from (8) gives us $23\alpha \in I$, and we thus have $23\alpha^2 \in I$ as well by subtracting the appropriate terms from (5) or (6). This verifies $\{23, 23\alpha, 23\alpha^2\} \in I$ and so $23S = (23, \alpha - 10)^2(23, \alpha - 3)$.

13. (b) Show that $(23, \alpha - 10, \alpha - 3) = S$. Conclude that $(23, \alpha - 10)$ and $(23, \alpha - 3)$ are relatively prime.

Since $-10 \cdot [(\alpha - 10) - (\alpha - 3)] - 3 \cdot 23 = 1$, $(23, \alpha - 10, \alpha - 3) = S$. Since $(23, \alpha - 10) \mid 23S$ and $(23, \alpha - 3) \mid 23S$, neither is the whole ring S and so they must be relatively prime ideals in S .

14. Let K and L be number fields, $K \subset L$, $R = \mathbb{A} \cap K$, $S = \mathbb{A} \cap L$. Assume L is normal over K and let G be the Galois group of L over K . Let $|G| = [K : L] = n$.

14. (a) Suppose Q and Q' are two primes of S lying over a prime P of R . Show the number of automorphisms σ such that $\sigma(Q) = Q$ is the same number of $\sigma \in G$ such that $\sigma(Q) = Q'$. Conclude this number is $e(Q|P)f(Q|P)$.

Enumerate the distinct automorphisms fixing Q as $\sigma_0, \dots, \sigma_k$, and the automorphisms taking Q to Q' as τ_0, \dots, τ_l . Let τ be one of the automorphisms taking Q to Q' (by Theorem 23, this must exist) and consider the automorphisms $\sigma_0\tau, \dots, \sigma_k\tau$. These are k distinct automorphisms taking Q to Q' (if $\sigma_i\tau = \sigma_j\tau$ then $\sigma_i = \sigma_j$), so $k \leq l$. Conversely, consider the automorphisms $\tau\tau_0^{-1}, \dots, \tau\tau_l^{-1}$ taking Q to Q . Each of these must be one of the σ_k , and each must be distinct; if $\tau\tau_i^{-1} = \tau\tau_j^{-1}$ then $\tau_i = \tau_j$, so $l \leq k$. Therefore $l = k$.

We count the number of permutations in G so as to determine the number of permutations fixing Q (call this number k). For each prime P , there are r distinct primes Q_1, \dots, Q_r lying over P , and so there are k automorphisms taking Q_1 to Q_1 , k automorphisms taking Q_1 to Q_2 , etc. Therefore $n = kr$; since $n = re(Q|P)f(Q|P)$, $k = e(Q|P)f(Q|P)$.

14. (b) For an ideal $I \subset S$, define $N_K^L(I)$ to be the ideal $R \cap \prod_{\sigma \in G} \sigma(I)$. Show that for a prime Q lying over P , $N_K^L(Q) = P^{f(Q|P)}$.

Let $e = e(Q|P)$, $f = f(Q|P)$. and Q_1, \dots, Q_r be the ideals of S lying over P . By (a) there are ef automorphisms sending Q to Q_1 , Q to Q_2 , etc. Therefore

$$\begin{aligned} N_K^L(I) &= R \cap (Q_1^{ef} \cdots Q_r^{ef})S \\ &= R \cap (Q_1 \cdots Q_r)^{ef}S \\ &= R \cap P^f S \\ &= P^f \end{aligned}$$

□

14. (c) Let I be an ideal of S . Show $\prod_{\sigma \in G} \sigma(I) = (N_K^L(I))S$.

Let $I = Q_1 \cdots Q_r S$; then $\prod_{\sigma \in G} \sigma(I) = \prod_{\sigma \in G} \sigma(Q_1) \cdots \sigma(Q_r)S$. With the product taken over all $\sigma \in G$, $\prod \sigma(Q_i) = P_i$ for some prime ideal P_i of R lying under I ; therefore $\prod_{\sigma \in G} \sigma(I) = P_1 \cdots P_r S = N_K^L(I)S$.

14. (d)

$$\begin{aligned}
N_K^L(IJ) &= R \cap \prod_{\sigma \in G} \sigma(IJ) \\
&= R \cap \prod_{\sigma \in G} \sigma(I) \prod_{\sigma \in G} \sigma(J) \\
&= R \cap \left(\prod_{\sigma \in G} \sigma(I) \right) \left(\prod_{\sigma \in G} \sigma(J) \right) \\
&= R \cap (N_K^L(I) N_K^L(J)) \\
&= N_K^L(I) N_K^L(J)
\end{aligned}$$

The final equality holds since $N_K^L(I)$ and $N_K^L(J)$ are ideals in R .

14. (e) If $\beta \in N_K^L((\alpha))$, then $\beta = \sigma_1(\alpha) \cdots \sigma_k(\alpha) \gamma = N_K^L(\alpha) \gamma$; since $\beta \in R$ and $N_K^L(\alpha) \in R$, γ must also be in R . Thus $N_K^L((\alpha))$ is the ideal generated by $N_K^L(\alpha)$.
15. (a) Show for three fields $K \subset L \subset M$, that $N_K^M(I) = N_K^L N_L^M(I)$ for an ideal $I \subset \mathbb{A} \cap M$.

We show the result for a prime U of $T = \mathbb{A} \cap M$. Let R, S be the ring of integers of K, L, M respectively, and let P and Q be the primes of R and S lying under U . Then using the multiplicativity of towers as shown in exercise 10,

$$N_K^M(U) = P^{f(U|P)} = P^{f(U|S)f(S|P)} = N_K^L N_L^M(U)$$

If $I = U_1 \cdots U_r$, then

$$N_K^M(I) = \prod_{i=1}^r N_K^M(U_i) = \prod_{i=1}^r N_K^L N_L^M(U_i) = N_K^L N_L^M(I)$$

15. (b) Let $K \subset L$, where L is not necessarily normal. Extend L to a normal extension M . Let $[M : L] = n$. We then have:

$$\begin{aligned}
N_K^M((\alpha)) &= (N_K^M(\alpha)) && \text{(exercise 14. (e))} \\
&= (N_K^L(N_L^M(\alpha))) && \text{Definition of relative norm} \\
&= (N_K^L(\alpha^n)) && \alpha \in L \text{ and } L \subset M \\
&= (N_K^L(\alpha))^n && \text{Factorization of ideals}
\end{aligned}$$

We also have the following transformation on the norm ideal of M over K :

$$\begin{aligned}
N_K^M((\alpha)) &= N_K^L N_L^M((\alpha)) && \text{part (a)} \\
&= N_K^L((\alpha^n)) && \text{Exercise 14. (e), } M \text{ is normal over } L \\
&= N_K^L((\alpha)^n) && \text{Factorization of ideals} \\
&= N_K^L((\alpha))^n && \text{Exercise 14. (d)}
\end{aligned}$$

We therefore have

$$(N_K^L(\alpha))^n = N_K^L((\alpha))^n$$

and conclude that $N_K^L(\alpha) = N_K^L((\alpha))$.

15. (c) For the case where $K = \mathbb{Q}$, show that $N_{\mathbb{Q}}^L(I)$ is the principal ideal in \mathbb{Z} generated by the number $\|I\|$.

For a prime Q of L lying over a prime ideal $P \subset \mathbb{Z}$ containing the prime $p \in P$, $N_{\mathbb{Q}}^L(Q) = P^f$, and $\|I\| = |R/Q| = p^f$. Next, suppose $I = Q_1 \cdots Q_r$ where Q_i lies over a prime P_i . By Theorem 22 (a), $\|I\| = \prod_{i=1}^r \|Q_i\| = \prod_{i=1}^r (P_i)^{f(Q_i|P_i)}$, this number then generates the principal ideal $N_{\mathbb{Q}}^L(I) = \prod_{i=1}^r N_{\mathbb{Q}}^L(Q_i) = \prod_{i=1}^r (P_i)^{f(Q_i|P_i)}$.

16. Let K and L be number fields, $K \subset L$, $R = \mathbb{A} \cap K$, $S = \mathbb{A} \cap L$. Denote by $G(R)$ and $G(S)$ the ideal class groups of R and S respectively.
16. (a) Show that the mapping $\psi : G(S) \rightarrow G(R)$ defined by taking any I in a given class C and sending C to the class containing $N_K^L(I)$ is a homomorphism.

We first show that ψ homomorphism is well-defined. Take $I, J \in C$, so there is some element α, β such that $\alpha I = \beta J$. Therefore

$$\begin{aligned} N_K^L(\alpha I) &= N_K^L(\beta J) \\ N_K^L((\alpha))N_K^L(I) &= N_K^L((\beta))N_K^L(J) \\ N_K^L(\alpha)N_K^L(I) &= N_K^L(\beta)N_K^L(J) \end{aligned}$$

Therefore the image of I and J are in the same ideal class, ψ does not depend on the choice of ideal in the class C .

$\psi((\alpha)) = N_K^L((\alpha))$ and so the identity element of the class group maps to the identity element. $\psi(IJ) = N_K^L(IJ) = N_K^L(I)N_K^L(J)$ and so the mapping respects operation. Therefore it is a homomorphism.

16. (b) Let Q be a prime of S lying over a prime P of R . Let d_Q denote the order of the class containing Q in $G(S)$, d_P denote the order of the class containing P in $G(R)$. Prove that $d_P \mid d_Q f$, where $f = f(Q|P)$.

Take $\psi : G(S) \rightarrow G(R)$ be the homomorphism defined in 1. Then $|\psi(Q)| \mid |Q|$. $\psi(Q) = P^f$; if $f \mid d_P$, $|\psi(Q)| = d_P/f$; otherwise $|\psi(Q)| = d_P$. In both cases we have $d_P \mid d_Q f$.

17. Let $K = \mathbb{Q}[\sqrt{23}]$, $L = \mathbb{Q}[\omega]$, where $\omega = e^{2\pi i/23}$. Let P be one of the primes of K lying over 2; take $P = (2, \theta)$ where $\theta = (1 + \sqrt{-23})/2$, and let Q a prime of $\mathbb{Q}[\omega]$ lying over P .
17. (a)] By Theorem 25, $f(Q|2)$ is the multiplicative order of 2 mod 23; $2^{11} = 2048 \equiv 1 \pmod{23}$. Since $f(P|2) = 1$ ($\text{ref} = [K : \mathbb{Q}] = 2$ and $r = 2$) and f is multiplicative in towers, $f(Q|P) = 11$.

17. (b) $P^3 = (\theta - 2)$:

$$P = (2, \theta), P^2 = (4, 2\theta, \theta - 6) = (4, \theta + 2)$$

and

$$P^3 = (8, 4\theta, 2\theta + 4, 3(\theta - 2)) = (\theta - 2)$$

First $\theta - 2 \in P^3 : 4\theta - 3(\theta - 2) - 8 = \theta - 2$. Then, we have $8 = (\theta - 2)(-\theta - 1)$, $4\theta = 4(\theta - 2) + 8$, $2\theta + 4 = 2(\theta - 2) + 8$, so every element of P^3 is representable as $\theta - 2$ and this is a principal ideal.

However, P is not principal: since $(2, \theta)(2, \bar{\theta}) = (2)$ and the norm of (2) is 4, the ideal $(2, \theta)$ must have norm 2. For it to be generated by a single α we would need some $(a + b\sqrt{-23})/2 \in \mathbb{Z}[\theta]$ where $a^2 + 23b^2 = 8$. This has no integer solution so $(2, \theta)$ is not a principal ideal.

Since P^3 is a principal ideal the ideal class group of $\mathbb{Q}[\sqrt{-23}]$ must have an order dividing 3.

17. (c) By 16. (b), the order of P divides the order of Q multiplied by $f(Q|P)$; therefore $3 \mid d_Q 11$ and so $3 \mid d_Q$. Therefore Q must also not be a principal ideal.
17. (d) Suppose $2 = \alpha\beta$ in $\mathbb{Z}[\omega]$ and neither α nor β is a unit, therefore $2\mathbb{Z}[\omega] = (\alpha)(\beta) = (2, \theta)(2, \bar{\theta})$. By the uniqueness of ideal factorization, $(2, \theta)$ must be principal; however, we have seen that this is not the case in part (c). This is a contradiction; therefore either α or β must be a unit.
18. (a) Show $\text{disc}(r\alpha_1, \alpha_2, \dots, \alpha_n) = r^2 \text{disc}(\alpha_1, \dots, \alpha_n)$.

