Theoretical Guide
QueryTree 2.0

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1 Bitwise

Turn on bit i x & (1 << i)
Turn off bit i x & (~(1 << i))

1.1 XOR from 1 to N

$$f(n) = \begin{cases} n & n \equiv 0 \pmod{4} \\ 1 & n \equiv 1 \pmod{4} \\ n+1 & n \equiv 2 \pmod{4} \\ 0 & n \equiv 3 \pmod{4} \end{cases}$$

1.2 XOR Basis

An XOR basis is a minimal set of linearly independent binary vectors that can represent any vector in a given set through XOR combinations. In computational problems, constructing an XOR basis involves iteratively adding vectors to the basis while ensuring each new vector remains independent by reducing it with existing basis vectors. This basis allows efficient representation and manipulation of binary vector spaces, enabling quick determination of linear independence and facilitating solutions to various optimization and combinatorial problems.

XOR basis involves two parts:

- Represent each given number in its base 2 form, considering it as a vector in the $\mathbb{Z}_{\nvDash}$ vector space, where d is the maximum possible number of bits. The XOR operation on these numbers is equivalent to the addition of the corresponding vectors in the vector space $\mathbb{Z}_{\nvDash}$.

- Relate the answers to the queries of the second type with the basis of the vectors found in Part 1.

By constructing an XOR basis from the set of vectors, we can efficiently answer various queries about linear independence, redundancy, and other properties related to the XOR combinations of the given numbers. This basis provides a compact representation that allows for quick computation and manipulation of the vector space.

2 Identities

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} \frac{1}{i} \approx \log n \qquad \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$

3 Math

$\wedge$ = and = conjunction $\quad$ $\vee$ = or = disjunction

3.1 Trigonometry

3.2 Logarithm

$$\log_b mn = \log_b m + \log_b n \qquad log_b \frac{m}{n} = \log_b m - \log_n n \qquad \log_b n^p = p \log_b n$$

$$\log_b \sqrt[q]{n} = \frac{1}{q} \log_b n \qquad \log_b n = \log_a n \log_b a \qquad b^{\log_b k} = k$$

$$\log_b a = \frac{\log_c a}{\log_c b} \qquad \log_b a = \frac{1}{\log_a b} \qquad \log_b a \ \log_a c = \log_b c$$

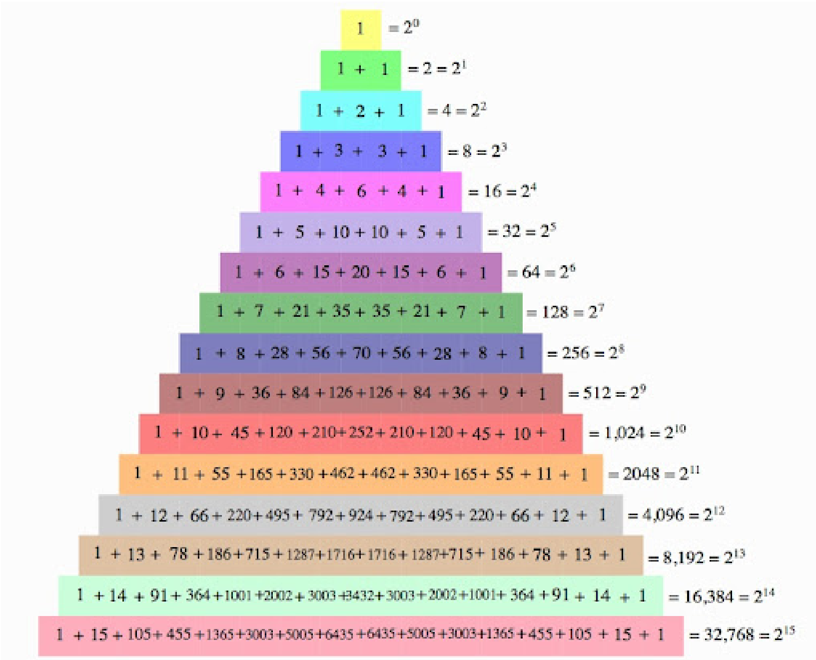
$$\log_b 1 = 0 \qquad \log_b b = 1$$

3.3 Truth Tables

a	b	$a \Rightarrow b$
0	0	1
0	1	1
1	0	0
1	1	1

a	b	$a \oplus b$
0	0	0
0	1	1
1	0	1
1	1	0

3.4 Pascal Triangle



3.5 De Morgan

$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$

3.6 2-SAT

Check and finds solution for boolean formulas of the form:

$$(a \vee b) \wedge (\neg a \vee c) \wedge (a \vee \neg b)$$

As $a \vee b \iff \neg a \Rightarrow b \wedge \neg b \Rightarrow a$, we construct a directed graph of these implications. It's possible to construct any truth table of 1 or 2 variables with only and's from pairs of or's.

