Transferring Entanglement Between Different Dimensions

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Submitted: April 21, 2021

Abstract

Entangled qudit states, which share higher dimensional entanglement, enable quantum computers to unlock greater advantages in the computations that they are able to perform. As such, the generation of these entangled states is a worthy problem to consider as quantum computing technology becomes more and more advanced. Many methods of generating Bell pair entangled qubits have been proposed and realised, but schemes for qudit pairs of arbitrary dimension are not as well-researched. A protocol utilising quantum walk dynamics has been proposed to repeatedly transfer bipartite entanglement from Bell pairs to be accumulated in qudit pairs. Analysis of this protocol shows that this transfer is not optimal when using true quantum walk dynamics. Due to its non-unitary nature, it is also not easily reversed to retrieve entanglement from higher dimensional entangled states. Post-selection is also needed, where states are sometimes discarded and the entire protocol has to be repeated. An adapted version of the protocol, given as a circuit of gates conforming to ancilla-based quantum computing constraints rather than quantum walk ones, is proposed as an alternative which is able to achieve optimal entanglement transfer. It allows for entanglement to be transferred freely between qubit and qudit pairs as it is fully unitary and hence reversible. Furthermore, it is shown that with the addition of a single gate, the circuit can be used to turn qudits into a form of quantum random access memory.

Contents

1 Introduction

Quantum computing is a field of research that has gained increasing attention over the course of the 21st Century. Loosely speaking, it is a field that seeks to examine how quantum phenomena can be exploited to solve computation problems. Quantum computers are of particular interest due to the class of problems they can solve faster than possible classically. For example, Shor's factorising algorithm [1] gives a recipe for a quantum computer to perform integer factorisation. This is of particular significance as there are cryptographic protocols that secure information by relying on the difficulty of factorising large numbers which are the products of two primes, a difficulty that could be bypassed relatively easily with a powerful enough quantum computer. Furthermore, as classical computers are made more and more computationally powerful, their physical power requirements go up also. Quantum computers offer a potential avenue to performing computations that require this additional classical computing power at a reduced energy cost.

Many schemes for quantum computation utilise qubits [2], the quantum analogue of the bit, which is the unit of information in classical computing. The main difference between the bit and the qubit, is that the bit can only exist in one of two states (0 or 1) at any one time, whereas the qubit, due to its quantum nature, can exist in superposition of both the possible states. Other models utilising qudits, which exist in superposition of d states rather than just two, have been proposed [3] as they can unlock further advantages at the cost of being more complex to implement physically.

In addition to superposition, another well known quantum phenomena that is often taken advantage of in quantum computing is *entanglement*, correlations present in quantum systems that are far stronger than possible to find in classical systems. There are a variety of protocols that require the presence of entangled qubits in order to achieve results not possible with classical computers, for example superdense coding [4], quantum key distribution [5] and quantum teleportation [6]. Higher dimensional entanglement, which in this report is taken to be entanglement between qudits, further enhances the power of quantum algorithms such as superdense coding [7]. As such the ability to possess and manipulate entangled states in higher dimensions has further benefits but again comes with its own challenges.

Quantum walks (QWs) are powerful tools in the landscape of quantum computing. Much like their classical analogue in the classical random walk, they exhibit many properties that are desirable for computations and are an extremely useful building block for many algorithms designed for quantum computers [8]. Specific research interest into the quantum variant stems from their very significant divergences from the classical random walk, including different spreading speeds and ability to traverse multiple paths at once. Their power is such that QWs can simulate any quantum computation and therefore are a model for universal quantum computing [9]. There is also evidence of robust performance even when the quantum computer is not perfectly isolated from its environment, and in certain situations it has been shown that decoherences due to interactions with the environment is beneficial for a given computation [10]. Quantum walks are divided into two categories, *discrete* and *continuous* time, these labels describing the nature of the evolution of the walker as the quantum walk progresses. Continuous time QWs have been shown to solve a wide range of problems in a

number of different settings, in some cases exponentially faster than a classical computer is able to [11], but in this report it will be discrete time QWs that are of interest.

An alternative universal quantum computing model is *ancilla-based quantum computing* (AQC), which is a model that aims to reconcile the conflicting demands in building a quantum computer. Qubits need to be well isolated to prevent decoherence, but doing so also makes them harder to interact with. AQC resolves this by utilising two different kinds of qubits, ones which are well isolated as a main register, and an ancilla register of qubits that are easier to manipulate but whose states decohere faster. By delocalising information across the ancilla and main register, computations can be performed on the ancilla register before the information is relocalised on the long-lived main register qubits, and the ancilla qubits can be reset or replaced to be used for further computations.

A potential solution to the demanding task of generating higher dimensional entanglement has been proposed by Giordani et al. [12]. This proposal uses the dynamics of discrete QWs to transfer lower dimensional entanglement between qubits, which is simpler to generate, into the high dimensional qudits. Whilst this scheme can be used to some moderate degree of success, it has issues in transferring entanglement optimally and cannot be easily reversed. However, the underlying principles of the QW protocol can be adapted to create a scheme based on AQC, which helps overcome some of these issues with the QW based protocol.

In this report, a primer on quantum computing, entanglement and two models of quantum computing, quantum walks and ancilla-based quantum computing, is given in section 2. Following this, section 3 focusses on the protocol that uses discrete QW dynamics to facilitate the transfer of entanglement, in particular analysing its efficiency in achieving the aim of entanglement transfer. The AQC scheme for entanglement transfer is presented in section 4. Further uses of the AQC scheme beyond the transfer of entanglement are presented in section 5. Finally, a discussion comparing the two schemes, potential directions for further work and concluding remarks are presented in section 6.

2 Background

2.1 Quantum Computation

2.1.1 Qubits

A qubit can be described by a state belonging to a Hilbert space $\mathcal{H}$ of dimension two. A standard basis for $\mathcal{H}$ is the *computational basis*, labelled by $|0\rangle$ and $|1\rangle$. Since a qubit is a quantum system, it can exist in a superposition of the basis states,

$$|\psi\rangle = \cos(\theta)\,|0\rangle + e^{i\phi}\sin\theta\,|1\rangle\,, \tag{1}$$

where the coefficients are the probability amplitudes of each state and via the trigonometric identity give a total probablity of 1, as expected. Naturally we can also use other bases for describing our qubit state. A common alternative is the basis $\{|+\rangle\,, |-\rangle\}$. This basis is best known as the *Hadamard basis*, as $|+\rangle$ and $|-\rangle$ are equal to $H\,|0\rangle$ and $H\,|1\rangle$, where H is the

Hadamard transform given by

$$|+\rangle = H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \tag{2}$$

$$|-\rangle = H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{3}$$

Qubits can also be written out in vector form, and the natural choice for this is to assign the computational basis states as

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \ |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}. \tag{4}$$

In this basis the Hadamard transform is given by

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}. \tag{5}$$

2.1.2 Qudits

In quantum computing, or indeed classical computing, there is no physical constraint that necessitates the use of two level qubits or bits. In classical computing, the bit can be generalised to a unit that takes on one of d states, known as the *dit*. Similarly, the quantum dit is the *qudit*, which can be in superposition of up to d states. Many aspects of quantum computing specific to qubits generalise nicely to qudits. The computational basis is now extended from $\{|0\rangle, |1\rangle\}$ to $\{|j\rangle\}_{j=0}^{d-1}$. To distinguish kets belonging to Hilbert spaces of different dimension, the notation $|\cdot\rangle_d$ will be used to denote a state belonging to $\mathcal{H}_d$, a Hilbert space of dimension d. For compactness, operators will not have their dimension explicitly labelled, since they belong to the same Hilbert space as the state they are acting on. Similar to qubits, there are many different bases that can be used to describe our qudit states. The d-dimensional extension of the Hadamard transform is the *quantum Fourier transform*, given by

$$F|j\rangle_d = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}\omega^{jk}|k\rangle_d, \tag{6}$$

where $\omega = e^{i\frac{2\pi}{d}}$, i.e. it is the d^{th} root of unity. From this it can be seen that for $d = 2$, $F = H$. Therefore the Hadamard basis $\{|+\rangle, |-\rangle\}$ can be generalised to the set of states $\{|+_j\rangle\}_{j=0}^{d-1}$, where

$$|+_j\rangle_d = F|j\rangle_d. \tag{7}$$

Again, qudits can also be represented by vectors with d entries, with the computational basis states again assigned as

$$|0\rangle = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \ |1\rangle = \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \ldots, \ |d-1\rangle = \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}. \tag{8}$$

In this basis, the Fourier transform is given by the $d \times d$ matrix

$$F = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{d-1} & \cdots & \omega^{(d-1)^2} \end{pmatrix}, \tag{9}$$

and the associated Fourier basis states are found by applying the matrix representation of F to the computational basis state vectors.

2.1.3 Collections of Qudits

Qudits considered in isolation are not overly useful for quantum computations and in general a collection of qudits is needed for a given algorithm. The Hilbert space of a collection of n qudits, which are not necessarily all of the same dimension, is found by taking the tensor product, $\otimes$, of the Hilbert spaces of each of the individual qudits

$$\mathcal{H} = \mathcal{H}_{d_0} \otimes \mathcal{H}_{d_1} \otimes \cdots \otimes \mathcal{H}_{d_{n-1}}. \tag{10}$$

The computational basis for the combined Hilbert space is found by taking all possible tensor product combinations of the basis states for each constituent Hilbert space. The state

$$|0\rangle_{d_0} \otimes |0\rangle_{d_1} \otimes \cdots \otimes |0\rangle_{d_{n-1}} \tag{11}$$

is an example of such a basis state. Notationally, this state can be further simplified by the omission of the tensor product symbol,

$$|0\rangle_{d_0} |0\rangle_{d_1} \ldots |0\rangle_{d_{n-1}} . \tag{12}$$

If the kets are of the same dimension then this can be further compacted by contracting them,

$$|0\rangle_d |0\rangle_d \ldots |0\rangle_d = |00\ldots0\rangle_d . \tag{13}$$

Since tensor products distribute over sums, when combining states in superposition they must also be distributed over. For example with two qudits,

$$(|0\rangle_d + |1\rangle_d) \otimes |2\rangle_d = (|0\rangle_d + |1\rangle_d) |2\rangle_d \tag{14}$$

$$= |02\rangle_d + |12\rangle_d . \tag{15}$$

Recall also from the properties of the tensor product that it is not commutative, therefore care must be taken in ensuring that the order of qudits is preserved throughout, e.g.

$$|01\rangle_2 \neq |10\rangle_2 . \tag{16}$$

In this report all three notational methods are used, with clarity given due preference over compactness. The subscripts are often dropped when the meaning is clear.

