Assignment 1 of AMA1500

Due date: 5pm, 10 March 2025 (Monday)

1. (10 marks) In the following questions, x is a real number. Find the solution sets of the inequalities in the form of unions of intervals.

   (a) $\dfrac{x+3}{x-2} \geq 0.$

   (b) $2x - 1 \leq |x+4|.$

   (c) $|x-2| < |x+1|.$

(a) $\begin{cases} x+3 \geq 0 \\ x-2 > 0 \end{cases}$ or $\begin{cases} x+3 \leq 0 \\ x-2 < 0 \end{cases}$ $\Rightarrow \begin{cases} x \geq -3 \\ x > 2 \end{cases}$ or $\begin{cases} x \leq -3 \\ x < 2 \end{cases}$

$\frac{x+3}{x-2} \geq 0 \Leftrightarrow x \in (2, +\infty) \cup (-\infty, -3]$

(b) ① $x \geq -4$
$2x-1 \leq x+4 \Rightarrow [-4, 5]$
$x \leq 5$
② $x < -4$
$2x-1 \leq -4-x \Rightarrow (-\infty, -4)$
$3x \leq -3$
$x \leq -1$
$\therefore x \in (-\infty, 5]$

(c)
$|x-2| < |x+1|$
$\Leftrightarrow x^2 - 4x + 4 < x^2 + 1 + 2x$
$3 < 6x$
$x > \frac{1}{2}$
$\therefore x \in (\frac{1}{2}, +\infty)$

2. (6 marks) Determine whether the following functions are one-to-one or not. If so, find $f^{-1}(y)$.

   (a) $f(x) = \frac{x-1}{x-2}$, $x \in \mathbb{R} \setminus \{2\}$.

   (b) $y = f(x) = x - 1/x$, $0 < x < \infty$.

2. (a) $y = \frac{x-1}{x-2}$
$y(x-2) = x-1$
$xy - 2y = x - 1$
$1 - 2y = (1-y)x$
$x = \frac{2y-1}{y-1}$
$\therefore f^{-1}(x) = \frac{2y-1}{y-1}$

(b) $y = x - \frac{1}{x} = \frac{x^2-1}{x}$
$xy = x^2 - 1$ $\quad \Delta = y^2 + 4$
$x^2 - y \cdot x - 1 = 0$ $\quad x_1 = \frac{y + \sqrt{y^2+4}}{2}$ $\quad x_2 = \frac{y - \sqrt{y^2+4}}{2} < 0$, expel. $\therefore f^{-1}(x) = \frac{x + \sqrt{x^2+4}}{2}$

3. (12 marks)

   (a) Find the coefficients of the the constant term in the expansions of $(2x - 1/x)^8$. 1120.

   (b) Let n be a positive integer greater than 5. Find the coefficient of x^3y^2 of the trinomials $(1 + 2x + 3y)^n$. $C_n^5 (2x)^3 (3y)^2 (1)^{n-1} = \frac{n(n-1)(n-2)}{6} \times 8 \times 9 = 12n(n-1)(n-2)$

   (c) There are 10 electronics manufacturers, 7 banks and 5 insurance companies listed in the City of Redcliff Stock Exchange. In how many ways can we construct two investment portfolios, one for Alice and one for Bob, each consisting of 2 electronics manufacturers, 2 banks and 2 insurance companies, such that no company appears in both portfolios?

(a)
$C_8^4 \cdot (2x)^4 \cdot (-\frac{1}{x})^4$
$= \frac{8 \times 7 \times 6 \times 5}{24} \times 16$
$= 70 \times 16 = 1120$

$(C_{10}^2 \cdot C_8^2) \cdot (C_7^2 \cdot C_5^2) \cdot (C_5^2 \cdot C_3^2)$
$= (45 \times 28) \times (21 \times 10) \times (10 \times 3)$
$= 7938000$

4. (12 marks) Solve the equations for x.

   (a) $2^{4x} + 5 \cdot 4^x = 6$.

   (b) $\log_{10}(x-1) + \log_{10}(x-3) = \log_{10}(9+x)$.

   (c) $\log_3(x-5) = \log_9(2x+5)$.

