Beliefs (theory)

Probability Refresher, Bayesian Updating

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Why Beliefs Matter

Beliefs describe what people *think* might happen.

Many decision situations do not feature *risk* (obj. probability) but *uncertainty*.

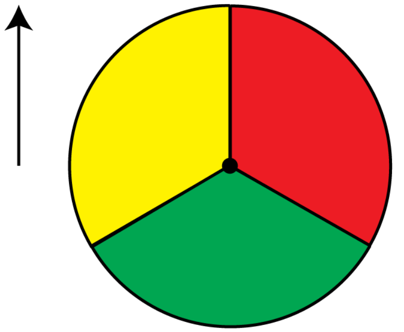
- Voting for a political party.
- Investing in stocks.
- Choosing a field of study.

In **SEU**, beliefs adhere to probability rules.

We will refresh them now.

Setup

- *State space* $\Omega = \{\omega_1, \ldots, \omega_n\}$ – exhaustive & mutually exclusive.
  - Example: spinning wheel $\Omega = \{R, G, Y\}$
  - $\omega = R$ is a possible **state of the world.**
- A **probability measure** P assigns each state a *probability* in $[0, 1]$ with $\sum_{\omega \in \Omega} P(\omega) = 1$.



Core Rules

Sum rule

If A and B are subsets of Ω with no elements in common, then $P(A \cup B) = P(A) + P(B)$.

For example, the probability of spinning Red or $Green$ on the spinning wheel is equal to $P(R) + P(G)$.

What about nonindependent events?

If Ω contains all human knowledge, then

$$P(\text{Knowledge I have} \cup \text{Knowledge required in exam}) < P(\text{Knowledge I have}) + P(\text{Knowledge required in exam}).$$

What about nonindependent events?

Consider two subsets $A, B \subset \Omega,$ with $A \cap B \neq \varnothing$ (they are not independent).

Then define:

- $C \equiv A \cap B,\ A' \equiv A \setminus C,\ B' \equiv B \setminus C.$

Observe that $P(A \cup B) = P(A' \cup B' \cup C).$

By the sum rule $P(A' \cup B' \cup C) = P(A') + P(B') + P(C).$

Plug in $P(C) = P(A \cap B),\ P(A') = P(A) - P(C),\ P(B') = P(B) - P(C)$ to arrive at $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Product rule

A and B are *independent* if

$$P(A \cap B) = P(A)P(B).$$

E.g., the prob. of simultaneously tossing tails on a coin and rolling six on a die is $\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$.

Conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \qquad P(B) > 0.$$

This is a generalization of the product rule. To see this, rearrange the def. to $P(A \cap B) = P(A|B)P(B)$.

If A is independent of B, then $P(A|B) = P(A)$ and we are back to the product rule.

The conditioning order matters

Consider the following statement:

> *"One third of sports injuries in Germany are football-related. No other sport generates so many injuries. Therefore, football is the most dangerous sport"*

The line of argument is: (i) $P(\mathrm{Sport}|\mathrm{Injury})$ is maximal for $\mathrm{Sport} = \mathrm{Football}.$ (ii) Therefore, $P(\mathrm{Injury}|\mathrm{Sport})$ is maximal for $\mathrm{Sport} = \mathrm{Football}.$ But (ii) does not necessarily follow from (i)!

The conditioning order matters

Suppose that this is the true joint distribution of sports activity and injuries:

	Play football	**Play other sport**
Injury	3.3%	6.7%
No injury	45%	45%

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The probability of *play football* conditional on *injury* is

$$P(\text{Play football}|\text{Injury}) = \frac{3.3\%}{3.3\% + 6.7\%} = \frac{1}{3}$$

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	Play football	**Play other sport**
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No injury	**45%**	45%

The probability of *injury* conditional on *play football* is

$$P(\text{Injury}|\text{Play football}) = \frac{3.3\%}{3.3\% + 45\%} \approx 6.8\%$$

The conditioning order matters

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	Play football	**Play other sport**
Injury	3.3%	**6.7%**
No injury	45%	**45%**

The probability of *injury* conditional on *play other sport*

is $P(\text{Injury}|\text{Play other sport}) = \frac{6.7\%}{6.7\% + 45\%} \approx 13.7\%$

