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FAST TRACK COMMUNICATION

Bounding quantum gate error rate based on reported average fidelity

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Abstract

Remarkable experimental advances in quantum computing are exemplified by recent announcements of impressive average gate fidelities exceeding 99.9% for single-qubit gates and 99% for two-qubit gates. Although these high numbers engender optimism that fault-tolerant quantum computing is within reach, the connection of average gate fidelity with fault-tolerance requirements is not direct. Here we use reported average gate fidelity to determine an upper bound on the quantum-gate error rate, which is the appropriate metric for assessing progress towards fault-tolerant quantum computation, and we demonstrate that this bound is asymptotically tight for general noise. Although this bound is unlikely to be saturated by experimental noise, we demonstrate using explicit examples that the bound indicates a realistic deviation between the true error rate and the reported average fidelity. We introduce the Pauli distance as a measure of this deviation, and we show that knowledge of the Pauli distance enables tighter estimates of the error rate of quantum gates.

1. Introduction

An ideal quantum computer could outperform any classical computer for certain computational problems in the sense that resource costs such as time or space scale better than for the best-known classical algorithm [1]. Famous examples include the provable quadratic speedup for search [2], the presumed exponential speedup for the abelian hidden subgroup problem [3, 4], and the possible speedup for stoquastic Hamiltonians using adiabatic quantum computing [5]. If a problem instance with an ℓ-bit input is solved within bounded error ε by an algorithm employing n bits or qubits (space cost) and ν Boolean or unitary gates (time cost), the algorithm is considered efficient if $n,\ \nu \in \mathrm{poly}(\ell)$ [6] and, if ε is treated as an asymptotic variable and not as a constant, $n,\ \nu \in \mathrm{polylog}(1/\varepsilon)$ [7].

In practice, preparation, processing and measurement are faulty, but the threshold theorem for fault-tolerant quantum computation ('threshold theorem') [8–12] guarantees that a noisy device can perform scalable fault-tolerant quantum computations under certain conditions. Specifically, the threshold theorem guarantees the existence of a threshold error rate η_0 $(0 < \eta_0 < 1)$ such that a faulty computer whose error rate η satisfies $\eta < \eta_0$ can perform universal quantum computations efficiently, namely with polylog (n, ν) additional overhead. The threshold theorem is the key to establishing that faulty quantum computers can be as efficient as ideal quantum computers. A key drawback of the threshold theorem is that η_0 is established existentially, not constructively [11]; consequently, this scalability figure of merit is elusive in practice. A practical approach to assessing fault-tolerance is to establish a lower bound $\eta_0^{\mathrm{lb}} \leqslant \eta_0$ by devising a code that is robust against errors that occur at a rate lower than η_0^{lb}; the C4/C6 code, for example, is known to have a threshold of $\eta_0^{\mathrm{lb}} \leqslant 3\%$ [13].

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Current experimental characterizations of quantum gates do not report η. Instead the average gate fidelity φ [14] is the typical figure of merit for gate performance because it can be reliably and scalably estimated using a procedure called randomized benchmarking [15]. Recent reports of φ exceeding 99.9% for one-qubit gates and 99% for two-qubit gates [16] generate strong optimism about the feasibility of scalable quantum computing. But despite its experimental convenience, φ is not the correct quantity to assess scalability via the threshold theorem.

Our aim is to convert reported φ to an upper bound η^{ub} for η. This upper bound provides a sufficient condition for fault-tolerant quantum computing: errors can be efficiently corrected if

$$\eta \leqslant \eta^{\mathrm{ub}} < \eta_0^{\mathrm{lb}} \leqslant \eta_0. \tag{1}$$

Thus, a code-derived η_0^{lb} can be used to determine whether the fidelity-derived η^{ub} suffices for scalability. The quantity η^{ub} can therefore be used to assess scalability based upon the experimentally convenient average fidelity φ.

Given the fidelity φ, the best known upper bound η^{ub} is

$$\eta^{\mathrm{ub}} := d\sqrt{\left(1 + d^{-1}\right)(1 - \varphi)}, \tag{2}$$

where d is the dimension of the system being acted on [17–19]. This bound is unfortunate because the square-root ensures that superficially impressive gate fidelities do not, by themselves, guarantee high-quality gate performance. A two-qubit gate with 99% fidelity is, for example, only guaranteed to have an error rate below 45%. Indeed, we have an explicit example (example 3) of a two-qubit gate with fidelity 99% but an error rate slightly under 13%. Furthermore, we demonstrate (example 1) that assessments of gate performance based on fidelity can mislead about the relative importance of different noise sources.

Our main claim is that η^{ub} is an asymptotically tight approximation to the least upper bound to η in terms of φ and d. The least upper bound is a function $\eta^{\mathrm{lub}}(\varphi, d)$ satisfying the following two properties:

(i) for any noise channel acting on a d-dimensional system with average fidelity φ and error rate η, $\eta \leqslant \eta^{\mathrm{lub}}(\varphi, d)$; and

(ii) $\eta^{\mathrm{lub}}(\varphi, d) \leqslant f(\varphi, d)$ for any function $f(\varphi, d)$ satisfying the first property.

We show that η^{lub} must scale as $\sqrt{1 - \varphi}$ for fixed d (proposition 1) and must scale as d for fixed φ (proposition 2). We conjecture that $\eta^{\mathrm{ub}} = \eta^{\mathrm{lub}}$.

We suggest one potentially useful kind of additional information about gate performance: a quantity we call the 'Pauli distance' δ^{Pauli}. This quantity is motivated by the fact that Pauli channels with average fidelity ϕ have an error rate of $\eta^{\mathrm{Pauli}} = (1 + d^{-1})(1 - \varphi)$, saturating a lower bound on the error rate in terms of the fidelity and dimension [19, 20]. We show that an arbitrary noise channel satisfies $\eta \leqslant \eta^{\mathrm{Pauli}} + \delta^{\mathrm{Pauli}}$ (proposition 3), so that smaller upper bounds η^{ub} to the error rate η of a quantum gate that avoid the $\sqrt{1 - \varphi}$ scaling can be found if the noise is promised to be nearly Pauli.

Our message is not that impressive reported average gate fidelities fail to demonstrate real progress towards fault-tolerant quantum computing, but that these reports are *insufficient* to claim that fault tolerance is now within reach. Our argument is that reported fidelity alone implies only loose bounds on the quantum gate error rate, and that tighter bounds on error rate are possible only if performance metrics other than average fidelity are also considered. In addition to our suggestion of the Pauli distance as a useful additional figure-of-merit, an intriguing quantity known as 'unitarity' [21–23] may also prove to be useful for assessing the performance of quantum gates.

Our paper proceeds as follows. We establish the definition of error rate in section 2. We give a brief review of the average gate fidelity and its relationship to the error rate in section 3. Our asymptotic tightness result is presented in section 4, and we introduce the Pauli distance in section 5. We use our results in section 6 to assess reported progress towards fault-tolerance, and we conclude in section 7.