Writing the discriminant as the determinant of each of the σ_j conjugates of α_n , we have:

$$\text{disc}(r\alpha_1, \alpha_2, \dots, \alpha_n) = \begin{vmatrix} \sigma_1(r\alpha_1) & \cdots & \sigma_k(\alpha_n) \\ \sigma_2(r\alpha_1) & \cdots & \sigma_k(\alpha_n) \\ \vdots & \ddots & \vdots \\ \sigma_k(r\alpha_1) & \cdots & \sigma_k(\alpha_n) \end{vmatrix}^2$$

Let A_{ij} be the matrix minor corresponding to row i , column j . Since $r \in \mathbb{Q}$, $\sigma_k(r\alpha_1) = r\sigma_k(\alpha_1)$ for all k . Taking the determinant along the first column, we have:

$$\begin{aligned} \text{disc}(r\alpha_1, \alpha_2, \dots, \alpha_n) &= \left(\sum_{i=0}^n (-1)^i \sigma_i(r\alpha_1) A_{1i} \right)^2 \\ &= \left(\sum_{i=0}^n (-1)^i r \sigma_i(\alpha_1) A_{1i} \right)^2 \\ &= r^2 \left(\sum_{i=0}^n (-1)^i \sigma_i(\alpha_1) A_{1i} \right)^2 \\ &= r^2 \text{disc}(\alpha_1, \dots, \alpha_n) \end{aligned}$$

18. (b) Let β be a linear combination of $\alpha_2, \dots, \alpha_n$ with coefficients in \mathbb{Q} . Show $\text{disc}(\alpha_1 + \beta, \alpha_2, \dots, \alpha_n) = \text{disc}(\alpha_1, \dots, \alpha_n)$.

For all σ_k , $\sigma_k(\alpha_1 + \beta) = \sigma_k(\alpha_1) + \sigma_k(\beta)$. If $\beta = p_2\alpha_2 + \dots + p_n\alpha_n$, then $\sigma_k(\beta) = p_2\sigma_k(\alpha_2) + \dots + p_n\sigma_k(\alpha_n)$ for $p_i \in \mathbb{Q}$. Writing $\text{disc}(\alpha_1 + \beta, \alpha_2, \dots, \alpha_n)$ in matrix form, the k -th row of the first column has the form $\sigma_k(\alpha_1) + p_2\sigma_k(\alpha_2) + \dots + p_n\sigma_k(\alpha_n)$.

Subtracting a column times a linear factor has no effect on the determinant of the matrix, so by subtracting p_i multiplied by column i from the first column for each i , we see $\text{disc}(\alpha_1 + \beta, \alpha_2, \dots, \alpha_n) = \text{disc}(\alpha_1, \dots, \alpha_n)$.

19. Let K and L be number fields, $K \subset L$, and let $R = \mathbb{A} \cap K$, $S = \mathbb{A} \cap L$. Let P be a prime of R .

19. (a) Show that if $\alpha \in S$, $\beta \in R$, and $\alpha\beta \in PS$, then either $\alpha \in PS$ or $\beta \in P$.

Let $\psi : S/PS \rightarrow P/R$ be the homomorphism defined by taking a coset in S/PS to the corresponding coset in P/R . If $\alpha\beta \in PS$, then $\psi(\alpha\beta) = 0$. Since P is maximal, R/P is an integral domain, so as ψ is a homomorphism into R/P either $\psi(\alpha) = 0$, or $\psi(\beta) = 0$; equivalently, $\alpha \in PS$ or $\beta \in P$.

22. ($\alpha^5 = 2\alpha + 2$) Let $\alpha^5 = 2\alpha + 2$ and $R = \mathbb{A} \cap \mathbb{Z}[\alpha]$. By Exercise 43, $\text{disc}(\alpha) = 4^4 \cdot (-2)^5 + 5^5 2^4 = 2^4 \cdot 3 \cdot 13 \cdot 67$.

As 2 is the prime with power greater than 1 dividing $\text{disc}(\alpha)$, we focus on its factorization. (If $\text{disc}(\alpha)$ were not the whole ring of integers, the order $|R/\mathbb{Z}[\alpha]|$ would be divisible by a power of 2.)

By 43 (d), $\alpha + 1$ is a unit, so we can determine the factorization of 2 by factoring the minimum polynomial of $\alpha + 1$ over $\mathbb{Q} \bmod 2$. Sage gives this as $x^5 - 5x^4 + 10x^3 - 10x^2 + 3x - 1$; mod 2 this is $x^5 + x^4 + x + 1 = (x - 1)^5$; therefore the prime 2 has the factorization $2R = (2, \alpha)^5$ and the inertial degree of the primes lying over 2 is 1. Using the improvement of Theorem 24, $2^4 \mid \text{disc}(R)$; therefore $\text{disc}(R) = \text{disc}(\alpha)$ and $\mathbb{A} \cap \mathbb{Z}[\alpha] = \mathbb{Z}[\alpha]$.

22. ($\alpha^5 = 2\alpha^4 + 2$) We proceed in a similar method to the previous case: we calculate the discriminant of α , take the prime factors that have square terms, and factor the minimum polynomial of $\alpha^4 + 1$ (a unit by 44 (d)) modulo those prime factors.

By 44 (a), $\text{disc}(\alpha) = (-2)^3(4^4(-2)^5 + 5^5(-2)) = 2^4 \cdot 3 \cdot 29 \cdot 83$, so again we only need to focus on $p = 2$. Sage gives the minimum polynomial of $\alpha^4 + 1$ as $x^5 - 21x^4 + 10x^3 - 10x^2 + 5x - 1$; mod 2 this is $x^5 + x^4 + x + 1 = (x - 1)^5$, so $2R = (2, \alpha^4)^5$ and so $\sum f_i = 1$ and by the improvement of Theorem 24, $2^4 \mid \text{disc}(R)$. Therefore $\text{disc}(\alpha) = \text{disc}(R)$ and $\mathbb{A} \cap \mathbb{Z}[\alpha] = \mathbb{Z}[\alpha]$.

24. Let R, K, S, L be as usual. A prime $P \subset R$ is *totally ramified* if $PS = Q^n$, $n = [L : K]$.

24. (a) Suppose P is totally ramified in S ; then $PS = Q^n$. Let M be an extension field such that $K \subset M \subset L$ with $\mathbb{A} \cap M = T$. and U be a prime of M

lying over P . Then $U \subset Q$ and $US = Q^{[M:L]}$. Since the ramification degree is multiplicative in towers, $[L : K] = e(Q|P) = e(Q|U)e(U|P) = [L : M]e(U|P)$; therefore $e(U|P) = [M : K]$ and so P is totally ramified in M .

24. (b) If P is totally ramified in some extension of L and unramified in L' , then take $L \cap L'$. By (a), if $L \cap L' \subset L$ then $L \cap L'$ must be totally ramified. However $L \cap L' \subset L'$ and so must be unramified by assumption. We conclude $[L \cap L' : K] = 1$ so $L \cap L' = K$.

24. (c) Let $m = p_1^{e_1} \dots p_r^{e_r}$. We prove $[\mathbb{Q}[\omega] : \mathbb{Q}] = \phi(m)$ by induction on r ; TODO

26. Let $\alpha = \sqrt[3]{m}$ where m is a cubefree integers, $K = \mathbb{Q}[\alpha]$, $R = \mathbb{A} \cap K$.

26. (a) Let p be a prime $\neq 3$ and $p^2 \nmid m$. By Exercise 2.41, $\text{disc}(\alpha) = -27m^2$, so $p \nmid \text{disc}(\alpha)$. Therefore $p \nmid |R/R[\alpha]|$ and so the prime decomposition of p in R is determined by factoring the polynomial $x^3 - m \pmod{p}$.

26. (b) Let $p \neq 3$ and suppose $p^2 \mid m$ and write $m = hk^2$. We set $\gamma = \alpha^2/k$. Note $\gamma^2 = h\alpha$.

By Exercise 2.41, There are two possible integral bases for R : either $\{1, \alpha, \alpha^2/k\}$ ($m \not\equiv \pm 1 \pmod{9}$) or $\{1, \alpha, (\alpha^2 \pm k^2 + k^2)/3k\}$ ($m \equiv \pm 1 \pmod{9}$).

$|R/R[\gamma]| = h$ in the first scenario, $|R/R[\gamma]| = 3h$ in the second. h is squarefree and so $p \nmid |R/R[\gamma]|$ and so the prime decomposition of p is determined by factoring the minimal polynomial for $\gamma \pmod{p}$. $\gamma^3 = \alpha^6/k^3 = m^2/k^3 = h^2k$ so the minimal polynomial for γ is $x^3 - h^2k$.

Since $p \mid k$, this reduces to factoring the equation $x^3 \pmod{p}$ and so there is one prime lying over p with a ramification degree of 3; therefore $pR = (p, \gamma)^3 = (p, \alpha^2/k)^3$.

26. (c) If $m \not\equiv \pm 1 \pmod{9}$, the integral basis for R is $\{1, \alpha, \alpha^2/k\}$ and so $|R/R[\gamma]| = h$. We split into two cases, one where $3 \mid k$, one where $3 \nmid k$.

Case 1: $3 \mid h$: If $3 \mid h$ then $(3, \alpha)^3$ has generators $(27, 3\alpha^2, 9\alpha, m)$. Because m is cubefree, $9 \nmid m$, so $\gcd(m, 27) = 3 \in (3, \alpha)$. Therefore $3R = (3, \alpha)^3$.

Case 2: $3 \nmid h$: By Theorem 27, the prime decomposition of $3R$ can be determined by factoring $x^3 - h^2k \pmod{3}$; $\pmod{3}$, $x^3 - h^2k \equiv (x - h^2k)^3 \pmod{3}$ and so 3 is totally ramified in R with $3R = (3, \gamma - h^2k)^3$.

26. (d) If $m = 10$, then the integral basis for R is $\{1, \alpha, (\alpha^2 + \alpha + 1)/3\}$. Taking $\beta = (\alpha - 1)^2/3$ we note $(\alpha^2 + \alpha + 1)/3 - \alpha = \beta$ and $\beta^2 = 2(\alpha^2 + \alpha + 1)/3 - 5$. Therefore, we have the series of equivalences (using the transformations

developed in Exercise 3.18):

$$\begin{aligned}
\text{disc}(R) &= \text{disc}\left(1, \alpha, \frac{\alpha^2 + \alpha + 1}{3}\right) \\
&= \text{disc}\left(1, \frac{\alpha^2 + \alpha + 1}{3} - \alpha, \frac{\alpha^2 + \alpha + 1}{3}\right) \\
&= \frac{1}{4} \text{disc}\left(1, \beta, \frac{2(\alpha^2 + \alpha + 1)}{3}\right) \\
4\text{disc}(R) &= \text{disc}\left(1, \beta, \frac{2(\alpha^2 + \alpha + 1)}{3} - 5\right) \\
&= \text{disc}(1, \beta, \beta^2)
\end{aligned}$$

By Exercise 2.27, $\text{disc}(\beta) = |R/R[\beta]|^2 \text{disc}(R)$ so we conclude that $|R/R[\beta]| = 2$ and so can apply Theorem 27.

By Exercise 2.41, the minimal polynomial for β if $m = 10$ is $x^3 - x^2 + 7x - 3$; mod 3, this reduces to $x(x^2 - 2x + 1) = x(x - 1)^2$. Therefore by Theorem 27, the prime decomposition of $3R = (3, \beta)(3, \beta - 1)^2$.

(Not done: consider for general $m \equiv \pm 1 \pmod{9}$.)

26. (e) When $m \equiv \pm 1 \pmod{9}$, an integral basis of R is $\{1, \alpha, (\alpha^2 \pm k^2 \alpha + k)/3k\}$; we have $|R/R[\alpha]| = 3h$ and by Exercise 2.27 and 2.41, $\text{disc}(\alpha) = |R/R[\alpha]|^2 \text{disc}(R) = -27m^2$, so $\text{disc}(R) = -3h^2k^2$. Since $m \equiv \pm 1 \pmod{9}$, $3 \nmid h$ (otherwise $m \equiv 0, 3, 6 \pmod{9}$) and $3 \nmid k$ (otherwise $m \equiv 0 \pmod{9}$).

By exercise 21, $p^{n-\sum f_i} \mid \text{disc}(R)$ and so $\sum f_i = 2$. Since $\sum f_i e_i = 3$ we must have at least 2 prime ideals lying over 3. Since $3 \mid \text{disc}(R)$, by the converse to Theorem 24, $3R$ must be ramified with degree greater than 1. Since the sum of the inertial degrees is also greater than 1, the only possibility satisfying both conditions is that $3R = P^2Q$ for prime ideals P and Q , with both P and Q having inertial degree 1.

27. Let $\alpha^5 = 5(\alpha + 1)$. From Exercise 2.43, we know $\text{disc}(\alpha) = 4^4(-5)^5 + 5^5(-5)^4 = 5^5(5^4 - 4^4) = 5^5 \cdot 3^2 \cdot 41$. This polynomial has the form in exercise 28 ($p = 5, r = 1$), and so by Exercise 3.28 (c), $5^4 \mid \text{disc}(R)$, therefore 3 is the only square prime dividing $|\text{disc}(R)/\text{disc}(R[\alpha])|$ and for all other primes, Theorem 27 applies.

