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WS 11/12
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Numerische Mathematik für Ingenieure II
Homework 9

Exercise 15: (6 points)
Let $V = H_0^1(\Omega)$ and $f \in L^2(\Omega)$. Furthermore, let $a : V \times V \to \mathbb{R}$ be the bilinear form defined by $a(u,v) = \int_\Omega \nabla u \cdot \nabla v d\Omega$ and $g : V \to \mathbb{R}$ be the linear form defined by $g(v) = \int_\Omega f v d\Omega$. Let $u \in V$ be the unique solution of the variational problem "find $u \in V$ s.t. $a(u,v) = g(v)$ for all $v \in V$" and let $u \in V_h$ be the Galerkin approximation with respect to the approximation space $V_h \subseteq V$, that is $a(u_h, v_h) = g(v_h)$ for all $v_h \in V_h$. Show that

$$\|\nabla(u - u_h)\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 - \|\nabla u_h\|_{L^2}^2.$$

Exercise 16: (6 points)
Assume the CG method (see below for an algorithm) applied to a linear system with a SPD matrix $A \in \mathbb{R}^{N,N}$ proceeds to the iteration $J \in \mathbb{N}$. Let $r_j \in \mathbb{R}^N$ and $p_j \in \mathbb{R}^N$ be the vectors generated by the CG method for $j = 0, \ldots, J$. Show that

(a) $r_i^\mathsf{T} r_j = 0$ for all $i, j = 0, \ldots, J$ with $i \neq j$.

(b) $p_i^\mathsf{T} A p_j = 0$ for all $i, j = 0, \ldots, J$ with $i \neq j$.

Programming exercise 12: (8 points)
Consider the following algorithm for the CG method:

Require: $A \in \mathbb{R}^{N\times N}$, $b, x_0 \in \mathbb{R}^N$, $maxit \in \mathbb{N}$, $tol \in \mathbb{R}_+$
  $r_0 = b - Ax_0$, $p_0 = r_0$ ▷ initialization
  for $j = 1, 2, \ldots, maxit$ **do**
    $\gamma_{j-1} = (r_{j-1}^T r_{j-1})/(p_{j-1}^T A p_{j-1})$
    $x_j = x_{j-1} + \gamma_{j-1} p_{j-1}$ ▷ update of approximation
    $r_j = r_{j-1} - \gamma_{j-1} A p_{j-1}$ ▷ update of residual
    if $\|r_j\|_2 / \|r_0\|_2 \leq tol$ **then** ▷ test for convergence
      return x_j
    end if
    $\beta_j = r_j^T r_j / r_{j-1}^T r_{j-1}$
    $p_j = r_j + \beta_j p_{j-1}$ ▷ update of search direction
  end for

(a) Implement the CG method in a `function [xj,r2u,r2,e2,eA] = p12cg(A, b, x0, maxit, tol, x)`. The parameter `x` is the exact solution of $Ax = b$ (which we don't have in practice, but we will use it for analyzing the method here). Let $J \in \mathbb{N}$ be the iteration number where the convergence criterion is satisfied (that is, $\|r_{J-1}\|_2/\|r_0\|_2 > tol$ and

$\|r_J\|_2/\|r_0\|_2 \le tol$) or where the maximal number of iterations is reached, $J = \texttt{maxit}$. The first return value `xj` then should be the approximation computed in the J-th iteration: $\texttt{xj} = x_J$. Additionally, in each iteration $j = 0, 1, 2, \ldots, J$ the following quantities have to be computed:

- $\texttt{r2u(}j\texttt{+1)} = \|r_j\|_2/\|r_0\|_2$ (relative Euclidean norm of the *updated* residual as computed in the `for`-loop).
- $\texttt{r2(}j\texttt{+1)} = \|b - Ax_j\|_2/\|b - Ax_0\|_2$ (relative Euclidean norm of the *explicitly computed* residual).
- $\texttt{e2(}j\texttt{+1)} = \|x - x_j\|_2/\|x - x_0\|_2$ (relative Euclidean norm of the error).
- $\texttt{eA(}j\texttt{+1)} = \|x - x_j\|_A/\|x - x_0\|_A$ (relative A-norm of the error; $\|v\|_A := \sqrt{v^\top Av}$).

The return values `r2u`, `r2`, `e2` and `eA` should be column vectors of length $J + 1$.

(b) Write a `function p12diag()`, that tests your implementation of the CG method with the following parameters and plots the return values `r2u`, `r2`, `e2` and `eA` with the `semilogy` command:

$$A = \texttt{diag([1:48])}, \quad x = \texttt{ones(48,1)}, \quad b = Ax,$$
$$x_0 = \texttt{zeros(48,1)}, \quad maxit = 200, \quad tol = 10^{-12}.$$

Annotate the plots appropriately.

(c) Write a `function p12laplace()` that tests your implementation of the CG method with the parameters $maxit = 400$, $tol = 10^{-6}$ and the $M^2 \times M^2$ matrix (compare Homework 4)

$$A_M = \begin{pmatrix} T_M & -E_M & & & \\ -E_M & T_M & -E_M & & \\ & \ddots & \ddots & \ddots & \\ & & -E_M & T_M & -E_M \\ & & & -E_M & T_M \end{pmatrix},$$

where $M = 200$, $E_M \in \mathbb{R}^{M\times M}$ is the identity matrix and $T_M = tridiag(-1, 4, -1) \in \mathbb{R}^{M\times M}$. Note that the matrix A_M is *sparse*. Therefore use the appropriate MATLAB/Octave functions (for example `kron`, `gallery`, `speye`) for the construction of A_M.

The right hand side is

$$b = \frac{1}{(M+1)^2}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

and the "exact" solution can be obtained by using the backslash-operator, that is `x=A\b`.

Plot the return values and annotate your plots appropriately as in (b).