



Constrained Optimization

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The Consumer's Problem: Review

- The **consumer's constrained optimization problem** is:

1) Choose: < some alternative >



2) In order to maximize: < some objective >



3) Subject to: < some constraints >





The Consumer's Problem: Tools

- **Constraints:**
 - Income and market prices
 - Market tradeoffs off between two goods
- **Utility function:**
 - Preferences or utility to maximize
 - Individual tradeoff off between two goods





The Consumer's Problem: Tools

- The **consumer's constrained optimization problem:**
 - Choose a bundle to maximize utility
 - Subject to income and market prices





The Consumer's Problem: Mathematically

- $\max_{x, y} u(x, y)$
- s.t. $p_x x + p_y y = m$
 - This requires calculus to solve. We will focus on **graphs** instead!





The Consumer's Optimum

Rule 1

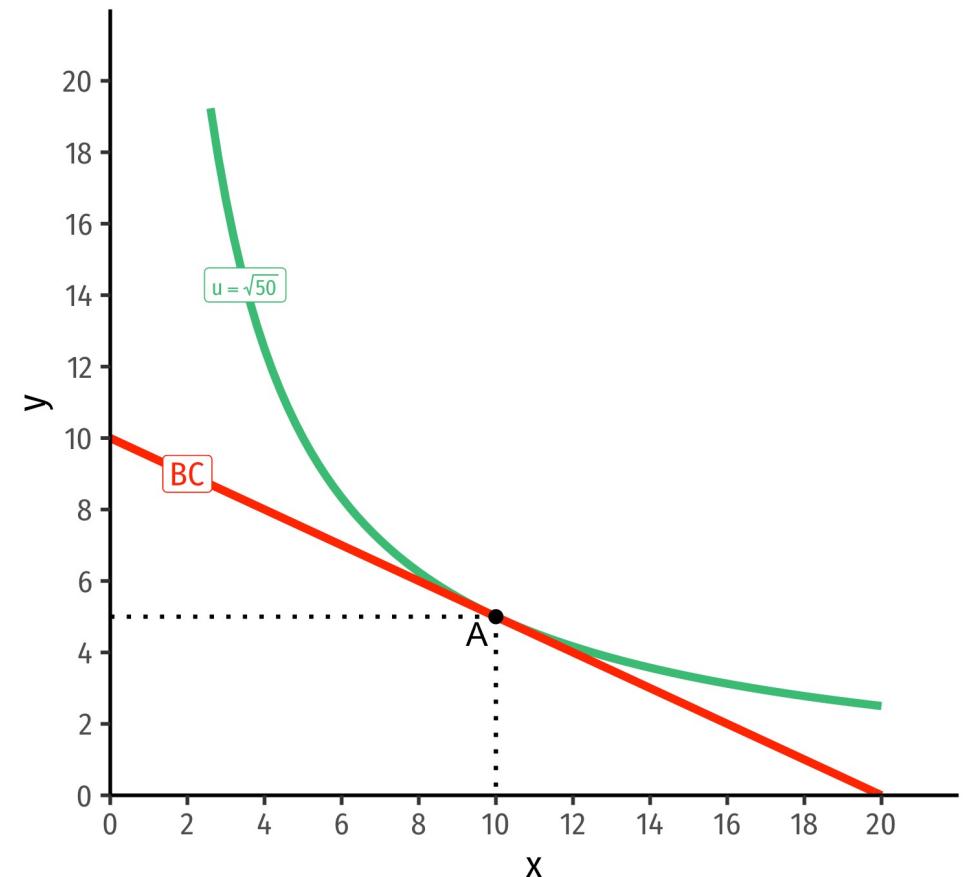
$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$

- Easier for calculation (slopes)

Rule 2

$$\frac{MU_x}{P_x} = \frac{MU_y}{p_y}$$

- Easier for intuition

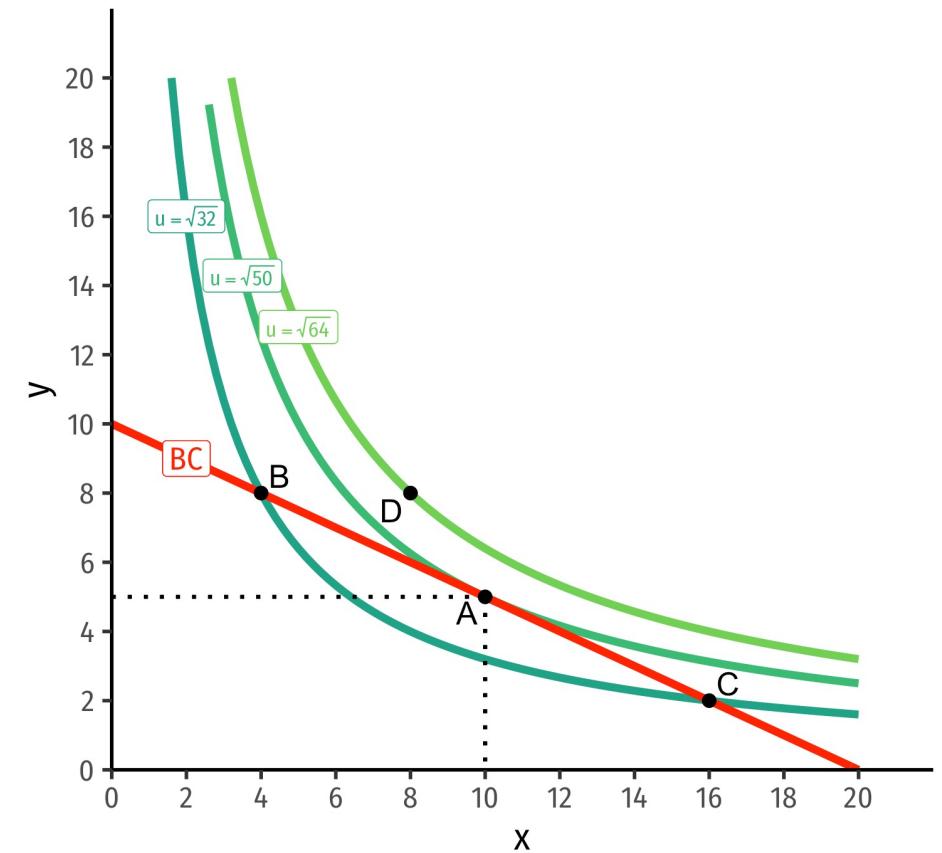




The Consumer's Optimum: Graphically

- **Graphical solution: Highest indifference curve tangent to budget constraint**

- Bundle A!

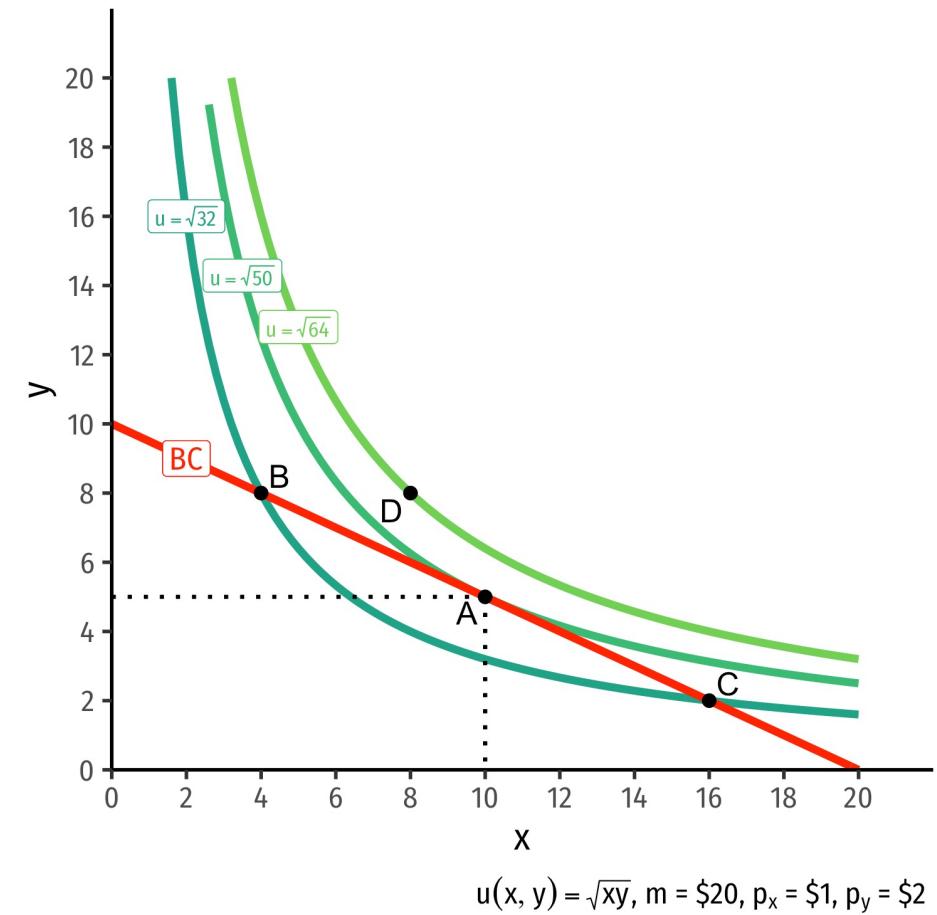


$$u(x, y) = \sqrt{xy}, m = \$20, p_x = \$1, p_y = \$2$$



The Consumer's Optimum: Graphically

- **Graphical solution: Highest indifference curve tangent to budget constraint**
 - Bundle A!
- B or C spend all income, but a better combination exists
 - Averages > extremes!



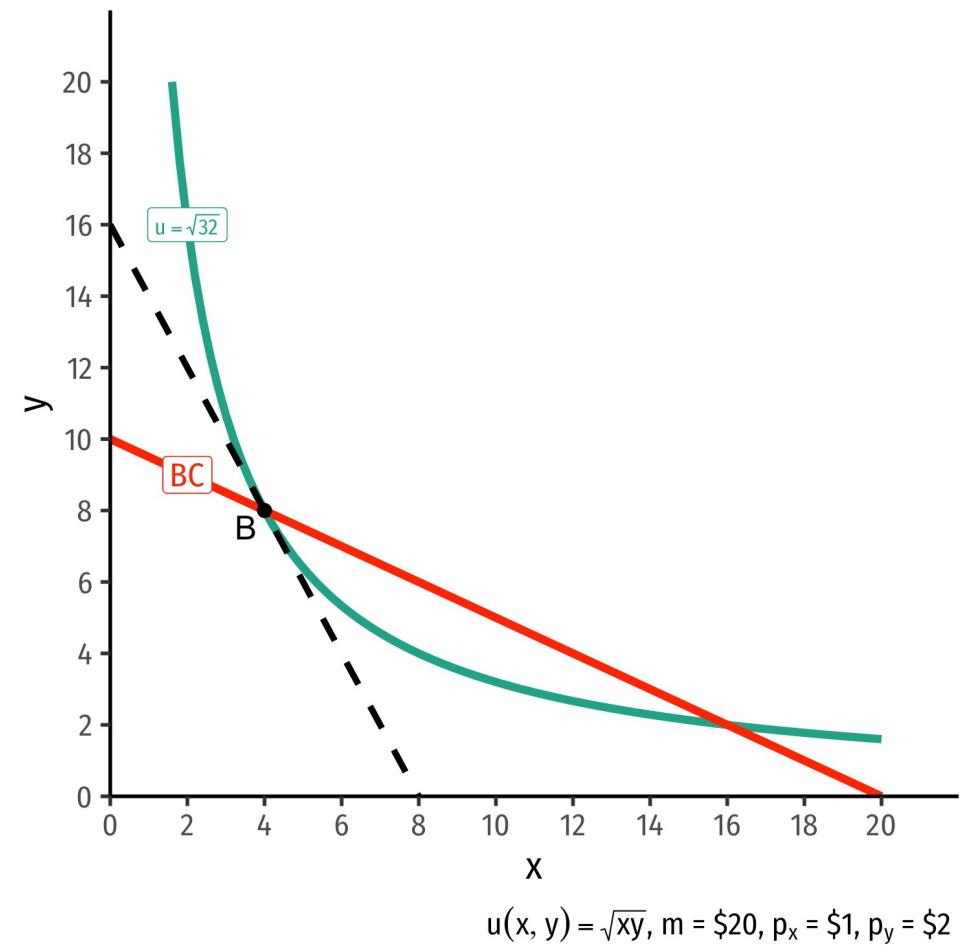


The Consumer's Optimum: Why Not B?

- indiff. curve slope > budget constr. slope

$$|MRS_{x,y}| > \left| \frac{p_x}{p_y} \right|$$

$$\left| \frac{MU_x}{MU_y} \right| > \left| \frac{p_x}{p_y} \right|$$





The Consumer's Optimum: Why Not B?

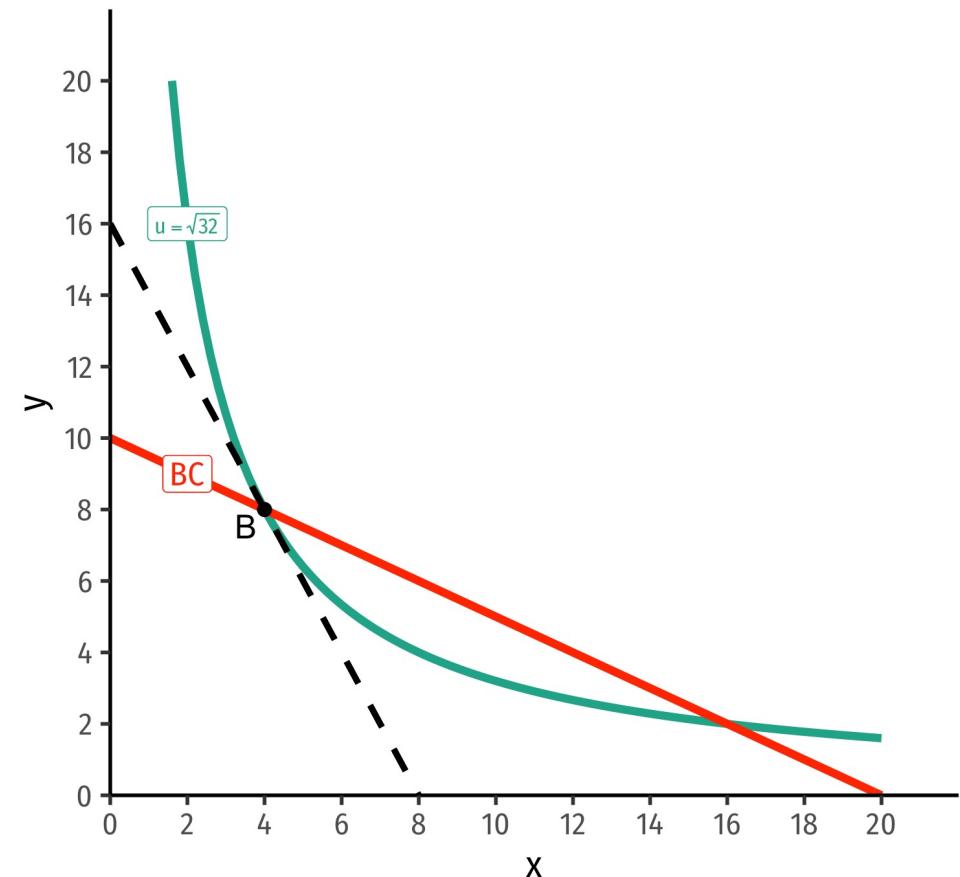
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$$\left| \frac{MU_x}{MU_y} \right| > \left| \frac{p_x}{p_y} \right|$$

$$|-2| > |-0.5|$$

- Consumer would exchange at $2Y:1X$
- Market exchange rate is $0.5Y:1X$

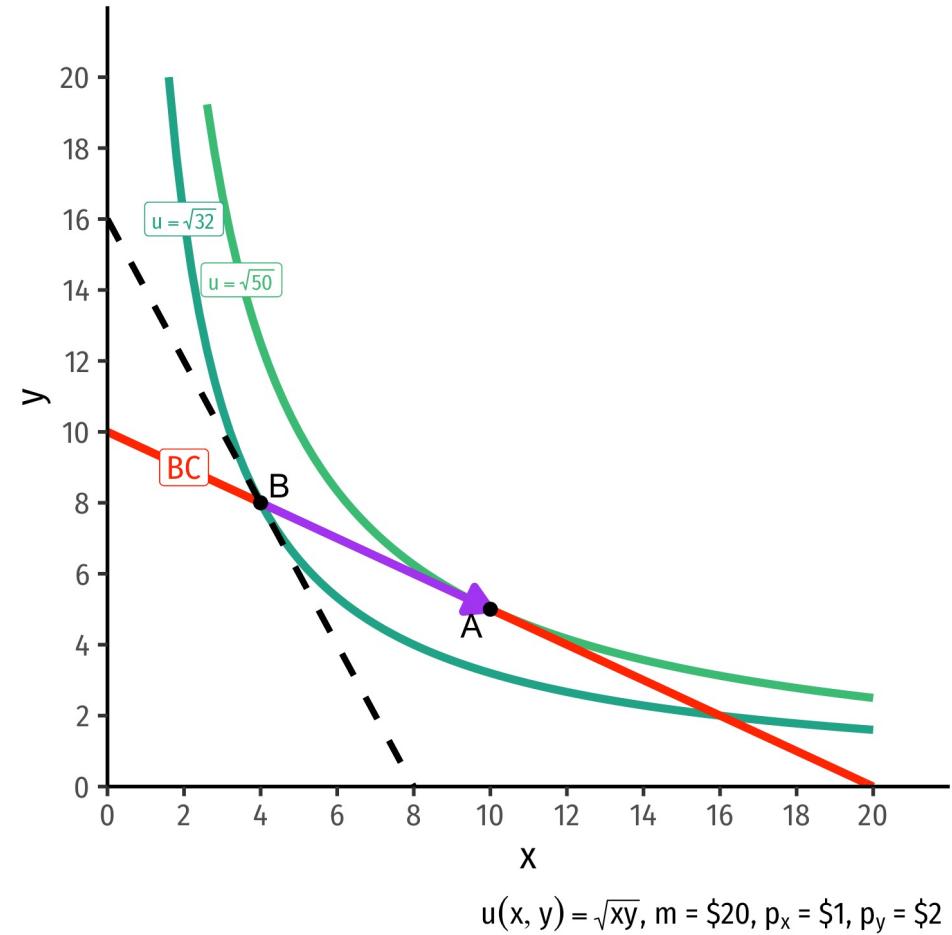


$$u(x, y) = \sqrt{xy}, m = \$20, p_x = \$1, p_y = \$2$$



The Consumer's Optimum: Why Not B?

- Can spend less on y more on x and get more utility!





The Consumer's Optimum: Why Not B?

