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Numerische Mathematik für Ingenieure II
Homework 8

Programming exercise 11: (20 points)
Consider the following Dirichlet boundary value problem: find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u + cu &= f \quad \text{in } \Omega \subset \mathbb{R}^3 \text{ and} \\ u &= g \quad \text{on } \Gamma := \partial\Omega \end{aligned}$$

with a constant $c > 0$, a right hand side $f : \Omega \to \mathbb{R}$ and a boundary value function $g : \Gamma \to \mathbb{R}$.

In this exercise, the solution to this problem is approximated with the Galerkin finite element method using a tetrahedral mesh and element-wise linear basis functions. To describe tetrahedrons and triangles we define the convex hull of the points $x^{(1)}, \ldots, x^{(n)} \in \mathbb{R}^3$:

$$\operatorname{conv}(x^{(1)}, \ldots, x^{(n)}) := \Big\{ \sum_{k=1}^{n} \lambda_k x^{(k)} \;\Big|\; \lambda_k \geq 0,\ \sum_{k=1}^{n} \lambda_k = 1 \Big\}.$$

The domain Ω and its boundary Γ are defined by three matrices:

- $\mathtt{p} \in \mathbb{R}^{3,\mathtt{np}}$ describes a set of points: $\mathtt{p(:,i)} \in \mathbb{R}^{3,1}$ are the coordinates of the $\mathtt{i}$-th point ($\mathtt{i} \in \{1, \ldots, \mathtt{np}\}$).
- $\mathtt{t} \in \mathbb{N}^{4,\mathtt{nt}}$ is an index matrix describing the elements that make up the domain Ω: for $j \in \{1, \ldots, \mathtt{nt}\}$ the columns of the matrix $\mathtt{pt} := \mathtt{p(:,t(:,}j\mathtt{))} \in \mathbb{R}^{3,4}$ are the vertices of the j-th element (a tetrahedron):

  $$\Omega_j := \operatorname{conv}(\mathtt{pt(:,1)}, \ldots, \mathtt{pt(:,4)}).$$

  The whole domain is $\overline{\Omega} := \bigcup_{j=1}^{\mathtt{nt}} \Omega_j$ and for $i, j \in \{1, \ldots, \mathtt{nt}\}$ with $i \neq j$ the tetrahedrons Ω_i and Ω_j may only share a face, a edge or a vertex.
- $\mathtt{e} \in \mathbb{N}^{3,\mathtt{ne}}$ is an index matrix describing the faces belonging to the boundary Γ: for $j \in \{1, \ldots, \mathtt{ne}\}$ the columns of the matrix $\mathtt{pe} := \mathtt{p(:,e(:,}j\mathtt{))} \in \mathbb{R}^{3,3}$ are the vertices of the j-th boundary face (a triangle):

  $$\Gamma_j := \operatorname{conv}(\mathtt{pe(:,1)}, \mathtt{pe(:,2)}, \mathtt{pe(:,3)}).$$

  The whole boundary is $\Gamma := \bigcup_{j=1}^{\mathtt{ne}} \Gamma_j$ and for $i, j \in \{1, \ldots, \mathtt{nt}\}$ with $i \neq j$ the triangles Γ_i and Γ_j may only share a edge or a vertex.

For $i \in \{1, \ldots, \mathtt{np}\}$ the global basis function $w_i : \overline{\Omega} \to \mathbb{R}$ is defined by the following conditions:

- w_i restricted to Ω_j is a linear polynomial for all $j \in \{1, \ldots, \mathtt{nt}\}$.
- $w_i(\mathtt{p(:,}j\mathtt{)}) = \delta_{ij}$ for all $j \in \{1, \ldots, \mathtt{np}\}$.

