2-5-2026

$V = \text{Span}\left\{\begin{bmatrix}3\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\end{bmatrix}\right\}$

Note: V is the graph of the equation $y+z=0$ — Desmos

Week 4 Problem videos

* \- Watch the conversion to Echelon form to review
* Watch RREF in Python

Determining if a vector is in the span of a set

$a\begin{bmatrix}3\\1\\-1\end{bmatrix} + b\begin{bmatrix}2\\-1\\1\end{bmatrix}$

All linear combinations of the vectors

Desmos

v = vector((0,0,0),(3,1,-1))
w = vector((0,0,0),(2,-1,1))

Is $\begin{bmatrix}7\\-1\\1\end{bmatrix}$ in V? Yes because $-1+1=0$

Is a vector in the span of the set?

$a\begin{bmatrix}3\\1\\-1\end{bmatrix} + b\begin{bmatrix}2\\-1\\1\end{bmatrix} + c\begin{bmatrix}7\\-1\\1\end{bmatrix}$ Is $\begin{bmatrix}7\\-1\\1\end{bmatrix}$ in V?

- Write down a vector equation

$$\left[\begin{array}{cc|c}3 & 2 & 7\\1 & -1 & -1\\-1 & 1 & 1\end{array}\right]$$

- Convert to augmented matrix
- solve

Need to practice by hand

$$\left[\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & 2\\0 & 0 & 0\end{array}\right]$$ ← RREF

$a = 1$
$b = 2$

- If there are solutions, Then the vector is in the span AND we know how to write it as a linear combination.
- If we get no solutions, it's not in the span

Does a set span $\mathbb{R}^n$?

Means that every vector is a linear combination of The vectors.

Ex. $\left\{\begin{bmatrix}-1\\0\\3\end{bmatrix}, \begin{bmatrix}5\\4\\-6\end{bmatrix}, \begin{bmatrix}3\\4\\0\end{bmatrix}, \begin{bmatrix}1\\-3\\10\end{bmatrix}\right\}$

a span of $\mathbb{R}^3$?

Ex. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 10 \end{bmatrix} \right\}$

a span of $\mathbb{R}^3$?

Steps: 1. Make a matrix with the vectors as the columns.

$$\begin{bmatrix} -1 & 5 & 3 & 1 \\ 0 & 4 & 4 & -3 \\ 3 & -6 & 0 & 10 \end{bmatrix}$$

2. Convert to echelon form (RREF works too)

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Is there a pivot in each row?

Yes? It spans $\mathbb{R}^n$

No? doesn't span $\mathbb{R}^n$

Ex. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ -9 \end{bmatrix} \right\}$

$$\begin{bmatrix} -1 & 5 & 3 & 9 \\ 0 & 4 & 4 & 8 \\ 3 & -6 & 0 & -9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

– Doesn't span $\mathbb{R}^3$

Ex. 3 $\left\{ \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \right\}$

Does not span $\mathbb{R}^3$ because there are not enough vectors.

Linear Independence

Ex. $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -7 \\ 3 \end{bmatrix}$

Steps

1. Make an augmented matrix with the vectors as columns

$$\begin{bmatrix} 1 & 3 & 4 \\ -1 & 0 & 2 \\ 2 & 6 & -7 \\ 3 & -1 & 3 \end{bmatrix}$$

2. Convert to echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

3. Is there a pivot in each column?

Yes: Linearly independent No: Linearly dependent

Does

$$-3\begin{bmatrix}1\\-1\\2\\3\end{bmatrix}+2\begin{bmatrix}3\\0\\6\\-1\end{bmatrix}+1\begin{bmatrix}4\\2\\-7\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

have a nontrivial solution?
(Equivalently, infinitely many)

Ex 2. $\left\{\begin{bmatrix}1\\-1\\2\\3\end{bmatrix},\begin{bmatrix}3\\0\\6\\-1\end{bmatrix},\begin{bmatrix}-3\\-3\\-6\\11\end{bmatrix}\right\}$

$$\begin{bmatrix}1&3&-3\\-1&0&-3\\2&6&-6\\3&-1&11\end{bmatrix}\to\begin{bmatrix}1&0&-3\\0&1&2\\0&0&0\\0&0&0\end{bmatrix}$$

No pivot in the third column

$$\begin{bmatrix}-3\\-3\\-6\\11\end{bmatrix}=3\begin{bmatrix}1\\-1\\2\\3\end{bmatrix}-2\begin{bmatrix}3\\0\\6\\-1\end{bmatrix}$$

Is a set a **basis** for $\mathbb{R}^n$?
(spans and is linear independent)
- means that each vector in $\mathbb{R}^n$ is a linear combination of the vectors in exactly one way.

