Theoretical Guide
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1 Counting Problems

1.1 Burnside's Lemma

Let G be a group that acts on a set X. The Burnside Lemma states that the number of distinct orbits is equal to the average number of points fixed by an element of G.

$$T = \frac{1}{|G|}\sum_{g \in G} |\mathtt{fix}(g)|$$

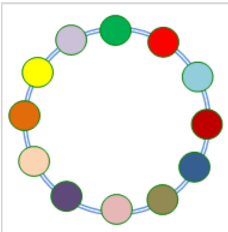
Where a orbit $\mathtt{orb}(x)$ is defined as

$$\mathtt{orb}(x) = \{y \in X : \exists g \in G \; gx = y\}$$

and $\mathtt{fix}(g)$ is the set of elements in X fixed by g

$$\mathtt{fix}(g) = \{x \in X : gx = x\}$$

Example: With k distinct types of beads how many distinct necklaces of size n can be made? Considering that two necklaces are equal if the rotation of one gives the other.



$$T = \frac{1}{n+1}\sum_{i=0}^{n} k^{gcd(i,n)} \qquad T = \frac{1}{n}\sum_{i=0}^{n-1} k^{gcd(i,n)}$$

2 Progressions

2.1 Geometric Progression

General Term: $a_1 q^{n-1}$

Sum: $\dfrac{a_1(q^n - 1)}{q - 1}$

Infinite Sum:

$$-1 < q < 1$$

$$\frac{a_1}{1-q}$$

3 C++

```
template<class T> using min_priority_queue = priority_queue<T, vecto
```

```
string(1, 'a')
```

3.1 Pragma optimize

```
#pragma GCC optimize("Ofast")
#pragma GCC target("avx,avx2,fma")
```

3.2 Ordered set and multiset

```
typedef tree<pair<ll, ll>, null_type, less<pair<ll, ll>>, rb_tree_ta
```

To change to multiset switch equal to less_equal.

3.3 Optimized unordered map

```
mp.reserve(8192);
mp.max_load_factor(0.25);
```

3.4 Interactive Problems

```
freopen("input.txt", "r", stdin);
freopen("output.txt", "w", stdout);
```

4 Bitwise

Turn on bit i x & (1 << i)
Turn off bit i x & (~(1 << i))

4.1 XOR from 1 to N

$$f(n) = \begin{cases} n & n \equiv 0 \pmod 4 \\ 1 & n \equiv 1 \pmod 4 \\ n+1 & n \equiv 2 \pmod 4 \\ 0 & n \equiv 3 \pmod 4 \end{cases}$$

5 Number Theory

$$(a+b) \bmod m = (a \bmod m + b \bmod m) \bmod m$$
$$(a-b) \bmod m = (a \bmod m - b \bmod m) \bmod m$$
$$(a \times b) \bmod m = ((a \bmod m) \times (b \bmod m)) \bmod m$$
$$a^b \bmod m = (a \bmod m)^b \bmod m$$
$$a \equiv b \pmod m \iff (b-a)|m$$

$$\gcd(a_1, a_2, a_3, a_4) = \gcd(a_1, \gcd(a_2, \gcd(a_3, a_4)))$$
$$\mathrm{lcm}(a,b) \times \gcd(a,b) = a \times b$$
$$\mathrm{lcm}(a,b) = \frac{a \times b}{\gcd(a,b)} = \frac{a}{\gcd(a,b)} \times b$$

5.1 Sum of digits of N written in base b

$$f(n,b) = \begin{cases} n & n < b \\ f\left(n, \left\lfloor \frac{n}{b} \right\rfloor + (n \mod b)\right) & n \geq b \end{cases}$$

5.2 Sum of Divisors

Let the sum of divisors when only considering the first i prime factors be S_i. The answer will be S_N.

$$S_i = S_{i-1} \sum_{j=0}^{k_i} x_i^j$$
$$= S_{i-1} \cdot \frac{x_i^{k_i+1} - 1}{x_i - 1}$$

We can calculate each S_i using fast exponentiation and modular inverses in $\mathcal{O}(N \log(\max(k_i)))$ time.