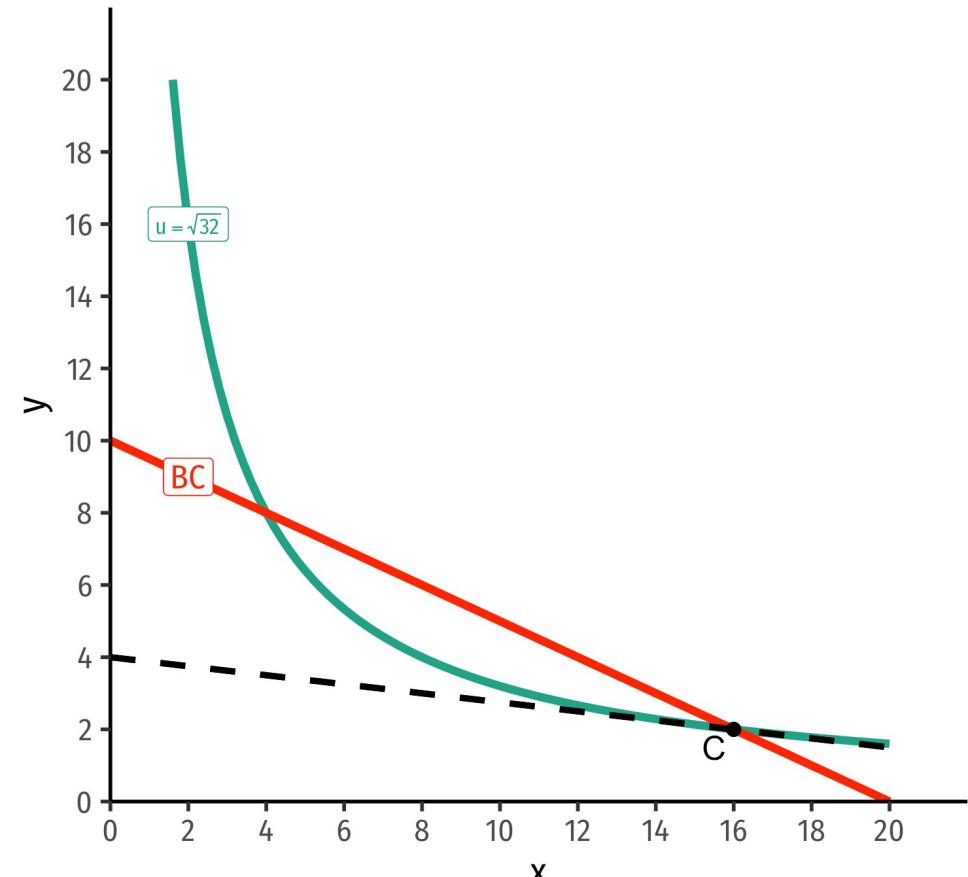
- indiff. curve slope > budget constr. slope

$$|MRS_{x,y}| < \left| \frac{p_x}{p_y} \right|$$

$$\left| \frac{MU_x}{MU_y} \right| < \left| \frac{p_x}{p_y} \right|$$

$$|-0.125| > |-0.5|$$

- Consumer would exchange at 0.125Y:X
- Market exchange rate is 0.5Y:X

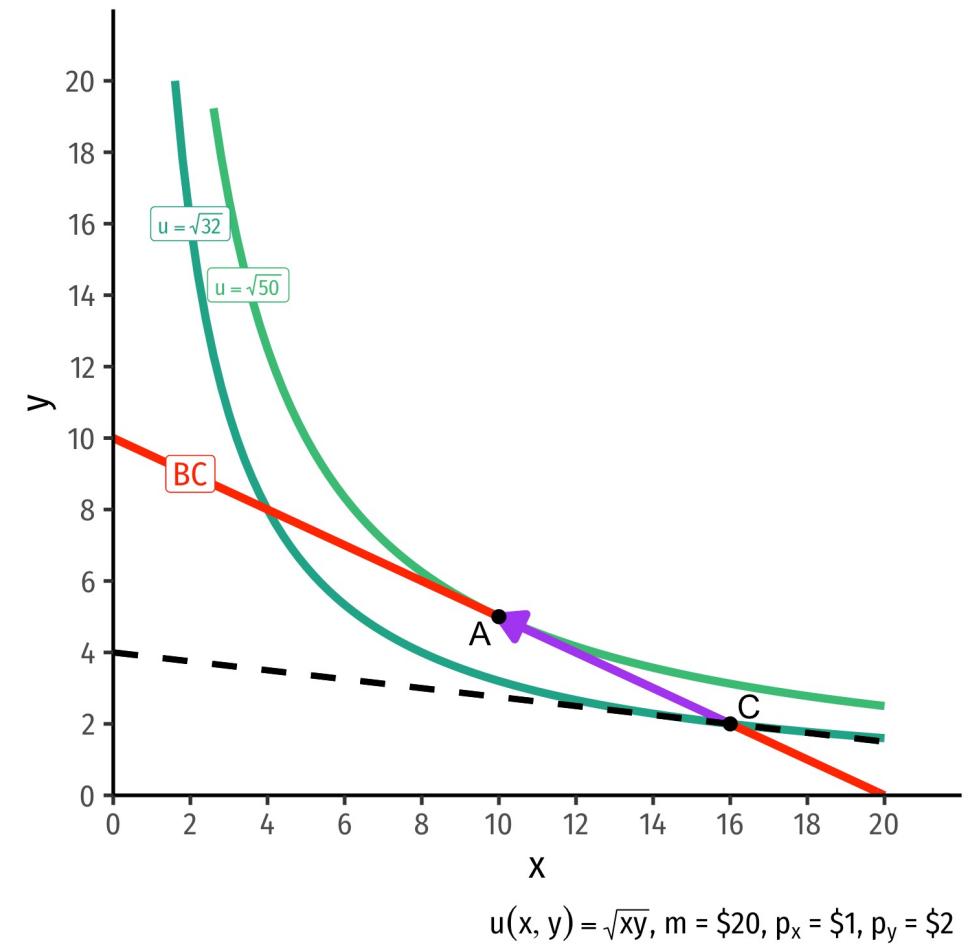


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The Consumer's Optimum: Why Not B?

- Can spend less on x, more on y and get more utility!





The Consumer's Optimum

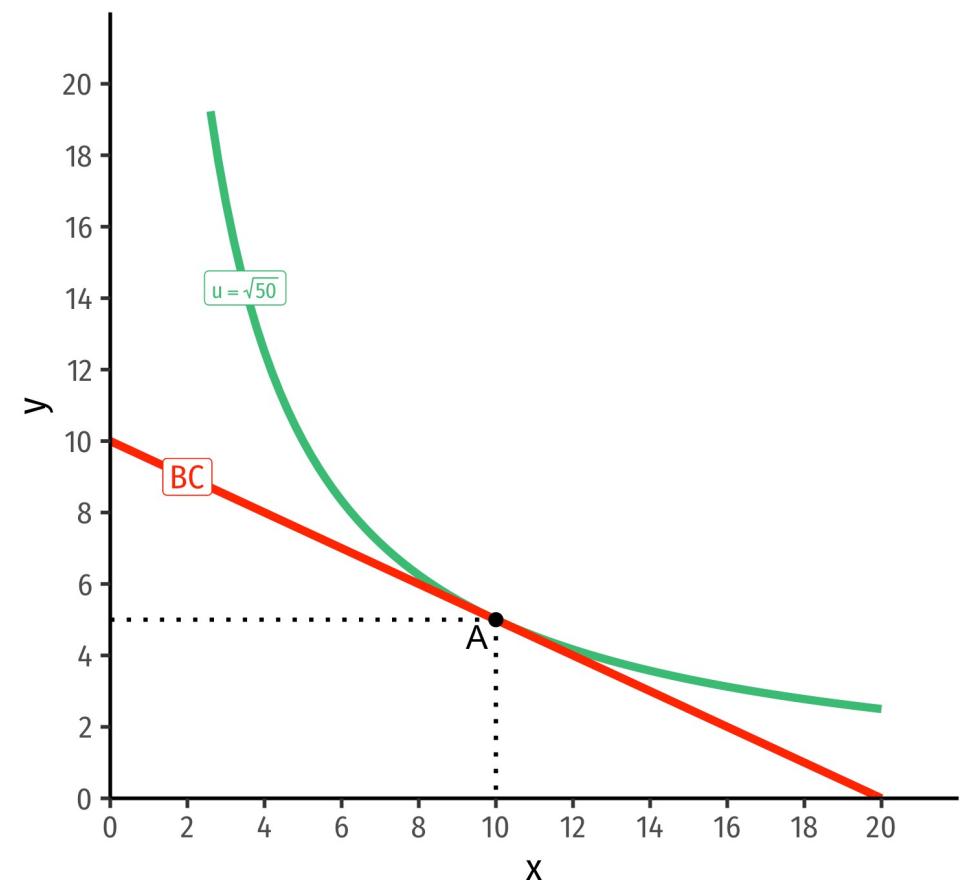
- indiff. curve slope > budget constr. slope

$$|MRS_{x,y}| = \left| \frac{p_x}{p_y} \right|$$

$$\left| \frac{MU_x}{MU_y} \right| = \left| \frac{p_x}{p_y} \right|$$

$$|-0.5| = |-0.5|$$

- Consumer** would exchange at same rate as **market**



$$u(x, y) = \sqrt{xy}, m = \$20, p_x = \$1, p_y = \$2$$



The Consumer's Optimum

Rule 1

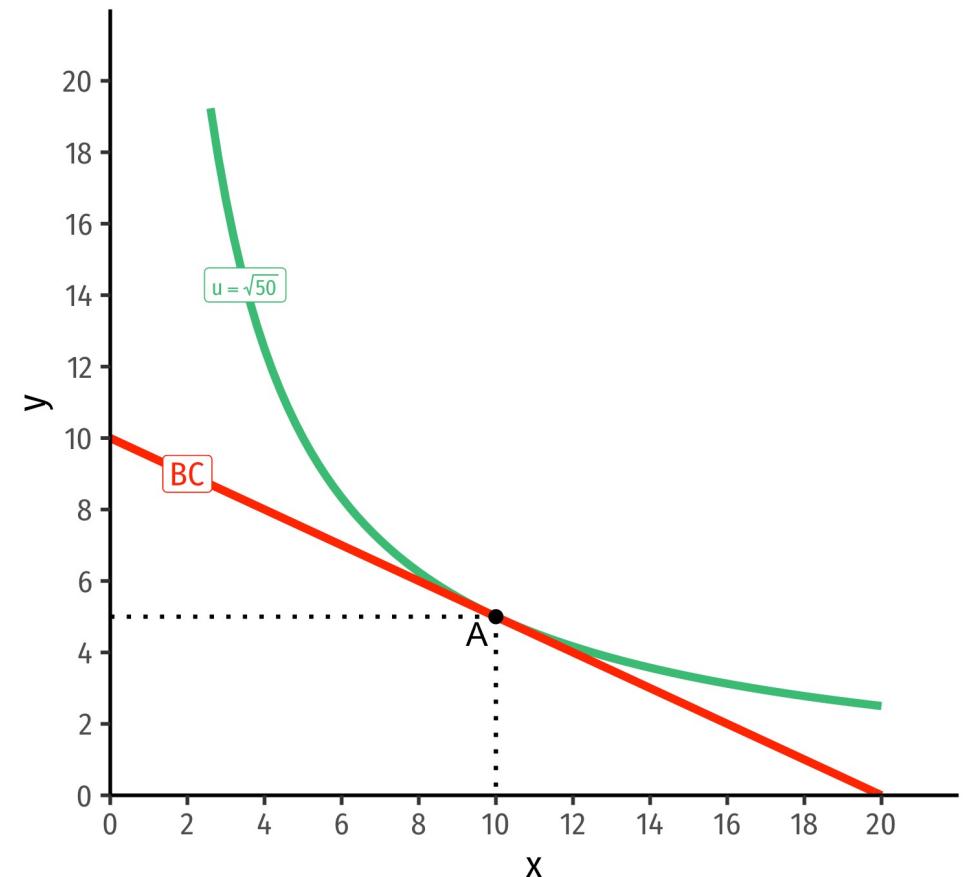
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Rule 2

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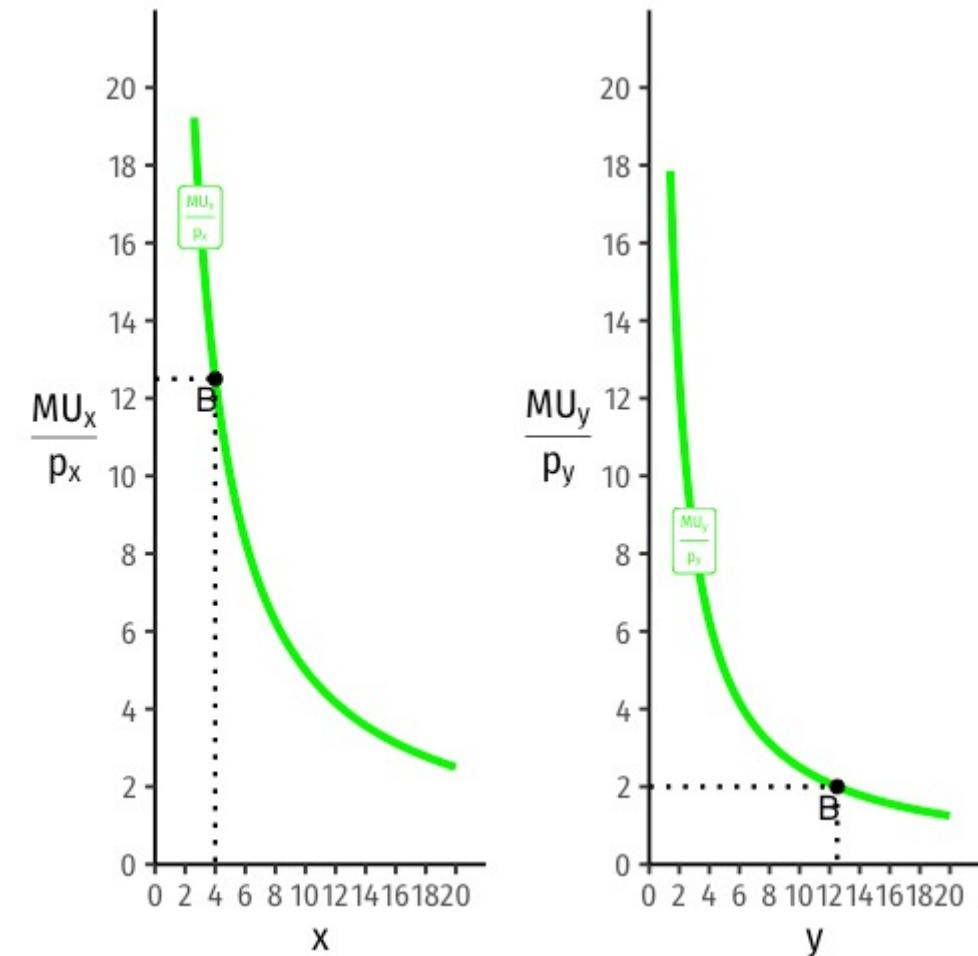




Visualizing the Equimarginal Rule

- Suppose you consume 4 of x and 12.5 of y (points B)

$$\frac{MU_x}{P_x} > \frac{MU_y}{p_y}$$



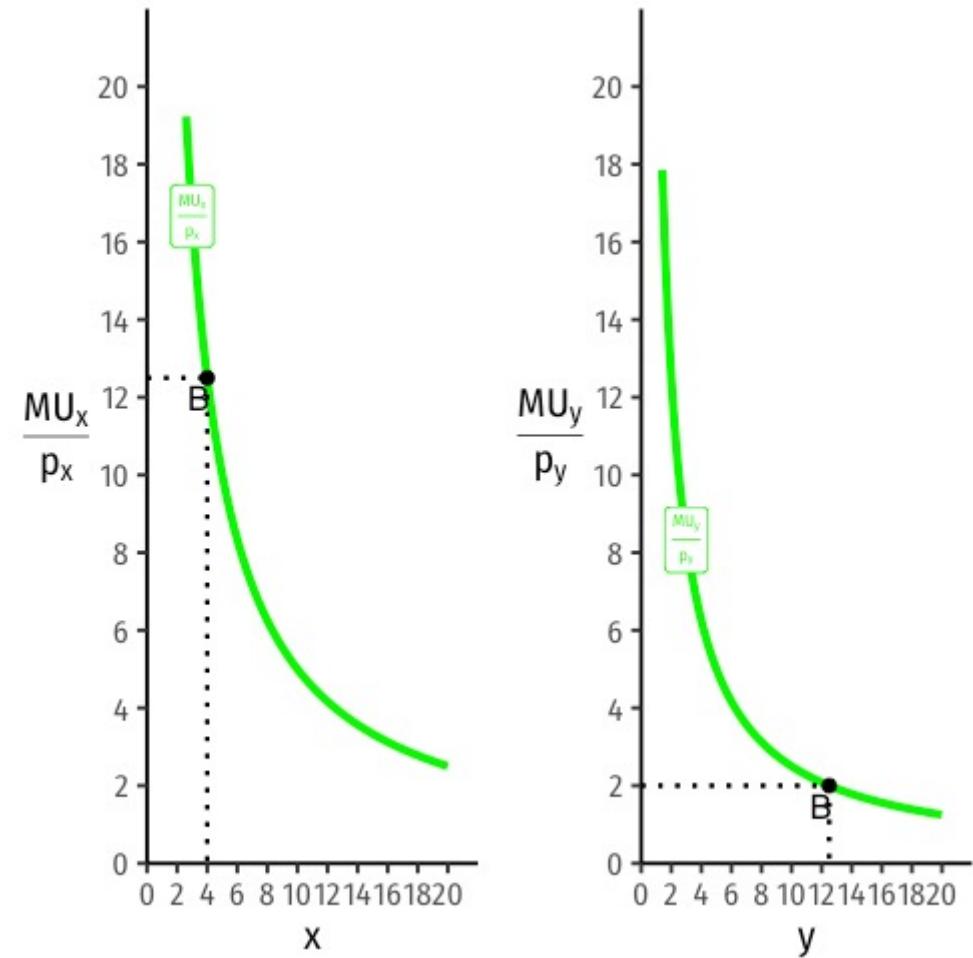


Visualizing the Equimarginal Rule

- Suppose you consume 4 of x and 12.5 of y (points B)

$$\frac{MU_x}{P_x} > \frac{MU_y}{p_y}$$

- More "bang for your buck" with x than y
- Consume more x, less y!



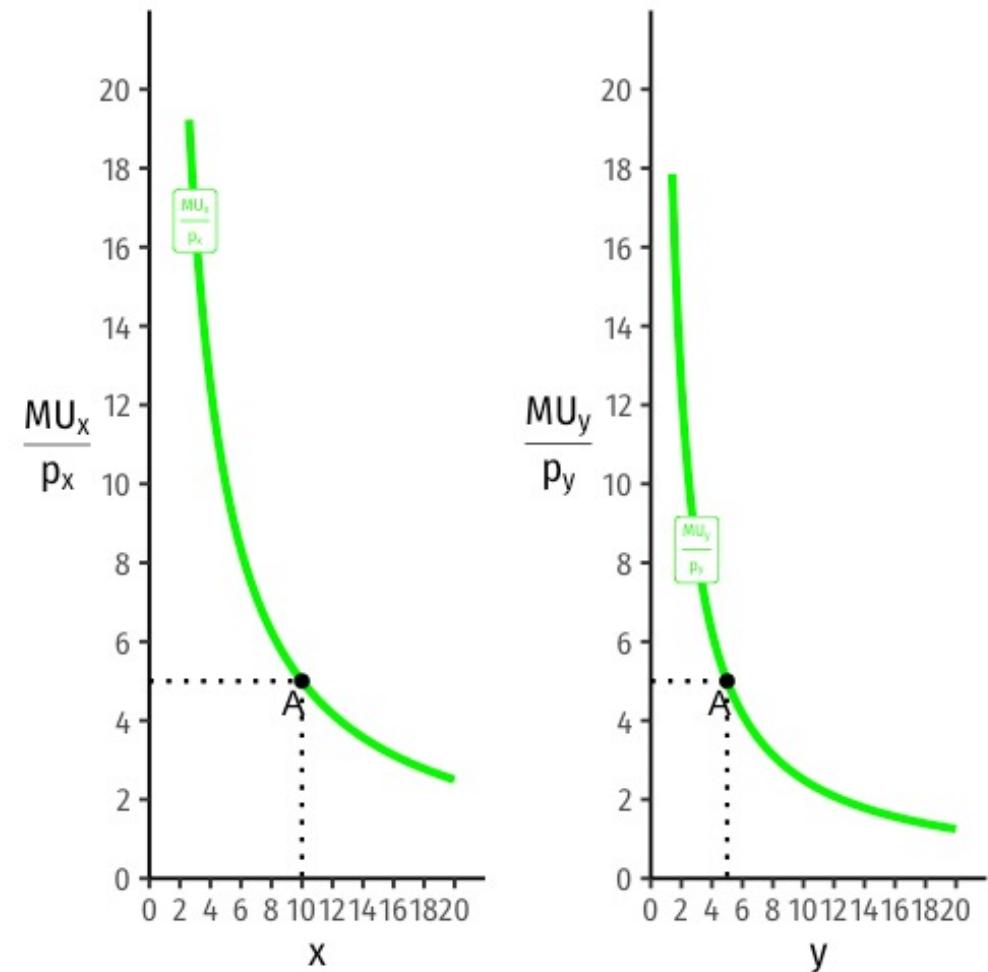


Visualizing the Equimarginal Rule

- At points A, consuming 10 of x and 5 of y

$$\frac{MU_x}{P_x} = \frac{MU_y}{P_y}$$

- No change (more x, less x, more y, less y) that could increase your utility!
- The optimum! Cost-adjusted marginal utilities are equalized





The Consumer's Optimum: The Equimarginal

$$\frac{MU_x}{P_x} = \frac{MU_y}{P_y} = \dots = \frac{MU_n}{P_n}$$

- **Equimarginal Rule:** consumption is optimized where the **marginal utility per dollar spent is equalized** across all n possible goods/decisions



The Consumer's Optimum: The Equimarginal

$$\frac{MU_x}{P_x} = \frac{MU_y}{P_y} = \dots = \frac{MU_n}{P_n}$$

- **Equimarginal Rule:** consumption is optimized where the **marginal utility per dollar spent is equalized** across all n possible goods/decisions
- You will always choose an option that gives higher marginal utility (e.g if $MU_x > MU_y$)
- But each option has a different cost, so we weight each option by its cost, hence $\frac{MU_x}{P_x}$



The Consumer's Optimum: The Equimarginal

- Any **optimum** in economics: no better alternatives exist under current constraints
- No possible change in your consumption that would increase your utility





