```python
import numpy as np
import scipy.stats
import matplotlib.pyplot as plt
```

Solutions to random variables and probability distributions

Inverse transform sampling

The PDF is

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}\sin x & \text{otherwise} \\ 0 & x > \pi \end{cases}$$

The CDF is

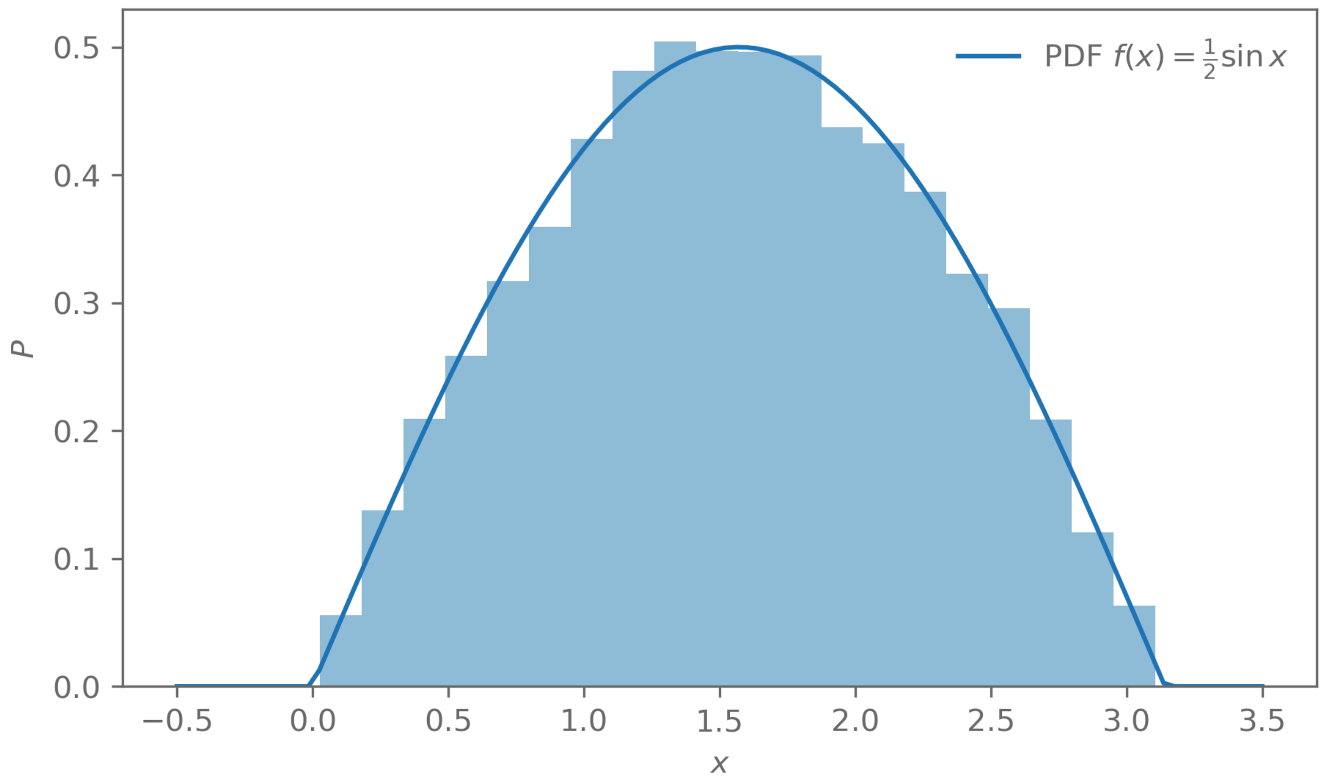
$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} - \frac{1}{2}\cos x & \text{otherwise} \\ 1 & x > \pi \end{cases} ,$$

with its inverse

$$F^{-1}(u) = \cos^{-1}(1 - 2u),\ u \in [0, \pi]\ .$$

```python
def sin_distr_pdf(x):
    return np.piecewise(x, [(0 <= x) & (x <= np.pi)], [lambda x: 0.5*np.sin(x), 0])

def sin_distr_cdf(x):
    return np.piecewise(x, [x < 0, (0 <= x) & (x <= np.pi)], [0, 0.5 - 0.5*np.cos(x),

def sin_distr_cdf_inv(u):
    if np.any((u < 0) | (1 < u)):
        raise ValueError("u outside domain of (0, 1)")
    return np.arccos(1- 2*u)
```

