Quantum Versions of Random Walks

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Abstract

Abstract abstract

Contents

1 Introduction

Quantum walks are powerful tools in the landscape of quantum computing. Much like their classical analogues, they exhibit many properties that are desirable for computations and are an extremely useful building block for many algorithms designed for quantum computers [4]. However, they also have some very significant divergences from classical random walks, including a quadratic speed up in their spreading and quantum correlations, known as entanglement, that have no classical comparison. Their power is such that quantum walks can simulate any quantum computation and therefore are a model for universal quantum computing [7]. There is also evidence of robust performance even when the quantum computer is not perfectly isolated from its environment, and in certain situations it has been shown that decoherences due to interactions with the environment is beneficial for a given computation [6]. Quantum walks are divided into two categories, discrete time and continuous time. The latter have been shown to solve a wide range of problems in a number of different settings. However, the focus of this report will be on entangled state generation and transfer which neccessitates the need for more than one Hilbert subspace in our system, a setting which lends itself much more readily to discrete time quantum walks.

Quantum walks have been shown to have great versitility in the generation and transfer of entanglement, quantum correlations which have no classical analogue, within a quantum system. Entanglement is a key resource for many quantum computing protocols [1][2][3], and is a key component in obtaining the speedup promised by quantum computers over their classical ancestors.

In this report we will first review discrete quantum walks on different graphs in section 2. Section 3 will then discuss using discrete quantum walks in order to transfer entanglement between subspaces within a quantum walk system. Finally, a short summary is presented in section 4.

As mentioned above, discrete time quantum walks are much more suited for the purposes of this report, therefore future references to quantum walks will be assumed to be the discrete variant unless stated otherwise. We will also use Q.W. to denote (discrete) quantum walk. The mathematical notation used in this report follows standard conventions, in particular we would like to make clear the eqivalency between $|u_1\rangle\,|u_2\rangle \equiv |u_1\rangle \otimes |u_2\rangle$, where both forms will be used interchangeably.

2 Walks on Different Graphs

2.1 Classical Random Walk on 1-D Lattice

Before discussing quantum random walks, we first outline the basic premise of the classical random walk on the 1-D lattice (a discrete number line). The walker starts at the origin and before taking a step to the left (-1) or the right (+1), they flip a (unbiased) coin to decide which direction to take a step in, moving to the right if the coin lands on heads and to the left if the coin lands on tails. By repeating this process we can plot a probability distribution of where the walker is likely to end up after n steps in the walk.

2.2 1-D Lattice

We now use a similar process to define our quantum counterpart to the classical walk on a 1-D lattice. In our walker system, we can divide the overall Hilbert space of the Q.W. $\mathcal{H}$ into two subspaces, the coin subspace $\mathcal{H}_C$ and the position subspace of the walker $\mathcal{H}_W$.

$$\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_W. \tag{1}$$

We note that whilst we do not place an constraints on the size of $\mathcal{H}_W$, we choose $dim(\mathcal{H}_C) = 2$, which is the natural thing to do since the classical coin has two possible states. To aid distiguishability between coin states and position states, we write that

$$\langle \mathcal{H}_C \rangle = \{ |\uparrow\rangle , |\downarrow\rangle \} \tag{2}$$
$$\langle \mathcal{H}_W \rangle = \{ |k\rangle \, | k \in \mathbb{Z} \} \tag{3}$$

where $\langle U \rangle$ denotes a set of vectors which span U. Therefore, the states $|\uparrow\rangle , |\downarrow\rangle$ take the place of heads and tails on our quantum 'coin'. Having defined the Hilbert space within which the walk will be conducted in, we can now define operators within our space that will dictate how the Q.W. will proceed. We first define the 'coin flip' operator $C \in \mathcal{H}_C$. There are several choices for C, details of which can be found here [5]. As detailed in [5], for walks on a line, if we restrict ourselves to choosing an unbiased coin with real coefficients the Hadamard coin

