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Subject Code AMA2111

Question 1

$y' = Ay$ and $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

Assume the eigenvalues of $A : \lambda$

$\det(A - I\lambda) = \det\left(\begin{bmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{bmatrix}\right) = 0$

span along 3rd row:

$\det(A) = 2 \cdot \begin{vmatrix} 2 & 2 \\ 4-\lambda & 2 \end{vmatrix} - 2 \begin{vmatrix} 4-\lambda & 2 \\ 2 & 2 \end{vmatrix} + (4-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}$

$= 8(\lambda-2) + (4-\lambda)(6-\lambda)(2-\lambda)$

$= (2-\lambda)(\lambda^2 - 10\lambda + 16) = 0$

$(\lambda-8)(\lambda-2)(2-\lambda) = 0$

$\therefore \lambda_1 = \lambda_2 = 2, \lambda_3 = 8$

Next solve eigen vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ by solving $(A - I\lambda)\vec{v} = \vec{0}$

① case $\lambda = 2$

$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{0}$

$\therefore a + b + c = 0$

$\vec{v_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\vec{v_2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

② case $\lambda = 8$

$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{0}$

$\begin{cases} -2a + b + c = 0 \\ a - 2b + c = 0 \\ a + b - 2c = 0 \end{cases}$ $\therefore \vec{v_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\Rightarrow a = b = c$

$\therefore$ general solution: $y = C_1 e^{\lambda_1 x}\vec{v_1} + C_2 e^{\lambda_2 x}\vec{v_2} + C_3 e^{\lambda_3 x}\vec{v_3}$

$= C_1 e^{2x}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 e^{2x}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_3 e^{8x}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

We have initial values.

$y(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $C_1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 + C_3 \\ -C_1 + C_3 \\ -C_2 + C_3 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

we solve: $\begin{cases} C_1 = 0 \\ C_2 = 1 \\ C_3 = 0 \end{cases}$

$\therefore$ solution.

$y = e^{2x}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{2x} \\ 0 \\ -e^{2x} \end{pmatrix}$

Question 2

$y'' + 2y' + 2y = 6\cos x$ general solution: $y = y_h + y_p$

$y_h = C_1 y_1 + C_2 y_2$ $m^2 + 2m + 2 = 0$. $m = -1 \pm i$. $\therefore y_h = C_1 y_1 + C_2 y_2$, and $y_1 = e^{-x}\cos x$, $y_2 = e^{-x}\sin x$

assume $y_p = A\cos x + B\sin x$ $y_p' = -A\sin x + B\cos x$ $y_p'' = -A\cos x - B\sin x$

Substitude into the equation: $-A\cos x - B\sin x - 2A\sin x + 2B\cos x + 2B\sin x + 2A\cos x = 6\cos x$

$(A + 2B)\cos x + (B - 2A)\sin x = 6\cos x$

$\Rightarrow B = 2A,\ A + 2B = 6$ $5A = 6$ $A = \frac{6}{5}, B = \frac{12}{5}$

$\therefore y_p = \frac{6}{5}\cos x + \frac{12}{5}\sin x$ $\quad y = y_h + y_p = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x + \frac{6}{5}\cos x + \frac{12}{5}\sin x$

Question 3

$y'' - y' - 2y = 3 + 5e^{2x}$, $y = y_p + y_h$

$m^2 - m - 2 = 0$ $(m-2)(m+1) = 0$, $m = 2/-1$ $\therefore y_h = C_1 e^{-x} + C_2 e^{2x}$

Solve y_p:

① Consider specific solution: $y_{p_1} = A$ $-2A = 3$, $A = -\frac{3}{2}$.

② general solution of y_p: $y_p = Axe^{2x}$

$\therefore y_p' = A(e^{2x} + 2xe^{2x})$

$= Ae^{2x} + 2Axe^{2x}$

$y_p'' = 2Ae^{2x} + 2A(e^{2x} + 2xe^{2x})$

$= 4Ae^{2x} + 4Axe^{2x}$

we solve $y_p'' - y_p' - 2y_p = 5e^{2x}$

$4Ae^{2x} + 4Axe^{2x} - Ae^{2x} - 2Axe^{2x} - 2Axe^{2x} = 5e^{2x}$

$3Ae^{2x} = 5e^{2x}$

$\therefore A = \frac{5}{3}$

$y_p = \frac{5}{3}xe^{2x} - \frac{3}{2}$

$\therefore y = y_p + y_h$

$= C_1 e^{-x} + C_2 e^{2x} + \frac{5}{3}xe^{2x} - \frac{3}{2}$

We have initial values.

