Quantum Versions of Random Walks

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Abstract

Quantum random walks have been an area of significant interest over the past two decades within quantum computation and information. They exhibit many desirable qualities that aid quantum computations and are extremely versatile for use in the design of quantum algorithms. In this report we analyse a recent publication that uses quantum walk dynamics in order to generate higher dimensional entangled states.

Contents

1 Introduction

Quantum walks are powerful tools in the landscape of quantum computing. Much like their classical analogues, they exhibit many properties that are desirable for computations and are an extremely useful building block for many algorithms designed for quantum computers [1]. Specific research interest into the quantum variant stems from their very significant divergences from the classical, including different spreading speeds and quantum correlations, known as entanglement, that have no classical comparison. Their power is such that quantum walks can simulate any quantum computation and therefore are a model for universal quantum computing [2]. There is also evidence of robust performance even when the quantum computer is not perfectly isolated from its environment, and in certain situations it has been shown that decoherences due to interactions with the environment is beneficial for a given computation [3]. Quantum walks are divided into two categories, *discrete* and *continuous* time, these labels describing the nature of the evolution of the walker as the quantum walk progresses. Continuous time quantum walks have been shown to solve a wide range of problems in a number of different settings, in some cases exponentially faster than a classical computer is able to [4]. However, the focus of this report will be on entangled state generation and transfer which neccessitates the need for more than one subspace in our system, a setting which lends itself much more readily to discrete time quantum walks.

Quantum systems often exhibit correlations that have no classical analogue. Such correlations are known as *entanglement*. A quantum state that has entanglement is known as an *entangled state*. Entanglement is a key resource for many quantum computing protocols [5–7], and is a key component in many algorithms that solve problems with greater efficiency than the best known classical algorithms. Higher dimensional entanglement, that is entanglement between qudits, is able to unlock even further benefits in quantum algorithms and as such its generation is extremely important but comes with its own challenges.

A potential solution to the demanding task of generating higher dimensional entanglement has been proposed [8] which uses the dynamics of quantum walks to transfer lower dimensional entanglement between qubits, which is far simpler to generate, into the high dimensional qudits. In this report, a brief primer on random walks, classical and quantum, and on measuring entanglement will be given in §2. §3 will focus on using quantum walk dynamics to facilitate the transfer of entanglement. Finally, a short summary is presented in §4.

As mentioned above, discrete time quantum walks are much more suited for the purposes of this report, therefore future references to quantum walks will be assumed to be the discrete variant unless stated otherwise. We will also use QW to denote (discrete) quantum walk. The mathematical notation used in this report follows standard conventions, in particular it should be made clear the equivalency between $|u_1, u_2\rangle \equiv |u_1\rangle\,|u_2\rangle \equiv |u_1\rangle \otimes |u_2\rangle$, where all forms will be used interchangeably.

2 Background

2.1 Random Walks

2.1.1 Classical Random Walk

Before discussing quantum random walks, we first outline the basic premise of the classical random walk on a discrete number line. The walker starts at the origin and before taking a step to the left (-1) or the right (+1), they flip an unbiased coin to decide which direction to take a step in, moving to the right if the coin lands on heads and to the left if the coin lands on tails. It is quite easy to show that as this classical walk progresses, we end up with a binomial distribution describing the probability distribution of the position of the walker.

2.1.2 Quantum Random Walk

We now use a similar process to define our quantum counterpart to the classical walk on a 1-D number line. In our walker system, we can divide the overall Hilbert space of the QW, $\mathcal{H}$, into two subspaces, the coin subspace $\mathcal{H}_C$ and the position subspace of the walker $\mathcal{H}_W$.

$$\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_W. \tag{1}$$

We note that whilst we do not place any constraints on the size of $\mathcal{H}_W$, we choose $dim(\mathcal{H}_C) = 2$, which is the natural thing to do when quantising our classical walk since the classical coin has two possible states. To aid distinguishability between coin states and position states, we write that

$$\langle \mathcal{H}_C \rangle = \{ |\uparrow\rangle, |\downarrow\rangle \} \tag{2}$$
$$\langle \mathcal{H}_W \rangle = \{ |k\rangle \, | k \in \mathbb{Z} \}, \tag{3}$$