$x^5 - 5x - 5 \equiv x^5 + x + 1 \equiv (x^2 + x + 1)(x^3 + x^2 + 1) \pmod{2}$, so by Theorem 27, $2R = (2, \alpha^2 + \alpha + 1)(2, \alpha^3 + \alpha^2 + 1)$.

I also worked this problem out using the fact that $\alpha + 1$ as a unit; its minimum polynomial over \mathbb{Q} is $x^5 - 5x^4 + 10x^3 - 10x^2 + 1$ and works for any prime (including $p = 3$). In particular for $p = 2$, $x^5 - 5x^4 + 10x^3 - 10x^2 + 1 \equiv x^5 + x^4 + 1 \pmod{2}$. This polynomial splits into two factors $x^2 + x + 1$ and $x^3 + x + 1$ over \mathbb{Z}_2 ; therefore $2R = (2, (\alpha + 1)^2 + \alpha)(2, (\alpha + 1)^3 + \alpha) = (2, \alpha^2 + \alpha + 1)(2, \alpha^3 + \alpha^2 + 1)$ which matches the other solution.

28. Let $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ where all $a_i \in \mathbb{Z}$ and let p be a prime divisor of a_0 with p^r the exact power of p dividing a_0 and suppose all a_i are divisible by p^r . Assume f is irreducible over \mathbb{Q} and let α be a root of f . Let $K = \mathbb{Q}[\alpha]$, $R = \mathbb{A} \cap K$.

28. (a) $\alpha^n = -(a_{n-1}\alpha^{n-1} + \dots + a_0) = p^r \left(\frac{-a_{n-1}}{p^r} \alpha^{n-1} + \dots + \frac{-a_0}{p^r} \right)$, and let $\beta = \frac{-a_{n-1}}{p^r} \alpha^{n-1} + \dots + \frac{-a_0}{p^r}$. Then $(\alpha^n) = (p^r)(\beta)$.

Let α have the factorization $\mathfrak{q}_1^{e_1} \dots \mathfrak{q}_m^{e_m}$ in R ; then $(\mathfrak{q}_1^{e_1} \dots \mathfrak{q}_m^{e_m})^n = (p^r)(\beta)$ and so

$$\mathfrak{q}_1^{ne_1} \dots \mathfrak{q}_m^{ne_m} = (p^r)(\beta)$$

If (p) is not relatively prime with (β) , then there is some $\alpha' \in K$ such that $\beta\alpha' = p$; therefore $\beta\alpha' - p = 0$ would give a linear dependence of the basis $\{1, \alpha, \dots, \alpha^{n-1}\}$ over \mathbb{Q} , but this set is linearly independent. Therefore (p) and (β) have mutually exclusive factors in R and so (reordering the \mathfrak{q}_i if necessary),

$$(p^r) = \mathfrak{q}_1^{ne_1} \dots \mathfrak{q}_k^{ne_k} = (\mathfrak{q}_1^{e_1} \dots \mathfrak{q}_k^{e_k})^n$$

. Therefore (p^r) is an n -th power in R .

28. (b) Given the factorization from part (a), we know r divides ne_i for all i ; if $(n, r) = 1$, then r must divide each of the e_i . Since p^r is an nr -th power, p is an n -th power.

Since the primes lying over p must have $ref = 1$, we conclude in the factorization of (p^r) must have $e_i = r$ and $f = 1$ and so we have the factorizations $(p^r) = (\mathfrak{q}^r)^n$ and $(p) = (\mathfrak{q})^n$; therefore p is totally ramified in R .

28. (c) If r is relatively prime to n , p is totally ramified in R and so $\sum f_i = 1$ and thus by Exercise 21 (b), $p^{n-1} \mid \text{disc}(R)$.

We now examine the scenario where $\gcd(n, r) = m$ and take the factorization from part (a). As in (b), we know r must divide ne_i for all i , and so $\frac{r}{m}$ divides $\frac{n}{m}e_i$ for each of the e_i . Since (p^r) is an n -th power, then

$$\begin{aligned} (p)^r &= (\mathfrak{q}_1^{e_1} \dots \mathfrak{q}_k^{e_k})^n \\ (p)^{r/m} &= (\mathfrak{q}_1^{e_1} \dots \mathfrak{q}_k^{e_k})^{n/m} \\ (p) &= \left(\mathfrak{q}_1^{\frac{me_1}{r}} \dots \mathfrak{q}_k^{\frac{me_k}{r}} \right)^{n/m} \end{aligned}$$

Therefore, if $d \neq n$, (p) is ramified with ramification degree at least n/d .

We know that for any prime of \mathbb{Z} , $ref = n$; therefore

$$\sum_{i=0}^k \frac{n}{m} \frac{me_i f_i}{r} = \frac{n}{m} \sum_{i=0}^k \frac{me_i f_i}{r}$$

and so we must have $\sum_{i=0}^k \frac{me_i f_i}{r} = m$. Each of the terms are integers and so $\sum f_i \leq m$; therefore by applying Exercise 21 (b), we have $p^{n-m} \mid \text{disc}(R)$.

This bound is as good as possible. Let $K = \mathbb{Q}[\alpha]$ where α is a root to the irreducible polynomial $x^4 + 3^2$ and let $R = \mathbb{A} \cap K$. $\text{disc}(K) = 2^8 \cdot 3^2$, so 3^2 is the greatest power dividing the discriminant ($2 = 4 - \gcd(4, 2)$). The prime 3 has the factorization $3R = (\alpha)^2$, and so the inertial degree of $(\alpha) = 2$.

28. (d) In both 43 (c) and 44 (d) we have α a root of a degree 5 polynomial satisfying the conditions of 28 (a) with the a_0 coefficient = a where a is squarefree.

For both equations, we have $p \mid a$, by (c) that $p^4 \mid \text{disc}(R)$. We have shown for both that $d_3 d_4 \mid a^2$, and we know $d_3 \mid d_4$.

43 (c): $\text{disc}(\alpha) = a^4(4^4 a + 5^5) = (d_3 d_4)^2 \text{disc}(R)$. By assumption $4^4 a + 5^5$ is squarefree. Suppose $p \mid d_3$ or $p \mid d_4$; then $p^6 \mid (d_3 d_4)^2 \text{disc}(R)$. This implies $a^2 \mid 4^4 a + 5^5$, contradicting $4^4 a + 5^5$ squarefree.

44 (d): $\text{disc}(\alpha) = a^4[(4a)^4 + 5^5] = (d_3 d_4)^2 \text{disc}(R)$. As in the previous case, $p^6 \mid (d_3 d_4)^2 \text{disc}(R)$ and so $a^2 \mid (4a)^4 + 5^5$, contradicting the assumption that this quantity is squarefree.

29. Let α be an algebraic integer and let f be a monic irreducible polynomial for α over \mathbb{Z} . Let $R = \mathbb{A} \cap \mathbb{Q}[\alpha]$ and suppose p is a prime in \mathbb{Z} such that f has a root r in \mathbb{Z}_p and $p \nmid |R/\mathbb{Z}[\alpha]|$.

29. (a) Show there is a ring homomorphism $R \rightarrow \mathbb{Z}_p$ that takes α to r .

Since $f(r) \equiv 0 \pmod{p}$ and $p \nmid |R/\mathbb{Z}[\alpha]|$, by Theorem 27, the prime ideal $Q = (p, \alpha - r)$ lies over P . As $x - r$ is a factor of $f(x) \pmod{p}$, the inertial degree of Q is 1, and so $|R/Q| = |p|$, so $R/Q \simeq \mathbb{Z}_p$. Let ψ be the mapping from R to its quotient ring R/Q : since $\alpha - r \in Q$, $\psi(\alpha) = r$.

29. (b) Let $\alpha^3 = \alpha + 1$. Show $\sqrt{\alpha} \notin \mathbb{Q}[\alpha]$.

By exercise 2.28, $\mathbb{A} \cap \mathbb{Q}[\alpha] = \mathbb{Z}[\alpha]$, so $|\mathbb{Z}[\alpha]/\mathbb{Z}[\alpha]| = 1$. Since $\mathbb{Z}[\alpha]$ is integrally closed in $\mathbb{Q}[\alpha]$ it suffices to show $\sqrt{\alpha} \notin \mathbb{Z}[\alpha]$: we will do this by finding appropriate r, p such that r is a root of $x^3 - x - 1 \pmod{p}$ and r is not a square mod p .

As suggested in the hint, we take $r = 2$ and $p = 5$. $2^3 - 2 - 1 \equiv 0 \pmod{5}$ and there is a ring homomorphism ψ from $\mathbb{Z}[\alpha] \rightarrow \mathbb{Z}_5$ where $\psi(\alpha) = 2$. If $\sqrt{\alpha} \in \mathbb{Z}[\alpha]$, then $\psi(\sqrt{\alpha})^2 = 2$; however, 2 is not a square mod 5. Therefore $\sqrt{\alpha} \notin \mathbb{Z}[\alpha]$ and so $\sqrt{\alpha} \notin \mathbb{Q}[\alpha]$.

29. (c) Show $\sqrt[3]{\alpha}$ and $\sqrt{\alpha + 2}$ are not in $\mathbb{Q}[\alpha]$.

$\sqrt[3]{\alpha} \notin \mathbb{Q}[\alpha]$: Let $r = 5$; then $5^3 - 5 - 1 = 119 \equiv 0 \pmod{7}$; however there is no element such that $x^3 \equiv 5 \pmod{7}$. Therefore $\sqrt[3]{\alpha} \notin \mathbb{Q}[\alpha]$.

$\sqrt{\alpha + 2}$: Let $r = 3$; then $3^3 - 3 - 1 = 23 \equiv 0 \pmod{23}$; however 5 is not a quadratic residue mod 23. Therefore $\sqrt{\alpha + 2} \notin \mathbb{Q}[\alpha]$.

29. (d) Let $\alpha^5 + 2\alpha = 2$. Prove $x^4 + y^4 + z^4 = \alpha$ has no solutions in $\mathbb{A} \cap \mathbb{Q}[\alpha]$. By exercise 2.43, $\text{disc}(\alpha) = 4^4(2)^5 + 5^5(-2)^4 = 58192 = 2^4 * 3637$ so all primes except 2 and 3637 satisfy $|\mathbb{A} \cap \mathbb{Q}[\alpha] : \mathbb{Z}[\alpha]|$.

Taking $r = 4$ and $p = 5$, we see $4^5 + 2 \cdot 4 - 2 = 130 \equiv 0 \pmod{5}$. Letting ψ be the homomorphism from (a), we observe that if there were x, y, z such that $x^4 + y^4 + z^4 = \alpha$, we would have $\psi(x^4 + y^4 + z^4) = \psi(\alpha) = 4$. However in \mathbb{Z}_5 , $x^4 \equiv 1$ for all x so $\psi(x^4 + y^4 + z^4) = \psi(x)^4 + \psi(y)^4 + \psi(z)^4 = 3 \not\equiv 4$. Therefore there are no $x, y, z \in \mathbb{Q}[\alpha]$ such that $x^4 + y^4 + z^4 = \alpha$.

30. (a) Let f a nonconstant polynomial $f(x)$ over \mathbb{Z} with $f(0) = 1$. Suppose there are only a finite number of primes such that $f(x) \equiv 0 \pmod{p}$ has a root; then there must be some largest prime P' for which a root exists. Consider the prime divisors of $f(P'!)$: for every prime p , $f(P'!) \equiv 1 \pmod{p}$, so it must have a prime divisor $q > P'$. However, then $f(P'!) \equiv 0 \pmod{q}$ contradicting that there were only a finite number of primes such that f had a root.

Next, suppose $f(x)$ is a nonconstant polynomial over \mathbb{Z} . If $f(0) = 0$ then $f(0) = 0$ and so f has a root for all primes p . Suppose $f(0)$ is nonzero, then the polynomial $g(x) = f(f(0)x)/f(0)$ is also in \mathbb{Z} (as $f(0)$ must divide each coefficient). $g(x)f(0) \equiv f(f(0)x) \pmod{p}$; as $\mathbb{Z}[x]$ is an integral domain, $g(x)$ has a root mod p if and only if $f(x)$ has a root. As $g(0) = 1$ it has a root for infinitely many primes and so does $f(x)$.

