AMA1500 - Foundation Mathematics for Accounting and Finance

Assignment 3

Due: 5pm on 21 April 2025, online submission via Blackboard

Questions

1. (48 marks, 4 marks each)) Evaluate the following integrals.

   (a) $\int x^5\sqrt{x^2-1}dx$

   (b) $\int_0^{\pi^2} x\cos(x^2)dx$

   (c) $\int \frac{1}{4x-x^2}dx$

   (d) $\int x^2e^{4x}dx$

   (e) $\int e^x\ln(1+e^x)dx$

   (f) $\int \cos(\sqrt{x})dx$

   (g) $\int \frac{\ln^3 x}{x}dx$

   (h) $\int \sin^2 x\cos^4 xdx$

   (i) $\int \sin 5x\cos 3xdx$

   (j) $\int \frac{1}{x^2-8x+15}dx$

   (k) $\int \frac{x^2+2x+7}{x^3+x^2-2}dx$

   (l) $\int_{-6}^{-3} \frac{\sqrt{x^2-9}}{x}dx$

2. (24 marks, 4 marks each) Evaluate the following integrals

   (a) $\int \frac{\cos(2x)}{\sin^2(2x)+4}\,dx\,.$

   (b) $\int x^2\,\cos(2x+3)\,dx\,.$

   (c) $\int \frac{5x^2-2x+42}{x^3-27}\,dx\,.$

   (d) $\int x^3\,(\ln(x))^2\,dx\,.$

   (e) $\int_0^{\frac{\pi}{4}} \sin^2(3x)\,dx\,.$

   (f) $\int_0^2 (2x^2-5)^6\,x\,dx\,.$

3. (12 marks, 4 marks each) Determine whether the following improper integrals are convergent or not.

   (a) $\int_0^\infty \frac{1}{\sqrt[3]{2x+7}}\,dx$ (b) $\int_0^\infty x\,\mathrm{e}^{1-3x}\,dx$ (c) $\int_{-\infty}^0 \frac{dx}{(2-3x)^3}$

4. (5 marks) Find the area of the region enclosed by the curve $y=(x+1)(x+2)(x-3)$ and the straight line $y-6x-6=0$.

5. (5 marks) Find the arc length along the curve $y=x^2-\frac{1}{8}\ln x$ from $(1,1)$ to $(3,f(3))$.

6. (6 marks) Let R be the region bounded between the x-axis and the curve $y=x^2-4x$. Find the volume generated by rotating R about the y-axis.

Question 1:

(a) $\int x^5 \sqrt{x^2-1}\, dx$

$t = x^2 - 1 \quad \frac{dt}{dx} = 2x$

$= \frac{1}{2}\int x^4 \sqrt{t}\, dt$

$= \frac{1}{2}\int (t+1)^2 \sqrt{t}\, dt$

$= \frac{1}{7}t^{\frac{7}{2}} + \frac{2}{5}t^{\frac{5}{2}} + \frac{1}{3}t^{\frac{3}{2}} + c$

$= \frac{1}{7}(x^2-1)^{\frac{7}{2}} + \frac{2}{5}(x^2-1)^{\frac{5}{2}} + \frac{1}{3}(x^2-1)^{\frac{3}{2}} + c$

(b) $t = x^2 \quad \frac{dx}{dt} = 2x$

$\therefore \int_0^{\pi^2} x\cos x^2\, dx$

$= \frac{1}{2}\int_0^{\pi^4} \cos t\, dt = \frac{1}{2}\sin t \Big|_0^{\pi^4}$

$= \frac{1}{2}\sin \pi^4$

(c) $\int \frac{1}{4x-x^2} dx = \int \frac{1}{4-(x-2)^2} dx$

Assume $x-2 = t \quad dx = dt$

$\int \frac{1}{4-t^2} dt = \frac{1}{4}\int \left(\frac{1}{2-t} - \frac{1}{2+t}\right) dt$

$= \frac{1}{4}\ln\left|\frac{t-2}{t+2}\right| + C = \frac{1}{4}\ln\left|\frac{x-4}{x}\right| + c$

