

# AMA2111 Mathematics I

## Homework 1 Solution

1. (2) Find all the roots of the equation  $(1+i)^5 z^4 + 2i = 2$  in polar form.

**Solution.**  $z^4 = \frac{2-2i}{(1+i)^5}$ . Since

$$2-2i = 2\sqrt{2}e^{-i\frac{\pi}{4}},$$

$$1+i = \sqrt{2}e^{i\frac{\pi}{4}},$$

using de Moivre's formula, we get

$$(1+i)^5 = (\sqrt{2})^5 e^{i\frac{5\pi}{4}} = 4\sqrt{2}e^{i\frac{5\pi}{4}},$$

$$z^4 = \frac{2\sqrt{2}e^{-i\frac{\pi}{4}}}{4\sqrt{2}e^{i\frac{5\pi}{4}}} = \frac{1}{2}e^{-i\frac{6\pi}{4}} = \frac{1}{2}e^{-i\frac{3\pi}{2}} = \frac{1}{2}e^{i(-\frac{3\pi}{2}+2k\pi)} \text{ for any integer } k.$$

The roots are

$$z_k = 2^{-\frac{1}{4}}e^{i(-\frac{3\pi}{8}+\frac{k\pi}{2})}, \quad k = 0, 1, 2, 3,$$

namely,

$$z_0 = 2^{-\frac{1}{4}}e^{-i\frac{3\pi}{8}},$$

$$z_1 = 2^{-\frac{1}{4}}e^{i\frac{\pi}{8}},$$

$$z_2 = 2^{-\frac{1}{4}}e^{i\frac{5\pi}{8}},$$

$$z_3 = 2^{-\frac{1}{4}}e^{i\frac{9\pi}{8}}.$$

2. (6) Consider the linear system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 2 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 = 3 \\ 3x_1 + 4x_2 + ax_3 + 6x_4 = 5 \\ 4x_1 + 5x_2 + 7x_3 + bx_4 = 4 \end{cases}$$

Find the conditions satisfied by  $a$  and  $b$  such that the system has

- (a) no solutions;
- (b) infinitely many solutions;
- (c) a unique solution.

Solve the system when it has solutions.

**Solution.**

Reduce the augmented matrix to the row-echelon form:

$$\begin{aligned}
 & \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 5 & 3 \\ 3 & 4 & a & 6 & 5 \\ 4 & 5 & 7 & b & 4 \end{array} \right] \xrightarrow{R_2-2R_1 \rightarrow R_2, R_3-3R_1 \rightarrow R_3, R_4-4R_1 \rightarrow R_4} \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 1 & a-3 & 3 & -1 \\ 0 & 1 & 3 & b-4 & -4 \end{array} \right] \\
 & \xrightarrow{\begin{array}{l} R_3-R_2 \rightarrow R_3, R_4-R_2 \rightarrow R_4 \\ R_3 \leftrightarrow R_4 \end{array}} \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 1 & b-7 & -3 \\ 0 & 0 & a-5 & 0 & 0 \end{array} \right] \\
 & \xrightarrow{R_4-(a-5)R_3 \rightarrow R_4} \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 1 & b-7 & -3 \\ 0 & 0 & 0 & (a-5)(7-b) & 3(a-5) \end{array} \right]
 \end{aligned}$$

- (i) The system has no solutions if  $a \neq 5$  and  $b = 7$ . In this case, the fourth row becomes inconsistent.
- (ii) The system has infinitely many solutions if  $a = 5$ . In this case, the fourth row is  $0 = 0$ , and the system reduces to three equations.

Let  $x_4 = t$  (free variable).

Then from the 3rd row:  $x_3 + (b-7)t = -3 \implies x_3 = -3 - (b-7)t$ .

From the second row:  $x_2 + 2x_3 + 3t = -1 \implies x_2 = -1 - 2(-3 - (b-7)t) - 3t = 5 + (2b-17)t$ .

From the first row:  $x_1 + x_2 + x_3 + t = 2 \implies x_1 = 2 - [5 + (2b-17)t] - [-3 - (b-7)t] - t = (9-b)t$ .

The general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} (9-b)t \\ 5 + (2b-17)t \\ -3 - (b-7)t \\ t \end{pmatrix},$$

where  $t \in \mathbb{R}$  is arbitrary.