30. (b) Let $K = \mathbb{Q}[\alpha]$ be a number field and take $f(x)$ be the minimal polynomial for α . Let $R = \mathbb{A} \cap K$ and consider the value $|R : \mathbb{Z}[\alpha]|$. For any prime $p \nmid |R : \mathbb{Z}[\alpha]|$ such that $f(x)$ has a root $r \pmod{p}$, by Theorem 27, there is a prime ideal of the form $(p, \alpha - r)$ lying above P . The inertial degree of this prime ideal is 1 (as $x - r$ has degree one). As $|R : \mathbb{Z}[\alpha]|$ is a finite value, only finitely many primes divide it. However f has a root for infinitely many primes p . Therefore there are infinitely many primes p such that $f(P|p) = 1$.

30. (c) Take the polynomial $x^m - 1$. If for any prime $x^m - 1 \equiv 0 \pmod{p}$ has a solution, then $x^m \equiv 1 \pmod{p}$ and so $m \mid p - 1$; thus there exists k such that $km = p - 1$ and so $1 \equiv p \pmod{m}$. By (a) the polynomial $x^m - 1$ has roots for infinitely many primes p , so for any m , an infinite number of primes p exist such that $p \equiv 1 \pmod{m}$.

30. (d) Let L and K be number fields. Take M the normal closure of K . Only finitely many primes are ramify in M ; however by (b) there are an infinite number of primes p such that $f(Q|p) = 1$ for Q a prime ideal lying over p . Taking away the primes that ramify, in M these primes have inertial degree 1 and ramification index 1, so they must split completely. As L is an intermediate field and inertial degree/ramification index is multiplicative, these primes must also have inertial degree/ramification index equal to 1 in L . Therefore there are an infinite number of primes p that split into $[L : K]$ distinct factors in the intermediate field L .

30. (e) TODO

31. (a) For fractional ideals A, B , let $A = \alpha I$ and $B = \beta I$. For $r \in A, s \in B$, then $r = \alpha i$ and $s = \beta j$ where $i \in I, j \in J$. Therefore $rs = \alpha i \beta j = \alpha \beta i j \in \alpha \beta I J$. Conversely assume $c \in \alpha \beta I J$ then $c = \alpha \beta c'$ where $c' \in I J$, so c' has the form $r s$ for $i \in I, j \in J$. Therefore $c = \alpha i \beta j$ and so is a member of $\alpha I \beta J = AB$. Therefore the product of fractional ideals is independent of the representation of its factors.

31. (b) Let $A = \alpha I$ for $\alpha \in K, I \subset R$; we will show $A^{-1}A = R$. By Theorem 15 there is some J such that IJ is principal, generated by some $\beta \in R$.

Claim: $A^{-1} = \alpha^{-1} \beta^{-1} J$

Take $a \in A^{-1}$. We have the following series of inclusions:

$$\begin{aligned} aA = a\alpha I &\subset R \\ a\alpha I J &\subset R J = J \\ a\alpha(\beta) &\subset J \\ (a) &\subset \alpha^{-1} \beta^{-1} J \end{aligned}$$

Therefore $a \in \alpha^{-1} \beta^{-1} J$. Conversely $\alpha^{-1} \beta^{-1} J \subset A^{-1}$ as $\alpha^{-1} \beta^{-1} I J = (1) \subset R$. This proves the claim.

Using the claim, $AA^{-1} = \alpha^{-1} \alpha \beta^{-1} I J = \beta^{-1}(\beta) = (1) = R$.

31. (c) Let A be a fractional ideal of the form αI for $\alpha \in K, I \subset R$. As K is the field of fractions of R , $\alpha = r/s$ for some $r \in R, s \in S$.

Let I have the factorization into prime ideals $P_1^{e_1} \dots P_k^{e_k}$, (r) have factorization $P_{k+1}^{e_{k+1}} \dots P_m^{e_m}$, and (s) have factorization $P_{m+1}^{e_{m+1}} \dots P_r^{e_r}$; combining the terms and removing the primes raised to the 0th power, gives the prime factorization of A as

$$A = P_1^{e_1} \dots P_k^{e_k} P_{k+1}^{e_{k+1}} \dots P_m^{e_m} P_{m+1}^{-e_{m+1}} \dots P_r^{-e_r}$$

Given two different factorizations of A , $A = P_1^{e_1} \dots P_k^{e_k} = Q_1^{e'_1} \dots Q_j^{e'_j}$, we have $R = (P_1^{e_1} \dots P_k^{e_k})^{-1} Q_1^{e'_1} \dots Q_j^{e'_j} = (P_1^{-e_1} \dots P_k^{-e_k}) Q_1^{e'_1} \dots Q_j^{e'_j}$. Since this product is equal to R , the P_i and Q_i terms must all cancel showing that the factorizations were identical.

31. (d-f) TODO

34. Let $\{\alpha_1, \dots, \alpha_n\}$ be a basis for L over K .

34. (a) Since the α_i s form a basis for L over K their discriminant is nonzero and so the matrix M corresponding to the trace product has nonzero determinant. Take $\beta_i = \sum_{k=1}^n \alpha_k M_{ik}^{-1}$. Then $\text{Tr}(\beta_i \alpha_j) = \sum_{k=1}^n \alpha_j \alpha_k M_{ik}^{-1} = \sum_{k=1}^n M_{ki} M_{ik}^{-1} = \delta_{ij}$ where δ_{ij} is the Kronecker delta function. Therefore the β_i form a dual basis to the α_i .

34. (b) Let $A = R\alpha_1 \oplus \cdots \oplus R\alpha_n$. Show that $A^* = B$ where B is the R -module generated by the β_i .

As the β_i were written as linear combinations of the α_i s, we know $B \subset A^*$; it remains to show the opposite direction.

Let $\gamma \in A^*$, define $m_i = \text{Tr}_L^K(\gamma\alpha_i)$; by assumption $m_i \in R$. Take $\beta = \sum_{i=1}^n m_i\beta_i$. For any $\alpha \in A$, $\alpha = r_1\alpha_1 \oplus r_n\alpha_n$ so $\text{Tr}_K^L(\gamma\alpha) = \sum_{i=1}^n r_i m_i = \text{Tr}_K^L(\beta\alpha)$. Therefore $\text{Tr}_L^K((\gamma - \beta)A) = 0$.

We claim $\gamma - \beta = 0$. Since A is a free R -module generated by the α_i , each $\alpha_i \in A$. If $(\gamma - \beta)^{-1} \neq 0 \in L$, it can be written as a sum of the α_i with coefficients in K . As K is the field of fractions of R there is some $r \neq 0$ such that r clears the denominators of the coefficients of the α_i and so $r(\gamma - \beta)^{-1} \in A$. Then $\text{Tr}_K^L(r(\gamma - \beta)(\gamma - \beta)^{-1}) = \text{Tr}_K^L(r) = rn$ where $n = [L : K]$. However $\text{Tr}_K^L((\gamma - \beta)\alpha) = 0$ for all $\alpha \in A$. Therefore $\gamma - \beta = 0$ and so $A^* \subset B$. We conclude $A^* = B$.

35. Let $\alpha \in L$, $L = K[\alpha]$. Let f be the monic irreducible polynomial for α over K , and write $f(x) = (x - a)g(x)$. Then we have

$$g(x) = \gamma_0 + \gamma_1 x + \cdots + \gamma_{n-1} x^{n-1}$$

We claim that

$$\left\{ \frac{\gamma_0}{f'(\alpha)}, \dots, \frac{\gamma_{n-1}}{f'(\alpha)} \right\}$$

is the dual basis to $\{1, \alpha, \dots, \alpha^{n-1}\}$.

35. (a) Let $\sigma_1, \dots, \sigma_n$ be the embeddings of L in \mathbb{C} fixing K pointwise. Then the $\sigma_i(\alpha)$ are the roots of f . Applying σ_i to $f(x)$ gives

$$\sigma_i(f(x)) = \sigma_i(x - \alpha)\sigma_i(g(x)) = (x - \alpha_i)g_i(x)$$

where $g_i(x)$ is $g(x)$ with σ_i applied to each coefficient and α_i is the conjugate $\sigma_i(\alpha)$.

35. (b) By the chain rule, we have

$$f'(x) = (x - \alpha_i)g'_i(x) + g_i(x)$$

So rearranging terms gives

$$g_i(\alpha_j) = f'(\alpha_j) - (\alpha_j - \alpha_i)g'_i(\alpha_j)$$

Clearly $g_i(\alpha_i) = f'(\alpha_i)$. If $i \neq j$, $f'(\alpha_j) = (\alpha_j - \alpha_i)g'_i(\alpha_j) + g_i(\alpha_j)$; $g_i(\alpha_j) = 0$ since g_i is a root of all conjugates of α except for α_i . Therefore $g_i(\alpha_j) = 0$.

35. (c) Let M be the matrix formed by $[\alpha_j^{i-1}]$ where i is the row and j is the column. Let N be the matrix $[\sigma_i(\gamma_{j-1}/f'(\alpha))]$.

Take NM_{ij} as the i th row and j th column of the matrix product NM . Then $NM_{ij} = \sum_{k=1}^n \alpha_j^{k-1} \sigma_i(\gamma_{k-1}/f'(\alpha)) = \frac{1}{f'(\alpha)} g'_i(\alpha)$. By (b) this value is 1 when $i = j$ and 0 otherwise; therefore NM is the identity matrix.

Since α is the root of a monic polynomial over K it must be such that $\alpha \notin K$ and $\text{disc}(\alpha) \neq 0$; therefore the matrix M is invertible and so $NM = I$ implies $N = M^{-1}$; so $MN = I$. $MN_{ij} = \sum_{k=1}^n \sigma_k(\alpha^{i-1} \gamma_{j-1}/f'(\alpha)) = \text{Tr}(\alpha^{i-1} \gamma_{j-1}/f'(\alpha))$.

Since MN is also the identity element the set $\{\frac{\gamma_0}{f'(\alpha)}, \dots, \frac{\gamma_{n-1}}{f'(\alpha)}\}$ is therefore the dual basis to $\{1, \alpha, \dots, \alpha^{n-1}\}$.

35. (d) Let a_i be the coefficient of the i th power of $f(x)$. Multiplying out $f(x) = (x - \alpha)g(x)$,

$$-\gamma_0\alpha + (\gamma_0 - \gamma_1\alpha)x + (\gamma_1 - \gamma_2\alpha)x^2 + \dots \gamma_{n-1}x^n$$

To show the γ_i as an R -module generate $R[\alpha]$ we prove the following lemma by induction:

Lemma 2. For $i \neq n-1$, $\gamma_i = \sum \alpha^{n-i-1} + a_{n-i-1}\alpha^{n-i-2} + \dots + a_{i+1}$.

Proof. From the above multiplication, $-\gamma_0\alpha = a_0 = -(\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_1\alpha)$ and so $\gamma_0 = \alpha^{n-1} + a_{n-1}\alpha^{n-2} + \dots + a_1$. This is a sum of powers with leading coefficient 1 and constant term equal to a_i . Therefore the base case is satisfied.

Next, assume $\gamma_i = \alpha^{n-i-1} + a_{n-i-1}\alpha^{n-i-2} + \dots + a_{i+1}$. From the above expansion we have $a_{i+1} = (\gamma_i - \alpha\gamma_{i+1})$, so $a_{i+1} = a_{i+1} + a_{i+2}\alpha + \dots + \alpha^{n-i-1} - \alpha\gamma_{i+1}$ and so $\alpha\gamma_{i+1} = \alpha(\alpha^{n-i-2} + \dots + a_{i+2})$. Therefore γ_{i+1} can also be written in the appropriate form. \square

For $i = n-1$, $\gamma_{n-1} = 1$ since $f(x)$ is a monic polynomial. There is then a translation matrix between the powers of α and the γ_i s with 1s on the diagonal. This translation matrix is upper triangular since the power of α in row i is $i-1$. This matrix is invertible over \mathbb{Z} and the powers of α must also be writable in terms of the γ_i s. Therefore the γ_i s generate $R[\alpha]$ as an the R -module.

35. (e) We have the following:

$$\begin{aligned} (R[\alpha])^* &= \left\{ \frac{\gamma_0}{f'(\alpha)}, \dots, \frac{\gamma_{n-1}}{f'(\alpha)} \right\} && \text{By 34. (b)} \\ &= \frac{1}{f'(\alpha)} \{\gamma_0, \dots, \gamma_{n-1}\} \\ &= \frac{1}{f'(\alpha)} R[\alpha] && \text{By 35. (d)} \end{aligned}$$

35. (f) Let $\beta \in \text{diff } R[\alpha]$; then $\beta(R[\alpha])^* \subset S$. By (e), $\beta(\frac{1}{f'(\alpha)})R[\alpha] \subset S$ so $\beta R[\alpha] \subset f'(\alpha)S$. Therefore $\beta \in f'(\alpha)S$. Therefore $\text{diff } R[\alpha] \subset f'(\alpha)S$. The reverse inclusion is straightforward. Therefore $\text{diff } R[\alpha] = f'(\alpha)S$.
35. (g) Let $R[\alpha] \subset S$ so $\text{diff } R[\alpha] = R[\alpha] \text{ diff } S \subset \text{diff } S$ by the final equality of Exercise 33. By (f), $\text{diff } R[\alpha] = f'(\alpha)S$. Since $f'(\alpha) \in \text{diff } f'(\alpha)S$ therefore $f'(\alpha) \in \text{diff } S$.

