7. $S(\xi) = e^{-\overbrace{\frac{1}{2}(\xi\hat{a}^{+2} - \xi^*\hat{a}^2)}^{\mathcal{X}}}$

a) $S^+(\xi)\hat{a}S(\xi) = \hat{a} + [\frac{1}{2}(\xi\hat{a}^{+2} - \xi^*\hat{a}^2), \hat{a}] + \frac{1}{2}[\mathcal{X}, \underbrace{[\mathcal{X}, \hat{a}]}_{\xi\hat{a}^+}] + \dots = \hat{a} - re^{2i\phi}\hat{a}^+ + \frac{r^2}{2}\hat{a} - \frac{1}{6}\frac{r^2}{2}re^{2i\phi}\hat{a}^+$

$= \cosh(r)\hat{a} - e^{2i\phi}\sinh(r)\hat{a}^+$

$\frac{1}{2}(\xi\underbrace{[\hat{a}^{+2}, \hat{a}]}- \xi^*\overbrace{[\hat{a}^2, \hat{a}]}^{=0})$

$= \frac{1}{2}(\xi \cdot -2\hat{a}^+) = \xi\hat{a}^+$

$\underbrace{\frac{1}{2}(\xi^2[\hat{a}^{+2}, \hat{a}^+] - \xi\xi^*\underbrace{[\hat{a}^2, \hat{a}^+]}_{2\hat{a}})}_{= -\frac{|\xi|^2}{2}\hat{a}}$

$\frac{1}{6}\underbrace{[A, \frac{|\xi|^2}{2}\hat{a}]}_{\frac{r^2}{2}\xi\hat{a}^+}$

$e^{-\frac{1}{2}(\xi\hat{a}^{+2} - \xi^*\hat{a}^2)}\hat{a}|n\rangle - \hat{a}e^{-\frac{1}{2}(\xi\hat{a}^{+2} - \xi^*\hat{a}^2)}|n\rangle$

$\sqrt{n}e^{-\frac{\xi}{2}\hat{a}^{+2}}e^{+\frac{\xi^*}{2}\hat{a}^2}e^{-\frac{|\xi|^2}{4}(n-1)}|n-1\rangle - \hat{a}e^{-\frac{\xi}{2}\hat{a}^{+2}}e^{+\frac{\xi^*}{2}\hat{a}^2}e^{-\frac{|\xi|^2}{4}(n-1)}|n\rangle$

O.b.d.A: $S^+(\xi)\hat{a}^+S(\xi) = \hat{a}^+\cosh(r) - ae^{-2i\phi}\sinh(r)$

fully symmetric calculation

b)

$S^+(r)aS(r) = e^{ra^+b^+ - rab}$

$= a + r[a^+b^+ - ab, a \otimes 1]$

$= a + r[a^+b^+, a] - r[ab, a]$

$[a^+b^+, a \otimes 1] = a^+$

$\Big(= a + r(a^+a \otimes b^+ - aa^+ \otimes b^+)|nn\rangle - r(a^2 \otimes b - a^2 \otimes b)|nn\rangle$

$= a + r(n|n\rangle\sqrt{n+1}|n+1\rangle - (n+1)\sqrt{n+1}|n\rangle|n+1\rangle)$

$r(-\sqrt{n+1}|n\rangle|n+1\rangle) = -r\sqrt{n+1}\,|n\rangle|n+1\rangle = -r|n\rangle b^+|n\rangle \Big)$

$= a + r(\underset{(-1)}{[a^+, a]} \otimes b^+) = a - r\underbrace{b^+} + \frac{r}{2}[a^+b^+ - ab, -rb^+] + [a^+b^+ - ab, a]$

$-b^+$

$\frac{r^2}{2}[-ab, -b^+]$

$\frac{r^2}{2}(-a \otimes -bb^+) - (-a \otimes -b^+b)$

$= \frac{r^2}{2}(a \otimes [b, b^+]) = \frac{r^2}{2}a$

$= a - rb^+ + \frac{r^2}{2}a - \frac{r^3}{6}b^+ + \dots$

$= a\cosh(r) - b^+\sinh(r)$

$S^+(r)a^+S(r) = a^+ + r(-[ab, a^+]) + \dots$

$= a^+ - rb + \underbrace{(-\frac{r^2}{2}\underbrace{[a^+b^+, b]}_{-a^+})}_{\frac{r^2}{2}a^+} + \dots$