$(a \vee b)$ turn of only the case $a = 0, b = 0$
$(a \vee \neg b)$ turn of only the case $a = 0, b = 1$
$(\neg a \vee b)$ turn of only the case $a = 1, b = 0$
$(\neg a \vee \neg b)$ turn of only the case $a = 1, b = 1$

Examples:

$a \oplus b = (a \vee b) \wedge (\neg a \vee \neg b)$
$a \wedge b = (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b)$

4 Number Theory

$$(a + b) \bmod m = (a \bmod m + b \bmod m) \bmod m$$

$$(a - b) \bmod m = (a \bmod m - b \bmod m) \bmod m$$

$$(a \times b) \bmod m = ((a \bmod m) \times (b \bmod m)) \bmod m$$

$$a^b \bmod m = (a \bmod m)^b \bmod m$$

$$a \equiv b \pmod{m} \iff (b - a) | m$$

$$\gcd(a_1, a_2, a_3, a_4) = \gcd(a_1, \gcd(a_2, \gcd(a_3, a_4)))$$

$$\operatorname{lcm}(a, b) \times \gcd(a, b) = a \times b$$

$$\operatorname{lcm}(a, b) = \frac{a \times b}{\gcd(a, b)} = \frac{a}{\gcd(a, b)} \times b$$

4.1 Sum of digits of N written in base b

$$f(n,b) = \begin{cases} n & n < b \\ f\left(n, \left\lfloor \frac{n}{b} \right\rfloor + (n \mod b)\right) & n \geq b \end{cases}$$

4.2 Sum of Divisors

Let the sum of divisors when only considering the first i prime factors be S_i. The answer will be S_N.

$$S_i = S_{i-1} \sum_{j=0}^{k_i} x_i^j$$

$$= S_{i-1} \cdot \frac{x_i^{k_i+1} - 1}{x_i - 1}$$

We can calculate each S_i using fast exponentiation and modular inverses in $\mathcal{O}(N \log(\max(k_i)))$ time.

4.3 Some Primes

999999937 1000000007 1000000009 1000000021 1000000033
$10^{18} - 11$ $10^{18} + 3$ $2305843009213693951 = 2^{61} - 1$
$998244353 = 119 \times 2^{23} + 1$ $10^6 + 3$
1055244485953076559 1392181229391707727 1178970662972334441
257900257981 584598951247 989509930063 105539556781
998244353 754974721 167772161 188244827 205587737
555130769 809747989 572255561 396588799 327208423
773840099 207936359 952818871 935456867 670948771

4.4 Product of Divisors

Let the product and number of divisors when only considering the first i prime factors be P_i and C_i respectively. The answer will be P_N.

$$P_i = P_{i-1}^{k_i+1} \left(x_i^{k_i(k_i+1)/2}\right)^{C_{i-1}}$$

Again, we can calculate each P_i using fast exponentiation in $\mathcal{O}(N \log(\max(k_i)))$ time, but there's a catch! It might be tempting to use C_{i-1} from your previously-calculated values in part 1 of this problem, but those values will yield wrong answers.

This is because $a^b \not\equiv a^{b \bmod p} \pmod p$ in general. However, by Fermat's little theorem, $a^b \equiv a^{b \bmod (p-1)} \pmod p$ for prime p, so we can just store C_i modulo $10^9 + 6$ to calculate P_i.

4.5 Prime counting function - $\pi(x)$

Expected to have $\frac{x}{\log x}$ primes within $[1, x]$. The prime counting function is asymptotic to $\frac{x}{\log x}$, by the prime number theorem.

x	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\pi(x)$	4	25	168	1 229	9 592	78 498	664 579	5 761 455

4.6 Number of Divisors

The number of divisors of n is about $\sqrt[3]{n}$.

n	6	60	360	5040	55440	720720	4324320	21621600
$d(n)$	4	12	24	60	120	240	384	576

Given the prime factorization of some number n:

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3}$$

The number of divisors will be $(a_1 + 1)(a_2 + 1)(a_3 + 1)$.