(a) assume $2^{2x} = t = 4^x > 0$
$t^2 + 5t - 6 = 0$
$(t+6)(t-1) = 0$
$\therefore 2^{2x} = 1$
$x = 0$
$x = 0$

(b) $\log_{10}(x-1)(x-3) = \log_{10}(9+x)$
$x^2 + 3 - 4x = x + 9$
$x^2 - 5x - 6 = 0$
$(x-6)(x+1) = 0$
$x = 6 / -1$,
But x should > 3,
$\therefore x = 6$
$x = 6$

(c) $(x > 5)$
$x - 5 = a$ $\quad 2x + 5 = a^2$
$\therefore x^2 + 25 - 10x = 2x + 5$
$x^2 - 12x + 20 = 0$
$(x-2)(x-10) = 0$
$\therefore x = 10$ or 2
But $x > 5$.
$\therefore x = 10$
$x = 10$

5. (36 marks) Find the limits if they exist.

   (a) $\lim\limits_{x \to 1} \tan(x^4 + x - 1)$.

   (b) $\lim\limits_{x \to 1} \ln(x^3 + 2)$.

   (c) $\lim\limits_{x \to 0} \cos^{-1}(x^2 - x)$.

(a) as $1 \in \tan(x^4+x-1)$'s continuous interval
$\lim\limits_{x \to 1} \tan(x^4+x-1) = \tan 1$

(b) $\ln(x^3+2)$'s continuous interval.
$(-\sqrt[3]{2}, +\infty)$
$\therefore \lim\limits_{x \to 1} \ln(x^3+2) = \ln(1+2) = \ln 3$.

(c)
As $x = 0$ belongs to $\arccos(x^2 - x)$'s continuous interval. $\lim\limits_{x \to 0} \arccos(x^2 - x) = \arccos 0 = \pi/2$.
$-1 \leq x^2 - x \leq 1$

记: $y = \tan^{-1} x$
$x = \tan y$
$y' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{1}{\cos^2 y}} = \cos^2 y$
$x = \frac{\sin y}{\cos y} = \frac{\sqrt{1 - \cos^2 y}}{\cos y}$
$x^2 \cos^2 y = 1 - \cos^2 y$
$\Rightarrow \cos^2 y = \frac{1}{x^2+1}$
$\therefore y' = \frac{1}{x^2+1}$

(d) $\lim_{x\to 0}\sqrt{\dfrac{\sin 6x}{\sin x}}$. $=\sqrt{\lim_{x\to 0}\dfrac{\sin 6x}{\sin x}} = \sqrt{\lim_{x\to 0}\dfrac{6\cos 6x}{\cos x}} = \sqrt{6}$

(e) $\lim_{x\to 0}\dfrac{\tan^{-1} 2x}{x}$. $= \lim_{x\to 0}\dfrac{\frac{2}{1+4x^2}}{1} = 2$ $y=\tan^{-1}x$, $y'=\frac{1}{1+x^2}$

(f) $\lim_{x\to 1}\dfrac{x^3-1}{\sin(x-1)}$. $=\lim_{x\to 1}\dfrac{(x-1)(x^2+x+1)}{\sin(x-1)} = \lim_{x\to 1}\dfrac{x-1}{\sin(x-1)}\cdot\lim_{x\to 1}(x^2+x+1) = (x^2+x+1)\big|_{x=1} = 3$

(g) $\lim_{x\to -1}[e^{-2x}+5\sin(2x+1)]$. $=\lim_{x\to -1}e^{-2x} + 5\lim_{x\to -1}\sin(2x+1) = e^2 - 5\sin 1$

(h) $\lim_{x\to\pi/2}[v^2-3v+1]$, where $v=\sin x$. $\lim_{x\to\frac{\pi}{2}}\sin x = 1$, $\lim_{x\to\frac{\pi}{2}}\sin^2 x = 1$ $\therefore \lim_{x\to\frac{\pi}{2}}v^2 - \lim_{x\to\frac{\pi}{2}}3v + 1 = -1$

(i) $\lim_{x\to 0}[w+\cos^{-1} w]$, where $w=\cos x$.