$[ra^+b^+, -rb]$

$= a^+ - rb + \frac{r^2}{2}a^+ + \frac{r^3}{6}\underbrace{[a^+b^+ - ab, a^+]}_{-b} = a^+\cosh(r) - b\sinh(r)$

$S^+(r)bS(r) = b + r\underbrace{[a^+b^+, b]}_{-ra^+} + \frac{r}{2}\underbrace{[-ab, -ra^+]}_{\frac{r^2}{2}b} + \frac{r^3}{6}\underbrace{[a^+b^+, b]}_{(-1)a^+} + \dots$

$= b\cosh(r) - a^+\sinh(r)$

$S^+(r)b^+S(r) = b^+ + r\underbrace{[-ab, b^+]}_{-a} + \underbrace{[ra^+b^+, -ra]}_{\frac{r^2}{2}b^+} + \frac{r^3}{6}\underbrace{[-ab, b^+]}_{-a} + \dots$

$= b^+\cosh(r) - a\sinh(r)$

$-b$

$$(\Delta X_a)^2 = \langle TMS|X_a^2|TMS\rangle - \langle TMS|X_a|TMS\rangle^2$$

$$= \langle TMS|X_a^2|TMS\rangle - \underbrace{\langle 00|\frac{S^\dagger a S + S^\dagger a^\dagger S}{\sqrt{2}}|00\rangle}_{\langle 00|(a+a^\dagger)\cosh(r) - (b+b^\dagger)\sinh(r)|00\rangle = 0}$$

$$\downarrow$$

$$= \langle TMS|\tfrac{1}{2}(a^2 + a^{\dagger 2} + a^\dagger a + a a^\dagger)|TMS\rangle$$

$$\downarrow$$

$$= \tfrac{1}{2}\langle TMS|\underbrace{(S^\dagger a S)(S^\dagger a S) + (S^\dagger a^\dagger S)^2}_{=0 \text{ because no } a^\dagger, a \text{ or } b^\dagger, b \text{ term mixing}} + (S^\dagger a^\dagger S)(S^\dagger a S) + (S^\dagger a S)(S^\dagger a^\dagger S)|TMS\rangle$$

$$(\cosh(r)a^\dagger - \sinh(r)b)(\cosh(r)a - \sinh(r)b^\dagger) + (\cosh(r)a - \sinh(r)b^\dagger)(\cosh(r)a^\dagger - \sinh(r)b)$$

$$\cosh^2(r)a^\dagger a + \underbrace{\sinh^2(r)bb^\dagger + \cosh^2(r)aa^\dagger}_{\text{only remaining terms}} + \sinh^2(r)b^\dagger b$$

$$\sinh^2(r) + \cosh^2(r) = 2\sinh^2(r) + 1$$

$$= \tfrac{1}{2}(2\sinh^2(r) + 1) = \sinh^2(r) + \tfrac{1}{2} \quad \square$$

$$\underbrace{\langle X_b^2\rangle} - \underbrace{\langle X_b\rangle}_{=0}^2 \quad \text{bcause no mixing terms}$$

$$= \tfrac{1}{2}\langle 00|S^\dagger(b^2 + b^{\dagger 2} + b^\dagger b + bb^\dagger)S|00\rangle$$

$$= \tfrac{1}{2}\langle 00|(S^\dagger b^\dagger S)(S^\dagger b S) + (S^\dagger b S)(S^\dagger b^\dagger S)|00\rangle$$

$$= \tfrac{1}{2}\langle 00|(b^\dagger\cosh(r) - a\sinh(r))(b\cosh(r) - a^\dagger\sinh(r)) + (b\cosh(r) - a^\dagger\sinh(r))(b^\dagger\cosh(r) - a\sinh(r))|00\rangle$$

$$= \tfrac{1}{2}\langle 00|\cancel{b^\dagger b}\cosh(r) + aa^\dagger\sinh^2(r) + bb^\dagger\cosh^2(r)|00\rangle$$

$$= \sinh^2(r) + \tfrac{1}{2} \quad \square$$

$$(\Delta X_+)^2 = \langle X_+\rangle^2 - \underbrace{\langle X_+\rangle}_{\frac{1}{\sqrt{2}}\underbrace{\langle X_a + X_b\rangle}_{\frac{1}{\sqrt{2}}\underbrace{\langle a + a^\dagger + b + b^\dagger\rangle}_{0}}}^2$$

