Constrained Optimization

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The Consumer's Problem: Review

- The **consumer's constrained optimization problem** is:

1) Choose: < some alternative >

2) In order to maximize: < some objective >

3) Subject to: < some constraints >

The Consumer's Problem: Tools

- **Constraints**:
  - Income and market prices
    - Market tradeoffs off between two goods

- **Utility function**:
- Preferences or utility to maximize
  - Individual tradeoff off between two goods

The Consumer's Problem: Tools

- The **consumer's constrained optimization problem**:

  - Choose a bundle to maximize utility
  - Subject to income and market prices

The Consumer's Problem: Mathematically

- $\max_{x,y} u(x,y)$

- $s.t. \quad p_x x + p_y y = m$

  - This requires calculus to solve. We will focus on **graphs** instead!

The Consumer's Optimum

Rule 1

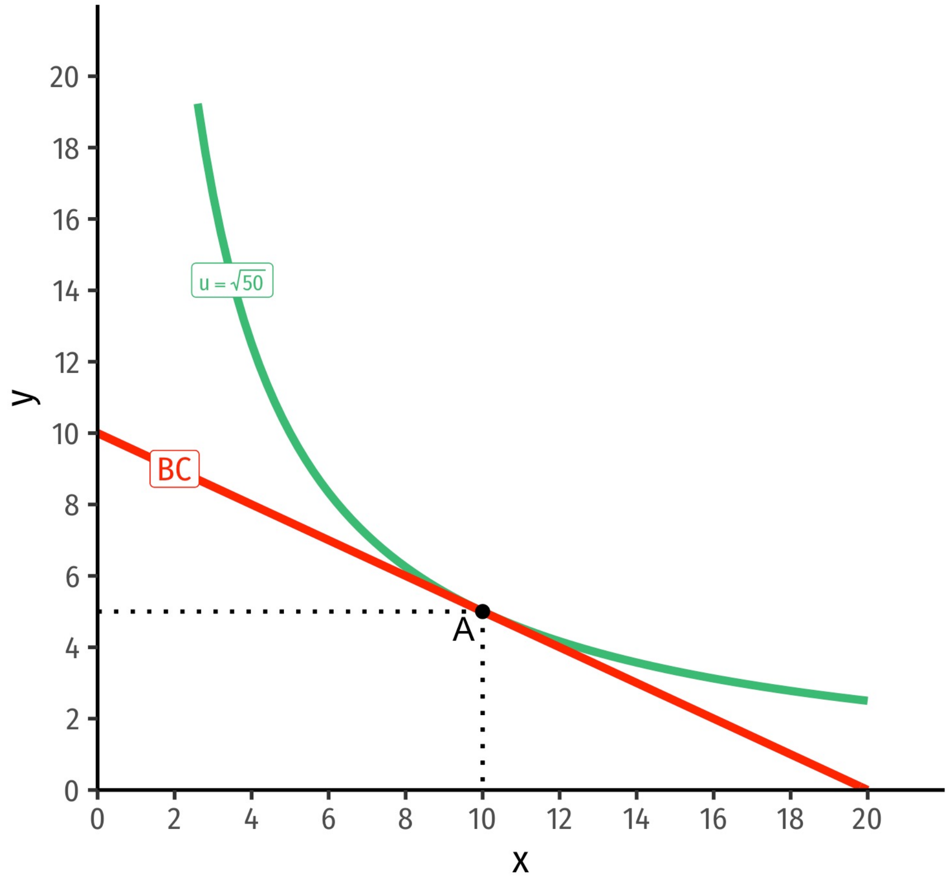
$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$

- Easier for calculation (slopes)

Rule 2

$$\frac{MU_x}{P_x} = \frac{MU_y}{p_y}$$

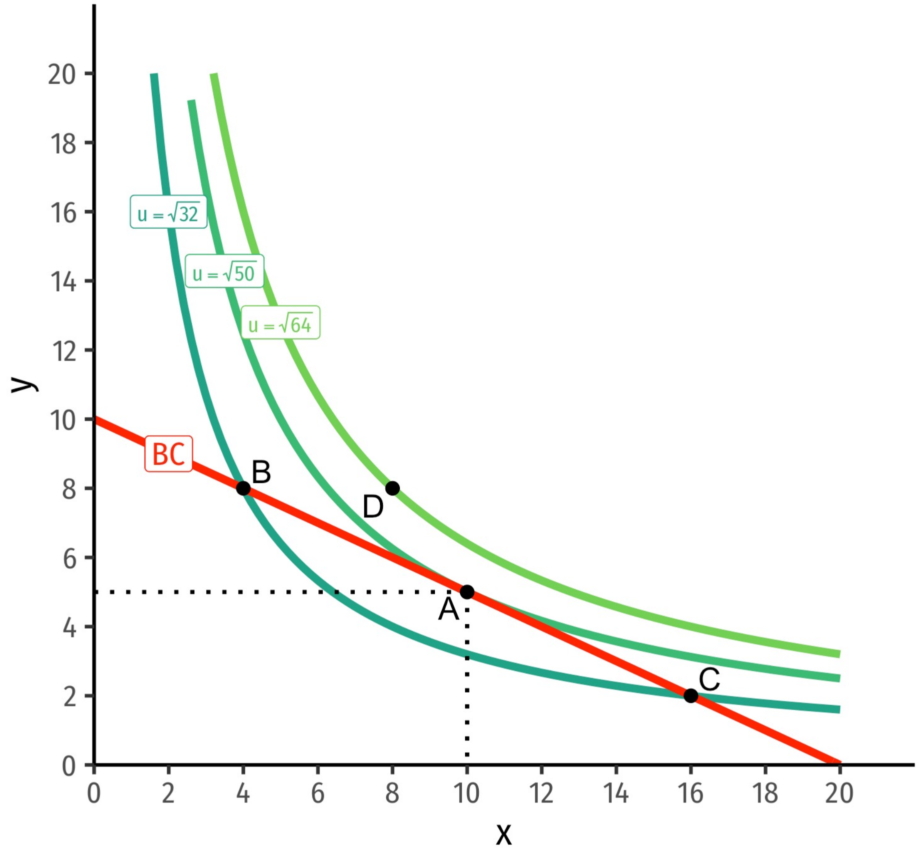
- Easier for intuition



$u(x, y) = \sqrt{xy}$, m = \$20, p_x = \$1, p_y = \$2

The Consumer's Optimum: Graphically

- **Graphical solution: Highest indifference curve *tangent* to budget constraint**
  - Bundle A!



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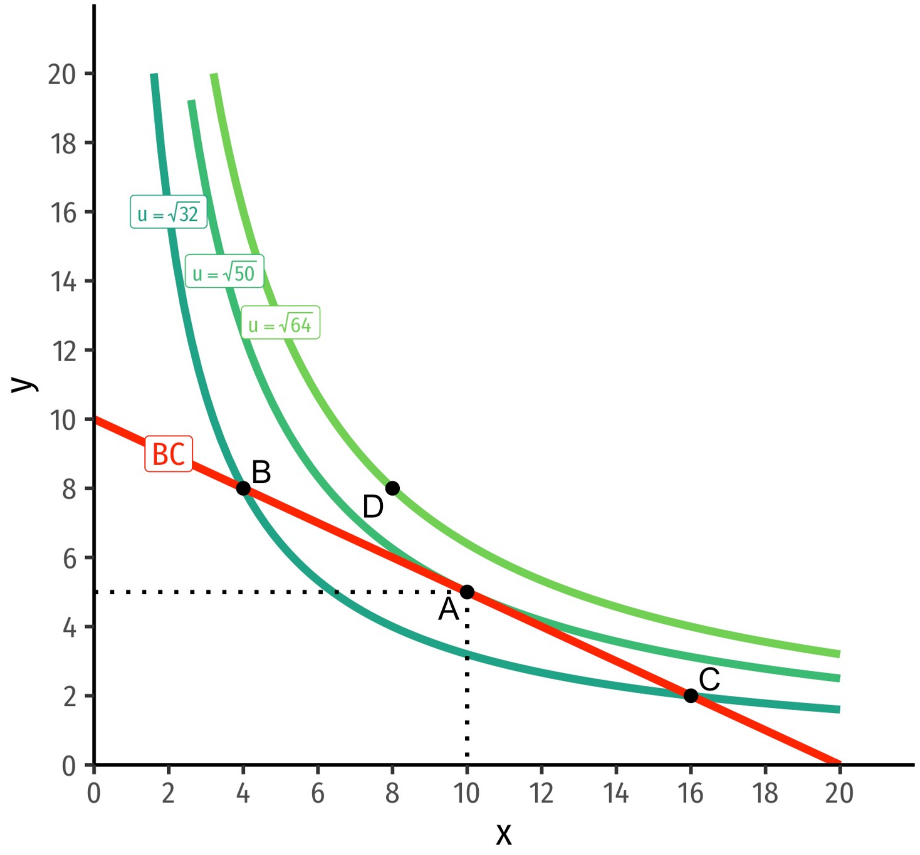
The Consumer's Optimum: Graphically

- **Graphical solution: Highest indifference curve *tangent* to budget constraint**
  - Bundle A!

- B or C spend all income, but a better combination exists
  - Averages ≻ extremes!



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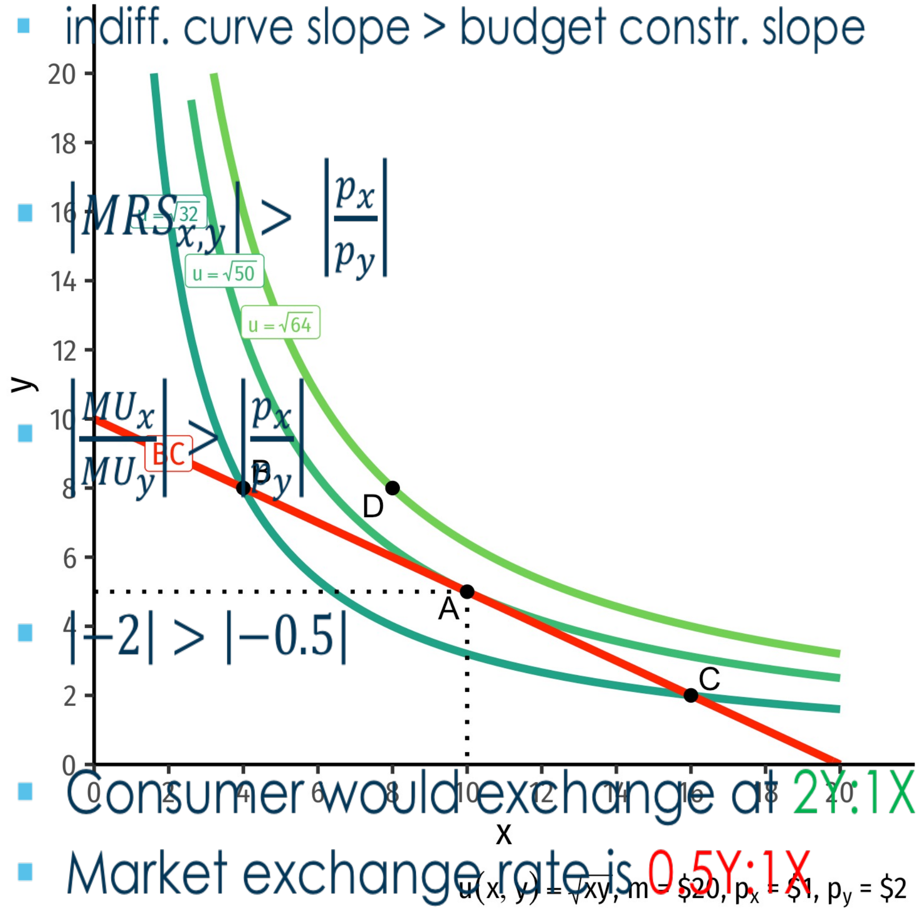
The Consumer's Optimum: Why Not B?

- indiff. curve slope > budget constr. slope

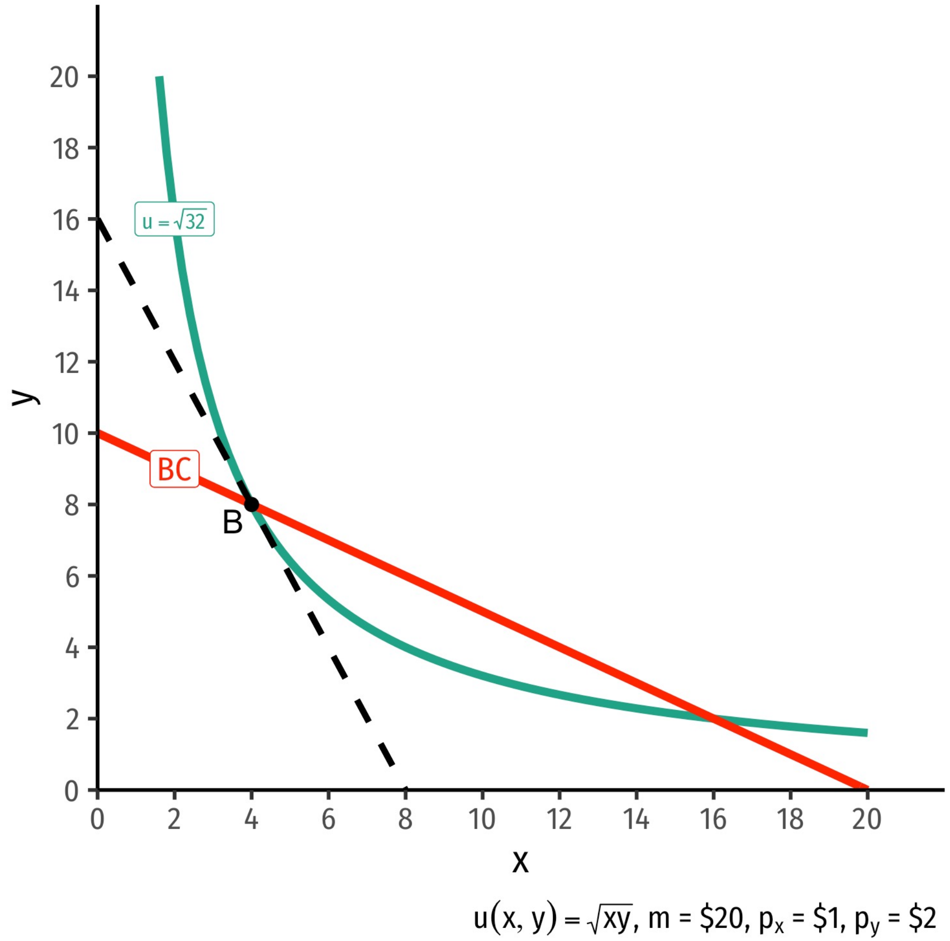


The Consumer's Optimum: Why Not B?

- indiff. curve slope > budget constr. slope

$$\left|MRS_{x,y}\right| > \left|\frac{p_x}{p_y}\right|$$

$$\left|\frac{MU_x}{MU_y}\right| > \left|\frac{p_x}{p_y}\right|$$



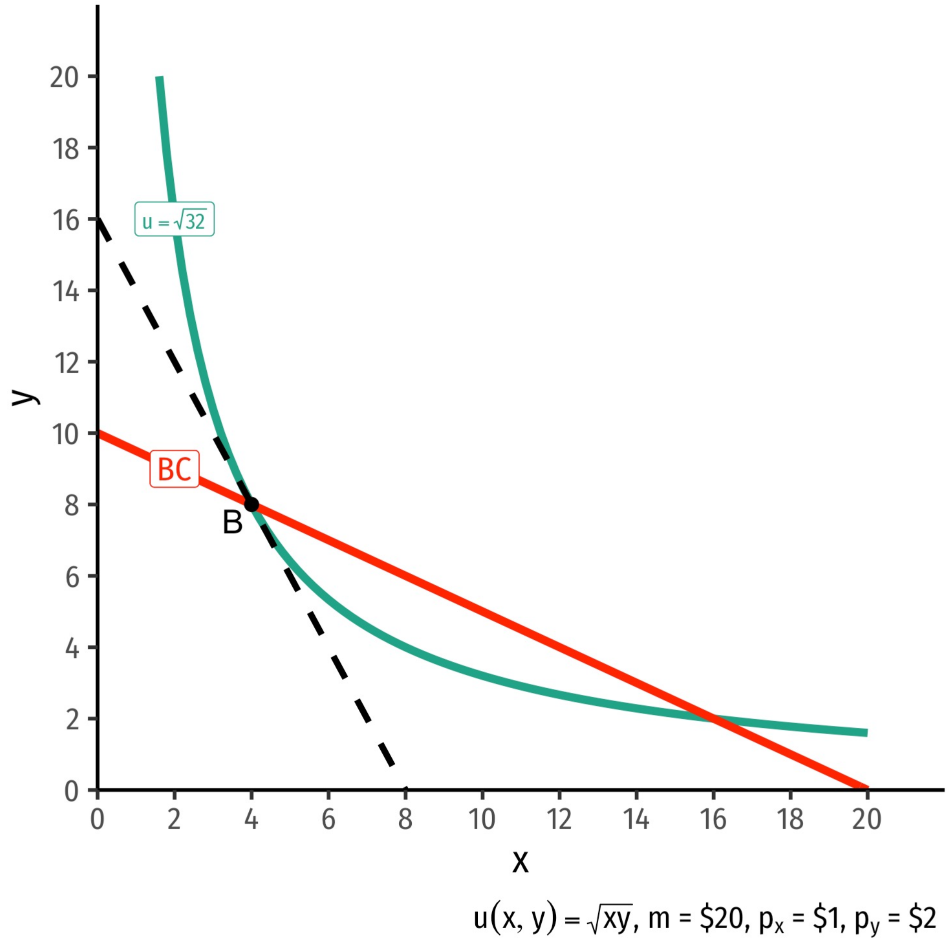
The Consumer's Optimum: Why Not B?

- indiff. curve slope > budget constr. slope

$$\left|MRS_{x,y}\right| > \left|\frac{p_x}{p_y}\right|$$

$$\left|\frac{MU_x}{MU_y}\right| > \left|\frac{p_x}{p_y}\right|$$

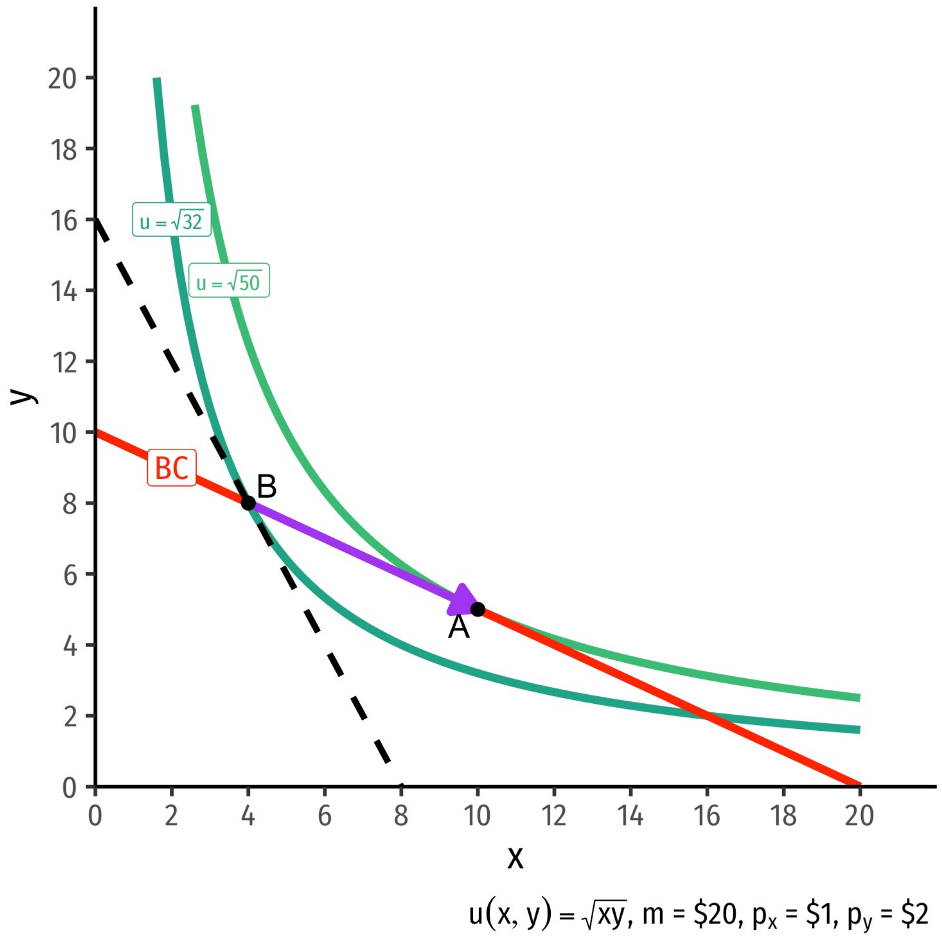
$$|-2| > |-0.5|$$



- Consumer would exchange at 2Y:1X
- Market exchange rate is 0.5Y:1X

The Consumer's Optimum: Why Not B?