5.3 Some Primes

999999937 1000000007 1000000009 1000000021 1000000033 $10^{18} - 11$ $10^{18} + 3$ $2305843009213693951 = 2^{61} - 1$ $998244353 = 119 \times 2^{23} + 1$ $10^6 + 3$

5.4 Product of Divisors

Let the product and number of divisors when only considering the first i prime factors be P_i and C_i respectively. The answer will be P_N.

$$P_i = P_{i-1}^{k_i+1} \left(x_i^{k_i(k_i+1)/2}\right)^{C_{i-1}}$$

Again, we can calculate each P_i using fast exponentiation in $\mathcal{O}(N \log(\max(k_i)))$ time, but there's a catch! It might be tempting to use C_{i-1} from your previously-calculated values in part 1 of this problem, but those values will yield wrong answers.

This is because $a^b \not\equiv a^{b \bmod p} \pmod p$ in general. However, by Fermat's little theorem, $a^b \equiv a^{b \bmod (p-1)} \pmod p$ for prime p, so we can just store C_i modulo $10^9 + 6$ to calculate P_i.

5.5 Prime counting function - $\pi(x)$

Expected to have $\frac{x}{\log x}$ primes within $[1, x]$. The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

5.6 Number of Divisors

The number of divisors of n is about $\sqrt[3]{n}$.

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

Given the prime factorization of some number n:

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3}$$

The number of divisors will be $(a_1 + 1)(a_2 + 1)(a_3 + 1)$.

5.7 Large Prime Gaps

For numbers until 10^9 the largest gap is 400.
For numbers until 10^{18} the largest gap is 1500.

5.8 Fermat's Theorems

Let P be a prime number and a an integer, then:

$$a^p \equiv a \pmod p$$

$$a^{p-1} \equiv 1 \pmod p$$

2	The last digit is even
3	The sum of the digits is divisible by 3
4	The last 2 digits are divisible by 4
5	The last digit is 0 or 5
7	Double the last digit and subtract it from a number made by the other digits. The res
8	The last three digits are divisible by 8
9	The sum of the digits is divisible by 9
11	Add and subtract digits in an alternating pattern (add digit, subtract next digit, add r
13	Multiply the last digit of N with 4 and add it to the rest truncate of the number. If th

Lemma: Let p be a prime number and a and b integers, then:

$$(a + b)^p \equiv a^p + b^p \pmod p$$

Lemma: Let p be a prime number and a an integer. The inverse of a modulo p is a^{p-2}:

$$a^{-1} \equiv a^{p-2} \pmod p$$

5.9 Divisibility Criteria

5.9.1 Other bases

Claim 1:

The divisibility rule for a number a to be divided by n is as follows. Express the number a in base $n + 1$. Let s denote the sum of digits of a expressed in base $n + 1$. Now $n|a \iff n|s$. More generally, $a \equiv s \pmod n$.

Example:

Before setting to prove this, we will see an example of this. Say we want to check if $13|611$. Express 611 in base 14.

$$611 = 3 \times 14^2 + 1 \times 14^1 + 9 \times 14^0 = (319)_{14}$$

where $(319)_{14}$ denotes that the decimal number 611 expressed in base 14. The sum of the digits $s = 3 + 1 + 9 = 13$. Clearly, $13|13$. Hence, $13|611$, which is indeed true since $611 = 13 \times 47$.

5.10 Diophantine Equations

6 Notes

- number of digits in $n!$

$$\log_b n! = \log_b(1 \times 2 \times 3 \times ... \times n) = \log_b 1 + \log_b 2 + \log_b 3 + ... + \log_b n$$

7 Constants

```
LLINF = 0x3f3f3f3f3f3f3f3fLL
```

```
MOD = 998'244'353
```

```
PI = acos(-1)
```

INT_MIN INT_MAX INT64_MIN INT64_MAX

7.1 PI

$\pi \approx 22/7$. Trust me.