0 because no $aa^\dagger$, $a^\dagger a$, $b^\dagger b$, $bb^\dagger$ terms

$$= \langle X_+^2\rangle$$

$$= \tfrac{1}{2}\langle (X_a + X_b)^2\rangle$$

$$= \tfrac{1}{2}\langle (X_a^2 + X_b^2 + X_a X_b + X_b X_a)\rangle$$

$$= \sinh^2(r) + \tfrac{1}{2} + \underbrace{\tfrac{1}{2}\langle X_a X_b + X_b X_a\rangle}_{\frac{1}{4}[(a+a^\dagger)(b+b^\dagger) + (b+b^\dagger)(a+a^\dagger)]} = \tfrac{1}{2}(\underbrace{\sinh^2(r) + \cosh^2(r)}_{\cosh(2r)}) - 4\cdot\tfrac{1}{4}\underbrace{\sinh(r)\cosh(r)}_{\frac{1}{2}\sinh(2r)} = \tfrac{1}{2}e^{-2r} \quad \square$$

$$\tfrac{1}{4}[ab + \cancel{ab^\dagger} + \cancel{a^\dagger b} + a^\dagger b^\dagger + ba + \cancel{ba^\dagger} + \cancel{b^\dagger a} + b^\dagger a^\dagger]$$

$$\langle ab\rangle = \underbrace{\langle 00|(S^\dagger a S)(S^\dagger b S)|00\rangle}_{(\cosh(r)a - \sinh(r)b^\dagger)(\cosh(r)b - \sinh(r)a^\dagger)}$$

$$-\sinh(r)\cosh(r)b^\dagger b = 0$$

$$-\cosh(r)\sinh(r)aa^\dagger$$

$$\langle ab^\dagger\rangle = \langle 00|(S^\dagger a S)(S^\dagger b^\dagger S)|00\rangle$$

$$= \langle 00|(\cosh(r)a - \sinh(r)b^\dagger)(\cosh(r)b^\dagger - \sinh(r)a)|00\rangle$$

$$= 0 = \langle a^\dagger b\rangle$$

$$\langle ba\rangle = \langle 00|(\cosh(r)b - \sinh(r)a^\dagger)(\cosh(r)a - \sinh(r)b^\dagger)|00\rangle$$

$$= -\cosh(r)\sinh(r)bb^\dagger$$

$$\langle b^\dagger a^\dagger\rangle = -\cosh(r)\sinh(r)aa^\dagger$$

$$\langle a^\dagger b^\dagger\rangle = \langle 00|(\cosh(r)a^\dagger - \sinh(r)b)(\cosh(r)b^\dagger - \sinh(r)a)|00\rangle$$

$$= -\sinh(r)\cosh(r)bb^\dagger$$

$$\bar{H}_R(\alpha) = \langle\alpha|H_R|\alpha\rangle = \hbar\omega_{eg}|e\rangle\langle e| + \hbar\omega_c\langle\alpha|a^\dagger a|\alpha\rangle + \frac{\hbar g}{2}(\langle\alpha|a|\alpha\rangle + \langle\alpha|a^\dagger|\alpha\rangle)(\sigma_+ + \sigma_-)$$

$$= \hbar\omega_{eg}|e\rangle\langle e| + \hbar\omega_c|\alpha|^2 + \frac{\hbar g}{2}(\alpha + \alpha^*)(\sigma_+ + \sigma_-)$$

$$= \hbar\omega_{eg}\begin{pmatrix}0&0\\0&1\end{pmatrix} + \hbar\omega_c|\alpha|^2\begin{pmatrix}1&0\\0&1\end{pmatrix} + \hbar g Re(\alpha)\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$= \hbar\begin{pmatrix}\omega_c|\alpha|^2 & g Re(\alpha)\\ g Re(\alpha) & \omega_{eg} + \omega_c|\alpha|^2\end{pmatrix}$$