Bayes' theorem. If A and B are subsets of Ω and $P(A), P(B) > 0, \quad$ then

$$P(A \mid B) = \frac{P(A)\ P(B \mid A)}{P(B)}.$$

This is an implication of the cond. probability definition. We know:

$$P(A|B) = \frac{P(A \cap B)}{P(B)},\ P(B|A) = \frac{P(B \cap A)}{P(A)}.$$

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Plug these definitions into Bayes' theorem and confirm that the rhs is equal to the lhs:

$$\frac{P(A \cap B)}{P(B)} = \frac{P(B \cap A)}{P(A)} \frac{P(A)}{P(B)} \Rightarrow P(A \cap B) = P(B \cap A).$$

Bayes' theorem as a foundation of learning

Consider the following example of a medical test.

> Sasha gets screened for a rare form of cancer that around 1 in 1000 people in Sasha's demographic have.
>
> - The test identifies a sick person as sick with 99%.
> - The test identifies a healthy person as healthy with 98%.
>
> The test result is positive. What's the probability that Sasha has cancer?

Sasha's example contd.

The test can be positive ($+$) or negative($-$).

Sasha can have cancer (C) or not ($\bar{C}$).

Possible states of the world: $\Omega = \{(C,+),(C,-),(\bar{C},+),(\bar{C},-)\}$.

Applying Bayes theorem:

$$P(C|+) = \frac{P(C)P(+|C)}{\underbrace{P(C)P(+|C) + P(\bar{C})P(+|\bar{C})}_{=P(+)}} = \frac{0.001 \cdot 0.99}{0.001 \cdot 0.99 + 0.999 \cdot 0.02} \approx 4.8\%.$$

Expectations

- If the states of the world have a numerical ordering, we can calculate the **expected value.**
  - E.g., die rolls have a numerical ordering ($\Omega = \{1, 2, 3, 4, 5, 6\}$).
- In such cases the **unconditional expectation** is equal to $\mathbb{E}[\omega] = \sum_{\omega \in \Omega} P(\omega)\omega$.
- **Conditional expectations** work similarly as conditional probabilities: $\mathbb{E}[\omega|B] = \frac{\sum_{\omega \in B} P(\omega)\omega}{\sum_{\omega \in B} P(\omega)}$.
  - E.g., the expected die roll conditional on being larger than 3 is: $\mathbb{E}[\omega|\omega > 3] = \frac{\frac{1}{6}(4+5+6)}{\frac{1}{2}} = 5$.

Martingale Property (Law of Iterated Expectations)

$$\mathbb{E}\left[\mathbb{E}[\omega|B]\right] = \mathbb{E}[\omega].$$

Conditioning **then** averaging returns the unconditional mean.

Implication. I cannot expect to change my expectation in any particular direction in the future.

Law of Iterated Expectations (Martingale Property)

$$\mathbb{E}\left[\mathbb{E}[\omega|B]\right] = \mathbb{E}[\omega].$$

E.g., denote tomorrow's expected temperature by $\mathbb{E}[T]$. This expectation might change depending on a future weather report, $\mathbb{E}[T|\text{weather rep.}]$. However, not knowing the future weather report, $\mathbb{E}[\mathbb{E}[T|\text{weather report}]] = \mathbb{E}[T]$.

I cannot expect that listening to the weather report will make me more optimistic about tomorrow's weather.

Martingale Property, implications

- This property underlies rational learning: you cannot systematically expect to revise your beliefs in a particular direction.
- If you become more optimistic after some information, there must be some other kind of information that makes you more pessimistic.

↑ **Inconsistent with martingale property**

Beliefs = Subjective Probabilities

SEU implies individual beliefs **must** obey probability laws.

This has profound implications:

- When learning new information, beliefs are updated according to Bayes’ rule.
- Beliefs adhere to the martingale property.

Bayesian Updating

Bayesian updating describes the process of learning from information using Bayes' theorem.

General structure:

$$\text{Prior belief} \rightarrow \ \text{Information} \rightarrow \text{Posterior belief.}$$

We will learn the mechanics of Bayesian updating in two cases that are often used in applied economic research.