- Can **spend less on y more on x** and get **more utility!**



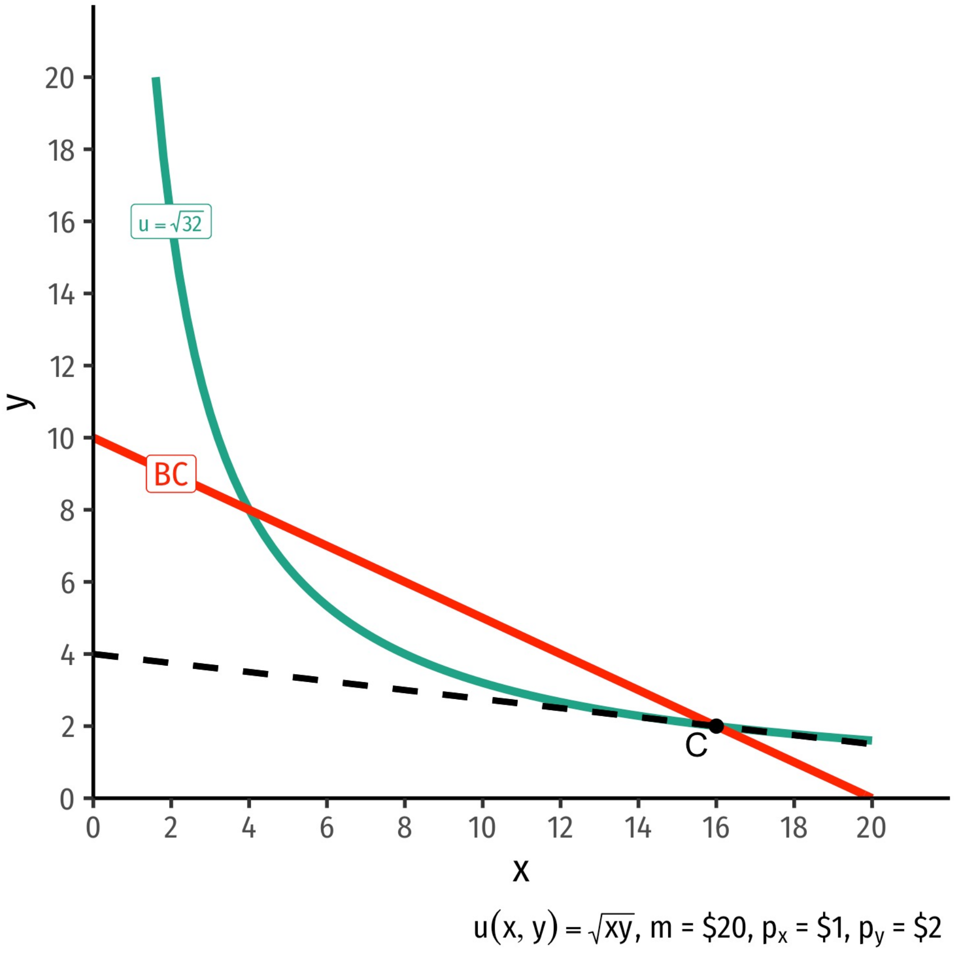
The Consumer's Optimum: Why Not B?

- indiff. curve slope > budget constr. slope

$$|MRS_{x,y}| < \left|\frac{p_x}{p_y}\right|$$

$$\left|\frac{MU_x}{MU_y}\right| < \left|\frac{p_x}{p_y}\right|$$

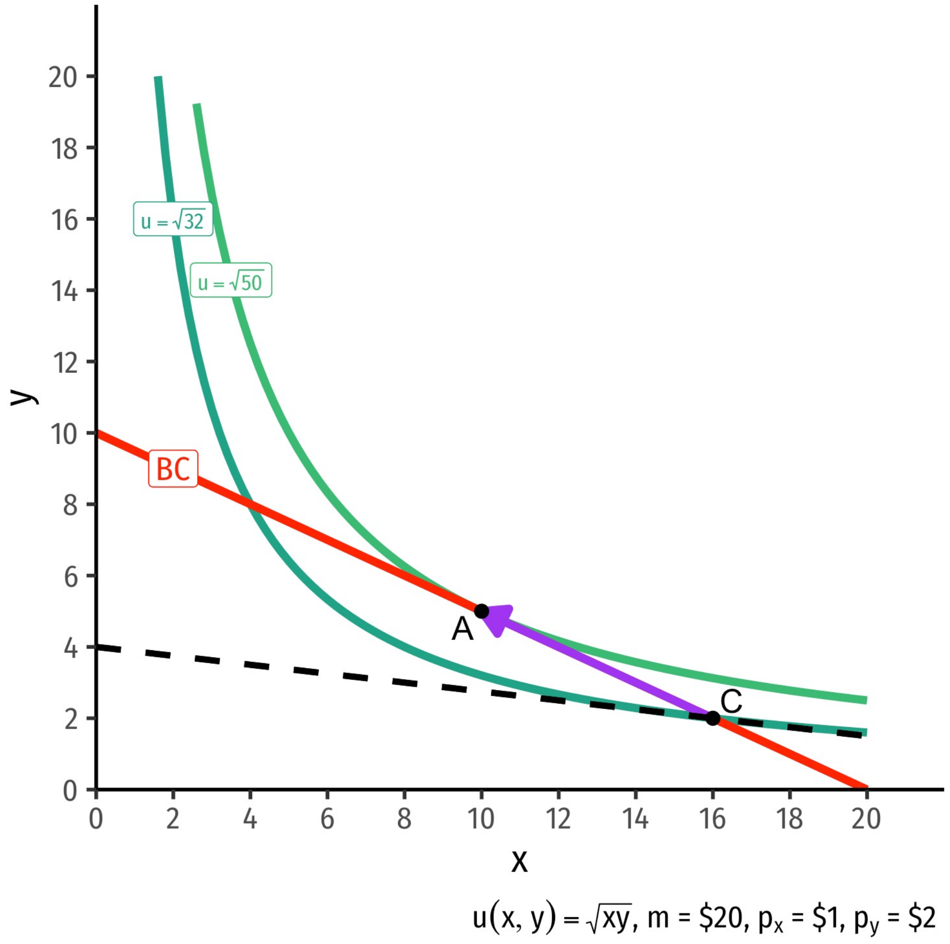
$$|-0.125| > |-0.5|$$



- Consumer would exchange at 0.125Y:X
- Market exchange rate is 0.5Y:X

The Consumer's Optimum: Why Not B?

- Can **spend less on x, more on y** and get **more utility!**



The Consumer's Optimum

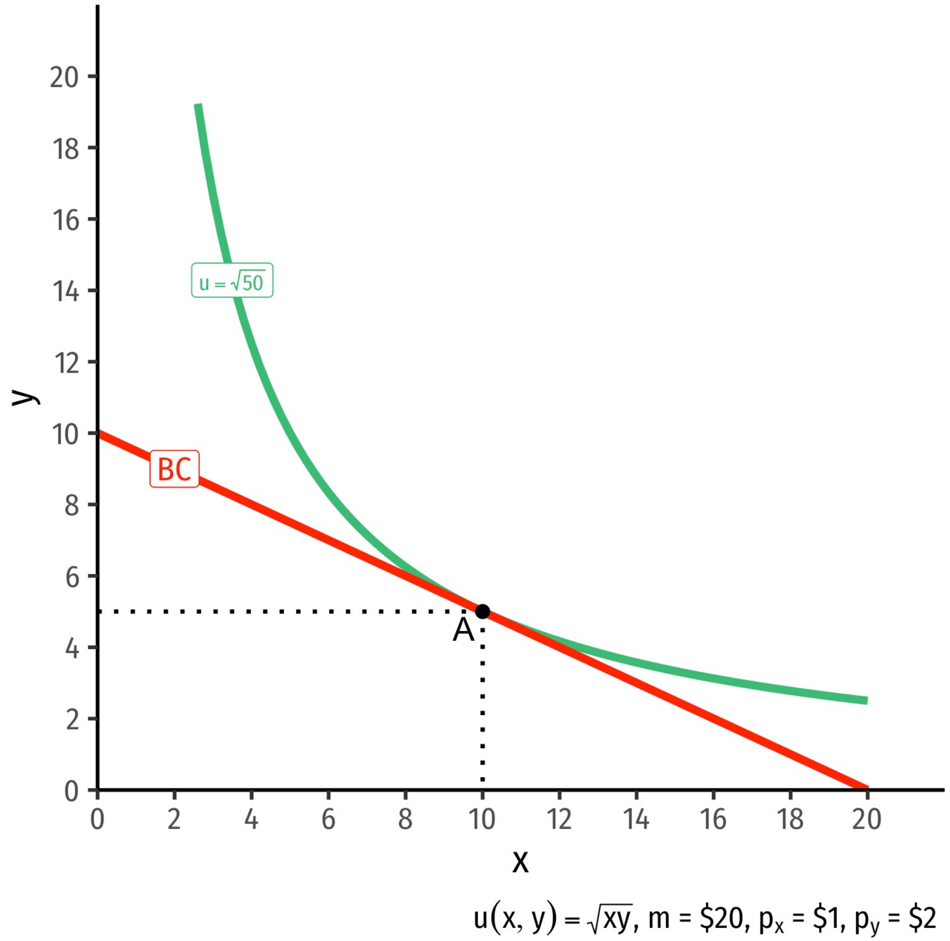
- indiff. curve slope > budget constr. slope

$$|MRS_{x,y}| = \left|\frac{p_x}{p_y}\right|$$

$$\left|\frac{MU_x}{MU_y}\right| = \left|\frac{p_x}{p_y}\right|$$

$$|-0.5| = |-0.5|$$

- **Consumer** would exchange at same rate as **market**



The Consumer's Optimum

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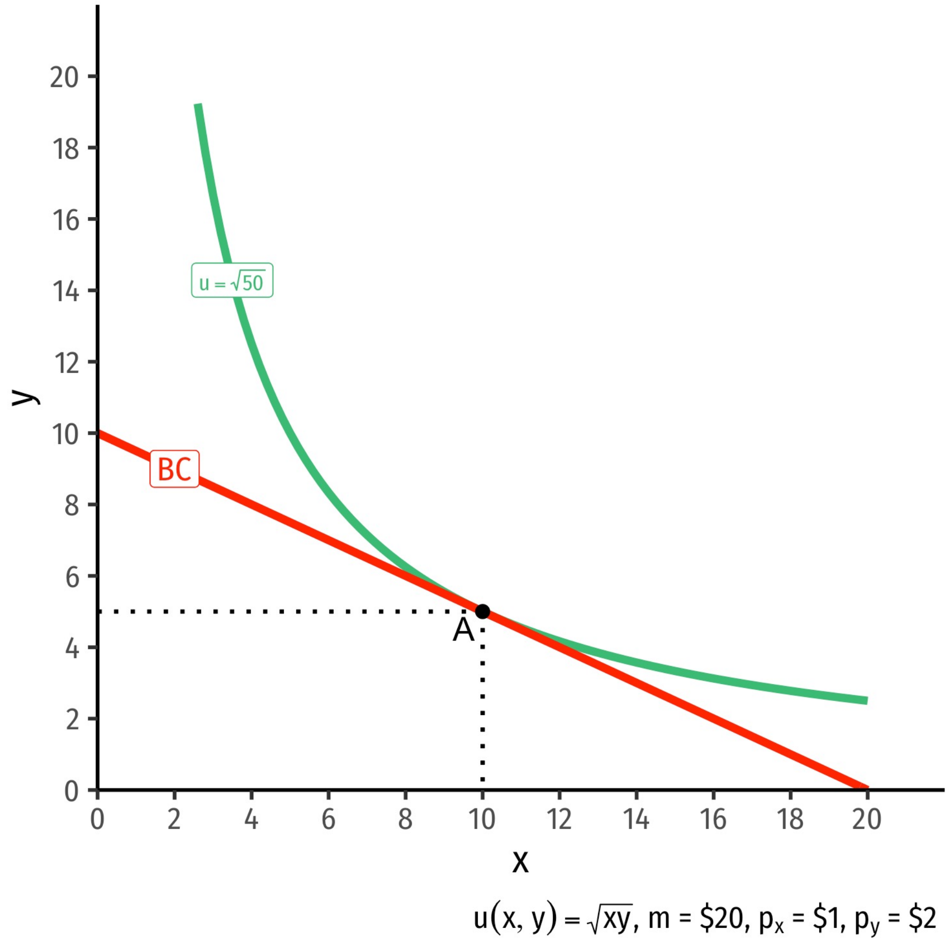
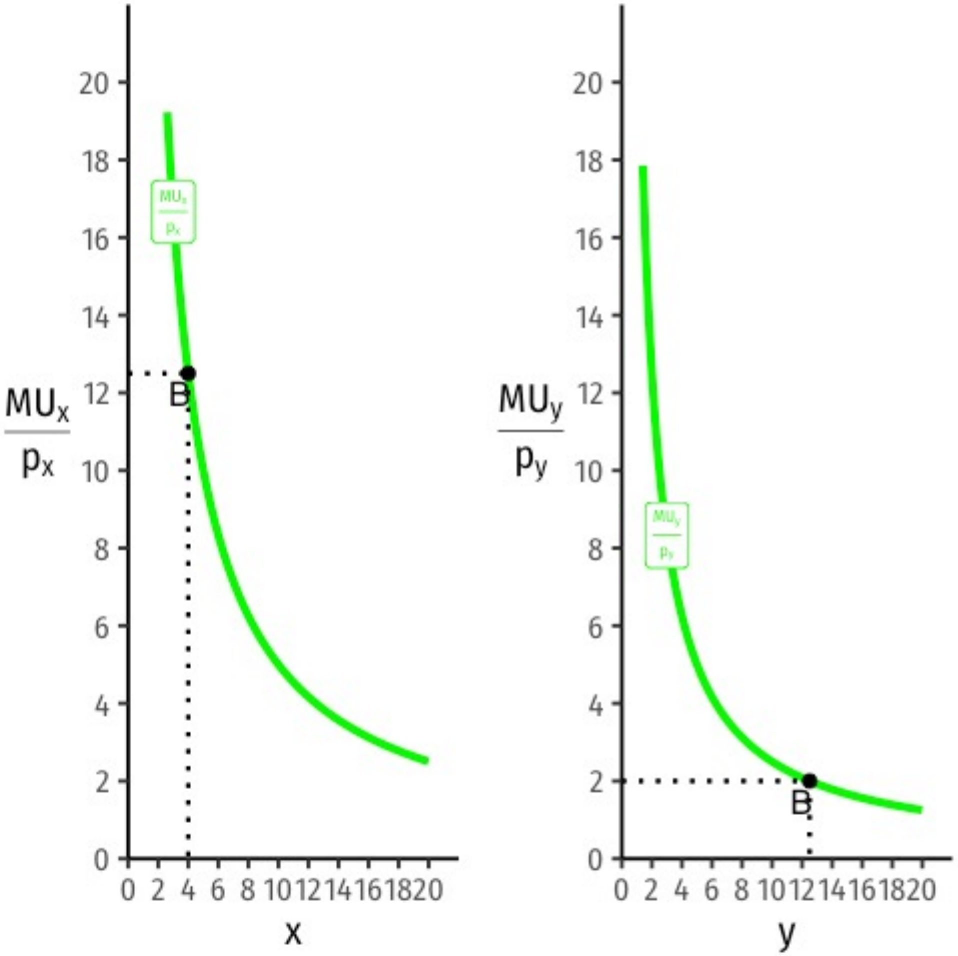


Visualizing the Equimarginal Rule

- Suppose you consume 4 of x and 12.5 of y (points B)

$$\frac{MU_x}{P_x} > \frac{MU_y}{p_y}$$

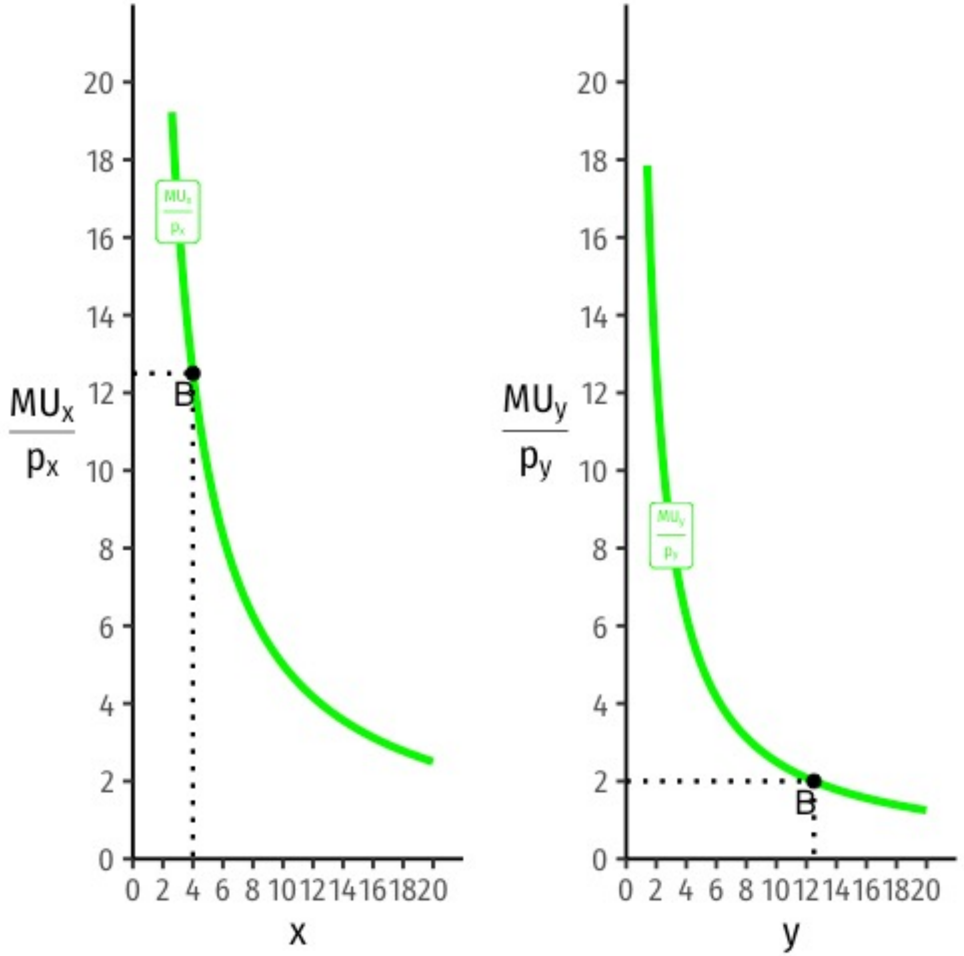


Visualizing the Equimarginal Rule

- Suppose you consume 4 of x and 12.5 of y (points B)

$$\frac{MU_x}{P_x} > \frac{MU_y}{p_y}$$

  - More "bang for your buck" with x than y
  - Consume more x, less y!

