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WS 11/12
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Numerische Mathematik für Ingenieure II
Homework 7

Exercise 14: (6 points)
Consider the Dirichlet boundary value problem

$$\left.\begin{aligned} -u''(x) + cu(x) &= f(x) \quad \text{in } \Omega := (0,1) \subset \mathbb{R}, \\ u(0) = u(1) &= 0, \end{aligned}\right\} \tag{1}$$

with $c \in \mathbb{R}$, $c \geq 0$.

(a) Let $V = C_0^1(\overline{\Omega}) := \left\{ v \in C^1(\overline{\Omega}) \;\middle|\; v(0) = v(1) = 0 \right\}$. Determine a symmetric bilinear form $a : V \times V \to \mathbb{R}$ and a linear form $g : V \to \mathbb{R}$ such that (1) is transformed into the variational formulation:

$$\text{Find } u \in V \text{ with } a(u,v) = g(v) \text{ for all } v \in V.$$

(b) Let $N \in \mathbb{N}$ and $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1$ be a discretization of $\overline{\Omega}$ with variable grid sizes $h_i := x_i - x_{i-1}$ for $i \in \{1, \ldots, N+1\}$. Consider the elements $\overline{\Omega}_i := [x_{i-1}, x_i]$ for $i \in \{1, \ldots, N+1\}$ and the finite element ansatz space

$$V_h^1 := \left\{ \varphi \in C^0(\overline{\Omega}) \;\middle|\; \varphi|_{\Omega_i} \text{ is linear } \quad \forall i \in \{1, \ldots, N+1\}, \;\; \varphi(0) = \varphi(1) = 0 \right\}$$

with the basis $B_h^1 := \{w_1, \ldots, w_N\}$ where w_i is defined as in the script (Chapter 5.4, page 54). Determine with a and g from (a) the matrix $A_h \in \mathbb{R}^{N \times N}$ and the right hand side $f_h \in \mathbb{R}^N$ for the Galerkin approximation $u_h = \sum_{i=1}^N \alpha_i w_i \in V_h^1$:

$$\begin{aligned} &\text{Find } u_h \in V_h^1 \text{ with } a(u_h, v_h) = g(v_h) \text{ for all } v_h \in V_h^1 \\ \Leftrightarrow \quad &\text{Find } \alpha_h = (\alpha_1, \ldots, \alpha_N)^\top \in \mathbb{R}^N \text{ with } A_h \alpha_h = f_h. \end{aligned}$$

Compute the entries of A_h explicitly, that is the occurring integrals have to be calculated.

Programming exercise 10: (12 points)

Implement the finite element method derived in exercise 14 in MATLAB. The inner grid points are passed as a vector $\texttt{G} \in \mathbb{R}^{N\times 1}$.

(a) Write a `function Lh = p10getLS(G, c)`, that returns the sparse (!) matrix $A_h \in \mathbb{R}^{N\times N}$ corresponding to the basis B_h^1 of V_h^1 with respect to the grid points given by `G`. $\texttt{c} \in \mathbb{R}$ is the constant factor for the $u(x)$ term in (1).

(b) Write a `function y = p10evalBasis(x, i, G)` that evaluates for the given grid `G` the `i`-th basis function $w_{\texttt{i}}$ at the point $\texttt{x} \in [0, 1]$.

(c) Implement the `function fh = p10getRS(f, G, m)` that returns the right hand side vector $\texttt{fh} \in \mathbb{R}^{N\times 1}$ for the function handle `f` and the basis B_h^1 defined by the grid given by `G`.

The parameter $\texttt{m} \in \mathbb{N}$ denotes the number of inner points that are used for numerical integration. Use the "summarized trapezoid rule" to compute the occurring integrals:

$$I := \int_\alpha^\beta g(x)\, dx \approx \gamma\left(\frac{1}{2}g(\alpha) + \sum_{k=1}^{m} g(\alpha + k\gamma) + \frac{1}{2}g(\beta)\right),$$

where $\gamma := \frac{\beta-\alpha}{m+1}$. Note that the support of the basis functions is small, that is they are nonzero on maximal two elements. Thus the integration should be carried out element-wise and only on the elements where the integrand can be nonzero. Choose α and β accordingly.

(d) Write a `function y = p10evalFunc(x, alphah, G)` that evaluates the function

$$u_h : \Omega \longrightarrow \mathbb{R},\ \texttt{x} \longmapsto \texttt{y} = \sum_{i=1}^{N} \texttt{alphah}(i)\, w_i(\texttt{x}),$$

for the given grid `G` and the coefficients $\texttt{alphah} \in \mathbb{R}^{N,1}$ at the point $\texttt{x} \in [0, 1]$.

(e) Verify your implementation for $c = 1$ and $f(x) = e^x 4x$. The exact solution is

$$u(x) = e^x(x - x^2).$$

Write a `function [err, eoc, errhalf, eochalf] = p10test(m)` that returns 4 vectors of length 14. The function should compute the following quantities on *uniform* grids with grid sizes $h_p = \frac{1}{2^p+1}$, $p = \{1, \dots, 14\}$:

- The error in the ∞-norm in the grid points of the p-th grid should be stored in `err(p)`.
- The EOC (experimental order of convergence, compare programming exercise 9) with respect to the $(p-1)$ and p-th grid should be stored in `eoc(p)`.
- `errhalf(p)` should contain the ∞-norm of the error evaluated at points of the grid with half grid size $\frac{1}{2}h_p$.
- Store the EOC with respect to these finer grids in `eochalf`.

The parameter `m` is the number of inner points for the numerical integration in part (c). Monitor the results for $\texttt{m} \in \{0, 1, 2, 3\}$.