2.a) $H = \int_0^L dx \left[\frac{\mu^2\dot{\xi}^2}{2\mu} + \frac{T}{2}\left(\frac{\partial\xi}{\partial x}\right)^2\right]$

Ansatz: $\xi(x,t) = \sum_n \xi_{1,n}(x)\,\xi_{2,n}(t)$

must obey: 1. $(\Delta + k^2)\xi_1(x) = 0$
$\rightarrow \frac{\partial^2}{\partial x^2}\xi_1(x) = -k^2\xi_1(x)$

2. $\left(\frac{\partial^2}{\partial t^2} - \frac{T}{\mu}\frac{\partial^2}{\partial x^2}\right)\xi(x,t) = 0$
$\rightarrow \sum_n \xi_1 \underbrace{\frac{\partial^2}{\partial t^2}\xi_2(t)}_{\omega_n^2\xi_2(t)} - \frac{T}{\mu}\underbrace{\left(\frac{\partial^2}{\partial x^2}\xi_1(x)\right)}_{-k^2\xi_1(x)}\xi_2(t) \overset{!}{=} 0 \qquad \Rightarrow \omega_n^2 = \frac{T}{\mu}k^2$

$\frac{T}{2}\int_0^L\left(\frac{\partial\xi}{\partial x}\right)^2 dx = -\frac{T}{2}\Big(\underbrace{\left[\xi(x,t)\frac{\partial\xi}{\partial x}\right]_0^L}_{=0 \text{ from boundary conditions}} - \int_0^L \xi(x,t)\frac{\partial^2\xi}{\partial x^2}dx\Big) = \left(-\int_0^L \xi^2(x,t)(-k^2)\,dx\right)\cdot\frac{T}{2}$

$\rightarrow H = \int_0^L dx\left[\frac{\mu^2\dot{\xi}^2}{2\mu} + \frac{T}{2}k^2\xi^2(x,t)\right] = \frac{1}{2}\int_0^L dx\left[\frac{\pi^2}{\mu} + \omega_n^2\xi^2\cdot\mu\right]$

$\xi_{1,n}(x)$ satisfies Helmholtz equation as above

Ansatz: $\xi_{1,n} = A\sin\left(\frac{n\pi x}{L}\right)_{n\in\mathbb{N}}$ $\quad A^2\int_0^L dx \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = A^2\left[\frac{1}{2}\int_0^L dx\cos\left(\frac{(n-m)\pi}{L}x\right) - \cos\left(\frac{(n+m)\pi}{L}x\right)\right]$
(to get rid of integral)

satisfies boundary conditions

$= \frac{A^2}{2}\left[\frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi}{L}x\right) - \frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi}{L}x\right)\right]_0^L$

$\xi_{1,n} = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$

$= \frac{A^2}{2}\left[0 - 0 - 0 + 0\right] = 0 \quad \forall n \neq m$

$\frac{\partial^2}{\partial x^2}\xi_{1,n} = \left(\frac{n\pi}{L}\right)^2\xi_{1,n}$
$= k^2$
$\rightarrow k = \frac{n\pi}{L}$

$n = m: \quad = \frac{A^2}{2}\left[\int_0^L dx\; 1 - \cos\left(\frac{2\pi n}{L}x\right)\right] = \frac{A^2}{2}L - \frac{A^2}{2}\int_0^L\cos\left(\frac{2\pi n}{L}x\right)dx$
$= \frac{A^2 L}{2} - \frac{A^2}{2}\left[\frac{L}{2\pi n}\sin\left(\frac{2\pi n}{L}x\right)\right]_0^L$
$= \frac{A^2 L}{2} \overset{!}{=} 1 \rightarrow A = \sqrt{\frac{2}{L}}$

$\omega_n = \sqrt{\frac{T}{\mu}}\frac{n\pi}{L}$

$H = \sum_n \frac{1}{2}\left[\mu\dot{\xi}_{2,n}^2 + \mu\omega_n^2\xi_{2,n}^2\right]$ since the spacial dependence cancels out through the orthonormality

$q_n = \sqrt{\mu}\,\xi_{2,n}$

$\Rightarrow H = \frac{1}{2}\sum_{n=1}\left[\dot{q}_n^2 + \omega_n^2 q_n^2\right]$

with $q_n = \sqrt{\frac{\hbar}{2\omega_n}}(\hat{a}_n^+ + \hat{a}_n)$ $\quad p_n = \dot{q}_n = i\sqrt{\frac{\hbar\omega_n}{2}}(\hat{a}_n^+ - \hat{a}_n)$

