

Quantum Versions of Random Walks

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Submitted: January 2, 2021

Abstract

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1 Introduction

Quantum walks are powerful tools in the landscape of quantum computing. Much like their classical analogues, they exhibit many properties that are desirable for computations and are an extremely useful building block for many algorithms designed for quantum computers [4]. However, they also have some very significant divergences from classical random walks, including a quadratic speed up in their spreading, and quantum correlations that have no classical comparison. Their power is such that it has been shown that quantum walks can simulate any quantum computation and therefore are a model for universal quantum computing [7]. There is also evidence of robust performance even when the quantum computer is not perfectly isolated from its environment, leading to situations where decoherences due to interactions with the environment is beneficial for a given computation [6]. It is possible to divide quantum walks into two categories, discrete time and continuous time. The latter have been shown to solve a wide range of problems in a number of different settings. However, the focus of this report will be on entangled state generation and transfer which necessitates the need for more than one Hilbert subspace in our system, a setting which lends itself much more readily to discrete time quantum walks.

Quantum walks have been shown to have great versatility in the generation and transfer of entanglement, quantum correlations which have no classical analogue, within a quantum system. Entanglement is a key resource for many quantum computing protocols [1][2][3], and is the source of the quadratic speedup evident in so many quantum algorithms that best their classical counterparts.

In this report we will first review discrete quantum walks on different graphs in section 2. Section 3 will then discuss using discrete quantum walks in order to transfer entanglement between subspaces within a quantum walk system. Finally, a short summary is presented in section 4.

As mentioned above, discrete time quantum walks are much more suited for the purposes of this report, therefore future references to quantum walks will be assumed to be the discrete variant unless stated otherwise. We will also use Q.W. to denote (discrete) quantum walk. The mathematical notation used in this report follows standard conventions, however it's worth making clear the equivalency between $|u_1\rangle |u_2\rangle \equiv |u_1\rangle \otimes |u_2\rangle$, where both forms will be used interchangeably.

2 Walks on Different Graphs

2.1 Classical Random Walk on 1-D Lattice

Before discussing quantum random walks, we first outline the basic premise of the classical random walk, in this case on the 1-D lattice (a discrete number line). The walker starts at the origin and before taking a step to the left (-1) or the right (+1), they flip a (unbiased) coin to decide which direction to take a step in, moving to the right if the coin lands on heads

and to the left if the coin lands on tails. By repeating this process we can plot a probability distribution of where the walker is likely to end up after n steps in the walk.

2.2 1-D Lattice

We now use a similar process to define our quantum counterpart to the classical walk on a 1-D lattice. In our walker system, we can divide the overall Hilbert space of the Q.W. \mathcal{H} into two subspaces, the coin subspace \mathcal{H}_C and the position subspace of the walker \mathcal{H}_W .

$$\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_W. \quad (1)$$

We note that whilst we do not place an constraints on the size of \mathcal{H}_W , we choose $\dim(\mathcal{H}_C) = 2$, which is an obvious thing to do since the classical coin has two possible states. To aid distinguishability between coin states and position states, we write that

$$\langle \mathcal{H}_C \rangle = \{|\uparrow\rangle, |\downarrow\rangle\} \quad (2)$$

$$\langle \mathcal{H}_W \rangle = \{|k\rangle | k \in \mathbb{Z}\} \quad (3)$$

where $\langle U \rangle$ denotes a set of vectors which span U . Therefore, the states $|\uparrow\rangle, |\downarrow\rangle$ take the place of heads and tails on our quantum "coin". Having defined the Hilbert space within which the walk will be conducted in, we can now define operators within our space that will dictate how the Q.W. will proceed. We first define the "coin flip" operator $C \in \mathcal{H}_C$. There are several choices for C , details of which can be found here [5]. As detailed in [5], for walks on a line, if we restrict ourselves to choosing an unbiased coin with real coefficients the Hadamard coin

$$C = \frac{1}{\sqrt{2}} [|\uparrow\rangle \langle\uparrow| + |\uparrow\rangle \langle\downarrow| + |\downarrow\rangle \langle\uparrow| - |\downarrow\rangle \langle\downarrow|] \quad (4)$$

$$= \frac{1}{\sqrt{2}} [(|\uparrow\rangle + |\downarrow\rangle) \langle\uparrow| + (|\uparrow\rangle - |\downarrow\rangle) \langle\downarrow|] \quad (5)$$

is the only choice of coin available. Equation (5) makes obvious the action of C , if the coin state is $|\uparrow\rangle$ then it becomes an equal superposition of $|\uparrow\rangle + |\downarrow\rangle$, if the coin state is in $|\downarrow\rangle$ then we get an equal superposition of $|\uparrow\rangle - |\downarrow\rangle$. These two equal superpositions are often denoted as $|+\rangle$ and $|-\rangle$ respectively.

We then define our shift operator $S \in \mathcal{H}$ which allows the position of our walker to change, dependent on the state of the coin.

$$S = \sum_k |\uparrow\rangle \langle\uparrow| \otimes |k+1\rangle \langle k| + |\downarrow\rangle \langle\downarrow| \otimes |k-1\rangle \langle k| \quad (6)$$

Again, this representation of S makes manifest its effect on our walker. If the coin is in the $|\uparrow\rangle$, then we take a step in the $+1$ direction, if in the $|\downarrow\rangle$ then we take a step in the -1 direction. The probability distribution of such a walk is plotted in Fig [FIG], where the initial coin state is $|\downarrow\rangle$, and is compared to a classical random walk. Whilst this highlights the faster spreading of the Q.W. away from the origin, there are not as many interesting properties of this variant of Q.W. compared to walks on other graphs.

2.3 N -Cycle

Having now introduced the Q.W. on a line, we can easily now discuss the Q.W. on the N -Cycle, a graph obtained by taking a 1-D lattice of size $N + 1$ and attaching the ends together. To study Q.W.s on N -Cycles we are able to use an identical formulation of C and an almost identical formulation of S as used for the Q.W. on a 1-D lattice, the only change for S being the summation range of k .

$$S = \sum_{k=0}^{N-1} |\uparrow\rangle \langle \uparrow| \otimes |k+1\rangle \langle k| + |\downarrow\rangle \langle \downarrow| \otimes |k-1\rangle \langle k|. \quad (7)$$

There has been much interesting analysis concerning quantum walks on N -Cycles, but we will focus here on a property that they possess known as *perfect state transfer*, when the initial state of the walker at some vertex on the graph is reproduced completely at another vertex on the graph after T steps in the quantum walk.

2.4 Hypercube

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3 Entanglement Transfer

3.1 Transfer using Two Walks

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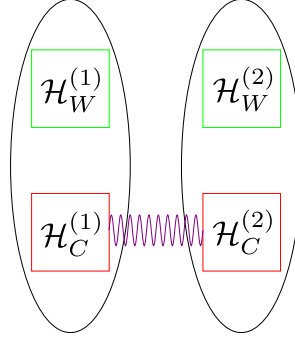


Figure 1: The initial prepared state has entanglement solely between the two coin subspaces.

3.2 Transfer using Multiple Coins

3.2.1 Three Coins

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3.2.2 Parrondo Sequences

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4 Conclusions

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Acknowledgements

The author would like to thank...

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Appendices

A Example long derivation