(d) $\int x^2 e^{4x} dx$

$= \frac{1}{4}e^{4x}x^2 - \frac{1}{2}\int x e^{4x} dx$

$= \frac{1}{4}xe^{4x} - \frac{1}{8}e^{4x}x + \frac{1}{32}e^{4x} + c$

(e) $\int e^x \ln(1+e^x) dx$

$= e^x\ln(1+e^x) - \int \frac{e^x}{1+e^x} e^x dx$

Assume $t = e^x \quad \frac{dt}{dx} = e^x$

$= \ln(1+e^x)e^x - e^x + \ln(e^x+1) + c$

(f) $\int \cos\sqrt{x}\, dx = \int \cos\sqrt{x} \cdot 1\, dx$

$= x\cos\sqrt{x} + \int x \sin\sqrt{x} \frac{1}{2\sqrt{x}} dx$

$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + c$

$\frac{1}{2}\int \sin\sqrt{x} \cdot \sqrt{x}\, dx \quad$ assume $\sqrt{x} = u \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$= \int \sin\sqrt{x} \cdot x\, du = \int \sin u\, u^2\, du = -\cos u\, u^2 + 2\int \cos u\, u\, du$

$= -\cos u\, u^2 + 2\sin u\, u - 2\int \sin u$

$= -\cos\sqrt{x} \cdot x + 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x}$

(g) $\int \frac{\ln^3 x}{x} dx \quad$ Assume $t = \ln x \quad \frac{dt}{dx} = \frac{1}{x}$

$= \int t^3 dt$

$= \frac{1}{4}t^4 + c = \frac{1}{4}\ln^4 x + c$

(h) $\int \sin^2 x \cos^4 x\, dx$

$\cos^2 x - \sin^2 x = \cos 2x$

$\sin x\cos x = \frac{\sin 2x}{2} \quad 2\cos^2 x - 1 = \cos 2x$

$\int \cos^2 x \frac{\sin^2 2x}{4} dx \quad \cos^2 x = \frac{\cos 2x + 1}{2}$

$= \frac{1}{4}\int \sin^2 2x \cdot \cos^2 x\, dx$

$= \frac{1}{8}\int \sin^2 2x(1+\cos 2x)\, dx$

$= \frac{1}{8}\int \sin^2 2x\, dx + \frac{1}{8}\int \cos 2x \sin^2 2x\, dx$

$1 - 2\sin^2 2x = \cos 4x$

$\frac{1-\cos 4x}{2} = \sin^2 2x$

$\frac{1}{8}\int \frac{1-\cos 4x}{2} dx + \frac{1}{8}\int \cos 2x \sin^2 2x\, dx$

$u = \sin 2x \quad \frac{du}{dx} = 2\cos 2x$

$\frac{1}{16}\int u^2 du = \frac{1}{16}\cdot\frac{1}{3}u^3$

$= \frac{1}{16}\int 1\, dx - \frac{1}{16}\int \cos 4x + \frac{1}{48}\sin^3 2x + c$

