Orbit Rational Transducers

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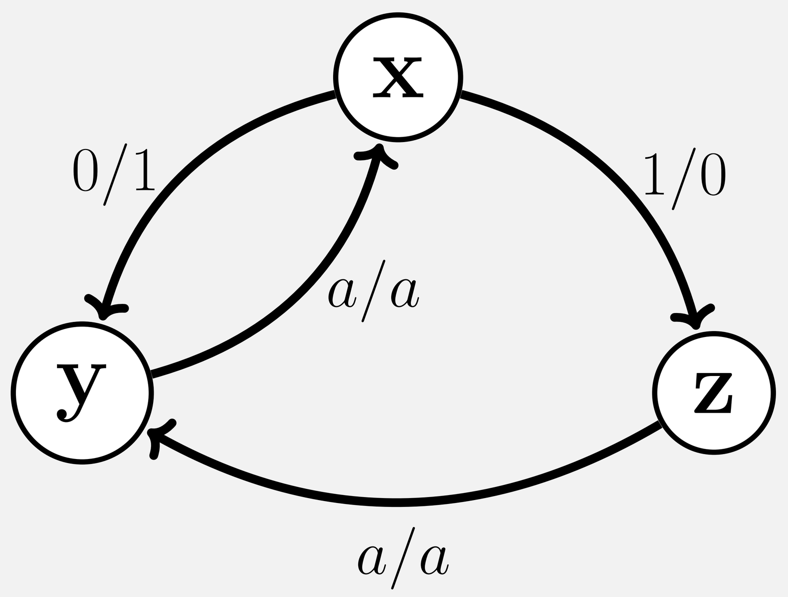
Abstract

We study the complexity arising from iterating transductions in invertible transducers. In particular, we give an algorithm to decide if an abelian transducer has rational orbit relations. The algorithm leverages a connection between abelian automaton groups and algebraic number fields.

Invertible Transducers

We are interested in transducers over the binary alphabet $\mathbf{2}$ whose transductions are invertible — the output function for each state is a permutation of $\mathbf{2}$. A state is called *odd* if its output function is the transposition and *even* otherwise.



The transductions of such a machine are length-preserving bijections on $\mathbf{2}^*$. These transductions form a semigroup under composition. When they commute with one another, we call the transducer *abelian*.

The *residuals* of a transduction f, denoted f_0 and f_1, are the induced transductions after reading one character. In the above example $\mathbf{x}_0 = \mathbf{y}$ and $\mathbf{z}_0 = \mathbf{z}_1 = \mathbf{y}$.

Orbit Relation

The *orbit relation* of a transduction f is defined as

$$\mathbf{orb}(f) = \{(x, y) \mid \exists t \in \mathbb{N} \,.\, f^t(x) = y\}$$

Because f is a length-preserving bijection on $\mathbf{2}^*$, $\mathbf{orb}(f)$ is an equivalence relation.

Orbit Rationality

We studied the complexity of deciding the orbit relation and found that in some cases, the relation is *rational*, meaning it can be decided by a finite state machine.

Theorem: One can efficiently decide if an abelian transducer is orbit-rational.

Proof Sketch: We will work in the semigroup of transductions. Then, $\mathbf{orb}(f)$ being rational is equivalent to the finiteness of the transitive closure of the following map applied to f

$$\varphi(f) = \begin{cases} f_0 & \text{if } f \text{ is even} \\ (f^2)_0 & \text{if } f \text{ is odd} \end{cases}$$

To each abelian transducer, we can associate (and efficiently compute) an algebraic number field $\mathcal{F} = \mathbb{Q}(\alpha)$ and view the semigroup as elements of $\mathcal{F}$.

Residuation in the semigroup corresponds to an affine map in $\mathcal{F}$: the transition $p \xrightarrow{a/b} q$ in the transducer corresponds to $\alpha p + (b - a) = q$ in $\mathcal{F}$, where b and a are interpreted as integers. In the number field, φ becomes

$$\varphi(f) = \begin{cases} \alpha f & \text{if } f \text{ is even} \\ 2\alpha f & \text{if } f \text{ is odd} \end{cases}$$

Now one can show that the closure $\varphi^*(f)$ is finite if and only if $(2\alpha^k)^n = 1$ for some integers n and k. This property can be efficiently checked because we can upper bound n and k in terms of the degree of α over $\mathbb{Q}$ using tricks from field theory. $\square$

Example: The machine to the left is orbit-rational. We can associate the field $\mathbb{Q}(\alpha)$ where α has minimal polynomial $\chi(z) = z^2 + z + 1/2$. Then $(2\alpha^2)^4 = 1$.

Acknowledgements

Thank you to Klaus Sutner for his guidance on this project.