Theoretical Guide
As Meninas Superpoderosas

Docinho, Lindinha & Florzinha

1 Bitwise

Turn on bit i x & (1 << i)
Turn off bit i x & (~(1 << i))

1.1 XOR from 1 to N

$$f(n) = \begin{cases} n & n \equiv 0 \pmod 4 \\ 1 & n \equiv 1 \pmod 4 \\ n+1 & n \equiv 2 \pmod 4 \\ 0 & n \equiv 3 \pmod 4 \end{cases}$$

2 Identities

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} \frac{1}{i} \approx \log n \qquad \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$

3 Math

$\wedge$ = and = conjunction $\qquad$ $\vee$ = or = disjunction

3.1 Trigonometry

3.2 Logarithm

$$\log_b mn = \log_b m + \log_b n \qquad log_b \frac{m}{n} = \log_b m - \log_n n \qquad \log_b n^p = p \log_b n$$

$$\log_b \sqrt[q]{n} = \frac{1}{q} \log_b n \qquad \log_b n = \log_a n \log_b a \qquad b^{\log_b k} = k$$

$$\log_b a = \frac{\log_c a}{\log_c b} \qquad \log_b a = \frac{1}{\log_a b} \qquad \log_b a \ \log_a c = \log_b c$$

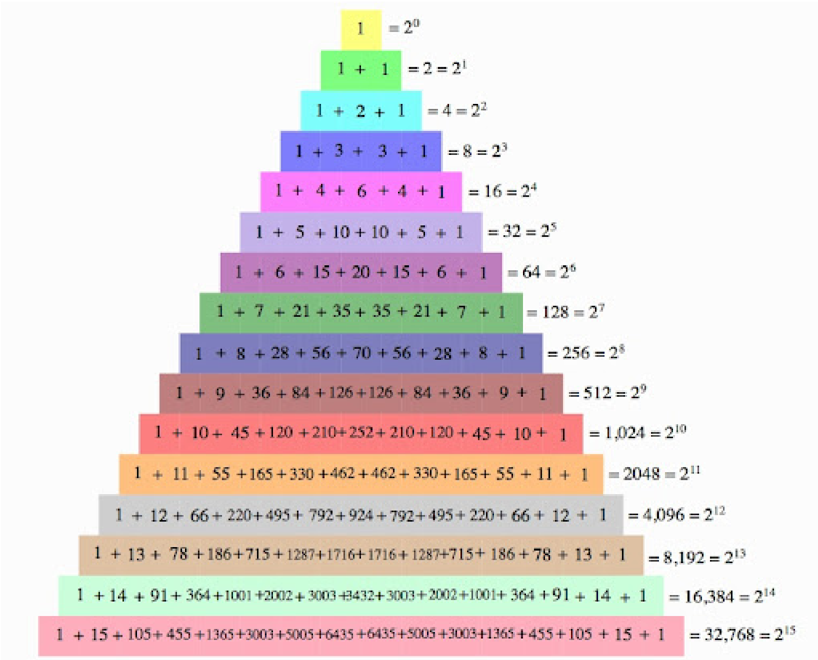
$$\log_b 1 = 0 \qquad \log_b b = 1$$

3.3 Truth Tables

a	b	$a \Rightarrow b$
0	0	1
0	1	1
1	0	0
1	1	1

a	b	$a \oplus b$
0	0	0
0	1	1
1	0	1
1	1	0

3.4 Pascal Triangle



3.5 De Morgan

$$\neg(p \wedge q) \iff \neg p \vee \neg q$$
$$\neg(p \vee q) \iff \neg p \wedge \neg q$$

3.6 2-SAT

Check and finds solution for boolean formulas of the form:

$$(a \vee b) \wedge (\neg a \vee c) \wedge (a \vee \neg b)$$

As $a \vee b \iff \neg a \Rightarrow b \wedge \neg b \Rightarrow a$, we construct a directed graph of these implications. It's possible to construct any truth table of 1 or 2 variables with only and's from pairs of or's.