Markets Equalize Everyone's MRS

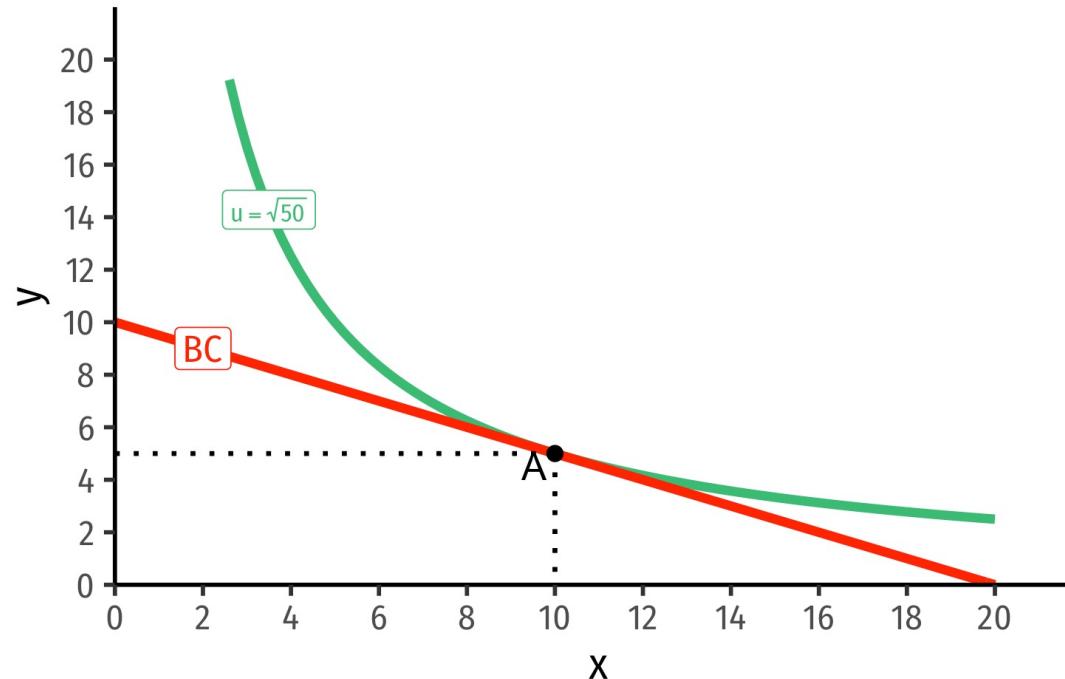
- Markets make it so everyone faces the same relative prices
 - Budget constraint. slope, $-\frac{p_x}{p_y}$
 - Note individuals' incomes, m , are certainly different!
- A person's optimal choice \Rightarrow they make same tradeoff as the market
 - Their MRS = relative price ratio
- markets equalize everyone's MRS



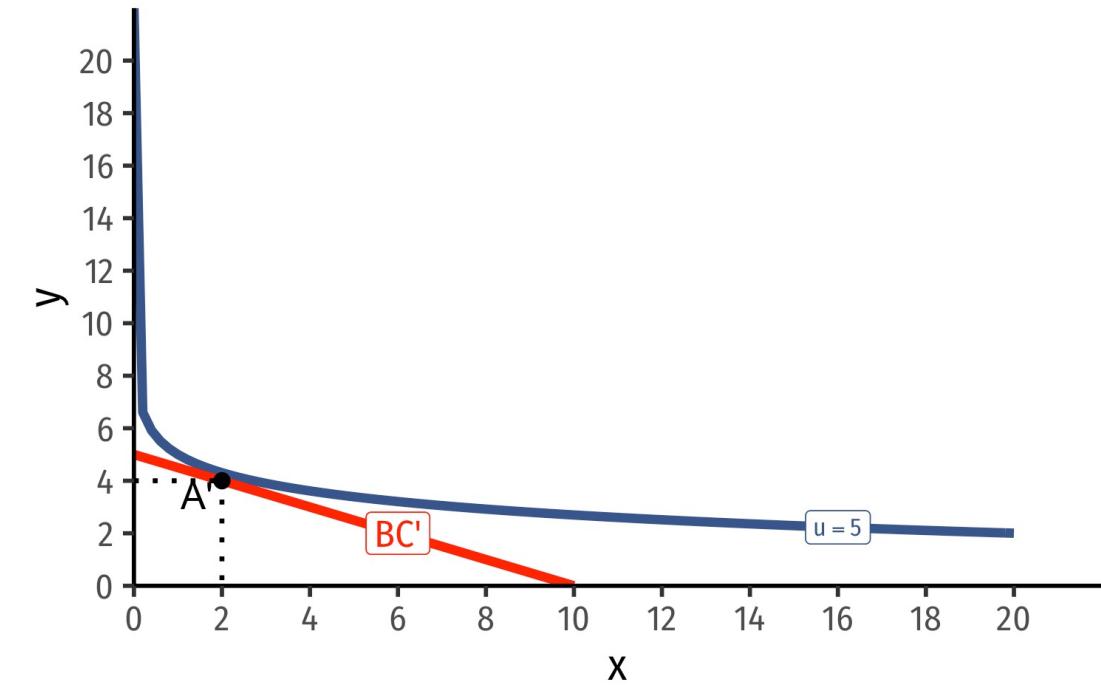


Markets Equalize Everyone's MRS

- Two people with very different income and preferences face the same market prices, and choose optimal consumption (points A and A') at an exchange rate of 0.5Y:1X



$$u(x, y) = \sqrt{xy}, m = \$20, p_x = \$1, p_y = \$2$$

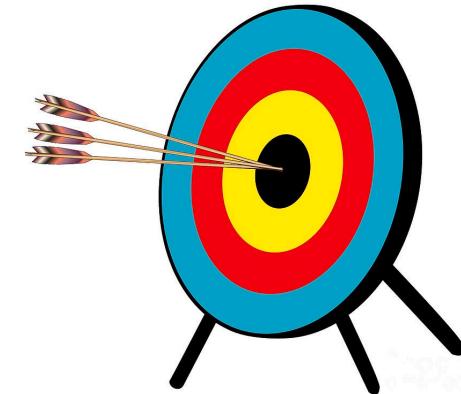


$$u(x, y) = \ln(x) + y, m = \$10, p_x = \$1, p_y = \$2$$



The Consumer's Optimum: The Equimarginal

- If possible people will always switch to a higher-valued option
- If a person has no better choices (under current constraints), they are at an **optimum**
- **If everyone is at an optimum, the system is in equilibrium**



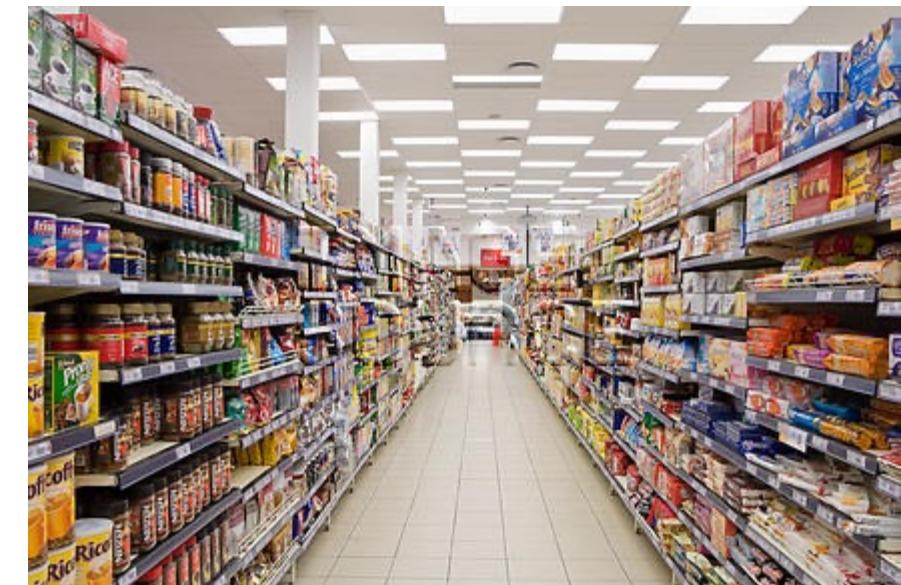


Deriving Demand



The Consumer's Problem

- The **consumer's constrained optimization problem** is:
- Choose: < a consumption bundle >
- In order to maximize: < utility >
- Subject to: < income and market prices >





A Demand Function

- The **consumer's demand for good x**

$$q_x^D = q_x^D(m, p_x, p_y)$$

- How does **demand for x** change?

- Income effects = $\frac{\Delta q_x^D}{\Delta m}$
- Cross-price effects = $\frac{\Delta q_x^D}{\Delta p_y}$
- Own price effects = $\frac{\Delta q_x^D}{\Delta p_x}$





Income Effect



Income Effect

- **Income effect:** change in optimal consumption of a good associated with a change in (nominal) income, holding relative prices constant

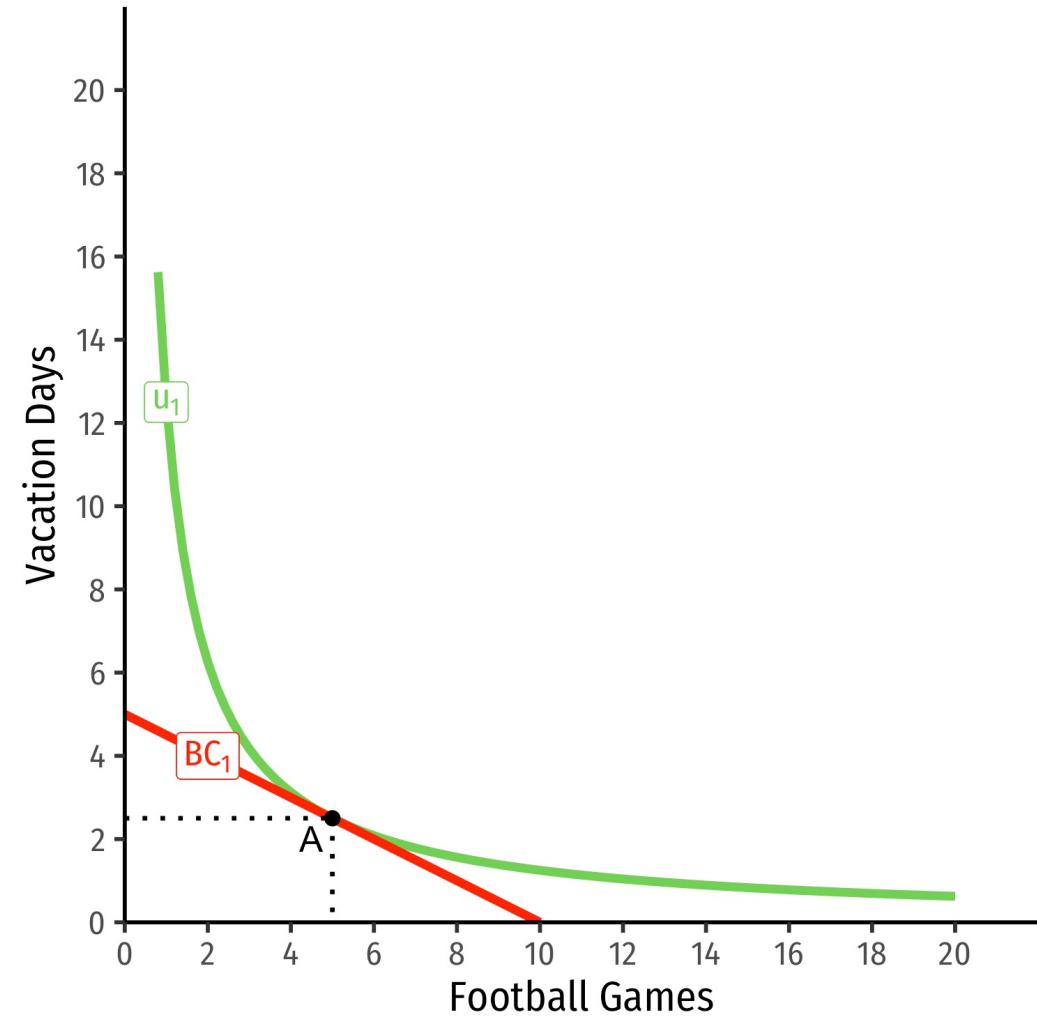
$$\frac{\Delta q^D}{\Delta m} < ? > 0$$





Income Effect (Normal)

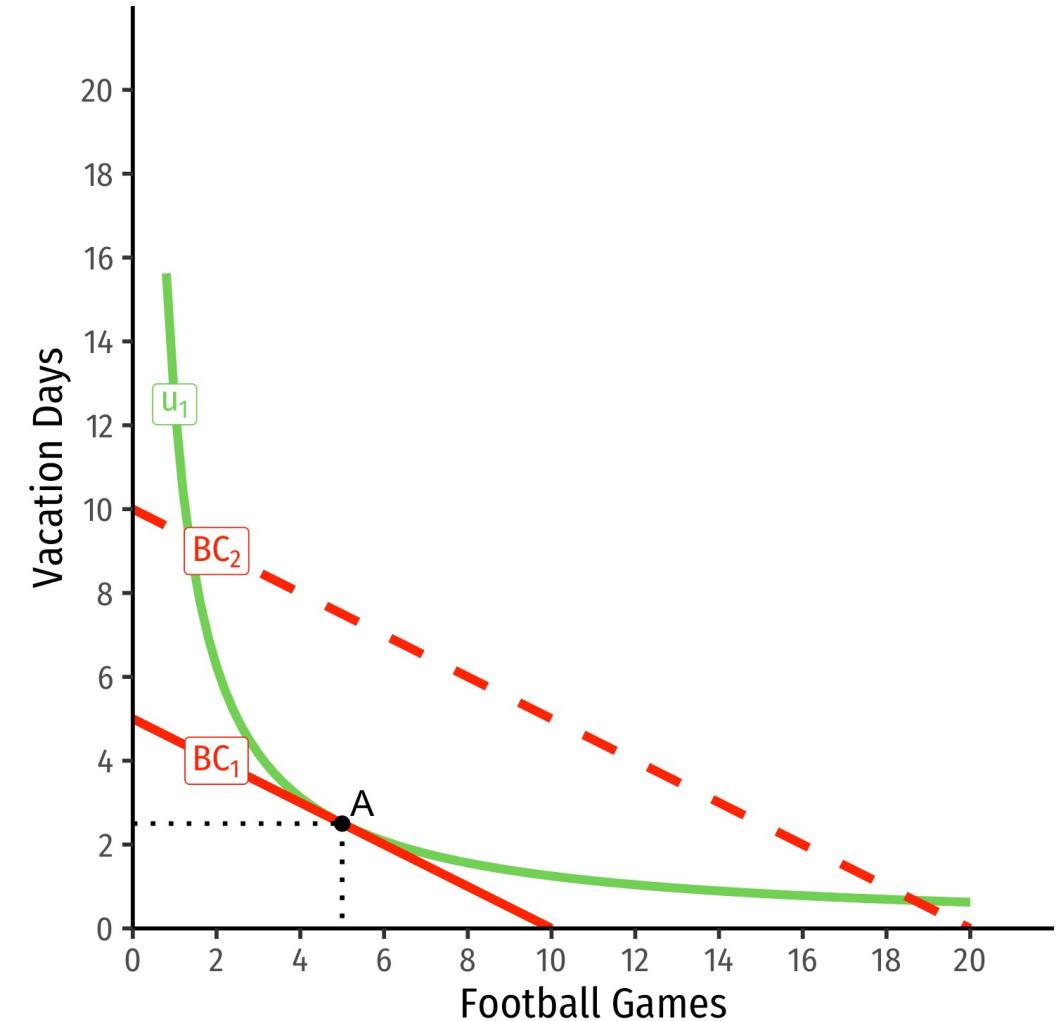
- Consider football tickets and vacation days





Income Effect (Normal)

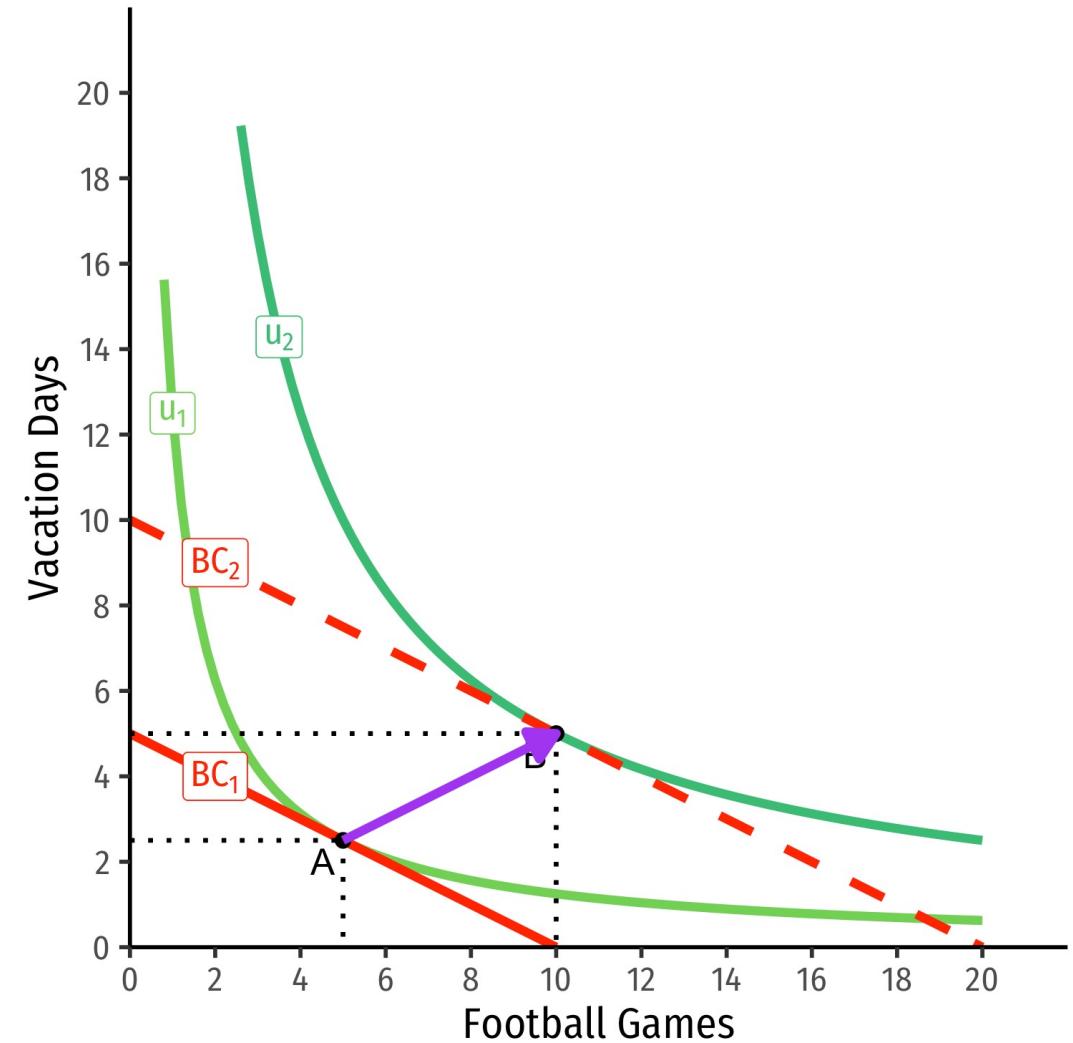
- Consider football tickets and vacation days
- Suppose income (m) increases





Income Effect (Normal)