Steps:
1. Make a matrix with the vectors as columns
2. Convert to echelon form
3. Is there a pivot in each row and column?

   Yes: basis

   No: Not a basis

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ -8 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 6 \\ -1 & -3 & -8 \\ 1 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$Rep_B(\vec{v})$

Ex. $P_3 = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \}$

P_3 "looks like" $\mathbb{R}^4$

"isomorphic as a vector space to"

$B = \langle 1, x, x^2, x^3 \rangle$

$Rep_B(1 + x + 3x^3) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}$

$Rep_B(x + 3x^2 - 5x^3) = \begin{bmatrix} 0 \\ 1 \\ 3 \\ -5 \end{bmatrix}$

Are $x^3 + 3x - 1$,
$x^3 + 2x^2 - x - 5$,
and $x^3 - 4x^2 + 11x + 7$
linearly independent?

Check if these are linearly independent

$\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 11 \\ -4 \\ 1 \end{bmatrix}$

V has a basis $B = \left\langle \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\rangle$

$\mathrm{Rep}_B\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ because $\begin{bmatrix} 7 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

If we have a matrix A Three spaces

row(A) = span of the rows of A
"orthogonal compliment of null(A)"

col(A) = span of the columns of A
= set of vectors such that $[A \mid \vec{v}]$
has a solution

null(A) = set of solutions to $[A \mid \vec{0}]$

Finding bases for row(A),
col(A), and null(A).

Finding bases for row(A), col(A), and null(A)

1. Convert to echelon form for row(A) & col(A), RREF for null(A)
   - Basis for row(A): nonzero rows in echelon form
   - Basis for col(A): pivot columns in original matrix
   - Basis for null(A): solve $[A|\vec{0}]$, write solution as vectors

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis

$$\mathrm{row}(A) = \mathrm{span}\left(\underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \right)$$

$$\mathrm{col}(A) = \mathrm{span}\left(\underbrace{\begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 2 \end{bmatrix}}_{\text{Basis}} \right)$$

$$\text{null}(A) = \quad \begin{aligned} x_1 &= 2s_1 - 3s_2 \\ x_2 &= s_1 \\ x_3 &= 5s_2 \\ x_4 &= s_2 \\ x_5 &= 0 \end{aligned} \quad = s_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -3 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{null}(A) = \text{span}\Bigg(\underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}}_{\text{Basis}} \Bigg)$$

Rank-nullity Theorem

If A is a matrix:

\# of pivots $\leftarrow$ $\dim(\text{col}(A)) = \dim(\text{row}(A)) = \text{rank}(A)$

$\dim(\text{null}(A)) = \text{nullity}(A)$

$$\underbrace{\text{rank}(A)}_{= \text{ \# of pivots}} + \underbrace{\text{nullity}}_{= \text{ \# of free variables}} = \text{\# of columns}$$

Find a basis for subspaces

Ex:

$$V = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ 9 \end{bmatrix} \right)$$

Option 1: Columns

$$\begin{bmatrix} -1 & 5 & 3 & 9 \\ 0 & 4 & 4 & 8 \\ 3 & -6 & 0 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -6 \end{bmatrix} \right\}$$ is a basis

Option 2:

$$\begin{bmatrix} -1 & 0 & 3 \\ 5 & 4 & -6 \\ 3 & 4 & 0 \\ 9 & 8 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & \frac{9}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{9}{4} \end{bmatrix}$$ is a basis

$\mathcal{P}_2 = \{a_0 + a_1 x + a_2 x^2\}$

$W = \{p \in \mathcal{P}_2 \mid p(1) = 0\}$ $\qquad \begin{bmatrix} 9 \\ 8 \\ 9 \end{bmatrix} = 9 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + 8 \begin{bmatrix} 0 \\ 1 \\ \frac{9}{4} \end{bmatrix}$

is a subspace
- what dimension?
- what is a basis?

Elements of W:

$x^2 - 1$

$x^2 - 2x + 1$

$x^2 - 4x + 3$

$W = \text{span}\left(x^2-1,\ x^2-2x+1,\ x^2-4x+3\right)$

Find a useful basis for W:

$B = \langle x^2, x, 1 \rangle$ basis for $\mathcal{P}_2$

$\text{Rep}_B(x^2 - 1) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\text{Rep}_B(x^2 - 2x + 1) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\text{Rep}_B(x^2 - 4x + 3) = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$

Rows

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 1 & -4 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{x^2 - 1,\ x - 1} \leftarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} \Leftarrow$$

2-19-2026

Function between spaces That maintain property

Ex. Suppose $T: \mathbb{R}^2 \to \mathbb{R}^3$ is linear with

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -7 \\ 1 \\ 8 \end{bmatrix}$$

what is $T\left(\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right)$?