2.1.4 Operators

There are a wide variety of other operators beyond those introduced above that equate to a change of bases. One of the most common is the (Pauli) X, or NOT, operator. As the second name implies, in the qubit setting this acts in the same way as the classical NOT operator:

$$|0\rangle \mapsto |1\rangle\,, \tag{17}$$
$$|1\rangle \mapsto |0\rangle\,. \tag{18}$$

More generally in the qudit setting,

$$X\,|j\rangle_d = |(j+1) \bmod d\rangle_d\,. \tag{19}$$

However, there are other gates that do not have a classical analogue such as the (Pauli) Z gate. In the qubit setting Z acts as the identity on $|0\rangle$ and applies a relative phase to $|1\rangle \mapsto -|1\rangle$. Again this can be generalised to the qudit setting:

$$Z\,|j\rangle_d = \omega^j\,|j\rangle_d\,, \tag{20}$$

where ω is again the d^{th} root of unity. Both of these operators are sometimes referred to as Pauli operators as their matrix representations in the computational basis are given by the Pauli matrices σ_x and σ_z. There is also the Y operator corresponding to the Pauli σ_y matrix but this is often omitted from consideration as

$$ZX = iY. \tag{21}$$

Hence, when designing a set of operators for a quantum computer, there is no theoretical need for Y as it already exists up to a global phase, which is essentially irrelevant. The final operation to be introduced in this section is the C-U, or controlled-U operator, where U is any unitary operation. This operator is distinct from the others previously introduced in that it acts on two qudits rather than just one, although it only directly alters the state of one of the two qudits. In the qubit setting, one qubit is designated as the control qubit and the other the target. If the control qubit is in the state $|0\rangle$, then nothing happens to the target qubit. If the control qubit is in the state $|1\rangle$, then the operator U is applied to the target qubit. Written mathematically, where the left-hand qubit is the control and the right hand qubit is the target in some arbitrary qubit state $|\psi\rangle$,

$$C\text{-}U\,(|0\rangle \otimes |\psi\rangle) = |0\rangle \otimes |\psi\rangle\,, \tag{22}$$
$$C\text{-}U\,(|1\rangle \otimes |\psi\rangle) = |1\rangle \otimes U\,|\psi\rangle\,. \tag{23}$$

To extend this to the qudit setting, first note that equations 22-23 can be compactly written as

$$C\text{-}U\,(|j\rangle \otimes |\psi\rangle) = |j\rangle \otimes U^j\,|\psi\rangle\,. \tag{24}$$

This equation now is well-defined for qudits, where instead of just taking values 0 or 1, j can now take values from $\{0, 1, \ldots, d-1\}$. In fact, equation 24 is well-defined even when

the dimensions of the control and target are not equal, provided that the action of U on the target is well-defined. Explicitly,

$$C\text{-}U\left(|j\rangle_d \otimes |\psi\rangle_{d'}\right) = |j\rangle_d \otimes U^j |\psi\rangle_{d'}\,. \tag{25}$$

In this report, unless otherwise stated, it is assumed that control qudits are always on the left and target qubits on the right.

It is sometimes necessary to have operators that act on single qudits in a collection of n qudits. These can be expressed by taking the tensor product with the identity acting on the other qudits in the collection. For example, the operator that acts with an X on the qudit of index 1 in the state $|\psi_0\rangle \otimes |\psi_1\rangle \otimes |\psi_2\rangle$ is given by

$$I \otimes X \otimes I. \tag{26}$$

2.1.5 Circuits and Gates

When describing a circuit that implements a series of operators on our qudits, it is often much clearer and simpler to use a circuit diagram. Straight lines in the circuits represent qudits. Operations on qudits are represented by gates placed on the straight lines corresponding to the qudits we wish to act on. A table of common gates relevant for this report is given in table 1.

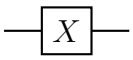
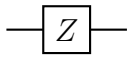
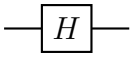
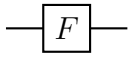
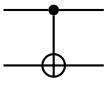
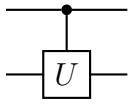
Operator	Gate
(Pauli) X	
(Pauli) Z	
Hadamard	
QFT	
C-X	
C-U	

Table 1: Table of operators and their gate representations in quantum circuits. The C-X gate can also be written in general C-U form.

When reading a circuit, time flows from left to right and a gate must finish its operation on a qudit before the next gate on the same qudit is enacted. Figure 1 is an example of a two qubit circuit implementing the following series of operations

$$C\text{-}X(Z \otimes I)(X \otimes I)\left(|0\rangle_2 \otimes |\psi\rangle_2\right). \tag{27}$$

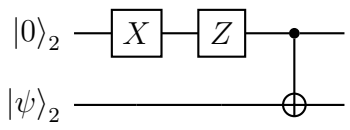


Figure 1: An example of a circuit representing the expression given in equation 27.

2.2 Entanglement

Entanglement is a property of quantum systems that is the source of many of the remarkable results displayed by quantum algorithms. Loosely, it is the presence of strong correlations in a quantum system, much stronger than possible classically. Strictly speaking, an entangled state is one that cannot be expressed as a separable state. A separable state is one that can be written as a tensor product of states in the Hilbert subspaces of a multipartite system. All of the basis states formed in section 2.1.3 are examples of separable states, as they are formed by taking the tensor product of basis states in each of the constituent Hilbert spaces. An example of an entangled state would be this state between two qubits,

$$\frac{1}{\sqrt{2}}(|00\rangle_2 + |11\rangle_2). \tag{28}$$

It is impossible to write this state in the form $(a\,|0\rangle_2 + b\,|1\rangle_2)\otimes(c\,|0\rangle_2 + d\,|1\rangle_2)$, hence it is not separable. This is one of the four *Bell states* or *Bell pairs* which are the *maximally entangled* two qubit states.

Definition 2.1 (Maximally Entangled State)**.** *A state $|\psi\rangle$ describing a bipartite system $\mathcal{H} = \mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$ is maximally entangled if the reduced density matrix representing $|\psi\rangle$ in either subspace is maximally mixed, that is, it is directly proportional to the identity.*

Taking the Bell state given in equation 28, its density matrix is given by

$$\rho = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right) \tag{29}$$

$$= \frac{1}{2}\left(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\right). \tag{30}$$

The reduced density matrix in $\mathcal{H}^{(A)}$ is therefore

$$\rho^{(A)} = \mathrm{Tr}_B(\rho) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{2}I, \tag{31}$$

where Tr_B denotes the partial trace over subspace $\mathcal{H}^{(B)}$.

2.2.1 Higher Dimensional Entanglement

Generalising entanglement to higher dimensions in the bipartite setting is done by having entanglement between qudits as opposed to qubits. There is no requirement for the dimensions of the entangled qudits to be matching. For example,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_2 \otimes |0\rangle_3 + |1\rangle_2 \otimes |1\rangle_3\right), \tag{32}$$

is a entangled state between a qubit and a qutrit ($d = 3$). Whilst higher dimensional entanglement can also refer to entanglement between multipartite systems, such as GHZ states or W states [13], in this report only bipartite entanglement is of concern.

Higher dimensional entanglement is useful as it contains stronger correlations than present in a Bell pair and can be utilised to further enhance the power of quantum computations. For example, the superdense coding protocol can transmit two classical bits of information for every Bell pair available. If higher dimensional entangled states are used instead, then more bits of classical information can be transmitted per entangled state [7], making the coding even denser. However, whilst higher dimensional entanglement unlocks further benefits, it comes with further challenges in its creation. Many schemes for generating entangled Bell pairs exist (see for example [14–16]) which have been implemented experimentally, but schemes generating higher dimensional entanglement in a qudit setting have not been as greatly researched. If it is possible to transfer and accumulate entanglement from Bell pairs into qudit pairs, then this does away with the need for different entanglement generation schemes for each dimension of qudit, and instead the same scheme can be used to generate Bell pairs and transfer this entanglement to the qudit pairs. In a practical setting this is also desirable, as only a source of Bell pairs is needed to generate entangled qudits of any dimension, saving on the space required in a quantum computer should different dimensions of entanglement be required for different computations. Furthermore, if the entanglement can be freely transferred between dimensions, that is, not just accumulated in but also retrieved from higher dimensional entangled states, then this would further enhance the versatility of quantum computers, giving them a store of entanglement that can be drawn from when required.

2.2.2 Measuring Entanglement

In order to analyse the efficiency of an entanglement transfer protocol, it is useful to have a method of measuring entanglement. Numerous methods of doing so have been proposed and are used, and a full mathematical discussion on entanglement measures is given in [17]. In this report, the measure of entanglement used is the *logarithmic negativity*, presented by Vidal [18].

Definition 2.2 (Logarithmic Negativity)**.** *Let ρ be a density matrix describing a bipartite quantum system $\mathcal{H} = \mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$. The logarithmic negativity, $E_{\mathcal{N}}(\rho)$, of ρ is defined as*

$$E_{\mathcal{N}}(\rho) = \log_2 \left\| \rho^{\Gamma_A} \right\|_1 . \tag{33}$$

where X^{Γ_A} denotes the partial transpose of X with respect to the subsystem $\mathcal{H}^{(A)}$ and $\|X\|_1 = Tr(\sqrt{X^\dagger X})$ denotes the trace norm of X. Here the principal square root is taken.

The logarithmic negativity is also referred to as the log negativity. It is an appropriate choice for the purpose of this report as it is an entanglement measure that is computable for mixed as well as pure states, and satisfies (strong) additivity

$$E_{\mathcal{N}}(\rho \otimes \sigma) = E_{\mathcal{N}}(\rho) + E_{\mathcal{N}}(\sigma), \tag{34}$$

where ρ, σ are the density matrices describing two bipartite systems. A proof for this is given in [18]. Another useful property of log negativity is that it is easy to compute, particularly in

the case that a state is maximally entangled in a Hilbert space of some dimension $d \times d$ (not necessarily the dimension of the full Hilbert space in which the bipartite system resides).

Claim 2.3. *A maximally entangled state in $d \times d$ dimensions has a log negativity of* $\log_2 d$.

Proof. First, note that for a Hilbert space of dimension $d \times d$, the state

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle \otimes |j\rangle \in \mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)} \tag{35}$$

is maximally entangled, since its reduced density matrix in either subspace is maximally mixed. So to calculate the log negativity of any maximally entangled state it is sufficient to calculate the log negativity of the density matrix ρ associated with $|\psi\rangle$. Since $|\psi\rangle$ is a pure state,

$$\rho = |\psi\rangle \langle\psi| \tag{36}$$

$$= \frac{1}{d} \sum_{j,k} |j\rangle \langle k| \otimes |j\rangle \langle k| . \tag{37}$$

To calculate $E_{\mathcal{N}}(\rho)$, first find the partial transpose on $\mathcal{H}^{(A)}$ (the choice is not significant, the partial transpose on $\mathcal{H}^{(B)}$ yields the same result as the subspaces are equally entangled with one another),

$$\rho^{\Gamma_A} = \frac{1}{d} \sum_{j,k} |k\rangle \langle j| \otimes |j\rangle \langle k| . \tag{38}$$

From this the trace norm $\|X\|_1 = \mathrm{Tr}\left(\sqrt{XX^\dagger}\right)$ can be determined:

$$\rho^{\Gamma_A} \left(\rho^{\Gamma_A}\right)^\dagger = \frac{1}{d^2} \sum_{j,k} |k\rangle \langle k| \otimes |j\rangle \langle j| \tag{39}$$

$$= \frac{1}{d^2} I_d \otimes I_d \tag{40}$$

$$= \frac{1}{d^2} I_{d^2}. \tag{41}$$

$$\implies \|\rho^{\Gamma_A}\|_1 = \frac{1}{d} \mathrm{Tr}\left(I_{d^2}\right) \tag{42}$$

$$= \frac{1}{d} \cdot d^2 \tag{43}$$

$$= d. \tag{44}$$

Therefore, this gives

$$E_{\mathcal{N}}(\rho) = \log_2 \left\|\rho^{\Gamma_A}\right\|_1 \tag{45}$$

$$= \log_2 d. \tag{46}$$

□

This combined with the strong additivity condition is what makes the choice of log negativity an appropriate one as it allows for a direct comparison of entanglement in a state before and after transfer. A single Bell pair resides in a Hilbert space of dimension 2×2 and has log negativity 1. Therefore, a collection of n Bell pairs exist in a Hilbert space of 2^{2n} and have a log negativity of n (additivity). The minimum dimension of the Hilbert space that this entanglement is to be transferred into is also 2^{2n} (if it is bigger then it can always be reduced to a space of that dimension). Therefore the qudit pair resides in a Hilbert space of $d \times d = 2^{2n} = 2^n \times 2^n$. Using the log negativity, a maximally entangled state in a Hilbert space of dimension $2^n \times 2^n$ has a log negativity of n (claim 2.3). Therefore the maximal log negativity in the qudit pair is equal to the sum of the log negativity of the Bell states that entanglement is transferred from. This means that if three Bell pairs have their entanglement transferred, the maximal possible log negativity of the resulting qudit pair is 3. If this upper bound is achieved then optimal transfer has occurred. If it is not achieved, then entanglement has been lost in the transfer. This might seem quite an obvious statement, but it is not true for all entanglement measures. For example, negativity, which is an alternative entanglement measure closely related to log negativity, cannot be used to make direct comparisons in the same way.

