Mach Cutoff: Determining Supersonic Flight Limits

A Guide to Atmospheric Refraction and Flight Envelope Calculation

1. Introduction

Mach Cutoff is an acoustic phenomenon where the sonic boom generated by a supersonic aircraft refracts (bends) upward due to atmospheric gradients, preventing the shockwave from reaching the ground.

For flight planning, this creates a "Boomless Window"—a specific range of Mach numbers where an aircraft can fly supersonically without creating a ground-level disturbance. This document outlines the mathematical framework for calculating the maximum permissible Mach number (M_{max}) and optimal cruise altitude given a specific atmospheric column.

2. The Physics of Refraction

Acoustic waves naturally refract **away** from regions of high sound velocity and **toward** regions of low sound velocity.

In a standard atmosphere:

- **Ground Level:** Warm air $\rightarrow$ High speed of sound.
- **Cruise Altitude:** Cold air $\rightarrow$ Low speed of sound.

As the shockwave travels downward from the cold altitude into the warm lower atmosphere, the increasing speed of sound causes the wavefront to tilt upward. If the aircraft's ground speed is sufficiently low, the wave will eventually refract entirely horizontally and then back upward, never touching the surface.

3. The Governing Limit

The fundamental condition for Mach Cutoff is that the aircraft's **Ground Speed (V_G)** must be less than the **maximum effective speed of sound** found in the air column below the aircraft.

$$V_G < V_{eff,max}$$

3.1 Effective Speed of Sound (V_{eff})

Sound propagation is affected by temperature (local sound speed, a) and wind. We define the effective speed of sound at any altitude z as:

$$V_{eff}(z) = a(z) + u_{wind}(z)$$

Where:

- $a(z)$: Local speed of sound $\approx 20.05\sqrt{T_{kelvin}}$.
- $u_{wind}(z)$: The wind velocity component in the direction of flight (Tailwind is positive, Headwind is negative).

3.2 The Theoretical Threshold (Surface Limit)

The absolute "acoustic ceiling" is established by the fastest layer of air below the aircraft. In 99% of atmospheric profiles, the air is warmest at the ground. Therefore, the theoretical limit is:

$$V_{limit\_theory} = V_{eff,surface} = a_{surface} + u_{surface}$$

3.3 The Safety Buffer (Z_{safe})

In practical operations, relying on refraction exactly at ground level is risky. If the boom refracts exactly at $Z = 0$, a **Caustic** (a focused, amplified shockwave) sweeps the ground. Furthermore, terrain elevation changes and atmospheric turbulence can cause the boom to "leak" through.

To prevent this, flight paths are planned so the refraction occurs at a **Safety Altitude (Z_{safe})**, typically **3,000 to 5,000 ft** above ground level. This ensures the shockwave turns upward before it gets near people or structures.

The **Operational Limit** becomes the effective speed of sound at this safety altitude:

$$V_{limit} = V_{eff}(Z_{safe}) = a(Z_{safe}) + u(Z_{safe})$$

Note: Since air at 5,000 ft is typically cooler than at the surface, V_{limit} will be lower than the surface limit, resulting in a slightly lower, more conservative maximum Mach number.

4. Calculating Maximum Mach Number (M_{max})

To provide a pilot or autopilot with a target speed, we must convert the ground speed limit into a Mach number relative to the air at cruise altitude.

Step 1: Define Ground Speed The aircraft's ground speed is its True Airspeed (V_{TAS}) plus the wind at cruise altitude:

$$V_G = V_{TAS} + u_{cruise}$$

Step 2: Convert to Mach Since $V_{TAS} = M \times a_{cruise}$:

$$V_G = (M \cdot a_{cruise}) + u_{cruise}$$

Step 3: Apply the Limit Set V_G equal to the operational limit V_{limit} and solve for Mach (M):

$$(M_{max} \cdot a_{cruise}) + u_{cruise} = V_{limit}$$

$$M_{max} = \frac{V_{limit} - u_{cruise}}{a_{cruise}}$$

5. The Algorithm: Finding Optimal Altitude

To find the fastest possible "boomless" speed, one must scan the atmospheric column to find the altitude that maximizes M_{max}.

Inputs: A dataset of Altitude (z), Temperature (T), and Wind (u) for the air column.

1. **Determine the Operational Limit:** Select a safety altitude Z_{safe} (e.g., 5,000 ft). Find the effective speed of sound at that specific altitude.

$$V_{limit} = a(Z_{safe}) + u(Z_{safe})$$

2. **Scan the Column:** For every altitude layer z above Z_{safe}:
   - Calculate local sound speed: $a(z)$.
   - Identify local wind component: $u(z)$.
   - Compute the Mach limit for that altitude:

$$M_{limit}(z) = \frac{V_{limit} - u(z)}{a(z)}$$

3. **Identify Optimum:** Compare the M_{limit} values across all altitudes. The altitude yielding the highest M_{limit} is the optimal cruise flight level.

6. Worked Example

Scenario: A supersonic transport flying East.

- **Safety Buffer:** 5,000 ft.
- **Conditions at 5,000 ft:** Warm (20°C), Moderate Tailwind (15 m/s).
- **Conditions at Cruise (35,000 ft):** Tropopause ($ -55^\circ$C), Strong Headwind (20 m/s against flight path).

Step A: Establish the Limit (At Safety Altitude)

- Temperature $T_{5k} = 20^\circ C \approx 293K$.
- Sound Speed $a_{5k} \approx 343$ m/s.
- Wind $u_{5k} = +15$ m/s.

$$V_{limit} = 343 + 15 = \mathbf{358} \text{ m/s}$$

(Note: If we used the surface at $30°C$, V_{limit} might have been ~365 m/s. The buffer makes us slower/safer.)

Step B: Analyze Cruise Altitude (35,000 ft)

- Temperature $T_{cruise} = -55°C \approx 218K$.
- Sound Speed $a_{cruise} \approx 295$ m/s.
- Wind $u_{cruise} = -20$ m/s (Headwind).

Step C: Calculate Max Mach

$$M_{max} = \frac{V_{limit} - u_{cruise}}{a_{cruise}}$$

$$M_{max} = \frac{358 - (-20)}{295}$$

$$M_{max} = \frac{358 + 20}{295} = \frac{378}{295}$$

$$\mathbf{M_{max} \approx 1.28}$$

Conclusion

Under these weather conditions with a 5,000 ft safety buffer, the aircraft can fly at **Mach 1.28** at