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WS 11/12
To be submitted in the lecture on 09.11.2011

Numerische Mathematik für Ingenieure II
Homework 3

Exercise 8: (8 points)

Consider the following boundary value problem with constant coefficients $a, b, c \in \mathbb{R}$ and boundary values $\alpha, \beta \in \mathbb{R}$:

$$\left.\begin{aligned} -au''(x) + bu'(x) + cu(x) &= f(x) \qquad \forall x \in \Omega := (0,1), \\ u(0) = \alpha, \quad u(1) &= \beta. \end{aligned}\right\} \tag{1}$$

Let $\overline{\Omega}_h = \{x_0, \ldots, x_{n+1}\}$ with $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1$ be a discretization of the domain $\overline{\Omega}$. Let $J := \{1, \ldots, n\}$ be the index set for the inner grid points. Construct a finite difference scheme for (1) on the (possibly non-uniform) grid Ω_h with grid sizes $h_i := x_i - x_{i-1}$ for $i \in J$. The first derivative u' should be approximated by the following adapted differences:

(a) D^-u defined by $D^-u(x_i) = \frac{u(x_i)-u(x_{i-1})}{h_i}$ for $i \in J$.

(b) D^+u defined by $D^+u(x_i) = \frac{u(x_{i+1})-u(x_i)}{h_{i+1}}$ for $i \in J$.

(c) D^0u defined by $D^0u(x_i) = \frac{u(x_{i+1})-u(x_{i-1})}{h_i+h_{i+1}}$ for $i \in J$.

In all three cases, the second derivative u'' should be approximated by the generalized difference D^2u defined by

$$D^2u(x_i) = \frac{2}{h_i + h_{i+1}}(D^+u(x_i) - D^-u(x_i))$$

for $i \in J$. Specify for (a), (b) and (c) the resulting systems of linear equations similar to equation (4.3) in the script.

Programming exercise 5: (6 points)

Implement the finite difference schemes you derived above. Therefore, write a `function [Lh,fh] = p05getLS(a,b,c,alpha,beta,f,xh,d)` that accepts the following arguments:

- `a,b,c,alpha,beta` are the coefficients and boundary values.
- `f` is a function handle that evaluates the right hand side f at the given set of points, for example the evaluation of `fh=f([0:1/10:1])` is equivalent to the execution of `fh(i)=f(i*1/10)` for all `i`$\in \{0, \ldots, 10\}$. (Hint: entry-wise operations can be carried out by preceding them by a period; e.g. `[1,2,3,4].*[5,6,7,8]`.)
- `xh`$= [x_0, \ldots, x_{n+1}]^\mathsf{T} \in \mathbb{R}^{n+2,1}$ is a column vector containing the grid points.

- `d` is one character to pick an approximation of u' from exercise 8: either `'-'` for D^-, `'+'` for D^+ or `'0'` for D^0. (The `switch` statement may be helpful.)

The function should return the (sparse!) matrix `Lh`$\in \mathbb{R}^{n,n}$ and the right hand side vector `fh`$\in \mathbb{R}^{n,1}$ of the finite difference scheme you derived in exercise 8.

Programming exercise 6: (6 points)
Consider the following *singularly perturbed* boundary-value problem

$$\left.\begin{array}{ll} -\varepsilon u''(x) + u'(x) = 1 & \forall x \in \Omega := (0,1), \\ u(0) = 0, \quad u(1) = 0 & \end{array}\right\} \tag{2}$$

where ε is a small positive parameter. The exact solution of this PDE is

$$u_\varepsilon(x) = x - \frac{e^{-(1-x)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} \qquad \text{for} \quad 0 \le x \le 1.$$

(a) Write a `function uh = p06solve(epsi,xh,d)` that approximates the solution of (2) with $\varepsilon =$ `epsi` on the grid given by `xh` and where the parameter `d` has the same semantics as in the previous exercise. The function should return the computed approximation `uh` as a column vector *including* the boundary values.

(b) Write a `function err = p06plot(epsi,xh,uh)` that plots the given approximated solution `uh` of (2) with $\varepsilon =$ `epsi` as well as the exact solution u_ε evaluated at the grid points `xh`. Furthermore, the error `err`$= \|R_h u_\varepsilon - \texttt{uh}\|_\infty$ should be returned.

(c) Write a `function xh = p06shishkin(n,sigma)` that generates a column vector of size 2`n`+1 describing a "Shishkin" grid `xh` that is defined by

$$\texttt{xh(i)} = \begin{cases} (\texttt{i} - 1)H & \text{for} \quad \texttt{i} = 1, \dots, \texttt{n} \\ (1 - \texttt{sigma}) + (\texttt{i} - \texttt{n} - 1)h & \text{for} \quad \texttt{i} = \texttt{n} + 1, \dots, 2\texttt{n} + 1 \end{cases}$$

where $H = (1 - \texttt{sigma})/\texttt{n}$ and $h = \texttt{sigma}/\texttt{n}$.

(d) Write a `function p06experiment()` that plots the exact and approximated solution for `epsi`=0.001 and all $n \in \{50, 200, 300, 500\}$ on a

  (a) uniform grid with $h = 1/(2n)$ and forward difference operator D^+ (`d='+'`).
  (b) uniform grid with $h = 1/(2n)$ and central difference operator D^0 (`d='0'`).
  (c) uniform grid with $h = 1/(2n)$ and backward difference operator D^- (`d='-'`).
  (d) non-uniform Shishkin grid with `sigma=4*epsi*log(2*n)` and central difference operator D^0 (`d='0'`).

Use the `subplot` command to group plots corresponding to the same grid and difference operator. Thus 4 windows for (a)-(d) should be created where each window contains 4 plots for the different choices of n. Use the `title` command for each plot to print information about the used grid and difference operator and about the resulting error.