$= \frac{x}{16} - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + c$

(i) $\int \sin 5x \cos 3x\, dx$

$= \frac{1}{2}\int [\sin 8x + \sin 2x]\, dx$

$= \frac{1}{2}\int \sin 8x\, dx + \frac{1}{2}\int \sin 2x\, dx$

$= -\frac{1}{16}\cos 8x - \frac{1}{4}\cos 2x + c$

(j) $\int \frac{1}{x^2-8x+15} dx$

$= \int \frac{1}{(x-4)^2-1} dx \quad$ Assume $t = x-4$

$= \int \frac{1}{t^2-1} dt = \int \frac{1}{(t+1)(t-1)} dt$

$= \frac{1}{2}\int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt$

$= \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + c$

$= \frac{1}{2}\ln\left|\frac{x-5}{x-3}\right| + c$

(k) $\int \frac{x^2+2x+7}{x^3+x^2-2} dx$

$= \int \frac{x^2+2x+7}{(x-1)(x^2+2x+2)}$

$= \int \frac{1}{x-1} dx + 5\int \frac{1}{(x-1)(x^2+2x+2)} dx$

$= \int \frac{1}{x-1} dx + \int \frac{1}{x-1} dx - \int \frac{x+3}{x^2+2x+2} dx$

$= 2\ln|x-1| - \frac{1}{2}\ln|x^2+2x+2| - 2\tan^{-1}(x+1) + C$

(l) $\int \frac{\sqrt{x^2-9}}{x} dx = \int \frac{\tan u}{\sec u} dx$

$x = 3\sec u$

$\frac{dx}{du} = 3\sec u \tan u$

$= 3\int \frac{\tan u}{\sec u} \sec u \tan u\, du$

$= 3\int \tan^2 u\, du$

$= 3\tan u - 3u + c$

$\int \tan^2 u\, du$

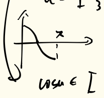
$= \int \frac{1-\cos^2 u}{\cos^2 u} du$

$= \int \sec^2 u - 1\, du$

$= \tan u - u + c$

$\sec u \in [-2, -1]$

$u \in [\frac{2}{3}\pi, \pi]$



$\cos u \in [-1, -\frac{1}{2}]$

$u \in [\frac{2}{3}\pi, \pi]$

$\therefore \int_{-6}^{-3} \frac{\sqrt{x^2-9}}{x} dx$

$= (3\tan u - 3u + c)\Big|_{\frac{2}{3}\pi}^{\pi}$

$= 3[(0-\pi) - (-\sqrt{3} - \frac{2}{3}\pi)]$

$= 3(\sqrt{3} - \frac{\pi}{3}) = 3\sqrt{3} - \pi$

Question 2

(a) $\int \frac{\cos 2x}{\sin^2 2x + 4} dx$

assume $t = \sin 2x \quad \frac{dt}{dx} = 2\cos 2x$

$= \frac{1}{2}\int \frac{dt}{t^2+4}$

$= \frac{1}{4}\tan^{-1}\frac{t}{2} + C$

$= \frac{1}{4}\tan^{-1}(\sin x \cos x) + C$

(b) $\int x^2 \cos(2x+3)\, dx$

assume $2x+3 = t$

$\frac{1}{8}\int (t-3)^2 \cos t\, dx$

$= \frac{1}{8}\sin t\,(t-3)^2 - \frac{1}{4}\sin t + \frac{t-3}{4}\cos t + C$

$= \frac{1}{2}x^2\sin(2x+3) - \frac{1}{4}\sin(2x+3) + \frac{1}{2}\cos(2x+3)x + C$

(c) $\int \frac{5x^2-2x+42}{x^3-27} dx$

$= \int \frac{5x^2-2x+42}{(x-3)(x^2+3x+9)} dx$

$= 3\int \frac{1}{x-3} dx + \int \frac{2x-5}{x^2+3x+9} dx$

assume $x^2+3x+9 = t \quad \frac{dt}{dx} = 2x+3$

$\int \frac{2x+3-8}{x^2+3x+9} dx$

$= \int \frac{2x+3}{x^2+3x+9} dx - 8\int \frac{1}{x^2+3x+9} dx$

$= \ln|t| - 8\int \frac{1}{(x+\frac{3}{2})^2 + \frac{27}{4}} dx \quad \frac{2}{3\sqrt{3}} \cdot 8\tan^{-1}\left(\frac{2}{3\sqrt{3}}(x+\frac{3}{2})\right)$

So final answer.

$\int \frac{5x^2-2x+42}{x^3-27} dx$

$= 3\ln|x-3| + \ln|x^2+3x+9| - \frac{16\sqrt{3}}{9}\tan^{-1}\left(\frac{2}{9}\sqrt{3}x + \frac{\sqrt{3}}{3}\right) + c$