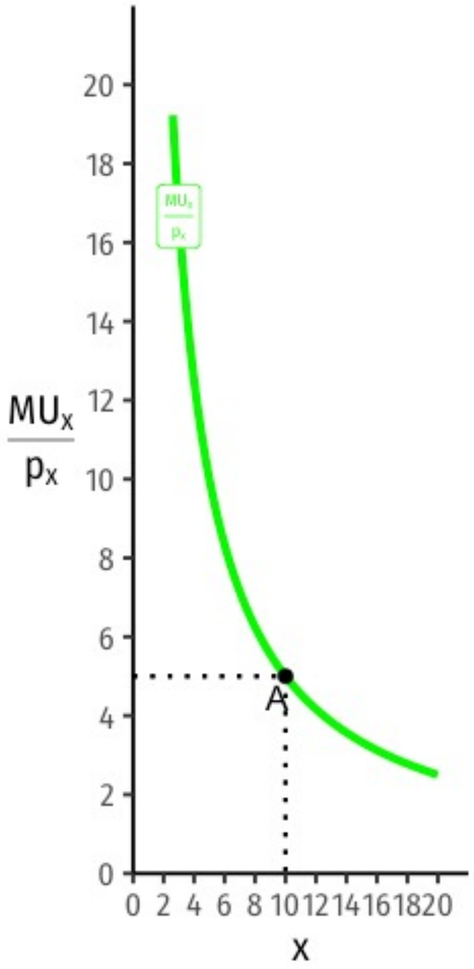


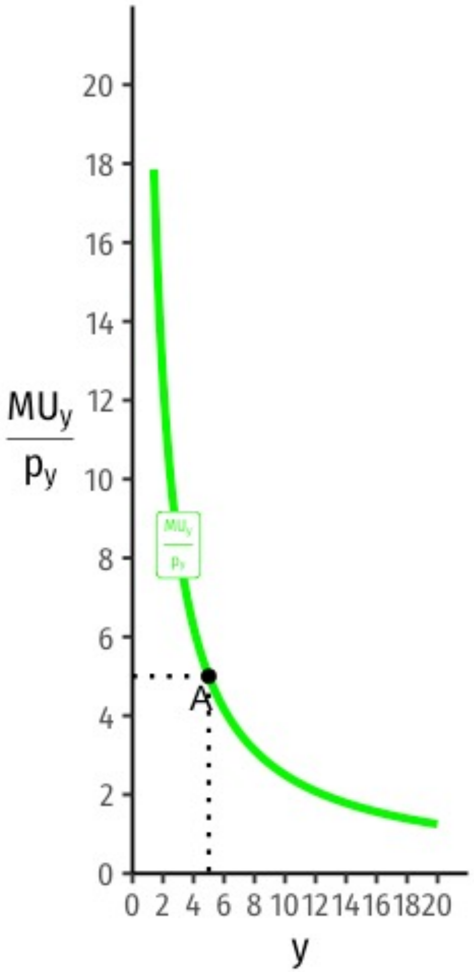
Visualizing the Equimarginal Rule

- At points A, consuming 10 of x and 5 of y

$$\frac{MU_x}{P_x} = \frac{MU_y}{p_y}$$

  - No change (more x, less x, more y, less y) that could increase your utility!
  - The optimum! Cost-adjusted marginal utilities are equalized





The Consumer's Optimum: The Equimarginal

$$\frac{MU_x}{P_x} = \frac{MU_y}{p_y} = \dots \frac{MU_n}{p_n}$$

- **Equimarginal Rule**: consumption is optimized where the **marginal utility per dollar spent** is **equalized** across all n possible goods/decisions

The Consumer's Optimum: The Equimarginal

$$\frac{MU_x}{P_x} = \frac{MU_y}{p_y} = \dots \frac{MU_n}{p_n}$$

- **Equimarginal Rule**: consumption is optimized where the **marginal utility per dollar spent** is **equalized** across all n possible goods/decisions
- You will always choose an option that gives higher marginal utility (e.g if $MU_x > MU_y$
- But each option has a different cost, so we weight each option by its cost, hence $\frac{MU_x}{P_x}$

The Consumer's Optimum: The Equimarginal

- Any **optimum** in economics: no better alternatives exist under current constraints

- No possible change in your consumption that would increase your utility

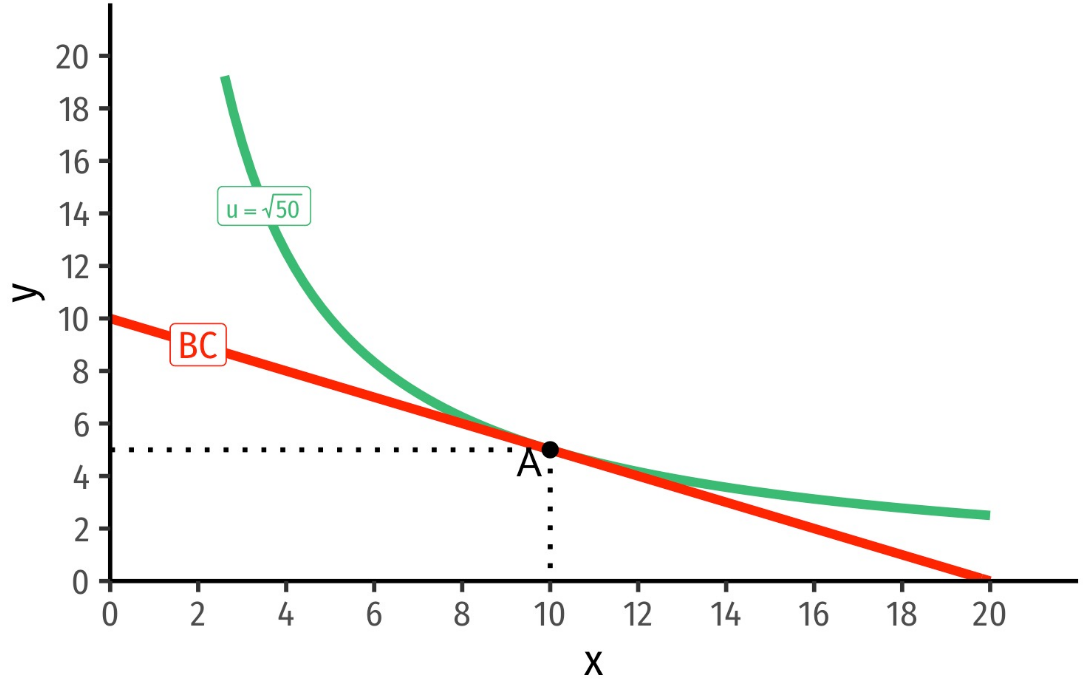
Markets Equalize Everyone's MRS

- **Markets make it so everyone faces the *same* relative prices**
  - Budget constraint. slope, $-\frac{p_x}{p_y}$
  - Note individuals' incomes, *m*, are certainly different!

- A person's optimal choice $\Longrightarrow$ they make same tradeoff as the market
  - Their MRS = relative price ratio
- **markets equalize everyone's MRS**

Markets Equalize Everyone's MRS

- Two people with very different income and preferences face the same market prices, and choose optimal consumption (points A and A') at an exchange rate of 0.5Y:1X



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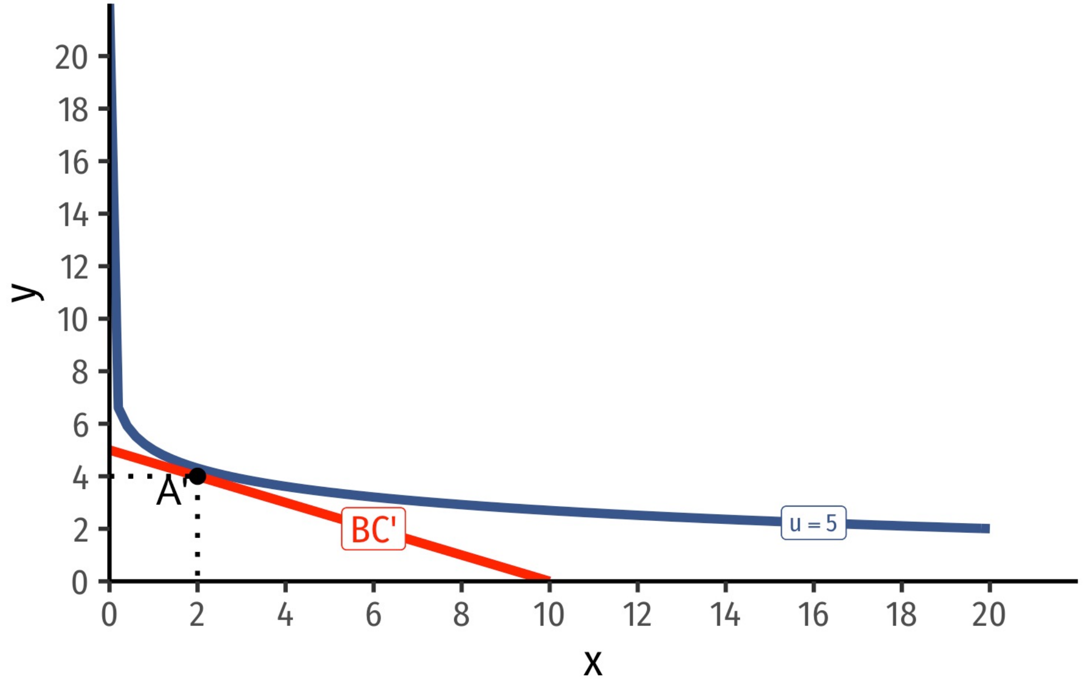
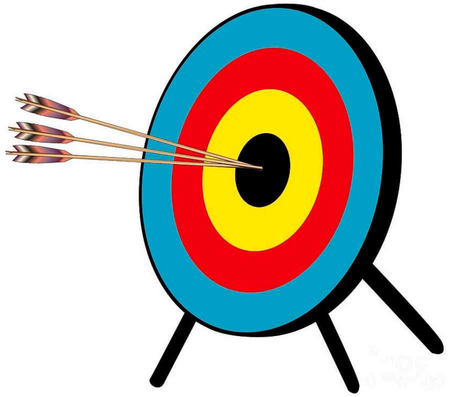


$u(x, y) = \ln(x) + y$, m = \$10, p_x = \$1, p_y = \$2

The Consumer's Optimum: The Equimarginal

- If possible people will always switch to a higher-valued option



- If a person has no *better* choices (under current constraints), they are at an **optimum**

- **If everyone is at an optimum**, the **system** is in **equilibrium**

Deriving Demand

The Consumer's Problem

- The **consumer's constrained optimization problem** is:

- **Choose: < a consumption bundle >**

- **In order to maximize: < utility >**

- **Subject to: < income and market prices >**

A Demand Function

- The **consumer's** **demand** **for good x**

$$q_x^D = q_x^D(m, p_x, p_y)$$

  - How does **demand for x** change?

1) Income effects = $\frac{\Delta q_x^D}{\Delta m}$
2) Cross-price effects = $\frac{\Delta q_x^D}{\Delta p_y}$
3) Own price effects = $\frac{\Delta q_x^D}{\Delta p_x}$

Income Effect

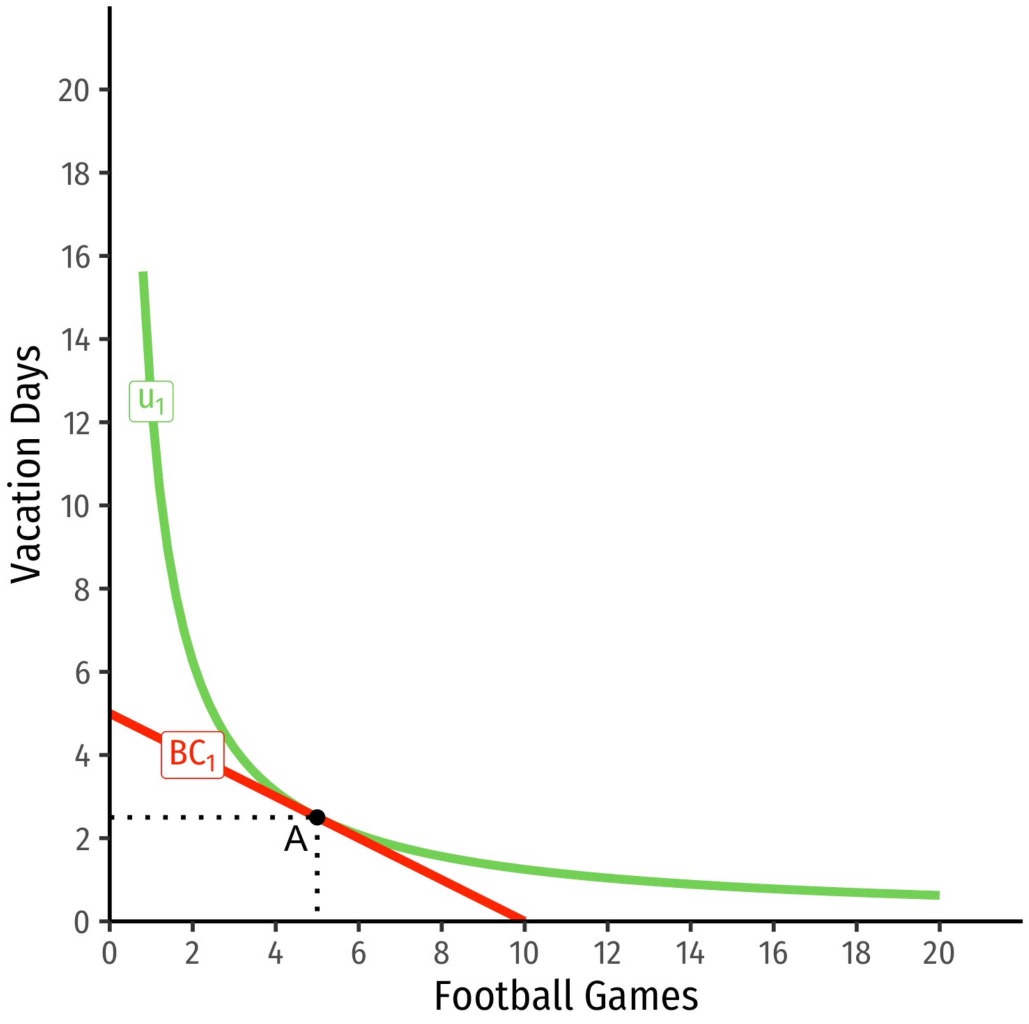
Income Effect

- **Income effect**: change in optimal consumption of a good associated with a change in (nominal) income, holding relative prices constant

$$\frac{\Delta q^D}{\Delta m} < ? > 0$$

Income Effect (Normal)

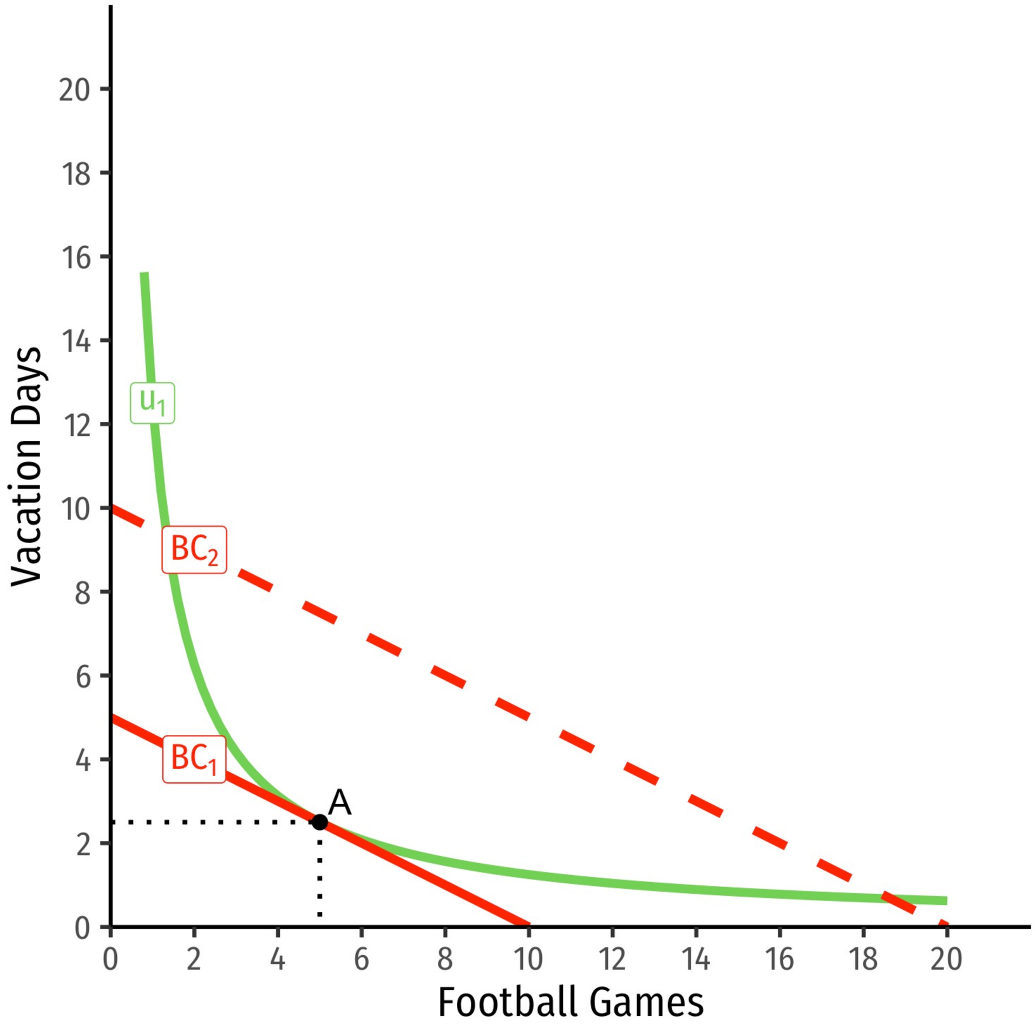
- Consider football tickets and vacation days



Income Effect (Normal)

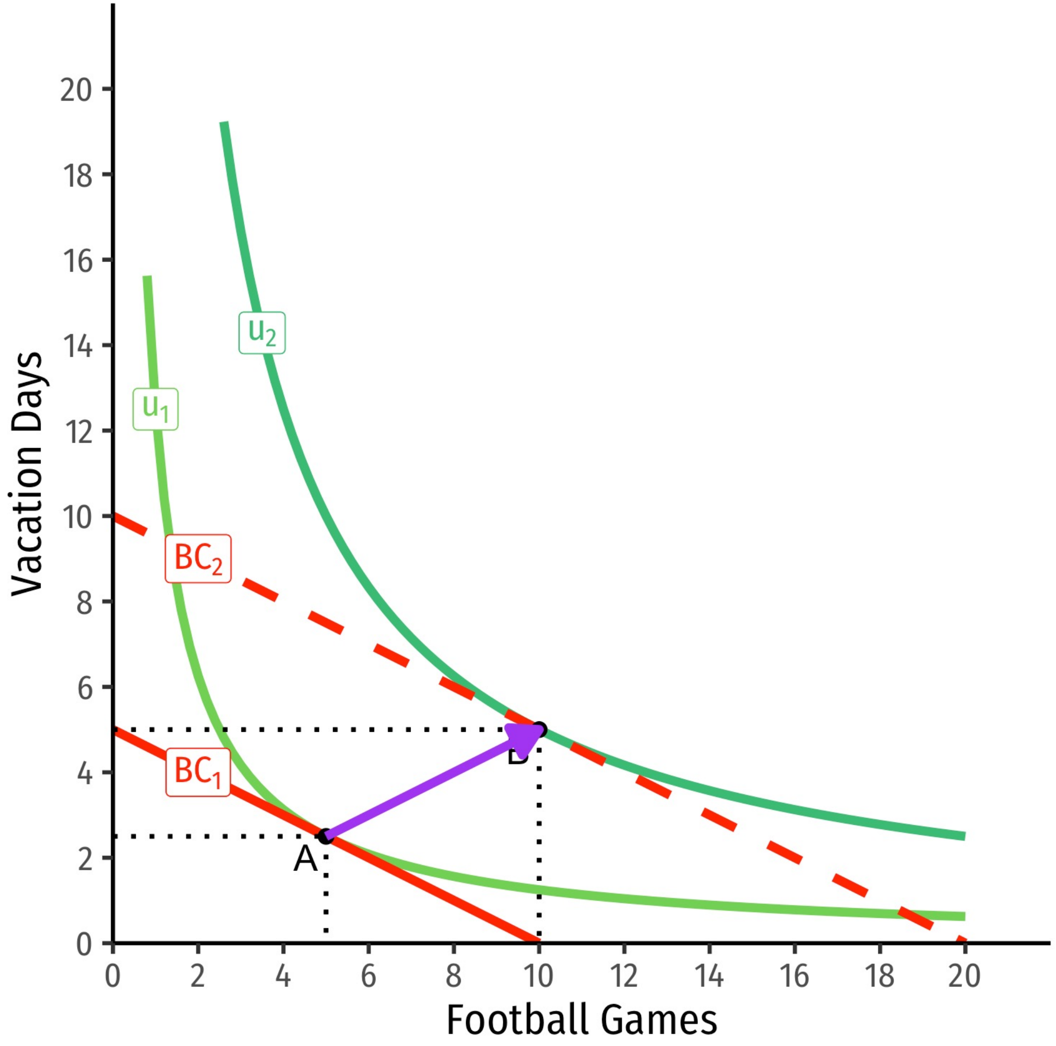
- Consider football tickets and vacation days