(i): $\lim_{x\to 0} w = 1$ $\therefore \lim_{x\to 0}(w+\arccos w) = 1+\arccos 1 = 1$

(j) $\lim_{x\to\infty}\dfrac{4x^3+x-2}{3x^3-5x^2+1} = \lim_{x\to\infty}\dfrac{4+\frac{1}{x^2}-\frac{1}{x^3}}{3-\frac{5}{x}+\frac{1}{x^3}} = \frac{4}{3}$

(k) $\lim_{x\to -\infty}\dfrac{4x^5+5x^3-6x^2-x-2}{3x^3-5x^2+1}$ $=-\lim_{x\to-\infty}\dfrac{20x^4+15x^2-12x-1}{9x^2-10x} = -\lim_{x\to-\infty}\dfrac{80x^3+30x-12}{18x-10} = -\lim_{x\to-\infty}\dfrac{240x^2+30}{18} = -\frac{\infty}{18} = -\infty$ (not exist)

(l) $\lim_{x\to -\infty}\dfrac{-3x^5+12x^3-2x^2+x+3}{5x^3+x^2-8}$ $=\lim_{x\to-\infty}\dfrac{-15x^4+36x^2-4x+1}{15x^2+2x} = \lim_{x\to-\infty}\dfrac{-60x^3+72x-4}{30x+2} = \dfrac{-180x^2+72}{30} = \dfrac{-\infty}{30} = -\infty$ (not exist)

6. (12 marks) Find if possible the one-sided limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ for the following functions.

(a) $f(x)=\dfrac{2x+3}{x(x-1)}$, $a=1$.

(a) $\lim_{x\to 1^+} f(x) = \frac{5}{0^+} = \infty$, $\lim_{x\to 1^-} f(x) = \frac{5}{0^-} = -\infty$

(b) $f(x)=\dfrac{\sin x}{x-1}$, $a=1$.

(b) $\lim_{x\to 1^+} f(x) = \frac{\sin 1}{0^+} = \infty$, $\lim_{x\to 1^-} f(x) = \frac{\sin 1}{0^-} = -\infty$

(c) $f(x)=\dfrac{e^x}{x^4}$, $a=0$.

(c) $\lim_{x\to 0^+}\frac{e^{0^+}}{0^+} = \frac{1}{0^+} = \infty$, $\lim_{x\to 0^-}\frac{1}{(0^-)^4} = \frac{1}{0^+} = \infty$

(d) $f(x)=\dfrac{1}{2x}+\dfrac{3}{\sin x}$, $a=0$.

(d): as $f(x)$ is odd function. $\lim_{x\to 0^-} f(x) = -\lim_{x\to 0^+} f(x)$. $\lim_{x\to 0^+} f(x) = \frac{1}{0^+}+\frac{3}{0^+} = \infty$ $\therefore \lim_{x\to 0^-} f(x) = -\infty$.

7. (6 marks) Let

$$f(x)=\begin{cases}\dfrac{2x^2-18}{x+3}, & \text{if } x<-3,\\ -12, & \text{if } x=-3,\\ x^2+7x, & \text{if } x>-3.\end{cases}$$

Find $\lim_{x\to -3^+} f(x)$, $\lim_{x\to -3^-} f(x)$, and $\lim_{x\to -3} f(x)$. Is f continuous at $x=-3$?

7. $\lim_{x\to -3^-} f(x)$:

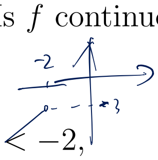
$f(x) = 2\cdot\dfrac{(x-3)(x+3)}{x+3}$

$\lim_{x\to -3^-} f(x) = 2(x-3)\big|_{x=-3} = -12$

$\lim_{x\to -3^+} f(x) = (x^2+7x)\big|_{x=-3} = -12$.

$\therefore \lim_{x\to -3} f(x)$ exists, as $\lim_{x\to -3^-} f(x) = \lim_{x\to -3^+} f(x) = -12$

$\therefore \lim_{x\to -3} f(x) = -12$, so $f(x)$ is continuous at point $x=-3$



8. (6 marks) Let

$$f(x)=\begin{cases}x-1, & \text{if } x<-2,\\ ax+b, & \text{if } -2\le x<0,\\ x+3, & \text{if } x\ge 0.\end{cases}$$

Find the values a and b so that the function f is continuous for all x.

8. As the function is continuous. We may assume: $\lim_{x\to(-2)^-} f(x) = \lim_{x\to(-2)^+} f(x)$

$\therefore -2a+b=-3$

Also, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \Rightarrow b=3$.

$\therefore -2a=-6$, $a=3$

$\therefore a=3$, $b=3$, $y=3x+3$. $x\in[-2,0]$