where $\langle U \rangle$ denotes a set of vectors which span U. Therefore, the states $|\uparrow\rangle, |\downarrow\rangle$ take the place of heads and tails on our quantum 'coin'. Having defined the Hilbert space within which the walk will be conducted in, we can now define operators within our space that will dictate how the QW will proceed. We first define the 'coin flip' operator $C \in \mathcal{H}_C$. There are several choices for C, details of which can be found here [9]. As detailed in [9], for walks on a line, if we restrict ourselves to choosing an unbiased coin with real coefficients the Hadamard coin

$$C = \frac{1}{\sqrt{2}} \Big[|\uparrow\rangle \langle\uparrow| + |\uparrow\rangle \langle\downarrow| + |\downarrow\rangle \langle\uparrow| - |\downarrow\rangle \langle\downarrow| \Big] \tag{4}$$
$$= \frac{1}{\sqrt{2}} \Big[(|\uparrow\rangle + |\downarrow\rangle) \langle\uparrow| + (|\uparrow\rangle - |\downarrow\rangle) \langle\downarrow| \Big] \tag{5}$$

is the only choice of coin available. Equation (5) makes obvious the action of C; if the coin state is $|\uparrow\rangle$ then it becomes an equal superposition of $|\uparrow\rangle + |\downarrow\rangle$, if the coin state is in $|\downarrow\rangle$ then we get an equal superposition of $|\uparrow\rangle - |\downarrow\rangle$. These two equal superpositions are often denoted as $|+\rangle$ and $|-\rangle$ respectively.

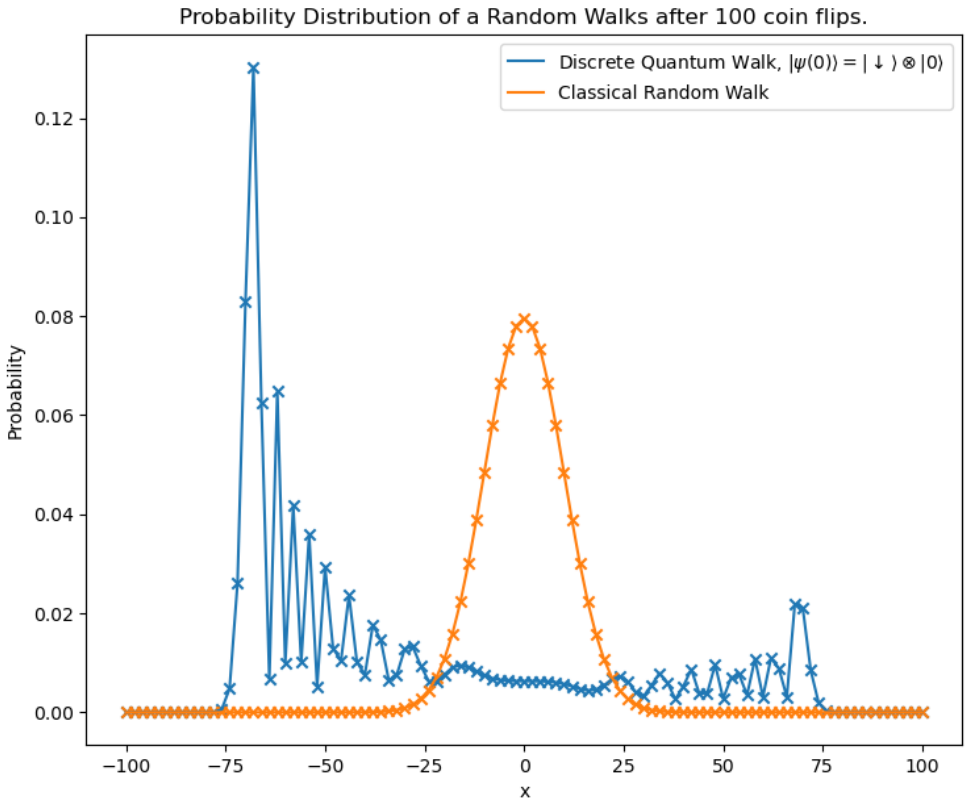

Figure 1: A comparison between the classical and quantum walk on a line. The quantum walk is initially in the $|\downarrow\rangle$ state in the coin subspace of the walk.

We then define our shift operator $S \in \mathcal{H}$ which allows the position of our walker to change, dependent on the state of the coin.

$$S = \sum_k |\uparrow\rangle \langle\uparrow| \otimes |k+1\rangle \langle k| + |\downarrow\rangle \langle\downarrow| \otimes |k-1\rangle \langle k| \,. \tag{6}$$

Again, this representation of S makes manifest its effect on our walker. If the coin is in the $|\uparrow\rangle$, then we take a step in the +1 direction, if in the $|\downarrow\rangle$ then we take a step in the -1 direction. The probability distribution of such a walk is plotted in Fig 1, where the initial coin state is $|\downarrow\rangle$, and is compared to a classical random walk.