2.3 Quantum Random Walks

Quantum random walks, as touched on in section 1, are a model for universal quantum computation, meaning that any computation that is possible for a quantum computer can be simulated by a QW. Like much of quantum computing, QWs can be motivated from a classical predecessor, the *classical random walk*. Here the classical random walk is used specifically to motivate the discrete QW since the protocol presented in section 3 makes use of such a QW. Hence, all future references to QWs are assumed to be of the discrete variant.

2.3.1 Motivation from the Classical Random Walk

In the classical random walk, a walker is constrained to moving up and down a discrete number line, starting their walk at the origin. To determine whether to take a step to the left (-1) or the right (+1), the walker flips an unbiased coin, moving to the right if the coin lands on heads and to the left if the coin lands on tails. This process of flipping a coin and taking a step can be repeated until a desired stopping point is reached, e.g. after a given number of coin flips. The walk can also continue on forever and in this limit, the walker will reach every point on the number line. In this report, however, only finite length walks are considered. Without much thought it can be concluded that after T coin flips the walker can be at most $\pm T$ steps away from the origin, corresponding to the case where every step taken is in the same direction. The probability distribution describing the probability of the walker being in any given position away from the origin is given by a binomial distribution, an example of which is shown in orange in figure 2 for 100 coin flips.

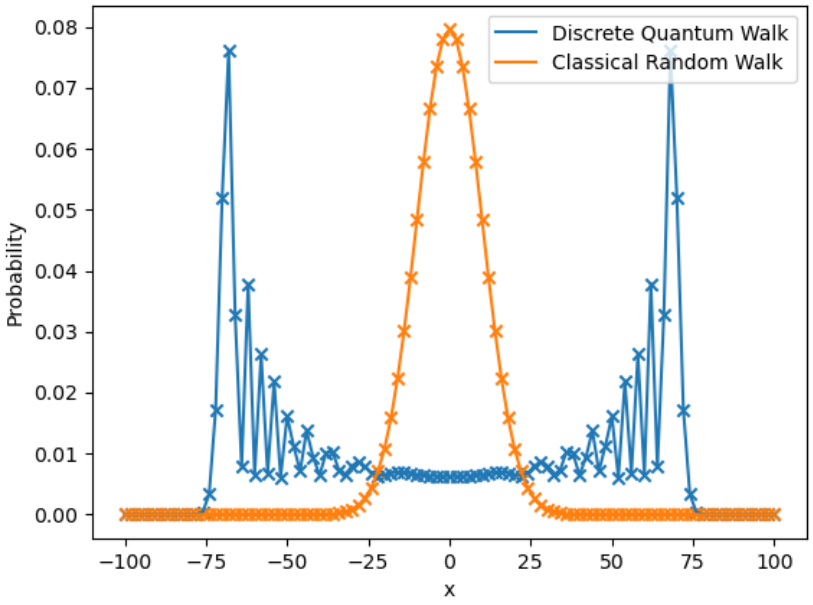


Figure 2: The probability distributions of a classical and Hadamard-coined quantum walk on a line, both originating at the origin. The quantum walk coin is initially in the $\frac{1}{\sqrt{2}}(|\uparrow\rangle + i\,|\downarrow\rangle)$ state. Odd points have been omitted as they all have zero probability.

2.3.2 Quantum Walk on a Line

Having described the classical random walk, it can now be 'quantised' into the *quantum random walk*. The QW is considered as a composite system of the coin, belonging to the Hilbert space $\mathcal{H}_C$, and walker $\mathcal{H}_W$,

$$\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_W. \tag{47}$$

Note that this use of subscript does not depart from the earlier usage to indicate the dimension of the Hilbert space, since $\mathcal{H}_C$ is spanned by two states (the 'heads' state and the 'tails' state), and $\mathcal{H}_W$ is spanned by all the points on the number line to be walked upon, so can be thought of as an unspecified d-dimensional space. Therefore the labels C and W, also serve to indicate the dimension of the labelled space.

To aid distinguishability between coin states and position states, they are labelled by

$$\langle \mathcal{H}_C \rangle = \{|\uparrow\rangle, |\downarrow\rangle\}, \tag{48}$$
$$\langle \mathcal{H}_W \rangle = \{|k\rangle \,|\, -T \leq k \leq +T, k \in \mathbb{Z}\}, \tag{49}$$

where $\langle U \rangle$ denotes a set of states which span U, and T again is the number of 'coin flips'. Therefore, the states $|\uparrow\rangle, |\downarrow\rangle$ take the place of heads and tails on our quantum 'coin'. Note in particular that there is no constraint placed on the dimensionality of $\mathcal{H}_W$, and therefore the walker can be taken to be a qudit, whereas $\dim \mathcal{H}_C = 2$ and the coin is taken to be a qubit. Now that the Hilbert space of the system has been contructed, operators are required in order

for the system to evolve. In the classical random walk, the first operation needed was a coin flip so in similar fashion the quantum walk too needs a coin flip operator, $C \in \mathcal{H}_C$ (not to be confused with the C-U controlled-unitary operator). There is a continuum of choices for C, and indeed QWs can be constructed on a whole host of different graphs [19]. For the walk on a line, a general coin of bias ρ and with real coefficients is given by the expression

$$C = \begin{pmatrix} \sqrt{\rho} & \sqrt{1-\rho} \\ \sqrt{1-\rho} & \sqrt{\rho} \end{pmatrix}. \tag{50}$$

Therefore, constraining the coin to be unbiased ($\rho = \frac{1}{2}$) and have real coefficients leaves the Hadamard coin as the only choice of coin available. This takes on the same form as the Hadamard transform defined in section 2.1.5 and is given here via an alternative expression,

$$H = \frac{1}{\sqrt{2}}\Big[|\uparrow\rangle \langle\uparrow| + |\uparrow\rangle \langle\downarrow| + |\downarrow\rangle \langle\uparrow| - |\downarrow\rangle \langle\downarrow| \Big] \tag{51}$$

$$= \frac{1}{\sqrt{2}}\Big[\Big(|\uparrow\rangle + |\downarrow\rangle \Big) \langle\uparrow| + \Big(|\uparrow\rangle - |\downarrow\rangle \Big) \langle\downarrow| \Big], \tag{52}$$

where $|0\rangle \to |\uparrow\rangle$ and $|1\rangle \to |\downarrow\rangle$. As before, if the coin state is $|\uparrow\rangle$ then it becomes an equal in phase superposition of $|\uparrow\rangle + |\downarrow\rangle$, if the coin state is in $|\downarrow\rangle$ then it becomes an equal antiphase superposition of $|\uparrow\rangle - |\downarrow\rangle$ and this is made particularly clear by equation 52. Following this, a shift operator $S \in \mathcal{H}$ is needed to move the walker dependent on the state of the coin.

$$S = \sum_k |\uparrow\rangle \langle\uparrow| \otimes |k+1\rangle \langle k| + |\downarrow\rangle \langle\downarrow| \otimes |k-1\rangle \langle k|. \tag{53}$$

Again, representing S in this way makes manifest its effect on the walker. If the coin state is $|\uparrow\rangle$, the walker takes a step in the +1 direction, if it is $|\downarrow\rangle$ then a step in the -1 direction is taken. The probability distribution of such a walk is plotted in blue in figure 2, where the initial coin state is $|\downarrow\rangle$, and is compared to a classical random walk.

Whilst the above choice of S is the most common on the number line, there are alternatives such as

$$\tilde{S} = \sum_k |\uparrow\rangle \langle\uparrow| \otimes |k\rangle \langle k| + |\downarrow\rangle \langle\downarrow| \otimes |k+1\rangle \langle k|. \tag{54}$$

The subtle difference between $\tilde{S}$ and S is that $\tilde{S}$ can only move in the +1 direction of the number line and has no 'left moving' part. This means that $\tilde{S}$ is restricted to the non-negative integers and can occupy all $|x\rangle$ for $0 \leq x \leq T$, where T is the number of time steps in our QW. This is compared with S which only occupies even or odd numbered positions dependent on T. In fact $\tilde{S}$ is an operator that was introduced earlier in section 2.1.4,

$$\tilde{S} = C\text{-}X. \tag{55}$$

2.4 Ancilla-Based Quantum Computing

The quantum walk model of quantum computation is but one of many universal quantum computing models. Another model is known as *ancilla-based quantum computing* (AQC), which, in particular, aims to resolve two conflicting demands when building quantum computers: qubits must be well isolated to prevent decoherence, yet it must also be possible for them to interact with one another in certain settings, such as implementing the C-X operator. However, a qubit cannot distinguish between unwanted and wanted interactions, resulting in the need for a balancing act that adequately fulfils both of these aims.

AQC aims to resolve this conflict by using additional qubits, known as ancilla qubits, which mediate interactions between the main register qubits. Using this model, it is possible to implement two complementary registers of qubits. The main register can be designed to be strongly isolated from interactions, bar a limited number of interactions with the ancilla register, whose qubits can be easily manipulated but decohere much faster. Quantum computations can be performed by *delocalising* quantum information from the main register across both the main and the ancilla registers. After performing a computation on the ancilla qubit, the quantum information is then *relocalised* back into the main register. The ancilla can then be reset to the initial state and used again for other computations. A simple example of a circuit designed for ancilla-based quantum computing is shown in figure 3.

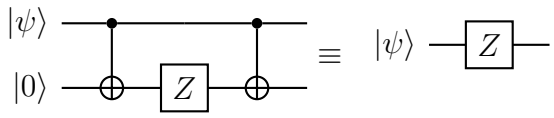


Figure 3: Two circuits that implement a Z gate acting on an arbitrary qubit $|\psi\rangle$. The circuit on the left does so indirectly via interactions with an ancilla qubit.

The circuit on the left-hand side of figure 3 does not directly implement any unitary transformations on an arbitrary qubit state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, but instead indirectly acts on $|\psi\rangle$ via the ancilla qubit, to which the quantum information of $|\psi\rangle$ is shared to via a C-X gate. Acting a Z gate on the ancilla and relocalising the quantum information with another C-X gate 'passes on' the effects of the Z gate. Note however that this would not directly work to implement an X gate on $|\psi\rangle$ if the Z gate were directly switched with an X gate. This is because only amplitudes can be delocalised and relocalised; computational basis states cannot be directly shifted by ancilla interactions. If however, $|\psi\rangle$ is mapped to the Hadamard basis, where the basis states differ by their amplitudes, before the Z circuit and mapped back afterwards, as in figure 4, then it is possible to effect an X operator in the AQC setting. These two circuits highlight the somewhat counterintuitive nature of AQC. A controlled operation is utilised, where the qudit is the control and the qubit is the target, yet it is the qudit state that is changed and the qubit is left 'untouched'.

This model of quantum computing is not exclusive to the qubit computing model and can be extended to qudits simply by using the general forms of operators, as described in section 2.1.4. A further advantage of AQC is that it is particularly well-placed to use registers of qudits with mismatched dimensions. A full introduction to utilising AQC in such a setting

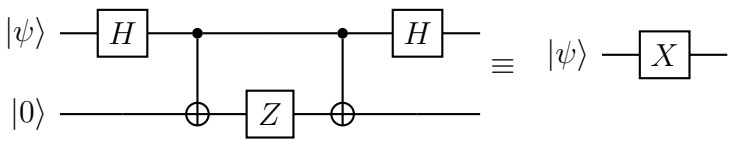


Figure 4: Implementing an X gate on $|\psi\rangle$ in the AQC setting by first mapping $|\psi\rangle$ to the Hadamard basis before and after the Z circuit given in figure 3.

is given by Proctor [20]. Due to this hybrid computing capability, it is an extremely suitable model of quantum computation for entanglement transfer between dimensions.