4.7 Large Prime Gaps

For numbers until 10^9 the largest gap is 400.
For numbers until 10^{18} the largest gap is 1500.

2	The last digit is even
3	The sum of the digits is divisible by 3
4	The last 2 digits are divisible by 4
5	The last digit is 0 or 5
7	Double the last digit and subtract it from a number made by the other digits. The result must be divisible by 7. (We can apply this rule to that answer again)
8	The last three digits are divisible by 8
9	The sum of the digits is divisible by 9
11	Add and subtract digits in an alternating pattern (add digit, subtract next digit, add next digit, etc). Then check if that answer is divisible by 11.
13	Multiply the last digit of N with 4 and add it to the rest truncate of the number. If the outcome is divisible by 13 then the number N is also divisible by 13.

4.8 Fermat's Theorems

Let P be a prime number and a an integer, then:

$$a^p \equiv a \pmod{p}$$

$$a^{p-1} \equiv 1 \pmod{p}$$

Lemma: Let p be a prime number and a and b integers, then:

$$(a+b)^p \equiv a^p + b^p \pmod{p}$$

Lemma: Let p be a prime number and a an integer. The inverse of a modulo p is a^{p-2}:

$$a^{-1} \equiv a^{p-2} \pmod{p}$$

4.9 Divisibility Criteria

4.9.1 Other bases

Claim 1:

The divisibility rule for a number a to be divided by n is as follows. Express the number a in base $n+1$. Let s denote the sum of digits of a expressed in base $n+1$. Now $n|a \iff n|s$. More generally, $a \equiv s \pmod{n}$.

Example:

Before setting to prove this, we will see an example of this. Say we want to check if $13|611$. Express 611 in base 14.

$$611 = 3 \times 14^2 + 1 \times 14^1 + 9 \times 14^0 = (319)_{14}$$

where $(319)_{14}$ denotes that the decimal number 611 expressed in base 14. The sum of the digits $s = 3 + 1 + 9 = 13$. Clearly, $13|13$. Hence, $13|611$, which is indeed true since $611 = 13 \times 47$.

4.10 Diophantine Equations

4.11 Chicken McNugget Theorem

The Chicken McNugget Theorem states that for any two relatively prime positive integers m, n, the greatest integer that cannot be written in the form $am + bn$ for nonnegative integers a, b is $mn - m - n$.

A consequence of the theorem is that there are exactly $\frac{(m-1)(n-1)}{2}$ positive integers which cannot be expressed in the form $am + bn$. The proof is based on the fact that in each pair of the form $(k, mn - m - n - k)$, exactly one element is expressible.

5 Progressions

5.1 Geometric Progression

General Term: $a_1 q^{n-1}$

Sum: $\dfrac{a_1(q^n - 1)}{q - 1}$

Infinite Sum:

$$-1 < q < 1$$

$$\frac{a_1}{1-q}$$

6 Notes

Number of digits in $n!$:

$$\log_b n! = \log_b(1 \times 2 \times 3 \times ... \times n) = \log_b 1 + \log_b 2 + \log_b 3 + ... + \log_b n$$

If $k \geq x$, then $(k\%x) < \frac{k}{2}$.

Every prime number greater than 3 is of the form $6k+1$ or $6k+5$.

Straight angle $= 180^{o}$, Right angle $= 90^{o}$

7 Geometry

7.1 Trigonometry

7.1.1 Sines Rule

$$\frac{a}{sin(\alpha)} = \frac{b}{sin(\beta)} = \frac{c}{sin(\gamma)}$$

7.1.2 Cossines Rule

$$a^2 = b^2 + c^2 - 2bccos(\alpha)$$

7.2 Triangle Existence Condition

$$a + b \geq c$$

$$a + c \geq b$$

$$b + c \geq a$$

7.3 Shoelace Formula

$$A = \frac{1}{2}|\sum_{i=1}^{n-1}(p_i \times p_{i+1}) = \frac{1}{2}|\sum_{i=1}^{n-1}(x_i y_{i+1} - x_{i+1} y_i)|$$

Where the points $p_1, pn, \ldots$ are in adjecent order and the first and last vertex is the same, that is, $p_1 = pn$

7.4 Pick's Theorem

$$A = a + \frac{b}{2} + 1$$

where A is the area of the polygon, a is the number of integer points inside the polygon and b is the number of integer points in the boundary of the polygon

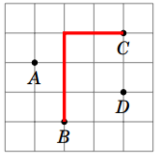
7.5 Distances

$$d(p,q) = \sqrt{(q.x - p.x)^2 + (q.y - p.y)^2}$$

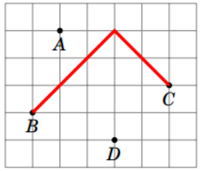
$$d(p,q)|p.x - q.x| + |p.y - q.y|$$

7.6 Maximum possible Manhattan distance between two points given n points

Given n points, for instance:



Rotate all coordinates 45^o do that (x,y) becomes $(x+y, y-x)$, so, p becomes p' and q becomes q'.