$$\rightarrow \det\begin{pmatrix}\lambda - \hbar\omega_c|\alpha|^2 & -g Re(\alpha)\\ -g Re(\alpha) & \lambda - \hbar(\omega_{eg} + \omega_c|\alpha|^2)\end{pmatrix} = 0$$

$$= (\lambda - \hbar\omega_c|\alpha|^2)(\lambda - \hbar(\omega_{eg} + \omega_c|\alpha|^2)) - g^2 Re^2(\alpha)\hbar^2 = 0$$

$$= \lambda^2 - \hbar\omega_c|\alpha|^2\lambda - \hbar(\omega_{eg} + \omega_c|\alpha|^2))\lambda + \hbar^2\omega_c|\alpha|^2(\omega_{eg} + \omega_c|\alpha|^2) - g^2 Re^2(\alpha)\hbar^2$$

$$2E_\pm = \hbar(2\omega_c|\alpha|^2 + \omega_{eg}) \pm \hbar\sqrt{(2\omega_c|\alpha|^2 + \omega_{eg})^2 - 4\cdot(\omega_c|\alpha|^2(\omega_{eg} + \omega_c|\alpha|^2) - \cancel{g^2 Re^2(\alpha)})}$$

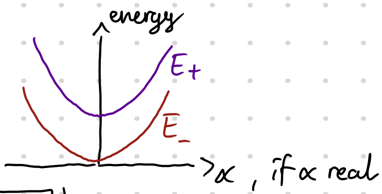
$g \ll \omega_c$:

$$(\cancel{4\omega_c^2|\alpha|^4} + \cancel{4\omega_{eg}\omega_c|\alpha|^2} + \omega_{eg}^2) - \cancel{4\omega_c^2|\alpha|^2\omega_{eg}} - \cancel{4\omega_c^2|\alpha|^4}$$

$$E_\pm = \frac{1}{2}[\hbar(2\omega_c|\alpha|^2 + \omega_{eg}) \pm \hbar\omega_{eg}]$$

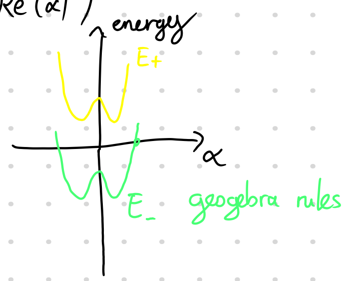
$$E_+ = \omega_c|\alpha|^2 + \omega_{eg}$$

$$E_- = \omega_c|\alpha|^2$$



$g > \omega_c$: $E_\pm = \frac{1}{2}(\hbar(2\omega_c|\alpha|^2 + \omega_{eg}) \pm \hbar\sqrt{\omega_{eg}^2 + 4g^2 Re^2(\alpha)})$

$\Rightarrow$ double well ?



b) $\omega_{eg} = \omega_c$: $E_\pm = \frac{1}{2}(\hbar(2\omega_c|\alpha|^2 + \omega_{eg}) - \hbar\sqrt{\omega_{eg}^2 + 4g^2 Re^2(\alpha)})$

$$E_- = \frac{1}{2}(\hbar\omega_c(2|\alpha|^2 + 1)) - \hbar\sqrt{\omega_c^2 + 4g^2 Re^2(\alpha)}$$

α_0: $\frac{d}{d\alpha}\left[\frac{1}{2}(\hbar\omega_c(2|\alpha|^2 + 1)) - \hbar\sqrt{\omega_c^2 + 4g^2 Re^2(\alpha)}\right] \overset{!}{=} 0$ take α real:

$= \frac{1}{2}\hbar\omega_c 4\alpha - \frac{1}{2\sqrt{\omega_c^2 + 4g^2\alpha^2}\hbar}\cdot 8g^2\alpha = 0$ if $\alpha_0 = 0 \rightarrow$ for $g < \omega_c$ $\rightarrow$ normal phase

else: $(2\sqrt{\omega_c^2 + 4g^2\alpha^2}\hbar)\, 2\hbar\omega_c\alpha = 8g^2\alpha$