Whilst the above choice of S is the most common on the number line, it is also possible to define an alternative choice of shift operator,

$$\tilde{S} = \sum_k |\uparrow\rangle \langle\uparrow| \otimes |k\rangle \langle k| + |\downarrow\rangle \langle\downarrow| \otimes |k+1\rangle \langle k| \,. \tag{7}$$

Th subtle difference between $\tilde{S}$ and S is that $\tilde{S}$ can only move in the +1 direction of the number line and has no 'left moving' part, so to speak. This means that $\tilde{S}$ is restricted to

the non-negative integers and unlike S, can occupy all $|x\rangle$ for $0 \leq x \leq T$, where T is the number of time steps in our QW.

2.2 Entanglement Measures

There are many different methods of quantifying entanglement, and there is no one standard method for doing so as different measures are preferable in different scenarios. Here we will introduce and define two common measures of entanglement.

We first define *von Neumann entropy*, S, used for pure states.

$$S(\rho) = -tr(\rho ln \rho), \tag{8}$$

where ρ is the density matrix representing the quantum system and ln is the natural matrix logarithm. It is often easier to compute this quantity using the spectral decomposition of ρ,

$$\rho = \sum_a \lambda_a |a\rangle \langle a| . \tag{9}$$

Using this form we can compute the entropy by

$$S(\rho) = -\sum_a \lambda_a ln \lambda_a. \tag{10}$$

Whilst QW dynamics are unitary, which ensures that pure states retain their purity as the walk progresses, we will see that this measure of entanglement is not appropriate for measuring how well our entanglement has been transferred as our protocol will also involve projections, which are not unitary. To this end, we introduce an alternative measure of entanglement called *negativity,* $\mathcal{N}$ [10].

$$\mathcal{N}(\rho) = \frac{||\rho^{\Gamma_A}||_1 - 1}{2}, \tag{11}$$

where ρ^{Γ_A} denotes the *partial transpose* of ρ with respect to the subsystem $\mathcal{H}_A$ and $||X||_1 = Tr(\sqrt{X^\dagger X})$ denotes the *trace norm* of X.

Similarly to von Neuman entropy, it is possible to rewrite this definition in terms of eigenvalues, this time eigenvalues of ρ^{Γ_A},

$$\mathcal{N}(\rho) = \sum_{\lambda_a < 0} |\lambda_a| = \sum_{\lambda_a} \frac{|\lambda_a| - \lambda_a}{2}. \tag{12}$$

Finally, we define the *ebit* as is the unit of bipartite entanglement. It is the entanglement contained in a Bell State [11]. Maximally entangled states in $d \times d$ dimensions have $log_2(d)$ ebits.

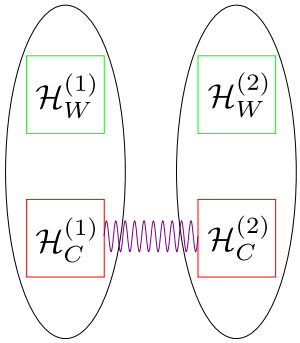

Figure 2: The initial prepared state has entanglement solely between the two coin subspaces. Figure is an edited version from FIG 3 from [8].

3 Entanglement Transfer

We now review the protocol presented in [8] which utilises quantum walk dynamics in order to generate higher dimensional entanglement.

In this section we use following notation:

- $|u\rangle^{(i)}$ is a state belonging to the subspace $\mathcal{H}^{(i)} = \mathcal{H}_C^{(i)} \otimes \mathcal{H}_W^{(i)}, i \in \{1, 2\}$.
- $|u\rangle_J$ is a state belonging to the subspace $\mathcal{H}_J = \mathcal{H}_J^{(1)} \otimes \mathcal{H}_J^{(2)}, J \in \{H, W\}$.
- $|u\rangle_J^{(i)}$ is a state belonging to the subspace $\mathcal{H}_J^{(i)}$.

