Supplementary material for the paper: **Looking within events: examining internal temporal structure with local relative rate**

Contents

A. Linear time warping procedure

A description of an interpolation-based LTW procedure is provided here and illustrated in Fig. A1. Assume that we have a signal $x(t)$ defined at evenly spaced sample times $t_{orig} \in \{t_0, \dots, t_{end}\}$. The procedure is as follows:

(i) determine a desired length (in samples) for the warped signal, L_{warp}. Equivalently, this length may be determined from a desired warped signal time interval $[\tilde{t}_0, \tilde{t}_{end}]$ and desired sampling rate. In many uses, the desired warped signal length/time interval may be chosen to be the median of a set of signals, or the length of a relevant reference signal, or sometimes just a convenient number like 1 or 100

(ii): define a vector t_{interp} of interpolation times, the vector being of length L_{warp}, with times equally spaced from t_0 to t_{end}. These are the "times", in the original time coordinate, at which we will interpolate signal values.

(iii): map the vector of interpolated times in the original time coordinate to a new vector $\tilde{t}_{norm}$ of times in the "normalized" or "warped" temporal coordinate. This vector contains L_{warp} equally spaced times from $\tilde{t}_0$ to $\tilde{t}_{end}$. This is the "time normalization" part of the procedure.

(iv): perform an interpolation of the signal at the interpolation times t_{interp}. The interpolation can be of any form (linear, cubic, spline, etc.)—note that this interpolation is not the time-warping itself and does not make use of the normalized time coordinates.

(v): the values obtained from the interpolation in the original coordinate are associated with the new time values in the normalized coordinate.

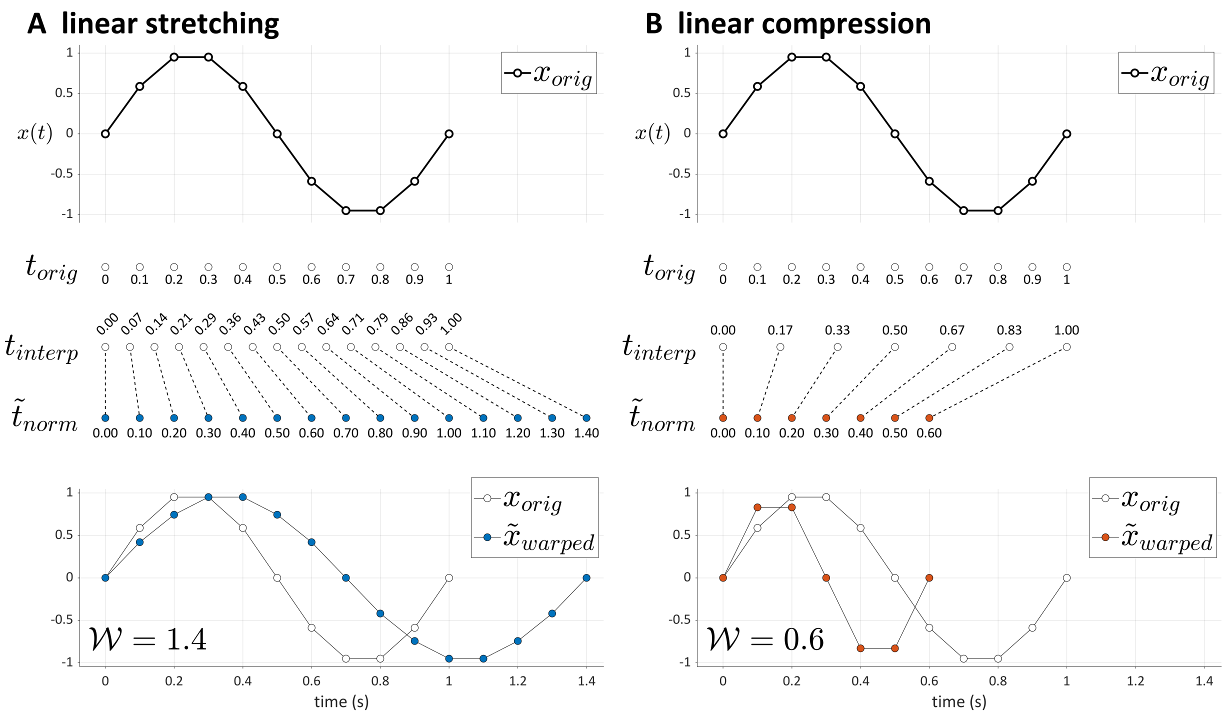

Fig. A1. Examples of linear stretching and compression. Top: original signals. Middle: original, interpolated, and normalized time indices. Bottom: linearly warped signals.