2. Error rates and the threshold theorem

The threshold theorem is currently our only rigorous guarantee that fault-tolerant quantum computing is viable if threshold operating conditions are met. The threshold operating conditions take two forms: noise must be restricted to a promised form and measurable errors must occur at a low enough rate. We first define the error rate of quantum gates by extension from the error rate of random processes (section 2.1). We then explain the threshold theorem and its connection to our definition of error rate and to numerical estimates of code-specific threshold bounds (section 2.2).

2.1. The error rate of a quantum logic gate

Our definition of the error rate of a quantum logic gate builds naturally on the concept of error rate for a random process, that is, a map from input to output states. For deterministic processes, we can say that an error has occurred if the process produces the 'wrong' output. However, no single output of a random process can be treated unambiguously as 'correct'. We therefore define the rate of error for a process by comparing the actual statistics of the process to its ideal statistics.

The statistics for an ideal process is governed by a probability distribution p_{id} over the set of possible outputs X; the ideal probability of output $x \in X$ is $p_{\mathrm{id}}(x)$. An error-prone process produces a different distribution p_{ac}, governing the actual statistics over the set of possible outcomes X. The total variation distance

$$d_{\mathrm{TV}}(\mu, \nu) \equiv \frac{1}{2} \sum_{x \in X} |\mu(x) - \nu(x)| \tag{3}$$

is a natural measure of the distance between two probability distributions μ and ν over a set of outcomes X.

The total variation distance $d_{\mathrm{TV}}(p_{\mathrm{ac}}, p_{\mathrm{id}})$ can be interpreted as the error rate of the process as follows. We can estimate p_{ac} by sampling the actual random process N times and counting the number $n(x)$ of occurrences of each possible output x; the fraction $n(x)/N$ approaches $p_{\mathrm{ac}}(x)$ as $N \to \infty$. By altering some fraction r of the samples so that the number of occurrences of each outcome x becomes $n'(x)$, we can ensure that $n'(x)/N \approx p_{\mathrm{id}}(x)$ rather than $p_{\mathrm{ac}}(x)$. The fraction r is not unique, but the minimum possible value $r_{\min}$ of r must be greater than zero for large N if $p_{\mathrm{ac}} \neq p_{\mathrm{id}}$. By proposition 4.7 of [24], $r_{\min} \to d_{\mathrm{TV}}(p_{\mathrm{ac}}, p_{\mathrm{id}})$ as $N \to \infty$. Thus, $d_{\mathrm{TV}}(p_{\mathrm{ac}}, p_{\mathrm{id}})$ approximates the fraction of a large sample that must be altered to ensure that the relative frequencies of each outcome match the ideal distribution p_{id}; each alteration can be interpreted as the correction of an error.

We claim that the total variation distance induces the diamond distance $d_\diamond$ on the space of quantum channels. Our argument follows from the work of Fuchs and van de Graaf [25], which shows that the error rate of quantum states is given by the trace distance between the quantum states, and the work of Kitaev [26], which shows that the diamond norm extends the trace norm to quantum channels. We begin by defining some terminology.

Quantum logic gates act reversibly on some fixed quantum register. Ideally, the state of this register can be represented by a unit vector in a fixed d-dimensional Hilbert space $\mathscr{H}$. The register is typically treated as a collection of n qubits, in which case $\mathscr{H}$ is canonically isomorphic with the n-fold tensor product of the Hilbert space $\mathscr{Q} \cong \mathbb{C}^2$ of a single qubit: $\mathscr{H} \cong \mathscr{Q}^{\otimes n}$. In this case, $d = 2^n$.

Whereas ideal register states are represented by unit vectors $|\psi\rangle \in \mathscr{H}$, realistic states are modeled by density operators ρ on $\mathscr{H}$. A measurement of a quantum state is described by a positive operator-valued measure (POVM), which is a set of positive operators $\{E_\ell\}$ acting on $\mathscr{H}$ such that $\sum_\ell E_\ell = I$, the identity operator. The probability of observing the outcome labeled ℓ is $\mathrm{Tr}\,(E_\ell \rho)$. Thus, the actual output ρ_{ac} of a gate acting on a specified input can be compared to the ideal output ρ_{id} by measuring with respect to some POVM. The error rate of this measurement is $d_{\mathrm{TV}}(p_{\mathrm{ac}}, p_{\mathrm{id}})$, where $p_{\mathrm{ac}}(\ell) = \mathrm{Tr}\,(E_\ell \rho_{\mathrm{ac}})$ and $p_{\mathrm{id}}(\ell) = \mathrm{Tr}\,(E_\ell \rho_{\mathrm{id}})$. Maximizing $d_{\mathrm{TV}}(p_{\mathrm{ac}}, p_{\mathrm{id}})$ over all possible choices of measurements yields [25]

$$d_{\mathrm{Tr}}(\rho_{\mathrm{ac}}, \rho_{\mathrm{id}}) \coloneqq \frac{1}{2} \|\rho_{\mathrm{ac}} - \rho_{\mathrm{id}}\|_{\mathrm{Tr}}, \tag{4}$$

where $\|A\|_{\mathrm{Tr}} \coloneqq \mathrm{Tr}\,\sqrt{A^\dagger A}$ for any linear operator A. Thus, the error rate of ρ_{ac} with respect to ρ_{id} is $d_{\mathrm{Tr}}(\rho_{\mathrm{ac}}, \rho_{\mathrm{id}})$.

Now that we have defined the error rate of the output of a quantum logic gate, we can define the error rate of the gate itself. We first present some mathematical notation for evaluating the difference between an ideal and real implementation of a quantum logic gate.

An ideal quantum logic gate, represented by G, acts as a unitary operator on $\mathscr{H}$. Whereas the operation on a pure state can be treated as direct (i.e. $|\psi\rangle \mapsto G|\psi\rangle$), the gate can act upon a mixed state ρ. In this instance, the gate performs the action $\rho \mapsto G\rho G^\dagger$. This action is represented by a quantum channel $\mathcal{G}_{\mathrm{id}}$; explicitly

$$\mathcal{G}_{\mathrm{id}}(\rho) \coloneqq G\rho G^\dagger. \tag{5}$$

This channel is compared with a non-ideal implementation $\mathcal{G}_{\mathrm{ac}}$ that is in general not represented by unitary conjugation but is a completely positive, trace preserving linear operator on the space of density operators over $\mathscr{H}$.

