Week 2 — Row Operations and Solving Systems

Building the Augmented Matrix

Definition: The **augmented matrix** [A | b] stores the coefficients and right-hand side of a linear system. Rows = equations, columns = variables (in order), last column = RHS.

Rules for construction:

- Variables must appear in a consistent order across all equations.
- If a variable is missing from an equation, its coefficient is **0**.
- The augmented matrix includes a vertical bar before the RHS column.

Example (Week 2 Quiz Problem 3)

```
-8x₁       - 7x₃ + 6x₄ = -1
  -x₁ + x₂ - 5x₃ - 7x₄ =  7
  -6x₁ + 2x₂      + 10x₄ = -10
   -x₁ + 5x₂ - 10x₃ + 9x₄ = -4
```

The second equation has x_2, so position (1,2) gets coefficient 1. The third has no x_3, so position (2,3) = 0. The correct matrix (answer D):

```
[ -8   0  -7   6 | -1 ]
[ -1   1  -5  -7 |  7 ]
[ -6   2   0  10 | -10]
[ -1   5 -10   9 | -4 ]
```

> ***Watch Out:*** *Answers A–C in the quiz shuffled the columns. Always use the variable order from the problem.*

Valid Elementary Row Operations (EROs)

Definition: An **elementary row operation** is one of:

1. **Swap** two rows: $R_i \leftrightarrow R_j$
2. **Scale** a row by a nonzero constant: $cR_i \to R_i$ ($c \neq 0$)
3. **Replace** a row with itself plus a multiple of another row: $cR_j + R_i \to R_i$

> ***Watch Out:*** *Multiplying a row by **0** is NOT a valid ERO — it destroys information and the operation is not reversible. (Week 2 Quiz, choice C was invalid.)*

> ***Watch Out:*** *"Replacing Row 2 with 4(Row 1) + Row 3" is NOT valid — this replaces R2 with a combination that doesn't include R2 itself. Valid replacement must have the form c*(some other row) + $R_i \to R_i$.*

> ***Watch Out:*** *You cannot perform two row operations simultaneously on the same row (e.g., adding R1 to R4 AND adding R4 to R1 at the same time is not valid).*

Valid EROs from Week 2 Quiz (answers ADFG):

- A: Adding Row 5 to Row 4 → valid (1·R5 + R4 → R4)
- D: Dividing Row 5 by 6 → valid (= multiplying by 1/6 ≠ 0)
- F: 3(Row 1) + Row 2 → Row 2 → valid replacement
- G: Switching Row 3 and Row 4 → valid swap

Echelon Form vs. Reduced Row Echelon Form (RREF)

Definition (Echelon Form): A matrix is in **echelon form** if:

1. All zero rows are at the bottom.
2. Each row's leading entry (pivot) is strictly to the right of the pivot in the row above.
3. All entries below a pivot are zero.

Definition (RREF): A matrix is in **reduced row echelon form** if additionally: 4. Each pivot is 1. 5. All entries **above** a pivot are also zero.

```
Echelon form:            RREF:
[ 1  3 -3 -10 ]        [ 1  0  0 | -3 ]
[ 0  1  9   7 ]        [ 0  1  0 |  3 ]
[ 0  0 -26 -26]        [ 0  0  1 |  2 ]
```

Reading Solutions from RREF

Step 1: Identify pivot columns (columns with a leading 1) and free columns. **Step 2:** Variables in pivot columns = **basic variables** (determined). **Step 3:** Variables in free columns = **free variables** (choose freely, set = parameter s_1, s_2, ...).

Case 1: Contradiction row → No Solution

If RREF contains a row of the form [0 0 0 | c] with c ≠ 0:

```
[ 1  7 -3 | -9 ]
[ 0  1 -5 | -4 ]
[ 0  0  0 |  7 ]   ← 0 = 7, contradiction
```

→ **Zero solutions**

Case 2: No free variables → Unique Solution

Every column except the RHS has a pivot. Read off values directly:

```
[ 1  0  0 | -3 ]     x = -3
[ 0  1  0 |  3 ]  →  y =  3
[ 0  0  1 |  2 ]     z =  2
```

→ **Exactly one solution**

Case 3: Free variables → Infinitely Many Solutions

(Week 2 WebWork Problem 2)

```
[ 1  -8  3 | -3 ]
[ 0   1 -1 |  1 ]
[ 0   0  0 |  0 ]
```

x_3 is free → set $x_3 = t$. Then $x_2 = 1 + t$, $x_1 = 5 + 5t$. Solution: $x = (5 + 5t, 1 + t, t)$ for any $t \in \mathbb{R}$.

Recognizing Free Variables from RREF (Week 2 Quiz Problem 4)

```
[ 1  -1   0  -2   0 | 17 ]
[ 0   0   1  15   0 |  9 ]
[ 0   0   0   0   1 | -8 ]
```

Variables: x_1, x_2, x_3, x_4, x_5 Pivot columns: 1 (x_1), 3 (x_3), 5 (x_5) → basic Free columns: 2 (x_2), 4 (x_4) → free variables → infinitely many solutions

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Homogeneous Systems

Definition: A system is **homogeneous** if every RHS entry is 0: Ax = 0.

Key facts:

- Homogeneous systems are **always consistent** (x = 0 is always a solution).
- If the system has **fewer equations than variables**: infinitely many solutions (guaranteed, because RREF cannot have pivots in every column, so free variables must exist).
- If #equations ≥ #variables: either exactly one solution (the trivial x=0) or infinitely many.

> ***Exam Pattern (Week 2 Quiz):***
>
> - *4 equations, 3 variables, homogeneous → 1 or ∞ solutions (correct answer A, not "no solution")*
> - *2 equations, 3 variables, homogeneous → guaranteed ∞ solutions (correct answer D)*

Gauss-Jordan Elimination Procedure

1. Write the augmented matrix.
2. Use EROs to reach echelon form (forward elimination):
   - Get a leading 1 in the first pivot position.
   - Zero out everything below that pivot.
   - Move to the next row/column.
3. Back-substitute (backward elimination) to reach RREF:
   - Zero out everything above each pivot.
4. Read the solution.

Example (Week 2 WebWork Problem 8)

```
x + 2y + 3z = 9
-3x − 2y + 4z = 11
-6x + 2y + 3z = 30
```

Augmented matrix → RREF:

```
[ 1  2  3 |  9 ]      [ 1  0  0 | -3 ]
[ -3 -2  4 | 11 ]  →  [ 0  1  0 |  3 ]
[ -6  2  3 | 30 ]      [ 0  0  1 |  2 ]
```

Solution: x = −3, y = 3, z = 2