Examples of linear stretching and compression are shown in Fig. A1-A and A1-B, respectively. The degree of stretching or compression of the original signal is the warping factor, $\mathcal{W} = \frac{(L_{warp}-1)}{(L_{orig}-1)}$. A value $\mathcal{W} > 1$ corresponds to stretching, and $\mathcal{W} < 1$ corresponds to compression. Equivalently, the warping factor is $\mathcal{W} = \frac{t_{end}-t_0}{\tilde{t}_{end}-\tilde{t}_0}$.

B. DTW alignment options

There are three main ways to use the DTW mapping to align signals. Note that these alignment procedures are associated more directly with signal transformation, as opposed to time normalization. However, differences between the three signal transformation approaches using DTW can be associated with differences in how the time normalization is interpreted, and in this way the time normalization and signal transformation components of DTW are relevant to one another.

To exemplify the three alignment methods, the panels of Fig. B1 show original and aligned signals, with mapping schemas in-between. In each case, the aligned signals are obtained by using the pairs of indices in the mapping to index the original signals. For example, in panel (A), the first, second, and third relations in the warping curve are shown in Table B1 below. These pairs of indices in the warping curve are used to index the original signal values, to obtain the aligned signal values, which are also shown in the table:

Table B1. Warping curve relations and aligned values for the example in Fig. B1, panel A

Relations of warping curve	**Aligned signal values**	
$\phi(1) = \left(\phi_x(1), \phi_y(1)\right) = (1,1)$	$\tilde{x}(1) = x\left(\phi_x(1)\right)$	$\tilde{y}(1) = y\left(\phi_y(1)\right)$
$\phi(2) = \left(\phi_x(2), \phi_y(2)\right) = (2,2)$	$\tilde{x}(2) = x\left(\phi_x(2)\right)$	$\tilde{y}(2) = y\left(\phi_y(2)\right)$
$\phi(3) = \left(\phi_x(3), \phi_y(3)\right) = (2,3)$	$\tilde{x}(3) = x\left(\phi_x(3)\right)$	$\tilde{y}(3) = y\left(\phi_y(3)\right)$

The first alignment method is an *expansive* alignment, shown in panel (A). The alignment method uses all of the L_ϕ relations in the mapping. It repeats the value of a signal when its index has a one-to-many relation with the other signal. For example, in the aligned signals $\tilde{x}$ and $\tilde{y}$ of (A), observe that the values of $\tilde{x}$ at $k = 2, 3, 4$ arise from repeating the value of $x(2)$, because index $i = 2$ is related to indices $j = 2$, 3, and 4. Similarly, y is stretched by repeating its values at normalized time indices $k = 7, 8, 9$. The alignment is called *expansive* because it will always result in aligned signals with lengths that are greater than or equal to $max\left(L_x, L_y\right)$. The expansive alignment can be interpreted as the alignment obtained from the perspective of an external observer who insists that both aligned signals retain every observed value—no values can be skipped. Another way to interpret the expansive alignment is to say that the external observer always accomplishes the warping of time by slowing down one time coordinate relative to the other: for example, the time coordinate of the warped signal $\tilde{x}$ is slowed down relative to $\tilde{y}$ at normalized time index $k = 2$.

