DARBOUX, MOSER AND WEINSTEIN THEOREMS FOR PREQUANTUM SYSTEMS

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ABSTRACT. We establish analogs of the Darboux, Moser and Weinstein theorems for prequantum systems. We show that two prequantum systems on a manifold with vanishing first cohomology, with symplectic forms defining the same cohomology class and homotopic to each other within that class, differ only by a symplectomorphism and a gauge transformation. As an application, we show that the Bohr-Sommerfeld quantization of prequantum system on a manifold with trivial first cohomology is independent of the choice of the connection.

1. INTRODUCTION

The Darboux theorem establishes that there are no local invariants in symplectic geometry. Namely, let M be a smooth manifold and let ω, ω' be two symplectic forms on M, then:

Theorem 1 (Darboux [1, 2]). *For every point $p \in M$ there exists a neighbourhood U of p and an embedding $\Phi : U \to M$ isotopic to the inclusion and fixing p such that*

$$\Phi^*\omega' = \omega|_U.$$

The global invariants of a symplectic manifold include the cohomology class defined by its symplectic form. This gives a complete classification of symplectic 2-manifolds. In general, there is the following theorem due to Moser.

Theorem 2 (Moser [6]). *Let M be a compact manifold endowed with two symplectic forms ω and ω'. Assume that $[\omega] = [\omega']$, and that there exists a path ω_t of symplectic forms such that $[\omega_t] = [\omega_0]$ for all t, and with $\omega_0 = \omega$ and $\omega_1 = \omega'$. Then, there exists a diffeomorphism isotopic to the identity $\Phi : M \to M$ such that $\Phi^*\omega' = \omega$.*

This proof based on Moser's method can be adapted when symmetries are present. Under the conditions above when a compact group acts on M preserving ω and ω', and where the path ω_t also consists of invariant forms, we have the following result.

Theorem 3 (Weinstein [7]). *There exists an equivariant diffeomorphism $\Phi : M \to M$ such that $\Phi^*\omega' = \omega$.*

Recall that if (M, ω) is a symplectic manifold, $\pi : L \to M$ is a (complex) line bundle, ∇ is a connection on L, and $\mathrm{curv}(\nabla) = \omega$, the quadruple (M, ω, L, ∇) is a *prequantum system*.

Both authors are partially supported by the Spanish State Research Agency grant PID2019-103849GB-I00 of MICIU/AEI / 10.13039/501100011033. E. Miranda is supported by the Catalan Institution for Research and Advanced Studies via an ICREA Academia Prize 2021 and by the Alexander Von Humboldt foundation via a Friedrich Wilhelm Bessel Research Award. E. Miranda is also supported by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (project CEX2020-001084-M) and partially supported by and by the AGAUR project 2021 SGR 00603.

J. Weitsman was supported in part by a Simons Collaboration Grant # 579801.

April 9, 2024.