$y(0) = 0$, $y'(0) = 1$

$y' = -C_1 e^{-x} + 2C_2 e^{2x} + \frac{5}{3}(e^{2x} + 2xe^{2x})$

$\Rightarrow -C_1 + 2C_2 + \frac{5}{3} = 1$

$C_1 + C_2 - \frac{3}{2} = 0$

We solve: $\begin{cases} C_1 = \frac{11}{9} \\ C_2 = \frac{5}{18} \end{cases}$

$\therefore y = \frac{11}{9}e^{-x} + \frac{5}{18}e^{2x} + \frac{5}{3}xe^{2x} - \frac{3}{2}$

Question 4

We use variation of parameters to solve.

$y'' - 2y' + y = \frac{e^x}{x}$ $\quad (m^2 - 2m + 1) = 0 \quad m = 1$

$\therefore y_h = c_1 e^x + c_2 x e^x$ and $y_1 = e^x$, $y_2 = xe^x$

Solve y_p: $y_p = v_1 y_1 + v_2 y_2$ $\quad W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$

$v_1 = -\int \frac{y_2 R(x)}{W(y_1, y_2)} dx = -\int \frac{xe^x e^x}{e^{2x} \cdot x} dx = -\int 1 dx = -x$

$v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx = \int \frac{e^{2x}}{x e^{2x}} dx = \ln|x|$

$\therefore y_p = v_1 y_1 + v_2 y_2$

$= -xe^x + \ln|x| \, xe^x$

$\therefore$ general solution:

$y = y_h + y_p$

$= c_1 e^x + c_2 xe^x - xe^x + xe^x \ln|x|$

Then we generalize c_2

we have $y = c_1 e^x + c_2 xe^x + xe^x \ln|x|$

Question 5

$y'' + y = 2x\sin x = R(x)$ $\qquad \cos^2 x - \sin^2 x$

$m^2 + 1 = 0$, $m = \pm i$ $\quad \therefore y_1 = \cos x \quad y_2 = \sin x$ $\qquad 1 - 2\sin^2 x = \cos 2x$

$y_h = c_1 \cos x + c_2 \sin x$ $\qquad y_p = v_1 y_1 + v_2 y_2$, $R(x) = 2x\sin x$

$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$v_1 = -\int \frac{y_2 R(x)}{W(y_1, y_2)} dx$
$= -\int 2\sin^2 x \cdot x \, dx$
$= -\int (1 - \cos 2x) x \, dx$ $\quad u \; v' \quad v = \frac{1}{2}\sin 2x$
$= -\frac{x^2}{2} + \int x \cos 2x \, dx$
$= -\frac{x^2}{2} + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x$

$v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$
$= \int 2\sin x \, x \cos x \, dx$
$= \int x \sin 2x \, dx$ $\quad u \; v'$
$= -\frac{1}{2} x \cos 2x + \frac{1}{2}\int \cos 2x \, dx$
$= \frac{1}{4}\sin 2x - \frac{x}{2}\cos 2x$

$\therefore$ general solution:

$y = y_h + y_p = c_1 \cos x + c_2 \sin x \underline{- \frac{x^2}{2}\cos x + \frac{x \sin 2x}{2}\cos x + \frac{1}{4}\cos 2x \cos x + \frac{1}{4}\sin 2x \cdot \sin x - \frac{x}{2}\cos 2x \sin x}$

For this part $= -\frac{x^2}{2}\cos x + x \sin x \cos^2 x - \frac{x}{2}(2\cos^2 x - 1)\sin x + \frac{1}{4}(\cos 2x \cos x + \sin 2x \sin x)$

$= -\frac{x^2}{2}\cos x + \frac{x}{2}\sin x + \frac{1}{4}\cos x$

$\therefore y = y_h + y_p = c_1 \cos x + c_2 \sin x - \frac{x^2}{2}\cos x + \frac{x}{2}\sin x + \frac{1}{4}\cos x \xrightarrow{\text{redefine } c_1} y = c_1 \cos x + c_2 \sin x - \frac{x^2}{2}\cos x + \frac{x}{2}\sin x$

we have $y(0) = 0$ $\quad 0 = c_1 \quad \Rightarrow y = c_2 \sin x - \frac{x^2}{2}\cos x + \frac{x}{2}\sin x$

$y' = c_2 \cos x - (x\cos x - \sin x \frac{x^2}{2}) + \frac{1}{2}(\sin x + x\cos x)$

we have $y'(0) = 1 \Rightarrow c_2 = 1$.

