

The prime-counting function:

The sieve of Eratosthenes is an easy method to generate primes.

You simply start at 2, and mark the multiples as composite numbers.

This is repeated with the next number, which was not marked as a composite, i.e. the next prime.

With this algorithm one can specify the prime-counting function as:

$$\pi(n) = |\{2, \dots, n\} \setminus \bigcup_{i=2}^{n-1} \bigcup_{k=2}^{n+1-i} i \cdot k|$$

If you shift the indices, the set of composites, which is subtracted from the set $\{2, \dots, n\}$ could also be written as:

$$\bigcup_{i=2}^{n-1} \bigcup_{k=2}^{n+1-i} i \cdot k = \bigcup_{i=3}^n \bigcup_{k=2}^{n+2-i} (i-1) \cdot k = \bigcup_{i=3}^n \bigcup_{k=2}^{i-1} (n+2-i) \cdot k =: \varphi(n)$$

Now consider the next function:

$$\bigcup_{i=3}^n \bigcup_{k=2}^{i-1} (i+1-k) \cdot k =: \psi(n)$$

The claim is: $\varphi(n) = \psi(n) \forall n \geq 2$, i.e. the two functions generate the same sets.

The following tables show the development of the sets of composites for $\varphi(n)$ and $\psi(n)$, starting at $n = 3$:

Proof by Mathematical Induction:

Base Case: $n = 3$

$$\varphi(3) = \bigcup_{i=3}^3 \bigcup_{k=2}^{i-1} (n+2-i) \cdot k = \{4\} = \bigcup_{i=3}^3 \bigcup_{k=2}^{i-1} (i+1-k) \cdot k = \psi(3)$$

Assumption:

$$\varphi(n-1) = \bigcup_{i=3}^{n-1} \bigcup_{k=2}^{i-1} (n+1-i) \cdot k = \bigcup_{i=3}^{n-1} \bigcup_{k=2}^{i-1} (i+1-k) \cdot k = \psi(n-1)$$

Induction Step: $n-1 \rightarrow n$

It shows that

| $\varphi(n)$ | | | | | |
|--------------|----|----|----|----|----|
| k \ i | 3 | 4 | 5 | 6 | 7 |
| 2 | 4 | | | | |
| 2 | 6 | 4 | | | |
| 3 | | 6 | | | |
| 2 | 8 | 6 | 4 | | |
| 3 | | 9 | 6 | | |
| 4 | | | 8 | | |
| 2 | 10 | 8 | 6 | 4 | |
| 3 | | 12 | 9 | 6 | |
| 4 | | | 12 | 8 | |
| 5 | | | | 10 | |
| 2 | 12 | 10 | 8 | 6 | 4 |
| 3 | | 15 | 12 | 9 | 6 |
| 4 | | | 16 | 12 | 8 |
| 5 | | | | 15 | 10 |
| 6 | | | | | 12 |

| $\psi(n)$ | | | | | |
|-----------|---|---|---|----|----|
| k \ i | 3 | 4 | 5 | 6 | 7 |
| 2 | 4 | | | | |
| 2 | 4 | 6 | | | |
| 3 | | 6 | | | |
| 2 | 4 | 6 | 8 | | |
| 3 | | 6 | 9 | | |
| 4 | | | 8 | | |
| 2 | 4 | 6 | 8 | 10 | |
| 3 | | 6 | 9 | 12 | |
| 4 | | | 8 | 12 | |
| 5 | | | | 10 | |
| 2 | 4 | 6 | 8 | 10 | 12 |
| 3 | | 6 | 9 | 12 | 15 |
| 4 | | | 8 | 12 | 16 |
| 5 | | | | 10 | 15 |
| 6 | | | | | 12 |

$$\varphi(n) = \bigcup_{i=3}^n \bigcup_{k=2}^{i-1} (n+2-i) \cdot k = \varphi(n-1) \cup \bigcup_{i=3}^n (n+2-i) \cdot (i-1) := \varphi(n-1) \cup \varphi'$$

and

$$\psi(n) = \bigcup_{i=3}^n \bigcup_{k=2}^{i-1} (i+1-k) \cdot k = \psi(n-1) \cup \bigcup_{k=2}^{n-1} (n+1-k) \cdot k := \psi(n-1) \cup \psi'$$

Now we show that the added sets are the equal:

$$\varphi' = \bigcup_{i=3}^n (n+2-i) \cdot (i-1) = \bigcup_{i=2}^{n-1} (n+2-(i+1)) \cdot ((i+1)-1) = \bigcup_{i=2}^{n-1} (n+1-i) \cdot i = \bigcup_{k=2}^{n-1} (n+1-k) \cdot k = \psi'$$

$$\Rightarrow \varphi(n) = \psi(n)$$

□

Applications:

The prime-counting function:

$$\pi(n) = |\{2, \dots, n\} \setminus \psi(n)|$$

The following function calculates the number of Goldbach partitions for the even number $2 \cdot n$:

$$G(n) = |\{2, \dots, n\} \setminus \bigcup_{b=0}^1 \bigcup_{i=3}^n \bigcup_{k=2}^{i-1} n + (-1)^b \cdot (n - (i + 1 - k) \cdot k)|$$

The function $\psi(n)$ could be further generalized to use a constant factor $c \in \mathbb{N}$:

$$\psi_c(n) = \bigcup_{j=2-c}^1 \bigcup_{i=3}^n \bigcup_{k=2}^{i-1} c \cdot (i + \frac{j}{c} - k) \cdot k$$

$$\pi_c(n) = |\{2, \dots, n\} \setminus \psi_c(n)|$$

$$G_c(n) = |\{2, \dots, n\} \setminus \bigcup_{b=0}^1 \bigcup_{j=2-c}^1 \bigcup_{i=3}^n \bigcup_{k=2}^{i-1} n + (-1)^b \cdot (n - c \cdot (i + \frac{j}{c} - k) \cdot k)|$$

It holds that:

$$\pi(n) = \pi_c(n) \forall_{n,c \in \mathbb{N}}$$

$$G(n) = G_c(n) \forall_{n,c \in \mathbb{N}}$$