Chapter 4

1. Prove from the definitions that $E(Q|P) \triangleleft D(Q|P)$.

Let $\sigma \in E(Q|P)$ and $\tau \in D(Q|P)$. We have the following series of equivalences:

$$\begin{aligned}\alpha &\equiv \alpha \pmod{Q} \\ \tau(\alpha) &\equiv \tau(\alpha) \pmod{Q} \\ \sigma(\tau(\alpha)) &\equiv \tau(\alpha) \pmod{Q} \\ \tau^{-1}(\sigma(\tau(\alpha))) &\equiv \tau^{-1}\tau(\alpha) \equiv \alpha \pmod{Q}\end{aligned}$$

Therefore E is normal in D .

3. (a) Let p be an odd prime. Since the multiplicative group of any prime is cyclic, it must have a generator g with order $p-1$. As $g^{p-1} \equiv 1 \pmod{p}$, $g^{(p-1)/2} \equiv -1 \pmod{p}$. Therefore if $p \equiv 1 \pmod{4}$ then $g^{(p-1)/4} \equiv -1 \pmod{p}$ and $\left(\frac{-1}{p}\right) = 1$. Conversely, if $\left(\frac{-1}{p}\right) = 1$, let a be such that $a^2 \equiv -1 \pmod{p}$; therefore $a^4 \equiv 1 \pmod{p}$ and so $4 \mid p-1$.
5. Let K and L be number fields, L a normal extension of K with Galois group G , and P a prime of K . By "intermediate field" we mean "intermediate field distinct from K and L ".
5. (a) If P is inert in L , then as $ref = n$, $f = n$. The decomposition group $D(Q|P)$ is thus all of G . As G is normal in itself, Corollary 1 to Theorem 28 gives us that G is cyclic of order n .
5. (b) Suppose P is totally ramified in every intermediate field, but not totally ramified in L . Take L_E to be the inertia field of P ; by Theorem 28, the ramification index of L_E is 1. By assumption L_E is totally ramified and so $[L_E : K] = 1$; therefore $L_E = K$ and so $f(Q|P) = 1$ and $r(Q|P) = 1$; therefore P must be totally ramified in L also, contradicting our assumption. Therefore no intermediate fields distinct from K and L must exist.
- Since no intermediate fields exist, G must have no proper subgroups and so be of prime order for some prime p , and so must also be cyclic.
5. (c) Suppose every intermediate field contains a unique prime lying over P but L does not. We argue in similar style to (c). Take L_D to be the

decomposition field of P . By Theorem 28, there must only be 1 unique prime lying over L_D and so $[L_D : K] = 1$ and therefore $r = 1$. Therefore $n = ef$ and so there is one unique prime lying over P in L , contradicting our assumption, and so no intermediate fields distinct from K and L exist. Therefore G is cyclic of prime order as in (b).

5. (d) If P is unramified in every intermediate field but ramified in L , then in particular P is unramified in L_E . Let $H \subset G$, then as L_H is unramified, $L_H \subset L_E$ and so $E \triangleleft H$. As L_E is also an intermediate field and is unramified, $[L : L_E] \neq 1$ and so E is nontrivial. Therefore E is the unique smallest nontrivial subgroup of G .

Since E has no subgroups, it must be of prime order for some prime p . As it is the unique subgroup of order p , it must be normal in G , as it is the sole element of its conjugacy class. Since $Z(G) \neq \emptyset$ in a group of prime power order and is a normal subgroup of G , E must also be contained in $Z(G)$.

Because every subgroup of G contains E , by the Sylow theorems, every subgroup of G must have prime power order, including G itself. (Otherwise there would be a q -subgroup H with $H \cap E \neq \emptyset$ which would give an element of H with order p , a contradiction.)

5. (e) If P splits completely in every intermediate field but not in L , then P must split completely in L_D and so $L_D \neq K$.

Let M be an intermediate field of L ; then $r_M = [L : M]$; but $r_M \leq r$. Therefore any intermediate field of L must be a subfield of L_D and there are no intermediate fields between L_D and L .

Therefore, for any nontrivial subgroup $H \subset G$, $D \subset H$, and D has no proper subgroups. Therefore it must be of prime order for some p and so cyclic. As in (d), G must be a group of order p^k , $D \triangleleft G$, and $D \subset Z(G)$.

An example over \mathbb{Q} : Let L be the cyclotomic field $\mathbb{Q}[\zeta_5]$. The Galois group of L has order 4 and is cyclic with generator σ .

Let $p = 19$; $19 \equiv -1 \pmod{5}$ so $19^2 \equiv 1 \pmod{5}$ and so 19 has multiplicative order 2 mod 5. By Theorem 26, its inertial degree in $\mathbb{Z}[\zeta_5]$ is 2 and as $\gcd(5, 19) = 1$, its ramification index is 1; since $ref = 4$, 19 must split into 2 primes in $\mathbb{Z}[\zeta_5]$.

Because there are 2 primes lying over 19 in $\mathbb{Z}[\zeta_5]$, and σ generates the Galois group, σ must permute the primes lying over 19, meaning its decomposition group $D = \{e, \sigma^2\}$. This is a normal subgroup in G and so Corollary 2 to Theorem 28 applies.

As there are no other subgroups of G , 19 splits completely in every proper subfield of $\mathbb{Q}[\zeta_5]$ but not in $\mathbb{Q}[\zeta_5]$, where it has inertial degree 2.

5. (f) Let P be inert in every intermediate field but not inert in L . By (b), for there to be an intermediate field, P must be ramified in L with degree e . By (d), G is a group of prime power order.

Let E be the inertia subgroup of P : since P remains inert in every subgroup, there E is a maximal subgroup in G . Applying (a) to L_E we have that E is cyclic; as E is a unique maximal subgroup, G is therefore also cyclic.

7. (a) Let $p = 3$, $K = \mathbb{Q}[\sqrt{-3}]$, $L = \mathbb{Q}[\sqrt{3}]$. Since $\mathbb{Q}[i] \subset KL$ and p is inert in $\mathbb{Q}[i]$, p is not totally ramified in KL despite being totally ramified in K and L .
7. (b) Let $p = 2$. By Theorem 25, p ramifies in any quadratic field $\mathbb{Q}[\sqrt{m}]$ with $m \equiv 3 \pmod{4}$; however, p splits in any quadratic field $1 \equiv 3 \pmod{4}$. Since $3^2 \equiv 1 \pmod{4}$, any two extensions of \mathbb{Q} $m, n \equiv 3 \pmod{4}$, $\gcd(m, n) = 1$, and $mn \not\equiv 5 \pmod{8}$ will contain a subfield where $p = 2$ splits into 2 distinct primes. Let $K = \mathbb{Q}[\sqrt{7}]$, $L = \mathbb{Q}[\sqrt{15}]$: then $7 \cdot 15 \equiv 1 \pmod{8}$ and so 2 does not remain inert and splits into two primes in its subfield $\mathbb{Q}[\sqrt{105}]$. Therefore 2 splits into two ramified prime ideals in KL .
7. (c) Let $p = 2$. By Theorem 25, p is inert in any quadratic field $\mathbb{Q}[\sqrt{m}]$ where $m \equiv 5 \pmod{8}$. Therefore p is inert in both $K = \mathbb{Q}[\sqrt{5}]$ and $L = \mathbb{Q}[\sqrt{13}]$. However, the composite field KL contains the subfield $\mathbb{Q}[\sqrt{65}]$ and $65 \equiv 1 \pmod{8}$, and by Theorem 25, 2 splits into two primes in $\mathbb{Q}[\sqrt{65}]$. Therefore 2 also splits into two primes in KL each with inertial degree 2.
7. (d) Let $p = 2$. We reverse the procedure we used in (b) and look for any two relatively prime integers m, n such that $mn \equiv 5 \pmod{8}$. Take $m = 3, n = 15$; then p is ramified in both $K = \mathbb{Q}[\sqrt{3}]$ and $L = \mathbb{Q}[\sqrt{15}]$ but remains inert in the subfield $\mathbb{Q}[\sqrt{5}] \subset KL$. Therefore p has a residue field extension of degree 2 in KL .
8. Let r, e, f be positive integers. I will assume Dirichlet's Theorem on primes in arithmetic progression: given integers a and d such that $\gcd(a, d) = 1$, there exists a prime p such that $p = a + nd$.
8. (a) The q th cyclotomic field has degree $q - 1$ over \mathbb{Q} . Since q is the only prime dividing $\text{disc}(\mathbb{Z}[\zeta_q])$, no distinct prime p will ramify in $\mathbb{Q}[\zeta_q]$; therefore, by Theorem 26, if p has multiplicative order $f \pmod{q}$, it will split into $(q - 1)/f$ distinct primes in $\mathbb{Q}[\zeta_q]$.

For any given r , take q a prime such that $q \equiv 1 \pmod{r}$; by Exercise 30 (c), there are an infinite number of primes that satisfy this property.

Since q is a prime, it has a primitive root g and g^r has order f in q . It remains to find a prime p such that $p \equiv g^r \pmod{q}$. Such a prime p always exists by Dirichlet's Theorem on primes in arithmetic progression.

- (b) Take q to be a prime such that $q \equiv rf \pmod{1}$; then there exists some k such that $krf + 1 = q$. Let g be a primitive root for q : then q^r is an element of order fk , so using Dirichlet's Theorem we find p such that $p \equiv g^r \pmod{q}$.

Since p has order $fk \bmod q$, the prime ideal (p) in \mathbb{Z} splits into r distinct prime ideals in $\mathbb{Q}[\zeta_q]$.

As $\text{Gal}(\mathbb{Q}[\zeta_q])$ is cyclic, the comment after Theorem 28 applies and $\mathbb{Q}[\zeta_q]$ splits into r distinct prime ideals in every subfield containing the decomposition field, which is of order r . Therefore p also splits into r distinct prime ideals in the subfield K' of degree rf .

- (c) When choosing p , we apply the Chinese Remainder Theorem to the system of equivalences $g^{kr} \equiv p \pmod{q}$ and $p \equiv 1 \pmod{e}$ (choosing a q such that $\gcd(q, e) = 1$). This gives an integer M , possibly composite, such that $M \equiv 1 \pmod{e}$ and $M \equiv g^{kr} \pmod{q}$. We know that $M \not\equiv 1 \pmod{q}$ since g was chosen to be a primitive root of q . Therefore $\gcd(M, qe) = 1$, and we can apply Dirichlet's Theorem on primes in arithmetic progression to find a prime p such that $p = M + neq$ for some n ; therefore $p \equiv 1 \pmod{e}$ and $p \equiv g^{kr} \pmod{q}$.

- (d) In the p th cyclotomic field, p is totally ramified. Since $\mathbb{Q}[\zeta_p]$ is normal over \mathbb{Q} , p is totally ramified in every intermediate field. Finally, $\text{Gal}(\mathbb{Q}[\zeta_p])$ is cyclic so it has a normal subgroup of order d for each divisor of $p - 1$. As p was chosen such that $p \equiv 1 \pmod{e}$. It follows that $\mathbb{Q}[\zeta_p]$ has a subfield K'' such that $[K'' : \mathbb{Q}] = e$.

The composition field $K'K'' \subset \mathbb{Q}[\zeta_q]$ and has degree ref over \mathbb{Q} . Since p splits into r distinct factors in K' and is ramified in K'' , it splits into r distinct factors, each with ramification index e .

- (e) Take $e = 2, f = 3, r = 5$. We start by finding a prime q such that $q \equiv 1 \pmod{fr}$; $q = 31$ works (here $k = 2$). 31 has several primitive roots - we want to choose one such that $g^r = g^5$ is smallest as this will make our search for a prime easiest. In particular 13 is a primitive root of 31 and $13^5 \equiv 6 \pmod{31}$, so we choose $p = 37$.

The $31 \cdot 37 = 1147$ th cyclotomic field therefore has a subfield of degree $ref = 30$ over \mathbb{Q} (by (d)) where the prime ideal (5) splits into 5 distinct primes, each with ramification index 2.

9. Let L be a normal extension of K , P a prime of K , and Q and Q' of L lying over P . Since L is normal there is some σ such that $Q' = \sigma Q$. Let D and E be the decomposition and inertia groups for Q over P and D' and E' the corresponding things for Q' over P .
9. (a) Prove that $D' = \sigma D \sigma^{-1}$ and $E' = \sigma E \sigma^{-1}$.