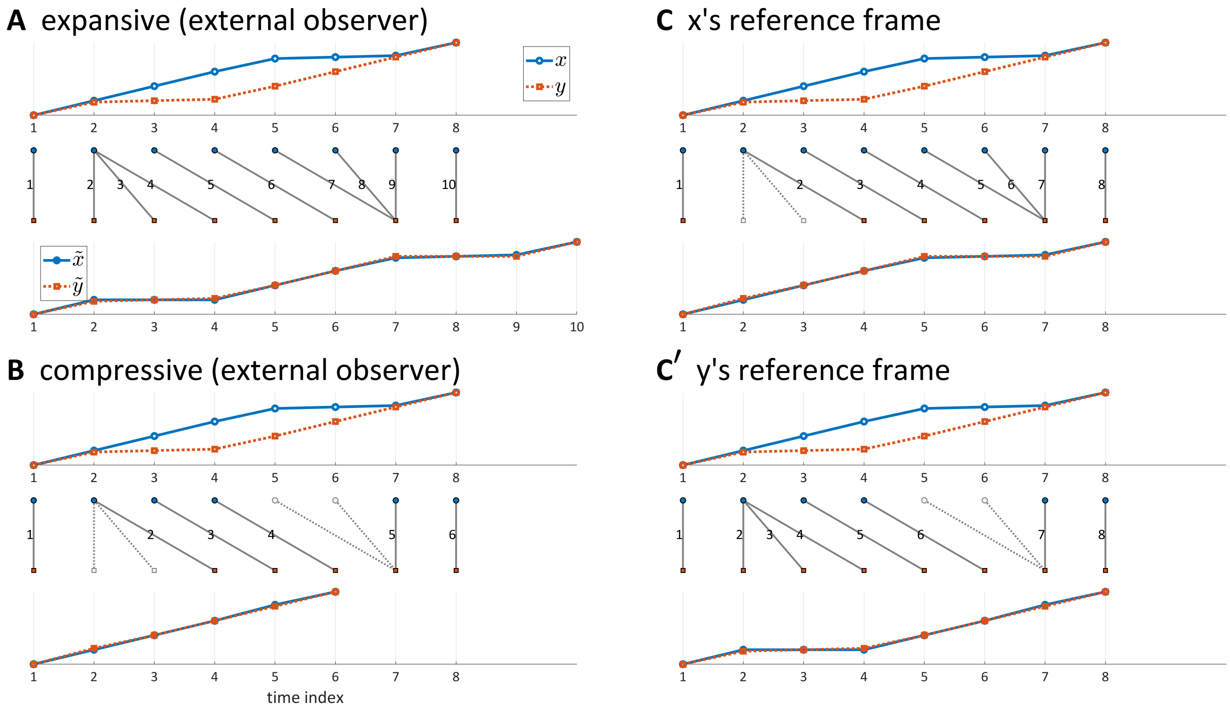

Fig. B1. Three different methods of alignment. (A) Expansive alignment: all original signal values are retained in the aligned signals. (B) Compressive alignment: original time indices are never repeated. (C/C′) Asymmetric alignments from the perspective of an observer moving with signal x or y.

In the *compressive* alignment of panel (B), the external observer always handles one-to-many relations by skipping all but the first or last relation in a sequence of relations that have the same index for one of the signals. For example, the second and third relations in the mapping (dashed lines in (B)) are ignored. This alignment strategy is called *compressive* because it will always result in a normalized time dimension whose length is less than or equal to $min(L_x, L_y)$. The external observer refuses to repeat any signal indices in compressive alignment; or, the warping of time is always accomplished by speeding up one time coordinate relative to the other: $\tilde{y}$ is sped up in the vicinity of $k = 2, 3, 4$ by skipping all but one of the original values.

In both expansive and compressive alignments, the external observer treats the input signals equivalently—there is an exchange symmetry such that the topology of the mapping and outcome of alignment does not change when the input signals x and y are exchanged.

Asymmetric alignments break the exchange symmetry by adopting the perspective of one or the other input signals, thereby treating that input as a "reference". In effect, the time indices of the reference signal are not allowed to change—the reference temporal coordinate becomes the temporal coordinate of both aligned signals. To interpret this, imagine that there are two observers, X and Y, who are traveling with the two signals, respectively. Both observers experience the passage of time uniformly, but in different, self-oriented coordinates. Observer X experiences the passage of time in indices of x, and observer Y experiences the passage of time in the time indices of y. In other words, the "clock" for X is based on the progression of states in x, and the clock for Y is based on the progression of states in y. Thus from the perspective of X, a one-to-many relation from x to y always appears be a case in which y "slows down"—in the example of panel (C), to X it appears that Y takes three time steps (from j = 2 to 5) to accomplish the same change of state that X experiences from i = 2 to 3. In order to align y to x, the observer X must speed up y over this period. Crucially, the observer X always interprets one-to-many mappings in this way, because they assume that the progression of time associated with their reference frame functions as "ground truth". Observer X thus has a consistent interpretation of the origins of the mismatch: the process generating y slowed down in this time period. Conversely, the many-to-one mapping over indices $k = 5, 6, 7$ is interpreted by X as a speeding up of the process generating y, and so to adjust y, it must be slowed down (its values are repeated). Essentially the same (but reversed) descriptions can be applied to the alignment obtained from Y's perspective, shown in panel (C'). Note that in the aligned signals of (C), $\tilde{x} = x$, while in (C'), $\tilde{y} = y$.

