Week 1 — Linear Systems

What Makes an Equation Linear?

Definition: An equation in variables $x_1, x_2, \ldots, x_n$ is **linear** if every variable appears only to the first power, is not multiplied by another variable, and is not inside any function (square root, trig, etc.). The general form is:

```
a₁x₁ + a₂x₂ + ... + aₙxₙ = b
```

where $a_1,\ldots,a_n$ and b are constants (can be zero).

Quiz Examples — Linear or Not?

Equation	Linear?	Reason
$3x + 6y - 4z = 9$	**YES**	each variable appears once, degree 1
$2x^2 + 4x - 7 = y$	**NO**	x^2 — variable raised to power 2
$7(x_1 + 6x_2) = 9(x_3 - x_2)$	**YES**	distributes to $7x_1 + 42x_2 - 9x_3 + 9x_2 = 0$
$(3x_1 - x_2)(5x_3 - x_4) = 9$	**NO**	product of variables (x_1x_3 appears)

> ***Watch Out:*** *Parentheses alone do not make an equation nonlinear — only if multiplication between variables results after distributing.*

Three Possible Solution Outcomes

1. **No solution** (inconsistent) — the equations contradict each other
2. **Exactly one solution** (unique) — all variables pinned down
3. **Infinitely many solutions** — at least one free variable

There is no such thing as "exactly two solutions" for a linear system.

Critical Insight: Equation/Variable Count Does NOT Determine Solutions

Exam Pattern: The Week 1 quiz tested this concept on every problem. Memorize the correct claims.

Situation	What you can conclude
#equations = #variables	Could be 0, 1, or ∞ solutions — **nothing guaranteed**
#equations < #variables	Either **no solution** OR **infinitely many** (never unique)
#equations > #variables	Could be 0, 1, or ∞ solutions — **nothing guaranteed**

> ***Watch Out:*** *"Same number of equations as variables" does NOT mean a unique solution exists. The correct answer is always "could be 0, 1, or ∞."*

> ***Watch Out:*** *"Fewer equations than variables" means you CAN rule out a unique solution — but you cannot rule out no solution (e.g., two parallel planes have no intersection).*

Worked Example (from Week 1 Quiz)

```
-7x + 8y - 6z = 1
  x       + z = 1
```

2 equations, 3 variables → could have 0 or ∞ solutions. The quiz says this system has ∞ solutions. Checking candidate ($x=1$, $y=1$, $z=0$):

- Eq 1: $-7(1) + 8(1) - 6(0) = 1$ ✓
- Eq 2: $1 + 0 = 1$ ✓ → (1, 1, 0) is a solution.

Checking ($x=1$, $y=7/4$, $z=1$):

- Eq 2: $1 + 1 = 2 \neq 1$ × → NOT a solution.

How to Verify a Candidate Solution

1. Substitute the candidate values into **every** equation.
2. Check that both sides are equal for **all** equations.
3. If any single equation fails, the candidate is not a solution.

Worked Example (Week 1 WebWork Problem 3)

System: $8x_1 + 3x_2 - 3x_3 = 21$ and $3x_1 + 9x_2 - 4x_3 = -48$.

Checking (6, −6, 3):

- $8(6) + 3(-6) - 3(3) = 48 - 18 - 9 = 21$ ✓
- $3(6) + 9(-6) - 4(3) = 18 - 54 - 12 = -48$ ✓ → **solution**

Checking (1, −4, −3):

- $8(1) + 3(-4) - 3(-3) = 8 - 12 + 9 = 5 \neq 21$ × → **not a solution**