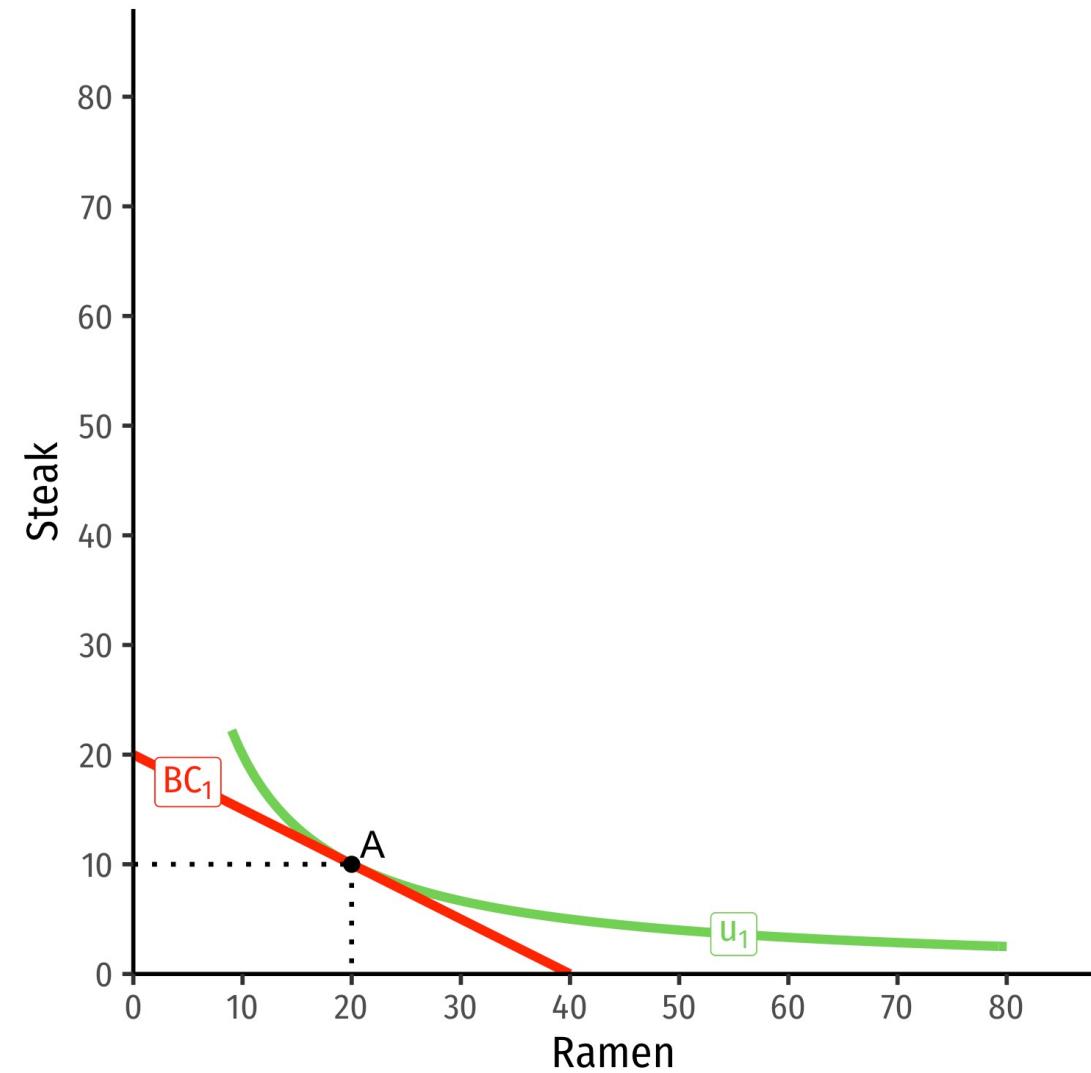
- Consider football tickets and vacation days
- Suppose income (m) increases
- At new optimum (B), consumes more of both
- Then both goods are **normal goods**





Income Effect (Inferior)

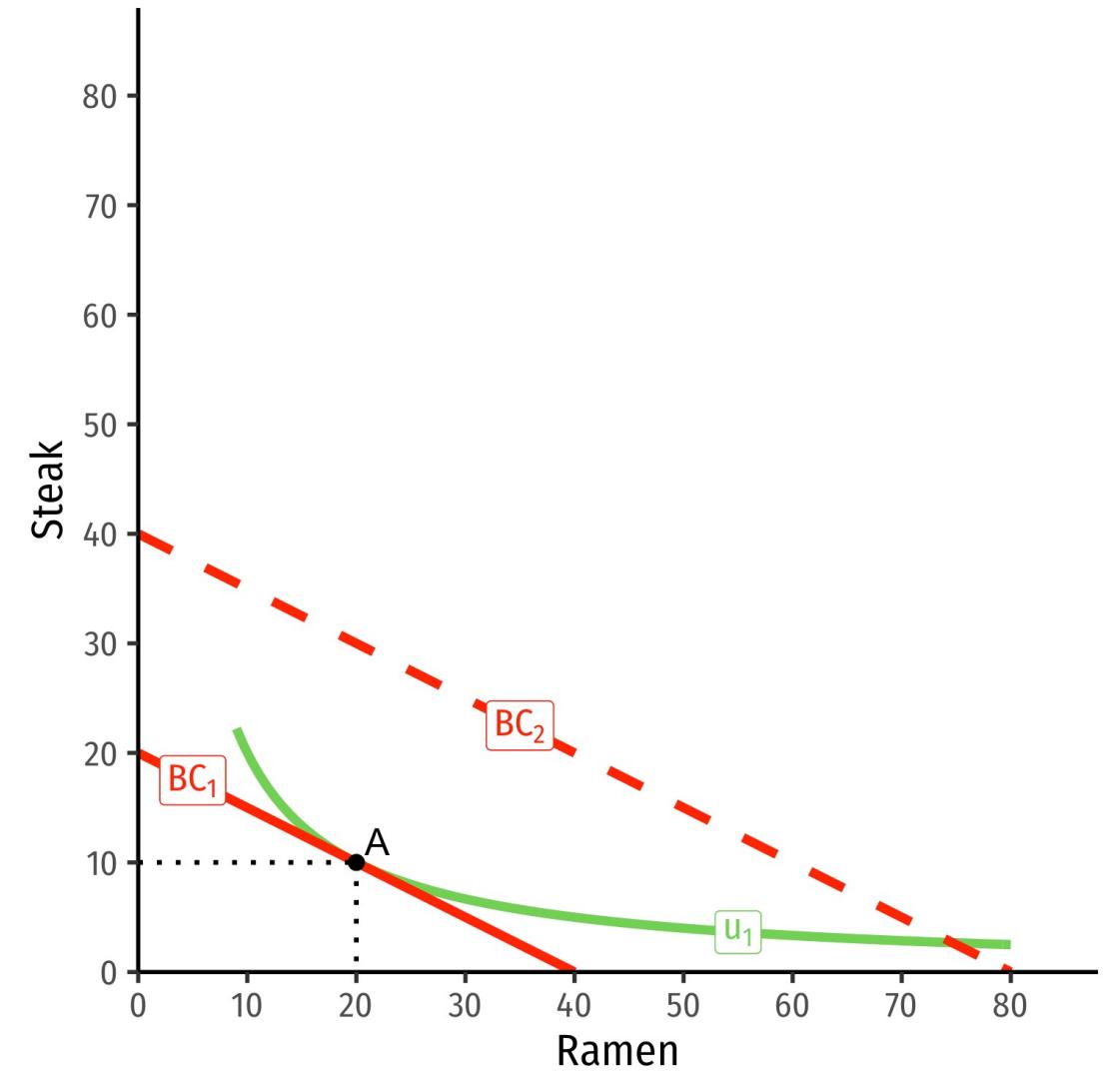
- Consider ramen and steak





Income Effect (Inferior)

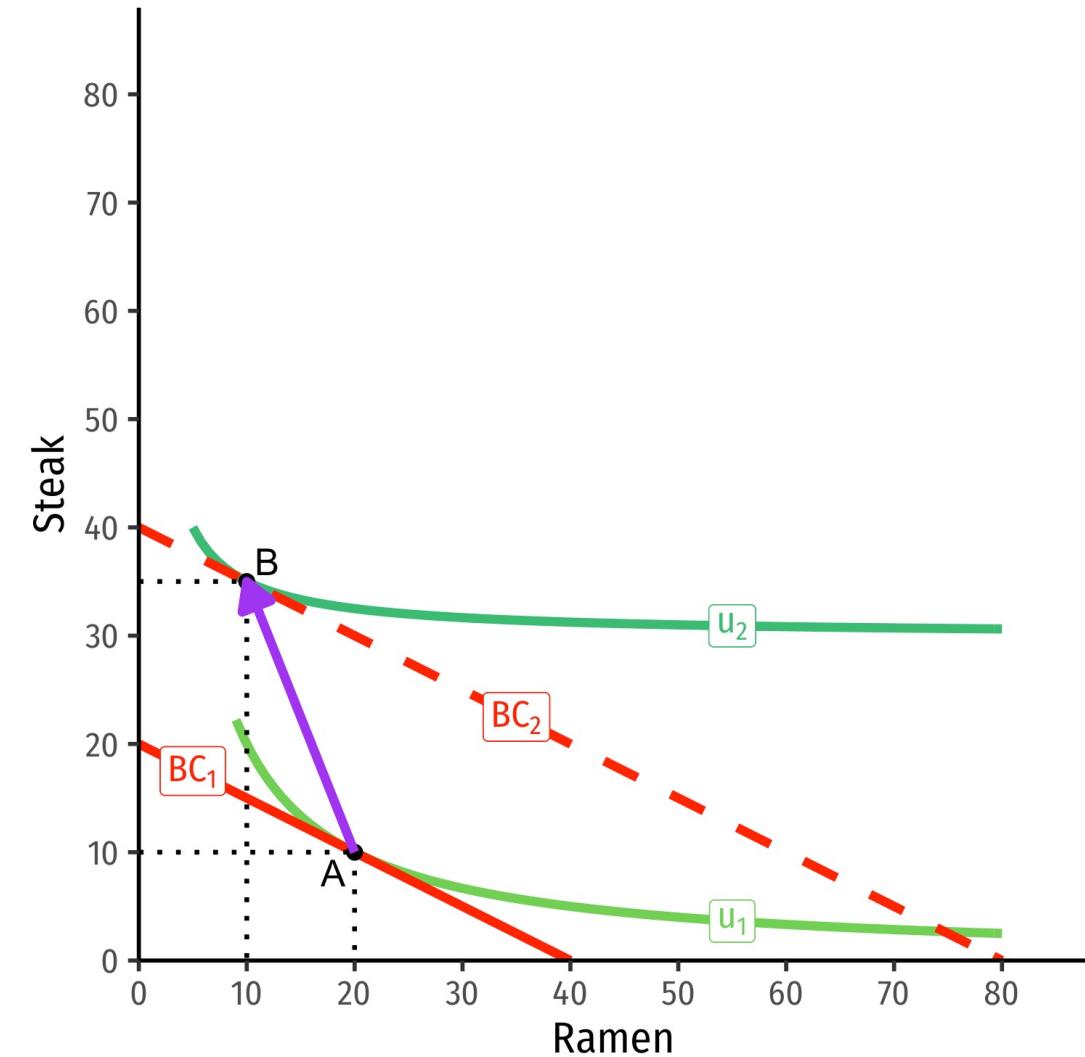
- Consider ramen and steak
- Suppose income (m) increases





Income Effect (Inferior)

- Consider ramen and steak
- Suppose income (m) increases
- At new optimum (B), consumes more steak, less ramen
- Steak is a **normal good**, ramen is an **inferior good**





Income Effect

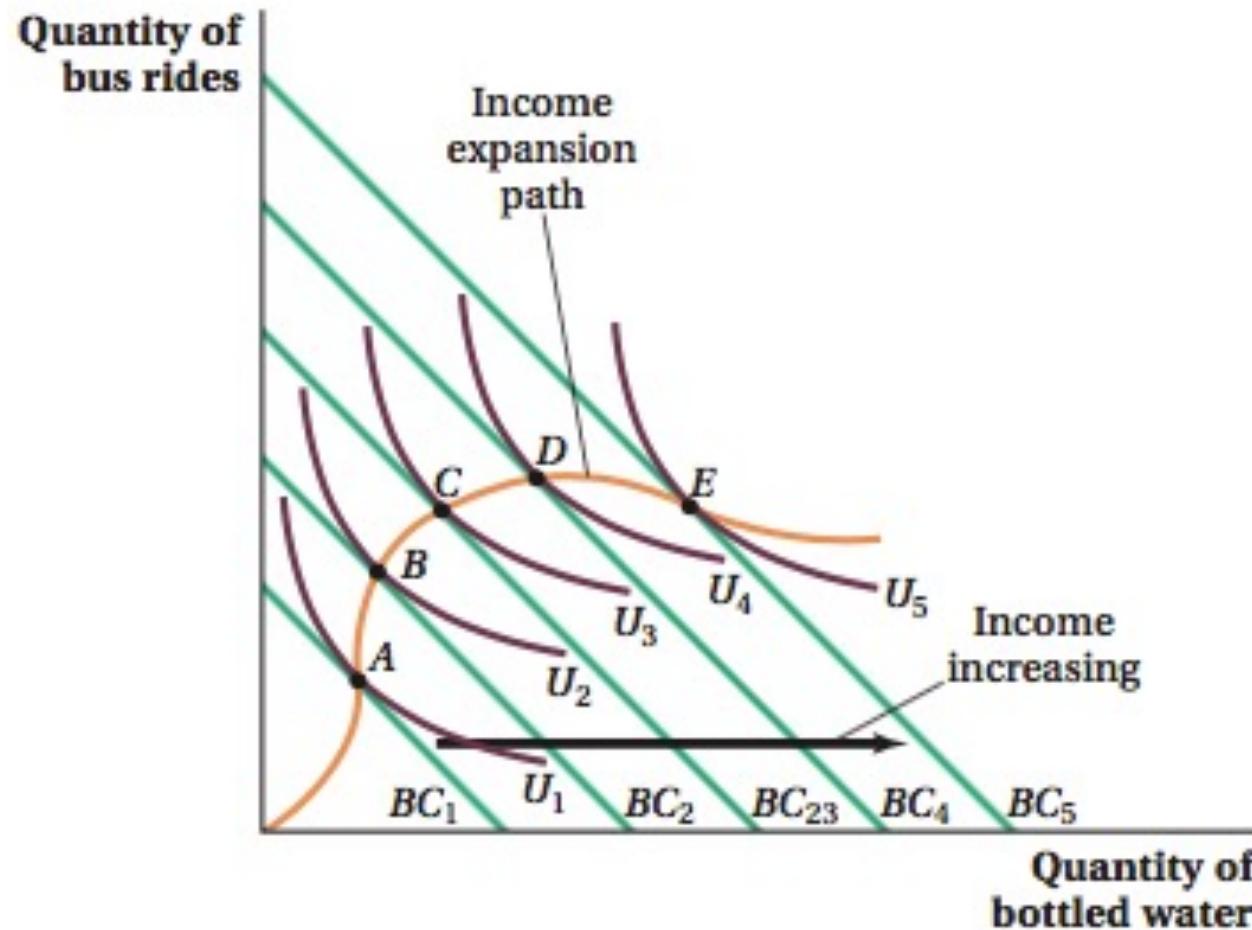
$$\frac{\Delta q^D}{\Delta m} < ? > 0$$

- **Normal goods:** consumption increases with more income (and vice versa)
- **Inferior goods:** consumption decreases with more income (and vice versa)





Income Expansion Path



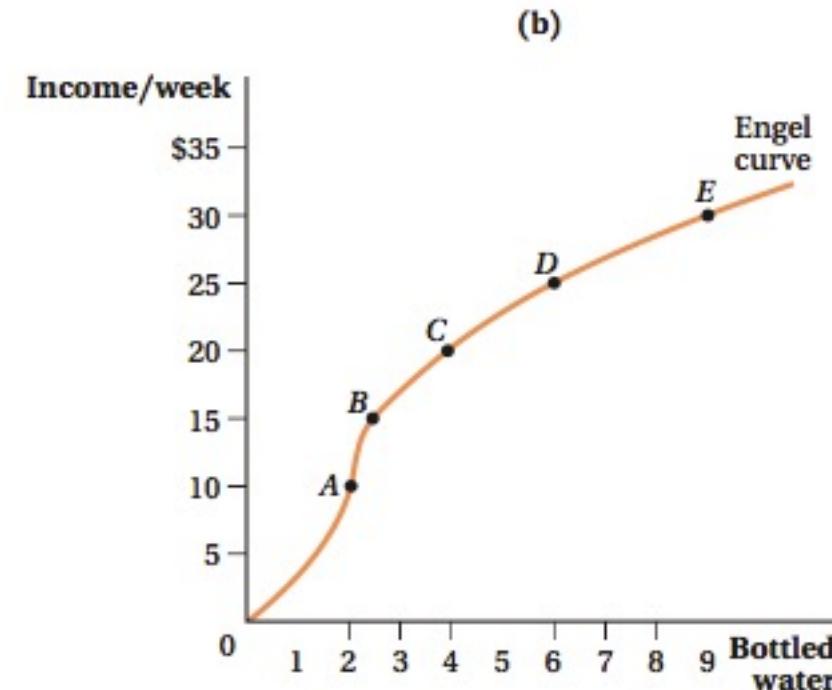
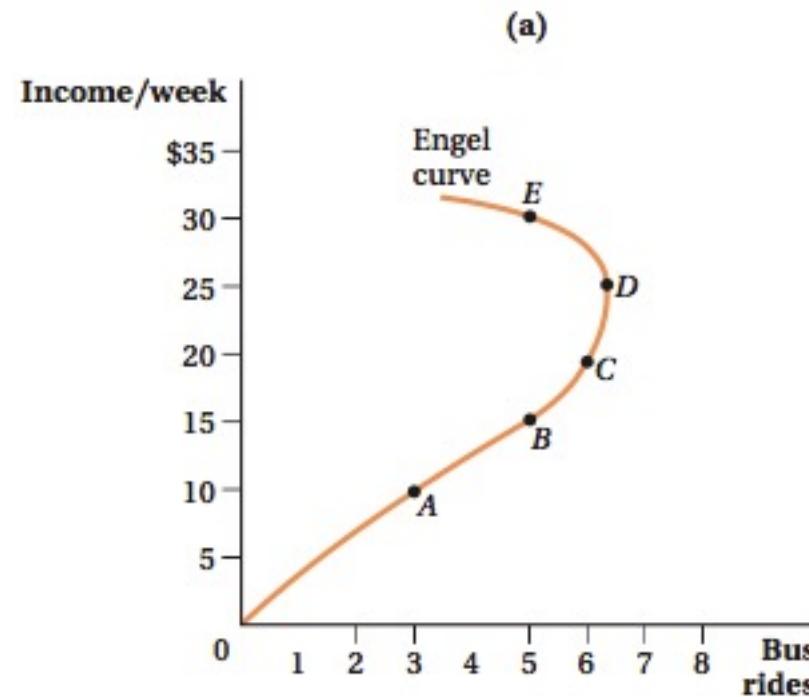
- **Income expansion path** describes how consumption of each good changes when income increases

Goolsbee, et al (2011:169)



Engel Curves

- **Engel curve** of each good is more helpful to visualize: shows how consumption of one good changes when income increases



Goolsbee, et al (2011:171)



Cross-Price Effects



Cross-Price Effects

- **Cross-price effect:** change in optimal consumption of a good associated with a change in price of another good income, holding the good's own price (and income) constant

$$\frac{\Delta q_x}{\Delta p_y} > ? < 0$$





Income and Substitution Effects



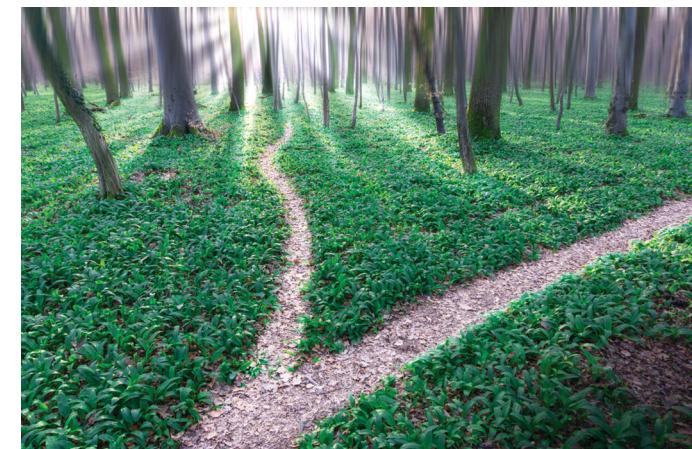
A Demand Function

- The **consumer's demand for good x**

$$q_x^D = q_x^D(m, p_x, p_y)$$

- How does **demand for x** change?

