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WS 11/12
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Numerische Mathematik für Ingenieure II
Homework 2

Exercise 4: (4 points)
Show the following inequalities for $A, B \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$ and $p \in \mathbb{N} \cup \{\infty\}$:

(a) $\|AB\|_p \leq \|A\|_p \|B\|_p$,

(b) $\|Ax\|_p \leq \|A\|_p \|x\|_p$.

Exercise 5: (5 points)
Determine the (maximum) domains Ω for which the following partial differential equations are elliptic, parabolic or hyperbolic. Explain your answers.

(a) $yu_{xx} + u_{yy} = 0, \quad (\Omega \subset \mathbb{R}^2)$,

(b) $u_{xx} + 4u_{yy} + 9u_{zz} - 4u_{xy} + 3u_x = u, \quad (\Omega \subset \mathbb{R}^3)$,

(c) $(x^2-1)u_{xx} + (y^2-1)u_{yy} = xu_x + yu_y, \quad (\Omega \subset \mathbb{R}^2)$.

Exercise 6: (2 points)
Let $u : (0,1) \longrightarrow \mathbb{R}$. Show that the following holds for all $h > 0$ and $x \in (0,1)$ with $x \pm h \in (0,1)$:

(a) $D^0u(x) = \frac{1}{2}(D^+u(x) + D^-u(x))$,

(b) $D^-D^+u(x) = D^+D^-u(x)$.

Exercise 7: (6 points)
Let $I \subset \mathbb{R}$ be an open interval and let $x \in I$ and $h > 0$ with $[x-h, x+h] \subseteq \overline{I}$. Show that

(a) $D^0u(x) = u'(x) + h^2 R_0 \quad \text{with} \quad |R_0| \leq \frac{1}{6} \max\limits_{\xi\in[x-h,x+h]} |u'''(\xi)|$, if $u \in C^3(\overline{I})$.

(b) $D^-D^+u(x) = u''(x) + h^2 R_1 \quad \text{with} \quad |R_1| \leq \frac{1}{12} \max\limits_{\xi\in[x-h,x+h]} |u^{(4)}(\xi)|$, if $u \in C^4(\overline{I})$.

(c) Try to derive a formula for $D^-D^0u(x)$ similar to the one in (b) and explain on the basis of your computation why this difference scheme is unsuitable for the approximation of the second derivative.

Programming exercise 3: (4 points)

Let $Lu = f$ be a linear PDE of second order with constant coefficients. Write a MATLAB `function t = p03getPDEType(A, b)` which accepts the coefficients of L's second order derivatives in the matrix `A` and the coefficients of L's first order derivatives in the vector `b`. The output `t` should be

- `0` if the PDE cannot be classified,
- `1` if the PDE is elliptic,
- `2` if the PDE is parabolic,
- `3` if the PDE is hyperbolic.

Programming exercise 4: (6 points)

Consider the following boundary value problem:

$$\begin{aligned} -u''(x) - 3u'(x) + u(x) &= -2 + 10x^2 - x^3, \qquad && \forall\, x \in (0,1), \\ u(0) &= u(1) = 0. \end{aligned}$$

The exact solution is

$$u(x) = x^2 - x^3, \qquad \forall\, x \in (0,1).$$

Discretize this PDE with finite differences with grid size $h = (2^p + 1)^{-1}$, $p \in \{1, \ldots, 20\}$. Use the approximation (4.2) in the script (see ISIS website).

(a) Write a `function [Ah,fh] = p04getLS(p)` that sets up the sparse matrix `Ah` and right hand side `fh` of the corresponding linear system for the refinement level `p`. Useful commands are `speye` and `gallery('tridiag',...)`. If you use Octave instead of MATLAB, use the gallery.m file from the ISIS website.

(b) Write a `function errors = p04plot()` that solves the discretized problem for all `p` $\in \{1, \ldots, 20\}$. Determine for each `p` the error between the computed approximation and the restricted exact solution $R_h u$ in the maximum norm $\|\cdot\|_\infty$ and store it in `errors(p)`. At the end of the function the errors should be plotted with `semilogy`.