$H = \sum_n \hbar\omega_n(\hat{a}_n^+\hat{a}_n + \frac{1}{2})$

b) $\hat{\xi}(x,t) = \sum_n \xi_{1,n}(x)\xi_{2,n}(t) = \sum_n\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\sqrt{\frac{\hbar}{2\omega_n\mu}}(\hat{a}_n^+ + \hat{a}_n)$, $n\in\mathbb{N}$ and $\omega_n =$

$\dot{a}_n = \frac{i}{\hbar}[H, a_m] = \frac{i}{\hbar}\left(\sum_n\hbar\omega_n[N, \hat{a}_m]\right) = -i\omega_n\hat{a}_n$ for $n=m$ otherwise zero since $[a_n, a_m^+] = \delta_{n,m}$
Ehrenfest

$\Rightarrow \hat{\xi}(x,t) = \sum_n\sqrt{\frac{\hbar}{L\omega_n\mu}}\left(\hat{a}_n(0)e^{-i\omega_n t} + \hat{a}_n^+(0)e^{i\omega_n t}\right)\sin\left(\frac{n\pi x}{L}\right)$

2c)

Foch states

$$\mathrm{Var}\,\xi^2 = \langle n|\xi^2|n\rangle - \underbrace{\langle n|\xi|n\rangle^2}$$

$$A\cdot\langle n|\hat{a}_n(0)e^{-i\omega_n t} + \hat{a}_n^\dagger(0)e^{i\omega_n t}|n\rangle$$

$$= 0$$

$$\to A^2\langle n|(\hat{a}_n e^{-i\omega_n t} + \hat{a}_n^\dagger e^{i\omega_n t})(\hat{a}_n e^{-i\omega_n t} + \hat{a}_n^\dagger e^{i\omega_n t})|n\rangle$$

$$= A^2\langle n|\hat{a}_n\hat{a}_n^\dagger + \hat{a}_n^\dagger\hat{a}_n|n\rangle$$

$$= A^2\langle n|n+1+n|n\rangle = A^2(2n+1) = \left|\sin^2\left(\frac{n\pi x}{L}\right)\right| \frac{\hbar}{\mu\omega_n L}(2n+1)$$

$$\leq \frac{\hbar}{\mu\omega_n L}(2n+1)$$

$$\Delta\xi \leq \sqrt{\frac{\hbar}{\mu\omega_n L}(2n+1)} = (2.9\cdot 10^{-19}\,m)\sqrt{2n+1}$$

3. a)

i) $\langle\beta|\alpha\rangle = e^{-\frac{|\beta|^2}{2}} \sum_n \langle n| \frac{(\beta^*)^n}{\sqrt{n!}} \; e^{-\frac{|\alpha|^2}{2}} \sum_{n'} \frac{\alpha^{n'}}{\sqrt{n'!}} |n'\rangle$

$= e^{-\frac{|\beta|^2+|\alpha|^2}{2}} \sum_n \frac{(\beta^*)^n}{n!} \alpha^n = e^{-\frac{|\beta|^2+|\alpha|^2}{2}} \sum_n \frac{(\alpha\beta^*)^n}{n!}$

$= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} e^{\alpha\beta^*} = e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} e^{-\frac{|\alpha|^2+|\beta|^2}{2}} e^{\frac{1}{2}(\alpha\beta^*+\alpha^*\beta)}$

$= e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} e^{-\frac{(|\alpha|^2+|\beta|^2 - \alpha\beta^* - \alpha^*\beta)}{2}} = e^{i(\frac{1}{2i}(\alpha\beta^*-\alpha^*\beta))} e^{-\frac{1}{2}(\alpha-\beta)(\alpha^*-\beta^*)} = e^{i\,\mathrm{Im}(\alpha\beta^*)} e^{-|\alpha-\beta|^2/2}$

with $\mathrm{Im}(z) = \frac{1}{2i}(z-\bar{z})$ $\square$

ii) $\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \frac{1}{\pi}\int d^2\alpha\, e^{-|\alpha|^2} \sum_{n,m=0} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle\langle m|$