$D' = \sigma D \sigma^{-1}$: Suppose $\tau \in D$; then $\tau(Q) = Q$. As $Q' = \sigma Q$, $\sigma^{-1}(Q') = Q$. Then $\sigma(\tau(\sigma^{-1}(Q')) = Q' = \sigma \tau \sigma^{-1}(Q')$, so $\sigma D \sigma^{-1} \subseteq D'$.

Conversely assume $\tau' \in D'$, so $\tau'(Q') = Q'$ and $\tau'(\sigma(Q)) = Q'$. Thus $\sigma^{-1}\tau'(\sigma(Q)) = Q$, so $\sigma^{-1}\tau'\sigma \in D$ and $\tau' \in \sigma D \sigma^{-1}$. Therefore $D' \subseteq \sigma D \sigma^{-1}$; we conclude $D' = \sigma D \sigma^{-1}$.

$E' = \sigma E \sigma^{-1}$: Let $\tau \in E$; then $\alpha \equiv \tau(\alpha) \pmod{Q}$ for all $\alpha \in S$, so $\alpha - \tau(\alpha) \in Q$. In particular $\sigma^{-1}(\alpha) - \tau(\sigma^{-1}(\alpha)) \in Q$, and so $\alpha - \sigma(\tau(\sigma^{-1}(\alpha))) \in \sigma Q = Q'$. Therefore $\sigma \tau \sigma^{-1} \in E'$ and $\sigma E \sigma^{-1} \subset E'$.

Conversely assume $\tau' \in E'$; then $\alpha - \tau'(\alpha) \in Q'$ for any $\alpha \in S$. So $\sigma(\alpha) - \tau'(\sigma(\alpha)) \in Q'$ and $\alpha - \sigma^{-1}(\tau'(\sigma(\alpha))) \in \sigma^{-1}Q' = Q$. Therefore $\sigma^{-1}E'\sigma \subset E$ and so $E' \subset \sigma E \sigma^{-1}$; we conclude $E = \sigma E' \sigma^{-1}$.

9. (b) Let $\psi(\alpha) \equiv \alpha^{\|P\|} \pmod{Q}$; then $\psi(\alpha) - \alpha^{\|P\|} \in Q$ for any $\alpha \in S$. In particular $\psi(\sigma(\alpha)) - \sigma(\alpha)^{\|P\|} \in Q$ and so $\sigma^{-1}\psi(\sigma(\alpha)) - \alpha^{\|P\|} \in \sigma^{-1}Q = Q'$. Therefore $\psi' = \sigma^{-1}\psi\sigma$.

- 12 (a) Let $\omega = e^{2\pi i/m}$, let $G = \text{Gal}(\mathbb{Q}[\omega]/\mathbb{Q}) \simeq \mathbb{Z}_m^\times$, K be any subfield of $\mathbb{Q}[\omega]$ and H the subgroup of G fixing K . Let $p \in \mathbb{Z}$ such that $p \nmid m$, and let f denote the least integer such that $\overline{p^f} \in H$. Show f is the inertial degree $f(P|p)$ for any P of K lying over p .

Since p is unramified, $\phi(P|p)$ exists and corresponds to some $b \in G$. A permutation in G corresponds to taking $\omega \mapsto \omega^b$. However the Frobenius automorphism has the property $b(\omega) = \omega^b \equiv \omega^p \pmod{P}$, so we have $b \equiv p \pmod{n}$.

Let $H\overline{a_1}, \dots, H\overline{a_n}$ be the cosets of H (with $\overline{a_1}$ being the permutation that takes $\omega \mapsto \omega^{a_1}$).

Let the cyclic group $\{1, p, p^2, \dots, p^{f-1}\}$ (with $p^f = 1$) act on the right cosets of H . For any coset Hx , $p^a(Hx) = Hxp^a$; if $xp^a = x$ then $\omega^{xp^a} = \omega^x$ and so $xp^a \equiv x \pmod{m}$. Therefore $p^a \equiv 1 \pmod{m}$; as $p \nmid m$, $a = 0$. By the orbit stabilizer theorem, the size of an orbit of Hx is the size of the whole group, i.e. f , by Theorem 33 the inertial degree of any prime P of K lying over p is f .

12. (b) Let $\mathbb{Q}[\omega + \omega^{-1}]$ be a subfield of \mathbb{Q} . The subgroup of G that fixes $\mathbb{Q}[\omega + \omega^{-1}]$ is the subgroup of order 2 consisting of the identity and complex conjugation τ that takes $\omega \mapsto \omega^{-1}$. This τ is identified with the permutation $m-1 \in \mathbb{Z}_m^\times$, so the subgroup H consists of $\{1, m-1\} = \{1, -1\}$.

Let $K = \mathbb{Q}[\omega + \omega^{-1}]$. By the tower law, $\phi(m) = [\mathbb{Q}[\omega] : \mathbb{Q}] = [\mathbb{Q}[\omega] : K][K : \mathbb{Q}] = 2 \cdot [K : \mathbb{Q}]$, so $[K : \mathbb{Q}] = \phi(m)/2$. For any odd prime such that $p \nmid m$, p is unramified in $\mathbb{Q}[\omega]$ and so also in K . By (a), p will split into $\phi(m)/2f$ primes in K , where f is the smallest integer such that $p^f \equiv \pm 1 \pmod{m}$.

12. (c) Letting p be a prime not dividing m , and take K to be any quadratic subfield $\mathbb{Q}[\sqrt{d}] \subset \mathbb{Q}[\omega]$. Let H be the subgroup fixing K . Since $p \nmid m$, p is unramified in $\mathbb{Q}[\omega]$ and so also unramified in $\mathbb{Q}[\sqrt{d}]$. Therefore either p remains inert or splits into two primes in $\mathbb{Q}[\omega]$.

Let p be odd. We will show that $\bar{p} \in H$ iff d is a square mod p ; this is equivalent to $f = 1$ iff d is a square mod p . As p is unramified, Theorem 25 gives that $f = 1$ iff d is a square mod p .

Now let $p = 2$. We will show that $\bar{p} \in H$ iff $d \equiv 1 \pmod{8}$. This is equivalent to $f = 1$ iff $d \equiv 1 \pmod{8}$. As p is unramified, Theorem 25 gives the only possibility where $f = 1$ is $d \equiv 1 \pmod{8}$.

13. Let $m \in \mathbb{Z}$. Assume m is not square and $m \neq -1$. Let $K = \mathbb{Q}[\sqrt[4]{m}]$ and $L = \mathbb{Q}[\sqrt[4]{m}, i]$; then L is a normal extension of \mathbb{Q} such that $K \subset L$. Denote the roots $\alpha, i\alpha, -\alpha, -i\alpha$ as 1, 2, 3, and 4.
13. (a) $[K : \mathbb{Q}] = 4$ and $[L : K] = 2$ so L is a degree 8 extension of \mathbb{Q} where $f(x) = x^4 - m$ splits. Let $G = \text{Gal}(L/K)$; so $|G| = 8$. Letting the roots of f be $\alpha, i\alpha, -\alpha, -i\alpha$, $G \subset S_4$ and so $G \simeq D_8$ (the dihedral group on 4 objects) as this is the only subgroup of S_4 of order 8. Therefore $G = \{1, \tau, \sigma, \tau\sigma, \sigma^2, \tau\sigma^2, \sigma^3, \tau\sigma^3\}$. In D_8 , $\tau^2 = 1$ and $\sigma^4 = 1$, with $\tau\sigma = \sigma^{-1}\tau$. Note τ corresponds to complex conjugation (switching $i\alpha$ with $-i\alpha$).
13. (b) Let p be an odd prime not dividing m . Prove p is unramified in L .
Let $S = \mathbb{A} \cap L$ and consider $\text{disc}(\alpha)$. $\text{disc}(\alpha) = N(f'(\alpha)) = \pm N(4\alpha^3) = \pm 4^8 N(\alpha)^3 = \pm 4^8 (-m)^3$; therefore $p \nmid \text{disc}(\alpha)$. Because $\text{disc}(R) \mid \text{disc}(\alpha)$, $p \nmid \text{disc}(R)$ also, so p is unramified in L .
13. (c) Let Q be a prime lying over p . Since p is unramified in L , the Frobenius automorphism $\phi(Q|p)$ exists. The subgroup H of G that fixes K is the subgroup corresponding to complex conjugation: $\{1, \tau\}$. The right cosets of H are $H, H\sigma, H\sigma^2, H\sigma^3$.
Suppose $\phi(Q|p) = \tau$. Since $H\sigma\tau = H\tau\sigma^3 = H\sigma^3$, these two cosets are in the same partition. $H\tau = H$ and $H\sigma^2\tau = H\tau\sigma^2\tau = H\sigma^2$. Therefore we have three partitions of cosets: $\{H\}, \{H\sigma, H\sigma^3\}, \{H\sigma^2\}$, and by Theorem 32, Q splits into 3 primes in K .
13. (d) For each permutation of G , we follow a similar process to (c) to give how Q splits. The subgroup H fixing K remains the same as in (c).
The partitions are straightforward to calculate since right-multiplication of any $H\sigma^n$ by any permutation gives another coset of the form $H\sigma^m$. This is straightforward for permutations σ^a , whereas for permutations $\sigma^a\tau$,

$$H\sigma^n(\sigma^a\tau) = H\sigma^{n+a}\tau = H\tau\sigma^{-(n+a)} = H\sigma^{-(n+a)}$$

$\psi(Q p)$	Partitions	Number of Primes
1	$\{H\}, \{H\sigma\}, \{H\sigma^2\}, \{H\sigma^3\}$	4
σ, σ^3	$\{H, H\sigma, H\sigma^2, H\sigma^3\}$	1
σ^2	$\{H, H\sigma^2\}, \{H\sigma, H\sigma^3\}$	2
τ	$\{H\}, \{H\sigma, H\sigma^3\}, \{H\sigma^2\}$	3
$\sigma\tau$	$\{H, H\sigma^3\}, \{H\sigma, H\sigma^2\}$	2
$\sigma^2\tau$	$\{H, H\sigma^2\}, \{H\sigma\}, \{H\sigma^3\}$	3
$\sigma^3\tau$	$\{H, H\sigma\}, \{H\sigma^2, H\sigma^3\}$	2

18. (a) TODO
18. (b) Suppose $\sigma \in \cap V_m$, then for any $\alpha \in S$, $\sigma(\alpha) - \alpha \in \cap Q^m = 0$. Since α was chosen arbitrarily σ must be the identity permutation. Therefore the descending chain of subgroups eventually stabilizes at the identity.

19. Let $\sigma \in V_{m-1}$, $\pi \in Q - Q^2$, and $\sigma(\pi) \equiv \sigma(Q^{m+1})$.
19. (a) Suppose $\alpha \in \pi S$, so it must have form $\alpha = \pi\beta$ for $\beta \in S$. By the definition of σ , $\sigma(\beta) \equiv \beta(Q^m)$. As $\pi \in Q - Q^2$, $\pi\sigma(\beta) \equiv \pi(\beta)(Q^{m+1})$. Since $\pi \equiv \sigma(\pi)(Q^{m+1})$, $\sigma(\pi)\sigma(\beta) = \sigma(\pi\beta) = \sigma(\alpha) \equiv \alpha(Q^{m+1})$.
19. (b) Since $\pi \notin Q^2$, the principal ideal (π) has a decomposition JQ where $J \notin Q$. If the ideal (α) has a decomposition IQ^n where $I \notin Q$, then $IJQ \subset \pi S$ and $IJ \notin Q$.
 IJ also contains an integer m ($m = \|IJ\|$), take $\beta = m$. Then $\sigma(\beta) = \beta$ and $\beta\alpha \in \pi S$.
 Given $\alpha \in Q$, $\beta\alpha \in \beta Q = \pi S$. By (a), $\sigma(\beta\alpha) \equiv \beta\alpha(Q^{m+1})$. As $\sigma(\beta) = \beta$, $\beta\sigma(\alpha) \equiv \beta\alpha(Q^{m+1})$. Since $\beta \notin Q$ we can cancel both sides and so $\sigma(\alpha) \equiv \alpha(Q^{m+1})$.
19. (c) Following the hint, $S = S_E + Q$; so every $\alpha \in S$ can be written as a sum of an element of the number ring of the fixed field of E and an element of Q . Let $\alpha = \beta + \gamma$ where $\beta \in S_E$ and $\gamma \in Q$. By (b), $\sigma(\gamma) \equiv \gamma(Q^{m+1})$, so $\sigma(\gamma) + \beta \equiv \gamma + \beta(Q^{m+1})$. Since $\beta \in S_E$, $\sigma(\beta) = \beta$ and so, $\sigma(\gamma + \beta) \equiv \gamma + \beta(Q^{m+1})$. This is the required result.