$(a \vee b)$ turn of only the case $a = 0, b = 0$
$(a \vee \neg b)$ turn of only the case $a = 0, b = 1$
$(\neg a \vee b)$ turn of only the case $a = 1, b = 0$
$(\neg a \vee \neg b)$ turn of only the case $a = 1, b = 1$

Examples:

$a \oplus b = (a \vee b) \wedge (\neg a \vee \neg b)$
$a \wedge b = (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b)$

4 Number Theory

$$(a + b) \bmod m = (a \bmod m + b \bmod m) \bmod m$$
$$(a - b) \bmod m = (a \bmod m - b \bmod m) \bmod m$$
$$(a \times b) \bmod m = ((a \bmod m) \times (b \bmod m)) \bmod m$$
$$a^b \bmod m = (a \bmod m)^b \bmod m$$
$$a \equiv b \pmod{m} \iff (b - a) | m$$

$$\gcd(a_1, a_2, a_3, a_4) = \gcd(a_1, \gcd(a_2, \gcd(a_3, a_4)))$$
$$\mathrm{lcm}(a, b) \times \gcd(a, b) = a \times b$$
$$\mathrm{lcm}(a, b) = \frac{a \times b}{\gcd(a, b)} = \frac{a}{\gcd(a, b)} \times b$$

4.1 Sum of digits of N written in base b

$$f(n, b) = \begin{cases} n & n < b \\ f\left(n, \left\lfloor \frac{n}{b} \right\rfloor + (n \mod b)\right) & n \geq b \end{cases}$$

4.2 Sum of Divisors

Let the sum of divisors when only considering the first i prime factors be S_i. The answer will be S_N.

$$S_i = S_{i-1} \sum_{j=0}^{k_i} x_i^j$$

$$= S_{i-1} \cdot \frac{x_i^{k_i+1} - 1}{x_i - 1}$$

We can calculate each S_i using fast exponentiation and modular inverses in $\mathcal{O}(N \log(\max(k_i)))$ time.

4.3 Some Primes

999999937 1000000007 1000000009 1000000021 1000000033 $10^{18} - 11$ $10^{18} + 3$ $2305843009213693951 = 2^{61} - 1$ $998244353 = 119 \times 2^{23} + 1$ $10^6 + 3$

4.4 Product of Divisors

Let the product and number of divisors when only considering the first i prime factors be P_i and C_i respectively. The answer will be P_N.

$$P_i = P_{i-1}^{k_i+1} \left(x_i^{k_i(k_i+1)/2} \right)^{C_{i-1}}$$

Again, we can calculate each P_i using fast exponentiation in $\mathcal{O}(N \log(\max(k_i)))$ time, but there's a catch! It might be tempting to use C_{i-1} from your previously-calculated values in part 1 of this problem, but those values will yield wrong answers.

This is because $a^b \not\equiv a^{b \bmod p} \pmod{p}$ in general. However, by Fermat's little theorem, $a^b \equiv a^{b \bmod (p-1)} \pmod{p}$ for prime p, so we can just store C_i modulo $10^9 + 6$ to calculate P_i.

4.5 Prime counting function - $\pi(x)$

Expected to have $\frac{x}{\log x}$ primes within $[1, x]$. The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

4.6 Number of Divisors

The number of divisors of n is about $\sqrt[3]{n}$.

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

Given the prime factorization of some number n:

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3}$$

The number of divisors will be $(a_1 + 1)(a_2 + 1)(a_3 + 1)$.

4.7 Large Prime Gaps

For numbers until 10^9 the largest gap is 400.
For numbers until 10^{18} the largest gap is 1500.

