Übung zur Vorlesung

Maschinelles Lernen 2

Sommersemester 2013

Abteilung Maschinelles Lernen
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Exercise sheet 3

Due May 6, 2013, at 10:00 a.m. local time in electronic form via the ISIS website. See the corresponding submission page for formatting details (.zip archives, source code without fancy non-ASCII symbols etc.)

3. **The SSA Cost Function (30 + 30 P)**

(a) Let $X_1 \sim \mathcal{N}(\mu_1, \Sigma_1)$ and $X_2 \sim \mathcal{N}(\mu_2, \Sigma_2)$ be Gaussian random variables with values in $\mathbb{R}^n$, with mean $\mu_i \in \mathbb{R}^n$ and covariance matrix $\Sigma_i \in \mathbb{R}^{n\times n}$, i.e., the probability density function of X_i is

$$p_i(x) = (2\pi \det \Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu_i)^\top \Sigma^{-1}(x-\mu_i)\right).$$

Derive the following explicit formula for the KL-divergence between X_1 and X_2:

$$D_{\mathrm{KL}}\left(X_2 \,||\, X_1)\right) = \frac{1}{2}\left(\log\left(\frac{\det \Sigma_1}{\det \Sigma_2}\right) + \operatorname{tr}\left(\Sigma_1^{-1}\Sigma_2\right) + (\mu_1-\mu_2)^\top \Sigma_1^{-1}(\mu_1-\mu_2) - n\right). \tag{1}$$

(b) *Derive* the cost function for gradient based SSA. That is, consider the following scenario:

Let D be the dimension of the input data, $d < D$ the number of stationary sources and N the number of epochs. Denote the estimates of the means and covariances in each of the N epochs by $\hat{\boldsymbol{\mu}}_1, \ldots, \hat{\boldsymbol{\mu}}_N \in \mathbb{R}^D$ and $\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N \in \mathbb{R}^{D\times D}$. In SSA, a centering and whitening is performed on the data; that is, you may assume that

$$\sum_{i=1}^N \hat{\boldsymbol{\mu}}_i = 0 \quad\text{and}\quad \sum_{i=1}^N \hat{\Sigma}_i = I, \tag{2}$$

where I is the $(D \times D)$ identity matrix. Given the rotational component $R \in \mathbb{R}^{D\times D}$ (with $R^\top R = I$) of the estimated demixing matrix, the mean and covariance matrix of the estimated stationary sources of epoch i can be formulated as

$$\hat{\boldsymbol{\mu}}_i^{\mathfrak{s}} = I^d R \hat{\boldsymbol{\mu}}_i \quad\text{and}\quad \hat{\Sigma}_i^{\mathfrak{s}} = I^d R \hat{\Sigma}_i (I^d R)^\top, \tag{3}$$

where $I^d \in \mathbb{R}^{d\times D}$ is the identity matrix, truncated to the first d rows.

Show the following explicit formula for the SSA cost function:

$$\begin{aligned} L(R) &= \sum_{i=1}^N D_{\mathrm{KL}}\left[\mathcal{N}(\hat{\boldsymbol{\mu}}_i^{\mathfrak{s}}, \hat{\Sigma}_i^{\mathfrak{s}}) \,||\, \mathcal{N}(0, I)\right] \\ &= \frac{1}{2}\sum_{i=1}^N \left(-\log\det \hat{\Sigma}_i^{\mathfrak{s}} + \hat{\boldsymbol{\mu}}_i^{\mathfrak{s}\top} \hat{\boldsymbol{\mu}}_i^{\mathfrak{s}}\right) - \frac{N-1}{2} d. \end{aligned} \tag{4}$$

This means: *show* that the first line, written out, equates to the second.

Hint: Use the explicit formula for the Kullback-Leibler divergence derived in (a).

4. **Finding the stationary subspace (40 P)**
   On ISIS, you can find a data set and a MATLAB implementation of SSA. *Determine* the number of and the projection onto the stationary sources using the SSA algorithm. *Write* a brief report on what you have done. That is, *describe* which experiments you have done, *make* suitable plots, and *discuss* your findings.

Please ask questions in the ISIS discussion forums for Machine Learning 2:
`https://www.isis.tu-berlin.de/course/view.php?id=6602`