Chapter 5

6. Show that $\mathbb{A} \cap \mathbb{Q}[\sqrt{m}]$ is principal for $m = 2, 3, 5, 6, 7, 173, 293, 437$. As each of these m is positive, these are real quadratic fields and so the number of complex embeddings $s = 0$. Therefore the bound given by Minkowski's Theorem is that every ideal class contains an ideal with $\|J\| < \frac{2!}{2^2} \cdot \sqrt{|\text{disc}(R)|}$, where $\text{disc}(R) = m$ if $m \equiv 1 \pmod{4}$ and $\text{disc}(R) = 4m$ otherwise. Therefore

$$\|J\| < \begin{cases} \frac{\sqrt{m}}{2} & m \equiv 1 \pmod{4} \\ \sqrt{m} & m \equiv 2, 3 \pmod{4} \end{cases}$$

- $m = 2$: $\|J\| < \sqrt{2} \approx 1.4$ so every ideal class contains an ideal with norm 1. Therefore every ideal must be principal.
- $m = 3$: $\|J\| < \sqrt{3} \approx 1.7$ so every ideal is principal.
- $m = 5$: $\|J\| < \sqrt{5}/2 \approx 1.1$ so every ideal is principal.
- $m = 6$: $\|J\| < \sqrt{6} \approx 2.4$ so we must check that the prime ideal containing 2 is principal. By Theorem 25, 2 factors as $(2, \sqrt{6})^2$ in $\mathbb{Q}[\sqrt{6}]$. $(2, \sqrt{6})$ is principal, generated by $(2 + \sqrt{6})$: $2 = -1 \cdot (2 + \sqrt{6})(2 - \sqrt{6})$ and $\sqrt{6} = 2 + \sqrt{6} - 2$. Therefore every ideal is principal.
- $m = 7$: $\|J\| < \sqrt{7} \approx 2.6$ so we must check that the prime ideal containing 2 is principal. By Theorem 25, 2 factors as $(2, 1 + \sqrt{7})^2$. $(2, \sqrt{7})$ is generated by $3 + \sqrt{7}$; $2 = (3 + \sqrt{7})(3 - \sqrt{7})$ and $1 + \sqrt{7} = 3 + \sqrt{7} - 2$. Therefore every ideal is principal.

- $m = 173$: $173 \equiv 1 \pmod{4}$, so $\|J\| < \sqrt{173}/2 \approx 6.5$, so we must check that the prime ideals containing 2, 3, 5 are all principal. Since $173 \equiv 5 \pmod{8}$, 2 remains prime in $\mathbb{Q}[\sqrt{m}]$ and so its ideal is principal. 173 is a prime number. Since $173 \equiv 1 \pmod{4}$, by quadratic reciprocity, $\left(\frac{173}{p}\right) = \left(\frac{p}{173}\right)$, so $\left(\frac{3}{173}\right) = \left(\frac{173}{3}\right) = \left(\frac{2}{3}\right) = -1$. Similarly $\left(\frac{5}{173}\right) = \left(\frac{173}{5}\right) = \left(\frac{3}{5}\right) = -1$. So both 3 and 5 remain inert in $\mathbb{Q}[\sqrt{m}]$ and so every ideal is principal.
- $m = 293$: $293 \equiv 1 \pmod{4}$ so $\|J\| < \sqrt{293}/2 \approx 8.5$ so we must check the prime ideals containing 2, 3, 5, 7 are all principal. $293 \equiv 5 \pmod{8}$ so 2 remains prime in $\mathbb{Q}[\sqrt{293}]$ and its ideal is principal. 293 is a prime number. Calculation with Sage shows that 3, 5, 7 each are not squares mod 293, so these prime ideals remain inert in $\mathbb{Q}[\sqrt{293}]$ and so are principal. Therefore every ideal is principal.
- $m = 437$: $437 \equiv 1 \pmod{4}$ so $\|J\| < \sqrt{437}/2 \approx 10.4$ so we must check that the prime ideals 2, 3, 5, 7 are all principal. $437 \equiv 5 \pmod{8}$ so 2 remains prime in $\mathbb{Q}[\sqrt{437}]$. $437 = 19 \cdot 23$ so none of 3, 5, 7 ramify in $\sqrt{437}$. Using Sage we can calculate the Jacobi symbol $\left(\frac{3, 5, 7}{437}\right) = -1$. As the Jacobi symbol is -1 each of these are nonresidues mod 437 and so by Theorem 25 remain prime in $\mathbb{Q}[\sqrt{437}]$. Therefore every ideal is principal.

10. (b) First, the problem in the book has a typo (both editions). The problem statement should be "Suppose p is an odd prime such that $4p < -m$. Show m is a non-square mod p ."

Since $m \equiv 5 \pmod{8}$, $m \equiv 1 \pmod{4}$. Let $R = \mathbb{A} \cap \mathbb{Q}[\sqrt{m}]$. If m is a square mod p , then $pR = (p, \frac{n+\sqrt{m}}{2})(p, \frac{n-\sqrt{m}}{2})$ (by Theorem 25). Suppose R is a PID; therefore the ideal $(p, \frac{n+\sqrt{m}}{2})$ is generated by $\frac{n+\sqrt{m}}{2}$ with $p = (\frac{n+\sqrt{m}}{2})(\frac{n-\sqrt{m}}{2})$. Therefore

$$n^2 - m = 4p \implies n^2 - m < -m \implies n^2 < 0$$

This is a contradiction $n^2 \equiv m \pmod{p}$, so p must be inert in R , meaning m is a nonsquare mod p .

10. (c) From (a) we know $m \equiv 5 \pmod{8}$, and so $m \equiv 1 \pmod{4}$. If m is to be a principal ideal domain, it must be prime: otherwise any prime $p \mid m$ would ramify in $\mathbb{Q}[\sqrt{m}]$ as the ideal (p, \sqrt{m}) which is nonprincipal if \sqrt{m} has a norm $\neq p$.

The first prime above 19 such that $m \equiv 5 \pmod{8}$ is 29; therefore by (b) any $m < -19$ such that $\mathbb{A} \cap \mathbb{Q}[\sqrt{m}]$ is a principal ideal domain must have 3, 5, and 7 be inert.

For an odd prime p to be inert, m must be a non-residue modulo p . We already know that 2 is inert if $m \equiv 5 \pmod{8}$. For 3 to be inert, $m \equiv 2 \pmod{3}$. For 5 to be inert, $m \equiv \pm 2 \pmod{5}$. For 7 to be inert, $m \equiv 3, 5, 6 \pmod{7}$. Using the Chinese Remainder Theorem on each of these combinations we have

$$m \equiv -403, -163, -43, -67, -667, -547 \pmod{840}$$

Relevant computation with Sage:

```
sage: modulae_list = [( [2], 3), ([2, 3], 5),
...                  ([3, 5, 6], 7), ([5], 8)]
sage: prime_list = [a[1] for a in modulae_list]
sage: residues = [a[0] for a in modulae_list]
sage: [-(840-CRT_list(list(rs),prime_list))
...     for rs in itertools.product(*residues)]
[-403, -163, -43, -67, -667, -547]
```

10. (d) TODO (base off (c))

11. The Minkowski bound for $\mathbb{Z}[\sqrt{-6}]$ is $\frac{2!}{2^2} \left(\frac{4}{\pi}\right) \sqrt{24} \approx 3.11$. Both 2 and 3 ramify in $\mathbb{Z}[\sqrt{-6}]$ as they divide the discriminant; $(2, \sqrt{-6})^2 = (2)$ and $(3, \sqrt{-6})^2 = (3)$. Neither of these ideals is principal as $x^2 + 6y^2 = \{2, 3\}$ has no integer solutions. $(2, \sqrt{-6}) = \frac{\sqrt{-6}}{3}(3, \sqrt{-6})$, so they are in the same ideal class element. Therefore the ideal class group of $\mathbb{Z}[\sqrt{-6}]$ has order 2.

The Minkowski bound for $\mathbb{Z}[\sqrt{-10}]$ is $\frac{2!}{2^2} \left(\frac{4}{\pi}\right) \sqrt{40} \approx 4.02$, so again we only need to consider 2 and 3. 2 ramifies as $(2, \sqrt{-10})^2$; as in the previous case this is not a principal ideal due to $x^2 + 10y^2 = 2$ having no integer solutions. 3 remains inert; $-10 \equiv 2 \pmod{3}$ and $\left(\frac{2}{3}\right) = -1$, so 3 does not factor in $\mathbb{Z}[\sqrt{-10}]$.

12. The Minkowski bound for $\mathbb{Z}[\sqrt{-23}]$ is 3.053, so only 2 and 3 need to be considered. As $\left(\frac{2}{-23}\right) = 1$, 2 factors as $2 = \left(2, \frac{\sqrt{-23}-1}{2}\right)\left(2, \frac{\sqrt{-23}+1}{2}\right)$. $x^2 + 23y^2 = 8$ has no integer solutions, so this must be a non-principal ideal. $\left(2, \frac{\sqrt{-23}-1}{2}\right)^2 = (4, \sqrt{-23}-1, \frac{\sqrt{-23}-11}{2}) = (4, \frac{\sqrt{-23}+3}{2})$. This is also not a principal ideal since $x^2 + 23y^2 = 16$ has no solutions with $y \neq 0$ (since $\frac{\sqrt{-23}+3}{2}$ is in the ideal it can't just be generated by an integer). However, $\left(2, \frac{\sqrt{-23}-1}{2}\right)^3 = (8, \sqrt{-23}-1, \sqrt{-23}+3, \frac{\sqrt{-23}-13}{2})$. First observe $\frac{\sqrt{-23}+3}{2}$ is in this ideal. Next, observe $8 = N(\frac{\sqrt{-23}+3}{2})$, so the ideal is principal. Therefore there are 3 elements in the ideal class group.

I am (for now) omitting the part of the exercise which asks that we do the same for $-31, -83, -139$.

(Skip a bunch of quadratic field exercises.)

17. Let $\omega = e^{2\pi i/7}$. $\text{disc}(\mathbb{Z}[\omega]) = -7^5$, and $\mathbb{Z}[\omega]$ has degree 6 over \mathbb{Q} . There are 6 embeddings from $\mathbb{Z}[\omega]$ into \mathbb{Q} ; each taking ω to a different root of unit. Therefore the Minkowski bound is $\frac{6!}{6^6} \left(\left(\frac{4}{\pi}\right)^3 \sqrt{7^5}\right) = \frac{720 \cdot 64 \cdot 49 \sqrt{7}}{\pi^3 6^6} = \frac{3920 \sqrt{7}}{81 \pi^3} \approx 4.12$. So the only prime ideals we need to examine contain 2 or 3.

By the Corollary to Theorem 26, 2 splits into $\frac{\varphi(7)}{3} = 2$ ideals mod 7 ($2^3 \equiv 1 \pmod{7}$), while 3 remains inert. Mod 2, $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

factors as $(x^3 + x + 1)(x^3 + x^2 + 1)$, so $2 = (2, \omega^3 + \omega + 1)(2, \omega^3 + \omega^2 + 1)$. As $(\omega^3 + \omega + 1)(\omega^3 + \omega^2 + 1)(\omega^4) = 2$, so both of these ideals is principal. Therefore $\mathbb{Z}[\omega]$ is a PID.

Next, let $\mathbb{Z}[\omega + \omega^{-1}]$. By Exercise 35, Chapter 2, the discriminant over this number field is $7^2 = 49$. This is a real subfield of order 3 over \mathbb{Q} (no embeddings in \mathbb{C}) so its Minkowski bound is $\frac{3!}{3^3} \sqrt{(49)} = \frac{14}{9} < 2$, so $\mathbb{Z}[\omega + \omega^{-1}]$ is a PID.

18. Let $\omega = e^{2\pi i/11}$. This is a real subfield of degree 5 over \mathbb{Q} with discriminant 11^4 , so its Minkowski bound is $\frac{5!}{5^5} \cdot 11^2 = \frac{2904}{625} \approx 4.6$. So we must investigate the prime ideals 2 and 3.

By Exercise 12 Chapter 4 primes in $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ for ζ_p a p th root of unity split into $\frac{p-1}{2f}$ ideals, where f the smallest integer so that $p^f \equiv \pm 1 \pmod{p}$. Letting $p = 11$ and looking at 2 and 3, we have both $2^5 \equiv -1 \pmod{11}$ and $3^5 \equiv 1 \pmod{11}$, so both 2 and 3 remain inert in $\mathbb{Z}[\omega]$. Therefore $\mathbb{Z}[\omega]$ is a PID.