- Income effects = $\frac{\Delta q_x^D}{\Delta m}$
- Cross-price effects = $\frac{\Delta q_x^D}{\Delta p_y}$
- Own price effects = $\frac{\Delta q_x^D}{\Delta p_x}$





Own price Effect



The (Own) Price Effect

- **Price effect:** change in optimal consumption of a good associated with a change in its price
- holding income and other prices constant

$$\frac{\Delta q_x^D}{\Delta p_x} < 0$$

- **The law of demand**





Decomposing the Price Effect

- The **price effect** (law of demand) is actually the **net result of two effects**
- **(Real) income effect:** change in consumption due to change in real purchasing power
- **Substitution effect:** change in consumption due to change in relative prices
- **Price Effect = Real income effect + Substitution Effect**



(Real) Income Effect



(Real) Income Effect

- Income: \$100
- $P_x = \$10, Q_x = 10$
- Suppose the price of x falls
- $P_x = \$5, Q_x = 20$
- This is the **real income effect**





(Real) Income Effect

- **Real income effect:**
 - consumption mix changes due to the price change
 - changes your **real income or purchasing power** (the amount of goods you can buy)
- Note your ***actual (nominal) income (\$100) never changed!***



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(Real) Income Effect: Size

- The size of the income effect depends on how large a portion of your budget you spend on the good
- **Large-budget items:**
 - e.g. Housing/apartment rent, car prices
 - Price increase makes you much poorer
 - Price decrease makes you much wealthier





(Real) Income Effect: Size

- The size of the income effect depends on how large a portion of your budget you spend on the good
- **Small-budget items:**
 - e.g. pencils, toothpicks, candy
 - Price changes don't have much of an effect on your wealth or change your behavior much



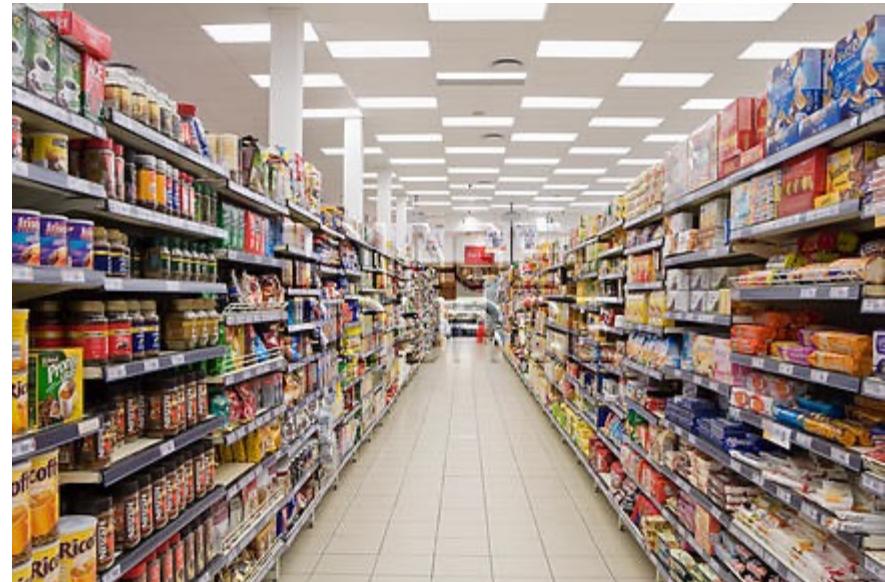


Substitution Effect



Substitution Effect: Demonstration

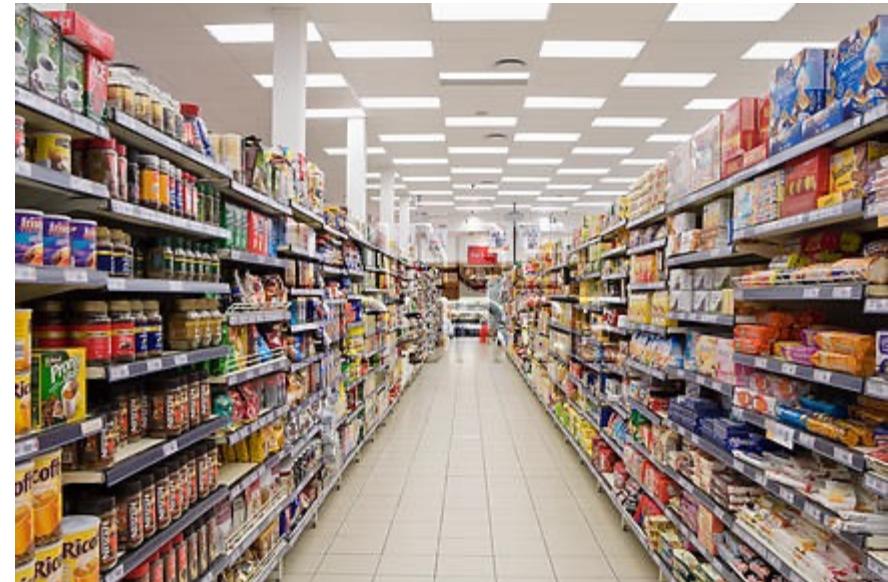
- 1000's of goods, and none of them a major part of your budget
 - So real income effect is insignificant
- The price of one good, x increases
- You consume **less** of x relative to other goods because x is now *relatively* more expensive
- That's the **substitution effect**





Substitution Effect: Demonstration

- **Substitution effect:** consumption mix changes because of a change in **relative prices**
- Buy more of the (now) relatively cheaper items
- Buy less of the (now) relatively more expensive item (x)





Putting the Price Effects Together



Putting the Effects Together

- **Real income effect:** change in consumption due to change in real purchasing power
- **Substitution effect:** change in consumption due to change in relative prices

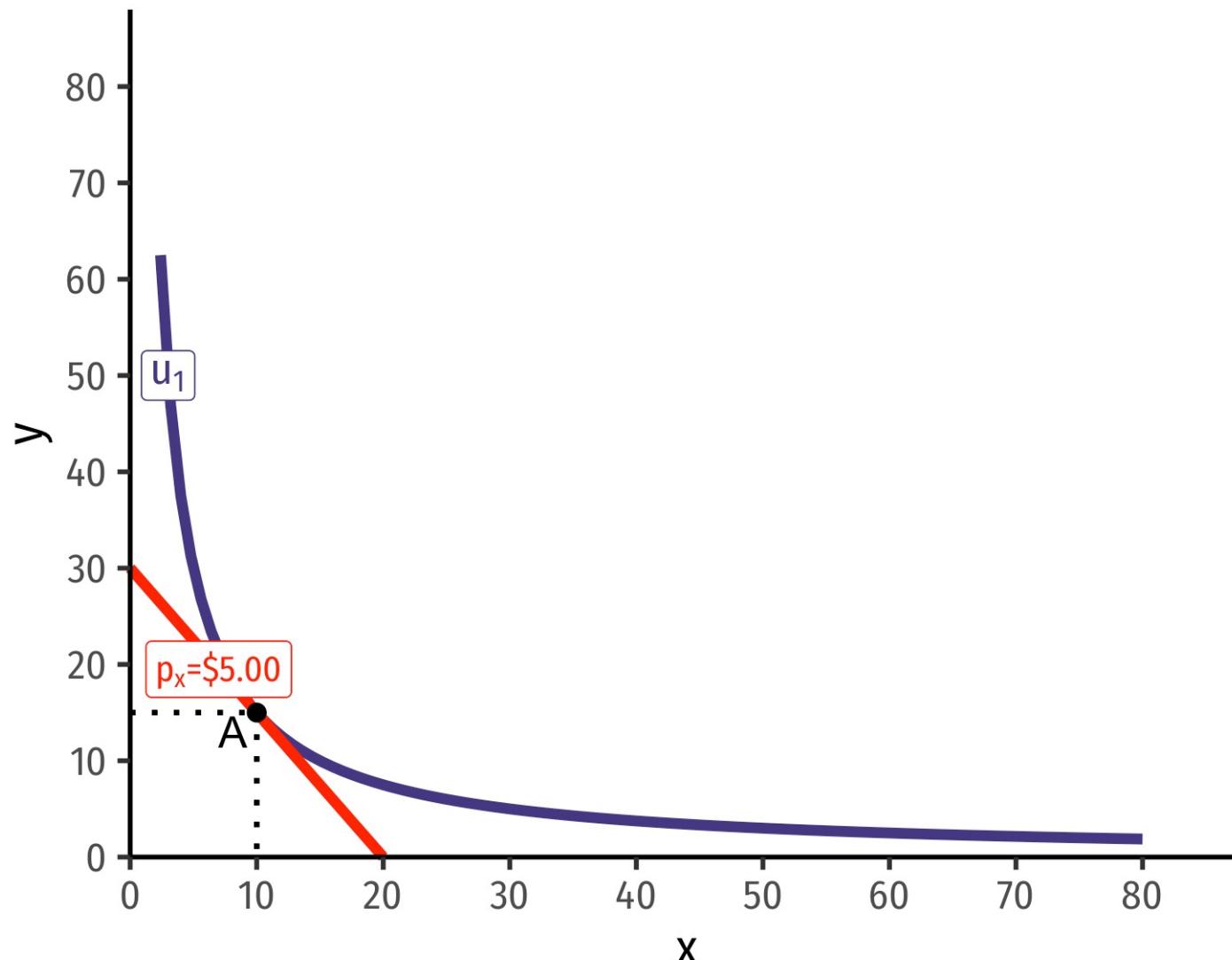


Putting the Effects Together

- **Real income effect:** change in consumption due to change in real purchasing power
 - Can be positive (**normal goods**) or negative (**inferior goods**)
 - Lower price of x means you can buy more x, y, or both (depending on your preferences between x and y)
- **Substitution effect:** change in consumption due to change in relative prices
 - If x gets cheaper relative to y, consume ↓y (and ↑x)
 - This is always the same direction! (↓ relatively expensive goods, ↑ relatively cheaper goods)
 - This is why demand curves slope downwards!
- **Price Effect** = **Real income effect** + **Substitution Effect**



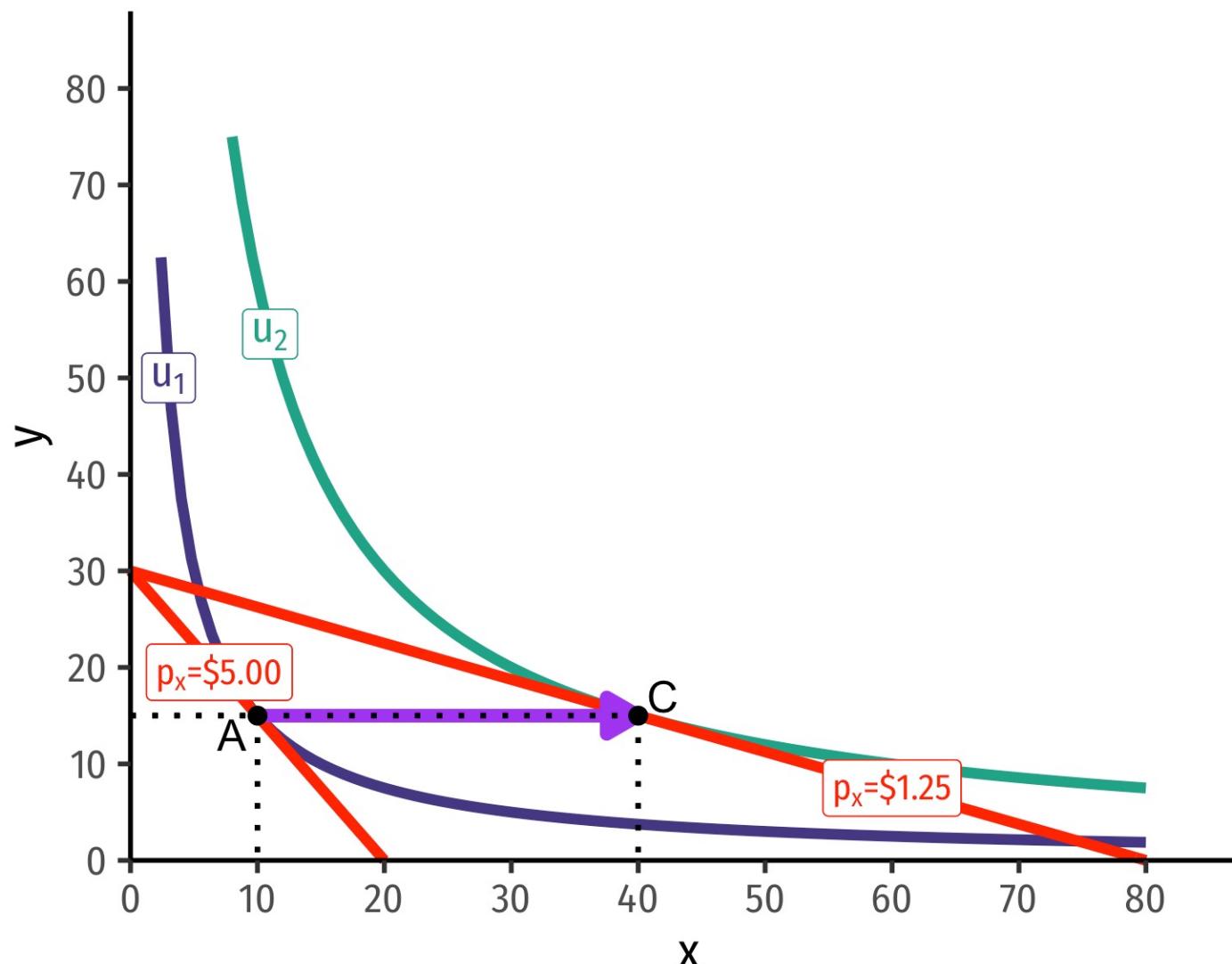
Substitution Effects, Graphically



Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 100$, $p_y = 3.33$



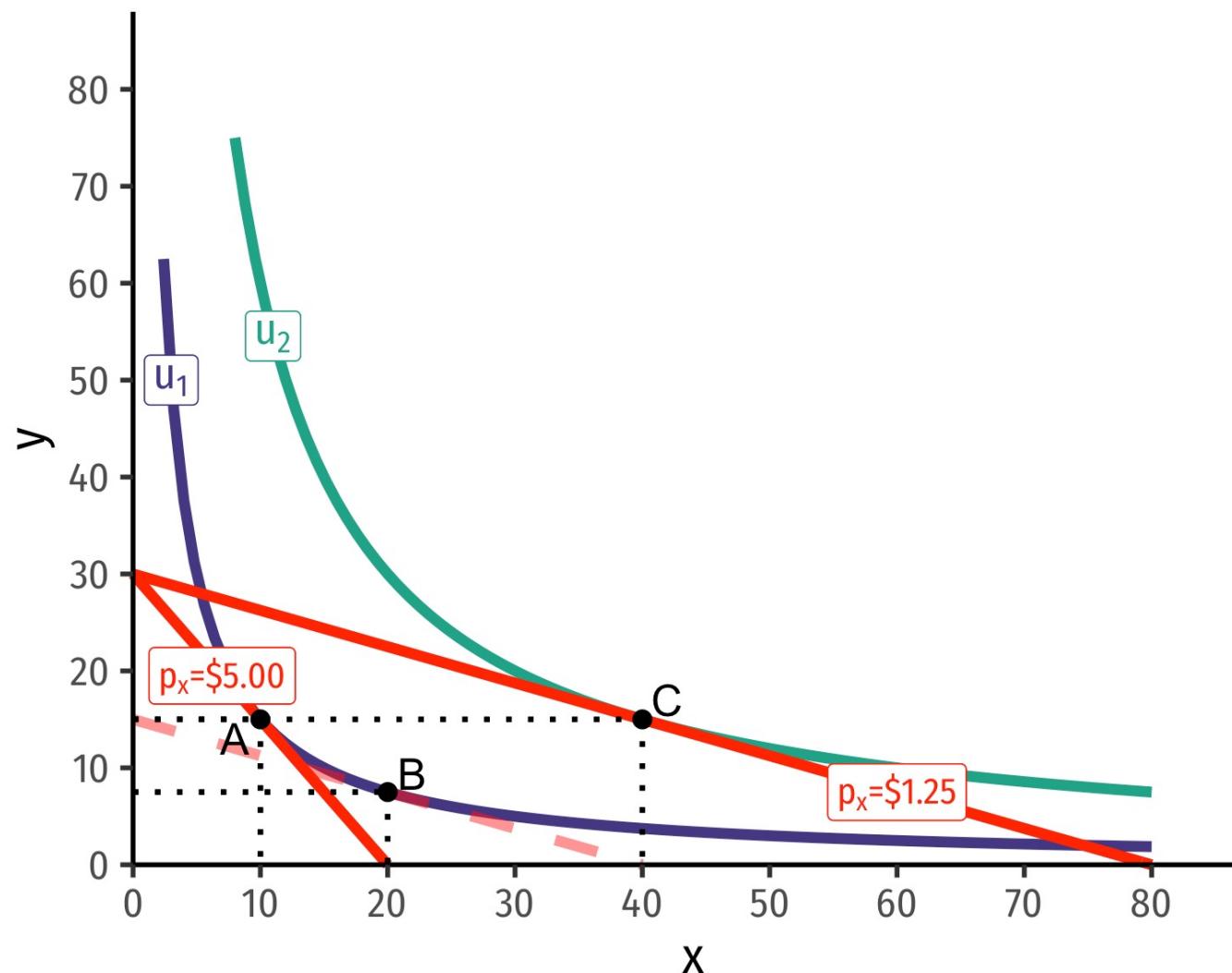
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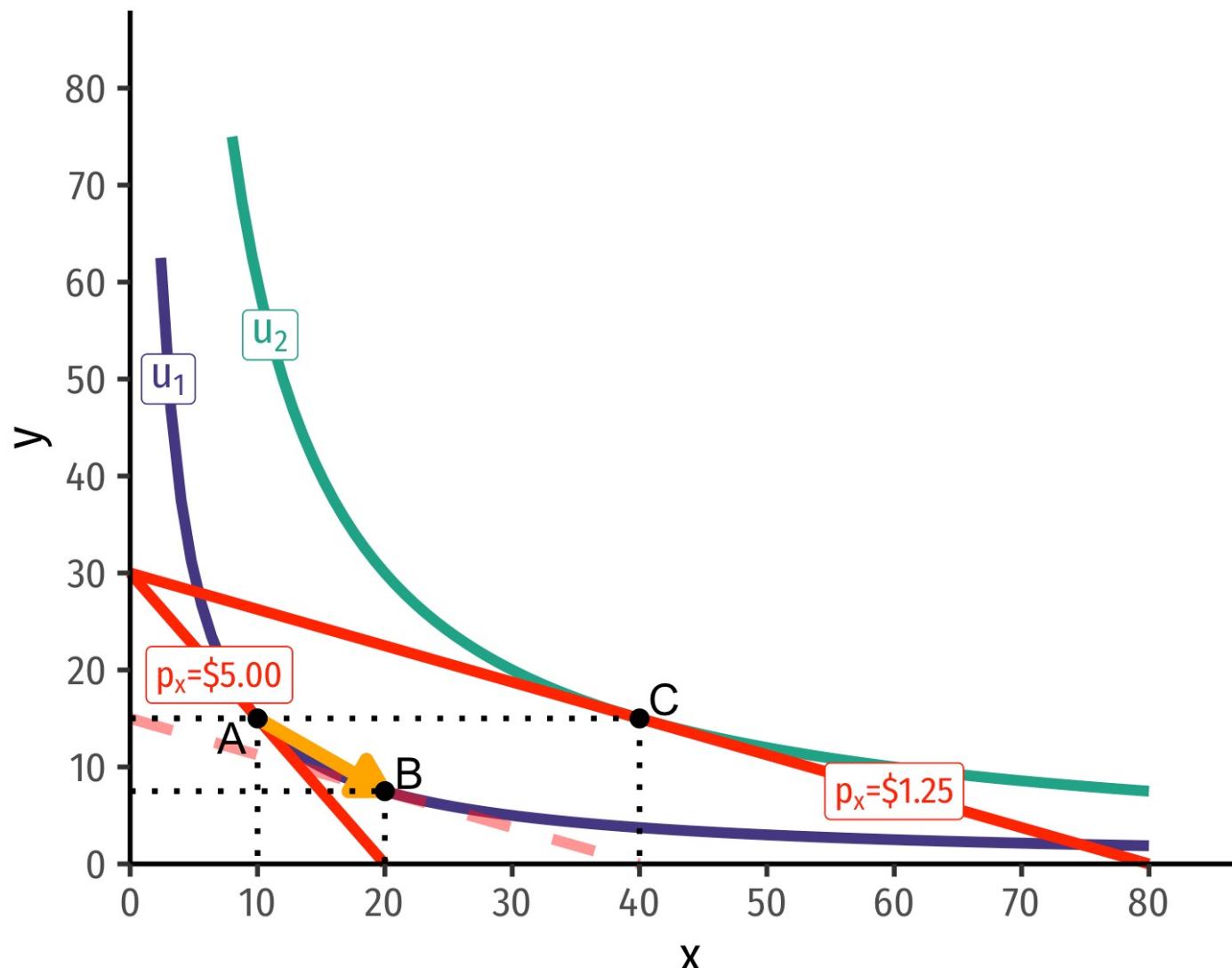
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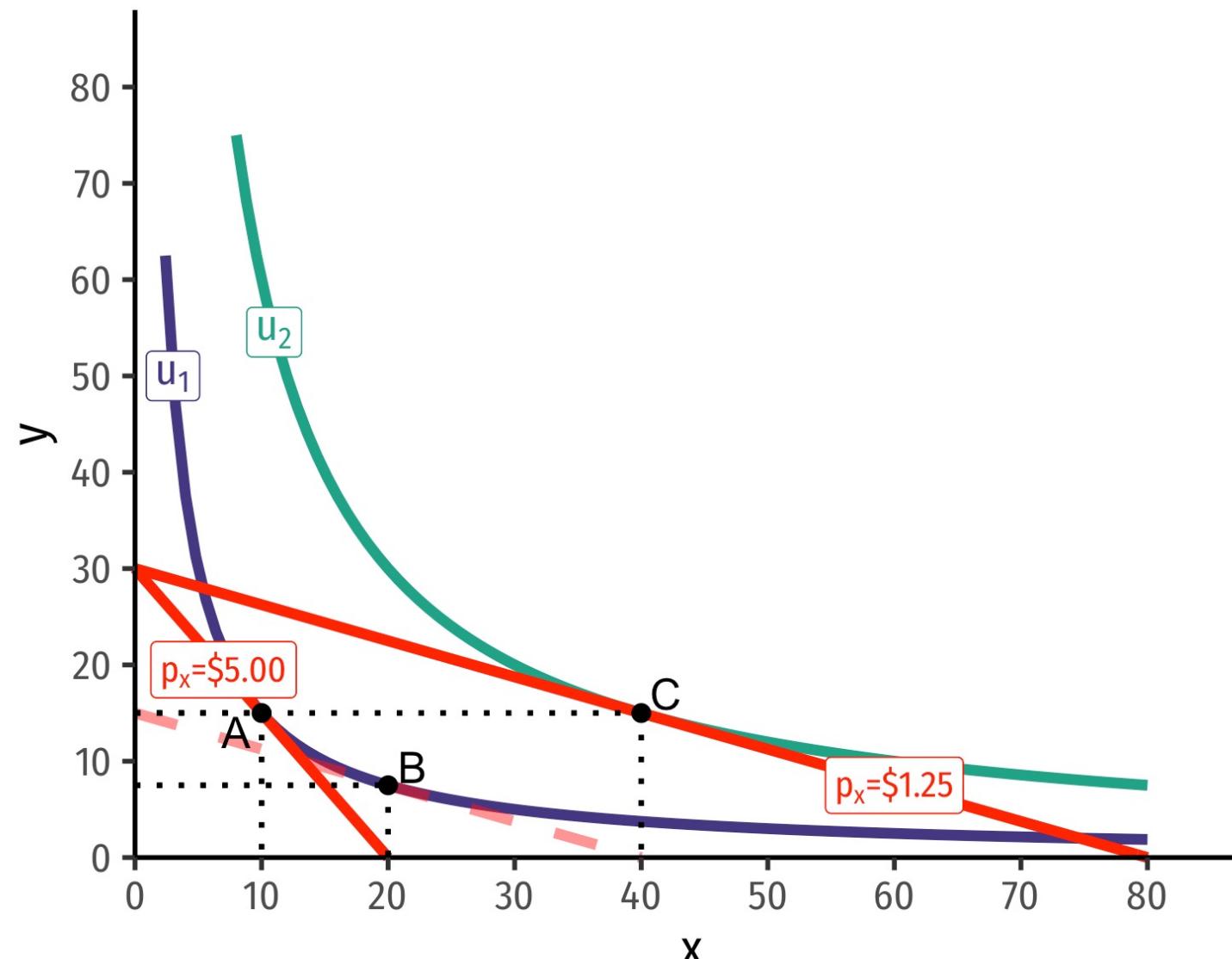
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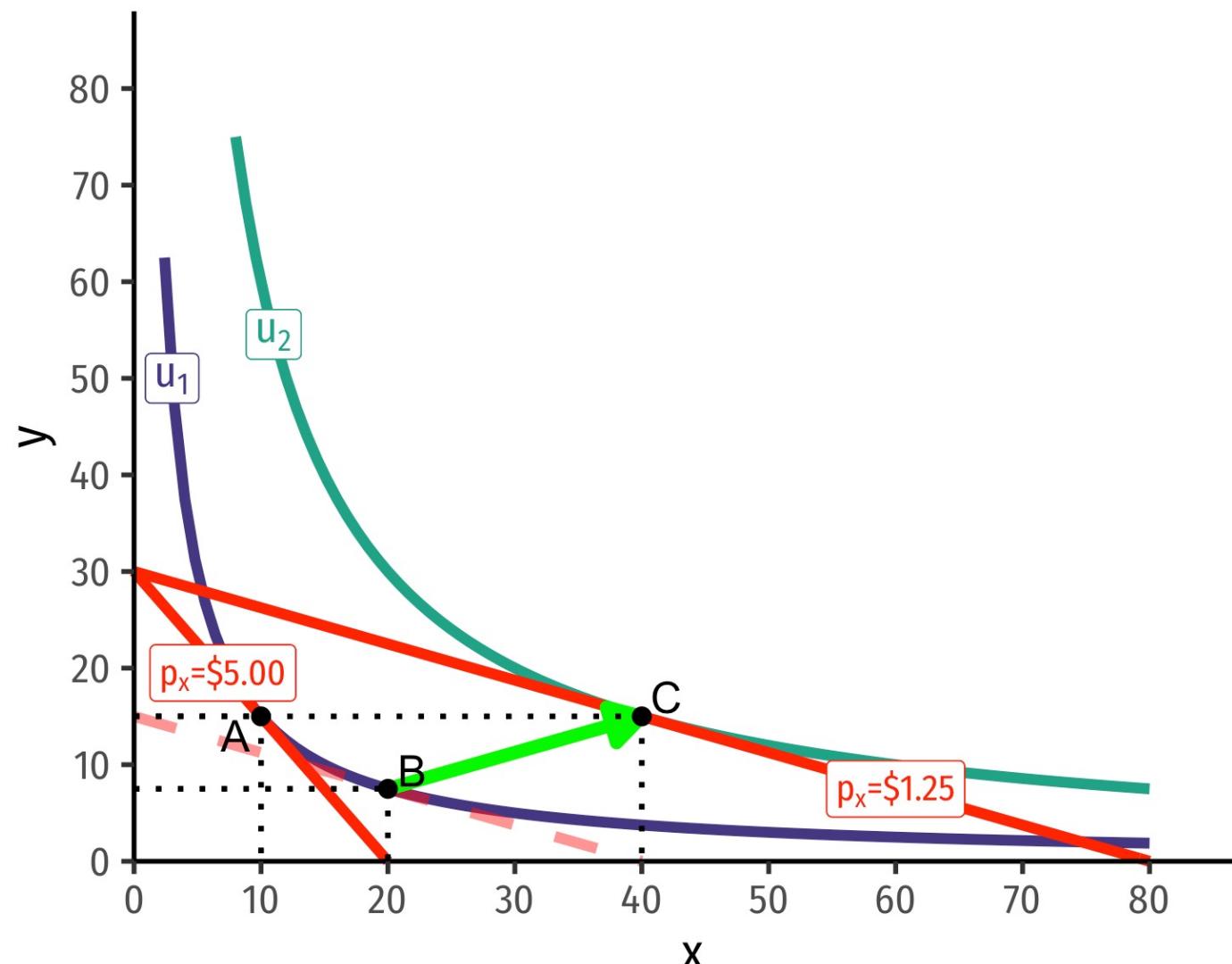
Real Income, Graphically



Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 100$, $p_y = 3.33$



Real Income, Graphically

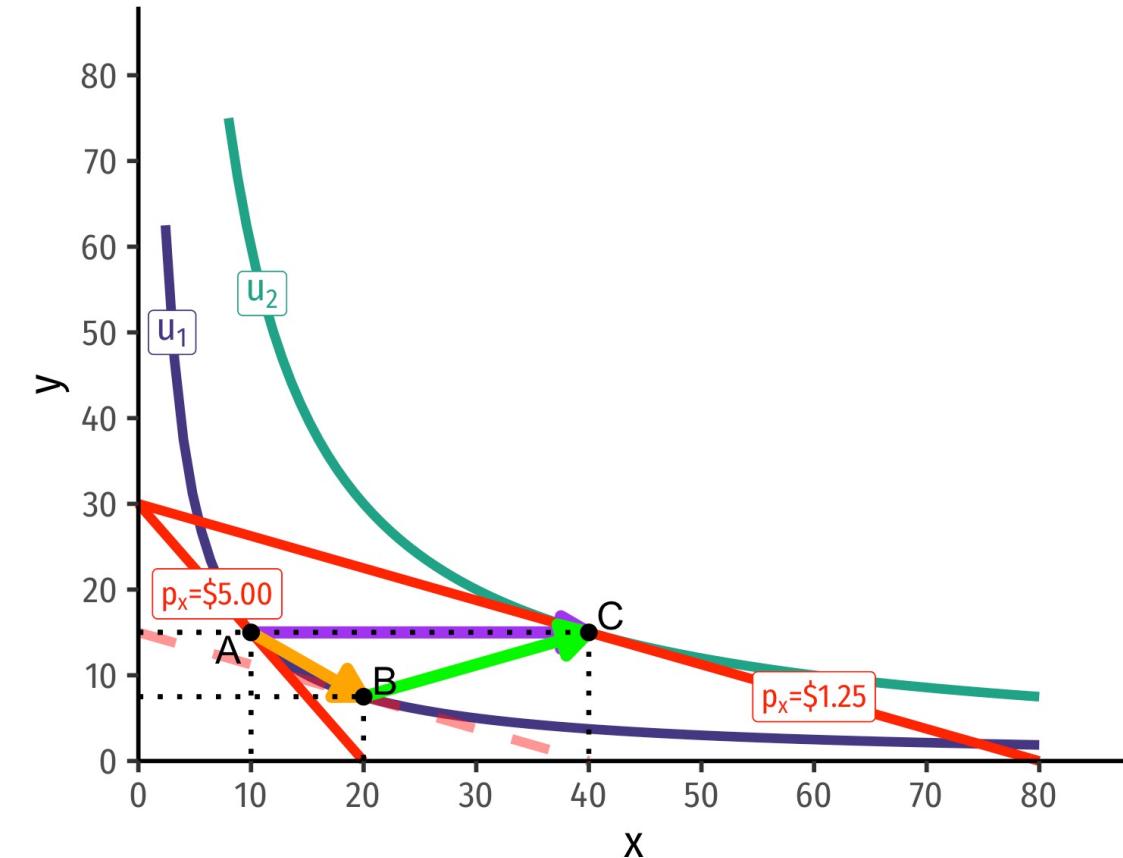


Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 100$, $p_y = 3.33$



Real Income and Substitution Effects, Graphically

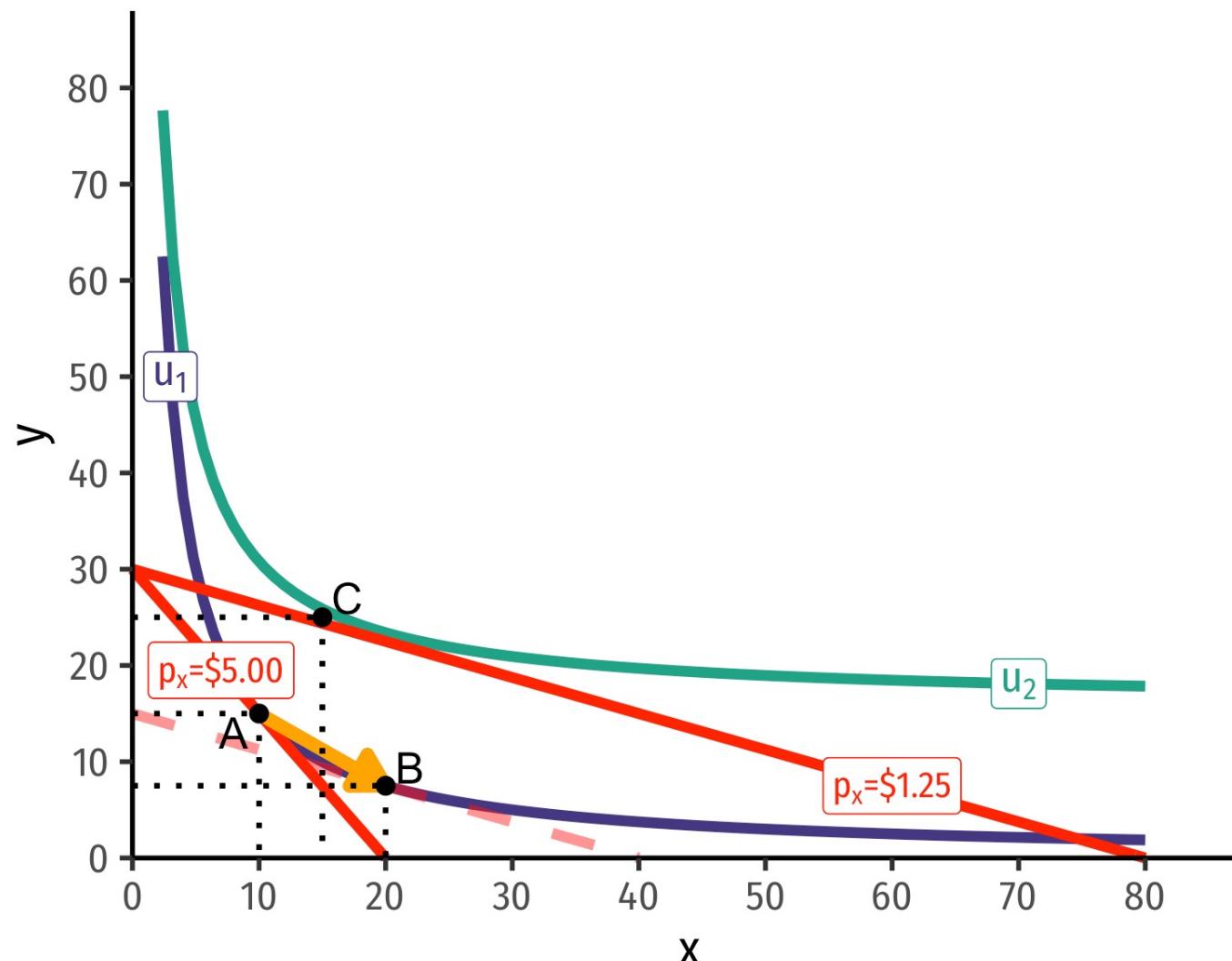
- Original optimal consumption (A)
- Price of x falls, new optimal consumption at (C)
- Substitution effect:** A→B on same I.C. (\uparrow cheaper x and \downarrow y)
- (Real) income effect:** B→C to new budget constraint (can buy more of x and/or y)
- (Total) price effect:** A→C



Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 100$, $p_y = 3.33$



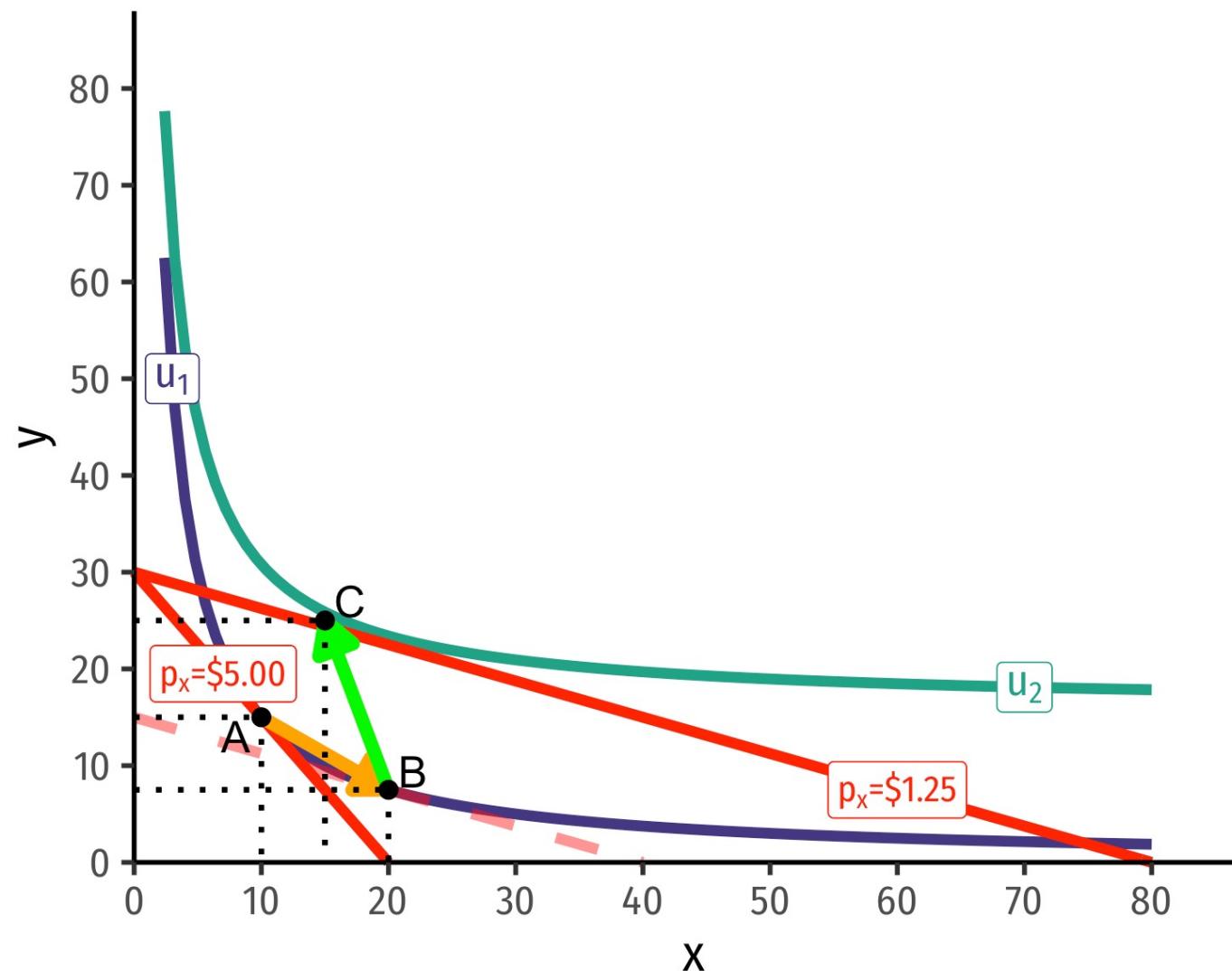
Substitution Effects, Graphically



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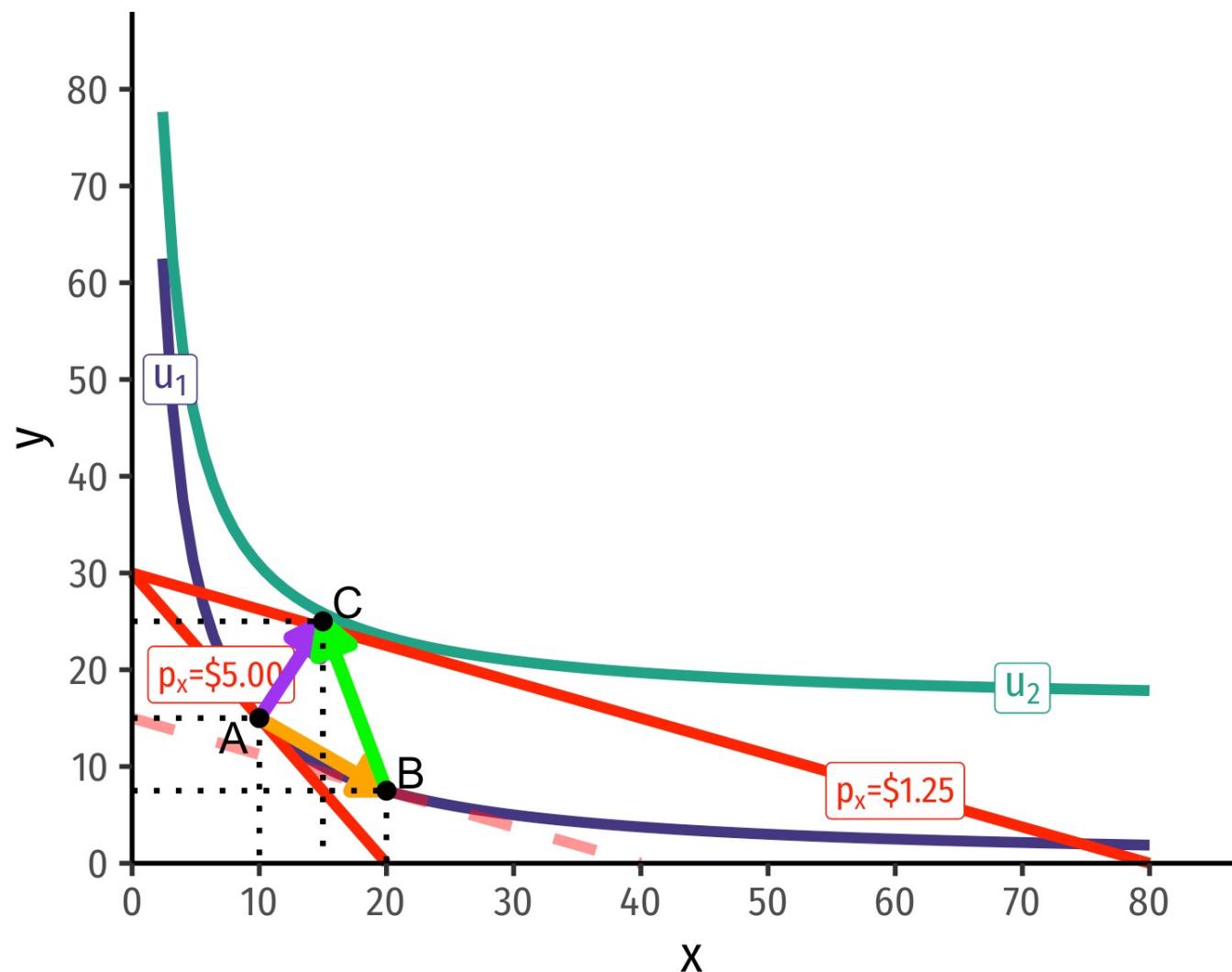
Real Income, Graphically



Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 100$, $p_y = 3.33$



Real Income and Substitution Effects, Graphically: Inferior good

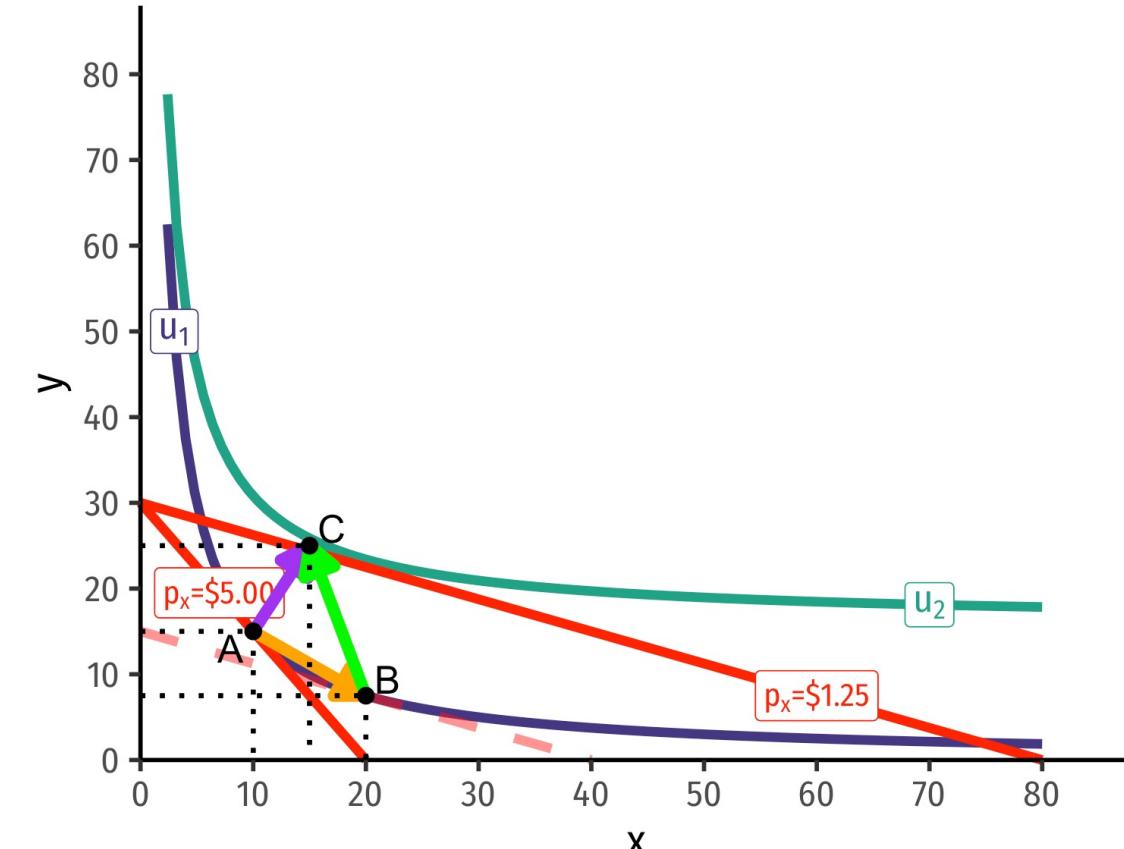


Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 100$, $p_y = 3.33$



Real Income and Substitution Effects, Graphically

- What about an **inferior** good (Ramen)?
- **Substitution effect:** A→B on same I.C. (\uparrow cheaper x and \downarrow y)
- **(Real) income effect:** B→C to new budget constraint (can buy more of x and/or y) **is negative**
- **(Total) price effect:** A→C



Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 100$, $p_y = 3.33$



Recap: Real Income and Substitution Effects

- **Price Effect** = **Real income effect** + **Substitution Effect**
- **Substitution effect**: is always in the direction of the cheaper good
- **Real Income effect**: can be positive (normal) or negative (inferior)
- **Law of Demand**/Demand curves slope downwards (**Price effect**) mostly because of the substitution effect
 - Even (inferior) goods with negative real income effects overpowered by substitution effect
- Exception in the theoretical **Giffen good**: negative R.I.E. > S.E.
 - An upward sloping demand curve!



Optimal Decision to Demand



Demand Schedule

- **Demand schedule** expresses the quantity of good a person would be willing to buy (q_d) at any given price (p_x)
- Note: **each of these is a consumer's optimum at a given price!**

| price | quantity |
|-------|----------|
| 10 | 0 |
| 9 | 1 |
| 8 | 2 |
| 7 | 3 |
| 6 | 4 |
| 5 | 5 |
| 4 | 6 |
| 3 | 7 |
| 8 | 2 |
| 9 | 1 |



Demand Function

- **Demand function** relates quantity to price

$$q_d = 10 - p$$

- Not graphable (wrong axes)!

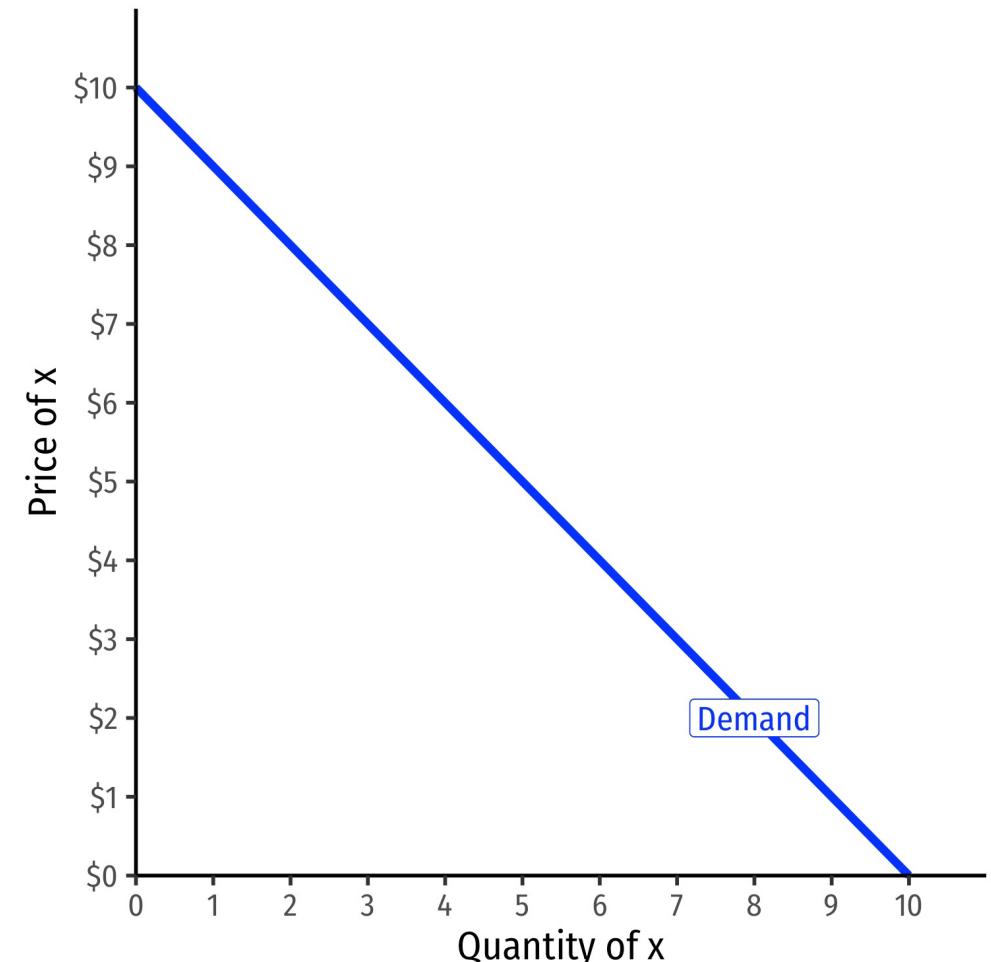


Inverse Demand Function

- **Inverse demand function** relates price to quantity
 - Take demand function and solve for p

$$p = 10 - q_d$$

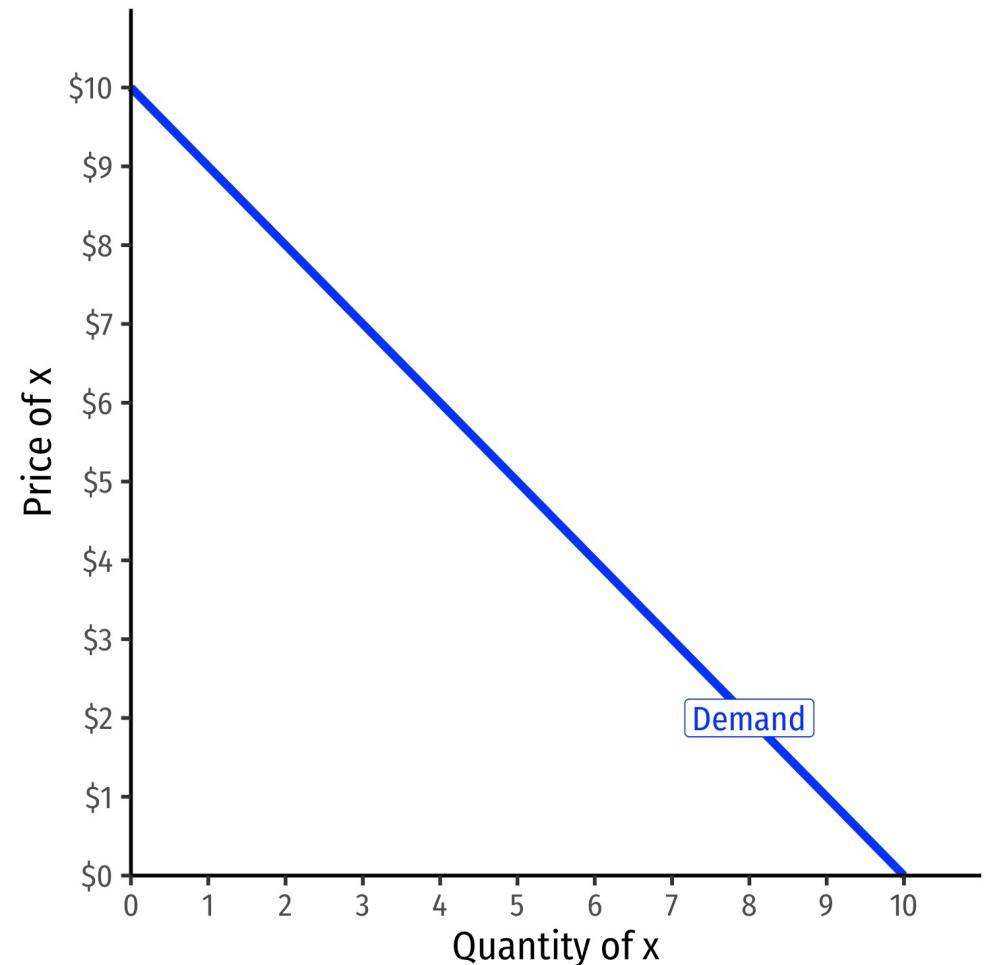
- Graphable (price on vertical axis)!
- Vertical intercept ("Choke price"): just high enough to discourage any purchases





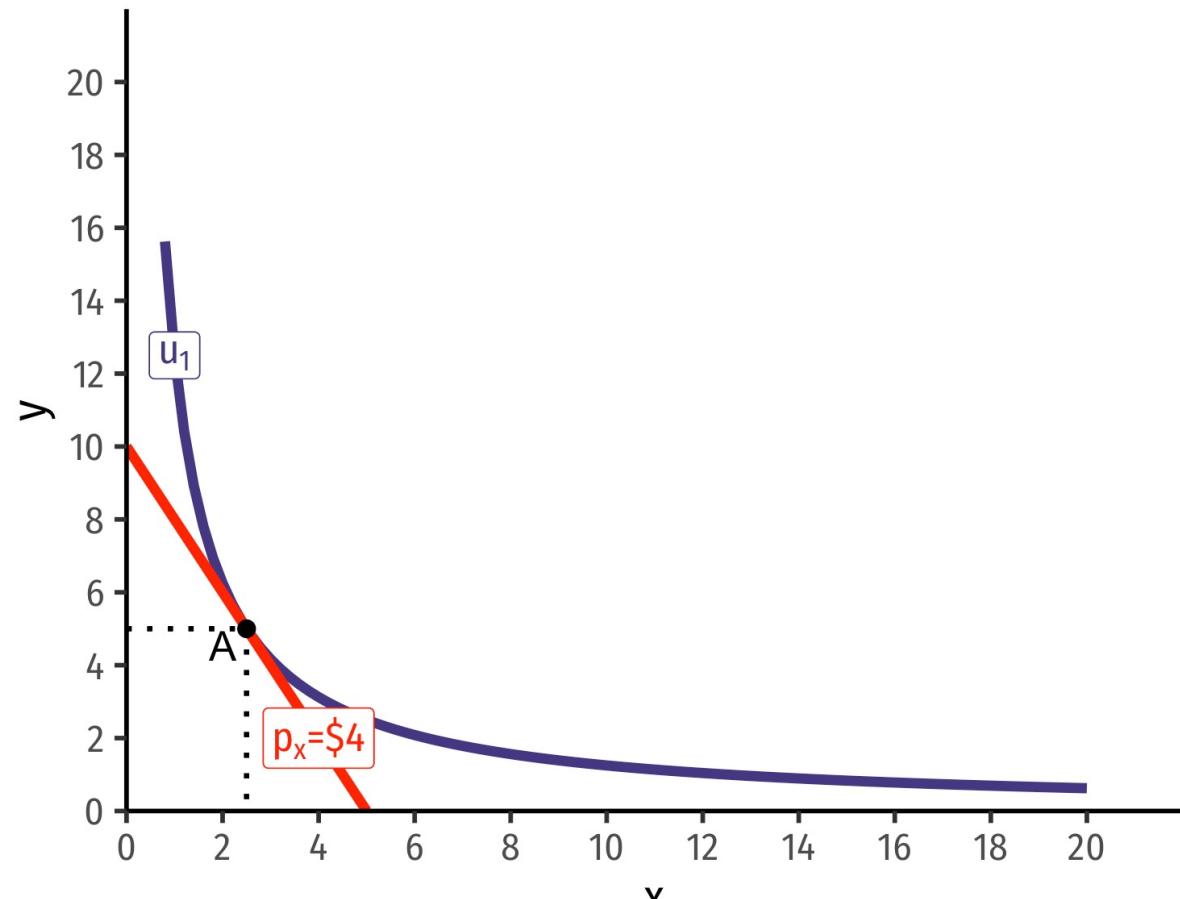
Inverse Demand Function

- Read two ways:
- Horizontally: at any given price, how many units person wants to buy
- Vertically: at any given quantity, the **maximum willingness to pay (WTP)** for that quantity
 - This way will be very useful later

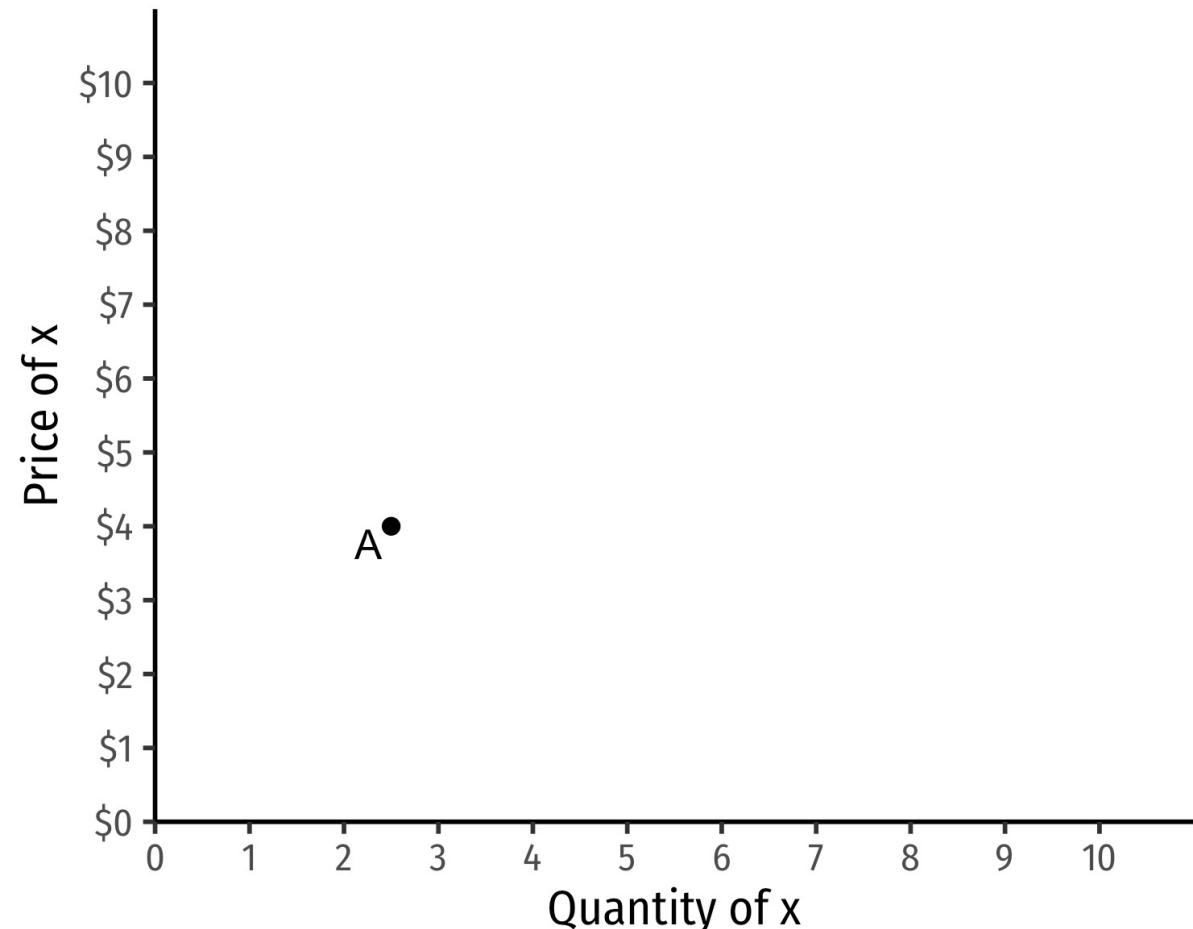




Deriving Demand



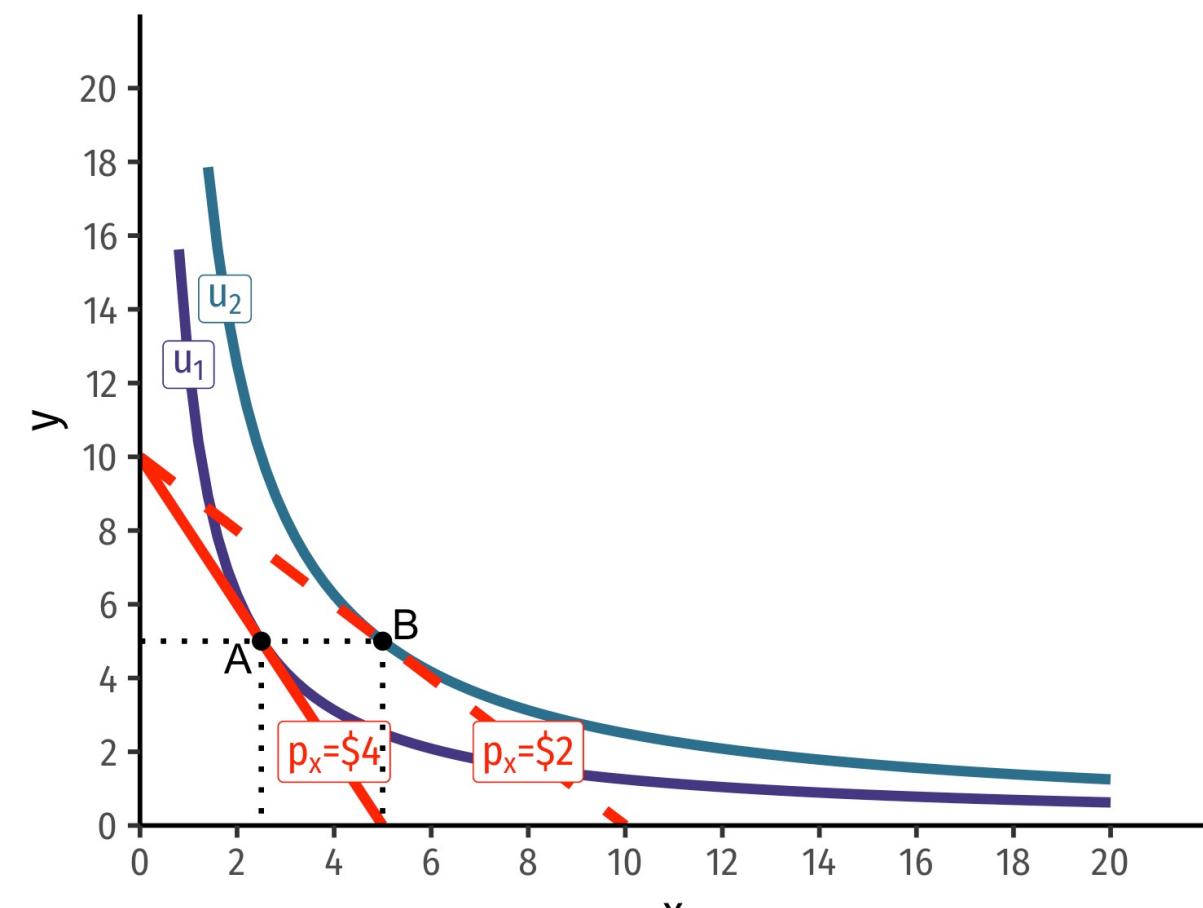
Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 20$, $p_y = 2$