In this mathematical framework the overall Hilbert state of the quantum system is comprised of two quantum walk subspaces,

$$\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \tag{13}$$

$$= \mathcal{H}_C^{(1)} \otimes \mathcal{H}_W^{(1)} \otimes \mathcal{H}_C^{(2)} \otimes \mathcal{H}_W^{(2)}. \tag{14}$$

The basic premise of this protocol is this:

1. Entangle the two coin spaces of the walkers $\mathcal{H}_C^{(i)}$. (Fig 2.)
2. Proceed with the random walk.
3. Use a projection $\mathcal{P}_\gamma = |\gamma\rangle \langle\gamma|, |\gamma\rangle \in \mathcal{H}_C^{(1)}$ to then transfer the entanglement so that it solely exists in the subspace $\mathcal{H}_W^{(1)} \otimes \mathcal{H}_C^{(2)} \otimes \mathcal{H}_W^{(2)}$.
4. In similar fashion, find a projection $\mathcal{P}_\delta = |\delta\rangle \langle\delta|, |\delta\rangle \in \mathcal{H}_C^{(2)}$ to transfer the entanglement to exist between the two walker subspaces, $\mathcal{H}_W^{(i)}$, only.
5. Accumulate entanglement in the walker subspaces by once more entangling the two coin spaces and repeating the protocol.

In this way, we are able to generate arbitrary amounts of high dimensional entanglement. As is the case with many quantum walk based protocols, particular attention must be paid to the choice of coin used for the quantum walk, as it will have a large impact on the success

of the protocol. In the presentation of the protocol in [8], the shift operator $\tilde{S}$ (the operator with no left moving part) is used, and as such we will too present this overview using the same shift operator. It should be noted however, that the choice of shift operator, S or $\tilde{S}$, has no real impact upon the workings of this protocol.

3.1 Transfer using I

We first use the example of a quantum walk with coin I, the identity. We prepare a state $|\psi(0)\rangle$ with the coin states entangled and walkers at the origin

$$|\psi(0)\rangle = \underbrace{\frac{1}{\sqrt{2}}\Big[\, |\uparrow\rangle_C^{(1)} |\uparrow\rangle_C^{(2)} + |\downarrow\rangle_C^{(1)} |\downarrow\rangle_C^{(2)} \,\Big]}_{\text{Bell State}} \otimes |0\rangle_W^{(1)} |0\rangle_W^{(2)} . \tag{15}$$

Following this we apply our coin, I, and then use our shift operator S to advance the quantum walk. Explicitly (dropping the indices and combining some of our kets together) we obtain

$$|\psi(1)\rangle = \tilde{S}I\,|\psi\rangle = \frac{1}{\sqrt{2}}\Big[\, |\uparrow,\uparrow\rangle\,|0,0\rangle + |\downarrow,\downarrow\rangle\,|1,1\rangle \,\Big]. \tag{16}$$

We then project the part of $|\psi(1)\rangle$ residing in the $\mathcal{H}_C^{(1)}$ subspace onto the vector $|\gamma\rangle$, using the projective operator $\mathcal{P}_\gamma = |\gamma\rangle\langle\gamma| \in \mathcal{H}_C^{(1)}$. Choose $|\gamma\rangle = \frac{1}{\sqrt{2}}\big[\, |\uparrow\rangle + |\downarrow\rangle \,\big]$ which then gives us

$$\mathcal{P}_\gamma\,|\psi(1)\rangle = \frac{1}{2}\Big[\, |\gamma\rangle \otimes \big(\, |\uparrow,0,0\rangle + |\downarrow,1,1\rangle \,\big)\Big]. \tag{17}$$

Similarly, we project the other walker subspace to $|\delta\rangle$ which we can in this instance take to be the same state as $|\gamma\rangle = \frac{1}{\sqrt{2}}\big[\, |\uparrow\rangle + |\downarrow\rangle \,\big]$,

$$\mathcal{P}_\delta\mathcal{P}_\gamma\,|\psi(1)\rangle = \frac{1}{2\sqrt{2}}\Big[\, |\gamma\rangle \otimes |\delta\rangle \otimes \frac{1}{\sqrt{2}}\Big(\, |0,0\rangle + |1,1\rangle \,\Big)\Big]. \tag{18}$$

Renormalising we see that we have the state

$$|\gamma\rangle_C^{(1)} \otimes |\delta\rangle_C^{(2)} \otimes \underbrace{\frac{1}{\sqrt{2}}\Big[\, |0,0\rangle + |1,1\rangle \,\Big]_W}_{\text{Bell State}}, \tag{19}$$

which has a Bell State in the $\mathcal{H}_W$ subspace, and the states in $\mathcal{H}_C$ are separable. Therefore we have transferred the entanglement that originally resided in the coin subspace to the walker one.