First 1044 decimal places of π:

$\pi \approx$ 3.1415926535897932384626433832795028841971693993751058209749445923
0781640628620899862803482534211706798214808651328230664709384460955058
2231725359408128481117450284102701938521105559644622948954930381964428
8109756659334461284756482337867831652712019091456485669234603486104543
2664821339360726024914127372458700660631558817488152092096282925409171
5364367892590360011330530548820466521384146951941511609433057270365759
5919530921861173819326117931051185480744623799627495673518857527248912
2793818301194912983367336244065664308602139494639522473719070217986094
3702770539217176293176752384674818467669405132000568127145263560827785
7713427577896091736371787214684409012249534301465495853710507922796892
5892354201995611212902196086403441815981362977477130996051870721134999
9998372978049951059731732816096318595024459455346908302642522308253344
6850352619311881710100031378387528865875332083814206171776691473035982
5349042875546873115956286388235378759375195778185778053217122680661300
1927876611195909216420198938095257201065485863278865936153381827968230

7.2 Some Powers of Two

$2^{0} \approx 10^{0}$	$2^{1} \approx 10^{0}$	$2^{2} \approx 10^{0}$	$2^{3} \approx 10^{0}$	$2^{4} \approx 10^{1}$	$2^{5} \approx 10^{1}$
$2^{6} \approx 10^{1}$	$2^{7} \approx 10^{2}$	$2^{8} \approx 10^{2}$	$2^{9} \approx 10^{2}$	$2^{10} \approx 10^{3}$	$2^{11} \approx 10^{3}$
$2^{12} \approx 10^{3}$	$2^{13} \approx 10^{3}$	$2^{14} \approx 10^{4}$	$2^{15} \approx 10^{4}$	$2^{16} \approx 10^{4}$	$2^{17} \approx 10^{5}$
$2^{18} \approx 10^{5}$	$2^{19} \approx 10^{5}$	$2^{20} \approx 10^{6}$	$2^{21} \approx 10^{6}$	$2^{22} \approx 10^{6}$	$2^{23} \approx 10^{6}$
$2^{24} \approx 10^{7}$	$2^{25} \approx 10^{7}$	$2^{26} \approx 10^{7}$	$2^{27} \approx 10^{8}$	$2^{28} \approx 10^{8}$	$2^{29} \approx 10^{8}$
$2^{30} \approx 10^{9}$	$2^{31} \approx 10^{9}$	$2^{32} \approx 10^{9}$	$2^{33} \approx 10^{9}$	$2^{34} \approx 10^{10}$	$2^{35} \approx 10^{10}$
$2^{36} \approx 10^{10}$	$2^{37} \approx 10^{11}$	$2^{38} \approx 10^{11}$	$2^{39} \approx 10^{11}$	$2^{40} \approx 10^{12}$	$2^{41} \approx 10^{12}$
$2^{42} \approx 10^{12}$	$2^{43} \approx 10^{12}$	$2^{44} \approx 10^{13}$	$2^{45} \approx 10^{13}$	$2^{46} \approx 10^{13}$	$2^{47} \approx 10^{14}$
$2^{48} \approx 10^{14}$	$2^{49} \approx 10^{14}$	$2^{50} \approx 10^{15}$	$2^{51} \approx 10^{15}$	$2^{52} \approx 10^{15}$	$2^{53} \approx 10^{15}$
$2^{54} \approx 10^{16}$	$2^{55} \approx 10^{16}$	$2^{56} \approx 10^{16}$	$2^{57} \approx 10^{17}$	$2^{58} \approx 10^{17}$	$2^{59} \approx 10^{17}$
$2^{60} \approx 10^{18}$	$2^{61} \approx 10^{18}$	$2^{62} \approx 10^{18}$	$2^{63} \approx 10^{18}$	$2^{64} \approx 10^{19}$	$2^{65} \approx 10^{19}$
$2^{66} \approx 10^{19}$	$2^{67} \approx 10^{20}$	$2^{68} \approx 10^{20}$	$2^{69} \approx 10^{20}$	$2^{70} \approx 10^{21}$	$2^{71} \approx 10^{21}$