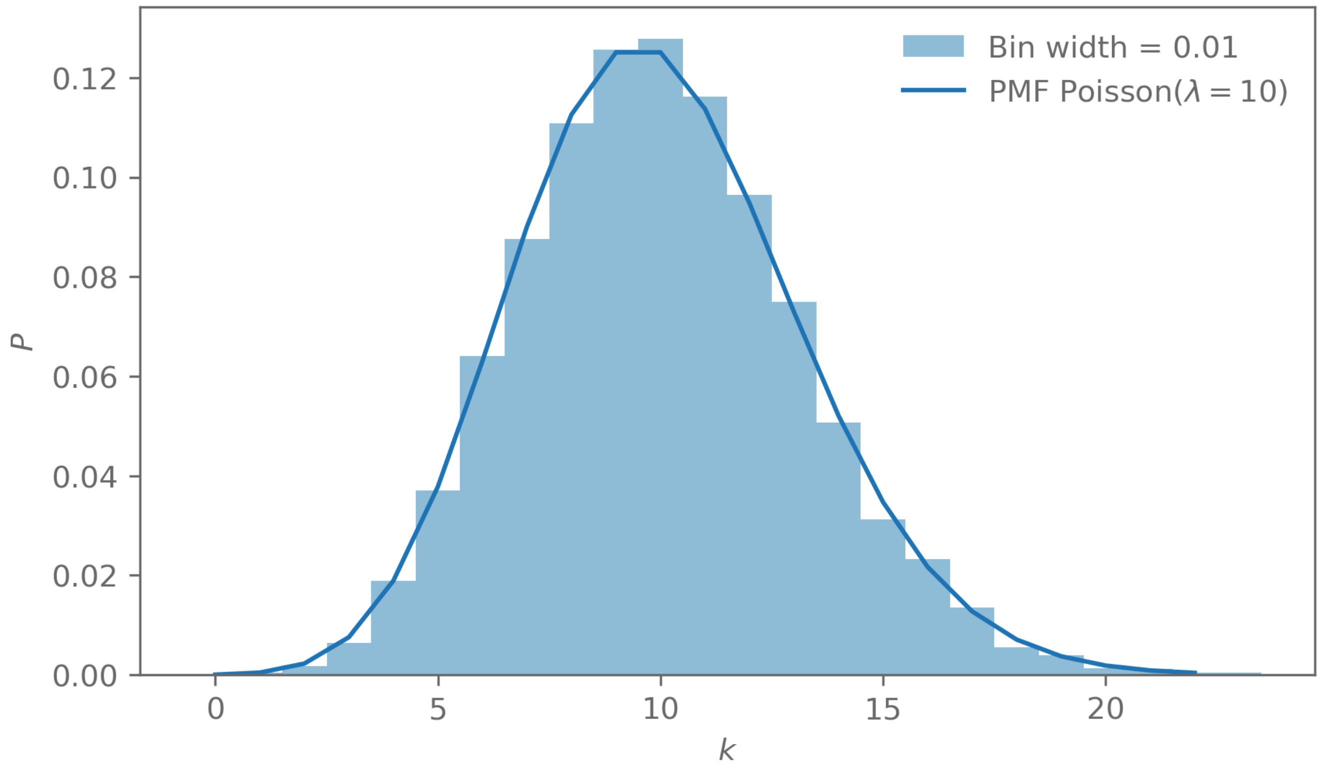
```
n = 10000
u = np.random.uniform(size=n)

grid = np.linspace(-0.5, 3.5, 100)

plt.hist(sin_distr_cdf_inv(u), bins=20, density=True, alpha=0.5)
plt.plot(grid, sin_distr_pdf(grid), c="C0", label=r"PDF $f(x)=\frac{1}{2}\sin x$")
plt.legend(frameon=False)
plt.xlabel("$x$")
plt.ylabel("$P$");
```