(d) $\int x^3 (\ln x)^2 dx$

$= \frac{1}{4}x^4\ln^2 x - \frac{1}{4}\int x^4 \frac{2\ln x}{x} dx$

$\frac{1}{2}\int x^3 \ln x\, dx$

$= \frac{1}{8}x^4\ln x - \frac{1}{32}x^4$

$= \frac{1}{4}x^4\ln^2 x - \frac{1}{8}\ln x\, x^4 + \frac{1}{32}x^4 + c$

(e) $\int_0^{\frac{\pi}{4}} \sin^2 3x\, dx$

$\int \sin^2 3x\, dx$

$= \frac{1}{2}\int 1 - \cos 6x\, dx$

$= \frac{1}{12}\int (1-\cos t)\, dt$

$= \frac{1}{12}t - \frac{1}{12}\sin t$

$= \frac{1}{2}x - \frac{1}{12}\sin 6x + c$

$\therefore \int_0^{\frac{\pi}{4}} \sin^2 3x\, dx$

$= (\frac{1}{2}x - \frac{1}{12}\sin 6x)\Big|_0^{\frac{\pi}{4}}$

$= (\frac{\pi}{8} - \frac{1}{12}) - (0-0)$

$= \frac{\pi}{8} - \frac{1}{12}$

(f) $\int_0^2 (2x^2-5)^6 x\, dx$

$2x^2 - 5 = t \quad \frac{dt}{dx} = 4x$

$t \in [-5, 3]$

$\frac{1}{4}\int_{-5}^{3} t^6 dt = (\frac{1}{28}t^7 + c)\Big|_{-5}^{3}$

$= \frac{80312}{28} = \frac{20078}{7}$

Question 3

(a) $\int_0^{\infty} \frac{1}{\sqrt[3]{2x+7}} dx$

assume $2x+7 = t$

$\frac{1}{2}\int_7^{\infty} t^{-\frac{1}{3}} dt$

$= \frac{3}{4}\lim_{m\to\infty}(t^{\frac{2}{3}})\Big|_7^m$

$= \frac{3}{4}\lim_{m\to\infty}(m^{\frac{2}{3}} - 7^{\frac{2}{3}}) = \infty$

So (a) is divergent.

(b) $\int_0^{\infty} x e^{1-3x} dx$

assume $t = 1-3x$

$-\frac{1}{3}\int_{-\infty}^{1} \frac{1-t}{3} e^t dt$

$= -\frac{1}{9}\int_{-\infty}^{1} e^t(1-t)\, dt$

$= -\frac{1}{9}\left(\int_{-\infty}^{1} e^t dt - \int_{-\infty}^{1} e^t t\, dt\right)$

convergent

Let's consider $\int_{-\infty}^{1} e^t t\, dt$ only.

$\int_{-\infty}^{1} e^t t$

$= \lim_{n\to-\infty}(e^t t - e^t)\Big|_n^1$

$= 0 - (t-1)e^t = 0$

<convergent>

$\therefore \int_0^{\infty} x e^{1-3x} dx$ is convergent.

(c) $\int_{-\infty}^{0} \frac{dx}{(2-3x)^3}$

$= -\int_{-\infty}^{0} \frac{dx}{(3x-2)^3}$

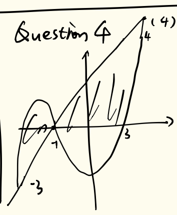
assume $3x-2 = t$

$= -\frac{1}{3}\int_{-\infty}^{-2} t^{-3} dt$

$= \frac{2}{3}t^{-2}\Big|_{-\infty}^{-2}$

So it's convergent.

Question 4



$S = \int_{-3}^{-1}(x^3 - 13x - 12)\, dx + \int_{-1}^{4}(12 + 13x - x^3)\, dx$

$= \left(\frac{1}{4}x^4 - \frac{13}{2}x^2 - 12x\right)\Big|_{-3}^{-1} + \left(12x + \frac{13}{2}x^2 - \frac{1}{4}x^4\right)\Big|_{-1}^{4}$

$= \left[\left(\frac{1}{4} - \frac{13}{2} + 12\right) - \left(\frac{27}{4} - \frac{13}{2}\cdot 9 + 36\right)\right] + \left[\left(48 + \frac{13}{2}\cdot 16 - 64\right) - \left(-12 + \frac{13}{2} - \frac{1}{4}\right)\right]$

$= 101.75$

Question 5

$y' = f'(x) = 2x - \frac{1}{8x}$

Arc Length

$= \int_1^3 \sqrt{1 + (f'(x))^2}\, dx$

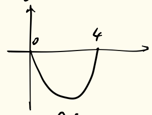
$= \int_1^3 \sqrt{4x^2 + \frac{1}{64x^2} + \frac{1}{2}}\, dx$

$= \int_1^3 (2x + \frac{1}{8x})\, dx$

$= (x^2 + \frac{1}{8}\ln x)\Big|_1^3$

$= 8 + \frac{1}{8}\ln 3$

Question 6



$V = \int_0^4 \pi f^2(x)\, dx$

$= \pi\int_0^4 (x^4 + 16x^2 - 8x^3)\, dx$

$= \pi\left(\frac{1}{5}x^5 + \frac{16}{3}x^3 - 2x^4\right)\Big|_0^4$

$= \frac{512}{15}\pi$