Step 1: $a\begin{bmatrix} 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ -1 & 3 & -5 \end{array}\right]$$

Solve: $a = -1$

$b = -2$

Check: $-1\begin{bmatrix} 1 \\ -1 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

Step 2: Use Linearity

$$T\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) = T\left(-1\begin{bmatrix} 1 \\ -1 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$$

$$= T\left(-1\begin{bmatrix}1\\-1\end{bmatrix} + T\left(-2\begin{bmatrix}-2\\3\end{bmatrix}\right)\right)$$

$$= -1\,T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) - 2\,T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right)$$

$$= -1\begin{bmatrix}3\\1\\1\end{bmatrix} - 2\begin{bmatrix}-7\\1\\8\end{bmatrix} = \begin{bmatrix}11\\-3\\-17\end{bmatrix}$$

Derivative is a linear transformation between <u>functions</u>

$$\frac{d}{dx} : \underset{\substack{\uparrow\\ \text{differential functions}}}{C^1(\mathbb{R})} \to \underset{\substack{\uparrow\\ \text{continuous functions}}}{C^0(\mathbb{R})}$$

Matrix Representation

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = 3\begin{bmatrix}3\\1\\1\end{bmatrix} + \begin{bmatrix}-7\\1\\8\end{bmatrix} = \begin{bmatrix}2\\4\\15\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 2\begin{bmatrix}3\\1\\1\end{bmatrix} + \begin{bmatrix}-7\\1\\8\end{bmatrix} = \begin{bmatrix}-1\\3\\10\end{bmatrix}$$

$$\begin{bmatrix}1\\0\end{bmatrix} = 3\begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}-2\\3\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} = 2\begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}-2\\3\end{bmatrix}$$

The matrix representation of T
(with respect to the standard basis)

is $\left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \right]$

More generally,

$\left[T(\vec{e}_1) \; T(\vec{e}_2) \ldots T(\vec{e}_m) \right]$

This is a n by m matrix

$= \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 11 & 10 \end{bmatrix}$ ← This tells us everything about T

Any 3x2 matrix represents some linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$

Computations with matrix representation

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 11 & 10 \end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}$$

Dot product

$$\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 11 & 10 \end{bmatrix} \begin{matrix} 2\cdot 1 + (-1)\cdot(-1) \\ 4\cdot 1 + 3(-1) \\ 11\cdot 1 + 10(-1) \end{matrix} = \begin{bmatrix}3\\1\\1\end{bmatrix}$$

$A \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 11 & 10 \end{bmatrix}$ is the matrix representation of T

$T(\vec{v}) = A\vec{v}$

What does A tell us about T?

- One to one, onto, bijective

If $f(x) = f(y)$
Then $x = y$

Every element of the codomain is in the image
(for all y there exists x with $f(x) = y$)

Both

$T: \mathbb{R}^2 \to \mathbb{R}^3$

domain   codomain

Range(T) = Subset of the codomain that actually appears as $T(\vec{v})$

Ker(T) = Set of elements in the domain that are mapped to the $\vec{0}$.

$\ker(T) = \{\vec{v} \in \mathbb{R}^2 \mid T(\vec{v}) = \vec{0}\}$

$A\vec{v} = \vec{0}$

$$\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 11 & 10 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2v_1 - v_2 = 0$
$4v_1 + 3v_2 = 0$
$11v_1 + 10v_2 = 0$

$\underline{\ker(T) = \text{null}(A)}$

$\text{Range}(T) = \{A\vec{v}\}$

$$A \cdot \vec{v} = v_1 \begin{bmatrix} 2 \\ 4 \\ 11 \end{bmatrix} + v_2 \begin{bmatrix} -1 \\ 3 \\ 10 \end{bmatrix}$$

All linear combinations of the columns

$\underline{\text{range}(T) = \text{col}(A)}$

<u>Rephrase rank nullity:</u>

$T: \mathbb{R}^m \to \mathbb{R}^n$

$\dim(\text{range}(T)) + \dim(\ker(T)) = m$

A

$$\text{RREF}(A) = \left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{array}\right]$$

* No free variables so at most one solution

Row of 0's means not onto (isn't a pivot in each row)

