Numerical Mathematics for Engineers II
Homework 1

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Exercise 1

Claim 1. *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be symmetric. $\boldsymbol{A}$ is positive definite iff all eigenvalues of $\boldsymbol{A}$ are positive.*

Proof. $\Rightarrow$ Let x be an eigenvector of $\boldsymbol{A}$ with $\lambda x = \boldsymbol{A}x$. $0 < x^T\boldsymbol{A}x = \lambda x^T x = \lambda |x|^2$. Since $|x|^2 > 0 \Rightarrow \lambda > 0$.

$\Leftarrow$ Since $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{A} = \boldsymbol{A}^T$ there exists an orthonormal basis $(b_1, \dots, b_n)$ consisting of the eigenvectors of $\boldsymbol{A}$. So for any vector exists c_i such that $x = \sum_{i=1}^n c_i b_i$. $x^T\boldsymbol{A}x = \langle x, Ax\rangle = \langle \sum_{i=1}^n c_i b_i, \sum_{i=1}^n \lambda_i c_i b_i\rangle = \sum_{i,j=1}^n \langle c_j b_j, \lambda_i c_i b_i\rangle = \sum_{i=1}^n \langle c_i b_i, \lambda_i c_i b_i\rangle = \sum_{i=1}^n \lambda_i c_i^2 > 0$.

In the next to last step, we have used that the basis vectors are orthogonal. In the last step we have used that all λ_i are positive and the sum of them multiplied by a positive factor c_i^2.

$\square$

Exercise 2

Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix.

(a)

Claim 2. *$\boldsymbol{A}$ is invertible and $\boldsymbol{A}^{-1}$ is SPD*

Proof. Let's assume that $\boldsymbol{A}$ is not invertible. That means $\exists x$ such that $\boldsymbol{A}x = 0$. This further implies $x^T Ax = x^T 0 = 0$, which contradicts the fact that $\boldsymbol{A}$ is positive definite.

Then it holds $\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{I}$ and $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$. With $(\boldsymbol{A}^{-1}\boldsymbol{A})^T = \boldsymbol{A}^T(\boldsymbol{A}^{-1})^T = \boldsymbol{A}(\boldsymbol{A}^{-1})^T = \boldsymbol{I}$. And thus it holds $\boldsymbol{A}^{-1} = (A^{-1})^T$.

The positive definiteness follows from: $0 < x^T\boldsymbol{A}x = x^T\boldsymbol{A}\boldsymbol{A}^{-1}\boldsymbol{A}x = (\boldsymbol{A}x)^T\boldsymbol{A}^{-1}(\boldsymbol{A}x)$. Since $\boldsymbol{A}$ is regular $\forall \tilde{x} : \exists x : \tilde{x} = \boldsymbol{A}x$ and thus with $\tilde{x} = \boldsymbol{A}x$ it holds that $\tilde{x}^T\boldsymbol{A}^{-1}\tilde{x} > 0$. $\square$

(b)

Claim 3. *The diagonal elements* $\boldsymbol{a}_{ii}$ *of* $\boldsymbol{A}$ *are positive.*

Proof. Choose $x = e_i$ with $i \in [1, n]$. $x^T \boldsymbol{A} x = x^T \boldsymbol{a}_i$ with $\boldsymbol{a}_i$ being the ith column. Then $x^T \boldsymbol{a}_i = \boldsymbol{a}_{ii} > 0$ because $\boldsymbol{A}$ is positive definite. □

Exercise 3

Let $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$ be an orthogonal matrix.

(a)

Claim 4. $||\boldsymbol{Q}x||_2 = ||x||_2 \ \forall x \in \mathbb{R}^n$

Proof. Since $\boldsymbol{Q}$ is an orthogonal matrix, its row vectors form a basis $(r_1, \ldots, r_n)$ of $\mathbb{R}^n$ with $||r_i||_2 = 1$. Thus for every $x \in \mathbb{R}^n$ exist c_i such that $x = \sum_i^n c_i r_i$. $||\boldsymbol{Q}x||_2 = ||\sum_i^n c_i \boldsymbol{Q} r_i||_2 = ||\sum_i^n c_i e_i||_2 = \sqrt{\sum_i^n c_i^2} = \sqrt{\langle \sum_i^n c_i r_i, \sum_i^n c_i r_i \rangle} = ||x||_2$ □

(b)

Claim 5. $\forall \lambda$ *of* $\boldsymbol{Q}$*:* $|\lambda| = 1$.

Proof. Let x be an eigenvector of $\boldsymbol{Q}$. $||\boldsymbol{Q}x||_2 = ||\lambda x||_2 = |\lambda| \cdot ||x||_2 = ||x||_2$. Thus $|\lambda| = 1$. □

(c)

Claim 6. $\kappa_2(\boldsymbol{Q}) = 1$

Proof. $||\boldsymbol{Q}||_2 := \sup\{||\boldsymbol{Q}y||_2 : y \in \mathbb{R}^n, ||y||_2 = 1\}$. According to (a): $||\boldsymbol{Q}||_2 = \sup\{||y||_2 = 1\} = 1$. Since $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$ and $\boldsymbol{Q}^T$ is an orthogonal matrix, it follows that $||\boldsymbol{Q}^{-1}||_2 = 1$ and thus $\kappa_2(\boldsymbol{Q}) = 1$. □

(d)

Claim 7. *Let* $\boldsymbol{A} \in \mathbb{R}^{n \times n}$. $||\boldsymbol{A}||_2 = ||\boldsymbol{QA}||_2 = ||\boldsymbol{AQ}||_2$.

Proof. $||\boldsymbol{A}||_2 = \sup\{||\boldsymbol{A}y||_2 : y \in \mathbb{R}^n, ||y||_2 = 1\} = \sup\{||x||_2 : x = \boldsymbol{A}y, y \in \mathbb{R}^n, ||y||_2 = 1\} = \sup\{||\boldsymbol{Q}x||_2 : x = \boldsymbol{A}y, y \in \mathbb{R}^n, ||y||_2 = 1\} = \sup\{||\boldsymbol{QA}y||_2 : y \in \mathbb{R}^n, ||y||_2 = 1\} = ||\boldsymbol{QA}||_2$

$||\boldsymbol{A}||_2 = \sup\{||\boldsymbol{A}y||_2 : y \in \mathbb{R}^n, ||y||_2 = 1\} = \sup\{||\boldsymbol{AQ}x||_2 : \boldsymbol{Q}x \in \mathbb{R}^n, ||\boldsymbol{Q}x||_2 = 1\} = \sup\{||\boldsymbol{AQ}x||_2 : x \in \mathbb{R}^n, ||\boldsymbol{Q}x||_2 = 1\} = \sup\{||\boldsymbol{AQ}x||_2 : x \in \mathbb{R}^n, ||x||_2 = 1\} = ||\boldsymbol{AQ}||_2$. □