Probabilities

Outline

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What are probabilities?

- Probability as frequency of outcome of events:
  - In this way of thinking about probabilities we look at the number of times a given event happens over a large number of tries.
  - It is difficult to define consistently however, without running into circular reasoning.

- Probability as degree of belief:
  - Subjective probability is associated with personal judgements about how likely something is to happen.
  - For example, 'I believe that team X will beat team Y, because teams Y's star player has an injury, while team X has been training really hard.' Such statements can be made even if teams X and Y have never played each other.
  - By requiring that two people arrive at the same conclusion if given the same assumptions and data, this definition of probability can be formalised into a mathematical system

equivalent to the other definitions.

- Probably derived from axioms:
  - Probability is a measure that satisfies a set of axioms derived from logic and set theory, such as the Kolmogorov or Cox axioms.
  - This sidesteps the frequentist vs Bayesian interpretation by sticking to purely mathematical concepts.

In this course we start out with this definition. In general we will follow the Bayesian degree-of-believe way of thinking about probability.

Notation and basic concepts

Set notation

- A set is a collection of elements, e.g.: $A = \{1, 2, 3\}$
- $e \in A$ means e is a member of the set A, e.g.: $1 \in \{1, 2, 3\}$
- A set can also be represented by a rule: $\{x | x \text{ follows a rule}\}$

  For example, the set E of even integers: $E = \{x | x = 2y,\ y \in \mathbb{Z}\}$
- Set inclusion ($\subseteq$). A is included in B (or is a subset of B) if all the elements of A are also elements of B.

  For example: $\{1, 2\} \subseteq \{1, 2, 3\}$

Set operators

Let $A = \{1, 3, 5\}$; $B = \{2, 3, 4\}$

- **Union** $\cup$ All elements of A and all elements of B

  $A \cup B = \{1, 2, 3, 4, 5\}$
- **Intersection** $\cap$: Elements that are in both A and B

  $A \cap B = \{3\}$
- **Difference** $\setminus$ Elements that are in A but not in B

  $A \setminus B = \{1, 5\}$

  $B \setminus A = \{2, 4\}$
- **Complement**:

The complement of A in reference to Ω includes all elements in Ω that are not in A. For the die example $\Omega = \{1, 2, 3, 4, 5, 6\}$,

$A^c = \{2, 4, 6\}$ or

$A^c = \{\omega : \omega \in \Omega \text{ and } \omega \notin A\}$

- **Empty set** $\varnothing$

  The empty set, $\varnothing$, is the complement of the universal set:

  $\Omega^c = \varnothing$ and $\varnothing^c = \Omega$.

  This means, $A \cup \varnothing = A$ and $A \cap \varnothing = \varnothing$.

- **Power set**

  Collection of all possible sets of a given set

  $A = \{1, 3, 5\}$

  $\mathcal{P}(A) = \{\varnothing, \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 3, 5\}\}$

Outcomes, events, probability

Outcomes and sample space

The outcomes ω of an experiment are elements of the set of all possible outcomes, called the sample space Ω.

Consider the experiment of tossing a (fair) coin twice:

- $\omega = \mathrm{HH}$ ("two heads")
- $\omega \in \Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$

Events and event space

An event F is a set of outcomes

- $F = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}\}$ ("at least one head")

Events are elements of the event space $\mathcal{F}$: the power set of the sample space (the set of all possible outcomes)

Note that Ω and $\mathcal{F}$ are not the same. The sample space contains the basic outcomes and event space contains sets of outcomes.

Probability

The probability function P assigns a probability (a number between 0 and 1) to events

- $F = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}\}$
- $\Pr(F) = \frac{3}{4}$

Kolmogorov's axioms of probability

- The probability measure of events is a real number equal or larger than 0:

  $0 \leq \Pr(A)$

- The probability measure of the universal set is 1.

  $\Pr(\Omega) = 1$

- If the sets A_1, A_2, A_3 ... $\in \mathcal{F}$ are disjoint, then

  $\Pr(A_1 \cup A_2 \cup \ldots) = \Pr(A_1) + \Pr(A_2) + \ldots$

Consequences of the axioms of probability

- Numeric bound:

  $0 \leq \Pr(A) \leq 1$

- Monoticity:

  $A \subseteq B$ then $\Pr(A) \leq \Pr(B)$

- Complement rule:

  $\Pr(A^c) = 1 - \Pr(A)$

- Sum rule:

  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Example, a single fair die:

- $\Omega = \{1, 2, 3, 4, 5, 6\}$
- $\Pr(\omega) = \frac{1}{6} \quad \forall \omega \in \Omega$
- Events $A = \{1, 3\}$ and $B = \{1, 2, 3, 4\}$

- Monoticity:

  $A \subseteq B$, $\Pr(A) = \frac{1}{3} \leq \Pr(B) = \frac{2}{3}$

- Complement rule:

  $\Pr(A^c) = \Pr(\{2,4,5,6\}) = \frac{2}{3} = 1 - \Pr(A)$

- Sum rule:

  $\Pr(A \cup B) = \Pr(\{1,2,3,4\}) = \frac{2}{3}$

  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = \frac{1}{3} + \frac{2}{3} - \Pr(\{1,3\}) = \frac{2}{3}$

Clicker

Two fair six-sided dice are rolled. What's the probability to obtain two sixes?