3 Entanglement Transfer using Quantum Walks

Here the protocol presented by Giordani et al. [12] is summarised. The aim of the protocol is to generate higher dimensional entanglement via the use of quantum walks. The imagined setup of this protocol is that there are two labs, A and B, which have a shared source of entangled qubits but are otherwise spatially separated and cannot interact with one another.

In this section the following notation is employed:

- $|\psi\rangle_J$ is a state belonging to the subspace $\mathcal{H}_J = \mathcal{H}_J^{(A)} \otimes \mathcal{H}_J^{(B)}, J \in \{C, W\}$. It is a state that describes the combined state of the coins or walkers.
- $|\psi\rangle^{(K)}$ is a state belonging to the subspace $\mathcal{H}^{(K)} = \mathcal{H}_C^{(K)} \otimes \mathcal{H}_W^{(K)}, K \in \{A, B\}$. It is a state that describes the combined state of one of the quantum walks.
- $|\psi\rangle_J^{(K)}$ is a state belonging to the subspace $\mathcal{H}_J^{(K)}$.

In this mathematical framework the overall Hilbert space comprises of two quantum walk subspaces,

$$\mathcal{H} = \mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)} \tag{56}$$

$$= \mathcal{H}_C^{(A)} \otimes \mathcal{H}_W^{(A)} \otimes \mathcal{H}_C^{(B)} \otimes \mathcal{H}_W^{(B)}. \tag{57}$$

Due to the difficulty in writing entangled states when the entangled spaces are not adjacent, the subspaces will instead be taken to be in the order

$$\mathcal{H} = \mathcal{H}_W^{(A)} \otimes \mathcal{H}_W^{(B)} \otimes \mathcal{H}_C^{(A)} \otimes \mathcal{H}_C^{(B)}, \tag{58}$$

i.e. with the walker subspaces adjacent to each other and the coin subspaces adjacent to each other.

The basic premise of this protocol is this:

1. Entangle the two coin spaces of the walkers $\mathcal{H}_C$ (figure 5).
2. Proceed with the quantum walk for some determined number of steps.

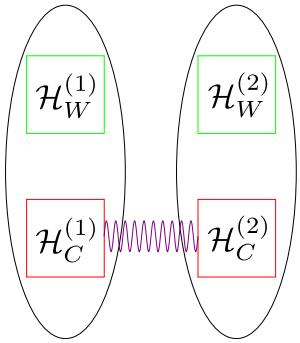


Figure 5: The initial prepared state has entanglement solely between the two coin subspaces. Figure is an edited version of FIG 3 from [12].

3. Use a projective measurement $\mathcal{P}_\gamma = |\gamma\rangle\langle\gamma| ; |\gamma\rangle \in \mathcal{H}_C^{(A)}$ to then transfer the entanglement so that it solely exists in the subspace $\mathcal{H}_W^{(A)} \otimes \mathcal{H}_W^{(B)} \otimes \mathcal{H}_C^{(B)}$. $|\gamma\rangle$ can be defined by two complex parameters, θ and ϕ (see equation 1), which therefore also define $\mathcal{P}_\gamma$.
4. In similar fashion, find a projection $\mathcal{P}_\delta = |\delta\rangle\langle\delta| ; |\delta\rangle \in \mathcal{H}_C^{(B)}$ to transfer the entanglement to exist between the two walker subspaces, $\mathcal{H}_W$, only. $|\delta\rangle$ is defined in the same way as $|\gamma\rangle$.
5. Accumulate entanglement in the walker subspaces by once more entangling the two coin spaces and repeating the protocol.

In this way, arbitrary amounts of higher dimensional entanglement can be generated.
The ordering of steps 3 and 4 is not overly important since the full operators describing the projective measurements are given by

$$I \otimes I \otimes \mathcal{P}_\gamma \otimes I \text{ and} \tag{59}$$
$$I \otimes I \otimes I \otimes \mathcal{P}_\delta, \tag{60}$$

which clearly commute. As is the case with many quantum walk based protocols, particular attention must be paid to the choice of coin used for the quantum walk, as it will have a large impact on the success of the protocol. The shift operator used to advance quantum walks is $\tilde{S}$, as outlined in section 2.3.2. (More accurately it is $\tilde{S}^{(A)} \otimes \tilde{S}^{(B)}$, one for each walk, but for the sake of brevity $\tilde{S}$ will be used.)

3.1 Transfer

To illustrate the basic principles of the protocol, consider the case where

$$C = I_d \otimes I_d \otimes \underbrace{I_2 \otimes I_2}_{\text{Coin operators}} = I. \tag{61}$$

Note that this is not actually an instance of using quantum walk dynamics, since an identity coin equates to no coin at all.

1. A state $|\psi(0)\rangle$ is prepared with the walkers at the origin and coin states entangled,

$$|\psi(0)\rangle = |0\rangle_W^{(A)} |0\rangle_W^{(B)} \otimes \underbrace{\frac{1}{\sqrt{2}}\Big[\, |\uparrow\rangle_C^{(A)} |\uparrow\rangle_C^{(B)} + |\downarrow\rangle_C^{(A)} |\downarrow\rangle_C^{(B)} \Big]}_{\text{Bell State}}. \tag{62}$$

2. The 'coin', I, is applied and followed by the shift operator $\tilde{S}$ to advance the quantum walks. Explicitly (dropping the indices and combining kets together) the two walks evolve to the state

$$|\psi(1)\rangle = \tilde{S} I |\psi\rangle = \frac{1}{\sqrt{2}}\Big[\, |0,0\rangle |\uparrow,\uparrow\rangle + |1,1\rangle |\downarrow,\downarrow\rangle \Big]. \tag{63}$$

   Considering $\tilde{S}$ as C-X then the control qubits are on the right and the target qudits are on the left here.

3. Using the operator $\mathcal{P}_\gamma = |\gamma\rangle\langle\gamma|$ the part of $|\psi(1)\rangle$ residing in the $\mathcal{H}_C^{(A)}$ subspace is projected onto the state $|\gamma\rangle$.

   For this example, choose $|\gamma\rangle = \frac{1}{\sqrt{2}}\big[\, |\uparrow\rangle + |\downarrow\rangle \,\big]$ which then gives

$$\mathcal{P}_\gamma |\psi(1)\rangle = \frac{1}{2}\Big[\, |0,0\rangle |\gamma,\uparrow\rangle + |1,1\rangle |\gamma,\downarrow\rangle \Big]. \tag{64}$$

   Here it is assumed that the projective measurement has given the correct state, but there is a 50% chance that this projective measurement will project the coin onto $|\gamma\rangle^\perp$, the orthogonal state to $|\gamma\rangle$. It does not matter too much in this example if this happens, as the same amount of entanglement is transferred in either case. However this is not true in general, and post selection is needed to correct for this.

4. Now, project the other coin onto $|\delta\rangle$ which in this instance is taken to be the same state, $|\delta\rangle = \frac{1}{\sqrt{2}}\big[\, |\uparrow\rangle + |\downarrow\rangle \,\big]$,

$$\mathcal{P}_\delta \mathcal{P}_\gamma |\psi(1)\rangle = \frac{1}{2}\Big[\frac{1}{\sqrt{2}}\Big(\, |0,0\rangle + |1,1\rangle \,\Big) |\gamma\rangle |\delta\rangle \Big]. \tag{65}$$

   Again, it is assumed the projective measurement has given the correct state.

Renormalising gives the final state

$$\underbrace{\frac{1}{\sqrt{2}}\Big[\, |0,0\rangle + |1,1\rangle \,\Big]_W}_{\text{Bell State}} \otimes |\gamma\rangle_C^{(A)} \otimes |\delta\rangle_C^{(B)}, \tag{66}$$

which has a Bell state in the $\mathcal{H}_W$ subspace, and the coin states are separable. Therefore the entanglement that originally resided in the coin subspace has been transferred to the walker one.

3.2 Accumulation

The true motivation behind this protocol is the ability to accumulate the entanglement transferred from the lower dimensional coin subspace to the higher dimensional walker one. This is done by repeating the entire process with some small changes. Again I is used as the coin operator and it is assumed the projective measurements succeed.

1. Starting with the final state obtained from the first iteration of the protocol (equation 66), the coin subspaces are re-entangled, obtaining a new initial state $|\psi(0)\rangle$,

$$\frac{1}{\sqrt{2}}\Big[\,|0,0\rangle + |1,1\rangle\,\Big]_W \otimes |\gamma\rangle_C^{(A)} \otimes |\delta\rangle_C^{(B)} \tag{67}$$

$$\xrightarrow{\text{Entangle coins}} \quad \frac{1}{2}\Big[\,|0,0\rangle + |1,1\rangle\,\Big]_W \otimes \Big[\,|\uparrow,\uparrow\rangle + |\downarrow,\downarrow\rangle\,\Big]_C := |\psi(0)\rangle\,. \tag{68}$$

2. Now take two steps instead of one in the walk.

$$|\psi(2)\rangle = (\tilde{S}I)^2\,|\psi(0)\rangle \tag{69}$$

$$= \frac{1}{2}\Big[\big(\,|0,0\rangle + |1,1\rangle\,\big)\,|\uparrow,\uparrow\rangle + \big(\,|2,2\rangle + |3,3\rangle\,\big)\,|\downarrow,\downarrow\rangle\,\Big]. \tag{70}$$

3–4. Using the same projection operators in the two coin subspaces, $\mathcal{P}_\gamma \in \mathcal{H}_C^{(A)}, \mathcal{P}_\delta \in \mathcal{H}_C^{(B)}$, and renormalising gives the final state

$$\frac{1}{2}\Big[\,|0,0\rangle + |1,1\rangle + |2,2\rangle + |3,3\rangle\,\Big] \otimes |\gamma\rangle \otimes |\delta\rangle\,. \tag{71}$$

Using claim 2.3, the log negativity of

$$\frac{1}{2}\Big[\,|0,0\rangle + |1,1\rangle + |2,2\rangle + |3,3\rangle\,\Big] \tag{72}$$

is 2, setting $d = 4$ (since both of the walkers only have 4 states of non-zero amplitude they can be simulated by ququarts even if their true dimension is greater than 4). Therefore, 2 units of log negativity have been transferred from the Bell states (each of log negativity 1) to the qudit pair and optimal transfer has been achieved. The process can be repeated to accumulate arbitrarily large amounts of entanglement into our walker subspace. The number of steps needed in the quantum walk for each iteration is as follows.

Claim 3.1. *The n^{th} iteration (counting from 1) of the protocol requires 2^{n-1} steps in the quantum walk.*

A proof of this is given in appendix B.

3.3 Retrieval

Although accumulating the entanglement in higher dimensions is of significant use, it is also possible to imagine that retrieving the entanglement back into the qubits would also be of

interest. For example, consider the case where the method of generating entangled qubits is not deterministic, or takes a reasonably long time, but an algorithm requiring large numbers of Bell states is to be executed. If the accumulation of entanglement can be reversed to get Bell pairs back, then the entangled Bell pairs can be stored in the qudits when they are successfully generated and then retrieved all at once to ensure the algorithm can be performed. In principle, it is possible to use a similar quantum walk setup where the entangled qudits are the coins and the qubits are the walkers to get entanglement back out. However due to the non-unitary nature of the protocol (the projective measurements are irreversible), this setup is not simple to design.