4.8 Fermat's Theorems

Let P be a prime number and a an integer, then:

$$a^p \equiv a \pmod{p}$$

$$a^{p-1} \equiv 1 \pmod{p}$$

Lemma: Let p be a prime number and a and b integers, then:

$$(a + b)^p \equiv a^p + b^p \pmod{p}$$

Lemma: Let p be a prime number and a an integer. The inverse of a modulo p is a^{p-2}:

$$a^{-1} \equiv a^{p-2} \pmod{p}$$

2	The last digit is even
3	The sum of the digits is divisible by 3
4	The last 2 digits are divisible by 4
5	The last digit is 0 or 5
7	Double the last digit and subtract it from a number made by the other digits. The result must be divisible by 7. (We can apply this rule to that answer again)
8	The last three digits are divisible by 8
9	The sum of the digits is divisible by 9
11	Add and subtract digits in an alternating pattern (add digit, subtract next digit, add next digit, etc). Then check if that answer is divisible by 11.
13	Multiply the last digit of N with 4 and add it to the rest truncate of the number. If the outcome is divisible by 13 then the number N is also divisible by 13.

4.9 Divisibility Criteria

4.9.1 Other bases

Claim 1:

The divisibility rule for a number a to be divided by n is as follows. Express the number a in base $n+1$. Let s denote the sum of digits of a expressed in base $n+1$. Now $n|a \iff n|s$. More generally, $a \equiv s \pmod{n}$.

Example:

Before setting to prove this, we will see an example of this. Say we want to check if $13|611$. Express 611 in base 14.

$$611 = 3 \times 14^2 + 1 \times 14^1 + 9 \times 14^0 = (319)_{14}$$

where $(319)_{14}$ denotes that the decimal number 611 expressed in base 14. The sum of the digits $s = 3 + 1 + 9 = 13$. Clearly, $13|13$. Hence, $13|611$, which is indeed true since $611 = 13 \times 47$.

4.10 Diophantine Equations

5 Progressions

5.1 Geometric Progression

General Term: $a_1 q^{n-1}$

Sum: $\dfrac{a_1(q^n - 1)}{q - 1}$

Infinite Sum:

$$-1 < q < 1$$

$$\frac{a_1}{1-q}$$

6 Notes

- number of digits in $n!$

$$\log_b n! = \log_b(1 \times 2 \times 3 \times ... \times n) = \log_b 1 + \log_b 2 + \log_b 3 + ... + \log_b n$$

7 Geometry

7.1 Trigonometry

7.1.1 Sines Rule

$$\frac{a}{sin(\alpha)} = \frac{b}{sin(\beta)} = \frac{c}{sin(\gamma)}$$

7.1.2 Cossines Rule

$$a^2 = b^2 + c^2 - 2bccos(\alpha)$$

7.2 Triangle Existence Condition

$$a + b \geq c$$
$$a + c \geq b$$
$$b + c \geq a$$

7.3 Shoelace Formula

$$A = \frac{1}{2}|\sum_{i=1}^{n-1}(p_i \times p_{i+1}) = \frac{1}{2}|\sum_{i=1}^{n-1}(x_i y_{i+1} - x_{i+1} y_i)|$$

Where the points $p_1, pn, \ldots$ are in adjecent order and the first and last vertex is the same, that is, $p_1 = pn$

7.4 Pick's Theorem

$$A = a + \frac{b}{2} + 1$$

where A is the area of the polygon, a is the number of integer points inside the polygon and b is the number of integer points in the boundary of the polygon

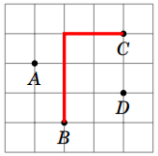
7.5 Distances

$$d(p,q) = \sqrt{(q.x - p.x)^2 + (q.y - p.y)^2}$$

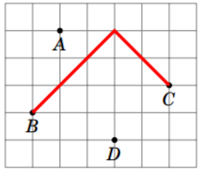
$$d(p,q)|p.x - q.x| + |p.y - q.y|$$

7.6 Maximum possible Manhattan distance between two points given n points

Given n points, for instance:



Rotate all coordinates 45^o do that (x,y) becomes $(x+y, y-x)$, so, p becomes p' and q becomes q'.