We have established in equation (4) that the error rate for a quantum logic gate acting on a specified input state ρ is given by $d_{\mathrm{Tr}}(\mathcal{G}_{\mathrm{ac}}(\rho), \mathcal{G}_{\mathrm{id}}(\rho))$. The error rate of $\mathcal{G}_{\mathrm{ac}}$ with respect to $\mathcal{G}_{\mathrm{id}}$ involves maximization over inputs. Whereas the error rate could be defined as $\max_\rho d_{\mathrm{Tr}}(\mathcal{G}_{\mathrm{ac}}(\rho), \mathcal{G}_{\mathrm{id}}(\rho))$, such a definition is undesirable because, in general, the error rate of $\mathcal{G}_{\mathrm{ac}} \otimes \mathbb{1}$ (where $\mathbb{1}$ acts on some ancillary space $\mathscr{H}'$) differs from that of $\mathcal{G}_{\mathrm{ac}}$ [26, 27]. We therefore amend this definition by maximizing over inputs and ancillary spaces using a construction called the diamond norm [26]:

$$\|\mathcal{A}\|_\Diamond := \sup_{\mathscr{H}'} \sup_{\rho \in \text{dens}(\mathscr{H} \otimes \mathscr{H}')} \|\mathcal{A} \otimes \mathbb{1}(\rho)\|_{\text{Tr}}, \tag{6}$$

where $\mathcal{A}$ is any superoperator over $\mathscr{H}$ and $\text{dens}(\mathscr{H} \otimes \mathscr{H}')$ is the set of density operators over the joint Hilbert space of the original register and some ancilla. We therefore define

$$d_\Diamond\left(\mathcal{G}_{\text{ac}}, \mathcal{G}_{\text{id}}\right) := \frac{1}{2} \|\mathcal{G}_{\text{ac}} - \mathcal{G}_{\text{id}}\|_\Diamond \tag{7}$$

to be the error rate η of $\mathcal{G}_{\text{ac}}$ with respect to $\mathcal{G}_{\text{id}}$: $\eta = d_\Diamond(\mathcal{G}_{\text{ac}}, \mathcal{G}_{\text{id}})$. However, we shall use a modified but equivalent form of this definition in the remainder of this paper. Our modification is purely for mathematical convenience.

Definition 1. If $\mathcal{G}_{\text{ac}}$ is some implementation of a gate G, define

$$\mathcal{D}_G := \mathcal{G}_{\text{ac}} \circ \mathcal{G}_{\text{id}}^{-1} \tag{8}$$

to be the *discrepancy channel* of G, where the channel $\mathcal{G}_{\text{id}}$ defined in equation (5) is unitary and hence invertible.

Definition 2. The *error rate* of an implementation of G is given by

$$\eta = d_\Diamond\left(\mathcal{D}_G, \mathbb{1}\right), \tag{9}$$

where $\mathcal{D}_G$ is the discrepancy channel of the implementation.

2.2. The threshold theorem

We now explain the threshold theorem. We begin by elaborating on the promised form of noise; namely, noise locality. We then identify a statement of the theorem that is appropriate for our needs. Finally, we review commonly quoted estimates of fault-tolerance thresholds.

We elaborate on the assumption of noise locality because it is required for defining the error rate of logic gates independent of the circuit in which they are employed. Briefly, a logic circuit is said to experience local noise if the noise acts separately on individual logic gates. To be precise, recall that a logic circuit is defined as a directed acyclic graph with nodes labeled by elements of some set of logic gates, where arrows into a node represent inputs and arrows out of a node represent outputs [28]. A quantum logic circuit can similarly be represented by a directed acyclic graph. The noise of a logic circuit is local if it can be represented as the composition of noise processes on individual nodes of the circuit graph.

As noise is assumed to affect each gate independently, we model noise by replacing the intended unitary gate G by some imperfect implementation $\mathcal{G}_{\text{ac}}$ represented as a quantum channel (i.e. a completely positive, trace-preserving linear map on density operators over the state space of the input register). Such a model is reasonable for an imperfect gate subject to local noise if the interaction between the register space and its environment obeys the Born–Markov approximation [29]. We therefore assign an error rate η to each gate $\mathcal{G}_{\text{ac}}$ in a circuit Q', which simulates Q efficiently and accurately in the presence of local noise if $\eta < \eta_0$.

The various formulations of the threshold theorem are distinguished by assumptions concerning noise. We prefer to employ the statement of Aharonov and Ben-Or because of its directness with a minimum of jargon.

Threshold theorem ([11]). There exists a threshold $\eta_0 > 0$ such that the following holds. Let $\varepsilon > 0$. If Q is a quantum circuit operating on n input qubits for t time steps using s two- and one-qubit gates, there exists a quantum circuit Q' with depth, size, and width overheads which are polylogarithmic in n, s, t, and $1/\varepsilon$ such that, in the presence of local noise of error rate $\eta < \eta_0$, Q' computes a function which is within ε total variation distance from the function computed by Q.

This theorem guarantees that a value $\eta_0 > 0$, called the 'threshold', exists such that a quantum circuit Q can be efficiently simulated by another circuit Q' to within an arbitrary error tolerance $\varepsilon > 0$ even if Q' is subject to 'local noise' at a rate $\eta < \eta_0$. Inequivalent statements of 'the' threshold theorem are inequivalent because they assume promises about noise that are different from that of noise locality.

There are two important limitations to the utility of the threshold η_0. Firstly, surpassing the threshold is sufficient but not necessary for fault-tolerance: error rates larger than η_0 could be acceptable if stronger promises can be made about noise. Conversely, devices subject to noise that does not satisfy the assumptions of the threshold theorem cannot be said to be fault-tolerant based on a demonstration that error rates fall below threshold; a stronger threshold $\eta_0' < \eta_0$ could apply. The second limitation is that the choice of Q' depends in practice upon the specified quantum-error-correcting code. Based on the choice of code, the appropriate performance target is η_0^{lb}, rather than η_0. As with the first limitation, the validity of η_0^{lb} as a performance target derives from the validity of the promises made about the noise affecting real devices.

Whereas some threshold estimates are obtained through rigorous analysis of the performance of a code in the presence of noise subject to promises of varying strength, others are obtained through numerical simulation of performance in the presence of a parametrized family of noise models. Estimates based on numerical simulation are more optimistic and are often used as performance targets for experimental fault-tolerant quantum computing research [16, 30].

Analytic estimates of the threshold can be produced based on details of the proof of the threshold theorem. Aharonov and Ben-Or, for example, can justify an estimate of $\eta_0^{\mathrm{lb}} \approx 10^{-6}$ [11] based on their choice of coding strategy. They report a value of $\eta_0^{\mathrm{lb}} \approx 10^{-3}$ [31, 32] as being the largest rigorously established value. Numerical estimates, by contrast, are produced by simulating the behavior of an error-correcting code in the presence of a restricted class of noise models, typically depolarizing [13, 33]. The relationship of these estimates with thresholds of the kind established by the threshold theorem is not clear [34, 35], but these simulations are nonetheless often seen as indicative [36] of true threshold values. The surface code, in particular, is often believed to have a threshold of around 1% [33, 36].