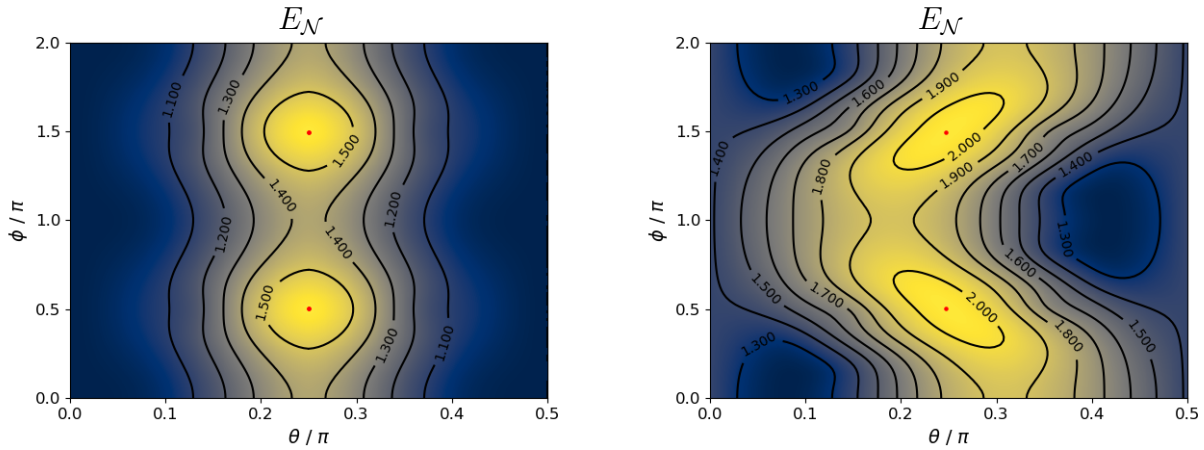
3.4 Results

Despite its simple construction, the state of a quantum walk becomes extremely complex after even just a few steps in most cases. This is doubly true for two quantum walks running in conjunction. Hence, numerical simulations were employed to analyse the protocol in settings that are not as trivial as the identity coin example. When transferring entanglement from only one Bell pair, the protocol was able to use Hadamard-coined quantum walks to optimally transfer the entanglement to the qudits (this is shown in appendix A figure A.1). When simulating the accumulation of two and three Bell pairs of entanglement, again using Hadamard-coined QWs, figure 6 shows that the protocol can generate states with log negativities of up to approximately 1.585 and 2.036 respectively if using the same $\mathcal{P}_\gamma$ for both coins. Simulations were also done using different projective operators $\mathcal{P}_\gamma$, $\mathcal{P}_\delta$ to transfer entanglement from two Bell states. These also showed ~ 1.585 is still the maximum possible when using the Hadamard coin. Sample plots from simulations with differing projective operators are found in appendix A figure A.2.

The relationship between the bias of a coin with real coefficients (equation 50) and the maximum log negativity of the combined walker state is plotted in figure 7. It highlights how entanglement transfer gets closer to being optimal as the coin gets closer to the identity, at which point there are no quantum walk dynamics at all - the protocol works better the less it resembles a quantum walk.

Furthermore, all of this analysis is done with the assumption that the projective measurements succeed. This is not true in general and the procedure has to be repeated if the projective measurements project onto the orthogonal states. For example, an experimental proposal using photons as both the coins and walkers is given in [12] which requires post-selection, where the final state of the walkers is discarded if the projective measurements result in undesired states.

In light of this, it is fair to conclude that the choice to facilitate entanglement transfer via quantum walks is an impractical one. However, given that the protocol has the potential to work optimally, it is worth reconstructing it in an alternative model of quantum computing, whose constraints can lead to optimal transfer being the natural outcome of the model.



(a) Transferring entanglement from two Bell states. Red points correspond to $E_{\mathcal{N}} \sim 1.585$.

(b) Transferring entanglement from three Bell states. Red points correspond to $E_{\mathcal{N}} \sim 2.036$.

Figure 6: Log negativity, $E_{\mathcal{N}}$, of combined walker states using the QW transfer protocol. θ, ϕ are the parameters defining $\mathcal{P}_\gamma, \mathcal{P}_\delta$ where $\mathcal{P}_\gamma = \mathcal{P}_\delta$ for both simulations. Both walks were conducted using the Hadamard coin. Red points indicate peak values of log negativity on the plots. θ and ϕ are each varied over 100 evenly spaced values.

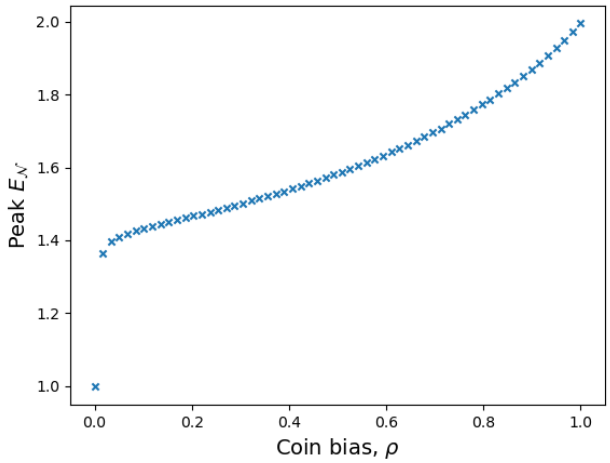


Figure 7: Peak $E_{\mathcal{N}}$ of walkers after transferring entanglement from two Bell pairs using quantum walks with coins of varying bias, ρ. The peak value was found by computing the largest value of log negativity for 100 evenly spaced values of both θ and ϕ, the parameters for the coin projection operators $\mathcal{P}_\gamma, \mathcal{P}_\delta$ where $\mathcal{P}_\gamma = \mathcal{P}_\delta$.

4 Entanglement Transfer using Ancilla-Based Quantum Computing

In this section an alternative entanglement transfer scheme is outlined. It is designed to operate in the same setting at the QW based protocol, where there are two labs, again called A and B, which share a common source of entangled qubits and each have their own qudits.

Analogously to the QW protocol, the qudit can be thought of as a walker driven by the ancilla qubit 'coin'. The ideas contained in the QW fit well with an AQC model because they emulate the design philosophy behind AQC near perfectly. Entangled coins have their entanglement transferred and are then replaced - these can be viewed as the ancilla qubits. The entanglement is transferred and accumulated in the walkers - these can be viewed as long-lived qudits.

The AQC protocol can be succinctly summarised by the circuit schematic given in figure 8, and essentially is two copies of the same sub-circuit.

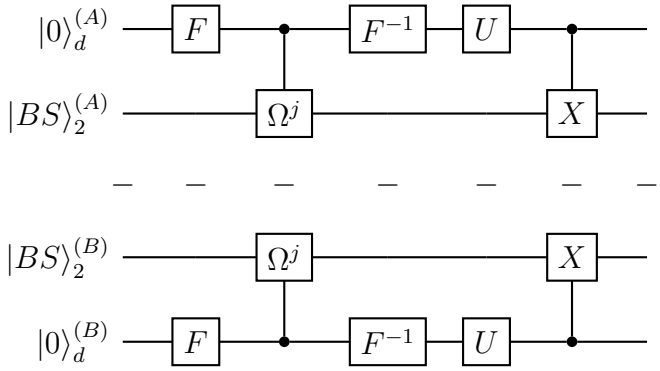


Figure 8: The AQC circuit for entanglement transfer. The schematic shows two separated circuits that exist in spatially separated labs, A and B. The qudits and qubits belonging to each individual lab are labelled by their superscripts. The qubits form an entangled Bell state (denoted by BS).

The majority of the gates shown in the schematic are discussed in section 2.1.5, except for the gate Ω^j. The action of Ω in the computational basis is given by the matrix

$$\Omega = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \tag{73}$$

$$\implies \Omega^j = \begin{pmatrix} 1 & 0 \\ 0 & \omega^j \end{pmatrix}. \tag{74}$$

Ω is similar to the Z operator in that it applies relative phase difference, but instead of a phase of -1, a phase corresponding to the d^{th} root of unity is introduced on the $|1\rangle_2$ states, with d corresponding to the dimension of the control qudits. (Indeed, when $d = 2$, $\Omega = Z$.) Therefore the C-Ω^j operator is one that acts as

$$C\text{-}\Omega^j(|x\rangle_d |y\rangle_2) = |x\rangle_d \otimes \Omega^{xj} |y\rangle_2 \tag{75}$$

$$= |x\rangle_d \otimes \omega^{xyj} |y\rangle_2 , \tag{76}$$

where $|x\rangle_d$ is the control qudit and $|y\rangle_2$ the target qubit.

Claim 4.1. *The C-Ω^j operator acts on the product state $|+_k\rangle_d \otimes |1\rangle_2$ to give $|+_{k+j}\rangle_d \otimes |1\rangle_2$*

A proof for this is given in appendix C. This exhibits again the counterintuitive nature of AQC discussed in section 2.4. Despite the qudit being the control and the qubit being the

target, the only change effected is on the qudit, shifting the Fourier basis states. Claim 4.1 also demonstrates the role of the index j, which essentially dictates by how many states each Fourier basis state is shifted.

The following sections demonstrate explicitly how the circuit operates.

4.1 Transfer

Assume that there are two qudits in the state $|0\rangle_d$ in labs A and B, and each lab shares a qubit each from an entangled Bell pair. The composite state is therefore given by

$$|00\rangle_d^{(A,B)} \otimes \frac{1}{\sqrt{2}} \left(|00\rangle_2^{(A,B)} + |11\rangle_2^{(A,B)} \right). \tag{77}$$

Note that in the above expression, the qudits and qubits in both labs, A and B, are in fact in the same states. This is true for the entire running of the circuit and as such there is no need to explicitly differentiate the qudits or ancilla qubits in either lab so the superscript labelling is dropped for the rest of this section.

The qudits are first Fourier transformed into the Fourier basis to give

$$|+_0+_0\rangle_d \otimes \frac{1}{\sqrt{2}} \left(|00\rangle_2 + |11\rangle_2 \right) \tag{78}$$

$$= \frac{1}{\sqrt{2}} \Big(|+_0+_0\rangle_d \otimes |00\rangle_2 + |+_0+_0\rangle_d \otimes |11\rangle_2 \Big). \tag{79}$$

The change of basis is needed because to have entanglement in the qudits, they must be in superposition of at least two states of non-zero amplitude. Since only amplitudes can be delocalised across the qudits and ancilla qubits, in order to give other qudit states a non-zero amplitude, it is necessary to be in a basis where the basis states are related to one another by differences in their amplitudes. The Fourier basis is one such basis where the basis states are in equal superposition of all the computational basis states but differ by the relative phases in the superposition. Hence the need to be in the Fourier basis when using the delocalised amplitudes to effect changes in the qudit basis states. The index j on the Ω has an analogy to the QW protocol where the iteration number dictates how many steps need to be taken in the QW. Using the same line of reasoning as outlined in claim 3.1, on the n^{th} iteration of the protocol, $j = 2^{n-1}$ so in this instance (where $n = 1$) $j = 1$. Using claim 4.1, acting C-Ω in both A and B gives the state

$$\frac{1}{\sqrt{2}} \left(|+_0+_0\rangle_d \otimes |00\rangle_2 + |+_1+_1\rangle_d \otimes |11\rangle_2 \right). \tag{80}$$

The qudits are then transformed back into the computational basis,

$$\frac{1}{\sqrt{2}} \left(|00\rangle_d \otimes |00\rangle_2 + |11\rangle_d \otimes |11\rangle_2 \right). \tag{81}$$

The operator U is a correctional gate that essentially pairs all the even numbered computational qudit basis states with $|0\rangle_2$ and the odd numbered computational basis states with

$|1\rangle_2$. (A more complete description is given in section 4.2.) In this instance, U can be ignored since no corrections are needed, so C-X is directly applied, giving

$$\frac{1}{\sqrt{2}}\left(|00\rangle_d \otimes |00\rangle_2 + |11\rangle_d \otimes |00\rangle_2\right) \tag{82}$$

$$= \underbrace{\frac{1}{\sqrt{2}}\left(|00\rangle_d + |11\rangle_d\right)}_{\text{Bell state}} \otimes |00\rangle_2 . \tag{83}$$

As shown in equation 83, the qudits now form a Bell state and the qubits are no longer entangled - the entanglement has been transferred to the qudits.