- Suppose income (m) increases



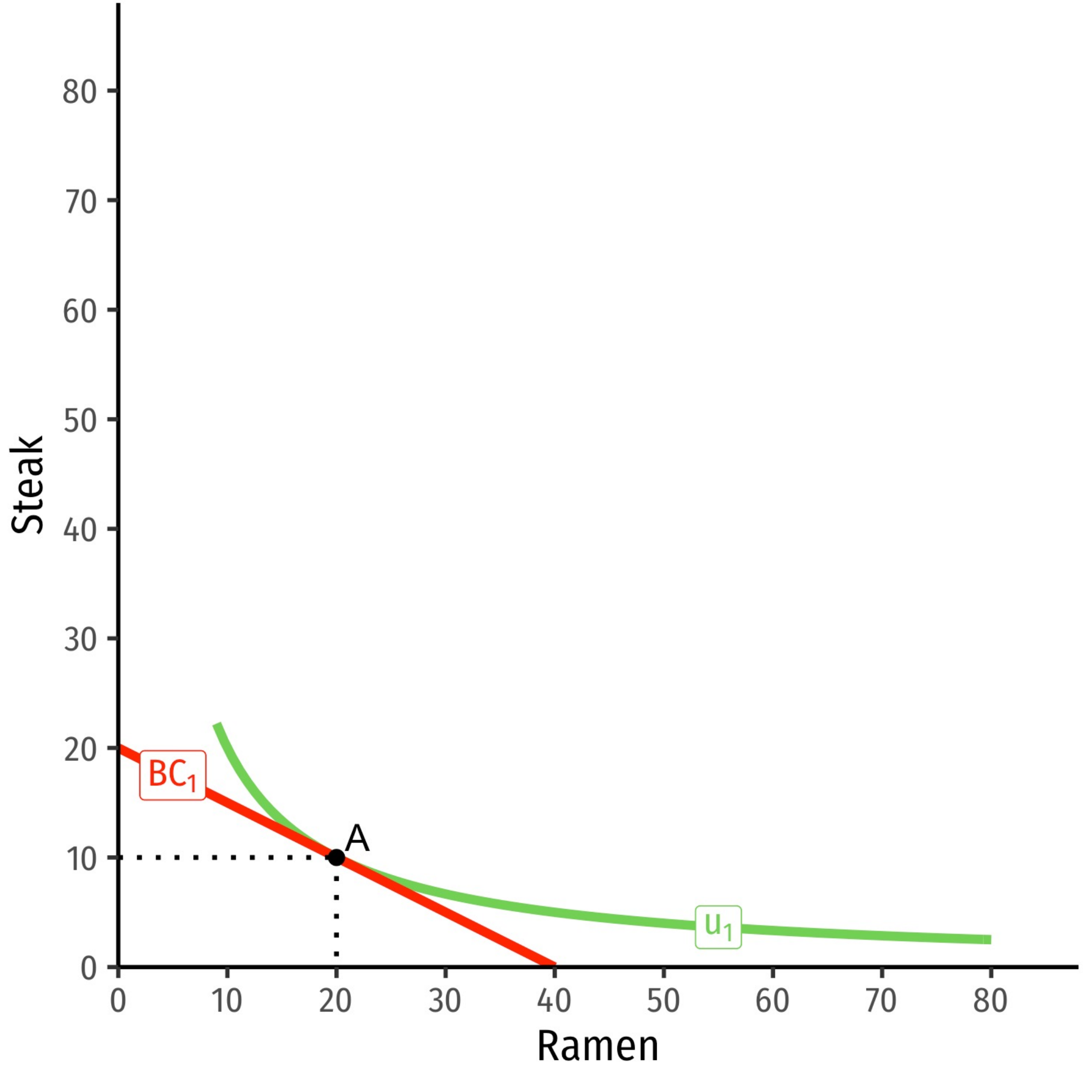
Income Effect (Normal)

- Consider football tickets and vacation days
- Suppose income (m) increases
- At new optimum (B), consumes more of both
- Then both goods are **normal goods**



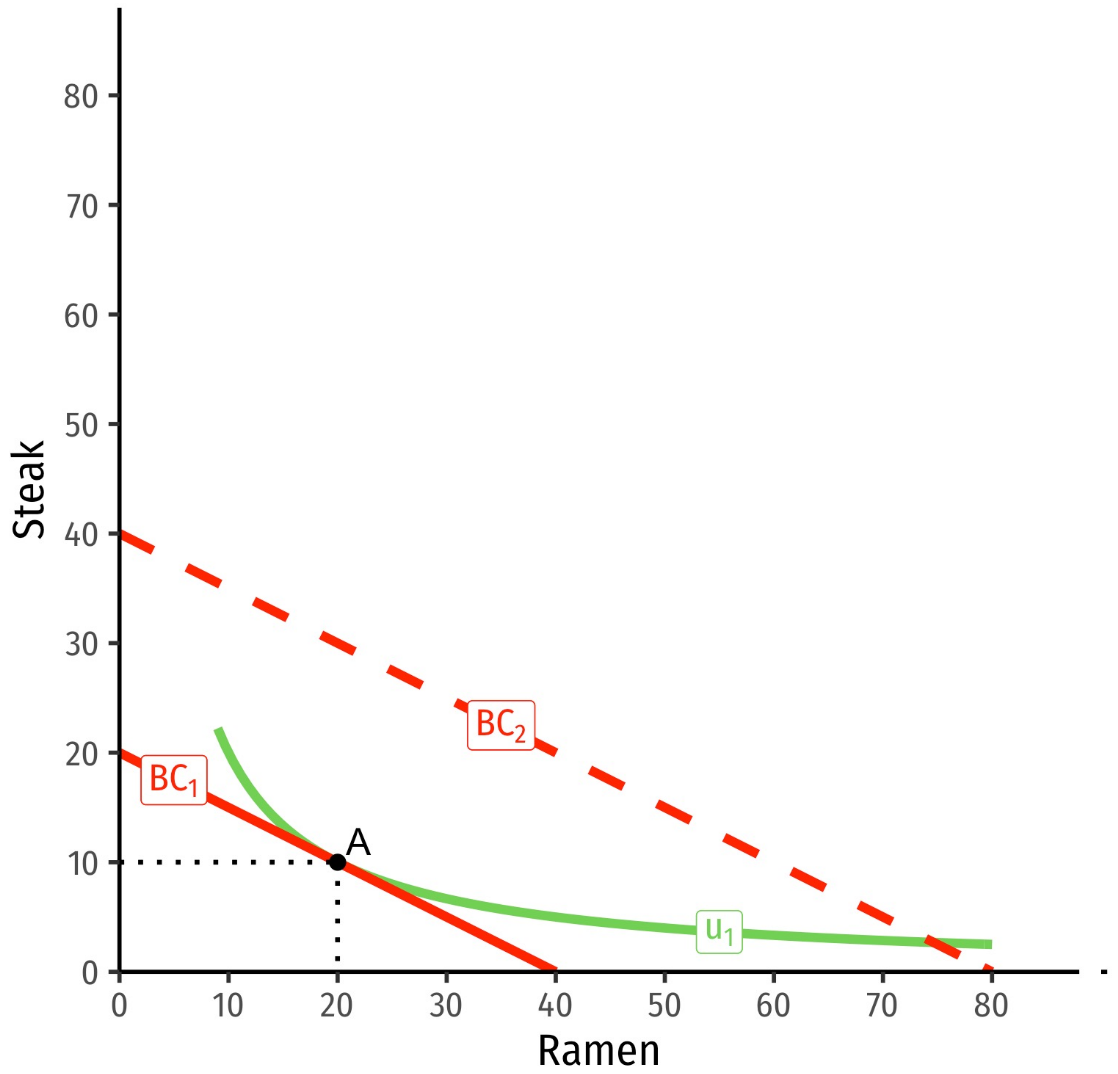
Income Effect (Inferior)

- Consider ramen and steak



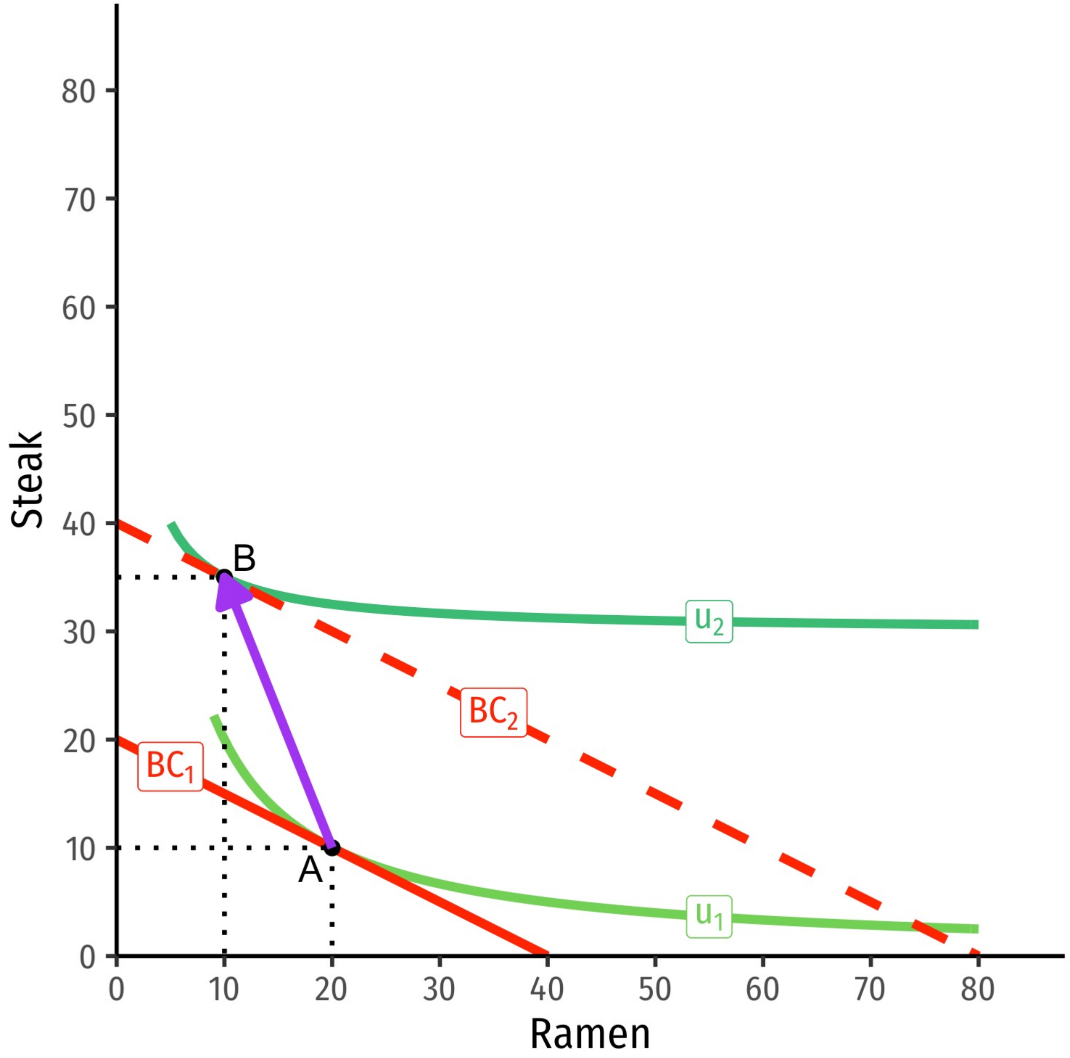
Income Effect (Inferior)

- Consider ramen and steak
- Suppose income (m) increases



Income Effect (Inferior)

- Consider ramen and steak
- Suppose income (m) increases
- At new optimum (B), consumes more steak, less ramen
- Steak is a **normal good**, ramen is an **inferior good**



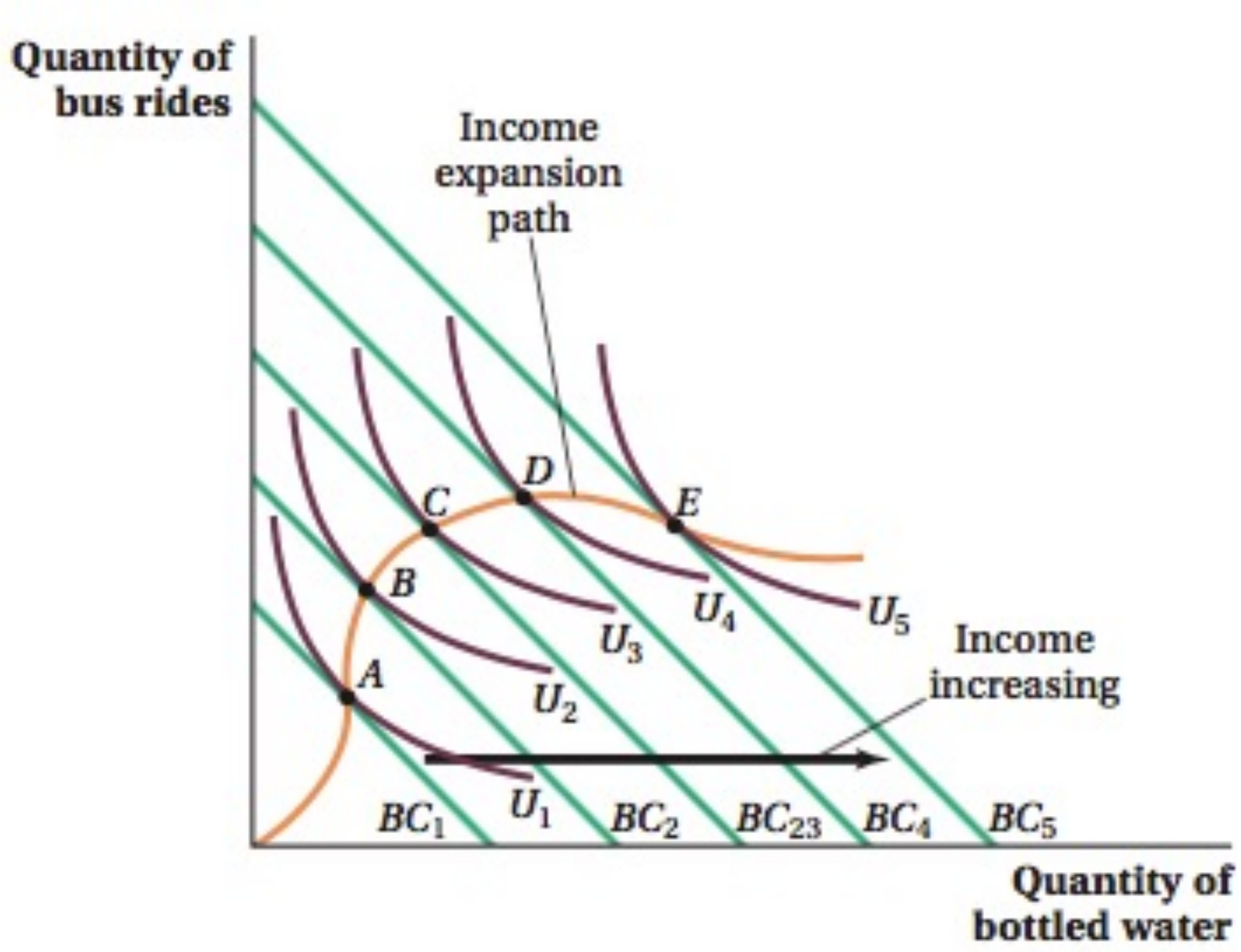
Income Effect

$$\frac{\Delta q^D}{\Delta m} < ? > 0$$

- **Normal goods**: consumption increases with more income (and vice versa)

- **Inferior goods**: consumption decreases with more income (and vice versa)

Income Expansion Path

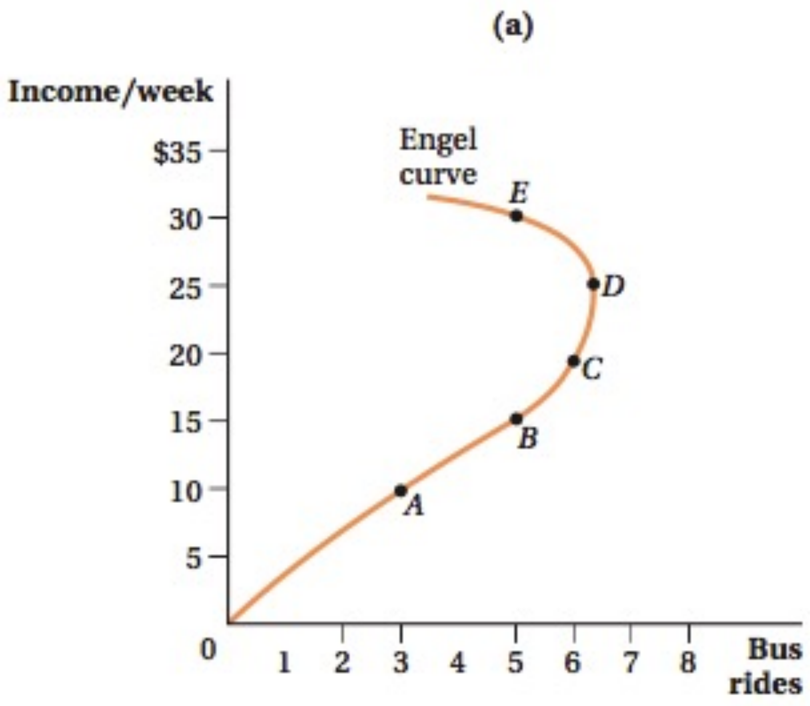


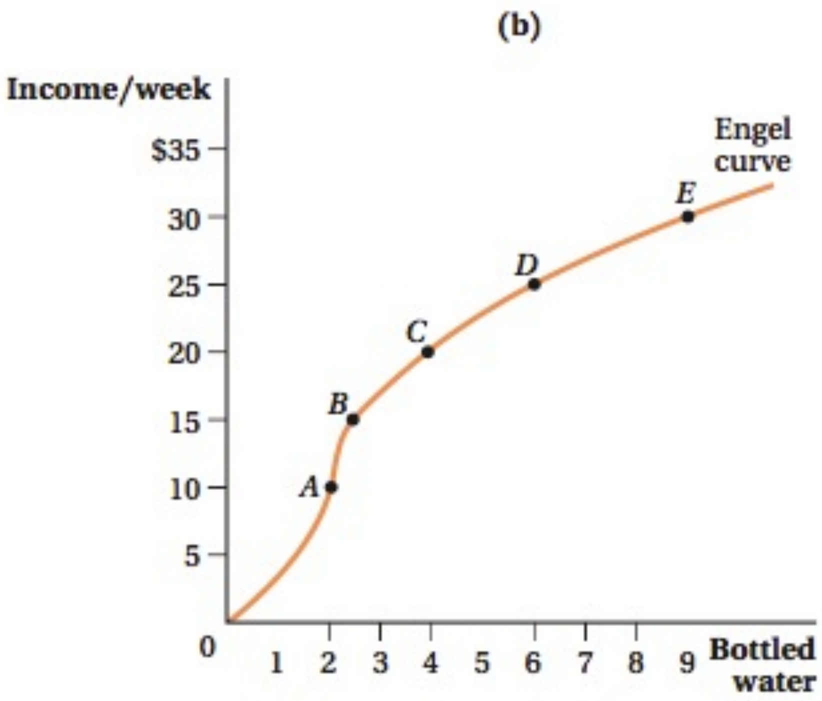
Goolsbee, et al (2011:169)

- **Income expansion path** describes how consumption of each good changes when income increases

Engel Curves

- **Engel curve** of each good is more helpful to visualize: shows how consumption of *one* good changes when income increases





Goolsbee, et al (2011:171)

Cross-Price Effects

Cross-Price Effects

- **Cross-price effect**: change in optimal consumption of a good associated with a change in price of *another* good income, holding the good's *own* price (and income) constant

$$\frac{\Delta q_x}{\Delta p_y} > ? < 0$$

Income and Substitution Effects

A Demand Function

- The **consumer's demand for good x**

$$q_x^D = q_x^D(m, p_x, p_y)$$

  - How does **demand for x** change?