$\therefore$ final answer: $y = \sin x - \frac{x^2}{2}\cos x + \frac{x}{2}\sin x$

Question 6

$y''(t) + 0 \cdot y'(t) + 4y(t) = 8\sin 2t = f(t)$

Take Laplace Transform to both sides.

$L\{y''(t)\} + 4L\{y(t)\} = 8L\{\sin 2t\}$

$\Rightarrow s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = 8 \cdot \frac{2}{s^2+4}$

we have $y(0) = 0$, $y'(0) = 1$

$\therefore s^2 Y(s) - 1 + 4Y(s) = \frac{16}{s^2+4}$

$\therefore Y(s) = \frac{1}{s^2+4} + \frac{16}{(s^2+4)^2}$

next, we take inverse transform:

$L^{-1}\{\frac{1}{s^2+4}\} = \frac{1}{2}L^{-1}\{\frac{2}{s^2+2^2}\} = \frac{1}{2}\sin 2t$ ①

$L^{-1}\{\frac{16}{(s^2+4)^2}\} = L^{-1}\{\frac{2 \cdot 2^3}{(s^2+2^2)^2}\} = \sin 2t - 2t\cos 2t$ ②

$\therefore L^{-1}\{Y(s)\} = ① + ② = y(t)$

$\therefore y(t) = \frac{3}{2}\sin 2t - 2t\cos 2t$

Question 7

$f(t) = y'' + 4y = \begin{cases} t & 0 \le t < 2 \\ 4 & t \ge 2 \end{cases}$

$f(t) = f_1(t) + [f_2(t) - f_1(t)] H(t-2)$

$= t + [4 - t] H(t-2)$

$= t - tH(t-2) + 4H(t-2)$

$= t - (t-2)H(t-2) + 2H(t-2)$

$L\{f(t)\}$

$= L\{y''\} + 4L\{y\}$

$= s^2 Y(s) + 4Y(s)$

$= L\{t\} - L\{(t-2)H(t-2)\} + 2L\{H(t-2)\}$

$= \frac{1}{s^2} - \frac{1}{s^2} \cdot e^{-2s} + 2\frac{e^{-2s}}{s} = \frac{1}{s^2} - \frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s}$

$\therefore Y(s) = \left(\frac{1}{s^2} - \frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s}\right) \frac{1}{s^2+4}$

we analyze by parts:

①
$L^{-1}\{\frac{1}{s^2(s^2+4)}\}$
$= \frac{1}{4}L^{-1}\{\frac{1}{s^2} - \frac{1}{s^2+4}\}$
$= \frac{1}{4}(L^{-1}\{\frac{1}{s^2}\} - \frac{1}{2}L^{-1}\{\frac{2}{s^2+2^2}\})$
$= \frac{1}{4}t - \frac{1}{8}\sin 2t$

②
$L^{-1}\{\frac{e^{-2s}}{s^2(s^2+4)}\}$
we have $e^{-2s} \cdot F(s)$.
$L^{-1}\{e^{-2s}F(s)\}$
$= f(t-2)H(t-2)$
By ①, we know that
$L^{-1}\{F(s)\} = f(t)$ $\qquad f(t-2)$
$= \frac{1}{4}t - \frac{1}{8}\sin 2t$ $\quad = \frac{1}{4}t - \frac{1}{2} - \frac{1}{8}\sin(2t-4)$

③
$L^{-1}\{\frac{2e^{-2s}}{s(s^2+4)}\} = 2L^{-1}\{\frac{e^{-2s}}{s(s^2+4)}\} = L^{-1}\{e^{-2s} \cdot F(s)\}$. $F(s) = \frac{1}{s(s^2+4)} = \frac{A}{s} - \frac{B_1 s + B_2}{s^2+4}$

Assume.

$= \frac{As^2 + 4A - B_1 s^2 - B_2 s}{(s^2+4)s}$ $\Rightarrow \begin{cases} 4A = 1 \\ A - B_1 = 0 \\ B_2 = 0 \end{cases}$ $\quad A = \frac{1}{4}$, $B_1 = \frac{1}{4}$, $B_2 = 0$

$\therefore F(s) = \frac{1}{4} \cdot \frac{1}{s} - \frac{1}{4}\frac{s}{s^2+2^2}$ $\quad L^{-1}\{F(s)\} = f(t)$
$= \frac{1}{4} - \frac{1}{4} \cdot \cos 2t$