- 1/25
- 1/36
- 2/36
- 1/6

Conditional probabilities and independence

Conditional probabilities

The conditional probability of event A happening, given that event B happened, is

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Instead of the Kolmogorov axioms, probability theory can also be defined in terms of conditional probabilities, using the Cox axioms.

Independence

If A is independent of B, $\Pr(A|B) = \Pr(A)$: the conditional probability of A given B does not depend on B. From this follows that

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

Law of total probability

Let $\{H_1, H_2, \ldots\}$ be a countable collection of sets which is a partition of Ω, where

$H_i \cap H_j = \varnothing$ for $i \neq j$

$H_1 \cup H_2 \cup \ldots = \Omega$

The probability of an event D can be calculated as

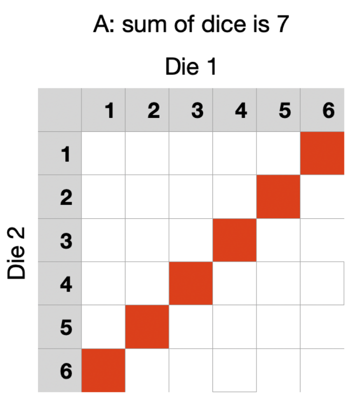
$$\Pr(D) = \Pr(D \cap H_1) + \Pr(D \cap H_2) + \ldots$$

or in terms of conditional probabilities

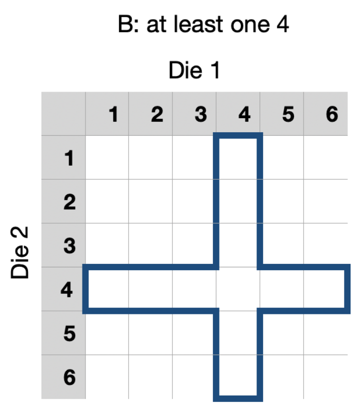
$$\Pr(D) = \Pr(D|H_1)\Pr(H_1) + \Pr(D|H_2)\Pr(H_2) + \ldots$$

Clicker

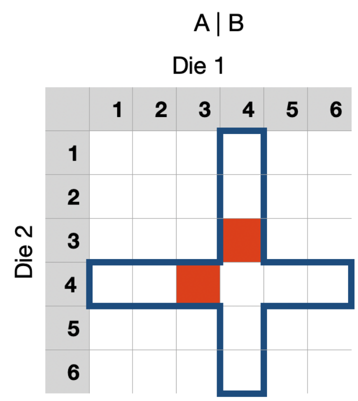
Two fair six-sided dice are rolled. Let A be the event that the sum of the dice is 7, and let B be the event that at least one of the two dice is a 4. Are A and B independent?



P(A) = 1/6

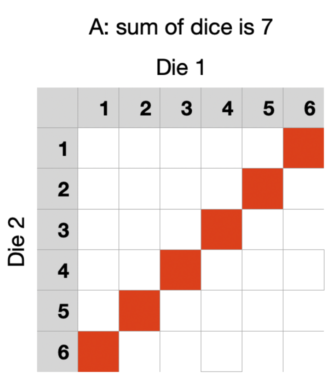


P(B) = 11/36

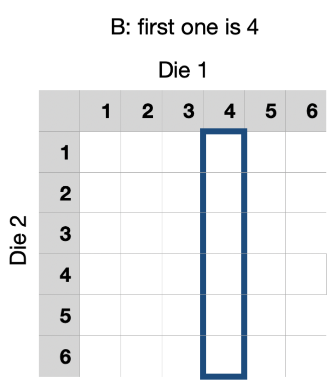


P(A|B) = 2/11

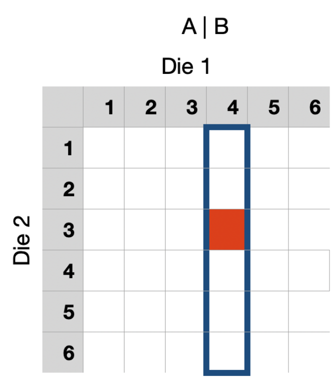
What about when B is the event that the first die is a 4?



P(A) = 1/6



P(B) = 1/6



P(A|B) = 1/6

Bayes' theorem

Applying the definition of the conditional probability twice we get Bayes' theorem:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$$

Named after Thomas Bayes, British clergyman, 1702-1761

Clicker: a test for rare events

Let us assume there is a rare disease that affects 0.1% of the population. There is a test that can detect this disease. It has a detection efficiency of 99% and a probability of error (false-positive) of 2%.

What is the probability $\Pr(D|+)$ of having the disease when receiving a positive test?

$\Pr(D|+) = \frac{\Pr(+|D)\Pr(D)}{\Pr(+)} = \frac{\Pr(+|D)\Pr(D)}{\Pr(+|D)\Pr(D)+\Pr(+|D^c)\Pr(D^c)}$ (law of total probability)

- $\Pr(+|D) = 0.99$: probability of a postive test result, given the disease is present (detection efficiency of 99%)
- $\Pr(D) = 0.001$: the disease affects 0.1% of the population
- $\Pr(+|D^c) = 0.02$: probability of a postive test result, given the disease is not present (false-positive rate of 2%)

$\Pr(D|+) = \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 + 0.02 \cdot 0.999} = 0.047$

The disease is only present in 5% of the cases where the test is positive!