4.2 Accumulation

Assume that one Bell pair has been transferred to our qudits and a second Bell pair is to be transferred. Taking the final state given by equation 83 and replacing the ancilla qubits with another entangled Bell pair gives

$$\frac{1}{2}\left(|00\rangle_d + |11\rangle_d\right) \otimes \left(|00\rangle_2 + |11\rangle_2\right). \tag{84}$$

Again, the qudits are transformed to the Fourier basis before applying the C-Ω^j gate. As noted in the discussion on C-Ω^j above, for this iteration (where $n = 2$) $j = 2$, giving the state

$$\frac{1}{2}\Big[\left(|+_0+_0\rangle_d + |+_1+_1\rangle_d\right) \otimes |00\rangle_2 + \left(|+_2+_2\rangle_d + |+_3+_3\rangle_d\right) \otimes |11\rangle_2\Big]. \tag{85}$$

After transforming back to the computational basis, the total state becomes

$$\frac{1}{2}\Big[\left(|00\rangle_d + |11\rangle_d\right) \otimes |00\rangle_2 + \left(|22\rangle_d + |33\rangle_d\right) \otimes |11\rangle_2\Big]. \tag{86}$$

Unlike the previous example for $n = 1$, here the correctional operator U is needed so that the C-X will ensure all the qubit states are $|0\rangle_2$, allowing our qubit state to be completely separable from its associated qudit. In order for the C-X to do this, it must act as the identity on $|0\rangle_2$, and implement an X operation on $|1\rangle_2$. Since X is self-inverse, the control must be in an even labelled basis state (turning X into the identity) when the target is $|0\rangle_2$ and in an odd labelled basis state (leaving X unchanged) when the target is $|1\rangle_2$. Hence in this case, U must implement

$$|1\rangle \mapsto |2\rangle \tag{87}$$

$$|2\rangle \mapsto |1\rangle , \tag{88}$$

which then gives the state

$$\frac{1}{2}\Big[\left(|00\rangle_d + |22\rangle_d\right) \otimes |00\rangle_2 + \left(|11\rangle_d + |33\rangle_d\right) \otimes |11\rangle_2\Big]. \tag{89}$$

As desired, U has paired even numbered qudit basis states with $|00\rangle_2$ and odd numbered qudit basis states with $|11\rangle_2$. Therefore after the C-X operation the state becomes

$$\frac{1}{2}(|00\rangle_d + |22\rangle_d + |11\rangle_d + |33\rangle_d) \otimes |00\rangle_2 . \tag{90}$$

The two qudits are now in an entangled state with a log negativity of 2 (claim 2.3). This protocol can be repeated again and again with the sole change needed being the index j of $C\text{-}\Omega^j$ and the operator U. For $n > 2$ there is no unique choice of U, as long as the end result is that even numbered states are paired with $|0\rangle_2$ and odd numbered states are paired with $|1\rangle_2$. This can be re-expressed as requiring odd numbered states less than $\frac{d}{2}$ and even numbered states greater than or equal to $\frac{d}{2}$ to be swapped in some way.

4.3 Retrieval

Given that this is a circuit that solely utilises unitary transformations, retrieval of entangled Bell pairs is trivially done by running the circuit backwards. Furthermore, there is no requirement to use the same ancilla qubits to retrieve the entanglement. By the end of the circuit the ancilla qubits are in the $|0\rangle$ state, therefore any pair of ancilla qubits in the $|0\rangle$ state may be used to retrieve the entanglement.

This ease of transfer means that entanglement can actually be transferred between qudit pairs of any dimension. The entanglement from the source qudit pair can first be transferred to a collection of intermediary ancilla qubits, and then transferred from the ancilla qubits to the target qudit pair.

5 Further Uses of the AQC Circuit

Although the goal of this research was to design a protocol in a similar setting to the QW based protocol that could optimally generate maximally entangled states, further probing of the AQC circuit shows that it has uses beyond storage of entanglement. In this section, a scenario is considered where the quantum information contained in a collection of qubits is to be stored. By taking one half of the circuit given for the two lab setting and adding a gate U^{-1} that undoes the basis state map seen before, as shown in figure 9, arbitrary qubit states can be stored in the qudit, where n is the number of qubit states to be stored. Furthermore, in the case that two or more qubit states are stored via this circuit then it is in fact possible to retrieve the qubit states in a different order to that in which they were stored, as long as the original order they were stored in is known. Therefore, this circuit can be utilised in turning qudits into quantum random access memory.

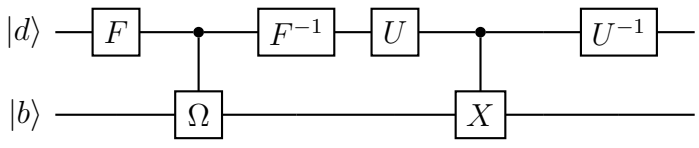


Figure 9: The AQC circuit with one qudit and qubit which can store arbitrary qubit states in the qudit.

5.1 Quantum Random Access Memory

The procedure for utilising the circuit as a quantum random access memory is as follows. Assume that there is a qudit in state $|0\rangle_d$ and two qubits, qubit 1 and qubit 2, in states $a|0\rangle_2 + b|1\rangle_2$ and $c|0\rangle_2 + d|1\rangle_2$ respectively. First run the circuit with the qudit and qubit 1,

$$|0\rangle_d \otimes (a|0\rangle + b|1\rangle) \longrightarrow (a|0\rangle_d + b|1\rangle_d) \otimes |0\rangle_2 . \tag{91}$$

Then replace qubit 1 with qubit 2 and run the circuit again,

$$(a|0\rangle_d + b|1\rangle_d) \otimes (c|0\rangle_2 + d|1\rangle_2) \longrightarrow (ac|0\rangle_d + bc|1\rangle_d + ad|2\rangle_d + bd|3\rangle_d) \otimes |0\rangle_2 . \tag{92}$$

In order to retrieve qubit 2 back, this can be done simply by running the circuit backwards. However, if the state of qubit 1 is to be retrieved then a specific unitary operator is first required. The form of the unitary transform can be found as follows. Rewrite each of the qudit basis state numbers in binary, i.e.

$$|0\rangle = |00\rangle \tag{93}$$
$$|1\rangle = |01\rangle \tag{94}$$
$$|2\rangle = |10\rangle \tag{95}$$
$$|3\rangle = |11\rangle . \tag{96}$$

Rewriting the final state of equation 92 in this way gives

$$ac|0\rangle + bc|1\rangle + ad|2\rangle + bd|3\rangle = ac|00\rangle + bc|01\rangle + ad|10\rangle + bd|11\rangle , \tag{97}$$

which can be rewritten as the product state

$$\underbrace{(c|0\rangle + d|1\rangle)}_{\text{Qubit 2}} \otimes \underbrace{(a|0\rangle + b|1\rangle)}_{\text{Qubit 1}} . \tag{98}$$

Quite remarkably the qudit has an intuitive alternative expression as the product state of qubit 1 and qubit 2. If the circuit is run backwards, qubit 2 will be retrieved, since it was the last qubit stored. Therefore, to retrieve qubit 1 instead, the qudit state should be expressible as

$$\underbrace{(a|0\rangle + b|1\rangle)}_{\text{Qubit 1}} \otimes \underbrace{(c|0\rangle + d|1\rangle)}_{\text{Qubit 2}} . \tag{99}$$

This can be thought of as switching the positions of our two qubits, which leads us to the unitary transformation needed. A map, M, is required which will switch the positions of our two qubits. M is found by mapping each of the binary qudit state representations $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ to the basis state with the binary digits switched.

$$|00\rangle \mapsto |00\rangle \tag{100}$$
$$|01\rangle \mapsto |10\rangle \tag{101}$$
$$|10\rangle \mapsto |01\rangle \tag{102}$$
$$|11\rangle \mapsto |11\rangle . \tag{103}$$

To verify that this does give the desired result, take the RHS of equation 97 and act M on it to obtain

$$M\left(ac\left|00\right\rangle + ad\left|01\right\rangle + bc\left|10\right\rangle + bd\left|11\right\rangle\right) = ac\left|00\right\rangle + ad\left|10\right\rangle + bc\left|01\right\rangle + bd\left|11\right\rangle \tag{104}$$

$$= \underbrace{\left(a\left|0\right\rangle + b\left|1\right\rangle\right)}_{\text{Qubit 1}} \otimes \underbrace{\left(c\left|0\right\rangle + d\left|1\right\rangle\right)}_{\text{Qubit 2}}, \tag{105}$$

as required. M can be expressed in terms of the original qudit state labelling by converting the binary back

$$\left|0\right\rangle \mapsto \left|0\right\rangle \tag{106}$$

$$\left|1\right\rangle \mapsto \left|2\right\rangle \tag{107}$$

$$\left|2\right\rangle \mapsto \left|1\right\rangle \tag{108}$$

$$\left|3\right\rangle \mapsto \left|3\right\rangle. \tag{109}$$

The matrix representation of M is given by

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{110}$$

Generalising this to larger numbers of qubits stored is done via the same thought process. Imagine that there is now third qubit to be stored in the qudit, qubit 3, in the state

$$e\left|0\right\rangle + f\left|1\right\rangle. \tag{111}$$

When qubits 1, 2 and 3 are stored in that same order, it results in the state

$$ace\left|0\right\rangle + ade\left|1\right\rangle + bce\left|2\right\rangle + bde\left|3\right\rangle + ace\left|4\right\rangle + adf\left|5\right\rangle + bcf\left|6\right\rangle + bdf\left|7\right\rangle, \tag{112}$$

which again, when converted to binary numbers, can be expressed as the product state

$$\underbrace{\left(e\left|0\right\rangle + f\left|1\right\rangle\right)}_{\text{Qubit 3}} \otimes \left(c\left|0\right\rangle + d\left|1\right\rangle\right) \otimes \underbrace{\left(a\left|0\right\rangle + b\left|1\right\rangle\right)}_{\text{Qubit 1}}. \tag{113}$$

If the state of qubit 1 is needed, then M must switch the first and last qubits, and therefore must implement the map

$$\left|1\right\rangle = \left|001\right\rangle \mapsto \left|100\right\rangle = \left|4\right\rangle \tag{114}$$

$$\left|3\right\rangle = \left|011\right\rangle \mapsto \left|110\right\rangle = \left|6\right\rangle \tag{115}$$

$$\left|4\right\rangle = \left|100\right\rangle \mapsto \left|001\right\rangle = \left|1\right\rangle \tag{116}$$

$$\left|6\right\rangle = \left|110\right\rangle \mapsto \left|011\right\rangle = \left|3\right\rangle \tag{117}$$

with all other basis states mapped to themselves as they are identical when switching the first and last digits of their binary representations. Having operated M on the qudit state, the circuit can be run backwards in order to retrieve qubit 1. As with retrieving Bell pairs in

section 4, any qubit in the state $|0\rangle$ can be used to retrieve qubit 1. A proof that this circuit can store any number of qubit states is given in appendix D.

In a practical setting it may be more efficient to build a gate, P, that permutes the order of the qubits instead of swapping two digits. One example of P would be the map that permutes each qubit one place the right,

$$|1\rangle = |001\rangle \mapsto |100\rangle = |4\rangle \tag{118}$$
$$|2\rangle = |010\rangle \mapsto |001\rangle = |1\rangle \tag{119}$$
$$|3\rangle = |011\rangle \mapsto |101\rangle = |5\rangle \tag{120}$$
$$|4\rangle = |100\rangle \mapsto |010\rangle = |2\rangle \tag{121}$$
$$|5\rangle = |101\rangle \mapsto |110\rangle = |6\rangle \tag{122}$$
$$|6\rangle = |110\rangle \mapsto |011\rangle = |3\rangle . \tag{123}$$

Again $|0\rangle , |7\rangle$ are mapped to themselves as the permutation does not affect them. In this way, any of the qubit states can be retrieved by continually applying P until the qubit state to be retrieved is in the right 'place'. Utilising a single gate would also be more appropriate for the ancilla-based quantum computing setting where a minimal gate set is desired for the control qudits. However, this would require multiple applications of the same gate which, without adequate error correction, would lead to greater decoherence of the quantum information than if just a single swapping gate M were applied.