7.3 Some Factorials

$6! \approx 10^2$	$7! \approx 10^3$	$8! \approx 10^4$	$9! \approx 10^5$	$10! \approx 10^6$	$11! \approx 10^7$
$12! \approx 10^8$	$13! \approx 10^9$	$14! \approx 10^{10}$	$15! \approx 10^{12}$	$16! \approx 10^{13}$	$17! \approx 10^{14}$
$18! \approx 10^{15}$	$19! \approx 10^{17}$	$20! \approx 10^{18}$	$21! \approx 10^{19}$	$22! \approx 10^{21}$	$23! \approx 10^{22}$

8 Math

$\wedge$ = and = conjunction $\vee$ = or = disjunction

8.1 Trigonometry

8.2 Logarithm

$$\log_b mn = \log_b m + \log_b n \qquad log_b \frac{m}{n} = \log_b m - \log_n n \qquad \log_b n^p = p \log_b n$$

$$\log_b \sqrt[q]{n} = \frac{1}{q} \log_b n \qquad \log_b n = \log_a n \log_b a \qquad b^{\log_b k} = k$$

$$\log_b a = \frac{\log_c a}{\log_c b} \qquad \log_b a = \frac{1}{\log_a b} \qquad \log_b a \ \log_a c = \log_b c$$

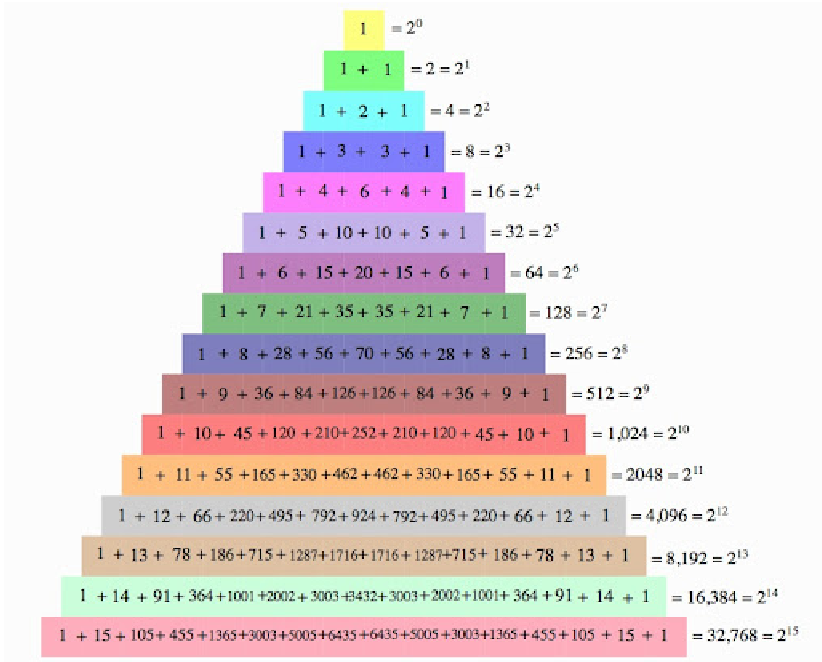
$$\log_b 1 = 0 \qquad \log_b b = 1$$

8.3 Truth Tables

a	b	$a \Rightarrow b$
0	0	1
0	1	1
1	0	0
1	1	1

a	b	$a \oplus b$
0	0	0
0	1	1
1	0	1
1	1	0

8.4 Pascal Triangle



8.5 De Morgan

$$\neg(p \wedge q) \iff \neg p \vee \neg q$$
$$\neg(p \vee q) \iff \neg p \wedge \neg q$$

8.6 2-SAT

Check and finds solution for boolean formulas of the form:

$$(a \vee b) \wedge (\neg a \vee c) \wedge (a \vee \neg b)$$

As $a \vee b \iff \neg a \Rightarrow b \wedge \neg b \Rightarrow a$, we construct a directed graph of these implications. It's possible to construct any truth table of 1 or 2 variables with only and's from pairs of or's.