Binary State + Binary Signal Model

This model has a binary state, $\Omega = \{H, L\}$, and agents receive binary signals, $s \in \{g, b\}$.

- Think: Stocks will either decrease or increase (state), news can either be good or bad (signal).

The model specifies the **prior belief,** $P(H) = \lambda$, and **signal precision:**

$$P(s = g|\omega = H) = q_g,\ P(s = b|\omega = L) = q_b, \quad q_g, q_b \geq \frac{1}{2}.$$

Binary State + Binary Signal Model

Proposition. In the binary-binary model,

$$P(H \mid g) = \frac{\lambda q_g}{\lambda q_g + (1-\lambda)(1-q_b)},$$

$$P(H \mid b) = \frac{\lambda(1-q_g)}{\lambda(1-q_g) + (1-\lambda)q_b}.$$

Likelihood-ratio Form

We can use Bayes' theorem to derive the following result:

$$\frac{P(H \mid g)}{P(L \mid g)} = \frac{\lambda}{1-\lambda} \frac{q_g}{1-q_b}.$$

- Posterior LR = Prior LR × Signal strength.
- More precise signals **move beliefs more**.

Likelihood-ratio Form

We can easily iterate the LR form to derive that, after γ good & β bad signals (with $q_g = q_b = q$):

$$\frac{P(H \mid \gamma g + \beta b)}{P(L \mid \gamma g + \beta b)} = \frac{\lambda}{1-\lambda}\left(\frac{q}{1-q}\right)^{\gamma-\beta}.$$

Only the **excess** of good over bad news matters.

Binary-Binary Model Implications

Posterior beliefs depend on:

- Prior belief: The higher $\lambda,$ the higher $P(H|I)$ after any finite number of signals I.
- Signal precision: The higher $q_g,$ the higher $P(H|g)$ (and vice versa for b).

As the number of signals grows to infinity, the belief converges to the truth.

Normal–Normal Model

This model has a real-valued state, $\Omega = \mathbb{R}$, and agents receive real-valued signals, $s \in \mathbb{R}$.

- Think: Company valuation (state), yearly revenue (signal).

The model specifies the **prior belief,** $\omega \sim \mathcal{N}(\bar{\omega}_0, \sigma_0^2)$, and **signal precision:** $s \mid \omega \sim \mathcal{N}(\omega, \sigma_s^2)$.