A direct comparison of the above threshold values is not justifiable because each value makes different assumptions about the behavior of noise and the choice of code. Thus, the estimate of $\eta_0^{\mathrm{lb}} \approx 1\%$ for the surface code under depolarizing noise does not make the surface code less desirable than Knill's C4/C6 code even though the C4/C6 code could have a threshold as high as 3% [13] because there are other practical reasons to prefer the surface code over the C4/C6 code. Similarly, actual gate performance should not be directly compared with these threshold values because those gates are certainly subject to noise that is not well-approximated by the depolarizing noise model. The connection between numerical simulations and fault-tolerance thresholds is a matter of active research [37, 38].

Whereas there are important open questions regarding the interpretation of threshold estimates produced by simulation, the term 'threshold' is unambiguously a reference to an upper bound of the error rate introduced by any given logic gate in a quantum circuit. The main point of this paper is to connect these theoretical characterizations of error to the experimentally convenient average gate fidelity.

3. Average gate fidelity

Whereas η_0, defined by the threshold theorem, and η_0^{lb}, defined in equation (1) and established by noise models and coding strategies, are appropriate quantities for analyzing scalability, average gate fidelity is employed in experimental studies because of its convenience. In this section, we define average gate fidelity and discuss the connection between average gate fidelity and error rate, first by reviewing the literature and then by constructing an example that shows that this connection is problematic: average gate fidelity and quantum gate error rate are not directly connected. Instead, only lower and upper bounds to the error rate can be derived from fidelity; the gap between these bounds is substantial in regimes of interest.

The fidelity of a state ρ to a pure state ψ is $\mathrm{Tr}(|\psi\rangle\langle\psi|\ \rho)$ [15, 39–41]. The fidelity of the output of the actual gate $\mathcal{G}_{\mathrm{ac}}$ to the output of the ideal gate $\mathcal{G}_{\mathrm{id}}$ for a given input state $|\psi\rangle$ is therefore

$$\mathrm{Tr}\left(\mathcal{G}_{\mathrm{id}}(|\psi\rangle\langle\psi|)\mathcal{G}_{\mathrm{ac}}(|\psi\rangle\langle\psi|)\right) = \langle\psi|\ \mathcal{D}_G(|\psi\rangle\langle\psi|)|\psi\rangle. \tag{10}$$

Averaging over pure state inputs with respect to the Haar measure then gives the average gate fidelity

$$\varphi \coloneqq \int \mathrm{d}\mu(\psi)\langle\psi|\ \mathcal{D}_G(|\psi\rangle\langle\psi|)|\psi\rangle, \tag{11}$$

where we have used the unitary invariance of the Haar measure and $\mathcal{D}_G = \mathcal{G}_{\mathrm{ac}} \circ \mathcal{G}_{\mathrm{id}}^{-1}$. The popular randomized benchmarking protocol [15] produces an estimate of this quantity averaged over a gate-set, though proposed extensions [42] produce estimates of the average gate fidelity for individual gates.

The state fidelity can be interpreted as the error rate of a particular measurement. If we define for each pure state $|\psi\rangle$ the POVM $\{|\psi\rangle\langle\psi|, \mathbb{1} - |\psi\rangle\langle\psi|\}$, the outcome of this measurement applied to a state ρ will be $|\psi\rangle\langle\psi|$ with probability $\langle\psi|\ \rho\ |\psi\rangle$, which is the state fidelity of ρ with respect to $|\psi\rangle$. Thus, the total variation distance of the actual statistics p_{ac} from the ideal statistics p_{id} of this measurement upon the output $\mathcal{D}_G(|\psi\rangle\langle\psi|)$ is the state infidelity of $\mathcal{D}_G(|\psi\rangle\langle\psi|)$ with respect to $|\psi\rangle\langle\psi|$.

However, the average gate infidelity $1 - \varphi$ cannot be so easily interpreted as an average error rate (despite the common habit [16, 30, 43]), as the measurement basis is not fixed in the integral (and so the infidelity is not an average error for a fixed measurement), yet neither is it averaged independently from the state.

To clarify the relationship between average gate fidelity and error rate, we consider two noise processes on a single qubit. The first is given by depolarizing noise

$$\mathcal{E}_r^{\mathrm{dep}}(\rho) \coloneqq (1 - r)\rho + rI/2, \tag{12}$$

where I is the identity operator on $\mathcal{Q}$, and the second is a unitary error

$$\mathcal{E}_\theta^U(\rho) := U\rho U^\dagger, \tag{13}$$

where U is some unitary operator on $\mathcal{Q}$ with eigenvalues $e^{\pm i\theta}$ for $0 \leqslant \theta \leqslant \pi$. The average gate fidelity for depolarizing noise is

$$\varphi^{\text{dep}}(r) = 1 - \frac{r}{2} \tag{14}$$

whereas the fidelity of the unitary error is [14]

$$\varphi^U(\theta) = \frac{1}{3} + \frac{2}{3}\cos^2\theta. \tag{15}$$

By contrast

$$\eta^{\text{dep}}(r) = \frac{3}{4}r \tag{16}$$

for depolarizing noise, which follows from the fact that depolarizing noise is Pauli [20], and

$$\eta^U(\theta) = \sin\theta \tag{17}$$

for unitary error [44]. Therefore

$$\eta^{\text{dep}} = \frac{3}{2}\left(1 - \varphi^{\text{dep}}\right) \tag{18}$$

for depolarizing noise but

$$\eta^U = \sqrt{\frac{3}{2}\left(1 - \varphi^U\right)} \tag{19}$$

for unitary error which means that there is no single function f such that $f(\varphi) = \eta$ for every possible noise channel.

We demonstrate this difficulty in the following example.

Example 1. Consider a single-qubit gate that is subject to the two noise processes of equations (12) and (13): depolarizing and unitary. The depolarizing rate is $r = 10^{-3}$, with corresponding fidelity

$$\varphi^{\text{dep}} = 1 - 5.0 \times 10^{-4}, \tag{20}$$

whereas the unitary error has angle $\theta = 10^{-2}$, with corresponding fidelity

$$\varphi^U = 1 - 6.7 \times 10^{-5}. \tag{21}$$

The combination

$$\mathcal{D}_G := \mathcal{E}_r^{\text{dep}} \circ \mathcal{E}_\theta^U \equiv \mathcal{E}_\theta^U \circ \mathcal{E}_r^{\text{dep}} \equiv (1 - r)\mathcal{E}_\theta^U + r\mathcal{E}_{r=1}^{\text{dep}} \tag{22}$$

has fidelity

$$\varphi^{\text{tot}} = (1 - r)\varphi^U + \frac{r}{2} = 1 - 5.3 \times 10^{-4}, \tag{23}$$

so the fidelity loss seems to arise mostly from depolarizing noise. Yet the error rate due to unitary error is

$$\eta^U = 10^{-2} \tag{24}$$

whereas the error rate due to depolarizing noise is

$$\eta^{\text{dep}} = 7.5 \times 10^{-4}. \tag{25}$$

The triangle inequalities imply that the error rate of the combined noise process is

$$\eta^{\text{tot}} = (1 \pm 0.08) \times 10^{-2}, \tag{26}$$

so the unitary error is in fact dominating over depolarizing even though the fidelity appears to imply the reverse.

