

AMA2111 Mathematics I

Homework 1 Solution

1. (2) Find all the roots of the equation $(1+i)^5 z^4 + 2i = 2$ in polar form.

Solution. $z^4 = \frac{2-2i}{(1+i)^5}$. Since

$$2-2i = 2\sqrt{2}e^{-i\frac{\pi}{4}},$$
$$1+i = \sqrt{2}e^{i\frac{\pi}{4}},$$

using de Moivre's formula, we get

$$(1+i)^5 = (\sqrt{2})^5 e^{i\frac{5\pi}{4}} = 4\sqrt{2}e^{i\frac{5\pi}{4}},$$

$$z^4 = \frac{2\sqrt{2}e^{-i\frac{\pi}{4}}}{4\sqrt{2}e^{i\frac{5\pi}{4}}} = \frac{1}{2}e^{-i\frac{6\pi}{4}} = \frac{1}{2}e^{-i\frac{3\pi}{2}} = \frac{1}{2}e^{i(-\frac{3\pi}{2}+2k\pi)} \text{ for any integer } k.$$

The roots are

$$z_k = 2^{-\frac{1}{4}}e^{i(-\frac{3\pi}{8}+\frac{k\pi}{2})}, \quad k = 0, 1, 2, 3,$$

namely,

$$\begin{aligned} z_0 &= 2^{-\frac{1}{4}}e^{-i\frac{3\pi}{8}}, \\ z_1 &= 2^{-\frac{1}{4}}e^{i\frac{\pi}{8}}, \\ z_2 &= 2^{-\frac{1}{4}}e^{i\frac{5\pi}{8}}, \\ z_3 &= 2^{-\frac{1}{4}}e^{i\frac{9\pi}{8}}. \end{aligned}$$

2. (6) Consider the linear system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 2 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 = 3 \\ 3x_1 + 4x_2 + ax_3 + 6x_4 = 5 \\ 4x_1 + 5x_2 + 7x_3 + bx_4 = 4 \end{cases}$$

Find the conditions satisfied by a and b such that the system has

- (a) no solutions;
- (b) infinitely many solutions;
- (c) a unique solution.

Solve the system when it has solutions.

Solution.

Reduce the augmented matrix to the row-echelon form:

$$\begin{array}{c}
 \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 5 & 3 \\ 3 & 4 & a & 6 & 5 \\ 4 & 5 & 7 & b & 4 \end{array} \right] \xrightarrow{R_2 - 2R_1 \rightarrow R_2, R_3 - 3R_1 \rightarrow R_3, R_4 - 4R_1 \rightarrow R_4} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 1 & a-3 & 3 & -1 \\ 0 & 1 & 3 & b-4 & -4 \end{array} \right] \\
 \xrightarrow[R_3 \leftrightarrow R_4]{R_3 - R_2 \rightarrow R_3, R_4 - R_2 \rightarrow R_4} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 1 & b-7 & -3 \\ 0 & 0 & a-5 & 0 & 0 \end{array} \right] \\
 \xrightarrow{R_4 - (a-5)R_3 \rightarrow R_4} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 1 & b-7 & -3 \\ 0 & 0 & 0 & (a-5)(7-b) & 3(a-5) \end{array} \right]
 \end{array}$$

- (i) The system has no solutions if $a \neq 5$ and $b = 7$. In this case, the fourth row becomes inconsistent.
- (ii) The system has infinitely many solutions if $a = 5$. In this case, the fourth row is $0 = 0$, and the system reduces to three equations.

Let $x_4 = t$ (free variable).

Then from the 3rd row: $x_3 + (b-7)t = -3 \implies x_3 = -3 - (b-7)t$.

From the second row: $x_2 + 2x_3 + 3t = -1 \implies x_2 = -1 - 2(-3 - (b-7)t) - 3t = 5 + (2b-17)t$.

From the first row: $x_1 + x_2 + x_3 + t = 2 \implies x_1 = 2 - [5 + (2b-17)t] - [-3 - (b-7)t] - t = (9-b)t$.

The general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} (9-b)t \\ 5 + (2b-17)t \\ -3 - (b-7)t \\ t \end{pmatrix},$$

where $t \in \mathbb{R}$ is arbitrary.

