Orbit Rational Transducers

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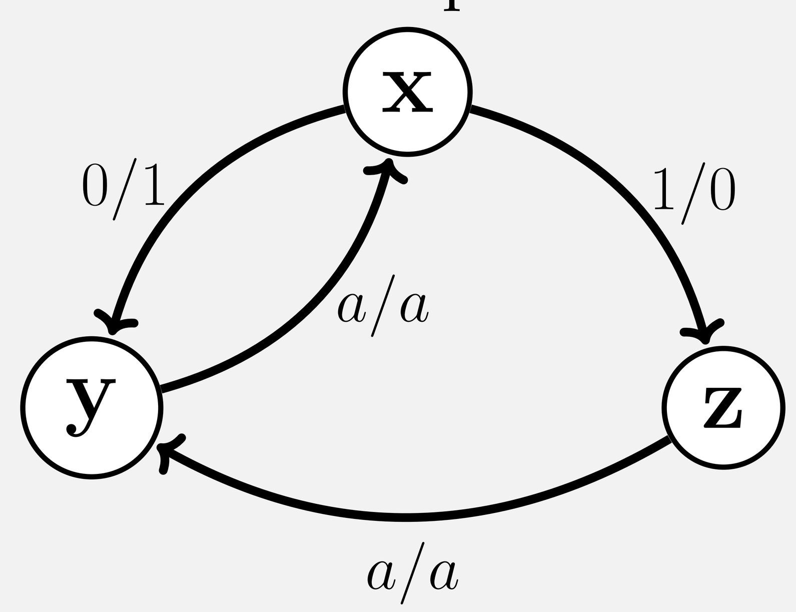
Abstract

We study the complexity arising from iterating transductions in invertible transducers and give an algorithm to decide if an abelian transducer has rational orbit relations. The algorithm leverages a connection between abelian automaton groups and algebraic number fields.

Invertible Transducers

We are interested in transducers over the binary alphabet $\mathbf{2}$ whose transductions are invertible — the output function for each state is a permutation of $\mathbf{2}$. A state is called *odd* if its output function is the transposition and *even* otherwise.



The transductions of such a machine are length-preserving bijections on $\mathbf{2}^*$. These transductions form a semigroup under composition. When they commute with one another, we call the transducer *abelian*.

The *residuals* of a transduction f, denoted f_0 and f_1, are the induced transductions after reading one character. In the above example $\mathbf{x}_0 = \mathbf{y}$ and $\mathbf{z}_0 = \mathbf{z}_1 = \mathbf{y}$.

Orbit Relation

The *orbit relation* of a transduction f is defined as

$$\mathbf{orb}(f) = \{(x, y) \mid \exists t \in \mathbb{N} \,.\, f^t(x) = y\}$$

Because f is a length-preserving bijection on $\mathbf{2}^*$, $\mathbf{orb}(f)$ is an equivalence relation.

Orbit Rationality

We studied the complexity of deciding the orbit relation and found that in some cases, the relation is *rational*, meaning it can be decided by a finite state machine.

Theorem: One can efficiently decide if an abelian transducer is orbit-rational.

Proof Sketch: We will work in the semigroup of transductions. Then, $\mathbf{orb}(f)$ being rational is equivalent to the finiteness of the transitive closure of the following map applied to f

$$\varphi(f) = \begin{cases} f_0 & \text{if } f \text{ is even} \\ (f^2)_0 & \text{if } f \text{ is odd} \end{cases}$$

To each abelian transducer, we can associate (and efficiently compute) an algebraic number field $\mathcal{F} = \mathbb{Q}(\alpha)$ and view the semigroup as elements of $\mathcal{F}$.

Residuation in the semigroup corresponds to an affine map in $\mathcal{F}$: the transition $p \xrightarrow{a/b} q$ in the transducer corresponds to $\alpha p + (b - a) = q$ in $\mathcal{F}$, where b and a are interpreted as integers. In the number field, φ becomes

$$\varphi(f) = \begin{cases} \alpha f & \text{if } f \text{ is even} \\ 2\alpha f & \text{if } f \text{ is odd} \end{cases}$$

Now one can show that the closure $\varphi^*(f)$ is finite if and only if $(2\alpha^k)^n = 1$ for some integers n and k. This property can be efficiently checked because we can upper bound n and k in terms of the degree of α over $\mathbb{Q}$ using tricks from field theory. □

Example: The machine to the left is orbit-rational. We can associate the field $\mathbb{Q}(\alpha)$ where α has minimal polynomial $\chi(z) = z^2 + z + 1/2$. Then $(2\alpha^2)^4 = 1$.

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