$\therefore f(t-2)$
$= \frac{1}{4} - \frac{1}{4}\cos(2t-4)$

$\therefore L^{-1}\{\frac{2e^{-2s}}{s(s^2+4)}\} = 2\left(\frac{1}{4} - \frac{1}{4}\cos(2t-4)\right) \cdot H(t-2)$

$$\therefore\ y(t) = \mathcal{L}^{-1}\{Y(s)\}$$
$$= ① - ② + ③$$
$$= \frac{1}{4}t - \frac{1}{8}\sin 2t - \left(\frac{1}{4}t - \frac{1}{2} - \frac{1}{8}\sin(2t-4)\right)H(t-2) + \left(\frac{1}{2} - \frac{1}{2}\cos(2t-4)\right)H(t-2)$$
$$= \frac{1}{4}t - \frac{1}{8}\sin 2t + H(t-2)\left[1 - \frac{1}{4}t - \frac{1}{2}\cos(2t-4) + \frac{1}{8}\sin(2t-4)\right]$$

Question 8

(a) verify the equation holds:

$$3x\, w_x = (2x+3y)\, w_y$$
$$w = f(x^2+3xy)$$
$$= f(g(x))$$
$$w_x = \frac{\partial f}{\partial g}\cdot\frac{\partial g}{\partial x}, \quad w_y = \frac{\partial f}{\partial g}\cdot\frac{\partial g}{\partial y} \qquad g(x,y) = x^2+3xy.$$

we just verify $\frac{\partial g}{\partial x}\cdot 3x = (2x+3y)\frac{\partial g}{\partial y}$ (*)

$$\frac{\partial g}{\partial x} = 2x+3y, \quad \frac{\partial g}{\partial y} = 3x.$$

$\therefore$ (*) holds, the equation also holds.

(b) $f(u) = e^u$

$$w = f(x^2+3xy) = e^{x^2+3xy}.$$
$$w_{xy} = \frac{\partial^2 w}{\partial x\partial y} \qquad w_x = \frac{\partial w}{\partial x} = e^{x^2+3xy}(2x+3y)$$
$$w_{xy} = e^{x^2+3xy}\left(\frac{\partial(2x+3y)}{\partial y}\right) + (2x+3y)\frac{\partial e^{x^2+3xy}}{\partial y}$$
$$= 3e^{x^2+3xy} + (2x+3y)\, e^{x^2+3xy}\cdot 3x$$
$$= e^{x^2+3xy}(6x^2+9xy+3)$$

Question 9

(a) write the above $f(x,y)$

to $g(x,y,w) = \ln w + w - xy^2 = 0$

when $(x=1, y=1)$, $w=1$

$$w_x(x,y) = \frac{\partial w}{\partial x} = -\frac{g_x}{g_w} \qquad w_y(x,y) = \frac{\partial w}{\partial y} = -\frac{g_y}{g_w}$$
$$g_x = \frac{\partial g}{\partial x} = -y^2 \qquad g_y = \frac{\partial g}{\partial y} = -2xy$$
$$g_w = \frac{\partial g}{\partial w} = \frac{1}{w}+1 \qquad w_y(x,y) = -\frac{-2xy}{\frac{1}{w}+1} = \frac{2xyw}{1+w}$$
$$\therefore\ w_x(y,x) = -\frac{-y^2}{\frac{1}{w}+1} = \frac{wy^2}{1+w}$$

$$w_x(x,y) = \frac{wy^2}{1+w} \qquad \text{assume } v = wy^2,\ u = 1+w.$$
$$w_{xy} = \frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left(\frac{v}{u}\right) = \frac{v_y u - u_y v}{u^2}$$
$$v_y = 2wy + y^2\cdot w_y, \quad u_y = w_y = \frac{2xyw}{1+w}$$
$$= 2wy + y^2\cdot\frac{2xyw}{1+w}$$
$$\therefore\ \frac{v_y u - u_y v}{u^2} = \frac{\left(2wy + y^2\cdot\frac{2xyw}{1+w}\right)(1+w) - \frac{2xyw}{1+w}\cdot wy^2}{(1+w)^2}$$
$$= \frac{2wy + \frac{2xy^3w}{1+w} + 2w^2y + \cancel{\frac{2xy^3w^2}{1+w}} - \cancel{\frac{2xy^3w^2}{1+w}}}{(1+w)^2} = \frac{2wy\left(1+\frac{y^2x}{1+w}+w\right)}{(1+w)^2}$$

(b) $w = f(x,y) = f(1.05, 0.95)$

$$f(x,y) \doteq f(1,1) + w_x(1,1)(1.05-1) + w_y(1,1)(0.95-1)$$
$$w_x(x,y) = w_x(1,1) = \frac{1}{2} \qquad w_y(1,1) = 1$$
($w=1$ here)

$$\therefore\ f(1.05, 0.95) \doteq 1 + \frac{1}{2}\times 0.05 - 0.05$$
$$= \underline{0.975}$$