Thus, information beyond fidelity is needed to assess the relative importance of various noise processes affecting the quantum computing device. Determination of the Pauli-distance (section 5) is one possible approach to characterizing the influence of different noise sources; extending randomized benchmarking to estimate unitarity [21, 22] is another.

Although no direct connection between average gate fidelity and error rate exists in general, average gate fidelity is clearly of some worth: if the fidelity of a quantum logic gate is precisely one, we are certain that the gate will always perform exactly as intended. More generally

$$\eta^{\text{Pauli}} := \left(1 + d^{-1}\right)(1 - \varphi) \leqslant \eta \leqslant d\sqrt{\left(1 + d^{-1}\right)(1 - \varphi)} =: \eta^{\text{ub}}, \tag{27}$$

where d is the dimension of $\mathscr{H}$. Note that the upper bound can exceed unity if

$$\varphi < 1 - \left(d^2 + d\right)^{-1}, \tag{28}$$

so a fidelity less than 83% for single-qubit gates or less than 95% for two-qubit gates does not ensure that $\eta < 1$; it is possible that the gate is performing incorrectly all the time for at least one input. Ensuring that $\eta < 1$ when φ fails to meet this threshold for non-triviality must involve additional promises about the form of noise.

To illustrate the gap between the above lower and upper bounds, consider a target fidelity of 99%. Then for one, two and three qubits, the above upper bound gives 25%, 45% and 85% respectively, whereas the lower bound is essentially 1%. For target fidelities of 99.9%, the upper bounds become 7.75%, 14.2% and 26.9% respectively, whereas the lower bound is approximately 0.1% in each case. Hence, these upper and lower bounds differ by orders of magnitude in regimes of experimental interest.

4. Tightness of the upper bound on the error rate

We now prove that the upper bound η^{ub} on the error rate is asymptotically tight with respect to fidelity for fixed dimension and asymptotically tight with respect to dimension for fixed fidelity. In addition, we prove by example that this bound cannot be improved by better than a factor varying as the square-root of dimension. To demonstrate these facts, we first define the variables and functions about which we make asymptotic statements.

Definition 3. The *least upper bound of error rate with respect to average gate fidelity* $\eta^{\text{lub}} = \eta^{\text{lub}}(\varphi, d)$ is the unique function of φ and d that satisfies the following. For any discrepancy channel $\mathcal{D}_G$ of dimension d with average gate fidelity φ and error rate η, $\eta^{\text{lub}}(\varphi, d) \geqslant \eta$. Furthermore, suppose that $\eta^{\text{ub}} = \eta^{\text{ub}}(\varphi, d)$ is any other function with the same property. Then $\eta^{\text{lub}}(\varphi, d) \leqslant \eta^{\text{ub}}(\varphi, d)$ for all φ and d.

We shall establish the scaling of $\eta^{\text{lub}}(\varphi, d)$ as a function of each variable when the other is fixed. Notationally, we distinguish fixed from variable quantities as follows. If the dimension d is fixed but φ is variable, we write $\eta^{\text{lub}}(\varphi)|_d$; if vice versa, $\eta^{\text{lub}}(d)|_\varphi$. We use similar notation for η^{ub}. We seek to establish the scaling of η^{lub} in the limit $\varphi \to 1$. To make asymptotic arguments about this scaling, we define two variables that go to infinity as $\varphi \to 1$.

Definition 4. Define the *inverse error rate* of a quantum logic gate to be $\zeta := \eta^{-1}$, where η is the error rate of the gate. Thus, $\zeta \to \infty$ as $\eta \to 0$. We also write

$$\zeta^{\text{lub}} := \left(\eta^{\text{lub}}\right)^{-1}, \zeta^{\text{ub}} := \left(\eta^{\text{ub}}\right)^{-1}, \tag{29}$$

which are lower bounds to ζ.

Definition 5. Define the *inverse average infidelity* of a quantum logic gate to be

$$\upsilon := (1 - \varphi)^{-1}. \tag{30}$$

Thus, $\upsilon \to \infty$ as $\varphi \to 1$.

Thus, we can write $\zeta^{\text{lub}} = \zeta^{\text{lub}}(\upsilon, d)$ and compare this function to

$$\zeta^{\text{ub}}(\upsilon, d) = \frac{\sqrt{\upsilon}}{d\sqrt{1 + \frac{1}{d}}}. \tag{31}$$

We show that

$$\zeta^{\text{lub}}(\upsilon)|_d \in \Theta\left(\sqrt{\upsilon}\right) \tag{32}$$

and

$$\zeta^{\text{lub}}(d)|_\upsilon \in \Theta\left(d^{-1}\right); \tag{33}$$

thus, ζ^{ub} has optimal scaling with respect to ϕ and d when the the other is fixed. We shall make use of a particular unitary gate, defined below.

Definition 6. Define the *generalized controlled-phase gate* by

$$\mathcal{G}_{\mathrm{id}}(\rho) := U_\theta \rho U_\theta^\dagger;\ U_\theta := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix}, \tag{34}$$

where U_θ is expressed in the computational basis.

Proposition 1. *For fixed dimension d,*

$$\zeta^{\mathrm{lub}}(\upsilon)|_d \in \Theta\left(\sqrt{\upsilon}\right). \tag{35}$$

Furthermore

$$\eta^{\mathrm{lub}}(\upsilon)|_d \geqslant \frac{1}{2}(d-1)^{-\frac{1}{2}}\eta^{\mathrm{ub}}(\varphi)|_d. \tag{36}$$

Therefore

$$\zeta^{\mathrm{ub}}(\upsilon)|_d \in \Theta\left(\zeta^{\mathrm{lub}}(\upsilon)|_d\right). \tag{37}$$

Proof. Suppose we have an implementation of the generalized controlled-phase gate given simply by $\mathcal{G}_{\mathrm{ac}}(\rho) = \rho$, the identity channel. The average gate fidelity is [14]

$$\varphi = \frac{d + \left|\mathrm{Tr}\left(U_{-\theta}\right)\right|^2}{d + d^2} = 1 - \frac{2(d-1)}{d(d+1)}(1 - \cos\theta). \tag{38}$$