The local basis functions $\widehat{w}_i : \widehat{\Omega} \to \mathbb{R}$ on the reference element $\widehat{\Omega} := \operatorname{conv}(0, e_1, e_2, e_3)$ are given for $\widehat{x} = [\widehat{x}_1, \widehat{x}_2, \widehat{x}_3]^\mathsf{T} \in \widehat{\Omega}$ by

$$\widehat{w}_1(\widehat{x}) := 1 - \widehat{x}_1 - \widehat{x}_2 - \widehat{x}_3, \qquad \widehat{w}_2(\widehat{x}) := \widehat{x}_1, \qquad \widehat{w}_3(\widehat{x}) := \widehat{x}_2, \qquad \widehat{w}_4(\widehat{x}) := \widehat{x}_3.$$

The local and global basis functions are related by `t`: for $j \in \{1, \ldots, \mathtt{nt}\}$ the relation is

$$w_{\mathtt{t}(i,j)}(F_j(\widehat{x})) = \widehat{w}_i(\widehat{x}) \qquad \forall \widehat{x} \in \widehat{\Omega} \quad \text{and} \quad i \in \{1, 2, 3, 4\},$$

where $F_j : \widehat{\Omega} \to \Omega_j$ is the transformation defined by $F_j(\widehat{x}) := \mathtt{pt(:,1)} + G_j \widehat{x}$ for all $\widehat{x} \in \widehat{\Omega}$ with `pt` as defined above and $G_j := [\mathtt{pt(:,2)} - \mathtt{pt(:,1)},\ \mathtt{pt(:,3)} - \mathtt{pt(:,1)},\ \mathtt{pt(:,4)} - \mathtt{pt(:,1)}] \in \mathbb{R}^{3,3}$.

(a) Implement a `function b=p11getBoundDOFs(p,e)` that returns a column vector $\mathtt{b} \in \mathbb{R}^{\mathtt{np},1}$ such that for $i \in \{1, \ldots, \mathtt{np}\}$ the following holds:

$$\mathtt{b}(i) = \begin{cases} 1 & \text{if} \quad \mathtt{p}(:,i) \in \Gamma \\ 0 & \text{else.} \end{cases}$$

(b) Write a `function [elS,elM,elfh]=p11getLoc(pt,c,f)` that computes the local contributions from an element $k \in \{1, \ldots, \mathtt{nt}\}$ that is given by its points $\mathtt{pt{=}p(:,t(:,}k\mathtt{))}$. As shown in the tutorial the following quantities have to be computed for all $i, j \in \{1, \ldots, 4\}$:

(a) element "stiffness matrix": $\mathtt{elS}(i,j) = \int_{\widehat{\Omega}} (\nabla \widehat{w}_j(\widehat{x}))^\mathsf{T} (G_k^\mathsf{T} G_k)^{-1} \nabla \widehat{w}_i(\widehat{x}) |\det G_k| d\widehat{x}$.

(b) element "mass matrix": $\mathtt{elM}(i,j) = \mathtt{c} \int_{\widehat{\Omega}} \widehat{w}_j(\widehat{x}) \widehat{w}_i(\widehat{x}) |\det G_k| d\widehat{x}$.

(c) element right hand side: $\mathtt{elfh}(i) = \int_{\widehat{\Omega}} \mathtt{f}(F_k(\widehat{x})) \widehat{w}_i(\widehat{x}) |\det G_k| d\widehat{x}$.

Thus `elS` and `elM` are both $\mathbb{R}^{4,4}$ matrices and `elfh` should be a $\mathbb{R}^{4,1}$ column vector. Use the quadrature formula

$$\int_{\widehat{\Omega}} h(\widehat{x}) d\widehat{x} \;\approx\; \frac{1}{6}\, h\Big(\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \Big)$$

for integrals that cannot be computed explicitly.

(c) Write a `function [S,M,D,fh]=p11getLS(p,e,t,c,f,g)` that uses `b=p11getBoundDOFs(p,e)` and `p11getLoc` to construct the linear system $(\mathtt{S} + \mathtt{M} + \mathtt{D})\alpha = \mathtt{fh}$, where $u_h = \sum_{i=1}^{\mathtt{np}} \alpha_i w_i$ is the globally defined approximate solution. **As it will be shown in the tutorial on 16.12.2011**, the matrices and right hand side have to be adapted to incorporate the Dirichlet boundary conditions. For $i, j \in \{1, \ldots, \mathtt{np}\}$ the following should hold:

(a) $\mathtt{S}(i,j) = \begin{cases} \int_\Omega \nabla w_j(x) \cdot \nabla w_i(x) dx & \text{if } \mathtt{p}(:,i) \notin \Gamma, \\ 0 & \text{else.} \end{cases}$

