

Numerical Mathematics for Engineers II

Homework 1

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Exercise 1

Claim 1. Let $\mathbf{A} \in \mathcal{R}^{n \times n}$ be symmetric. \mathbf{A} is positive definite iff all eigenvalues of \mathbf{A} are positive.

Proof. \Rightarrow Let x be an eigenvector of \mathbf{A} with $\lambda x = \mathbf{A}x$. $0 < x^T \mathbf{A}x = \lambda x^T x = \lambda |x|^2$. Since $|x|^2 > 0 \Rightarrow \lambda > 0$.

\Leftarrow Since $\mathbf{A} \in \mathcal{R}^{n \times n}$ and $\mathbf{A} = \mathbf{A}^T$ there exists an orthonormal basis (b_1, \dots, b_n) consisting of the eigenvectors of \mathbf{A} . So for any vector exists c_i such that $x = \sum_{i=1}^n c_i b_i$. $x^T \mathbf{A}x = \langle x, \mathbf{A}x \rangle = \langle \sum_{i=1}^n c_i b_i, \sum_{i=1}^n \lambda_i c_i b_i \rangle = \sum_{i,j=1}^n \langle c_j b_j, \lambda_i c_i b_i \rangle = \sum_{i=1}^n \langle c_i b_i, \lambda_i c_i b_i \rangle = \sum_{i=1}^n \lambda_i c_i^2 > 0$.

In the next to last step, we have used that the basis vectors are orthogonal. In the last step we have used that all λ_i are positive and the sum of them multiplied by a positive factor c_i^2 .

□

Exercise 2

Let $\mathbf{A} \in \mathcal{R}^{n \times n}$ be a symmetric and positive definite matrix.

(a)

Claim 2. \mathbf{A} is invertible and \mathbf{A}^{-1} is SPD

Proof. Let's assume that \mathbf{A} is not invertible. That means $\exists x$ such that $\mathbf{A}x = 0$. This further implies $x^T \mathbf{A}x = x^T 0 = 0$, which contradicts the fact that \mathbf{A} is positive definite.

Then it holds $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I}$ and $\mathbf{A}^{-1} \mathbf{A} = \mathbf{I}$. With $(\mathbf{A}^{-1} \mathbf{A})^T = \mathbf{A}^T (\mathbf{A}^{-1})^T = \mathbf{A} (\mathbf{A}^{-1})^T = \mathbf{I}$. And thus it holds $\mathbf{A}^{-1} = (\mathbf{A}^{-1})^T$.

The positive definiteness follows from: $0 < x^T \mathbf{A}x = x^T \mathbf{A} \mathbf{A}^{-1} \mathbf{A}x = (\mathbf{A}x)^T \mathbf{A}^{-1} (\mathbf{A}x)$. Since \mathbf{A} is regular $\forall \tilde{x} : \exists x : \tilde{x} = \mathbf{A}x$ and thus with $\tilde{x} = \mathbf{A}x$ it holds that $\tilde{x}^T \mathbf{A}^{-1} \tilde{x} > 0$.

□

(b)

Claim 3. *The diagonal elements \mathbf{a}_{ii} of \mathbf{A} are positive.*

Proof. Choose $x = e_i$ with $i \in [1, n]$. $x^T \mathbf{A} x = x^T \mathbf{a}_i$ with \mathbf{a}_i being the i th column. Then $x^T \mathbf{a}_i = \mathbf{a}_{ii} > 0$ because \mathbf{A} is positive definite. \square

Exercise 3

Let $\mathbf{Q} \in \mathcal{R}^{n \times n}$ be an orthogonal matrix.

(a)

Claim 4. $\|\mathbf{Q}x\|_2 = \|x\|_2 \ \forall x \in \mathcal{R}^n$

Proof. Since \mathbf{Q} is an orthogonal matrix, its row vectors form a basis (r_1, \dots, r_n) of \mathcal{R}^n with $\|r_i\|_2 = 1$. Thus for every $x \in \mathcal{R}^n$ exist c_i such that $x = \sum_i^n c_i r_i$.
 $\|\mathbf{Q}x\|_2 = \|\sum_i^n c_i \mathbf{Q}r_i\|_2 = \|\sum_i^n c_i e_i\|_2 = \sqrt{\sum_i^n c_i^2} = \sqrt{\langle \sum_i^n c_i r_i, \sum_i^n c_i r_i \rangle} = \|x\|_2$ \square

(b)

Claim 5. $\forall \lambda$ of \mathbf{Q} : $|\lambda| = 1$.

Proof. Let x be an eigenvector of \mathbf{Q} . $\|\mathbf{Q}x\|_2 = \|\lambda x\|_2 = |\lambda| \cdot \|x\|_2 = \|x\|_2$. Thus $|\lambda| = 1$. \square

(c)

Claim 6. $\kappa_2(\mathbf{Q}) = 1$

Proof. $\|\mathbf{Q}\|_2 := \sup\{\|\mathbf{Q}y\|_2 : y \in \mathcal{R}^n, \|y\|_2 = 1\}$. According to (a): $\|\mathbf{Q}\|_2 = \sup\{\|y\|_2 = 1\} = 1$. Since $\mathbf{Q}^{-1} = \mathbf{Q}^T$ and \mathbf{Q}^T is an orthogonal matrix, it follows that $\|\mathbf{Q}^{-1}\|_2 = 1$ and thus $\kappa_2(\mathbf{Q}) = 1$. \square

(d)

Claim 7. *Let $\mathbf{A} \in \mathcal{R}^{n \times n}$. $\|\mathbf{A}\|_2 = \|\mathbf{Q}\mathbf{A}\|_2 = \|\mathbf{A}\mathbf{Q}\|_2$.*

Proof. $\|\mathbf{A}\|_2 = \sup\{\|\mathbf{A}y\|_2 : y \in \mathcal{R}^n, \|y\|_2 = 1\} = \sup\{\|x\|_2 : x = \mathbf{A}y, y \in \mathcal{R}^n, \|y\|_2 = 1\} = \sup\{\|\mathbf{Q}x\|_2 : x = \mathbf{A}y, y \in \mathcal{R}^n, \|y\|_2 = 1\} = \sup\{\|\mathbf{Q}\mathbf{A}y\|_2 : y \in \mathcal{R}^n, \|y\|_2 = 1\} = \|\mathbf{Q}\mathbf{A}\|_2$

$\|\mathbf{A}\|_2 = \sup\{\|\mathbf{A}y\|_2 : y \in \mathcal{R}^n, \|y\|_2 = 1\} = \sup\{\|\mathbf{A}\mathbf{Q}x\|_2 : \mathbf{Q}x \in \mathcal{R}^n, \|\mathbf{Q}x\|_2 = 1\} = \sup\{\|\mathbf{A}\mathbf{Q}x\|_2 : x \in \mathcal{R}^n, \|\mathbf{Q}x\|_2 = 1\} = \sup\{\|\mathbf{A}\mathbf{Q}x\|_2 : x \in \mathcal{R}^n, \|x\|_2 = 1\} = \|\mathbf{A}\mathbf{Q}\|_2$. \square