By theorem 26 of [44],

$$\eta = \sqrt{\frac{1 - \cos\theta}{2}}; \tag{39}$$

hence

$$\zeta = \sqrt{\frac{4(d-1)}{d(d+1)}} \times \sqrt{\upsilon}. \tag{40}$$

By contrast

$$\zeta^{\mathrm{ub}} = \sqrt{\frac{1}{d(d+1)}} \times \sqrt{\upsilon}. \tag{41}$$

Furthermore, ζ^{lub} is defined so that $\zeta \geqslant \zeta^{\mathrm{lub}} \geqslant \zeta^{\mathrm{ub}}$; thus

$$\sqrt{\frac{4(d-1)}{d(d+1)}} \times \sqrt{\upsilon} \geqslant \zeta^{\mathrm{lub}}(\upsilon)|_d \geqslant \sqrt{\frac{1}{d(d+1)}} \times \sqrt{\upsilon}. \tag{42}$$

□

Example 2. All single-qubit unitary errors satisfy

$$\eta = \frac{1}{2}\eta^{\mathrm{ub}} = \sqrt{\frac{3}{2}(1-\varphi)}. \tag{43}$$

If $\mathcal{D}_G(\rho) = U\rho U^\dagger$ for some 2×2 unitary operator U, then the eigenvalues of U can be written as $\mathrm{e}^{\pm\mathrm{i}\theta/2}$ for some θ. The diamond distance $d_\diamond$ and the fidelity are unitarily invariant, so the error rate of $\mathcal{D}_G$ depends only on θ. Furthermore, $\mathcal{D}_G$ is equivalent to the generalized controlled-phase gate (definition 6) and hence η satisfies equation (39). Equation (38) therefore implies that $\eta^{\mathrm{ub}} = 2\eta$.

Example 3. There exists a two-qubit gate with fidelity 99.0% but error rate 12.9%. Consider the generalized controlled-phase gate (definition 6) acting on two qubits: one target qubit and one control qubit. Setting $\theta = 0.259$, we have $\varphi = 99.0\%$ by equation (38) and $\eta = 12.9\%$ by equation (39).

We now demonstrate that the generalized controlled-phase gate example used to prove proposition 1 does not yield the true value of η^{lub}. We prove that $\zeta^{\mathrm{lub}}(d)|_\varphi \in \Theta(d^{-1})$, whereas the generalized controlled-phase gate has $\zeta(d)|_\varphi \in \Theta\left(d^{-\frac{1}{2}}\right)$ by equation (42).

Proposition 2. *For fixed fidelity* φ,

$$\zeta^{\text{lub}}(d)|_{\varphi} \in \Theta\left(d^{-1}\right). \tag{44}$$

Therefore

$$\zeta^{\text{ub}}(d)|_{\varphi} \in \Theta\left(\zeta^{\text{lub}}(d)|_{\varphi}\right). \tag{45}$$

Proof. We consider a special case of the generalized controlled-phase gate in which $\theta = \pi$, so the unitary U_π has an eigenvalue of -1. In this case, $||\mathcal{G}_{\text{id}} - \mathbb{1}||_\diamond = 2$ by theorem 26 of [44]. The implementation we consider is

$$\mathcal{G}_{\text{ac}} := (1 - \lambda)\mathcal{G}_{\text{id}} + \lambda\mathbb{1}. \tag{46}$$

The error rate is

$$\eta = \frac{1}{2}\,\|\mathcal{G}_{\text{ac}} - \mathcal{G}_{\text{id}}\|_\diamond = \lambda \times \frac{1}{2}\,\|\mathcal{G}_{\text{id}} - \mathbb{1}\|_\diamond = \lambda. \tag{47}$$

We calculate the fidelity by applying Nielsen's formula [14] to the Kraus decomposition

$$\left\{\sqrt{1-\lambda}\,I,\ \sqrt{\lambda}\,U_\pi\right\} \tag{48}$$

of the discrepancy channel $\mathcal{D}_{U_\pi}$:

$$\varphi = \frac{d + (1-\lambda)|\text{Tr}(I)|^2 + \lambda\left|\text{Tr}\left(U_\pi\right)\right|^2}{d + d^2} = 1 - \frac{4(d-1)}{d(d+1)} \times \lambda. \tag{49}$$

Combining equation (47) with equation (49) yields

$$\zeta = \frac{4(d-1)}{d(d+1)}\upsilon. \tag{50}$$

By definition, $\zeta^{\text{ub}} \leqslant \zeta^{\text{lub}} \leqslant \zeta$, which implies

$$\frac{1}{d}\sqrt{\frac{\upsilon}{1+\frac{1}{d}}} \leqslant \zeta^{\text{lub}} \leqslant \frac{4(d-1)}{d(d+1)}\upsilon. \tag{51}$$

For fixed υ, we define the constants $c_1 = 2^{-\frac{1}{2}}$ and $c_2 = 4\upsilon$. Then

$$\frac{c_1}{d} \leqslant \zeta^{\text{lub}}(d)|_\upsilon \leqslant \frac{c_2}{d}, \tag{52}$$

hence $\zeta^{\text{lub}}(d) \in \Theta(d^{-1})$. □

We have established that η^{ub} is asymptotically tight with respect to fidelity (proposition 1) and dimension (proposition 2) if the other is fixed. Furthermore, we showed that η^{ub} differs from the tightest possible bound by at most a factor of $2\sqrt{d-1}$, where d is the dimension of the gate. Although we conjecture that η^{ub} is indeed the tightest possible bound on error rate of a d-dimensional gate based only upon fidelity, important quantitative statements are true (examples 2 and 3) even if our conjecture is false.

5. New bounds for approximate Pauli channels

Here we derive improved bounds based on an additional promise about noise. Specifically, we provide alternative lower and upper bounds on error rate in terms of gate fidelity and a quantity we call the 'Pauli distance'. We show that the connection between error rate and gate fidelity is strongly improved if the Pauli distance of the noise process is known. We give numerical examples for two important single-qubit noise processes: amplitude damping and unitary error.

The Pauli distance is defined to be the diamond distance between a channel and its Pauli-twirl. To be precise, we define the single-qubit Pauli operators as the unitary matrices

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \text{and } Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{53}$$

a multi-qubit Pauli operator is a tensor-product of single-qubit Pauli operators. A Pauli channel is a quantum channel that has a Kraus representation in which each Kraus operator is proportional to a Pauli operator.

Definition 7. The *Pauli-twirl* of an n-qubit channel $\mathcal{E}$ (i.e. $d = 2^n$) is

$$\mathcal{E}^{\mathrm{PT}}(\bullet) := \frac{1}{4^n}\sum_{k=1}^{4^n} P_k^\dagger\, \mathcal{E}\left(P_k \bullet P_k^\dagger\right) P_k, \tag{54}$$

where P_k represents a choice of n-qubit Pauli operator.