3.2 Accumulation

Of course the real interest in this protocol, as previously stated, is the ability to accumulate the entanglement transferred from the lower dimensional coin subspace to the higher dimensional walker one. We now demonstrate how we can do this, again using I as our coin. We

start with the final state obtained in the previous section (equation (22)) and re-entangle the coin subspaces, redefining the resulting state as our $|\psi(0)\rangle$,

$$|\gamma\rangle_C^{(1)} \otimes |\delta\rangle_C^{(2)} \otimes \frac{1}{\sqrt{2}}\Big[\,|0,0\rangle + |1,1\rangle\,\Big]_W \tag{20}$$

$$\xrightarrow{\text{Entangle } \mathcal{H}_C} \frac{1}{\sqrt{2}}\Big[\,|\uparrow,\uparrow\rangle + |\downarrow,\downarrow\rangle\,\Big]_C \otimes \frac{1}{\sqrt{2}}\Big[\,|0,0\rangle + |1,1\rangle\,\Big]_W \tag{21}$$

$$= |\psi(0)\rangle\,. \tag{22}$$

We then proceed with the walk, however taking two steps instead of one this time,

$$|\psi(2)\rangle = (\tilde{S}I)^2\,|\psi(0)\rangle \tag{23}$$

$$= \frac{1}{2}\Big[\,|\uparrow,\uparrow\rangle\,\big(\,|0,0\rangle + |1,1\rangle\,\big) + |\downarrow,\downarrow\rangle\,\big(\,|2,2\rangle + |3,3\rangle\,\big)\Big]. \tag{24}$$

We use the same projectors in the two coin subspaces, $\mathcal{P}_\gamma \in \mathcal{H}_C^{(1)}, \mathcal{P}_\delta \in \mathcal{H}_C^{(2)}$, and after renormalisation have the final state

$$|\gamma\rangle \otimes |\delta\rangle \otimes \frac{1}{2}\Big[\,|0,0\rangle + |1,1\rangle + |2,2\rangle + |3,3\rangle\,\Big]. \tag{25}$$

In this way we have now accumulated two ebits in the walker subspace, and can continue to repeat this process to accumulate arbitrarily large amounts of entanglement into our walker subspace, noting that each n^{th} iteration requires an additional 2^{n-1} steps in our quantum walk.

3.3 Using a general coin

4 Conclusions

Acknowledgements

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Appendices

A Proof of No-Cloning Theorem

Here we present a proof of the no-cloning theorem.

Proof. Let $\mathcal{H}$ be a Hilbert space with two quantum systems A and B. Assume there exists a unitary operation $U \in \mathcal{H}$ that can clone any arbitrary $|\psi\rangle_A$ onto any arbitrary $|\phi\rangle_B$.

$$U |\psi\rangle \otimes |\phi\rangle = |\psi\rangle \otimes |\psi\rangle \tag{26}$$
$$\implies U^{-1} |\psi\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |\phi\rangle \tag{27}$$
$$\implies U^{\dagger} |\psi\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |\phi\rangle \tag{28}$$
$$\tag{29}$$

Suppose we have any two arbitrary $|\psi\rangle_A, |\tilde{\psi}\rangle_A$ s.t. $0 \leq \langle\psi|\tilde{\psi}\rangle \leq 1$.

$$\langle\psi|\tilde{\psi}\rangle = \langle\psi|\tilde{\psi}\rangle \langle\phi|\phi\rangle \tag{30}$$
$$= (\langle\psi| \otimes \langle\phi|)(|\tilde{\psi}\rangle \otimes |\phi\rangle) \tag{31}$$
$$= (\langle\psi| \otimes \langle\psi|)UU^{\dagger}(|\tilde{\psi}\rangle \otimes |\tilde{\psi}\rangle) \tag{32}$$
$$= (\langle\psi| \otimes \langle\psi|)(|\tilde{\psi}\rangle \otimes |\tilde{\psi}\rangle) \tag{33}$$
$$= \langle\psi|\tilde{\psi}\rangle^2 \tag{34}$$
$$\implies \langle\psi|\tilde{\psi}\rangle = 0 \text{ or } \langle\psi|\tilde{\psi}\rangle = 1 \tag{35}$$

which is a contradiction. □