1) Income effects = $\frac{\Delta q_x^D}{\Delta m}$
2) Cross-price effects = $\frac{\Delta q_x^D}{\Delta p_y}$
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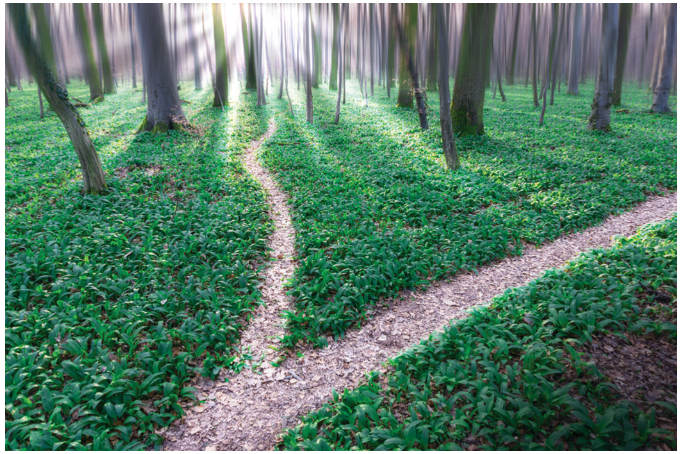


Own price Effect

The (Own) Price Effect

- **Price effect**: change in optimal consumption of a good associated with a change in its price
- holding income and other prices constant

$$\frac{\Delta q_x^D}{\Delta p_x} < 0$$

- **The law of demand**

Decomposing the Price Effect

- The **price effect** (law of demand) is actually the **net result of two effects**
- **(Real) income effect**: change in consumption due to change in real purchasing power
- **Substitution effect**: change in consumption due to change in relative prices
- **Price Effect** = **Real income effect** + **Substitution Effect**

(Real) Income Effect

(Real) Income Effect

- Income: $100
- $P_x = \$10$, $Q_x = 10$

- Suppose the price of x falls
- $P_x = \$5$, $Q_x = 20$

- This is the **real income effect**

(Real) Income Effect

- **Real income effect:**
  - consumption mix changes due to the price change
  - changes your **real income** or **purchasing power** (the amount of goods you can buy)

- Note your ***actual* (nominal) income** ($100) **never changed!**

(Real) Income Effect: Size

- The *size* of the income effect depends on how large a *portion of your budget* you spend on the good

- **Large-budget items**:
  - e.g. Housing/apartment rent, car prices
  - Price increase makes you much poorer
  - Price decrease makes you much wealthier

(Real) Income Effect: Size

- The *size* of the income effect depends on how large a *portion of your budget* you spend on the good

- **Small-budget items**:
  - e.g. pencils, toothpicks, candy
  - Price changes don't have much of an effect on your wealth or change your behavior much

Substitution Effect

Substitution Effect: Demonstration

- 1000's of goods, and none of them a major part of your budget
  - So real income effect is insignificant
- The price of one good, x increases
- You consume ***less*** of x relative to other goods because x is now *relatively* more expensive
- That's the **substitution effect**

Substitution Effect: Demonstration

- **Substitution effect**: consumption mix changes because of a change in **relative prices**
- Buy more of the (now) relatively cheaper items
- Buy less of the (now) relatively more expensive item (x)

Putting the Price Effects Together

Putting the Effects Together

- **Real income effect**: change in consumption due to change in real purchasing power
- **Substitution effect**: change in consumption due to change in relative prices

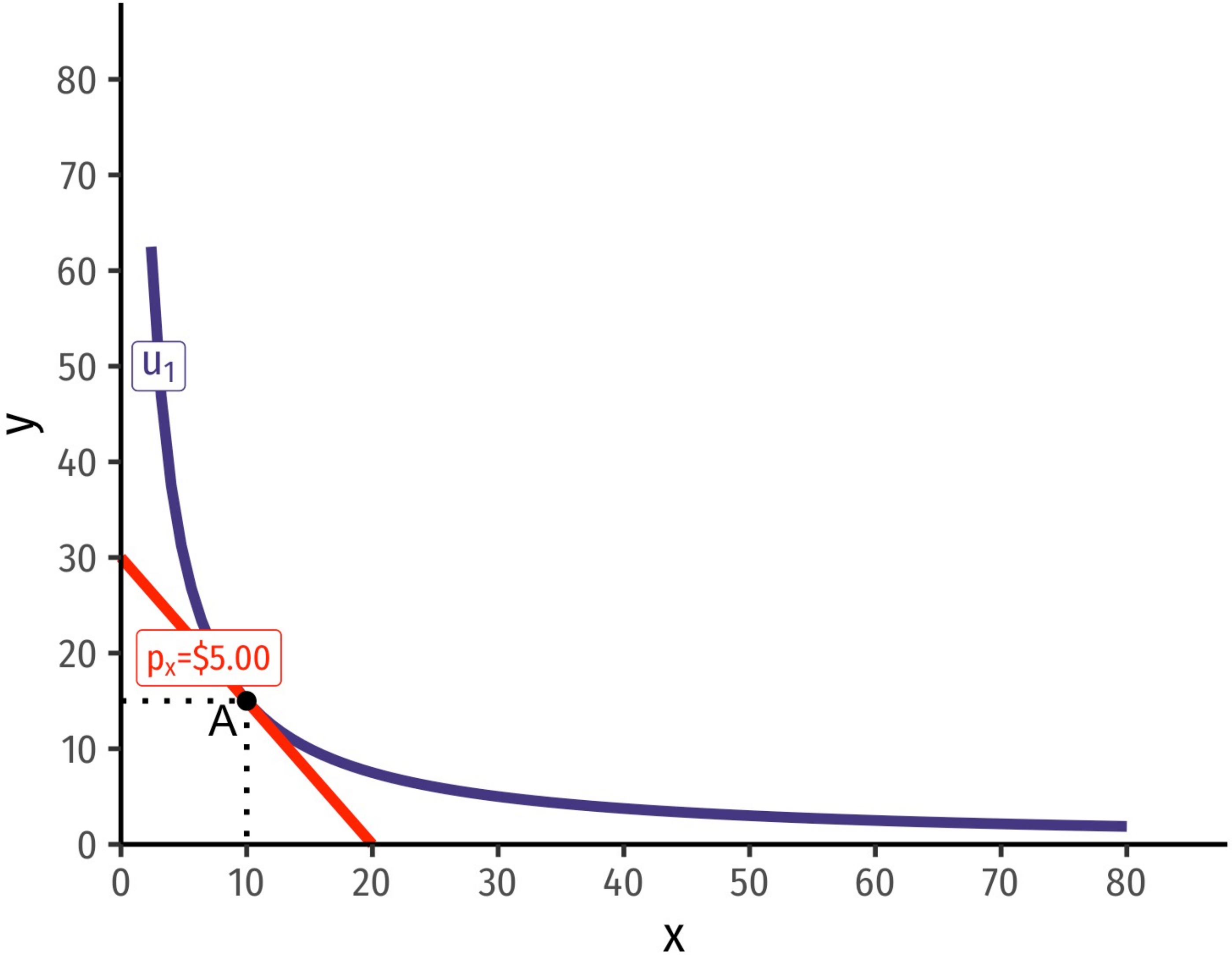
Putting the Effects Together

- **Real income effect**: change in consumption due to change in real purchasing power
  - Can be positive (**normal goods**) or negative (**inferior goods**)
  - Lower price of x means you can buy more x, y, or *both* (depending on your preferences between x and y)

- **Substitution effect**: change in consumption due to change in relative prices
  - If x gets cheaper relative to y, consume ↓y (and ↑x)
  - This is always the same direction! (↓ relatively expensive goods, ↑ relatively cheaper goods)
  - This is why demand curves slope downwards!

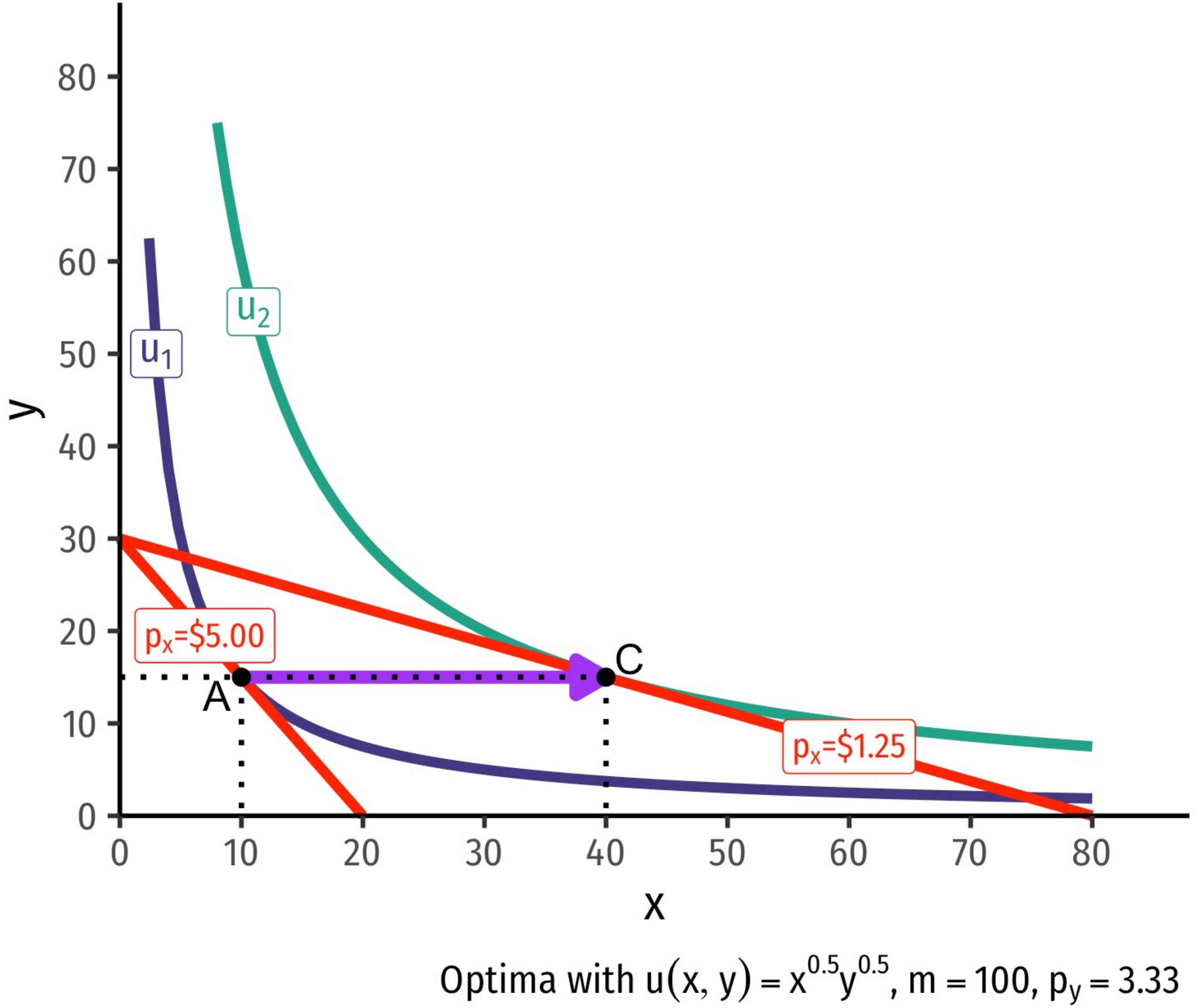
- **Price Effect** = **Real income effect** + **Substitution Effect**

Substitution Effects, Graphically



Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 100$, $p_y = 3.33$

Substitution Effects, Graphically



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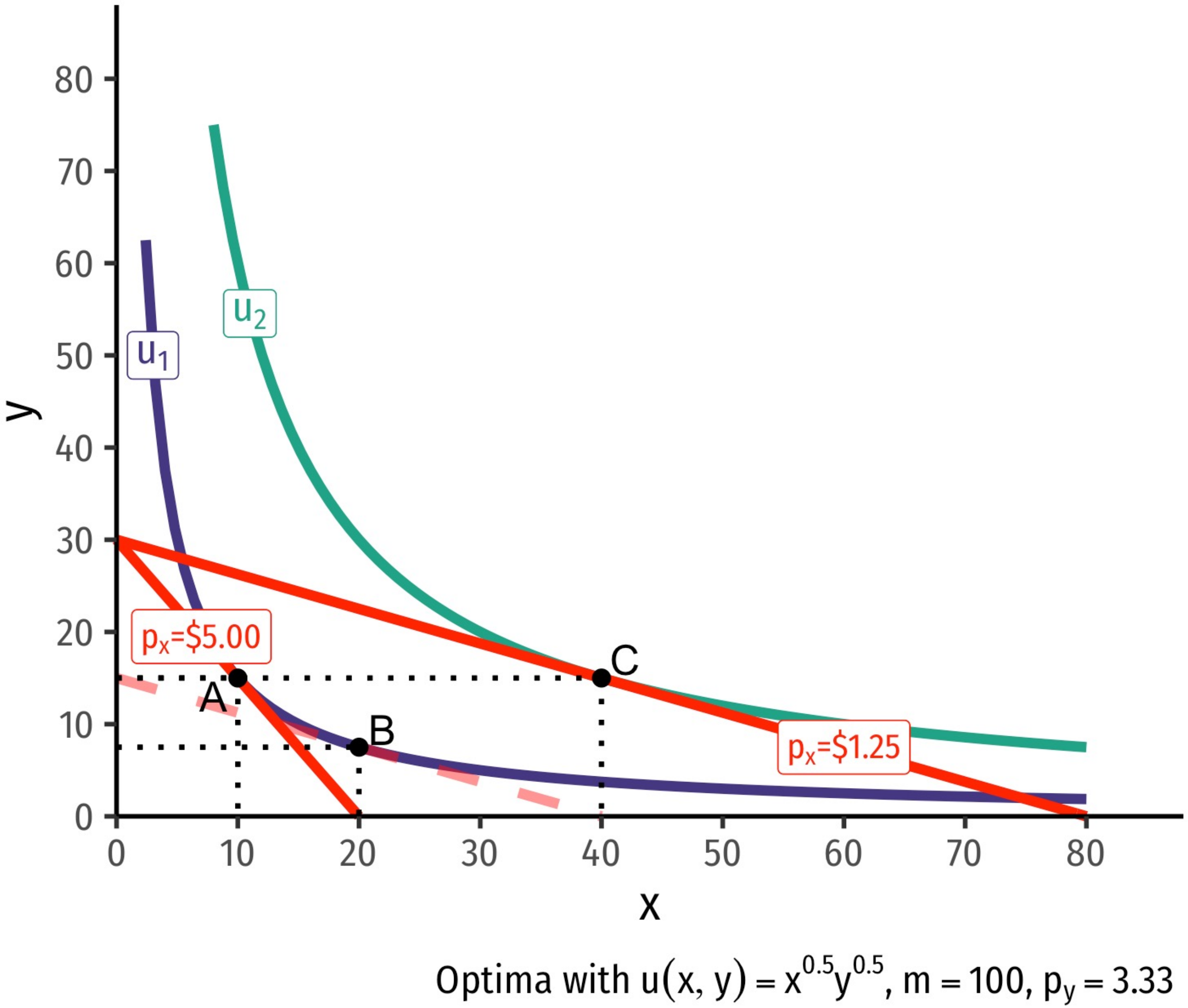
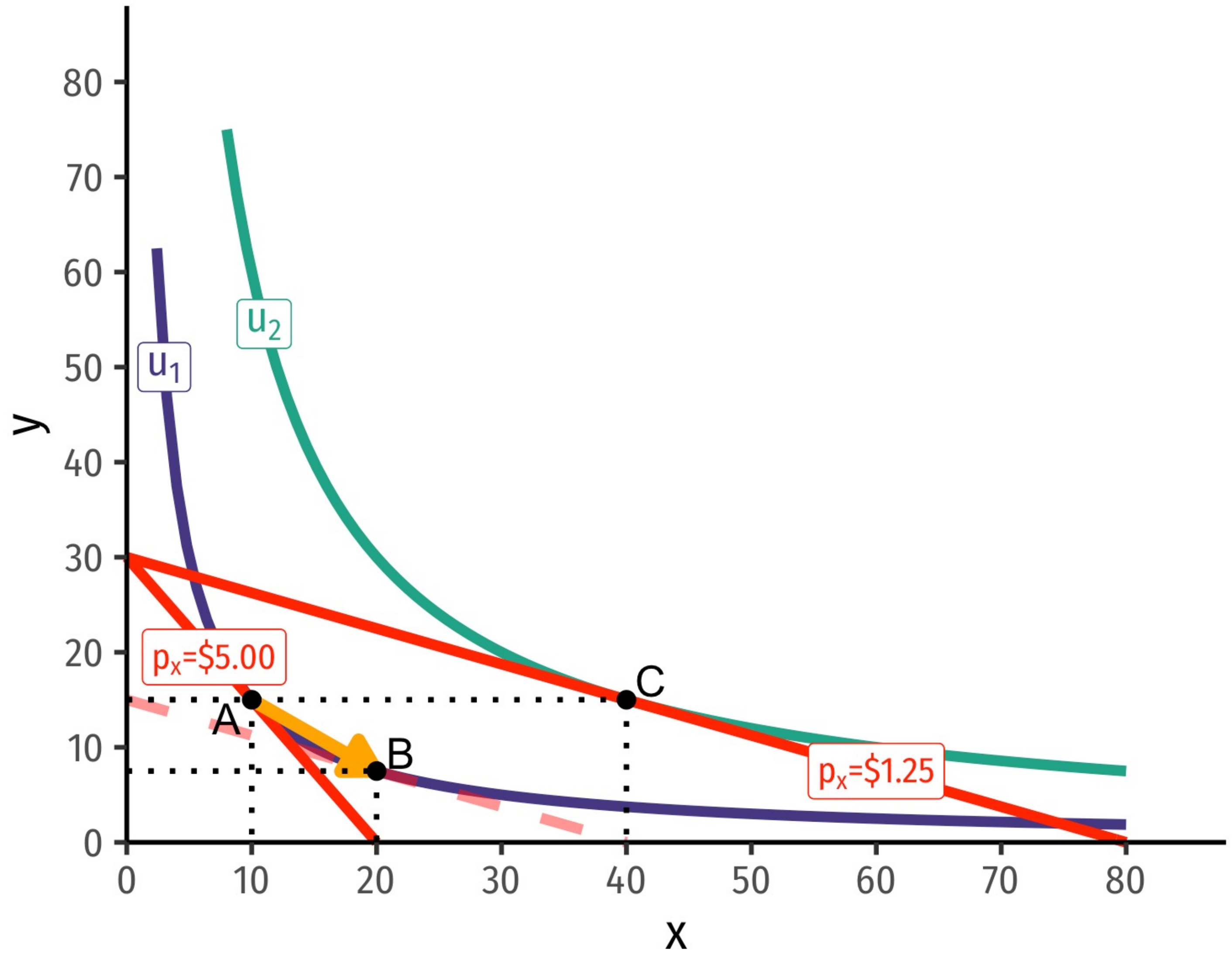