Definition 8. We define the *Pauli distance* of a gate implementation with discrepancy channel $\mathcal{D}_G$ to be

$$\delta^{\mathrm{Pauli}} := d_\diamond\left(\mathcal{D}_G, \mathcal{D}_G^{\mathrm{PT}}\right), \tag{55}$$

where $\mathcal{D}_G^{\mathrm{PT}}$ is the Pauli-twirl of $\mathcal{D}_G$.

The Pauli-twirl of any channel is a Pauli channel, and the Pauli-twirl of a Pauli channel is the same channel. For any channel $\mathcal{E}$, $\mathcal{E}$ and $\mathcal{E}^{\mathrm{PT}}$ have the same average gate fidelity as the average gate fidelity is linear and invariant under unitary conjugation. The diamond distance for channels of a fixed fidelity is minimized by Pauli channels, which satisfy $\eta^{\mathrm{Pauli}} = (1 + 2^{-n})(1 - \varphi)$ where n is the number of qubits [19, 20]. Several common sources of noise, such as depolarizing error and dephasing (T_2) processes, can be represented by Pauli channels [1]. Such noise processes have $\delta^{\mathrm{Pauli}} = 0$. Other sources of noise, such as amplitude-damping processes and unitary errors, cannot. In these cases, $\delta^{\mathrm{Pauli}} > 0$.

Proposition 3. *The error rate* η *of an n-qubit gate with gate fidelity* φ *and Pauli distance* δ^{Pauli} *satisfies*

$$\left|\delta^{\mathrm{Pauli}} - \eta^{\mathrm{Pauli}}\right| \leqslant \eta \leqslant \delta^{\mathrm{Pauli}} + \eta^{\mathrm{Pauli}}. \tag{56}$$

Proof. By the triangle inequality

$$\frac{1}{2}\|\mathcal{D}_G - \mathbb{1}\|_\diamond = \frac{1}{2}\|\mathcal{D}_G - \mathcal{D}_G^{\mathrm{PT}} + \mathcal{D}_G^{\mathrm{PT}} - \mathbb{1}\|_\diamond \leqslant \frac{1}{2}\|\mathcal{D}_G - \mathcal{D}_G^{\mathrm{PT}}\|_\diamond + \frac{1}{2}\|\mathcal{D}_G^{\mathrm{PT}} - \mathbb{1}\|_\diamond. \tag{57}$$

The left-hand side equals η and the right-hand side equals $\delta^{\mathrm{Pauli}} + \eta^{\mathrm{Pauli}}$. Similarly, the reverse triangle inequality implies $|\delta^{\mathrm{Pauli}} - \eta^{\mathrm{Pauli}}| \leqslant \eta$. □

Proposition 3 thus enables bounds to be placed on possible values of η in terms of φ and δ^{Pauli}. Indeed, a variation of this proposition can be applied to noise channels that have a known structure.

Proposition 4. *Suppose* $\mathcal{D}_G = \sum_k \mathcal{E}_k$*, where each* $\mathcal{E}_k$ *is some quantum channel. Let* $\delta_k^{\mathrm{Pauli}}$ *represent the Pauli distance of* $\mathcal{E}_k$*. Then the error rate* η *of* $\mathcal{D}_G$ *satisfies*

$$\eta \leqslant \eta^{\mathrm{Pauli}} + \sum_k \delta_k^{\mathrm{Pauli}}. \tag{58}$$

Proof. If δ^{Pauli} is the Pauli distance of $\mathcal{D}_G$, proposition 3 implies that

$$\eta \leqslant \eta^{\mathrm{Pauli}} + \delta^{\mathrm{Pauli}}, \tag{59}$$

so we only need to show that

$$\delta^{\mathrm{Pauli}} \leqslant \sum_k \delta_k^{\mathrm{Pauli}}. \tag{60}$$

As the Pauli-twirl operation on quantum channels is linear, i.e.

$$\mathcal{D}_G^{\mathrm{PT}} = \left(\sum_k \mathcal{E}_k\right)^{\mathrm{PT}} = \sum_k \mathcal{E}_k^{\mathrm{PT}}, \tag{61}$$

we apply the triangle inequality repeatedly to obtain

$$\frac{1}{2}\|\mathcal{D}_G - \mathcal{D}_G^{\mathrm{PT}}\|_\diamond = \frac{1}{2}\left\|\left(\sum_k \mathcal{E}_k\right) - \left(\sum_k \mathcal{E}_k^{\mathrm{PT}}\right)\right\|_\diamond \leqslant \sum_k \frac{1}{2}\|\mathcal{E}_k - \mathcal{E}_k^{\mathrm{PT}}\|_\diamond. \tag{62}$$

The left-hand side equals δ^{Pauli} and the right-hand side equals $\sum_k \delta_k^{\mathrm{Pauli}}$. □

Although proposition 4 yields weaker bounds than proposition 3 in general, it might be easier in practice to estimate δ^{Pauli} for individual sources of noise rather than for the overall noise process.

We consider two examples of single-qubit noise processes in which δ^{Pauli} is non-zero: unitary error, which is a model of control error, and an amplitude damping process, which is a model of thermalization with a

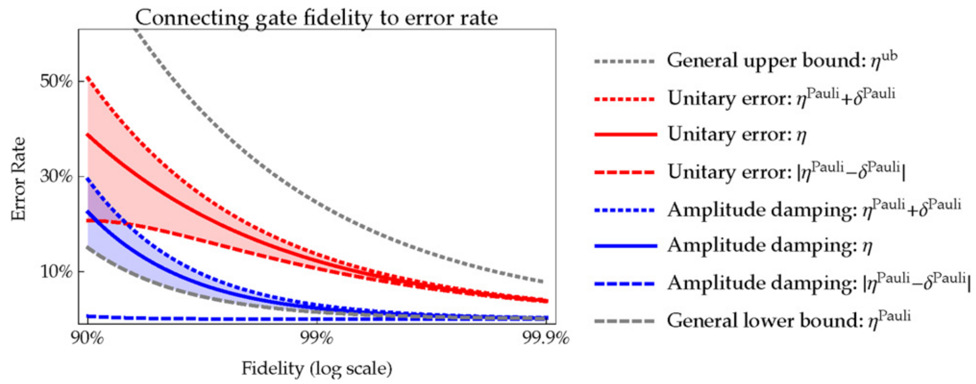

Figure 1. An illustration of the dichotomy between average gate fidelity and the error rate for single-qubit noise channels. The gray curves illustrate the generally applicable lower (dashed) and upper (dotted) bounds. The red curves pertain to unitary errors and the blue curves pertain to an amplitude damping ('a.d.') process. The solid red/blue curves are the numerical values of the error rate (vertical axis) given the average gate fidelity (horizontal axis) of the unitary/a.d. model. The dotted red/blue curves are the values of the Pauli-distance-based upper bound $\eta^{\mathrm{Pauli}} + \delta^{\mathrm{Pauli}}$ calculated for the unitary/a.d. model; the dashed are values for the lower bound $|\eta^{\mathrm{Pauli}} - \delta^{\mathrm{Pauli}}|$. The red/blue shading indicates region estimates for error rate based upon fidelity and Pauli distance; as $|\eta^{\mathrm{Pauli}} - \delta^{\mathrm{Pauli}}|$ falls below η^{Pauli} for the a.d. process, the blue region uses η^{Pauli} rather than $|\eta^{\mathrm{Pauli}} - \delta^{\mathrm{Pauli}}|$ as a lower bound for η. The calculations were performed using the QETLAB project [45].

zero-temperature bath. The unitary error can be entirely specified by the eigenvalues $e^{\pm i\theta}$ of the unitary operator, and the amplitude damping process may be specified by a rate parameter r. Both r and θ may be expressed in terms of the observed average gate fidelity φ and thus the error rate of each can be numerically evaluated as a function of φ. The results of this numerical evaluation are displayed in figure 1.