- (iii) The system has a unique solution if $a \neq 5$ and $b \neq 7$.

In this case, from the fourth row: $(7-b)x_4 = 3 \implies x_4 = -\frac{3}{b-7}$.

From the third row: $x_3 = 0$.

From the second row: $x_2 + 3x_4 = -1 \implies x_2 = -1 - 3\left(-\frac{3}{b-7}\right) = -1 + \frac{9}{b-7}$.

From the first row: $x_1 + x_2 + x_4 = 2 \implies x_1 = 2 - \left(-1 + \frac{9}{b-7}\right) - \left(-\frac{3}{b-7}\right) = 3 - \frac{6}{b-7}$.

3. (4) Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{ and } \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ in } \mathbb{R}_3.$$

- (a) Determine if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and \mathbf{v}_4 can span \mathbb{R}_3 or not. In other words, can any vector in \mathbb{R}_3 be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and \mathbf{v}_4 ? Explain the reason.
- (b) Find an example that one vector in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ can be expressed as a linear combination of other vectors.

Solution. Consider the following matrix

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4] = \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & 3 & 2 & 0 \end{array} \right] \xrightarrow{R_2-R_1 \rightarrow R_2, R_3+R_1 \rightarrow R_3} \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & -2 & 1 & -2 \\ 0 & 5 & 2 & 1 \end{array} \right]$$

$$\xrightarrow{-\frac{1}{2}R_2 \rightarrow R_2} \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 5 & 2 & 1 \end{array} \right] \xrightarrow{R_3-5R_2 \rightarrow R_3} \left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & \frac{9}{2} & -4 \end{array} \right]$$

therefore, $\text{Rank}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = 3$ and any three-dimensional column vector can be expressed as linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

b) An example: Consider the following system of linear equations $\mathbf{v}_1 = k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4$, that is

$$\begin{cases} 2k_2 + k_4 = 1 \\ k_3 - k_4 = 1 \\ 3k_2 + 2k_3 = -1 \end{cases}$$

then, we can get $k_2 = 5, k_3 = -8, k_4 = -9$, that is $\mathbf{v}_1 = 5\mathbf{v}_2 - 8\mathbf{v}_3 - 9\mathbf{v}_4$.

All possible solutions:

$$\mathbf{v}_1 = 5\mathbf{v}_2 - 8\mathbf{v}_3 - 9\mathbf{v}_4,$$

$$\mathbf{v}_2 = \frac{1}{5}\mathbf{v}_1 + \frac{8}{5}\mathbf{v}_3 + \frac{9}{5}\mathbf{v}_4,$$

$$\mathbf{v}_3 = -\frac{1}{8}\mathbf{v}_1 + \frac{5}{8}\mathbf{v}_2 - \frac{9}{8}\mathbf{v}_4,$$

$$\mathbf{v}_4 = -\frac{1}{9}\mathbf{v}_1 + \frac{5}{9}\mathbf{v}_2 - \frac{8}{9}\mathbf{v}_3.$$

4. (4) Let $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. Find a nonsingular matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Furthermore, evaluate A^7 .

Solution. We first find the eigenvalues and eigenvectors by solving the following equations:

$$\det(A - \lambda I) = 0$$

Then we have

$$(\lambda - 3)^2(\lambda - 6) = 0 \text{ and therefore, } \lambda_1 = \lambda_2 = 3 \text{ and } \lambda_3 = 6.$$

Find \mathbf{v} such that $(A - \lambda I)\mathbf{v} = 0$. For $\lambda_1 = \lambda_2 = 3$, we have $x_1 + x_2 + x_3 = 0$, we may choose two eigenvectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda_3 = 6$, we have $(A - 6I)\mathbf{v} = 0$. We may find one eigenvector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Therefore,

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \text{ and we have } A = PDP^{-1} = P \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} P^{-1}.$$