Poisson distribution

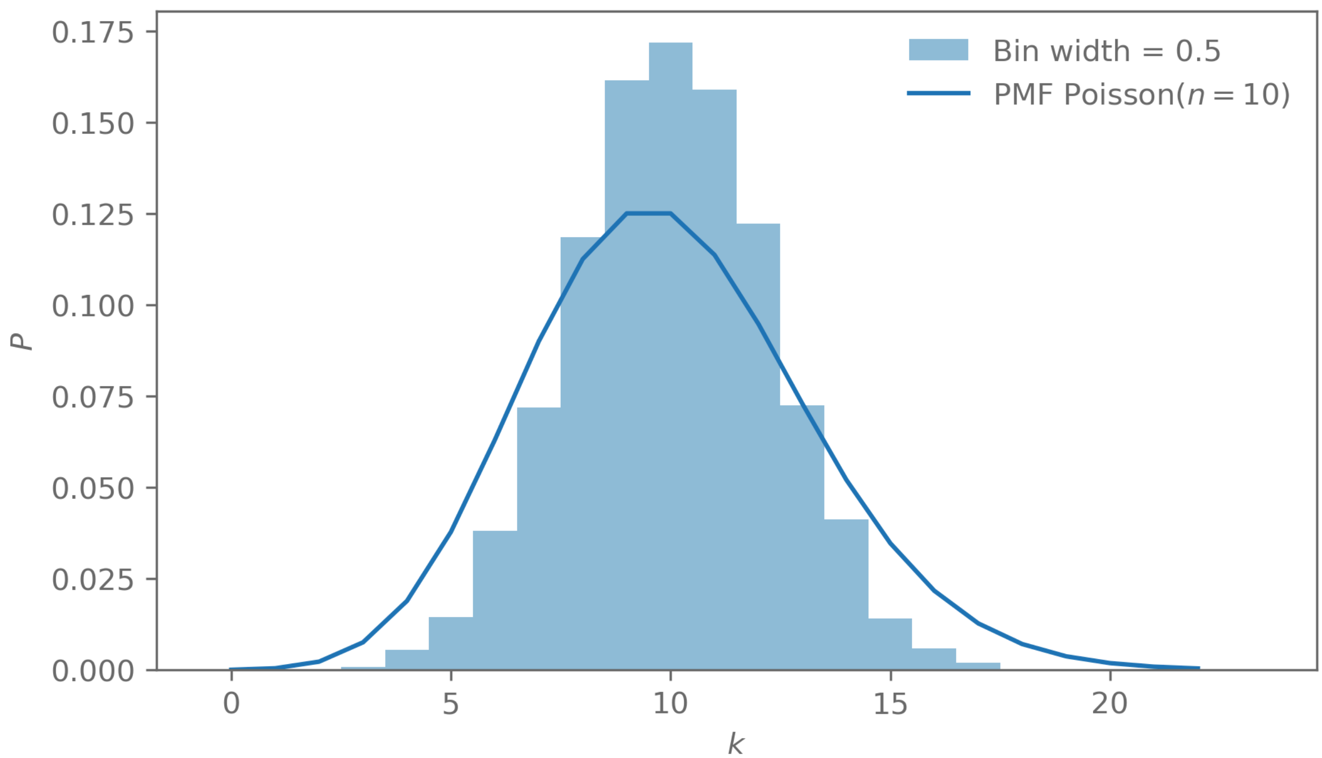
Uniform samples in small bins

```
n = 1000
bin_width = 0.01
rate = n*bin_width

x = np.random.uniform(size=(10000, n))

points_in_bin = np.sum((x < bin_width), axis=1)
plt.hist(points_in_bin, bins=np.arange(25)-0.5, density=True, alpha=0.5,
         label=f"Bin width = {bin_width}")
x = np.arange(23)
plt.plot(x, scipy.stats.poisson(rate).pmf(x), c="C0", label=r"PMF $\mathrm{Poisson}(\
plt.legend(frameon=False)
plt.xlabel("$k$")
plt.ylabel("$P$");
```