The maximum manhattan distance is obtaining by choosing the two points that maximize:

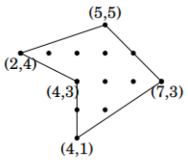
$$max(|p'.x - q'.x|, |p'.y - q'.y|)$$

7.7 Boundary points

The number of integer points in the boundary of a polygon is:

$$B = v + b$$

where v is the number of vertices (integer points as well) and b is the number of integer points situated between two vertices, like in the following figure:



b can be calculated for every line connecting two points (including the line between the last and the first point) as follows:

$$boundary\_points(p,q) = \begin{cases} |p.y - q.y| - 1 & \text{p.x = q.x} \\ |p.x - q.x| - 1 & \text{p.y = q.y} \\ gcd(|p.x - q.x|, |p.y - q.y|) - 1 & \end{cases}$$

7.8 Number of points with integer coordinates in a line

$$\gcd(|x_1 - x_2|, |y_1 - y_2|) + 1$$

7.9 3D Shapes

Volume of Sphere: $\frac{4}{3}\pi r^3$

Prism: $V = bh$

Pyramid: $\frac{bh}{3}$

Cone: $\frac{\pi r^2 h}{3}$

7.10 2D Shapes

Perimeter of circle: $2\pi r$

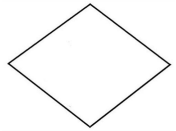
Area of circle: πr^2

Area of triangle: $\frac{b * h}{2}$

Square: l^2

Rectangle: hr

Rhombus:



D is the biggest diagonal and d is the smallest diagonal

$$A = \frac{1}{2} * D * d$$

8 C++

```
string(1, 'a')
```

8.1 Optimized unordered map

```
mp.reserve(8192);
mp.max_load_factor(0.25);
```

```
freopen("input.txt", "r", stdin);
freopen("output.txt", "w", stdout);
```

9 Constants

```
LLINF = 0x3f3f3f3f3f3f3f3fLL
```

```
MOD = 998'244'353
```

```
PI = acos(-1)
```

INT_MIN INT_MAX INT64_MIN INT64_MAX

9.1 PI

$\pi \approx 22/7$. Trust me.

First 1044 decimal places of π:

$\pi \approx$ 3.1415926535897932384626433832795028841971693993751058209749445923
0781640628620899862803482534211706798214808651328230664709384460955058
2231725359408128481117450284102701938521105559644622948954930381964428
8109756659334461284756482337867831652712019091456485669234603486104543
2664821339360726024914127372458700660631558817488152092096282925409171
5364367892590360011330530548820466521384146951941511609433057270365759
5919530921861173819326117931051185480744623799627495673518857527248912
2793818301194912983367336244065664308602139494639522473719070217986094
3702770539217176293176752384674818467669405132000568127145263560827785
7713427577896091736371787214684409012249534301465495853710507922796892
5892354201995611212902196086403441815981362977477130996051870721134999
9998372978049951059731732816096318595024459455346908302642522308253344
6850352619311881710100031378387528865875332083814206171776691473035982
5349042875546873115956286388235378759375195778185778053217122680661300
1927876611195909216420198938095257201065485863278865936153381827968230