The most important observation about figure 1 is that the Pauli-distance-based bounds on η yield excellent estimates of the error rate of a noise process as fidelity increases. In fact, fidelity indicates confidence interval if δ^{Pauli} is considered as an estimate of η. Therefore, the Pauli distance can be interpreted as a measure of the 'badness' of noise in the sense that it indicates the size of the gap between fidelity and error rate.

6. Assessing progress towards fault-tolerant quantum computing

The threshold theorem guarantees the possibility of fault-tolerant quantum computation in the presence of local errors that occur at a rate η below a threshold value η_0. Our aim has therefore been to convert gate fidelity φ, a commonly reported figure-of-merit for quantum logic operations, into an upper bound η^{ub} that can be compared, in principle, to η_0. Of course the noise assumptions underlying the threshold theorem could be either weaker or stronger than reasonable assumptions about the noise of real devices, but this subtlety is often overlooked: numerical simulations such as those of Knill [13] and Raussendorf and Harrington [33] are often considered to be indicative of a code-specific threshold value η_0^{lb} even though both papers are clear that only one well-behaved noise model is being simulated.

The proper interpretation of these results is, in the words of Knill, as 'evidence that accurate quantum computing is possible for [error rates] as high as three per cent'. Thus, Knill claims not that 3% is an *estimate* of η_0^{lb} for the C4/C6 code, but that it is an *upper bound*. The results of Raussendorf and Harrington can be interpreted similarly. As we stated at the end of section 2.2, the connection between such simulations and the estimation of threshold values for actual devices is the subject of ongoing research [34, 37, 38].

Whatever its actual value, the threshold error rate that is guaranteed to exist by the threshold theorem is often treated as a performance target for research efforts towards fault-tolerant quantum computing [16, 30, 43]. However, these authors quote the threshold not as a target error rate but as a target average gate fidelity. As we have shown in this paper, the error rate of a quantum gate cannot, in general, be computed as a function of fidelity. Therefore, the kind of threshold demonstrated to exist by the threshold theorem is not a fidelity threshold.

Of course we have agreed that bounds on the error rate of a quantum gate can be derived from the average gate fidelity [19, 20]. Whereas the lower bound was already known to be tight, we showed in section 4 that the upper bound is an asymptotically tight approximation to the tightest possible upper bound. We also agreed that the quantum gate error rate can be computed as a function of fidelity if the noise is guaranteed to be Pauli; indeed, we showed in section 5 that this relationship is approximately true if the noise can be represented by an approximate-Pauli channel. But noise is demonstrably non-Pauli in experiments [46–49] so the observed average gate fidelity is not necessarily indicative of the true error rate. Existing threshold results do not imply a practical performance target in terms of gate fidelity.

The wide-spread conflation of average gate fidelity with error rate has led to assertions that threshold fidelities for Pauli noise correspond to fidelity targets for general noise. One group [30], for example, claims that gate fidelities of 90%–99.5% ('depending on measurement errors') suffice for fault-tolerant quantum computation using the surface code. This is only known to be true if the relevant noise model is Pauli, which it is not. Another group [16] goes further by asserting that device performance has surpassed the fault-tolerance threshold for surface-code-based quantum computing. Their stated threshold value is 99% fidelity, which is derived from simulations of the code in the presence of depolarizing noise [50]. Yet depolarizing noise is Pauli and therefore saturates the lower bound on error rate as a function of fidelity, and the appendix of [16] makes it clear that there are non-Pauli sources of noise. Even if the quoted threshold value of 1% is trustworthy, it is a threshold error rate and not a threshold infidelity.

If the threshold error rate is indeed 1%, then η^{ub} yields rigorous, but relatively pessimistic, fidelity targets. If η_0^{est} is the error rate to be surpassed, a gate fidelity satisfying

$$\varphi > 1 - \frac{\left(\eta_0^{\mathrm{est}}\right)^2}{d(d+1)}, \tag{63}$$

where d is the dimension of the gate, is required to guarantee an error rate η below η_0^{est} without additional information. So if we assume that $\eta_0^{\mathrm{est}} = 1\%$, two-qubit gates ($d = 4$) need to have a fidelity greater than $1 - 5\times10^{-6} = 99.9995\%$ to ensure that the error rate falls below 1%. It is of course possible that lower fidelities suffice, but such a claim must be defended with information such as the Pauli distance (section 5) or unitarity [21] about gate performance additional to fidelity; the main point of our paper is that such additional information is *required*.

7. Conclusion

Reports of extremely high average gate fidelities engender optimism that current technology is near the threshold required for fault-tolerant quantum computation. Yet, although the average gate fidelity φ is an experimentally convenient figure of merit, it is not the proper metric, i.e. the worst-case quantum gate error rate η, for assessing progress towards fault-tolerance.

We have shown that $\eta^{\mathrm{ub}} = \sqrt{d(d+1)(1-\varphi)}$ is an asymptotically tight estimate of the tightest possible upper bound to η in terms of φ alone, and we conjecture that this is optimal. We have demonstrated that it is possible for a two-qubit gate with 99% fidelity to have an error rate of nearly 13%, and we have demonstrated that fidelity-based assessments of gate performance underestimate especially important noise sources such as unitary error. We have derived an alternative bound that can yield tighter estimates of gate performance if an additional piece of information we call the 'Pauli distance' of the noise channel is known, though other kinds of information can also be used to derive alternative bounds [21–23].

We have given a sobering assessment of reported progress towards fault-tolerant quantum computation by converting reported average gate fidelity to the worst-case quantum gate-error rate. Based on the best theoretical results currently available, we have shown that two-qubit gates must surpass 99.9995% gate fidelity to ensure that gates experience an error rate lower than 1%. We have used the Pauli distance to show that information additional to fidelity can be employed to justify tighter bounds on gate performance, and we argue that future attempts to verify quantum gate performance should include estimates of figures of merit additional to fidelity in order to circumvent the looseness of fidelity-based bounds.

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