$\dim(\text{col}(A)) = 2$

$T: \mathbb{R}^m \to \mathbb{R}^n$ is linear with matrix A $(T(\vec{v}) = A\vec{v})$

T is
- One to one if RREF(A) has a pivot in every column
- onto if RREF(A) has a pivot in each row
- bijective if the RREF is

$$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$ (The identity matrix)

$T: \mathbb{R}^4 \to \mathbb{R}^3$
$\vec{v} \longmapsto A\vec{v}$

$3 \times 4 =$ 3 rows
4 columns

(That is, $T(\vec{v}) = A\vec{v}$)

$$A = \begin{bmatrix} -1 & -2 & 1 & 6 \\ 2 & 4 & 3 & -17 \\ 1 & 2 & 1 & -5 \end{bmatrix}$$

$$\text{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & -7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$RREF(A) = \begin{bmatrix} 1 & 2 & 0 & -7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the bases for range(T) and ker(T)

$\ker(T) =$

$$\text{range}(T) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right)$$

$$\ker(T) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -2s_1 + 7 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 7 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Solve This

2-26-2025

Matrices / Matrix Multiplication

$$A = \begin{bmatrix} -.6 & .8 \\ .8 & .6 \end{bmatrix}$$

$$T(\vec{v}) = A\vec{v}$$

This a reflection across $y = 2x$

Matrix Addition and scalar multiplication

If $T, S : \mathbb{R}^m \to \mathbb{R}^n$

We can add $(T+S)(\vec{v}) = T(\vec{v}) + S(\vec{v})$

and $(cT)(\vec{v}) = cT(\vec{v})$

- We define $A+B$ and cA to match up with the linear transformation definition.

Matrix multiplication

If $T : \mathbb{R}^k \to \mathbb{R}^n$ and $S : \mathbb{R}^m \to \mathbb{R}^k$

$T \circ S : \mathbb{R}^m \to \mathbb{R}^n$ is defined by

"T composed with S" $(T \circ S)(\vec{v}) = T(S(\vec{v}))$

$\vec{v}$ is in $\mathbb{R}^m$

$S(\vec{v})$ is in $\mathbb{R}^k$

$T(S(\vec{v}))$ is in $\mathbb{R}^n$

$T \circ S$ makes sense if the domain of T equals the codomain of S.

$T(\vec{v}) = A\vec{v}$ and $S(\vec{v}) = B\vec{v}$

Then AB to be the matrix of $T \circ S$

$T: \mathbb{R}^4 \to \mathbb{R}^2$ $\qquad$ $S: \mathbb{R}^3 \to \mathbb{R}^4$

$\vec{v} \mapsto A\vec{v}$ $\qquad$ $\vec{v} \mapsto B\vec{v}$

$$A = \begin{bmatrix} 1 & -1 & 3 & 7 \\ 0 & 2 & 4 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -2 & 0 \\ 4 & 0 & 1 \\ 8 & 2 & -1 \\ 0 & 3 & 5 \end{bmatrix}$$

$T \circ S : \mathbb{R}^3 \to \mathbb{R}^2$

$$S\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 & -2 & 0 \\ 4 & 0 & 1 \\ 8 & 2 & -1 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} (3)(1) + (-2)(2) + (0)(3) \\ (4)(1) + (0)(2) + (1)(3) \\ (8)(1) + (2)(2) + (-1)(3) \\ (0)(1) + (3)(2) + (5)(3) \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 7 \\ 9 \\ 21 \end{bmatrix}$$

$$T\begin{bmatrix} -1 \\ 7 \\ 9 \\ 21 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 & 7 \\ 0 & 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \\ 9 \\ 21 \end{bmatrix} = \begin{bmatrix} 166 \\ 29 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 1 & -1 & 3 & 7 \\ 0 & 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 0 \\ 4 & 0 & 1 \\ 8 & 2 & -1 \\ 0 & 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} 23 & 25 & 31 \\ 40 & 5 & -7 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$(A \cdot B) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 23 & 25 & 31 \\ 40 & 5 & -7 \end{bmatrix}$$

Properties of Matrix Arithmetic

- AB is defined if # of columns of A equals the # of rows of B.
- $(AB)C = A(BC)$ (Associative)
  ABC is ambiguous
- $A(B+C) = AB + AC$
- $(B+C)A = BA + CA$ } Distributive
- Identity matrix $I = I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$AI = IA = A$

$\uparrow \quad \uparrow$

$I_4 \quad I_2$

A is a 2×4

$T(\vec{v}) = (\vec{v})$

Non Properties (not generally true)