The maximum manhattan distance is obtaining by choosing the two points that maximize:

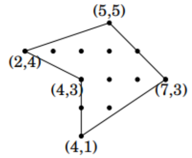
$$max(|p'.x - q'.x|, |p'.y - q'.y|)$$

7.7 Boundary points

The number of integer points in the boundary of a polygon is:

$$B = v + b$$

where v is the number of vertices (integer points as well) and b is the number of integer points situated between two vertices, like in the following figure:



b can be calculated for every line connecting two points (including the line between the last and the first point) as follows:

$$boundary\_points(p,q) = \begin{cases} |p.y - q.y| - 1 & \text{p.x = q.x} \\ |p.x - q.x| - 1 & \text{p.y = q.y} \\ gcd(|p.x - q.x|, |p.y - q.y|) - 1 \end{cases}$$

7.8 Number of points with integer coordinates in a line

$$\gcd(|x_1 - x_2|, |y_1 - y_2|) + 1$$

7.9 3D Shapes

Volume of Sphere: $\frac{4}{3}\pi r^3$

Prism: $V = bh$

Pyramid: $\frac{bh}{3}$

Cone: $\frac{\pi r^2 h}{3}$

7.10 2D Shapes

Perimeter of circle: $2\pi r$

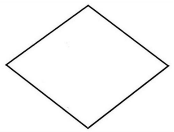
Area of circle: πr^2

Area of triangle: $\frac{b * h}{2}$

Square: l^2

Rectangle: hr

Rhombus:



D is the biggest diagonal and d is the smallest diagonal

$$A = \frac{1}{2} * D * d$$

8 C++

```
template<class T> using min_priority_queue = priority_queue<T, vector<T>, greater<T>>;
```

```
string(1, 'a')
```

8.1 Pragma optimize

```
#pragma GCC optimize("Ofast")
#pragma GCC target("avx,avx2,fma")
```

8.2 Ordered set and multiset

```
typedef tree<pair<ll , ll >, null_type , less<pair<ll , ll >>, rb_tree_tag , tree_order_statistics_node_update> ordered_set;
```

To change to multiset switch equal to less_equal.

8.3 Optimized unordered map

```
mp.reserve(8192);
mp.max_load_factor(0.25);
```

8.4 Interactive Problems

```
freopen("input.txt", "r", stdin);
freopen("output.txt", "w", stdout);
```

9 Constants

```
LLINF = 0x3f3f3f3f3f3f3f3fLL
```

```
MOD = 998'244'353
```

```
PI = acos(-1)
```

```
INT_MIN    INT_MAX    INT64_MIN    INT64_MAX
```

9.1 PI

$\pi \approx 22/7$. Trust me.

First 1044 decimal places of π:

$\pi \approx$ 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651328230664709384460955058223172535940812848111745028410270193852110555964462294895493038196442881097566593344612847564823378678316527120190914564856692346034861045432664821339360726024914127372458700660631558817488152092096282925409171536436789259036001133053054882046652138414695194151160943305727036575959195309218611738193261179310511854807446237996274956735188575272489122793818301194912983367336244065664308602139494639522473719070217986094370277053921717629317675238467481846766940513200056812714526356082778577134275778960917363717872146844090122495343014654958537105079227968925892354201995611212902196086403441815981362977477130996051870721134999999837297804995105973173281609631859502445945534690830264252230825334468503526193118817101000313783875288658753320838142061717766914730359825349042875546873115956286388235378759375195778185778053217122680661300192787661119590921642019893809525720106548586327886593615338182796823030