6 Discussion

In the example given in section 3, it is shown that the protocol proposed in the QW setting can optimally transfer all the entanglement to the qudit pair. However, this is with the significant caveat that the example does not use quantum walk dynamics in order to transfer the entanglement, as using the identity as a coin is akin to having no coin at all. In analysing the protocol with the Hadamard coin, it was only possible to transfer one Bell state's worth of entanglement optimally and numerical simulations of the protocol found that the protocol could not transfer all of the entanglement to the qudit pairs (figure 6). Analysing the protocol using coins of varying bias indicated the QW protocol only worked optimally with an identity coin. Furthermore, the projective measurements employed as part of the protocol mean that it is non-unitary. Therefore it is nontrivial to reverse the protocol in order to retrieve entanglement from the entangled qudits. They also lead to the need to post-select the final states in a practical setting, and the protocol must be repeated if the measurements did not result in the desired states. Overall, the QW protocol is rather unsuitable for practically generating higher dimensional entanglement, and serves more as a proof of concept in that it is possible to use quantum walk dynamics to transfer entanglement, but falls short in being a optimal method of doing so.

On the other hand, the ancilla-based quantum computing circuit allowed for optimal entanglement transfer to generate maximally entangled states. Exclusive utilisation of unitary operators means that the circuit can just be reversed with any ancilla qubit in the $|0\rangle$ state. The low gate depth of the circuit also makes it feasible to use ancilla qubits with relatively

low decoherence times, although the runtime of the circuit would depend on how the $C\text{-}\Omega^j$ gate is realised physically. If it required a single $C\text{-}\Omega$ to be repeated j times then this could significantly lengthen the time the circuit would take to run, and therefore affect the minimum decoherence time needed for the qubits. The AQC circuit can also turn qudit states into quantum random access memory, which furthers its claim as a practical circuit worth building and implementing.

6.1 Further Steps

This work leaves open the door for further analysis of the circuit to see if it can be extended to facilitate entanglement transfer from entangled qudit pairs of arbitrary dimension instead of just entangled Bell pairs. This would generalise it to a scheme where entanglement can be transferred between dimensions directly instead of via qubits. However, this would likely be more for academic rather than practical purposes and indeed takes it further away from the AQC environment in which it is designed to operate. It would also be of interest to examine if the circuit can be applied to generating higher dimensional multipartite entanglement to, for example using it to transfer entanglement from GHZ states or W states.

6.2 Conclusions

Overall, the QW protocol aims to solve an interesting problem with a suitable set of constraints that might be realistic ones to consider in the future as quantum computing technologies develop. In principle it does away with the need to repeatedly design different entanglement schemes for qudits of differing dimension, since the same scheme can used to accumulate entanglement in any dimension, just by using entangled Bell states from which to transfer the entanglement from. However, it was shown that in practice this protocol scaled inefficiently as more entanglement was transferred when actually using quantum walk dynamics, and optimal transfer is not possible. It also suffers from difficulties in transferring entanglement back to the qubits due to its non-unitary nature, and inefficiencies due to the need to post-select.

Instead, an alternative scheme was proposed to operate in the same physical setting, but based on an AQC model which came with a slightly different set of constraints. It was shown that this alternative scheme is able to achieve optimal transfer of entanglement, no matter the number of Bell states to be stored. Retrieval of entangled Bell pairs is also simply done by reversing the circuit. This allows the circuit to facilitate transfer of entanglement between qudits of any dimension, using intermediary ancilla qubits. Furthermore, it was also shown that the circuit had further uses outside of entanglement transfer and could be utilised to turn qudits into quantum random access memory. This increased versatility gives greater practical benefits to the AQC circuit as only one circuit is needed to achieve multiple aims. Its relative simplicity also makes it an effective circuit to implement on a practical quantum computer, with only a select few gates needed to implement it.

Acknowledgements

I would like to thank Dr Viv Kendon for her insight and advice during this research, especially during the challenging conditions we found ourselves having to work in.

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Appendices

A Additional Plots from Simulating Quantum Walk Entanglement Transfer

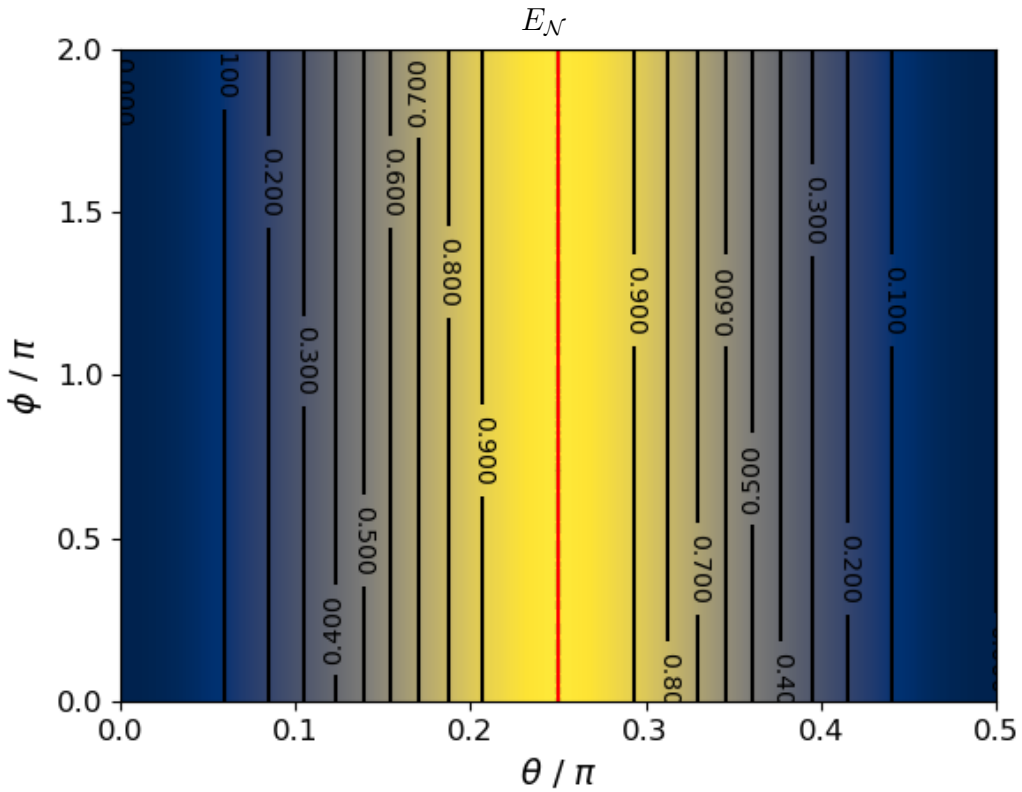
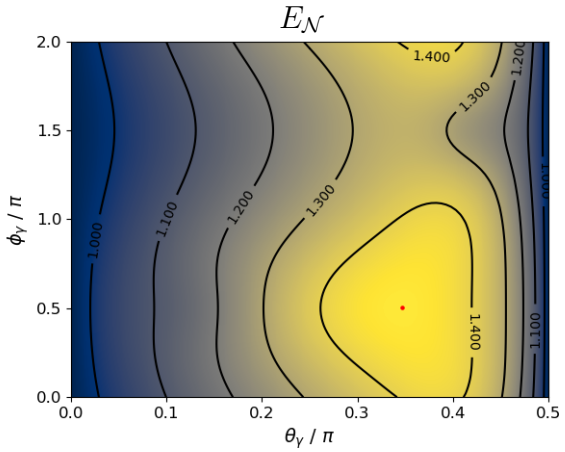
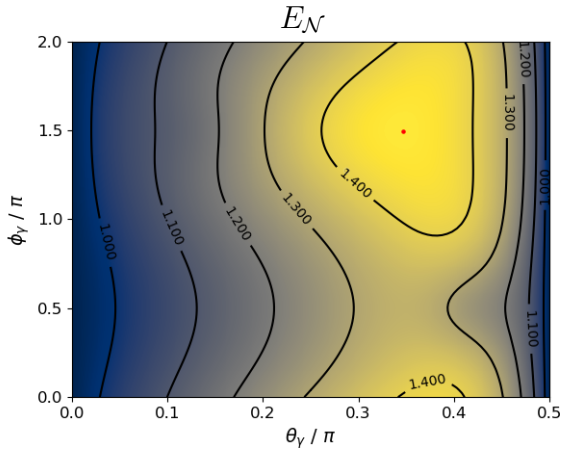


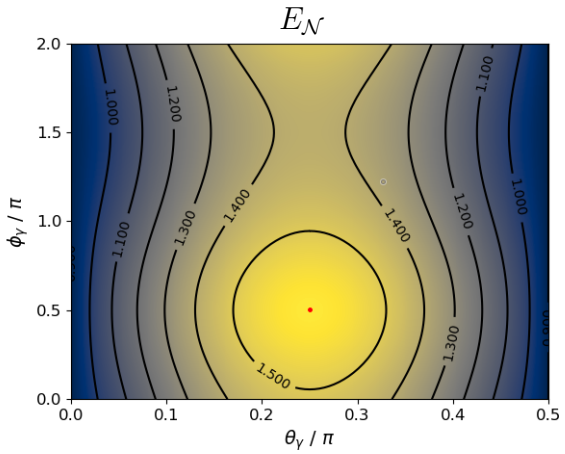
Figure A.1: Log negativity of walkers after transferring entanglement from one Bell pair using the same projective measurements $\mathcal{P}_\gamma = \mathcal{P}_\delta$. This was simulated for 100 evenly spaced values of both θ and ϕ. Every point along the red solid line has a peak value of 1.



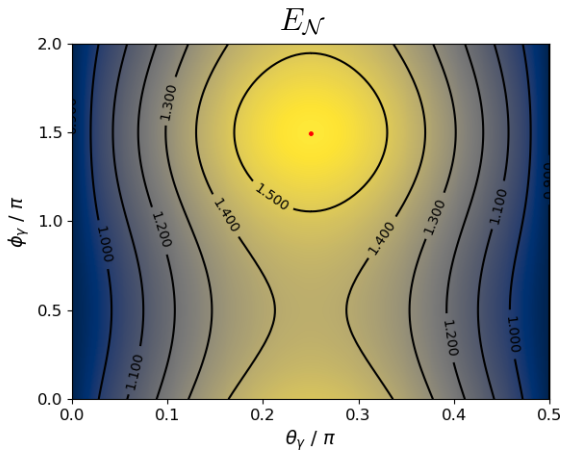
(a) $\theta_\delta = 0.125\pi$, $\phi_\delta = 0.5\pi$. Maximum $E_\mathcal{N} \sim 1.448$.



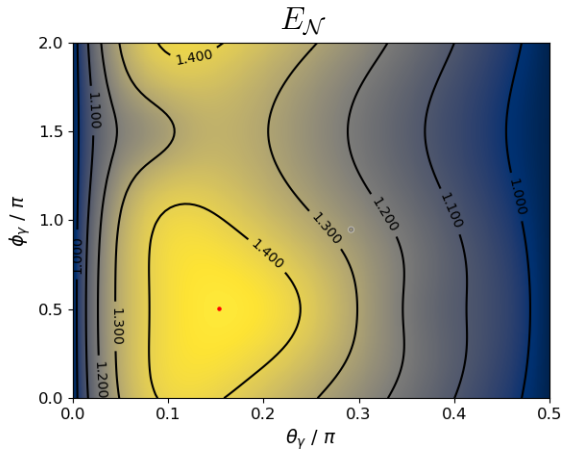
(b) $\theta_\delta = 0.125\pi$, $\phi_\delta = 1.5\pi$. Maximum $E_\mathcal{N} \sim 1.448$.