We repeating the same process for $\omega = e^{2\pi i/13}$. In this case the discriminant of $\mathbb{Z}[\omega + \omega^{-1}] = 13^5$ so the Minkowski bound is $\frac{6!169\sqrt{13}}{6^6} = \frac{845\sqrt{13}}{324} \approx 9.403$. We must examine the primes 2, 3, 5, 7. Here the ideal splitting formula that a prime splits into $\frac{6}{f}$ ideals.

2 and 7 remain inert in $\mathbb{Z}[\omega + \omega^{-1}]$ ($2^6, 7^6 \equiv -1 \pmod{13}$), but 3 splits into 2 ideals ($3^3 \equiv 1 \pmod{13}$) and 5 splits into 3 ideals ($5^2 \equiv -1 \pmod{13}$). Computation via Sage shows each of these ideals is principal. Therefore $\mathbb{Z}[\omega + \omega^{-1}]$ is a PID.

19. Let $K = \mathbb{Q}[\sqrt[3]{2}]$. $\mathbb{A} \cap K = \mathbb{Z}[\sqrt[3]{2}]$ by Exercise 41 of Chapter 2 ($2 \not\equiv \pm 1 \pmod{9}$), and $\text{disc}(\mathbb{Z}[\sqrt[3]{2}]) = -27(2)^2$. There are 2 complex embeddings for K so the Minkowski bound of K is $\frac{3!}{3^3} \left(\frac{4}{\pi}\right) \sqrt{|-27 \cdot 2^2|} = \frac{16\sqrt{3}}{3\pi} \approx 2.94$. The only prime ideal class we must investigate is the one lying over (2); however, 2 is fully ramified in $\mathbb{A} \cap AK$, and so its ideal is $(2, \sqrt[3]{2})^3 = (\sqrt[3]{2})^3$ is principal. Therefore $\mathbb{Z}[\sqrt[3]{2}]$ is a PID.

Let α be a root of $\alpha^3 - \alpha - 1$. By Exercise 28 of Chapter 2 $\mathbb{A} \cap \mathbb{Q}[\alpha] = \mathbb{Z}[\alpha]$, and $\text{disc}(\alpha) = -(-4 + 27) = -23$. This field also has 2 complex embeddings (there is only 1 real root), so the Minkowski bound is $\frac{8\sqrt{23}}{9\pi} \approx 1.356$. There are no primes within this bound so every ideal of $\mathbb{Z}[\alpha]$ is a principal ideal.

20. Let α be a root of $\alpha^3 - \alpha - 7$. By Exercise 28 of Chapter 2, $\text{disc}(\alpha) = (-4 + 27 \cdot 7^2) = -1319$. This is prime (so squarefree) and so $\mathbb{A} \cap \mathbb{Q}[\alpha] = \mathbb{Z}[\alpha]$ by Exercise 27 of Chapter 2. This field has 2 complex embeddings (only 1 real root) so the Minkowski bound is $\frac{8\sqrt{1319}}{9\pi} \approx 10.27$. We must investigate the primes 2, 3, 5, and 7. Since $\text{disc}(\alpha)$ is prime each of these factors in the Dedekind way, e.g. by seeing how the polynomial $\alpha^3 - \alpha - 7$ factors mod p . Additionally, since the minimum polynomial has degree 3 the prime remains inert if there is no root mod p .

- $p = 2 : x^3 - x - 7 \equiv x^3 + x + 1 \pmod{2}$. This has a root if there is an x so that $x(x^2 + 1) = 1$ (clearly not), so 2 remains inert.
- $p = 3 : x^3 - x - 7 \equiv x^3 + 2x + 2 \pmod{3}$. This has a root if $x(x^2 + 2) = 1$; however no options work. 3 remains inert.
- $p = 5 : x^3 - x - 7 \equiv x^3 + 4x + 3 \pmod{5}$. This has a root if $x(x^2 + 4) = 2$; again no options work.
- $p = 7$: as per the hint, $7 = \alpha^3 - \alpha = \alpha(\alpha + 1)(\alpha - 1)$, so $(7) = (7, \alpha)(7, \alpha + 1)(7, \alpha - 1)$. Since $N(\alpha) = 7$ it is a principal ideal. $\alpha + 1$ is the root of the polynomial $x^3 - 3x^2 + 2x - 7$ (determined via Sage) so $N(\alpha + 1) = 7$ and $(\alpha + 1)$ is also principal. Similarly $N(\alpha - 1) = 7$ and so $(7, \alpha - 1)$ is principal.

We have considered all ideals below the Minkowski bound; each of these are principal, so $\mathbb{Z}[\alpha]$ must be a PID.

21. By the binomial theorem, $(\sqrt[3]{m} + a)^3 = m + 3a(\sqrt[3]{m})^2 + 3a^2(\sqrt[3]{m}) + a^3$. Therefore the minimum polynomial for $\sqrt[3]{m} + a$ is $x^3 - \text{something} \cdot x^2 - \text{something} \cdot x - (m + a^3)$, so $N(\sqrt[3]{m} + a) = m + a^3$.
22. Any purely cubic field has 2 complex embeddings, so the Minkowski bound is $\frac{3!}{3^3} \left(\frac{4}{\pi}\right) \sqrt{|-27m^2|} = \frac{8m\sqrt{3}}{3\pi}$. By Exercise 41 of Chapter 2, if $m \neq \pm 1$ (9), $\mathbb{A} \cap K = \mathbb{Z}[\sqrt[3]{m}]$ (this is the case for $m = 3, 5, 6$). A real cubic field has degree 3 over \mathbb{Q} , so it has no non-trivial subfields and if a prime ramifies, it is totally ramified.

$m = 3$: the Minkowski bound is ≈ 4.4 and 3 ramifies as $(\sqrt[3]{3})^3$ (principal). $(\sqrt[3]{3}^2 + \sqrt[3]{3} + 1)(\sqrt[3]{3} - 1) = (2)$; by Exercise 21, $N(\sqrt[3]{3} - 1) = 3 + (-1)^3 = 2$ and so both the ideals lying over 2 are principal. Therefore $\mathbb{Q}[\sqrt[3]{3}]$ is a PID.

$m = 5$: the Minkowski bound is ≈ 7.3 . Since 5 ramifies as $(\sqrt[3]{5})^3$ (principal) we need to examine the ideal classes of 2, 3, and 7. Let $\alpha = \sqrt[3]{5}$.

The polynomial $x^3 - 5 \equiv x^3 + 1 \pmod{2}$ factors as $(x + 1)(x^2 + x + 1)$, so by Dedekind factoring (Theorem 25) 2 splits into two factors, $(2, \alpha + 1)(2, \alpha^2 + \alpha + 1)$. Sage indicates that these are principal ideals and equal to $(\alpha^2 - \alpha - 1)(\alpha^2 + 2\alpha + 3)$. [Not sure how to see this without a computer algebra program]

$\alpha - 2$ has norm -3 and so the ideal $(\alpha - 2)$ that lies over 3 is principal (in fact this ideal ramifies over 3).

Considering the ideal class of (7) : $x^3 - 5$ is irreducible mod 7; if it were reducible it would have a root, but 5 is not a cubic residue mod 7. Therefore 7 remains inert. Therefore $\mathbb{Q}[\sqrt[3]{5}]$ is a PID.

$m = 6$: the Minkowski bound is ≈ 8.8 , so we again must consider the ideal class groups of 2, 3, 5, and 7. Let $\alpha = \sqrt[3]{6}$.

By Exercise 21, $N(\alpha - 2) = -2$; as 2 is ramified, 2 factors as $(\alpha - 2)^3$. 3, 5, and 7 are seen as principal via Sage computation. 3 is ramified as

$(\alpha^2 + 2\alpha + 3)^3$, 5 has the principal ideals $(\alpha - 1)$ and $(\alpha^2 + \alpha + 1)$ both with norm 5 lying over it, and 7 factors as $(7) = (\alpha + 1)(2\alpha^2 + 4\alpha + 7)(\alpha^2 + \alpha - 5)$. Therefore $\mathbb{Z}[\alpha]$ is a PID.

23. (a) We let $K = \mathbb{Q}[\sqrt[3]{m}]$ and take $\alpha = \sqrt[3]{m}$. By Exercise 17, Chapter 2, the norm of $\alpha + b\alpha + c\alpha^2$ is the determinant of multiplying this element by each of the K basis $\{1, \alpha, \alpha^2\}$. Therefore:

$$N(a + b\alpha + c\alpha^2) = \det \begin{pmatrix} a & cm & bm \\ b & a & cm \\ c & b & a \end{pmatrix} = a^3 + c^3m^2 + b^3m - 3abcm$$

23. (b) If m is squarefree then $\{1, \alpha, \alpha^2\}$ is an integral basis (Exercise 41, Chapter 2). Therefore for all $\beta \in \mathbb{A} \cap K$, β has the form $a + b\alpha + c\alpha^2$ and so by (a), $N(\beta) \equiv a^3 \pmod{m}$.

24. Let $K = \mathbb{Z}[\sqrt[3]{7}]$ and $\alpha = \sqrt[3]{7}$. The Minkowski bound ≈ 10.29 so we must examine the ideals $(2), (3), (5), (7)$. (7) ramifies as the principal ideal $(\alpha)^3 = 7$. (2) and (5) factor as per Dedekind.

$x^3 - 7 \equiv x^3 + 1 \pmod{2}$ so $(2) = (2, \alpha + 1)(2, \alpha^2 + \alpha + 1)$. By Exercise 23, $N(\alpha + 1) = 1 + 7 = 8$, so $(2, \alpha + 1)$ is not principal. Similarly, $N(\alpha^2 + \alpha + 1) = 1 + 7^2 + 7 - 21 = 36$, and so $(2, \alpha^2 + \alpha + 1)$ is also not principal. $(2, \alpha + 1)^2 = (4, 2\alpha + 2, \alpha^2 + 2\alpha + 1) = (4, \alpha + 1)$. $(2, \alpha + 1)^3 = (\alpha + 1)$ which is principal. There are thus at least 3 ideal classes.

TODO: 5

Chapter 7

1. (a)

$$\begin{aligned} 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots - 2 \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \dots \right) \\ &= \zeta(s) - 2 \left(2^{-s} + \frac{2^{-s}}{2} + \frac{2^{-s}}{3} + \dots \right) \\ &= \zeta(s) - 2^{1-s} \zeta(s) = \zeta(s)(1 - 2^{1-s}) \end{aligned}$$

1. (b) $\frac{d}{ds}(1 - 2^{1-s}) = 2^{1-s} \ln(2)$ has the value $\ln(2)$ at $s = 1$ so it has a simple zero (order 1).

3.

$$\zeta_{K, A \cup B}(s) = \prod_{P \in A, B} \left(1 - \frac{1}{\|P\|^s} \right) = \prod_{P \in A} \left(1 - \frac{1}{\|P\|^s} \right) \prod_{P \in B} \left(1 - \frac{1}{\|P\|^s} \right)$$

The last equality follows as A, B are disjoint. Therefore $\zeta_{K, A \cup B} = \zeta_{K, A} \zeta_{K, B}$.

As the quotient and the product of two non-zero analytic functions on the same region is also analytic, the relationship $\zeta_{K,A \cup B} = \zeta_{K,A} \zeta_{K,B}$ shows that if any two of these functions are well-defined on the half plane $x > 1/2$, the third must also be.

Let $\zeta_{K,A \cup B}^n$ has a pole of order m (so $d(A \cup B) = m/n$). Therefore the product $\zeta_{K,A}^n \zeta_{K,B}^n$ has a pole of order m . Letting k and j be the largest negative numbers for the Laurent expansion of A and B , we must have $k+j = m$. Therefore $\zeta_{K,A}^n$ has polar density k/n and $\zeta_{K,B}^n$ has polar density j/n , and so $d(A \cup B) = d(A) + d(B)$.

4. Let H be the normal subgroup of $\{1, -1\} \in \mathbb{Z}_m^*$. By Corollary 3 of Theorem 43, this set has density $2/\varphi(m)$. Similarly, the subgroup $\{1\}$ (primes $p \equiv 1 \pmod{m}$) has density $1/\varphi(m)$. Applying the result of Exercise 3, $d(\text{primes} \equiv 1, -1 \pmod{m}) = d(\text{primes} \equiv 1 \pmod{m}) + d(\text{primes} \equiv -1 \pmod{m})$, so $2/\varphi(m) = 1/\varphi(m) + d(\text{primes} \equiv -1 \pmod{m})$. The result follows.
5. $\varphi(24) = 8$, and $\mathbb{Z}_{24}^* \simeq \mathbb{Z}_8^* \times \mathbb{Z}_3^* = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, so every element $a \in \mathbb{Z}_{24}^*$ has order 2. Applying Corollary 3 of Theorem 43, each subgroup $\{1, a\}$ has polar density $2/8$. $\{1\}$ has polar density $1/8$, so by applying Exercise 3, we see $\{a\}$ has polar density $1/8$ as desired.