The approximation breaks down when the bin width is too broad:

```
n = 20
bin_width = 0.5
rate = n*bin_width

x = np.random.uniform(size=(10000, n))

points_in_bin = np.sum((x < bin_width), axis=1)
plt.hist(points_in_bin, bins=np.arange(25)-0.5, density=True, alpha=0.5,
         label=f"Bin width = {bin_width}")
x = np.arange(23)
plt.plot(x, scipy.stats.poisson(rate).pmf(x), c="C0", label=r"PMF $\mathrm{Poisson}(n=
plt.legend(frameon=False)
plt.xlabel("$k$")
plt.ylabel("$P$");
```



Derivation of Poisson distribution from the binomial distribution

Let λ be the rate of events. This is, in an interval of $T = 1$, we expect λ events to occur.

Devide the interval into n sub-intervals, such that the probability of more than one event happening per sub-interval goes to 0.

The probability of k events happening in the n sub-intervals is then given by a binomial distribution, with $p = \frac{\lambda}{n}$:

$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

As $n \to \infty$, we have

$$\frac{n!}{(n-k)!n^k} = \frac{n(n-1)\dots(n-k+1)}{n^k} \to 1 \, ,$$

$$\left(1 - \frac{\lambda}{n}\right)^{-k} \to 1 \, ,$$

$$\left(1 - \frac{\lambda}{n}\right)^{n} \to \mathrm{e}^{-\lambda} \, .$$

Which leaves

$$\Pr(X=k) \to \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}\;(n\to\infty)\;,$$

the PMF of the Poisson distribution

Change of variables

Sum of Gaussians

The sum of two Gaussian RVs is also Gaussian distributed. We also know that a Gaussian is completely charaterised by its mean and variance. Let $X \sim \mathcal{N}(\mu_X, \sigma_X)$, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y)$, and $Z = X + Y$. Then

$$\mu_Z = \mathrm{E}[Z] = \mathrm{E}[X] + \mathrm{E}[Y] = \mu_X + \mu_Y$$

and

$$\sigma_Z^2 = \mathrm{Var}[Z] = \mathrm{Var}[X] + \mathrm{Var}[Y] = \sigma_X^2 + \sigma_Y^2\;.$$

Sum of RVs

Let $X \sim p_X$ and $Y \sim p_Y$. What is the distribution of $Z = X + Y$?

One way to solve this is to write the distribution of Z as the marginal of the joint distribution p_{XZ} of X and Z:

$$p_Z(z) = \int p_{XZ}(x,z)\mathrm{d}x\;.$$

We find p_{XZ} using a change of variables: $(x,y) = u\left((x,z)\right)$

$$p_{XZ}(x,z) = p_{XY}\left(u\left((x,z)\right)\right)|J_u|\mathrm{d}x\mathrm{d}y$$

The determinant of the Jacobian J_u is 1, so

$$p_{XZ}(x,z) = p_{XY}(x=x, y=z-x) = p_X(x)p_Y(y=z-x)\;,$$

where we used that X and Y are independent. Put together we get

$$p_Z(z) = \int p_X(x)p_Y(z-x)\mathrm{d}x\;.$$

The distribution of the sum of two independent RVs is a convolution of their respective distributions.

Chi-squared

$X \sim \mathcal{N}(0,1)$, $Z = X^2$, with $z = g(x)$. Note that $g(x)$ is not monotonic.

$$p_Z(z) = \sum_i p_X(g_i^{-1}(z)) \left| \frac{\mathrm{d}g_i^{-1}(z)}{\mathrm{d}z} \right| ,$$

with $g_i^{-1}(z) = \pm\sqrt{z}$ and $\left| \frac{\mathrm{d}g_i^{-1}(z)}{\mathrm{d}z} \right| = \frac{1}{2\sqrt{z}}$.

Which leaves

$$p_Z(z) = 2 \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}z} \frac{1}{2\sqrt{z}} = \frac{1}{\sqrt{2\pi z}} \mathrm{e}^{-\frac{1}{2}z}$$