9.2 Some Powers of Two

$2^{0} \approx 10^{0}$	$2^{1} \approx 10^{0}$	$2^{2} \approx 10^{0}$	$2^{3} \approx 10^{0}$	$2^{4} \approx 10^{1}$	$2^{5} \approx 10^{1}$
$2^{6} \approx 10^{1}$	$2^{7} \approx 10^{2}$	$2^{8} \approx 10^{2}$	$2^{9} \approx 10^{2}$	$2^{10} \approx 10^{3}$	$2^{11} \approx 10^{3}$
$2^{12} \approx 10^{3}$	$2^{13} \approx 10^{3}$	$2^{14} \approx 10^{4}$	$2^{15} \approx 10^{4}$	$2^{16} \approx 10^{4}$	$2^{17} \approx 10^{5}$
$2^{18} \approx 10^{5}$	$2^{19} \approx 10^{5}$	$2^{20} \approx 10^{6}$	$2^{21} \approx 10^{6}$	$2^{22} \approx 10^{6}$	$2^{23} \approx 10^{6}$
$2^{24} \approx 10^{7}$	$2^{25} \approx 10^{7}$	$2^{26} \approx 10^{7}$	$2^{27} \approx 10^{8}$	$2^{28} \approx 10^{8}$	$2^{29} \approx 10^{8}$
$2^{30} \approx 10^{9}$	$2^{31} \approx 10^{9}$	$2^{32} \approx 10^{9}$	$2^{33} \approx 10^{9}$	$2^{34} \approx 10^{10}$	$2^{35} \approx 10^{10}$
$2^{36} \approx 10^{10}$	$2^{37} \approx 10^{11}$	$2^{38} \approx 10^{11}$	$2^{39} \approx 10^{11}$	$2^{40} \approx 10^{12}$	$2^{41} \approx 10^{12}$
$2^{42} \approx 10^{12}$	$2^{43} \approx 10^{12}$	$2^{44} \approx 10^{13}$	$2^{45} \approx 10^{13}$	$2^{46} \approx 10^{13}$	$2^{47} \approx 10^{14}$
$2^{48} \approx 10^{14}$	$2^{49} \approx 10^{14}$	$2^{50} \approx 10^{15}$	$2^{51} \approx 10^{15}$	$2^{52} \approx 10^{15}$	$2^{53} \approx 10^{15}$
$2^{54} \approx 10^{16}$	$2^{55} \approx 10^{16}$	$2^{56} \approx 10^{16}$	$2^{57} \approx 10^{17}$	$2^{58} \approx 10^{17}$	$2^{59} \approx 10^{17}$
$2^{60} \approx 10^{18}$	$2^{61} \approx 10^{18}$	$2^{62} \approx 10^{18}$	$2^{63} \approx 10^{18}$	$2^{64} \approx 10^{19}$	$2^{65} \approx 10^{19}$
$2^{66} \approx 10^{19}$	$2^{67} \approx 10^{20}$	$2^{68} \approx 10^{20}$	$2^{69} \approx 10^{20}$	$2^{70} \approx 10^{21}$	$2^{71} \approx 10^{21}$

9.3 Some Factorials

$6! \approx 10^{2}$	$7! \approx 10^{3}$	$8! \approx 10^{4}$	$9! \approx 10^{5}$	$10! \approx 10^{6}$	$11! \approx 10^{7}$
$12! \approx 10^{8}$	$13! \approx 10^{9}$	$14! \approx 10^{10}$	$15! \approx 10^{12}$	$16! \approx 10^{13}$	$17! \approx 10^{14}$
$18! \approx 10^{15}$	$19! \approx 10^{17}$	$20! \approx 10^{18}$	$21! \approx 10^{19}$	$22! \approx 10^{21}$	$23! \approx 10^{22}$

10 Python

Remove python recursion limit:
import sys
sys.setrecursionlimit(100000010)

10.1 Sorting

sorted(student_tuples, key=lambda student: student[2])

11 Counting Problems

11.1 Burnside's Lemma

Let G be a group that acts on a set X. The Burnside Lemma states that the number of distinct orbits is equal to the average number of points fixed by an element of G.

$$T = \frac{1}{|G|} \sum_{g \in G} |\texttt{fix}(g)|$$

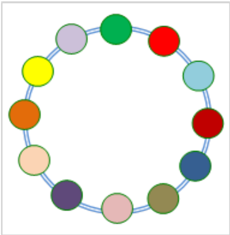
Where a orbit $\texttt{orb}(x)$ is defined as

$$\texttt{orb}(x) = \{y \in X : \exists g \in G \; gx = y\}$$

and $\texttt{fix}(g)$ is the set of elements in X fixed by g

$$\texttt{fix}(g) = \{x \in X : gx = x\}$$

Example: With k distinct types of beads how many distinct necklaces of size n can be made? Considering that two necklaces are equal if the rotation of one gives the other.



$$T = \frac{1}{n+1}\sum_{i=0}^{n} k^{gcd(i,n)}$$

$$T = \frac{1}{n}\sum_{i=0}^{n-1} k^{gcd(i,n)}$$