9.2 Some Powers of Two

$2^0 \approx 10^0$	$2^1 \approx 10^0$	$2^2 \approx 10^0$	$2^3 \approx 10^0$	$2^4 \approx 10^1$	$2^5 \approx 10^1$
$2^6 \approx 10^1$	$2^7 \approx 10^2$	$2^8 \approx 10^2$	$2^9 \approx 10^2$	$2^{10} \approx 10^3$	$2^{11} \approx 10^3$
$2^{12} \approx 10^3$	$2^{13} \approx 10^3$	$2^{14} \approx 10^4$	$2^{15} \approx 10^4$	$2^{16} \approx 10^4$	$2^{17} \approx 10^5$
$2^{18} \approx 10^5$	$2^{19} \approx 10^5$	$2^{20} \approx 10^6$	$2^{21} \approx 10^6$	$2^{22} \approx 10^6$	$2^{23} \approx 10^6$
$2^{24} \approx 10^7$	$2^{25} \approx 10^7$	$2^{26} \approx 10^7$	$2^{27} \approx 10^8$	$2^{28} \approx 10^8$	$2^{29} \approx 10^8$
$2^{30} \approx 10^9$	$2^{31} \approx 10^9$	$2^{32} \approx 10^9$	$2^{33} \approx 10^9$	$2^{34} \approx 10^{10}$	$2^{35} \approx 10^{10}$
$2^{36} \approx 10^{10}$	$2^{37} \approx 10^{11}$	$2^{38} \approx 10^{11}$	$2^{39} \approx 10^{11}$	$2^{40} \approx 10^{12}$	$2^{41} \approx 10^{12}$
$2^{42} \approx 10^{12}$	$2^{43} \approx 10^{12}$	$2^{44} \approx 10^{13}$	$2^{45} \approx 10^{13}$	$2^{46} \approx 10^{13}$	$2^{47} \approx 10^{14}$
$2^{48} \approx 10^{14}$	$2^{49} \approx 10^{14}$	$2^{50} \approx 10^{15}$	$2^{51} \approx 10^{15}$	$2^{52} \approx 10^{15}$	$2^{53} \approx 10^{15}$
$2^{54} \approx 10^{16}$	$2^{55} \approx 10^{16}$	$2^{56} \approx 10^{16}$	$2^{57} \approx 10^{17}$	$2^{58} \approx 10^{17}$	$2^{59} \approx 10^{17}$
$2^{60} \approx 10^{18}$	$2^{61} \approx 10^{18}$	$2^{62} \approx 10^{18}$	$2^{63} \approx 10^{18}$	$2^{64} \approx 10^{19}$	$2^{65} \approx 10^{19}$
$2^{66} \approx 10^{19}$	$2^{67} \approx 10^{20}$	$2^{68} \approx 10^{20}$	$2^{69} \approx 10^{20}$	$2^{70} \approx 10^{21}$	$2^{71} \approx 10^{21}$

9.3 Some Factorials

$6! \approx 10^2$	$7! \approx 10^3$	$8! \approx 10^4$	$9! \approx 10^5$	$10! \approx 10^6$	$11! \approx 10^7$
$12! \approx 10^8$	$13! \approx 10^9$	$14! \approx 10^{10}$	$15! \approx 10^{12}$	$16! \approx 10^{13}$	$17! \approx 10^{14}$
$18! \approx 10^{15}$	$19! \approx 10^{17}$	$20! \approx 10^{18}$	$21! \approx 10^{19}$	$22! \approx 10^{21}$	$23! \approx 10^{22}$

10 Counting Problems

10.1 Burnside's Lemma

Let G be a group that acts on a set X. The Burnside Lemma states that the number of distinct orbits is equal to the average number of points fixed by an element of G.

$$T = \frac{1}{|G|} \sum_{g \in G} |\texttt{fix}(g)|$$

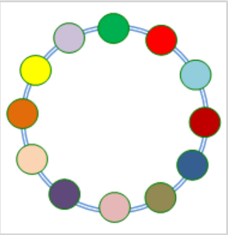
Where a orbit $\texttt{orb}(x)$ is defined as

$$\texttt{orb}(x) = \{y \in X : \exists g \in G\ gx = y\}$$

and $\texttt{fix}(g)$ is the set of elements in X fixed by g

$$\texttt{fix}(g) = \{x \in X : gx = x\}$$

Example: With k distinct types of beads how many distinct necklaces of size n can be made? Considering that two necklaces are equal if the rotation of one gives the other.



$$T = \frac{1}{n+1} \sum_{i=0}^{n} k^{gcd(i,n)} \qquad T = \frac{1}{n} \sum_{i=0}^{n-1} k^{gcd(i,n)}$$