Substitution Effects, Graphically



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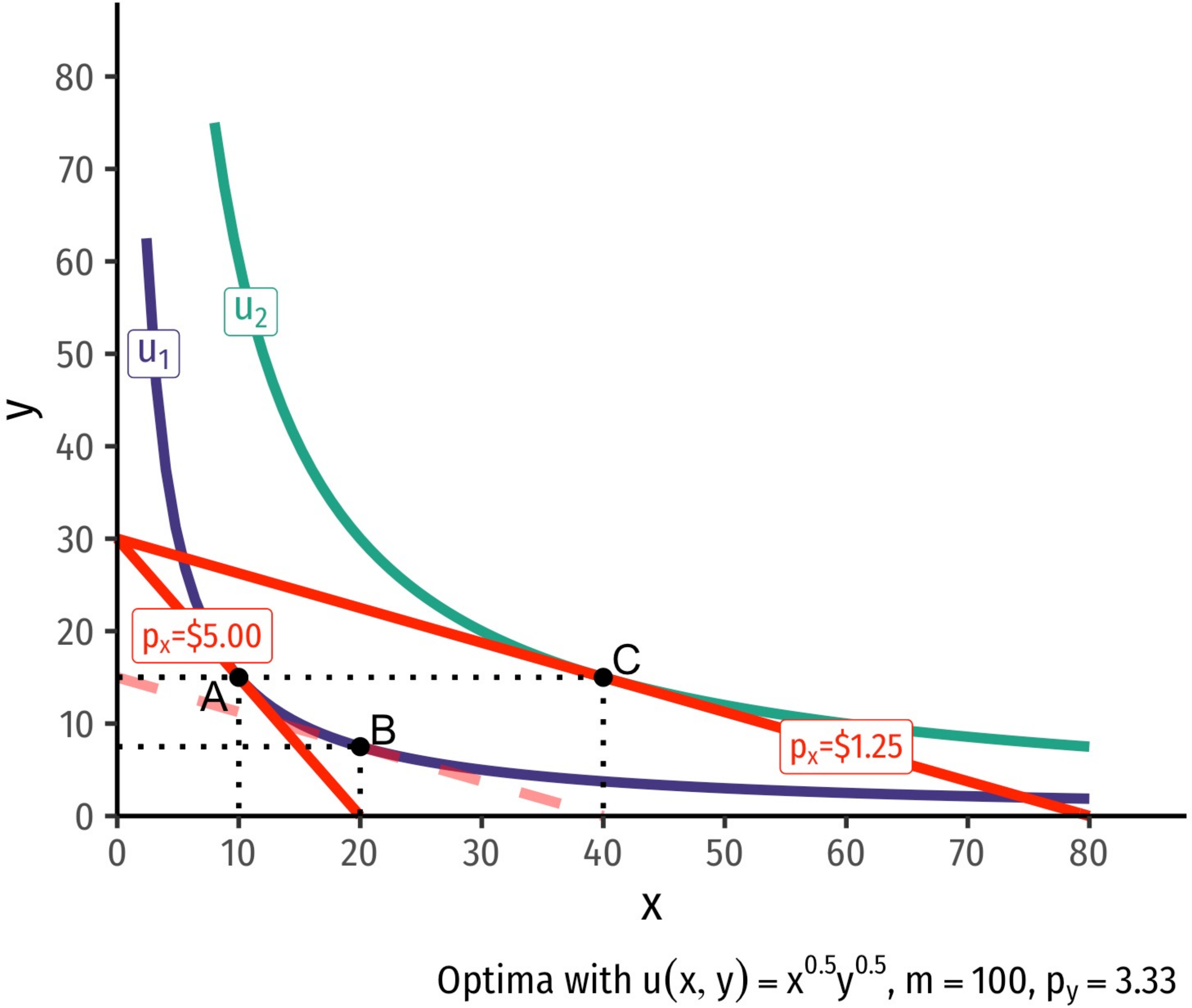
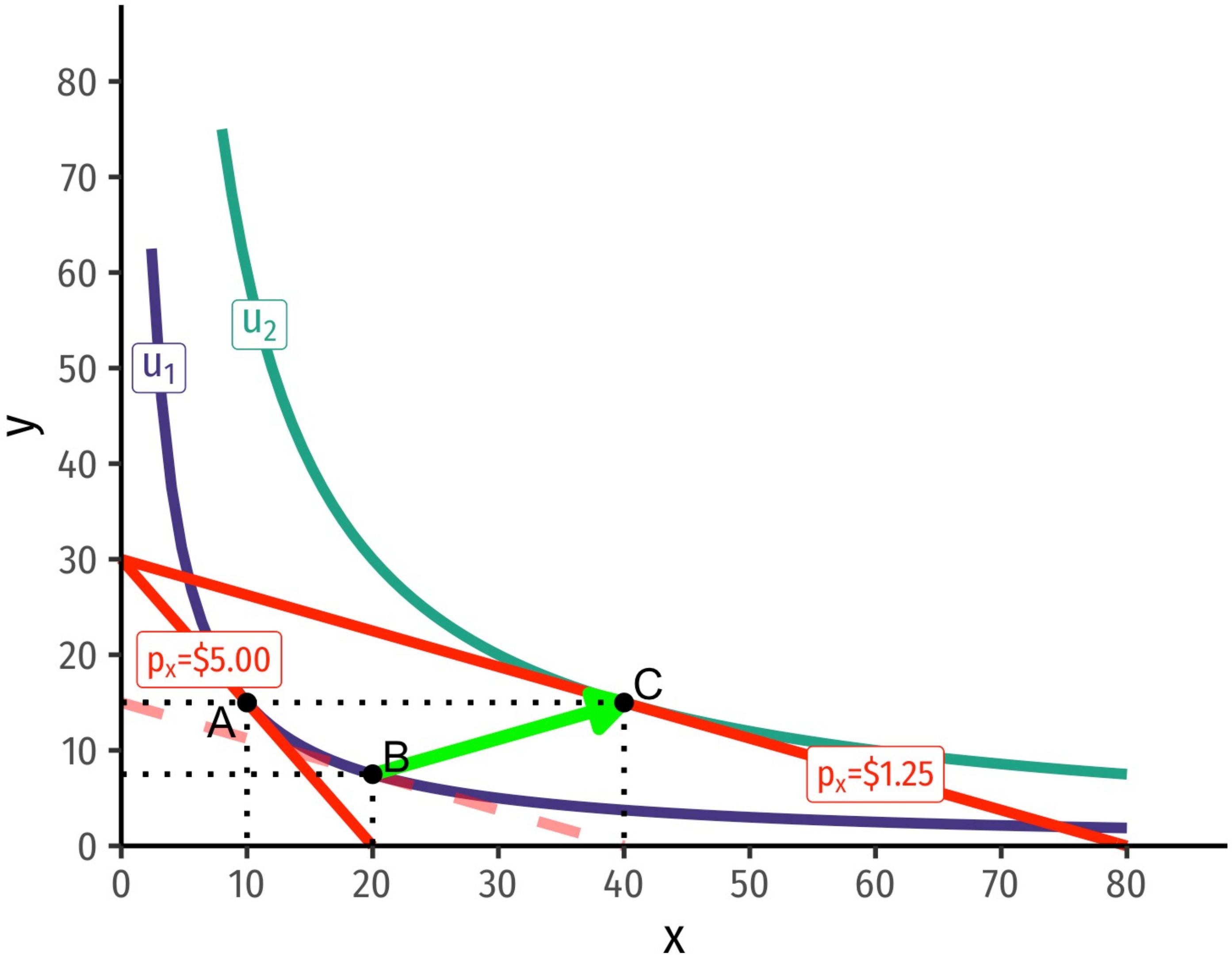
Real Income, Graphically



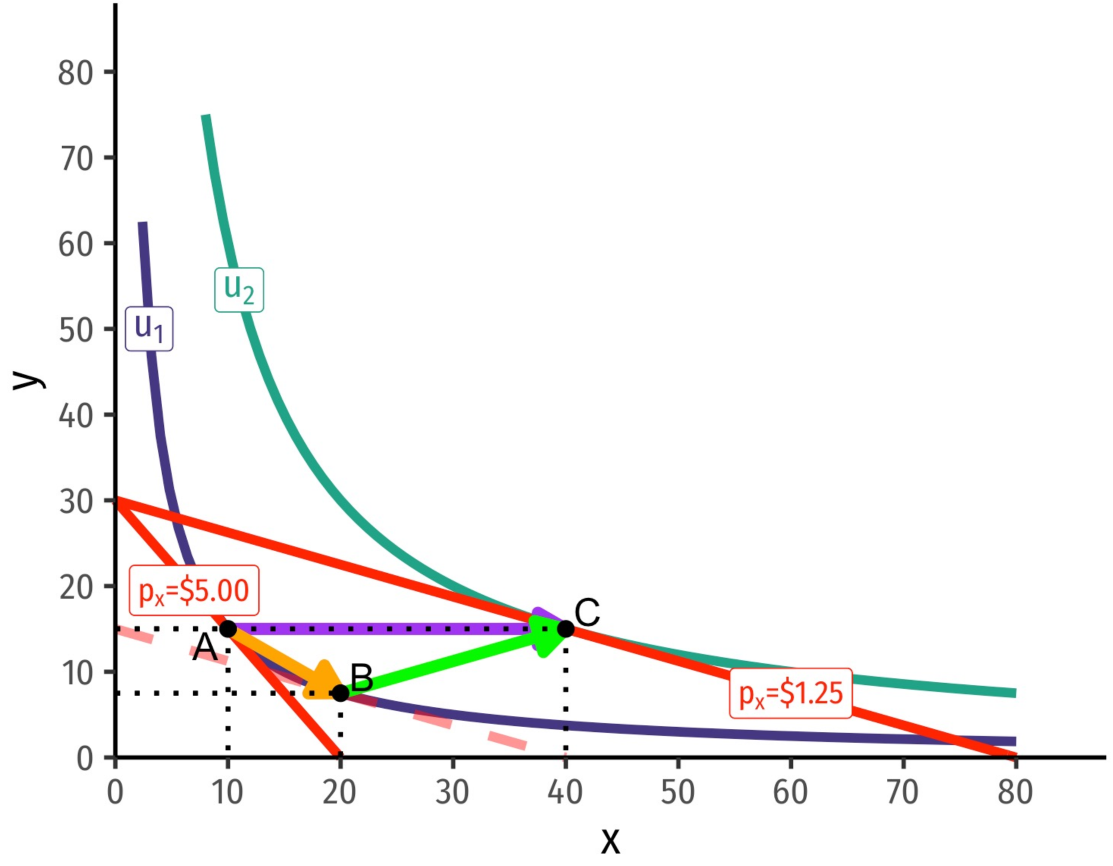
Real Income, Graphically



Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 100$, $p_y = 3.33$

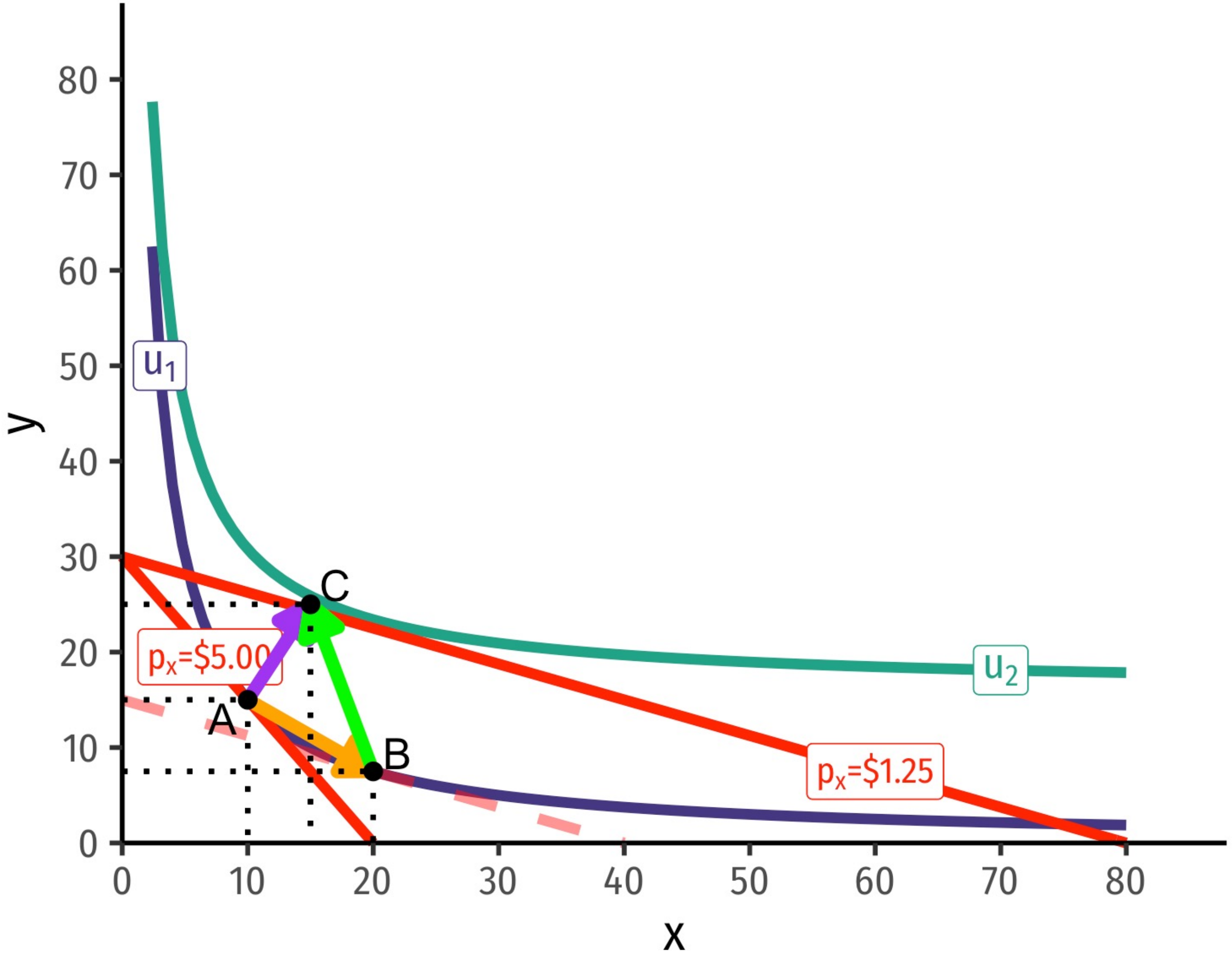
Real Income and Substitution Effects, Graphically

- Original optimal consumption (A)
- Price of x falls, new optimal consumption at (C)
- **Substitution effect:** A→B on same I.C. (↑ cheaper x and ↓ y)
- **(Real) income effect:** B→C to new budget constraint (can buy more of x and/or y)
- **(Total) price effect:** A→C



Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 100$, $p_y = 3.33$

Real Income and Substitution Effects, Graphically: Inferior good



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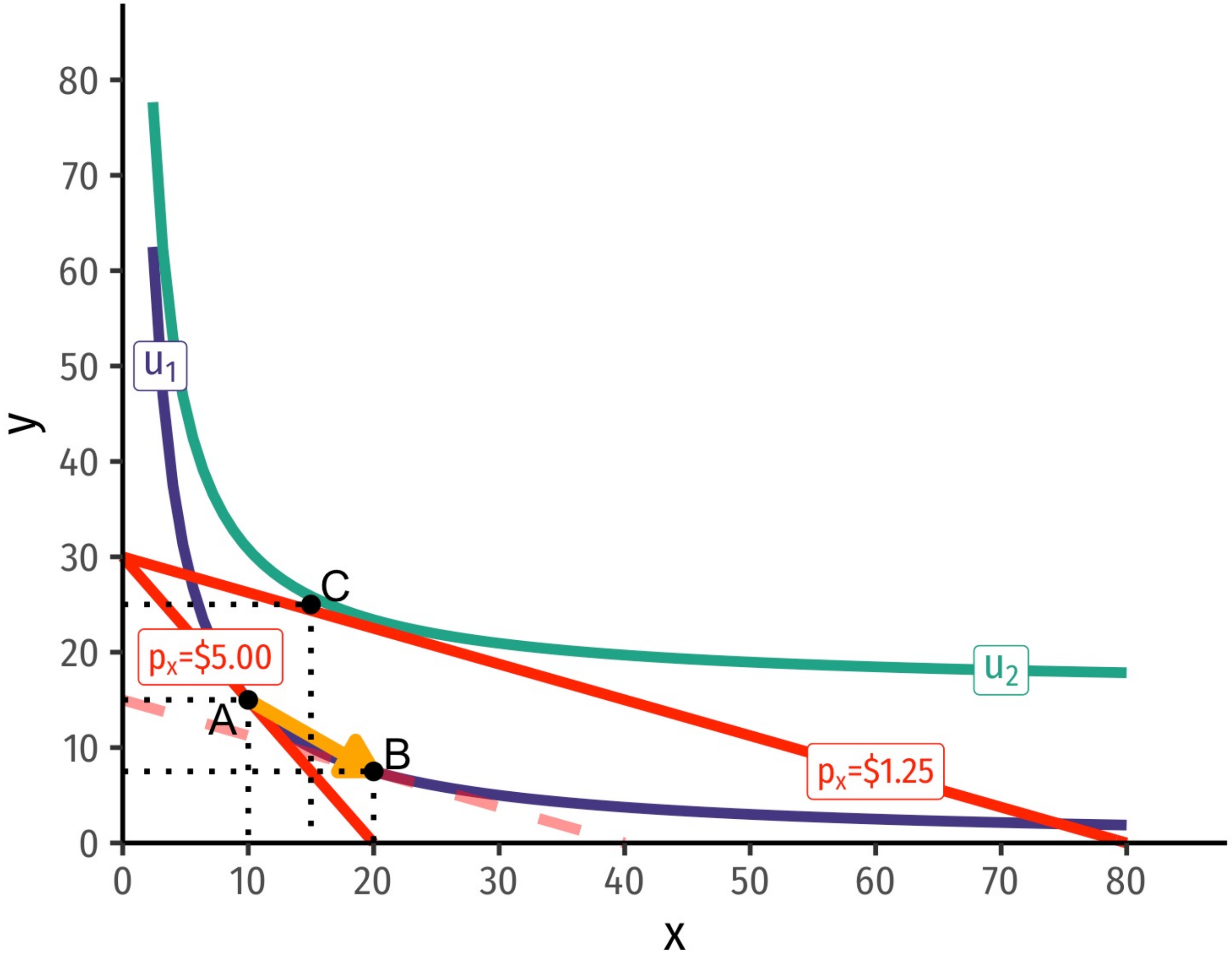
Substitution Effects, Graphically



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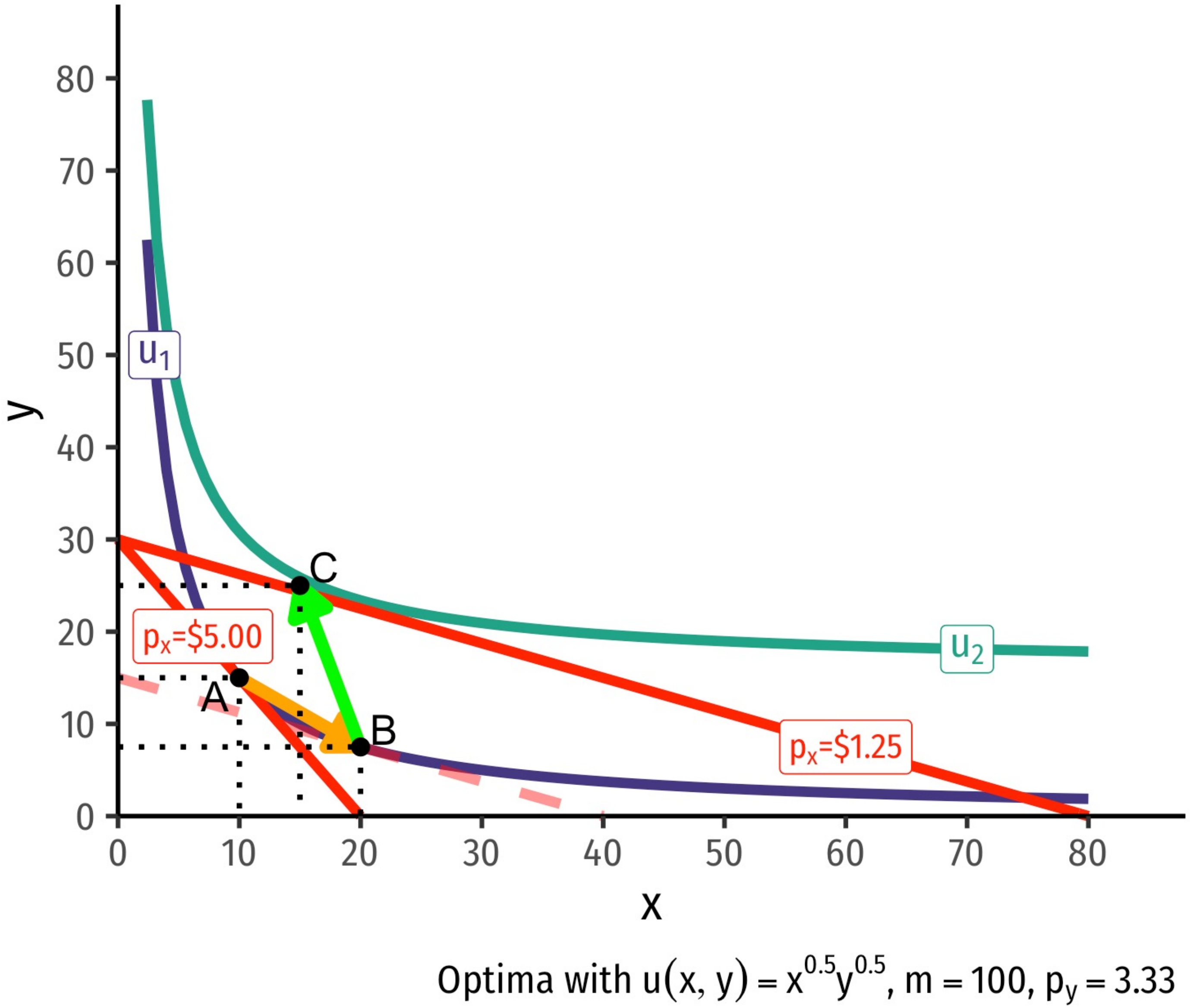
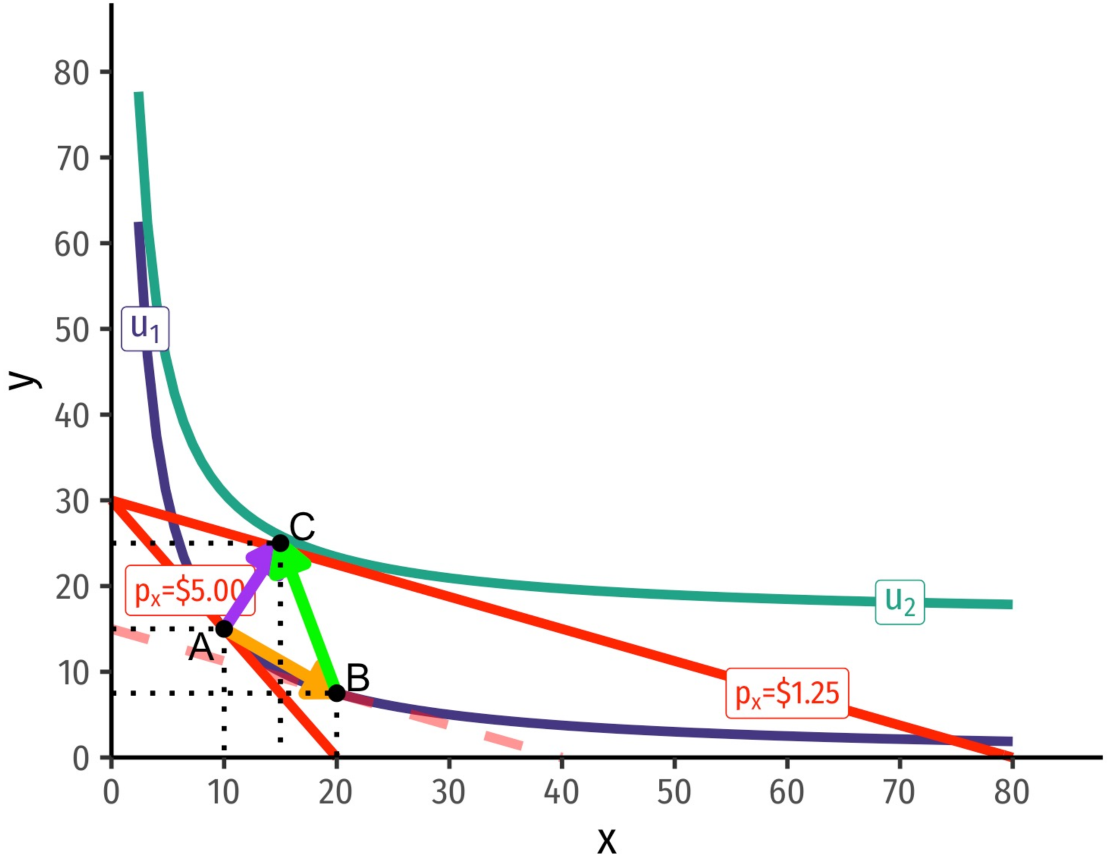
Real Income, Graphically