(b) $\mathtt{M}(i,j) = \begin{cases} \mathtt{c} \int_\Omega w_j(x) w_i(x) dx & \text{if } \mathtt{p}(:,i) \notin \Gamma, \\ 0 & \text{else.} \end{cases}$

(c) $\mathtt{D}(i,j) = \begin{cases} 0 & \text{if } \mathtt{p}(:,i) \notin \Gamma, \\ \delta_{ij} & \text{else.} \end{cases}$

(d) $\texttt{fh}(i) = \begin{cases} \int_\Omega \texttt{f}(x) w_i(x) dx & \text{if } \texttt{p(:,}i\texttt{)} \notin \Gamma, \\ \texttt{g(p(:,}i\texttt{))} & \text{else.} \end{cases}$

(Here and in the following $x = [x_1, x_2, x_3]^\mathsf{T} \in \overline{\Omega}$.) Thus `S`, `M` and `D` should be *sparse* $\mathbb{R}^{\texttt{np},\texttt{np}}$ matrices and `fh` is a $\mathbb{R}^{\texttt{np},1}$ column vector.

(d) Test and verify your implementation with $\overline{\Omega} = [0,1]^3$, $c = \frac{1}{2}$, the right hand side function

$$f(x) = \sin(2\pi x_1)\sin(2\pi x_2)\Big(\frac{5}{2} + 8\pi^2 x_3(1 - x_3)\Big) + x_1 x_2 x_3$$

and the Dirichlet boundary value function

$$g(x) = \exp\left(\frac{x_3}{\sqrt{2}}\right) + x_1 x_2 x_3.$$

The exact solution then is

$$u(x) = \sin(2\pi x_1)\sin(2\pi x_2) x_3 (1 - x_3) + \exp\left(\frac{x_3}{\sqrt{2}}\right) + x_1 x_2 x_3.$$

Write a `function [err,eoc]=p11eoc()` that computes the approximate solution u_r for a uniform tetrahedral mesh with mesh size $h_r = 1/2^r$ for $r \in \{1,2,3,4\}$. The output should be 2 column vectors with

$$\texttt{err(r)} = \max_{i \in \{1,\dots,np\}} |u(\texttt{p(:,i)}) - u_r(\texttt{p(:,i)})| \qquad \text{for } r \in \{1,2,3,4\} \qquad \text{and}$$

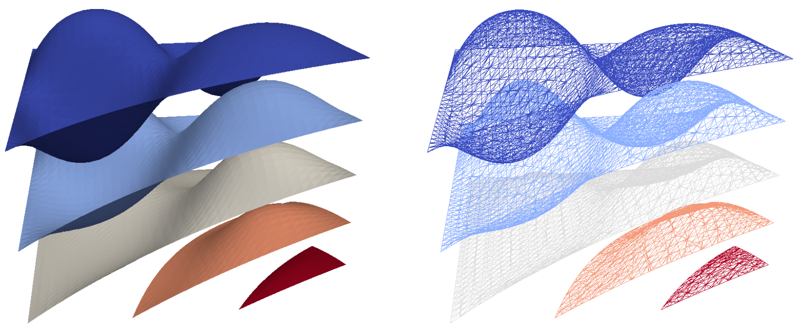
$$\texttt{eoc(r)} = \frac{\log(\texttt{err(r)/err(r-1)})}{\log(h_r/h_{r-1})} \qquad \text{for } r \in \{2,3,4\}.$$

A uniform mesh for the unit cube $\overline{\Omega}$ with mesh size $h_r = 1/2^r$ can be generated with the provided function `[p,e,t]=p11mshUnit(3,`$2^r - 1$`)`. The linear systems should be solved with the backslash operator. Verify that the EOC converges to 2.

Some notes for further experiments:

- An approximate solution given by a coefficient vector `alpha` can be written to a *VTK* (visualization toolkit) file (.vtu) with the provided function `p11vtkWrite(filename, p,e,t,alpha)`. The *VTK* files can then be plotted and post-processed by several tools, for example *ParaView* (free software).

- You can generate your own meshes with the tool *gmsh* (also free software). Use the provided function `[p,e,t]=p11mshRead(filename,3)` to read a *gmsh* mesh from a .msh file. Feel free to conduct experiments with your own geometry!



Level surfaces and corresponding part of the mesh for an approximation to the solution u visualized with *ParaView* (with the *Contour* filter).