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WS 11/12
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Numerische Mathematik für Ingenieure II
Homework 13

Programming exercise 16: (16 points)
Consider Poisson's equation with Dirichlet boundary condition:

$$\begin{aligned} -u'' &= f \qquad \text{in} \quad \Omega = (0,1) \\ u(0) &= u(1) = 0. \end{aligned}$$

In this exercise the solution is approximated with the multigrid method on a hierarchy of finite difference grids. Let $k \in \mathbb{N}$ be a grid index. We then define the number of inner grid points $N_k := 2^k - 1$ and the grid size $h_k := 1/(N_k + 1) = 1/2^k$ for the k-th grid $\Omega_k := \{i \cdot h_k | i = 1, \ldots, N_k\}$.

(a) Write a `function [A,b] = p16getLS(k)` that returns the matrix $A \in \mathbb{R}^{N_k,N_k}$ and right hand side $b \in \mathbb{R}^{N_k}$ corresponding to the finite difference discretization of the above PDE on the grid Ω_k. Make sure that b is returned as a column vector.

(b) Write a `function [Romega, omegaMb] = p16getRomega(A,b,omega)` that returns the iteration matrix $R(\omega) = \omega M^{-1}N + (1-\omega)I$ and the update vector $\omega M^{-1}b$ of the relaxed Jabobi method ($M := \operatorname{diag}(A)$ and $N := M - A$).

(c) Implement the relaxed Jacobi-method in the

```
function [xj,resvec] = p16jacobi(A,b,x0,omega,tol,maxit).
```

The method computes iterates $x_{j+1} = R(\omega)x_j + \omega M^{-1}b$ until $\|b - Ax_j\|_2/\|b\|_2 < tol$ or $j > maxit$. Here $R(\omega)$ and $\omega M^{-1}b$ are computed with `p16getRomega`. The function `p16jacobi` then returns x_j and a vector of residuals with `resvec(i+1)=`$\|b - Ax_i\|_2/\|b\|_2$.

(d) Write a `function p16eigRomega` that plots all eigenvalues of the iteration matrix $R(\omega)$ for $N = 128$ and $\omega \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$ in one window. Use the MATLAB functions `full` and `eig`.

(e) Write a `function [Rk,Pk] = p16getRP(k)` that assembles the restriction operator matrix $R_k \in \mathbb{R}^{N_{k-1},N_k}$ and the prolongation operator matrix $P_k = 2R_k^\mathsf{T}$. The restriction operator is defined by

$$(R_k)_{ij} = \begin{cases} 1/2 & \text{if} \quad j = 2i \\ 1/4 & \text{if} \quad |j - 2i| = 1 \\ 0 & \text{else.} \end{cases}$$

(f) Implement a multigrid cycle in the

```
function xj = p15mgcycle(A,b,x0,k,kmin,gamma,m1,m2,omega,tol).
```

Analogous to the scheme presented in the lecture the following steps have to be carried out:

if `k>kmin` **then**
  $x_j = \texttt{p16jacobi}(A, b, x_0, \omega, tol, m_1)$
  $d = b - Ax_j$
  $d_{\text{coarse}} = R_k d$
  $A_{\text{coarse}} = R_k A P_k$
  $y_{\text{coarse}} = [0, \ldots, 0]^T \in \mathbb{R}^N_{k-1}$
  for $j = 1, \ldots, \gamma$ **do**
    $y_{\text{coarse}} := \texttt{p16mgcycle}(A_{\text{coarse}}, d_{\text{coarse}}, y_{\text{coarse}}, k-1, kmin, \gamma, m_1, m_2, \omega, tol)$
  end for
  $x_j = x_j - P_k y_{\text{coarse}}$
  $x_j = \texttt{p16jacobi}(A, b, x_j, \omega, tol, m_2)$
else
  Solve $Ax_j = b$.
end if

(g) Test your implementation in the function `p16test` with the right hand side

$$f(x) := 12x^2 \sin(\pi x) + 8x^3 \cos(\pi x)\pi - x^4 \sin(\pi x)\pi^2.$$

The exact solution of the PDE is

$$u(x) := -x^4 \sin(\pi x).$$

Apply 20 multigrid cycles with the following parameters: $k = 14$, $k_{\min} = 4$, $x_0 = [0, \ldots, 0]^\mathsf{T} \in \mathbb{R}^{N_{14}}$, $\gamma = 1$, $m_1 = m_2 = 5$, $\omega = 2/3$, $tol = 10^{-12}$. Plot the residuals and the error in the grid points in the $\|\cdot\|_\infty$-norm with `semilogy` in one plot.

Use sparse matrices everywhere!