$= \frac{1}{\pi}\int_0^\infty \int_0^{2\pi} dr\, d\phi\, r e^{-r^2} \sum_{n,m=0} \frac{(re^{i\phi})^n (re^{-i\phi})^m}{\sqrt{n!m!}} |n\rangle\langle m|$

$= \frac{1}{\pi}\sum_{n,m=0} \left(\int_0^\infty dr\; e^{-r^2} r^{n+m+1} \underbrace{\int_0^{2\pi} d\phi\, (e^{i\phi})^{n-m}}_{\text{only} \neq 0 \text{ if } n=m} |n\rangle\langle m| \frac{1}{\sqrt{n!m!}} \right)$ o.b.d.A: $n \geq m$

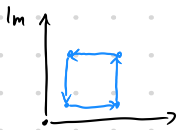
, same for $m \geq n$

$\Longrightarrow \quad = \frac{1}{\pi}\sum_{n=0} \quad e^{-|\alpha|^2} \frac{(|\alpha|^2)^n}{n!} |n\rangle\langle n| = e^{-|\alpha|^2} e^{|\alpha|^2} = \mathbb{1}$ $\square$

b) $D(-iy)D(-x)D(iy)D(x)$ from lecture: $D(\alpha)D(\beta) = e^{i\,\mathrm{Im}(\alpha\beta^*)} D(\alpha+\beta)$

$= e^{i\,\mathrm{Im}(-iy\cdot -x)} D(-(x+iy))\, e^{i\,\mathrm{Im}(iy\cdot x)} D(x+iy)$

$= e^{i2\,\mathrm{Im}(ixy)} D^{-1}(x+iy) D(x+iy) = e^{i2\,\mathrm{Im}(ixy)} = e^{2ixy}$ $\square$



geometric interpretation: e^{2ixy} is the phase acumulated over this set of displacements proportional to the area xy enclosed in the path.

c) $[Q_1, Q_2] = \cos(\sigma X)\cos(2\pi \frac{Y}{\sigma}) - \cos(2\pi\frac{Y}{\sigma})\cos(\sigma X)$

$= \frac{1}{2}(e^{i\sigma X} + e^{-i\sigma X})\frac{1}{2}(e^{i\frac{2\pi}{\sigma}Y} + e^{-i\frac{2\pi}{\sigma}Y}) - \frac{1}{2}(e^{i\frac{2\pi}{\sigma}Y} + e^{-i\frac{2\pi}{\sigma}Y})\frac{1}{2}(e^{i\sigma X} + e^{-i\sigma X})$

$= \frac{1}{4}\left(e^{i\sigma X} e^{i\frac{2\pi}{\sigma}Y} + e^{i\sigma X} e^{-i\frac{2\pi}{\sigma}Y} + e^{-i\sigma X} e^{i\frac{2\pi}{\sigma}Y} + e^{-i\sigma X} e^{-i\frac{2\pi}{\sigma}Y} \right) - \frac{1}{4}$

$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$

with $[X,Y] = 2i\sigma_z$

$i\sigma X \cdot i\frac{2\pi}{\sigma}Y - i\frac{2\pi}{\sigma}Y \cdot i\sigma X$

$= -2\pi\, XY + 2\pi\, YX = 2\pi[Y,X] = -2\pi\cdot 2iZ$

$= -i4\pi Z$

$= \frac{1}{4}\left[e^{i(\sigma X + \frac{2\pi}{\sigma}Y)} \underbrace{e^{-i2\pi Z}}_{\mathbb{1}} + e^{i(\sigma X - \frac{2\pi}{\sigma}Y)} \underbrace{e^{i2\pi Z}}_{\mathbb{1}} + e^{-i(\sigma X - \frac{2\pi}{\sigma}Y)} e^{i2\pi Z} + e^{-i(\sigma X + \frac{2\pi}{\sigma}Y)} e^{-i2\pi Z} \right]$

$-\frac{1}{4}\left[e^{i(\frac{2\pi}{\sigma}Y + \sigma X)} + e^{-i(\sigma X - \frac{2\pi}{\sigma}Y)} + e^{i(\sigma X - \frac{2\pi}{\sigma}Y)} + e^{-i(\frac{2\pi}{\sigma}Y + \sigma X)} \right] = 0$ all terms cancel