The expansive alignment is in some ways the most basic alignment method, because it uses the full set of relations in the output mapping of DTW. In the expansive approach, the continuity constraint is upheld in the use of warping functions for signal transformation. In contrast, the set of relations used for alignment in the compressive method can violate continuity for both signal transformations. For asymmetric/signal-based reference frames, the continuity constraint is upheld for whichever warping function is treated as the reference one.

An important point of the above exposition is that from the perspective of an external observer, there is an inherent ambiguity in whether the origins of one-to-many relations should be interpreted as slowing-downs or speeding-ups of the one signal relative to another—indeed, they might be interpreted as both. This ambiguity is entirely removed if we adopt the temporal reference frame of one signal or the other. In the reference frame of a particular signal, the normalized time coordinate of the warping curve/path is simply the original time coordinate of that signal.

C. Process vs. warping interpretations of DTW warping curve

It useful to recognize that there are two complementary interpretations of the warping curve. Say that we perform a DTW on two signals, *x* and *y*, indexed by *i* and *j*, respectively. The signals are generated by unknown processes *X* and *Y*. The two interpretations of the warping curve are:

- A process interpretation, in which the change in the indices of the warping curve are interpreted with respect to what happened in the underlying processes that generated the signal. In this case:

  Δi : change of indices of X / rate of change of process X

  Δj : change of indices of Y / rate of change of process Y

  $\frac{\Delta i}{\Delta j}$: rate of change of X relative to Y

  $\frac{\Delta i}{\Delta j} > 1$: process X sped up relative to Y

  $\frac{\Delta i}{\Delta j} < 1$: process X slowed down relative to Y

- A warping interpretation, in which the changes of indices of the warping curve are interpreted with respect to what an observer should do to align one signal to another, in which case:

  Δj : a to-be-imposed change of Y indices per change in X indices (Δi)

  $\frac{\Delta j}{\Delta i}$: the local slope of the warping curve

  $\frac{\Delta j}{\Delta i} > 1$: signal Y needs to be sped up/compressed relative to X

$\frac{\Delta j}{\Delta i} < 1$: signal Y X needs to be slowed down/stretched relative to X

Under these two interpretations, the terms Δi and Δj have different meanings. In the process (or "actual") interpretation, they are viewed to represent actual changes in process rates that generated in the signals. In the warping (or "counterfactual") interpretation, they represent what an observer should do to optimally align the signals. Because these interpretations are different, the following relation holds:

$$\frac{\Delta i}{\Delta j}\text{(process interp.)} = \frac{\Delta j}{\Delta i}\text{(warping interp.)}$$

This relation would seem to suggest that a value is always equal to its own reciprocal, i.e., $\frac{a}{b} = \frac{b}{a}$, which seems contradictory. However, the units of Δi and Δj have different meanings under the two interpretations, which more explicitly can be expressed with the equation:

$$\frac{\Delta i_{proc}}{\Delta j_{proc}} = \frac{\Delta j_{warp}}{\Delta i_{warp}}$$

Thus there are two closely related interpretations of the warping curve, which we refer to as the process (or *actual*) and warping (or *counterfactual*) interpretations. On one hand, the slope of the warping curve tells us the ratio of the actual rates of the processes that generated the signals. On the other hand, the curve can be interpreted counterfactually as the required amount of warping of that must be done, in order to align the signals. The process/actual interpretation describes a relation between generating process rates, while the warping/counterfactual interpretation describes a signal transformation that could be conducted to compensate for differences in process rates.