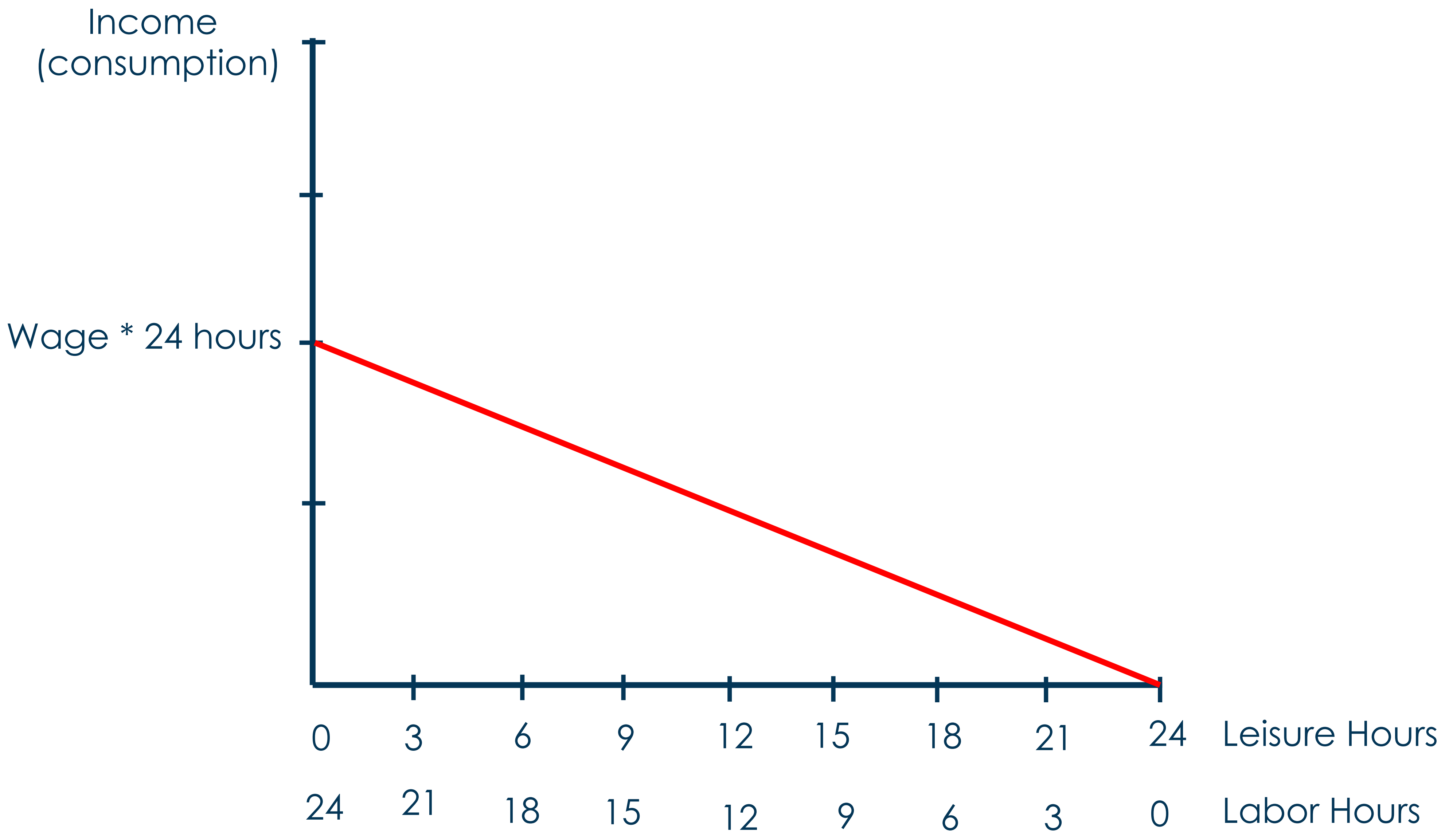
Real Income and Substitution Effects, Graphically

- What about an **inferior** good (Ramen)?
- **Substitution effect:** A→BA→B on same I.C. (↑ cheaper x and ↓ y)
- **(Real) income effect:** B→C to new budget constraint (can buy more of x and/or y) **is negative**
- **(Total) price effect:** A→C



Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 100$, $p_y = 3.33$

Example: Labor and Leisure



Recap: Real Income and Substitution Effects

- **Price Effect** = **Real income effect** + **Substitution Effect**

- **Substitution effect**: is always in the direction of the cheaper good

- **Real Income effect**: can be positive (normal) or negative (inferior)

- **Law of Demand**/Demand curves slope downwards (**Price effect**) mostly because of the substitution effect
  - Even (inferior) goods with negative real income effects overpowered by substitution effect

- Exception in the theoretical **Giffen good**: negative R.I.E. > S.E.
  - An upward sloping demand curve!

Optimal Decision to Demand

Demand Schedule

- **Demand schedule** expresses the quantity of good a person would be willing to buy (q_d) at any given price (p_x)

- Note: **each of these is a consumer's optimum at a given price!**

price	quantity
10	0
9	1
8	2
7	3
6	4
5	5
4	6
3	7
8	2
9	1

Demand Function

- **Demand function** relates quantity to price

$$q_d = 10 - p$$

- Not graphable (wrong axes)!

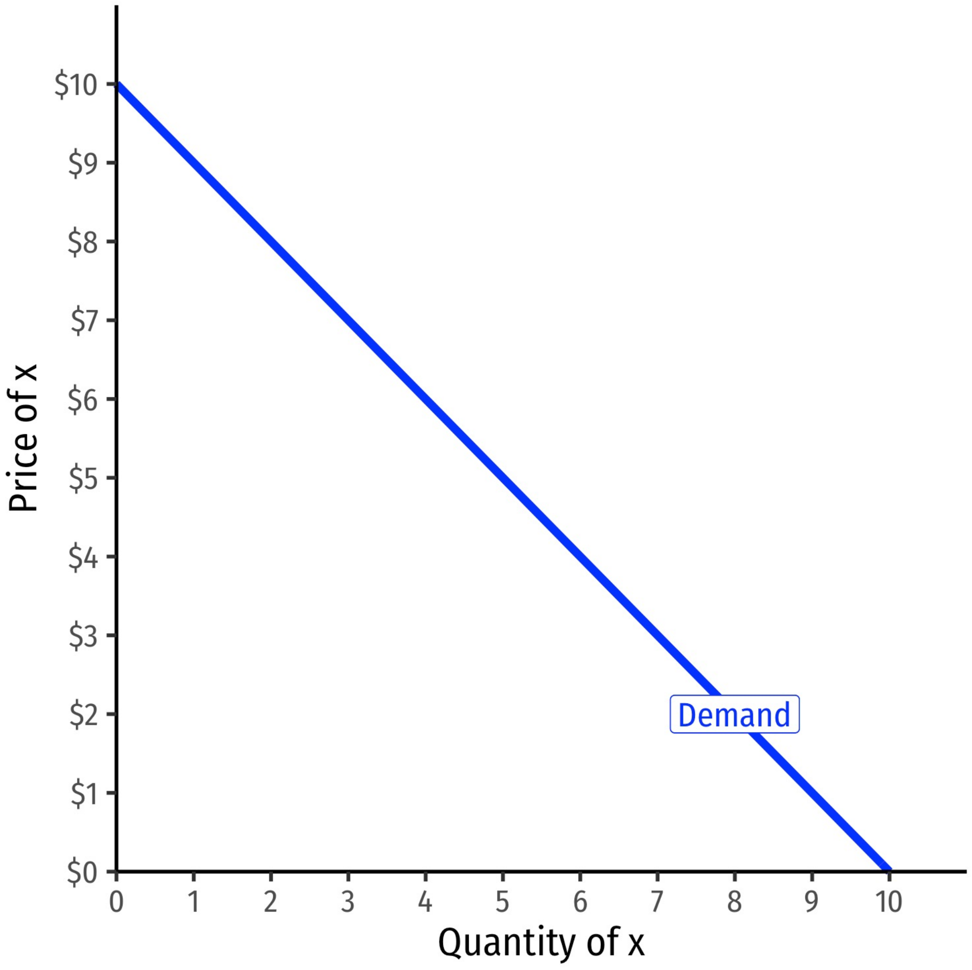
Inverse Demand Function

- ***Inverse* demand function** relates price to quantity
  - Take demand function and solve for p

$$p = 10 - q_d$$

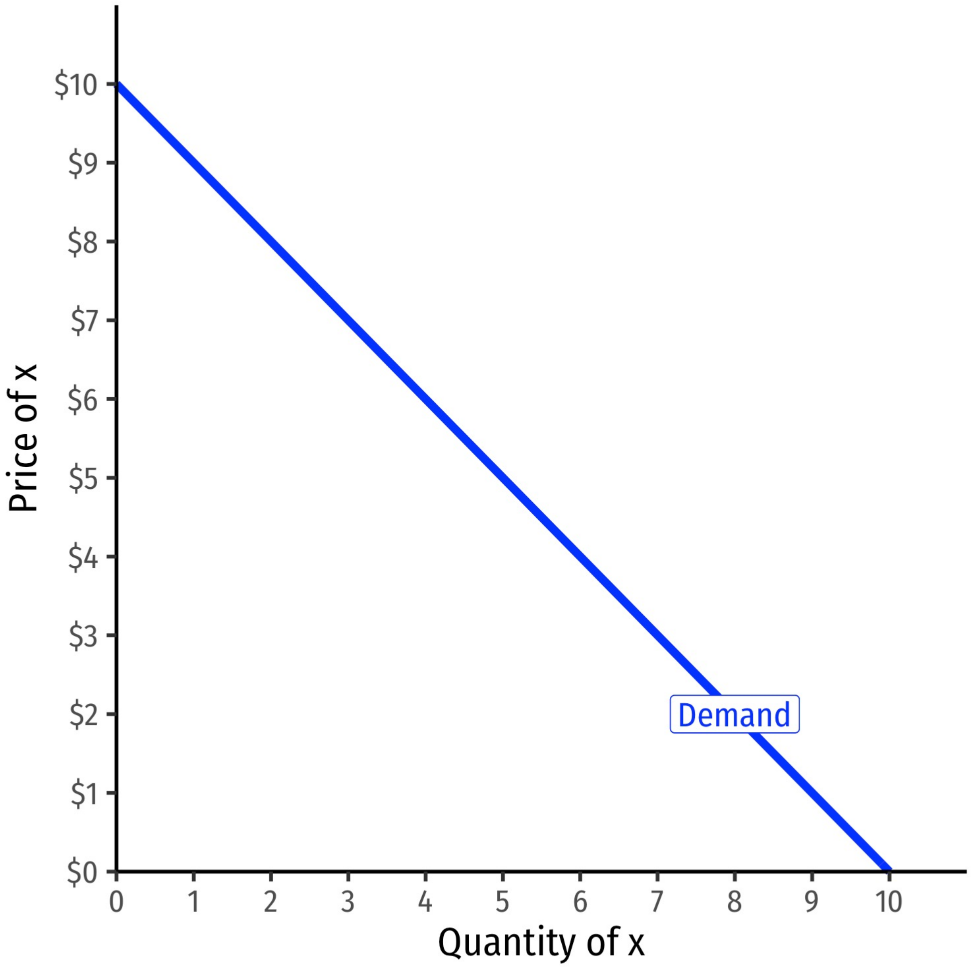
- Graphable (price on vertical axis)!

- Vertical intercept (**"Choke price"**): just high enough to discourage *any* purchases

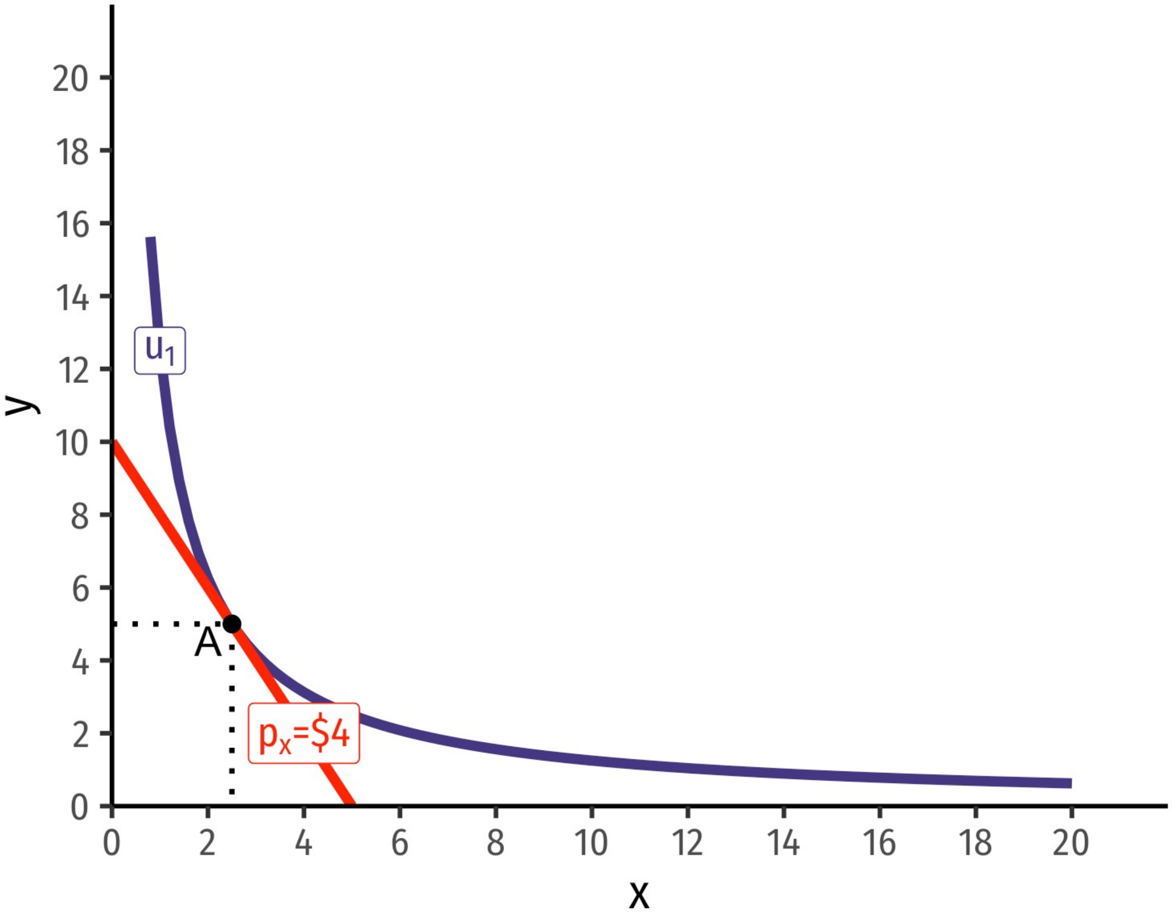


Inverse Demand Function

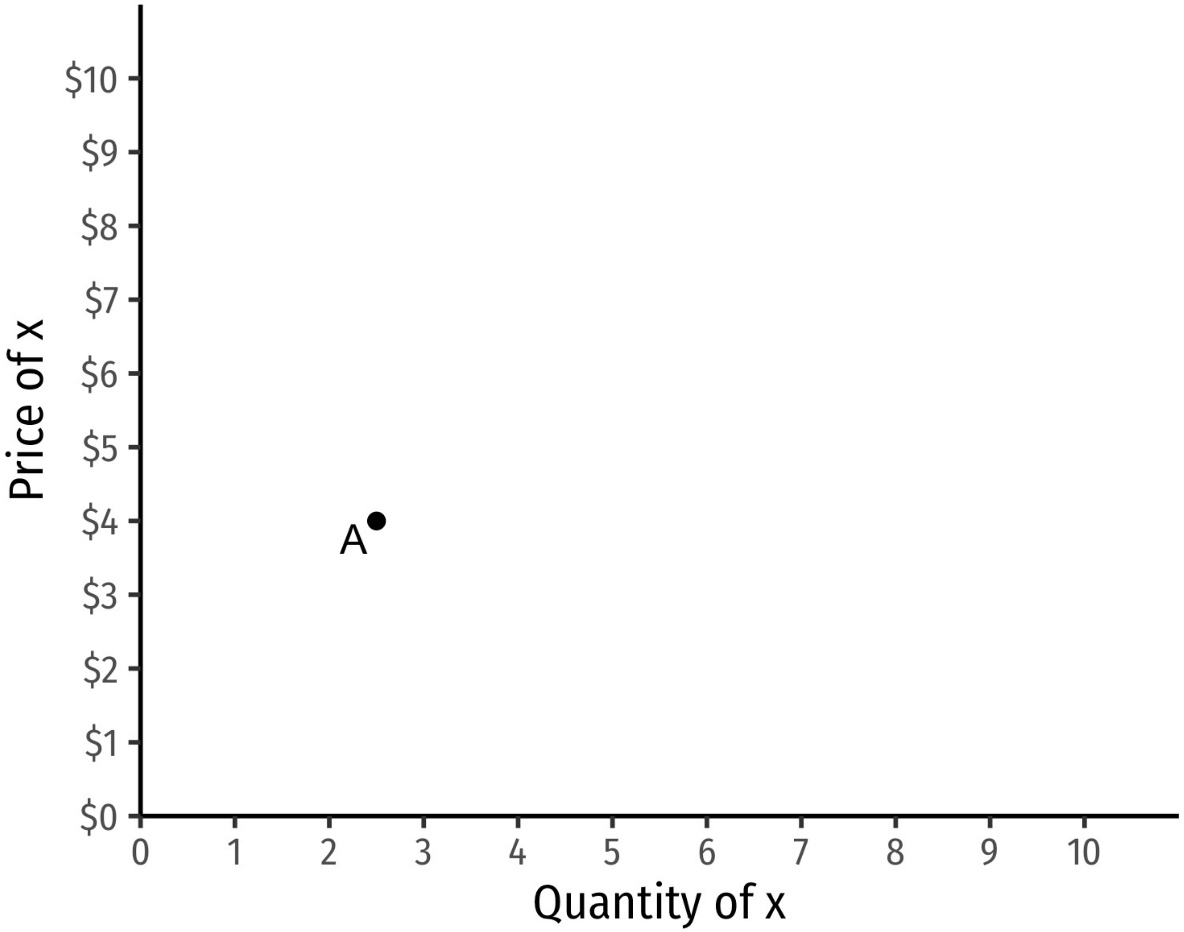
- Read two ways:
- Horizontally: at any given price, how many units person wants to buy
- Vertically: at any given quantity, the **maximum willingness to pay (WTP)** for that quantity
  - This way will be very useful later