- (iii) The system has a unique solution if  $a \neq 5$  and  $b \neq 7$ .

In this case, from the fourth row:  $(7-b)x_4 = 3 \implies x_4 = -\frac{3}{b-7}$ .

From the third row:  $x_3 = 0$ .

From the second row:  $x_2 + 3x_4 = -1 \implies x_2 = -1 - 3\left(-\frac{3}{b-7}\right) = -1 + \frac{9}{b-7}$ .

From the first row:  $x_1 + x_2 + x_4 = 2 \implies x_1 = 2 - \left(-1 + \frac{9}{b-7}\right) - \left(-\frac{3}{b-7}\right) = 3 - \frac{6}{b-7}$ .

3. (4) Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{ and } \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ in } \mathbb{R}_3.$$

- (a) Determine if  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  and  $\mathbf{v}_4$  can span  $\mathbb{R}_3$  or not. In other words, can any vector in  $\mathbb{R}_3$  be expressed as a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  and  $\mathbf{v}_4$ ? Explain the reason.
- (b) Find an example that one vector in  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  can be expressed as a linear combination of other vectors.

**Solution.** Consider the following matrix

$$\begin{aligned} [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4] &= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & 3 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1 \rightarrow R_2, R_3 + R_1 \rightarrow R_3} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & -2 & 1 & -2 \\ 0 & 5 & 2 & 1 \end{bmatrix} \\ &\xrightarrow{-\frac{1}{2}R_2 \rightarrow R_2} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 5 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 - 5R_2 \rightarrow R_3} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & \frac{9}{2} & -4 \end{bmatrix} \end{aligned}$$

therefore,  $\text{Rank}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = 3$  and any three-dimensional column vector can be expressed as linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ .

b) An example: Consider the following system of linear equations  $\mathbf{v}_1 = k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4$ , that is

$$\begin{cases} 2k_2 + k_4 = 1 \\ k_3 - k_4 = 1 \\ 3k_2 + 2k_3 = -1 \end{cases}$$

then, we can get  $k_2 = 5, k_3 = -8, k_4 = -9$ , that is  $\mathbf{v}_1 = 5\mathbf{v}_2 - 8\mathbf{v}_3 - 9\mathbf{v}_4$ .

All possible solutions:

$$\mathbf{v}_1 = 5\mathbf{v}_2 - 8\mathbf{v}_3 - 9\mathbf{v}_4,$$

$$\mathbf{v}_2 = \frac{1}{5}\mathbf{v}_1 + \frac{8}{5}\mathbf{v}_3 + \frac{9}{5}\mathbf{v}_4,$$

$$\mathbf{v}_3 = -\frac{1}{8}\mathbf{v}_1 + \frac{5}{8}\mathbf{v}_2 - \frac{9}{8}\mathbf{v}_4,$$

$$\mathbf{v}_4 = -\frac{1}{9}\mathbf{v}_1 + \frac{5}{9}\mathbf{v}_2 - \frac{8}{9}\mathbf{v}_3.$$

4. (4) Let  $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ . Find a nonsingular matrix  $P$  and a diagonal matrix  $D$

such that  $A = PDP^{-1}$ . Furthermore, evaluate  $A^7$ .

**Solution.** We first find the eigenvalues and eigenvectors by solving the following equations:

$$\det(A - \lambda I) = 0$$

Then we have

$$(\lambda - 3)^2(\lambda - 6) = 0 \text{ and therefore, } \lambda_1 = \lambda_2 = 3 \text{ and } \lambda_3 = 6.$$

Find  $\mathbf{v}$  such that  $(A - \lambda I)\mathbf{v} = 0$ . For  $\lambda_1 = \lambda_2 = 3$ , we have  $x_1 + x_2 + x_3 = 0$ , we may choose two eigenvectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For  $\lambda_3 = 6$ , we have  $(A - 6I)\mathbf{v} = 0$ . We may find one eigenvector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Therefore,

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \text{ and we have } A = PDP^{-1} = P \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} P^{-1}.$$

Furthermore,

$$A^7 = PD^7P^{-1} = P \begin{bmatrix} 3^7 & 0 & 0 \\ 0 & 3^7 & 0 \\ 0 & 0 & 6^7 \end{bmatrix} P^{-1} = \begin{bmatrix} \frac{1}{3}(6^7 + 2 \cdot 3^7) & \frac{1}{3}(6^7 - 3^7) & \frac{1}{3}(6^7 - 3^7) \\ \frac{1}{3}(6^7 - 3^7) & \frac{1}{3}(6^7 + 2 \cdot 3^7) & \frac{1}{3}(6^7 - 3^7) \\ \frac{1}{3}(6^7 - 3^7) & \frac{1}{3}(6^7 - 3^7) & \frac{1}{3}(6^7 + 2 \cdot 3^7) \end{bmatrix}$$

Note that

$$P^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

5. (6) Let  $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ .

- (a) Find the eigenvalues and eigenvectors of  $A$ .
- (b) Find an orthogonal matrix  $Q$  and a diagonal matrix  $D$  such that  $A = QDQ^T$ .
- (c) Use part (b) to diagonalize  $A^{-1}$ . Hence, find  $(A - 9A^{-1})^3$ .

**Solution.** (a) We can calculate the eigenvalues and eigenvectors of this matrix  $A$ :

$$\lambda_1 = \lambda_2 = 3, \mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \lambda_3 = 5, \mathbf{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

(b)

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \text{ and } Q^{-1} = Q^T, \text{ and } D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

In this case,

$$A = QDQ^T$$

(c)

$$A^{-1} = QD^{-1}Q^T,$$

$$(A - 9A^{-1})^3 = Q \left( \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} - 9 \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \right)^3 Q^T = Q \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 16/5 \end{bmatrix} \right)^3 Q^T$$

Therefore,

$$\begin{aligned} (A - 9A^{-1})^3 &= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4096/125 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2048/125 & 2048/125 & 0 \\ 2048/125 & 2048/125 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

6. (2) Find the explicit function  $y(x)$  of the initial value problem

$$y' + xy = y(\cos x + 3), \quad y(0) = 1.$$

**Solution.** We can see that

$$\begin{aligned} \int \frac{dy}{y} &= \int (\cos x + 3 - x) dx \Rightarrow \ln |y| = \sin(x) + 3x - \frac{1}{2}x^2 + C \\ \Rightarrow y &= \pm e^C e^{\sin(x) + 3x - \frac{1}{2}x^2} \xrightarrow{\text{re-define } C} y = C e^{\sin(x) + 3x - \frac{1}{2}x^2}, C \neq 0 \end{aligned}$$

Since  $y(0) = 1$ , we have  $C = 1$  and

$$y = e^{\sin(x) + 3x - \frac{1}{2}x^2}$$

7. (2) Find the general solution of the differential equation  $\frac{dy}{dx} + 3y = e^{-x}$ .

**Solution.** This is a 1st order linear ODE with  $p(x) = 3$  and  $q(x) = e^{-x}$ , with integrating factor

$$\mu(x) = e^{\int p(x) dx} = e^{\int 3 dx} = e^{3x}.$$

Multiplying the ODE by  $\mu(x)$ , we have

$$\frac{d}{dx} (e^{3x} y) = e^{2x} \Rightarrow e^{3x} y = \frac{1}{2} e^{2x} + C$$

So the general solution is  $y = \frac{1}{2} e^{-x} + C e^{-3x}$ ,  $C \in \mathbb{R}$ .

8. (2) Solve the initial value problem  $y' + 3xy = 3xy^3$ ,  $y(0) = \frac{1}{2}$ .

**Solution.** This is a Bernoulli equation with  $n = 3$ . If  $y \neq 0$ , divide the equation by  $y^3$  to get

$$y^{-3}y' + 3xy^{-2} = 3x.$$

Let  $v = y^{-2}$ , then  $v' = -2y^{-3}y'$ , and thus

$$v' - 6xv = -6x.$$

Let

$$\mu(x) = e^{\int (-6x)dx} = e^{-3x^2},$$

then

$$\int (-6x)\mu(x)dx = \int (-6x)e^{-3x^2}dx = e^{-3x^2},$$

and thus,

$$v(x) = \frac{e^{-3x^2} + C}{e^{-3x^2}} = 1 + Ce^{3x^2}.$$

Using the initial condition, we have  $v(0) = [y(0)]^{-2} = 4$ , then  $C = 3$ . Therefore,  $v(x) = 1 + 3e^{3x^2}$ , and thus,

$$y(x) = \frac{1}{\sqrt{1 + 3e^{3x^2}}}.$$