$(a \vee b)$ turn of only the case $a = 0, b = 0$

$(a \vee \neg b)$ turn of only the case $a = 0, b = 1$
$(\neg a \vee b)$ turn of only the case $a = 1, b = 0$
$(\neg a \vee \neg b)$ turn of only the case $a = 1, b = 1$

Examples:

$a \oplus b = (a \vee b) \wedge (\neg a \vee \neg b)$
$a \wedge b = (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b)$

9 Geometry

9.1 Trigonometry

9.1.1 Sines Rule

$$\frac{a}{sin(\alpha)} = \frac{b}{sin(\beta)} = \frac{c}{sin(\gamma)}$$

9.1.2 Cossines Rule

$$a^2 = b^2 + c^2 - 2bccos(\alpha)$$

9.2 Triangle Existence Condition

$$a + b \geq c$$

$$a + c \geq b$$

$$b + c \geq a$$

9.3 Shoelace Formula

$$A = \frac{1}{2} |\sum_{i=1}^{n-1} (p_i \times p_{i+1}) = \frac{1}{2} |\sum_{i=1}^{n-1} (x_i y_{i+1} - x_{i+1} y_i)|$$

Where the points $p_1, pn, \ldots$ are in adjecent order and the first and last vertex is the same, that is, $p_1 = pn$

9.4 Pick's Theorem

$$A = a + \frac{b}{2} + 1$$

where A is the area of the polygon, a is the number of integer points inside the polygon and b is the number of integer points in the boundary of the polygon

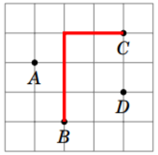
9.5 Distances

$$d(p, q) = \sqrt{(q.x - p.x)^2 + (q.y - p.y)^2}$$

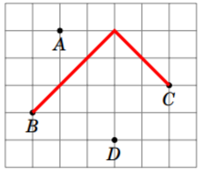
$$d(p, q)|p.x - q.x| + |p.y - q.y|$$

9.6 Maximum possible Manhattan distance between two points given n points

Given n points, for instance:



Rotate all coordinates 45^o do that (x, y) becomes $(x + y, y - x)$, so, p becomes p' and q becomes q'.



The maximum manhattan distance is obtaining by choosing the two points that maximize:

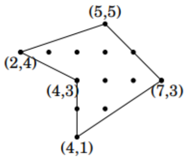
$$max(|p'.x - q'.x|, |p'.y - q'.y|)$$

9.7 Boundary points

The number of integer points in the boundary of a polygon is:

$$B = v + b$$

where v is the number of vertices (integer points as well) and b is the number of integer points situated between two vertices, like in the following figure:



b can be calculated for every line connecting two points (including the line between the last and the first point) as follows:

$$boundary\_points(p, q) = \begin{cases} |p.y - q.y| - 1 & \text{p.x = q.x} \\ |p.x - q.x| - 1 & \text{p.y = q.y} \\ gcd(|p.x - q.x|, |p.y - q.y|) - 1 & \end{cases}$$

9.8 Number of points with integer coordinates in a line

$$\gcd(|x_1 - x_2|, |y_1 - y_2|) + 1$$

9.9 3D Shapes

Volume of Sphere: $\frac{4}{3}\pi r^3$

Prism: $V = bh$

Pyramid: $\frac{bh}{3}$

Cone: $\frac{\pi r^2 h}{3}$

9.10 2D Shapes

Perimeter of circle: $2\pi r$

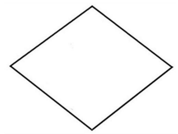
Area of circle: πr^2

Area of triangle: $\frac{b * h}{2}$

Square: l^2

Rectangle: hr

Rhombus:



D is the biggest diagonal and d is the smallest diagonal

$$A = \frac{1}{2} * D * d$$

10 Identities

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} \frac{1}{i} \approx \log n \qquad \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$