Deriving Demand

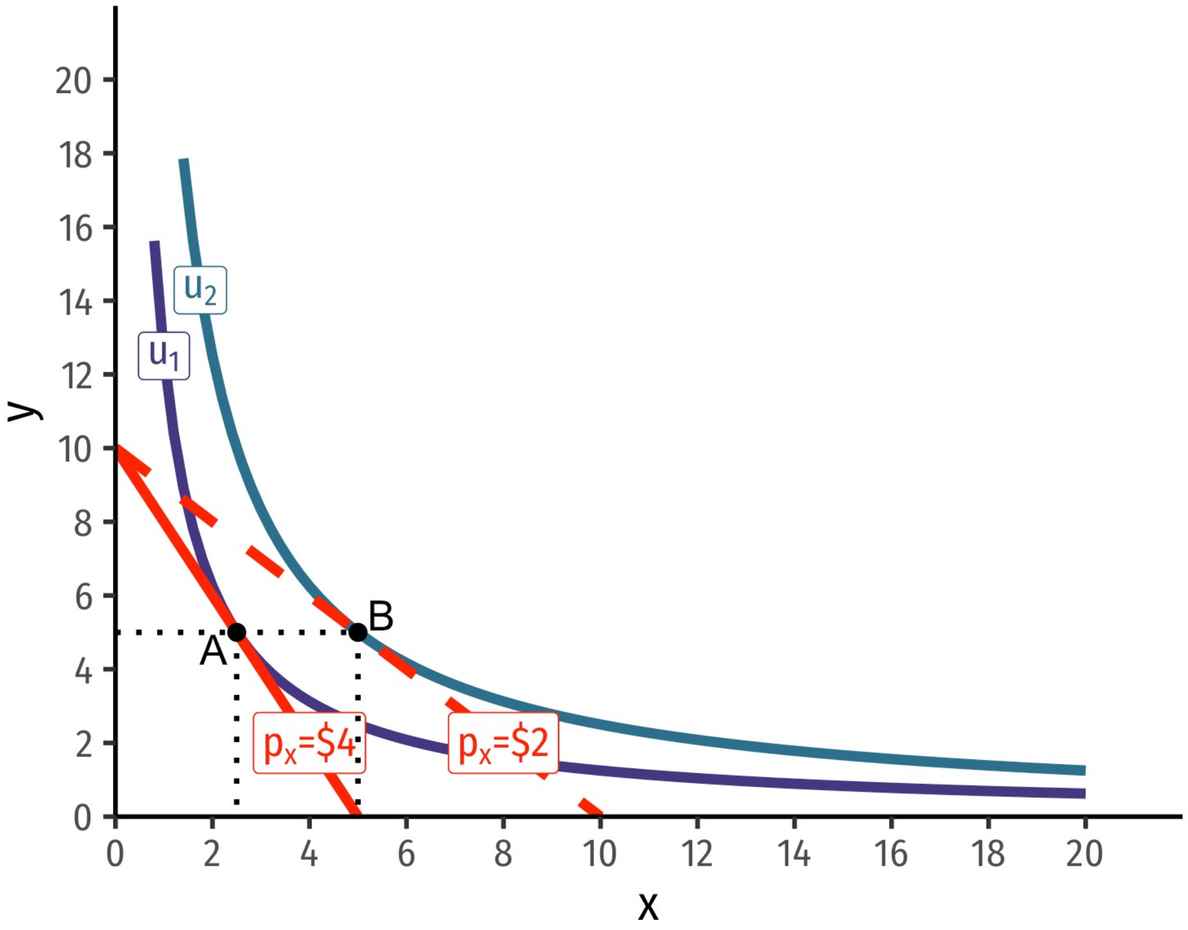


Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 20$, $p_y = 2$

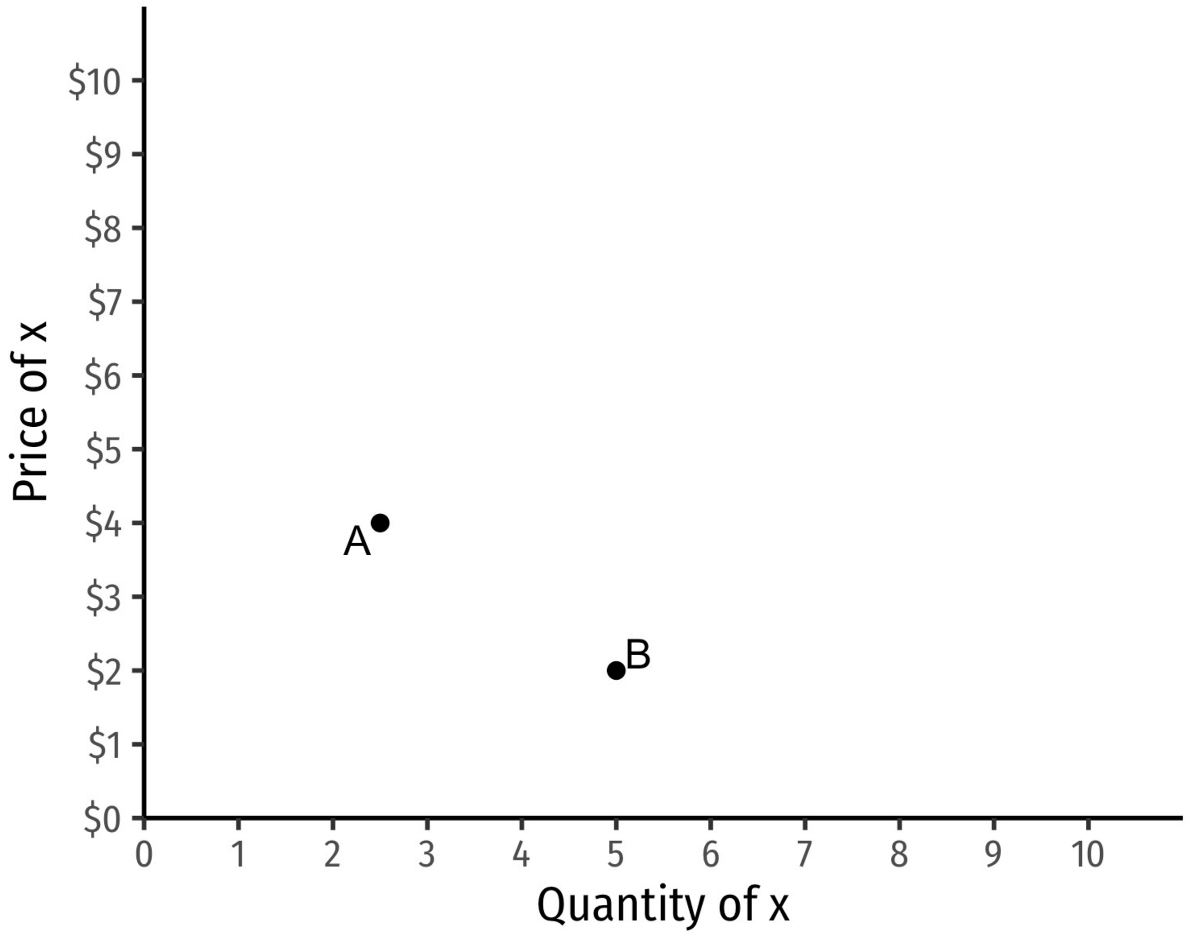


Demand function: $0.5m/p_x$; Inverse Demand function: $p_x = 0.5m/x$

Deriving Demand



Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 20$, $p_y = 2$



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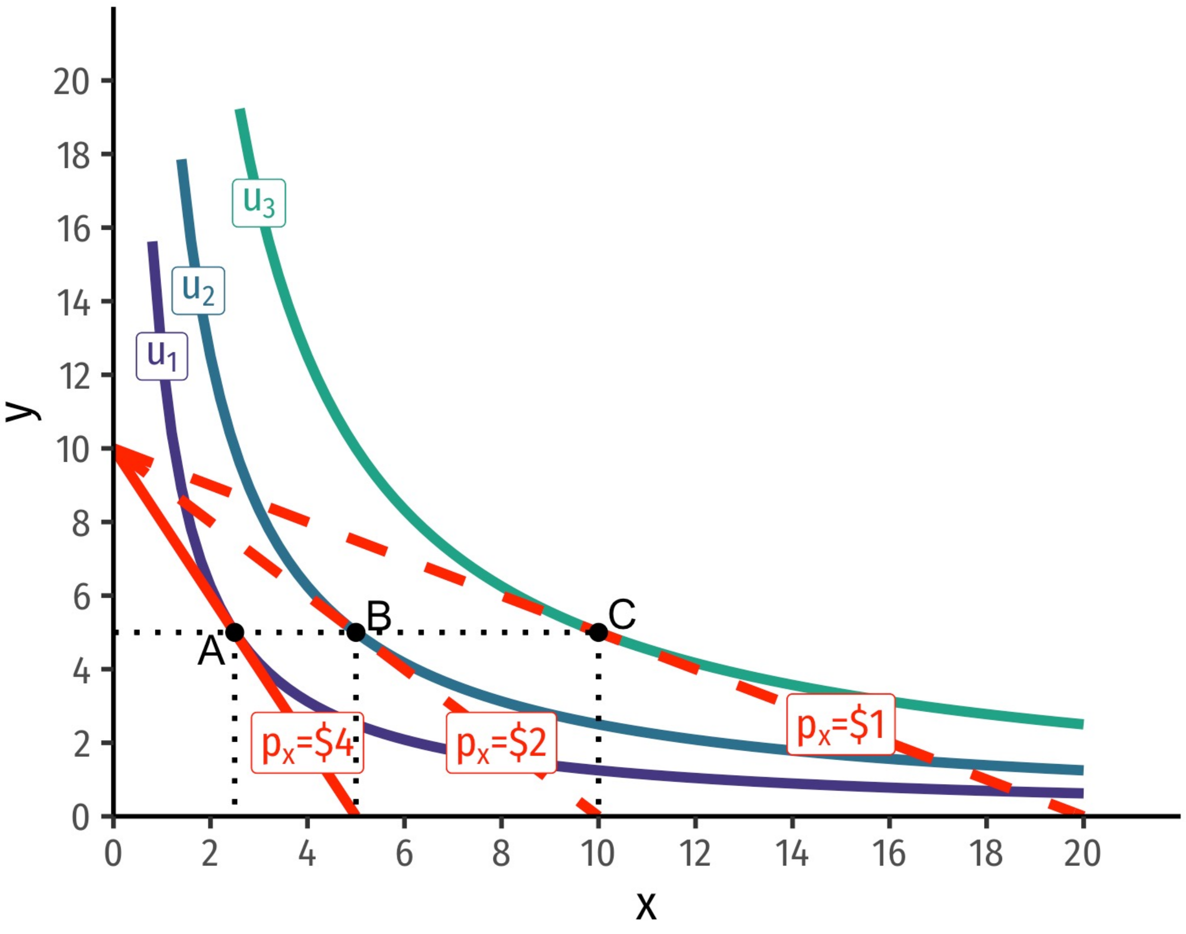
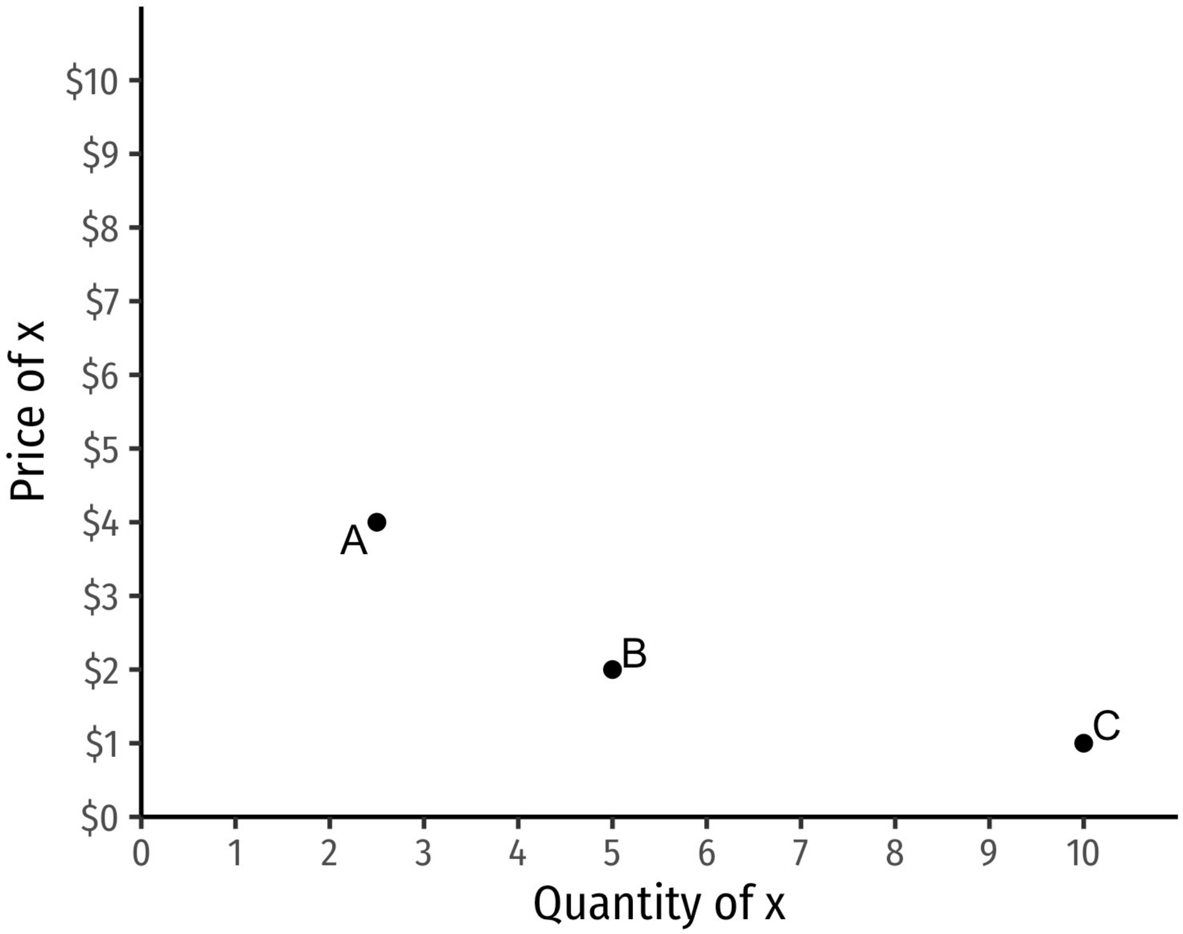
Deriving Demand



Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 20$, $p_y = 2$



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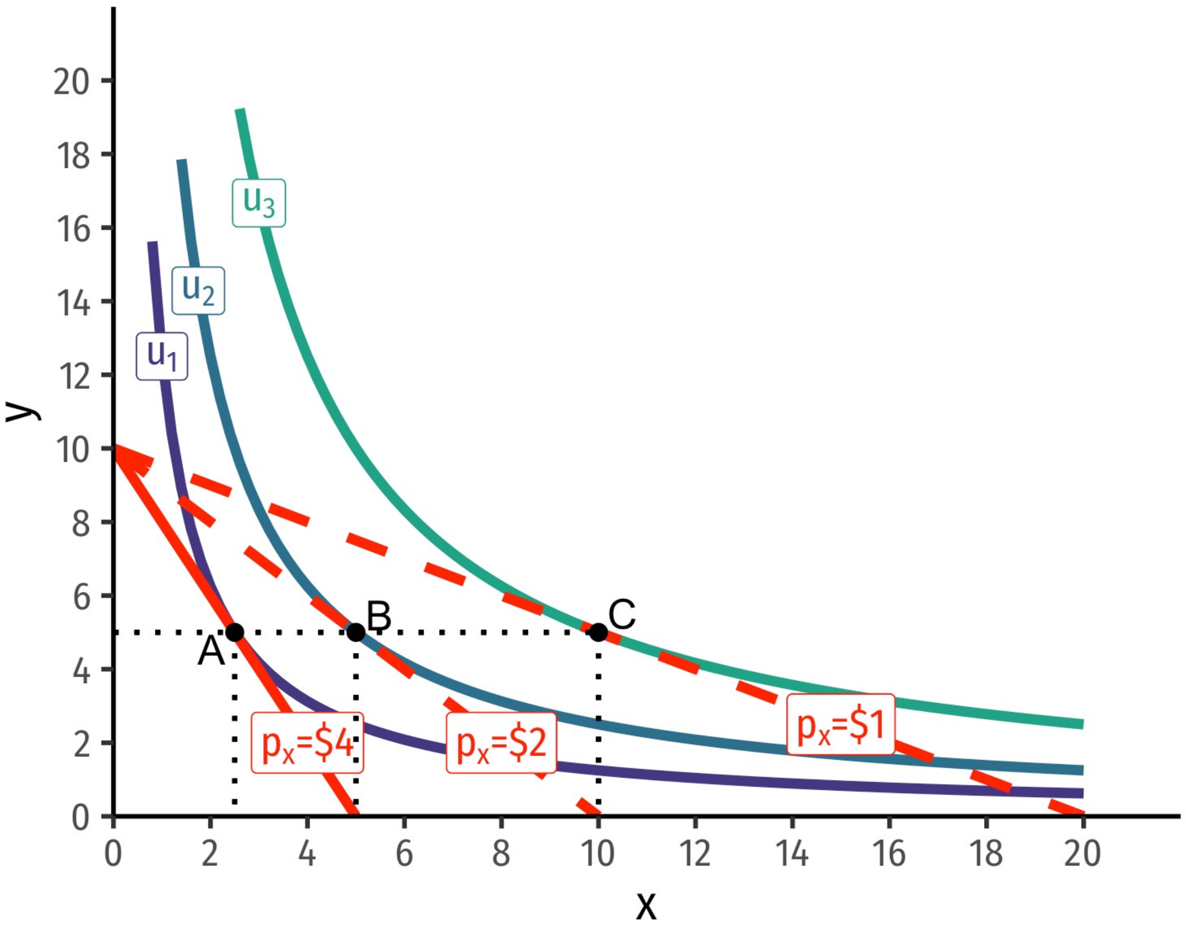

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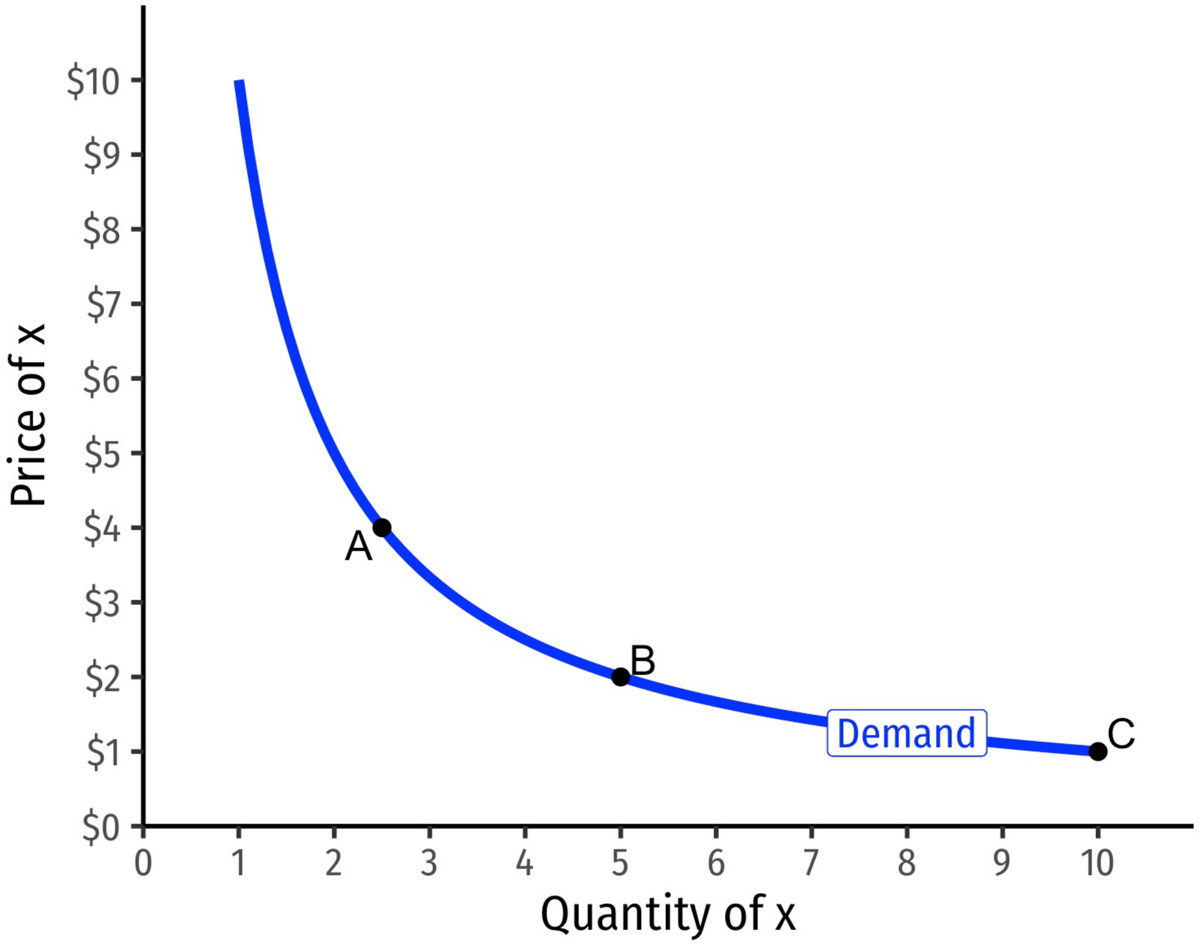

Demand function: $0.5m/p_x$; Inverse Demand function: $p_x = 0.5m/x$