Week 6 — Linear Transformations ⭐

Definition and Notation

Definition: A function $T: R^n \to R^m$ is a **linear transformation** if it satisfies:

1. **Additivity:** $T(u + v) = T(u) + T(v)$ for all $u, v \in R^n$
2. **Homogeneity:** $T(cu) = cT(u)$ for all $u \in R^n$, $c \in \mathbb{R}$

These two conditions together give the **combined linearity** property:

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T(au + bv) = aT(u) + bT(v)   for all u, v, and scalars a, b
```

More generally: $T(c_1u_1 + c_2u_2 + \ldots + c_ku_k) = c_1T(u_1) + c_2T(u_2) + \ldots + c_kT(u_k)$

Domain and Codomain:

- The domain of $T: R^n \to R^m$ is R^n (inputs are n-dimensional vectors)
- The codomain is R^m (outputs are m-dimensional vectors)

> ***Watch Out (Week 6 Quiz Problem 2):***
>
> - *"T(v) = w makes sense, so w must be in R^n" → **FALSE** — w must be in R^**m** (the codomain)*
> - *"T(v) = w makes sense, so v must be in R^m" → **FALSE** — v must be in R^**n** (the domain)*
> - *The matrix of T has **m rows and n columns** → **TRUE** (m = codomain dimension, n = domain dimension)*

T(0) = 0 Always

Theorem: For any linear transformation T, $T(0) = 0$.

Proof sketch: $T(0) = T(0 \cdot v) = 0 \cdot T(v) = 0$ for any v.

> ***Exam Pattern:*** *If $T(v) \neq 0$ for some v, T might still be linear (only 0 maps to 0 necessarily). But if $T(0) \neq 0$, then T is definitely NOT linear.*

Computing T of a Linear Combination from Known Outputs

If you know T on specific vectors, you can compute T on any linear combination of those vectors **without knowing the matrix**.

Method: Express the input as a linear combination of the known vectors, then apply linearity.

Week 6 Quiz Problem 1

$T: R^4 \to R^2$ with:

- $T([0,1,-9,8]^T) = [1,-5]^T$
- $T([9,1,0,-2]^T) = [5,-9]^T$

Find $T([54,11,-45,28]^T)$.

Step 1: Recognize $[54,11,-45,28]^T = 5\cdot[0,1,-9,8]^T + 6\cdot[9,1,0,-2]^T$ (Check: $5\cdot0 + 6\cdot9 = 54$ ✓, $5\cdot1 + 6\cdot1 = 11$ ✓, $5\cdot(-9) + 6\cdot0 = -45$ ✓, $5\cdot8 + 6\cdot(-2) = 28$ ✓)

Step 2: Apply linearity: $T([54,11,-45,28]^T) = 5\cdot T([0,1,-9,8]^T) + 6\cdot T([9,1,0,-2]^T) = 5\cdot[1,-5]^T + 6\cdot[5,-9]^T = [5,-25]^T + [30,-54]^T =$ $\mathbf{[35, -79]^T}$ ← answer B

Find $T([0,0,0,0]^T)$: → $T(0) = 0$ → $\mathbf{[0, 0]^T}$ ← answer C

The Matrix of a Linear Transformation

Theorem: Every linear transformation $T: R^n \to R^m$ can be represented as matrix multiplication: $T(x) = Ax$ for some m×n matrix A.

How to find A: The j-th column of A equals $T(e_j)$, where e_j is the j-th standard basis vector.

```
A = [T(e₁) | T(e₂) | ... | T(eₙ)]
```

> ***Exam Pattern (Week 6 Quiz Problem 2):*** *"If we know T on all standard basis vectors of R^n, we can figure out T on any vector"* → ***TRUE*** *(each column of the matrix comes from $T(e_j)$)*

Why this works: Any $x = x_1e_1 + x_2e_2 + \ldots + x_ne_n$, so by linearity: $T(x) = x_1T(e_1) + x_2T(e_2) + \ldots + x_nT(e_n) = A\cdot x$

Kernel and Image

Definition: The **kernel** of T (= null space of A) is: $\ker(T) = \{v \in R^n \mid T(v) = 0\} = \text{null}(A)$

Definition: The **image** of T (= column space of A) is: $\text{im}(T) = \{T(v) \mid v \in R^n\} = \text{col}(A)$

- ker(T) is a subspace of the **domain** (R^n)
- im(T) is a subspace of the **codomain** (R^m)

Derivative as a Linear Transformation

Theorem: The derivative $d/dx: P_3 \to P_2$ is a linear transformation.

$d/dx(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$

Why linear:

- $d/dx(f + g) = f' + g'$ ✓ (sum rule)
- $d/dx(cf) = cf'$ ✓ (constant multiple rule)

Second derivative d^2/dx^2: $P_3 \to P_1$ (Week 6 Quiz Problem 3): $d^2/dx^2(a_0 + a_1x + a_2x^2 + a_3x^3) = 2a_2 + 6a_3x$

The image (range) of d^2/dx^2 is all polynomials of degree ≤ 1, i.e., $\mathbf{P_1}$. Answer: A (P_1, not P_2 or P_0)

Reasoning: Taking the second derivative reduces degree by 2.

- Starting space: P_3 (degree ≤ 3)
- After one derivative: P_2 (degree ≤ 2)
- After two derivatives: $\mathbf{P_1}$ (degree ≤ 1)

Summary: Key Facts about $T: R^n \to R^m$

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Domain:          R^n       (inputs live here)
Codomain:        R^m       (outputs live here)
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Matrix A:        m × n     (m rows = codomain dim, n cols = domain dim)
T(0) = 0:        ALWAYS
ker(T):          subspace of R^n  (= null(A))
im(T):           subspace of R^m  (= col(A))
```

Linearity combo rule: $T(c_1v_1 + c_2v_2 + \ldots + c_kv_k) = c_1T(v_1) + \ldots + c_kT(v_k)$

This is the most-tested skill: given T on a few vectors, find T on a linear combination.

Worked Practice

Setup: $T: R^3 \to R^2$ is linear, with $T(e_1) = [1,2]^T$, $T(e_2) = [0,-1]^T$, $T(e_3) = [3,4]^T$.

Find the matrix A:

```
A = [T(e₁) | T(e₂) | T(e₃)] = [ 1   0   3 ]
                               [ 2  -1   4 ]
```

Compute $T([2, -1, 3]^T)$: Method 1 (matrix mult): $A \cdot [2,-1,3]^T = [1\cdot2 + 0\cdot(-1) + 3\cdot3,\ 2\cdot2 + (-1)\cdot(-1) + 4\cdot3]^T = [11, 17]^T$
Method 2 (linearity): $2T(e_1) - T(e_2) + 3T(e_3) = 2[1,2]^T - [0,-1]^T + 3[3,4]^T = [11,17]^T$ ✓