Normal-Normal Model

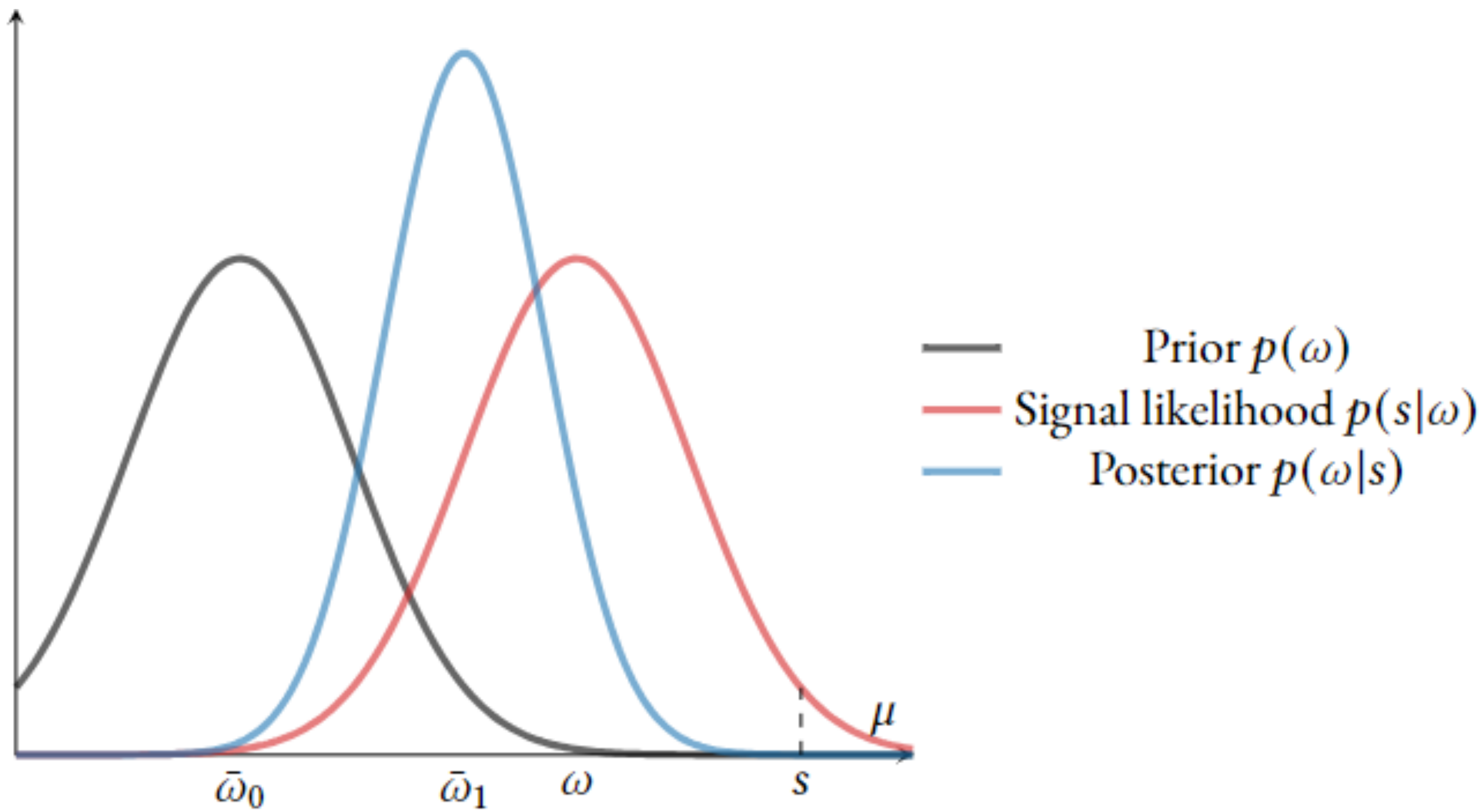
Proposition. In the binary-binary model,

$$\omega \mid s \;\sim\; \mathcal{N}\!\left(\bar{\omega}_1, \sigma_1^2\right) \text{ with}$$

$$\bar{\omega}_1 = \alpha s + (1-\alpha)\bar{\omega}_0, \qquad \sigma_1^2 = \frac{\sigma_s^2 \sigma_0^2}{\sigma_s^2 + \sigma_0^2}, \qquad \alpha = \frac{\sigma_0^2}{\sigma_0^2 + \sigma_s^2}.$$

- Posterior mean = **precision-weighted average** of prior mean & signal.
- Posterior variance < prior variance $\rightarrow$ beliefs tighten.

Graphical Intuition



Posterior lies between prior and likelihood; the variance shrinks when going from prior to posterior.

Normal-Normal Model Implications

The lessons we can draw from normal-normal are very related to the lessons we can draw from binary-binary.

Posterior beliefs depend on:

- Prior belief: The higher $\bar{\omega}_0,$ the higher $\bar{\omega}_1$ after any finite number of signals.
- Signal precision: The smaller $\sigma_s,$ the larger is the posterior belief movement towards the signal.

As the number of signals grows to infinity, the expected value $\bar{\omega}_1$ converges to the truth and the variance σ_1 goes to zero.

Summarizing the updating results

1. **Beliefs** in SEU are subjective probabilities; Bayesian agents obey probability laws.
2. **Binary model**: posterior LR = prior LR × signal likelihoods.
3. **Normal–normal**: posterior = precision-weighted average of prior and signal.
4. Precision determines learning speed; agents eventually learn the truth.

Takeaways

- **Beliefs** are subjective probabilities that guide decisions under uncertainty.
- **Probability rules** (sum, product, conditional, Bayes, martingale) are foundational for rational beliefs.
- **Bayesian updating**: Posterior beliefs depend on prior and signal precision; likelihood ratios summarize belief changes.
  - Tractable models of Bayesian updating exist. We learned about two such models, binary-binary and normal-normal.