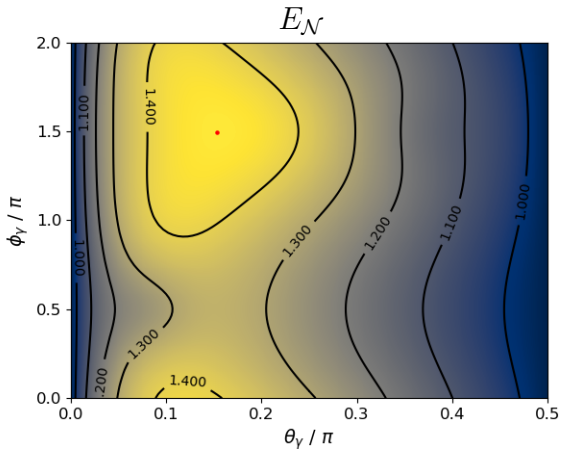
(c) $\theta_\delta = 0.25\pi$, $\phi_\delta = 0.5\pi$. Maximum $E_\mathcal{N} \sim 1.585$.



(d) $\theta_\delta = 0.25\pi$, $\phi_\delta = 1.5\pi$. Maximum $E_\mathcal{N} \sim 1.585$.



(e) $\theta_\delta = 0.375\pi$, $\phi_\delta = 0.5\pi$. Maximum $E_\mathcal{N} \sim 1.448$.



(f) $\theta_\delta = 0.375\pi$, $\phi_\delta = 1.5\pi$. Maximum $E_\mathcal{N} \sim 1.448$.

Figure A.2: Plots showing the log negativity, $E_\mathcal{N}$, of walkers after transferring entanglement from two Bell pairs with different projective measurements $\mathcal{P}_\gamma \neq \mathcal{P}_\delta$. θ_δ, ϕ_δ are fixed for $\mathcal{P}_\delta$, with the fixed values given in the captions. $\theta_\gamma, \phi_\gamma$ were each varied over 100 evenly spaced points for $\mathcal{P}_\gamma$. Peak values are indicated by red points.

B Proof of Claim 3.1

Here a proof is given for claim 3.1.

Proof. Each step in a quantum walk with shift operator $\tilde{S}$ increases the number of basis states with non-zero amplitude by 1, provided that the amplitude $|\downarrow\rangle$ coin basis state is non-zero. Therefore the dimension of each walker can be taken to be $s+1$, where s is the total number of steps taken in the walk, since each walker space can be reduced to the subspace spanned by the $s+1$ non-zero amplitude basis states. Using claim 2.3, the upper bound on the log negativity of the combined walker states after the n^{th} iteration of the walk is given by $\log_2(s+1)$. This implies the following condition on the total number of steps

$$\log_2(s+1) = n \implies s = 2^n - 1. \tag{B.1}$$

This is an equality rather than an inequality $\log_2(s+1) \geq n$, since the state of the walkers is to be maximally entangled. This means that the combined walker Hilbert space $\mathcal{H}_W$ must be reducible to one of dimension $2^n \times 2^n$ (the same as the total dimension of the n Bell pairs used for the protocol). This is only possible when taking the minimum number of steps in each quantum walk, if more steps are taken then there are more than 2^n basis states of non-zero amplitude for each walker.

Let s_{n-1} be the total number of steps taken up to the $(n-1)^{\text{th}}$ iteration of the protocol. If s_{n-1} satisfies equation B.1 this implies $s_{n-1} = 2^{n-1} - 1$. Therefore, the number of steps for the n^{th} iteration is given by

$$\begin{aligned} s - s_{n-1} &= 2^n - 1 - (2^{n-1} - 1) && \text{(B.2)} \\ &= 2^n - 2^{n-1} && \text{(B.3)} \\ &= 2 \times 2^{n-1} - 2^{n-1} && \text{(B.4)} \\ &= 2^{n-1}. && \text{(B.5)} \end{aligned}$$

For $n = 2$, it has been shown that $s_1 = 1$ therefore the claim is true by induction. □

C Proof of Claim 4.1

Here a proof is given for claim 4.1.

Proof. The Fourier basis state $|+_k\rangle_d$ is given by

$$|+_k\rangle_d = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \omega^{km} |m\rangle . \tag{C.1}$$

Therefore

$$C\text{-}\Omega^j \left(|+_k\rangle_d |1\rangle_2\right) = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} C\text{-}\Omega^j \left(\omega^{km} |m\rangle_d \otimes |1\rangle_2\right) \tag{C.2}$$

$$= \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \omega^{mj} \left(\omega^{km} |m\rangle_d \otimes |1\rangle_2\right) \tag{C.3}$$

$$= \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \omega^{(k+j)m} |m\rangle_d \otimes |1\rangle_2 \tag{C.4}$$

$$= \left(\frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \omega^{(k+j)m} |m\rangle_d\right) \otimes |1\rangle_2 \tag{C.5}$$

$$= |+_{k+j}\rangle_d \otimes |1\rangle_2 . \tag{C.6}$$

□

D Proof that the AQC Circuit Can Store Any Number of Qubits

Here it is proven by induction that the circuit presented in section 5 can store $n \in \mathbb{N}$ qubit states into the qudit, as long as the dimension of the Hilbert space $d \geq 2^n$.

Proof.
$n = 1$
It has already been shown in equation 91 that a single arbitrary qubit state can be stored giving

$$(a_1 |0\rangle_d + b_1 |1\rangle_d) \otimes |0\rangle_2 , \tag{D.1}$$

where the coefficients have been relabelled $a \to a_1, b \to b_1$.

$n = k$
Assume that n qubit states have been stored in the qudit, giving the qudit state

$$\left(\sum_{j=0}^{2^k-1} c_j |j\rangle_d \right) \otimes |0\rangle_2 . \tag{D.2}$$

Furthermore, assume that this can be re-expressed as a tensor product of n qubits by converting the computational basis state labels to their binary equivalents and 'expanding' to give

$$(a_k |0\rangle + b_k |1\rangle) \otimes (a_{k-1} |0\rangle + b_{k-1} |1\rangle) \otimes \cdots \otimes (a_1 |0\rangle + b_1 |1\rangle) \otimes |0\rangle_2 \tag{D.3}$$

$$= \bigotimes_{i=k}^{1} (a_i |0\rangle + b_i |1\rangle) \otimes |0\rangle_2 , \tag{D.4}$$

where the coefficients a_i and b_i match the amplitudes of the i^{th} qubit state that was stored. Note that the assumption that the state can be expanded in such a way implies equation D.2 where

$$c_j = \prod_{i=1}^{k} (a_i \delta^0_{j_{2_i}} + b_i \delta^1_{j_{2_i}}), \tag{D.5}$$

where j_{2_i} is the i^{th} digit of the binary expression of j, and δ^m_n is the Kronecker delta.

$n = k + 1$
The entire state prior to running the circuit with an additional qubit to be stored is given by

$$\left(\bigotimes_{i=k}^{1}(a_i |0\rangle + b_i |1\rangle)\right) \otimes (a_{k+1} |0\rangle_2 + b_{k+1} |1\rangle_2) \tag{D.6}$$

$$= \left(\bigotimes_{i=k}^{1}(a_i |0\rangle + b_i |1\rangle)\right) \otimes a_{k+1} |0\rangle_2 + \left(\bigotimes_{i=k}^{1}(a_i |0\rangle + b_i |1\rangle)\right) \otimes b_{k+1} |1\rangle_2 . \tag{D.7}$$

This state is now put through the circuit. Note that the summand which is tensored with $|0\rangle_2$ is not affected by the C-Ω^j since the target is $|0\rangle_2$. Therefore only the summand tensored with $|1\rangle_2$ is advanced, and by claim 4.1, the basis states are shifted up by j steps. From claim 3.1, for the C-Ω^j gate, $j = 2^k$. Consider what this means in terms of computational basis states. When written in terms of computational basis states, equation D.7 is given by the sum

$$\left(\sum_{i=0}^{2^k-1} c_i |i\rangle_d\right) \otimes a_{k+1} |0\rangle_2 + \left(\sum_{i=0}^{2^k-1} c_i |i\rangle_d\right) \otimes b_{k+1} |1\rangle_2 . \tag{D.8}$$

Shifting all of the computational basis states in the second summand by $j = 2^k$ yields

$$\left(\sum_{i=0}^{2^k-1} c_i |i\rangle_d\right) \otimes a_{k+1} |0\rangle_2 + \left(\sum_{i=0}^{2^k-1} c_i |i + 2^k\rangle_d\right) \otimes b_{k+1} |1\rangle_2 . \tag{D.9}$$

In binary, adding 2^k to a number that is less than 2^k can be thought of as placing an additional 1 at the start of the string of binary digits. For example, where $(x)_2$ denotes that the number x is written in binary,

$$7 + 8 = (2^3 - 1) + (2^3) = (111)_2 + (1000)_2 = (1111)_2. \tag{D.10}$$

In this way, the action of the C-Ω^j gate on the tensor representation of the qudit given in equation D.7 can be thought of as adding a $|1\rangle$ in front of all of the binary representations of the computational basis states tensored with $|1\rangle_2$. Similarly, a $|0\rangle$ can be added in front of the binary representations of the computational basis states tensored with $|0\rangle_2$, since it still represents the same number $(01 = 1)$.

$$|0\rangle \otimes \left(\bigotimes_{i=k}^{1}(a_i |0\rangle + b_i |1\rangle)\right) \otimes a_{k+1} |0\rangle_2 + |1\rangle \otimes \left(\bigotimes_{i=k}^{1}(a_i |0\rangle + b_i |1\rangle)\right) \otimes b_{k+1} |1\rangle_2 \tag{D.11}$$

$$= a_{k+1} |0\rangle \otimes \left(\bigotimes_{i=k}^{1}(a_i |0\rangle + b_i |1\rangle)\right) \otimes |0\rangle_2 + b_{k+1} |1\rangle \otimes \left(\bigotimes_{i=k}^{1}(a_i |0\rangle + b_i |1\rangle)\right) \otimes |1\rangle_2 . \tag{D.12}$$

It is only possible to add these additional kets if the $d \geq 2^{k+1}$, else the Hilbert space is not large enough to accommodate the additional states being represented, hence the need for that constraint. The next step of the circuit is to use some map U to pair the even numbered

computational basis states with $|0\rangle_2$ and the odd numbered computational basis states with $|1\rangle_2$ to convert the $|1\rangle_2$ to a $|0\rangle_2$ via a C-X operation, after which U is uncomputed. On the qudit state, since no amplitude changes are involved, this is akin to doing a $UU^{-1} = I$ and so leaves the qudit state untouched. Therefore the final state is given by

$$a_{k+1}|0\rangle \otimes \left(\bigotimes_{i=k}^{1}(a_i|0\rangle + b_i|1\rangle)\right) \otimes |0\rangle_2 + b_{k+1}|1\rangle \otimes \left(\bigotimes_{i=k}^{1}(a_i|0\rangle + b_i|1\rangle)\right) \otimes |0\rangle_2 \quad \text{(D.13)}$$

$$= (a_{k+1}|0\rangle + b_{k+1}|1\rangle) \otimes \left(\bigotimes_{i=k}^{1}(a_i|0\rangle + b_i|1\rangle)\right) \otimes |0\rangle_2 \quad \text{(D.14)}$$

$$= \left(\bigotimes_{i=k+1}^{1}(a_i|0\rangle + b_i|1\rangle)\right) \otimes |0\rangle_2 . \quad \text{(D.15)}$$

So the statement is true by induction. □