Furthermore,

$$A^7 = PD^7P^{-1} = P \begin{bmatrix} 3^7 & 0 & 0 \\ 0 & 3^7 & 0 \\ 0 & 0 & 6^7 \end{bmatrix} P^{-1} = \begin{bmatrix} \frac{1}{3}(6^7 + 2 \cdot 3^7) & \frac{1}{3}(6^7 - 3^7) & \frac{1}{3}(6^7 - 3^7) \\ \frac{1}{3}(6^7 - 3^7) & \frac{1}{3}(6^7 + 2 \cdot 3^7) & \frac{1}{3}(6^7 - 3^7) \\ \frac{1}{3}(6^7 - 3^7) & \frac{1}{3}(6^7 - 3^7) & \frac{1}{3}(6^7 + 2 \cdot 3^7) \end{bmatrix}$$

Note that

$$P^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

5. (6) Let $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

- (a) Find the eigenvalues and eigenvectors of A .
- (b) Find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^T$.
- (c) Use part (b) to diagonalize A^{-1} . Hence, find $(A - 9A^{-1})^3$.

Solution. (a) We can calculate the eigenvalues and eigenvectors of this matrix A :

$$\lambda_1 = \lambda_2 = 3, \mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \lambda_3 = 5, \mathbf{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

(b)

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \text{ and } Q^{-1} = Q^T, \text{ and } D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

In this case,

$$A = QDQ^T$$

(c)

$$A^{-1} = QD^{-1}Q^T,$$

$$(A - 9A^{-1})^3 = Q \left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} - 9 \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \right)^3 Q^T = Q \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 16/5 \end{bmatrix} \right)^3 Q^T$$

Therefore,

$$\begin{aligned} (A - 9A^{-1})^3 &= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4096/125 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2048/125 & 2048/125 & 0 \\ 2048/125 & 2048/125 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

6. (2) Find the explicit function $y(x)$ of the initial value problem

$$y' + xy = y(\cos x + 3), \quad y(0) = 1.$$

Solution. We can see that

$$\int \frac{dy}{y} = \int (\cos x + 3 - x) dx \Rightarrow \ln |y| = \sin(x) + 3x - \frac{1}{2}x^2 + C$$

$$\Rightarrow y = \pm e^C e^{\sin(x) + 3x - \frac{1}{2}x^2} \xrightarrow{\text{re-define } C} y = C e^{\sin(x) + 3x - \frac{1}{2}x^2}, C \neq 0$$

Since $y(0) = 1$, we have $C = 1$ and

$$y = e^{\sin(x) + 3x - \frac{1}{2}x^2}$$

7. (2) Find the general solution of the differential equation $\frac{dy}{dx} + 3y = e^{-x}$.

Solution. This is a 1st order linear ODE with $p(x) = 3$ and $q(x) = e^{-x}$, with integrating factor

$$\mu(x) = e^{\int p(x)dx} = e^{\int 3 dx} = e^{3x}.$$

Multiplying the ODE by $\mu(x)$, we have

$$\frac{d}{dx}(e^{3x}y) = e^{2x} \Rightarrow e^{3x}y = \frac{1}{2}e^{2x} + C$$

So the general solution is $y = \frac{1}{2}e^{-x} + Ce^{-3x}$, $C \in \mathbb{R}$.

8. (2) Solve the initial value problem $y' + 3xy = 3xy^3$, $y(0) = \frac{1}{2}$.

Solution. This is a Bernoulli equation with $n = 3$. If $y \neq 0$, divide the equation by y^3 to get

$$y^{-3}y' + 3xy^{-2} = 3x.$$

Let $v = y^{-2}$, then $v' = -2y^{-3}y'$, and thus

$$v' - 6xv = -6x.$$

Let

$$\mu(x) = e^{\int(-6x)dx} = e^{-3x^2},$$

then

$$\int(-6x)\mu(x)dx = \int(-6x)e^{-3x^2}dx = e^{-3x^2},$$

and thus,

$$v(x) = \frac{e^{-3x^2} + C}{e^{-3x^2}} = 1 + Ce^{3x^2}.$$

Using the initial condition, we have $v(0) = [y(0)]^{-2} = 4$, then $C = 3$. Therefore, $v(x) = 1 + 3e^{3x^2}$, and thus,

$$y(x) = \frac{1}{\sqrt{1 + 3e^{3x^2}}}.$$