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WS 11/12
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Numerische Mathematik für Ingenieure II
Homework 4

Exercise 9: (4 points)
Let $A \in \mathbb{R}^{N,N}$ be an M-Matrix. Show the following statements:

(a) If $v \in \mathbb{R}^N$ is a vector with $Av \geq 0$, then $v \geq 0$.

(b) If $v_1, v_2 \in \mathbb{R}^N$ are vectors such that $|Av_1| \leq Av_2$, then $|v_1| \leq v_2$.

Exercise 10: (4 points)
Consider the finite difference discretization of

$$\begin{aligned} -u''(x) + bu'(x) + cu(x) &= f(x), \qquad \forall\, x \in (0,1) \\ u(0) = u(1) &= 0 \end{aligned}$$

with $c > 0$ and the central difference approximation for u' on a uniform grid. For grid sizes satisfying $h \leq \frac{2}{|b|}$ the resulting matrix L_h is an M-Matrix (this does not have to be shown). Determine a vector v with $L_h v \geq (1, \dots, 1)^\top$. What can be said about the finite difference method for $h \longrightarrow 0$?

Exercise 11: (6 points)
Consider the 2-dimensional (stationary) convection-diffusion PDE

$$\begin{aligned} -\Delta u + v \cdot \nabla u &= f \qquad \text{in } \Omega := (0,1) \times (0,1) \subset \mathbb{R}^2 \\ u &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

with the convection vector $v \in \mathbb{R}^2$, the right hand side $f \in C(\Omega)$ and the boundary value function $g \in C(\partial\Omega)$. Derive a finite difference scheme on a uniform grid with grid size $h = 1/(n+1)$:

$$\Omega_h = \left\{ \begin{bmatrix} ih \\ jh \end{bmatrix} \in \mathbb{R}^2 \mid i, j = 1, \dots, n \right\}.$$

Please use the lexicographical order of the grid points:

$$x_{(j-1)n+i} = \begin{bmatrix} ih \\ jh \end{bmatrix} \quad \text{for } i, j = 1, \dots, n.$$

Use the 5-point stencil discussed in the lecture for $-\Delta u$ and the central difference formula for the convection term $v \cdot \nabla u$. What is the order of consistency for this discretization? Explain your answer!

Programming exercise 7: (10 points)

Consider the convection-diffusion equation (1) with the convection vector $v = \left[v_1, 0\right]^{\mathsf{T}}$, the zero right hand side $f \equiv 0$ and the boundary value function

$$g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{cases} \sin(\pi x_2) & \text{if } x_1 = 0, \\ 0 & \text{else.} \end{cases}$$

The exact solution then is

$$u\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \frac{1}{\sinh(\sigma)} \exp\left(\frac{v_1 x_1}{2}\right) \sinh\left(\sigma(1 - x_1)\right) \sin\left(\pi x_2\right) \quad \text{with} \quad \sigma := \frac{\sqrt{{v_1}^2 + 4\pi^2}}{2}.$$

The task in this exercise is to approximate the solution of (1) with the finite difference scheme derived in exercise 11.

(a) Write a `function [Lh,fh] = p07getLS(n,v,f,g)` which returns the discretized differential operator `Lh`$\in \mathbb{R}^{\mathtt{n}^2,\mathtt{n}^2}$ and the right hand side `fh`$\in \mathbb{R}^{\mathtt{n}^2,1}$ using the uniform grid with grid size $h = 1/(\mathtt{n}+1)$ and the lexicographical order as in exercise 11. `v`$\in \mathbb{R}^{2,1}$ is the convection vector and `f` and `g` are function handles.

Hint: useful functions are `speye`, `kron` and `gallery('tridiag',...)`.

(b) Write a `function [e2,einf,eoc2,eocinf] = p07getErr(r,v1)` that performs error and EOC (experimental order of convergence) computations for different grid sizes with the convection vector, right hand side, boundary value function and exact solution from above. For $p \in \{1, \ldots, \mathtt{r}\}$ let u_{h_p} be the solution of $L_{h_p} u_{h_p} = f_{h_p}$ with grid size $h_p := 1/(2^p + 1)$ and let $e_{h_p} := u_{h_p} - R_{h_p} u$ be the error. All returned vectors should be of size `r`$\times$1 and should contain the following data:

- $\mathtt{e2}(p) = \|e_{h_p}\|_2$ and $\mathtt{einf}(p) = \|e_{h_p}\|_\infty$ for $p = 1, \ldots, \mathtt{r}$.
- $\mathtt{eoc2}(p) = \dfrac{\log \frac{\|e_{h_p}\|_2}{\|e_{h_{(p-1)}}\|_2}}{\log \frac{h_p}{h_{(p-1)}}}$ and $\mathtt{eocinf}(p) = \dfrac{\log \frac{\|e_{h_p}\|_\infty}{\|e_{h_{(p-1)}}\|_\infty}}{\log \frac{h_p}{h_{(p-1)}}}$ for $p = 2, \ldots, \mathtt{r}$.

(c) Implement the `function p07err(r,v1)` which plots the errors computed with `p07getErr` for the given `r` and `v1`. Therefore create two figures with `subplot`. In the left figure both errors should be plotted with `semilogy` while in the right figure both EOCs should be plotted with `plot`. Please add informative annotations to your figures.

Verify your implementation for `v1`$= 0$ ($\Rightarrow$ Poisson equation!) and `r`$= 9$ on the basis of the convergence order result given in the lecture.

(d) Write a `function p07plot(p,v1)` that plots the finite difference solution for the grid with grid size $h = 1/(2^{\mathtt{p}} + 1)$ as well as the exact solution (evaluated at the grid points) for the given `v1`. The boundary values should be included in the plots. Please use the functions `meshgrid` and `surf`. Use `subplot` to plot the finite difference solution on the left and the exact solution on the right side.

Conduct experiments with `p`$= 4$ and the convection coefficients `v1`$\in \{0, 1, 5, 10, 25, 50, 100\}$.