- $AB \neq BA$ (not commutative)
- $AB = 0$ doesn't mean $A = 0$ or $B = 0$
- If you aren't sure if

* For property function, to have an inverse it must be a bijection

Inverse of Matrix

$n = 3$:

If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a bijection with $T(\vec{v}) = A\vec{v}$, then $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is also a linear with $T^{-1}(\vec{v}) = A^{-1}\vec{v}$

- $AA^{-1} = A^{-1}A = I_n$

Finding Inverses:

$n=1$: $\begin{bmatrix} a \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} \end{bmatrix}$

$n=2$: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}^{-1} = \frac{1}{(2)(3)-(4)(-1)} \begin{bmatrix} 3 & -4 \\ -(-1) & 2 \end{bmatrix}$$

$$\frac{1}{1} \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} .3 & -.4 \\ .1 & .2 \end{bmatrix}$$

* For any function to have an inverse it must be a bijection

$n=3$:

$$A = \begin{bmatrix} 1 & -6 & -2 \\ 3 & -5 & -2 \\ 3 & 1 & 0 \end{bmatrix}$$

Finding inverses

A is $n \times n$

- Augmented matrix $\begin{bmatrix} A \mid I_n \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -6 & -2 \\ 3 & -5 & -2 \\ 3 & 1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -6 & -2 & 1 & 0 & 0 \\ 3 & -5 & -2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -6 & -2 & 1 & 0 & 0 \\ 0 & 13 & 4 & -3 & 1 & 0 \\ 0 & 19 & 6 & -3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -6 & -2 & 1 & 0 & 0 \\ 0 & 1 & \frac{4}{13} & \frac{-3}{13} & \frac{1}{13} & 0 \\ 0 & 19 & 6 & -3 & 0 & 1 \end{array}\right]$$

$$\frac{4}{13} \cdot \frac{-19}{1} = \frac{-\cancel{74}}{13}$$

-3 19

$$\left[\begin{array}{ccc|ccc} 1 & -6 & -2 & 1 & 0 & 0 \\ 0 & 1 & \frac{4}{13} & \frac{-3}{13} & \frac{1}{13} & 0 \\ 0 & 0 & \frac{2}{13} & \frac{18}{13} & \frac{-19}{13} & 1 \end{array}\right]$$

$$\frac{\cancel{18}}{\cancel{13}} \cdot \frac{\cancel{13}}{2}$$

$$\frac{-19}{\cancel{13}} \cdot \frac{\cancel{13}}{2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -6 & -2 & 1 & 0 & 0 \\ 0 & 1 & \frac{4}{13} & -\frac{3}{13} & \frac{1}{13} & 0 \\ 0 & 0 & 1 & 9 & \frac{-19}{2} & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -6 & -2 & 1 & 0 & 0 \\ 0 & 1 & \frac{4}{13} & -\frac{3}{13} & \frac{1}{13} & 0 \\ 0 & 0 & 1 & 9 & -\frac{19}{2} & \frac{13}{2} \end{array}\right]$$

$-\frac{19}{\cancel{2}} \cdot \frac{\cancel{2}}{7}$

$$\left[\begin{array}{ccc|ccc} 1 & -6 & 0 & 19 & -19 & 13 \\ 0 & 1 & 0 & -3 & 3 & -2 \\ 0 & 0 & 1 & 9 & -\frac{19}{2} & \frac{13}{2} \end{array}\right]$$

$9 \cdot \frac{-4}{13} = -\frac{36}{13}$

$-\frac{39}{13}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -3 & 3 & -2 \\ 0 & 0 & 1 & 9 & -\frac{19}{2} & \frac{13}{2} \end{array}\right]$$

- Row reduce to $[I_n \mid A^{-1}]$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 3 & -2 \\ 9 & -\frac{19}{2} & \frac{13}{2} \end{bmatrix}}$$

If A is not invertible you will get a row of 0's on the left

$$\left[\begin{array}{c|c} A & \begin{matrix}1\\0\\0\end{matrix}\end{array}\right] \text{ means } A\vec{x} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$[\,A \mid I\,]$$

$$\downarrow$$

$$[\,E_1 A \mid E_1 I\,]$$

$$[\,E_2 E_1 A \mid E_2 E_1 I\,]$$

$$[\,E_k \cdots E_1 A \mid E_k \cdots E_1 I\,]$$

$$\left[\,\overset{\shortparallel}{I} \;\middle|\; \overset{\shortparallel}{A^{-1}}\,\right]$$