$$C = \frac{1}{\sqrt{2}} \left[|\uparrow\rangle \langle\uparrow| + |\uparrow\rangle \langle\downarrow| + |\downarrow\rangle \langle\uparrow| - |\downarrow\rangle \langle\downarrow| \right] \tag{4}$$
$$= \frac{1}{\sqrt{2}} \left[(|\uparrow\rangle + |\downarrow\rangle) \langle\uparrow| + (|\uparrow\rangle - |\downarrow\rangle) \langle\downarrow| \right] \tag{5}$$

is the only choice of coin available. Equation (5) makes obvious the action of C, if the coin state is $|\uparrow\rangle$ then it becomes an equal superposition of $|\uparrow\rangle + |\downarrow\rangle$, if the coin state is in $|\downarrow\rangle$ then we get an equal superposition of $|\uparrow\rangle - |\downarrow\rangle$. These two equal superpositions are often denoted as $|+\rangle$ and $|-\rangle$ respectively.
We then define our shift operator $S \in \mathcal{H}$ which allows the position of our walker to change, dependent on the state of the coin.

$$S = \sum_k |\uparrow\rangle \langle\uparrow| \otimes |k+1\rangle \langle k| + |\downarrow\rangle \langle\downarrow| \otimes |k-1\rangle \langle k| \tag{6}$$

Again, this representation of S makes manifest its effect on our walker. If the coin is in the $|\uparrow\rangle$, then we take a step in the +1 direction, if in the $|\downarrow\rangle$ then we take a step in the -1 direction. The probability distribution of such a walk is plotted in Fig [FIG], where the initial coin state is $|\downarrow\rangle$, and is compared to a classical random walk. Whilst this highlights the faster spreading of the Q.W. away from the origin, there are not as many interesting properties of this variant of Q.W. compared to walks on other graphs.

2.3 N-Cycle

Having now introduced the Q.W. on a line, we can easily now discuss the Q.W. on the N-Cycle, a graph obtained by taking a 1-D lattice of size $N+1$ and attaching the end vertices of degree 1 together. To study Q.W.s on N-Cycles we are able to use an identical formulation of C and an almost identical formulation of S as used for the Q.W. on a 1-D lattice, the only change for S being the summation range of k.

$$S = \sum_{k=0}^{N-1} |\uparrow\rangle \langle\uparrow| \otimes |k+1\rangle \langle k| + |\downarrow\rangle \langle\downarrow| \otimes |k-1\rangle \langle k| \,. \tag{7}$$

There has been much interesting analysis concerning quantum walks on N-Cycles, but we will focus here on a property that they possess known as *perfect state transfer*, when the initial state of the walker at the origin of the walk is reproduced completely at another vertex k on the graph after T steps in the quantum walk.

$$|\Psi(0)\rangle = |\psi(0)\rangle_C \otimes |0\rangle_W \tag{8}$$

$$|\Psi(T)\rangle = |\psi(0)\rangle_C \otimes |k\rangle_W \tag{9}$$

This behaviour only occurs for certain choices of C and N, see Table 1 from [8] for a clear table outlining the conditions on C to achieve perfect state transfer for a given N.

2.4 Hypercube

We now will consider the quantum walk on a hypercube of dimension N. What sets this graph apart from the two previously considered in this report is that the coin space is no longer of dimension 2, but of dimension N. Every vertex has N edges connected to it, so we need N linearly independent states in our coin space to determine which edge should be traversed in the walk. Generalised to N dimensions, our Hadamard coin in fact becomes a Fourier coin

3 Entanglement Transfer

3.1 Transfer using Two Walks

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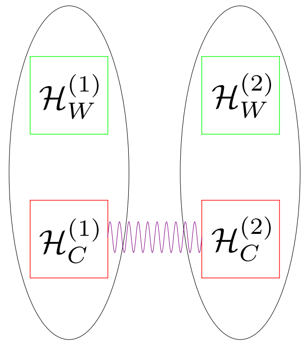

Figure 1: The initial prepared state has entanglement solely between the two coin subspaces.

3.2 Transfer using Multiple Coins

3.2.1 Three Coins

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3.2.2 Parrondo Sequences

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4 Conclusions

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Acknowledgements

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Appendices

A Example long derivation