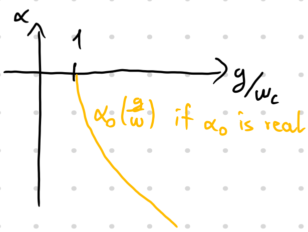
so for $g > \omega$

$$4\hbar^2(\omega_c^2 + 4g^2\alpha^2)\cdot 4\hbar^2\omega_c^2 = 64g^4$$

$$(4\hbar\omega_c^2)^2 + 4g^2\alpha^2 4\hbar^2\omega_c^2 = 64g^4$$

$$\alpha^2 = \frac{4g^4 - \hbar^4\omega_c^4}{g^2\hbar^2\omega_c^2}$$

$$\alpha_0 = -\sqrt{\frac{4g^4 - \hbar\omega_c^4}{g^2\hbar^2\omega_c^2}} = -\sqrt{\frac{4\frac{g^4}{\omega_c^4} - \hbar}{\frac{g^2}{\omega_c^2}\hbar^2}}$$



$\Rightarrow$ superradiant phase

9.a) $\tilde{H} = \boxed{U H_{JC} U^{\dagger} + i\hbar \dot{U} U^{\dagger}}$

$|0\rangle\langle 1| \; 1\rangle$

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\omega_c t} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\omega_c t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\omega_c t} \end{pmatrix}\begin{pmatrix} 0 & e^{-i\omega_c t} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{-i\omega_c t} \\ 0 & 0 \end{pmatrix}$$

$$= e^{i\omega_c t(a^{\dagger}a + |e\rangle\langle e|)} H_{JC}\, e^{-i\omega_c t(a^{\dagger}a + |e\rangle\langle e|)} + i\hbar i\omega_c (a^{\dagger}a + |e\rangle\langle e|) \cancel{UU^{\dagger}}$$

$$= e^{i\omega_c t \hat{n}} \hbar\omega_c \hat{n}\, e^{-i\omega_c t\hat{n}} + e^{i\omega_c t|e\rangle\langle e|} \hbar\omega_{eg} |e\rangle\langle e|\, e^{-i\omega_c t|e\rangle\langle e|} + \frac{\hbar}{2} g\Big[\underbrace{\left(e^{i\omega_c t a^{\dagger}a}\, a\, e^{-i\omega_c t a^{\dagger}a}\right)}\underbrace{\left(e^{i\omega_c t \overset{|e\rangle\langle e|}{\vee}} \sigma_+ e^{-i\omega_c t \overset{|e\rangle\langle e|}{\vee}}\right)}_{e^{i\omega_c t}\sigma_+} + \left(e^{i\omega_c t a^{\dagger}a}\, a^{\dagger} e^{-i\omega_c t a^{\dagger}a}\right)\underbrace{\left(e^{i\omega_c t \overset{|e\rangle\langle e|}{\vee}} \sigma_- e^{-i\omega_c t \overset{|e\rangle\langle e|}{\vee}}\right)}_{e^{-i\omega_c t}\sigma_-}\Big] + \ldots$$

$$= \hbar\omega_c \hat{n} + \hbar\omega_{eg} |e\rangle\langle e| + \ldots$$

$$a + i\omega_c t \underbrace{[a^{\dagger}a, a]}_{-a} + \underbrace{\frac{1}{2}(i\omega_c t)[a^{\dagger}a, -a i\omega_c t]}_{-\frac{(i\omega_c t)^2}{2}\underbrace{[a^{\dagger}a, a]}_{-a}} + \underbrace{\frac{1}{6}(i\omega_c t)[a^{\dagger}a, (i\omega_c t)^2 a]}_{-\frac{1}{6}(i\omega_c t)^3 a}$$

$$= \cosh(i\omega_c t)\, a - \sinh(i\omega_c t)\, a$$

$$= e^{-i\omega_c t} a$$

$$(\cosh(x) + \sinh(x))\, a^{\dagger} = e^{i\omega_c t} a^{\dagger}$$

from $[N, a^{\dagger}] = a^{\dagger}$

$$= \hbar\underbrace{(\omega_{eg} - \omega_c)}_{=\Delta} |e\rangle\langle e| + \frac{\hbar g}{2}(a\sigma_+ + a^{\dagger}\sigma_-)$$