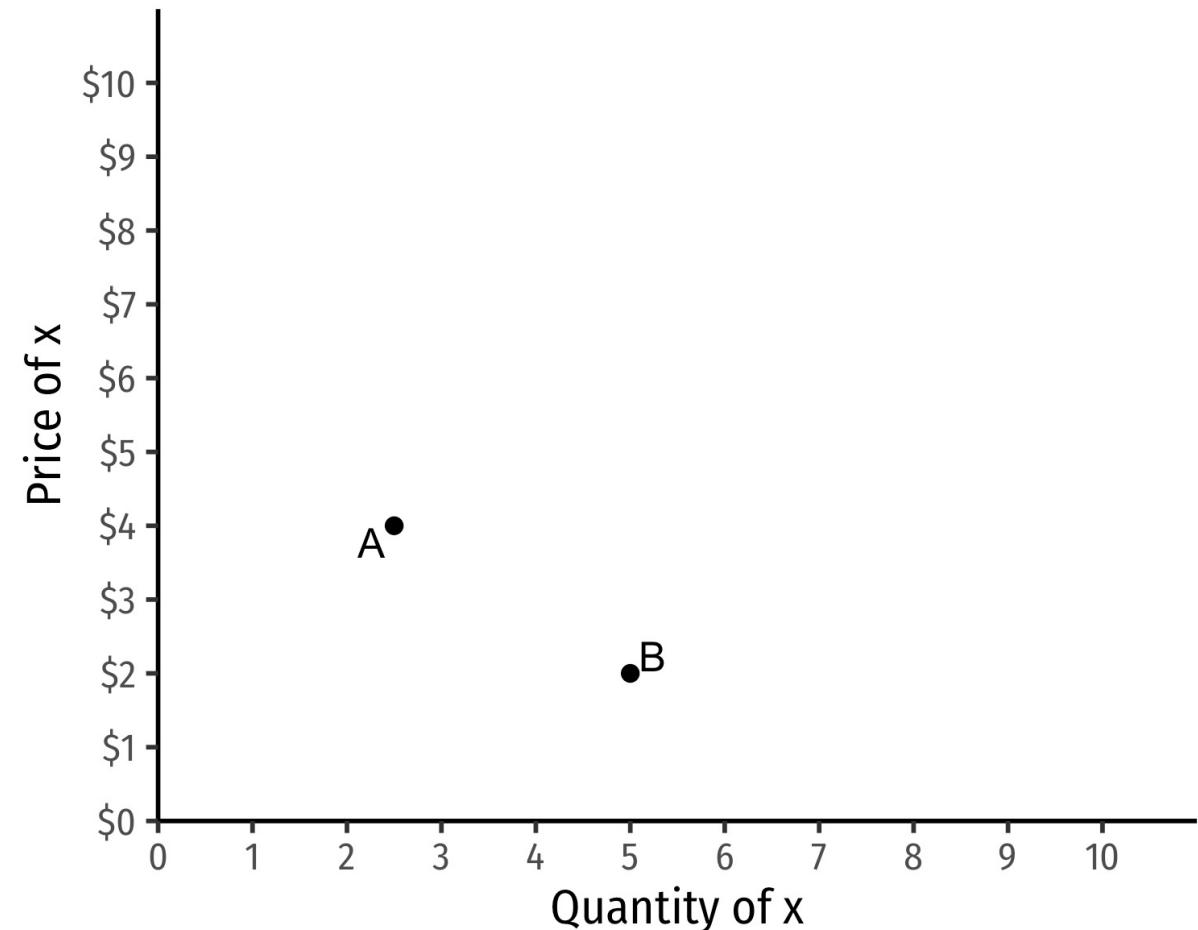
Demand function: $0.5m/p_x$; Inverse Demand function: $p_x = 0.5m/x$



Deriving Demand



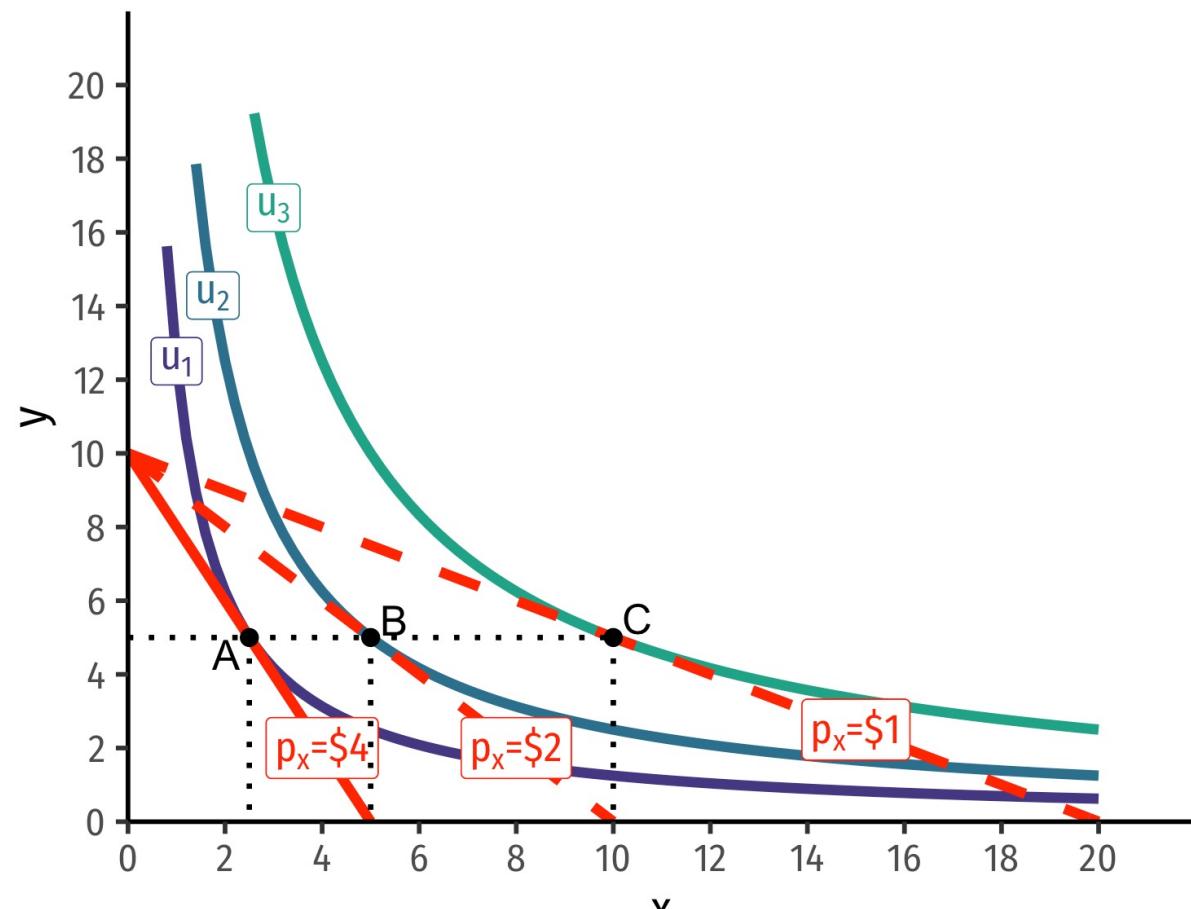
Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 20$, $p_y = 2$



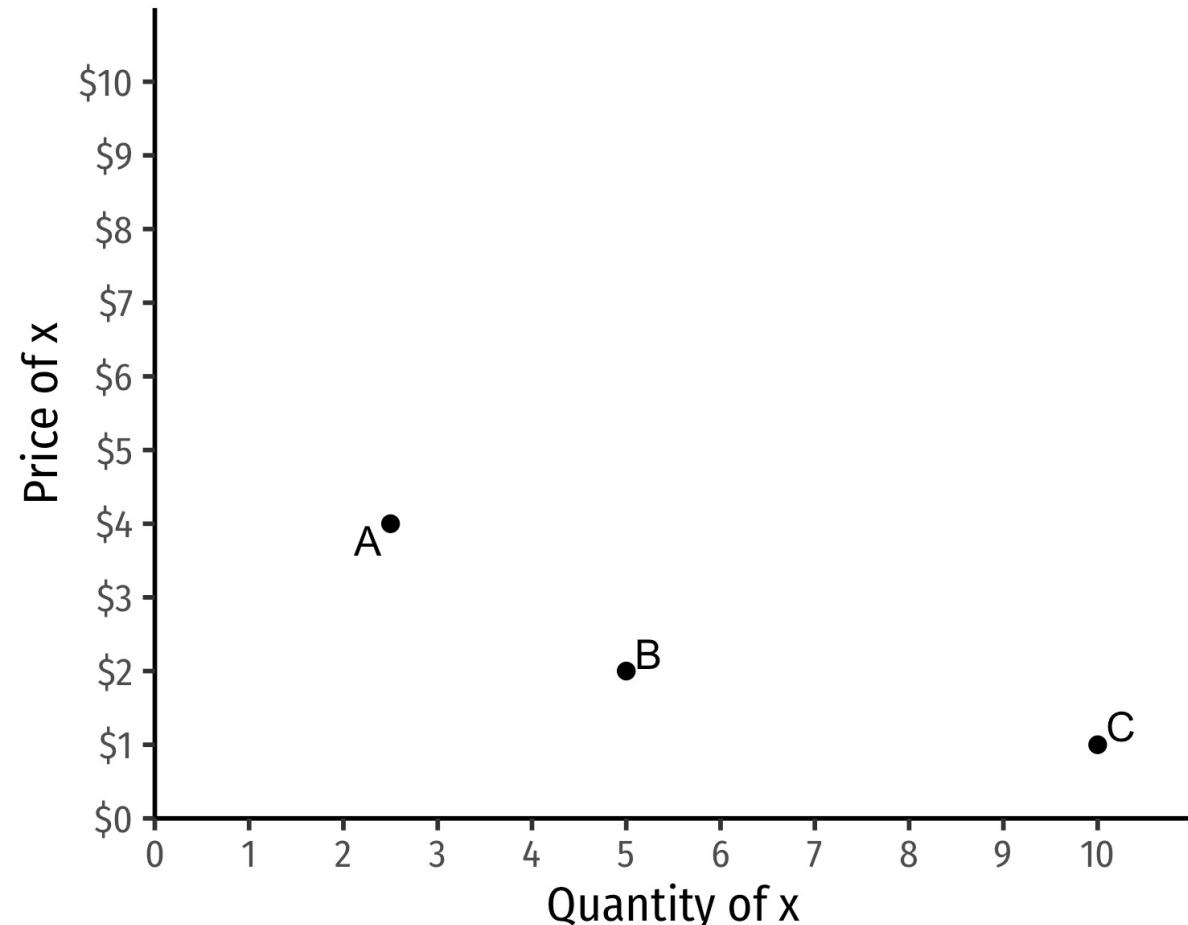
Demand function: $0.5m/p_x$; Inverse Demand function: $p_x = 0.5m/x$



Deriving Demand



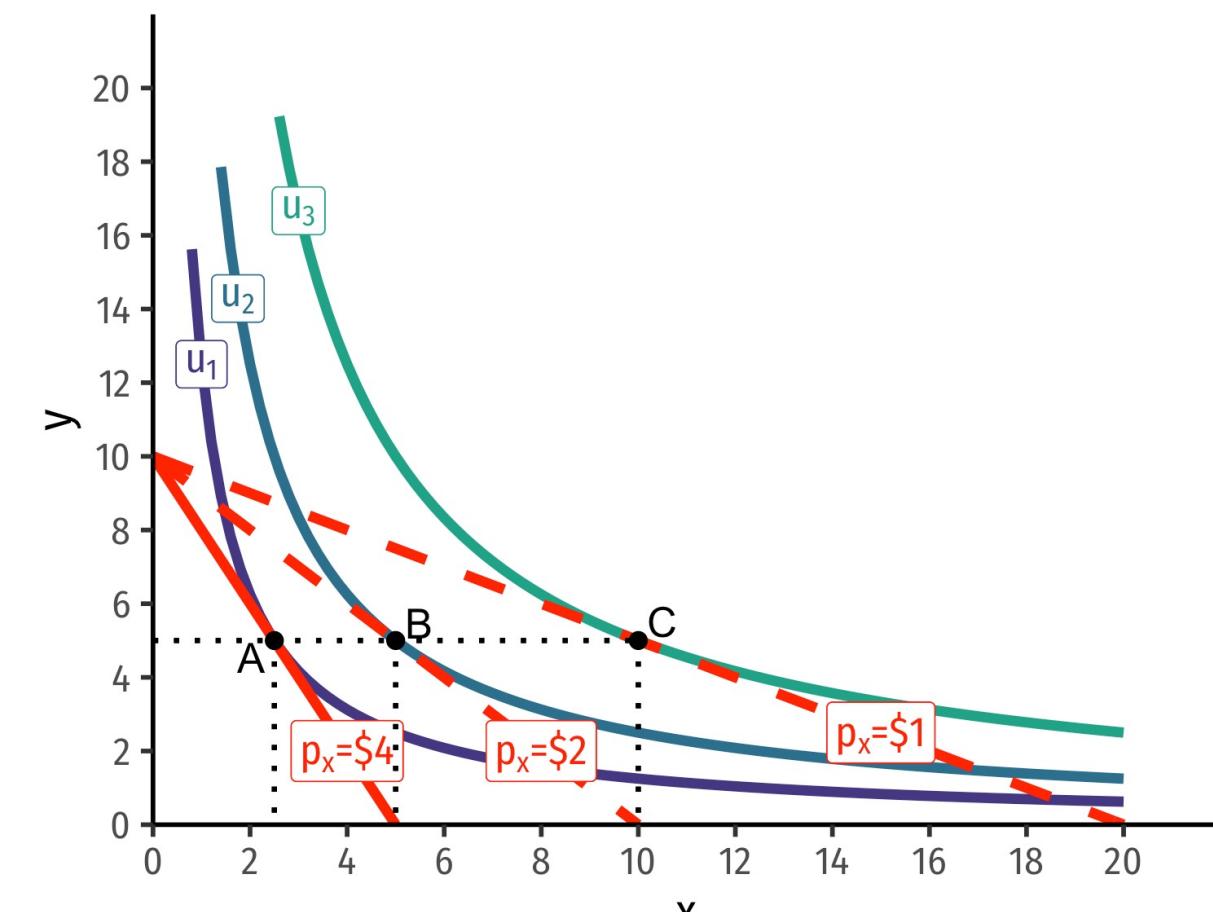
Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 20$, $p_y = 2$



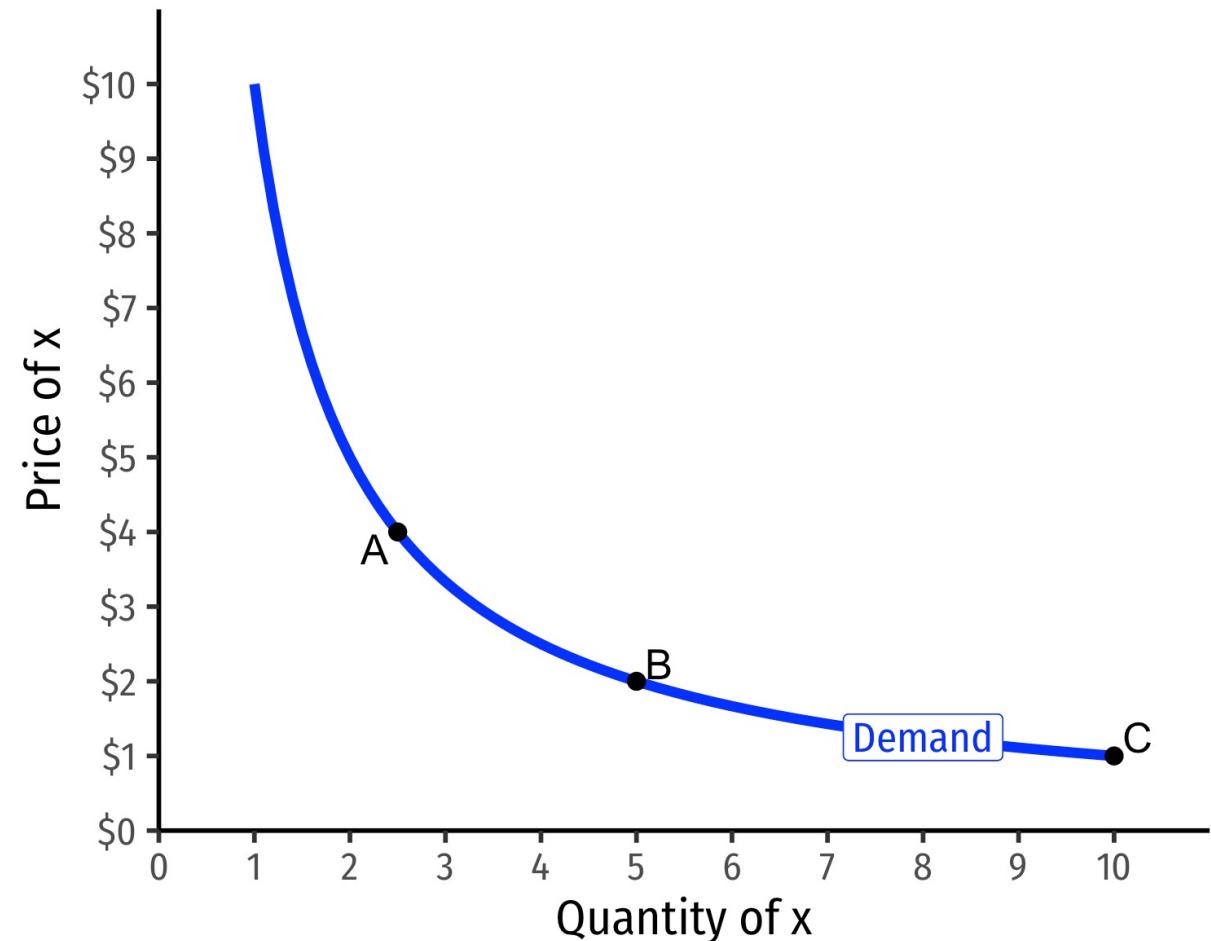
Demand function: $0.5m/p_x$; Inverse Demand function: $p_x = 0.5m/x$



Deriving Demand



Optima with $u(x, y) = x^{0.5}y^{0.5}$, $m = 20$, $p_y = 2$



Demand function: $0.5m/p_x$; Inverse Demand function: $p_x = 0.5m/x$



Example: Labor and Leisure

