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WS 11/12
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Numerische Mathematik für Ingenieure II
Homework 11

Exercise 16: (6 points)
Let $A \in \mathbb{C}^{n,n}$ and $\mathcal{S}_j, \mathcal{C}_j$ be two j-dimensional subspaces of $\mathbb{C}^n$. Suppose that one of the following conditions hold:

(a) A is Hermitian, positive definite and $\mathcal{S}_j = \mathcal{C}_j$,

(b) A is Hermitian and invertible, and $\mathcal{S}_j = A\mathcal{C}_j = \{Av : v \in \mathcal{C}_j\}$,

(c) A is invertible and $A\mathcal{S}_j = \mathcal{C}_j$.

Then $C_j^* A S_j$ is invertible for all matrices $S_j, C_j \in \mathbb{C}^{n,j}$, whose columns form a basis of $\mathcal{S}_j, \mathcal{C}_j$, respectively.
Note: Prove statements (b) and (c). (The proof of (a) was given in the lecture).

Exercise 17: (6 points)
The GMRES method for solving a linear algebraic system $Ax = b$ with a nonsingular matrix A is characterized by:

$$x_j \in x_0 + \mathcal{K}_j(A, r_0) \quad \text{such that} \quad r_j \perp A\mathcal{K}_j(A, r_0).$$

(a) Show that

$$\|r_j\| = \|b - Ax_j\| = \min_{y \in x_0 + \mathcal{K}_j(A, r_0)} \|b - Ay\|. \tag{1}$$

Here $\|\cdot\|$ denotes the Euclidean norm.

(b) Let A and b be real. Using the Arnoldi relation (see the lecture)

$$AV_j = V_{j+1}\underline{H}_j, \text{ where } V_j = [v_1, \ldots, v_j], \quad V_j^* V_j = I_j \text{ and}$$

$$\underline{H}_j = \begin{bmatrix} h_{1,1} & h_{1,2} & \cdots & \cdots & h_{1,j} \\ \|\widehat{v}_2\| & h_{2,2} & \cdots & \cdots & h_{2,j} \\ 0 & \|\widehat{v}_3\| & \ddots & & \vdots \\ & 0 & \ddots & \ddots & \ddots \\ & & & \|\widehat{v}_j\| & h_{j,j} \\ & & & & \|\widehat{v}_{j+1}\| \end{bmatrix},$$

show that the GMRES minimization problem (1) can be written as

$$\|r_j\| = \min_{y \in \mathbb{R}^j} \| \|r_0\| e_1 - \underline{H}_j y\|, \text{ with } y \in \mathbb{R}^j.$$

Recall in particular that $v_1 = r_0 / \|r_0\|$.

Programming exercise 14: (12 points)
The above problem is solved in the following GMRES algorithm:

Require: $A \in \mathbb{R}^{N\times N}$, $b, x_0 \in \mathbb{R}^N$, $maxit \in \mathbb{N}$, $tol \in \mathbb{R}_+$
  $r_0 = b - Ax_0$, $v_1 = r_0/\|r_0\|$ ▷ initialization
  for $j = 1, 2, \ldots, maxit$ **do**
    $\widehat{v}_{j+1} := Av_j$
    for $i = 1, \ldots, j$ **do** ▷ build $\widehat{v}_j \perp v_1, \ldots, v_{j-1}$
      $h_{i,j} := v_i^T \widehat{v}_{j+1}$
      $\widehat{v}_{j+1} := \widehat{v}_{j+1} - h_{i,j} v_i$
    end for
    $h_{j+1,j} = \|\widehat{v}_{j+1}\|$
    if $h_{j+1,j} = 0$ **then** go to $(\star)$
    end if
    $v_{j+1} = \widehat{v}_{j+1}/h_{j+1,j}$ ▷ normalization
    $(\star)$ Compute y_j, the minimizer of $\|\|r_0\|e_1 - \underline{H}_j y\|$,
    $x_j = x_0 + V_j y_j$
    $\|r_j\| = \|b - Ax_j\|$
    if $\|r_j\| < tol$ **then** ▷ Test for convergence
      return
    end if
  end for

(a) Implement the GMRES method in a

```
function [xj, r2] = p14gmres(A, b, x0, maxit, tol).
```

Solve the minimization problem $(\star)$ using the QR-decomposition of $\underline{H}_j$, namely

```
[Q,R] = qr(H(1:j+1,1:j)),
y = R(1:j,1:j)\(Q(1:j+1,1:j)'*norm(r0)*eye(j+1,1))
```

(b) Write a `function p14test()` that tests your GMRES implementation with matrices A and right hand sides b from Programming Exercise 7 (use `[A,b]=p07getLS(...)`). Conduct the experiments with `n`$= 2^5$ and the convection coefficients `v` $\in \{0, 1, 5, 10, 25, 50, 100\}$. Plot `r2` for different values of `v`. (See Homework 4 for more details).

(c) **Extra points**: Suggest a more efficient way to solve the minimization problem $(\star)$.