Week 3 — Vectors and Subspaces

Vector Arithmetic

Definition: A **vector** in R^n is a column of n real numbers. Vectors are usually written as column vectors.

Addition: Add component-wise: $(u + v)_i = u_i + v_i$

Scalar multiplication: Multiply each component: $(cv)_i = cv_i$

Geometric Interpretation (in R^2 and R^3)

To add vectors u and v geometrically: place the base of v at the tip of u; draw the arrow from the base of u to the tip of v. (Week 3 Quiz answer B)

Computation Example (Week 3 Quiz Problem 2)

```
v = [9, 10, 4, 5]ᵀ   and   w = [9, -8, 8, 5]ᵀ

-8v + 5w = -8[9,10,4,5]ᵀ + 5[9,-8,8,5]ᵀ
         = [-72,-80,-32,-40]ᵀ + [45,-40,40,25]ᵀ
         = [-27, -120, 8, -15]ᵀ   ← answer B
```

Linear Combinations

Definition: A **linear combination** of vectors $v_1, v_2, \ldots, v_k$ is any vector of the form:

```
c₁v₁ + c₂v₂ + ... + cₖvₖ
```

where $c_1, \ldots, c_k$ are scalars (real numbers).

The Three-Way Equivalence

These three representations describe exactly the same mathematical object:

```
Linear System          Vector Equation              Augmented Matrix
a₁₁x₁ + a₁₂x₂ = b₁   x₁[a₁₁] + x₂[a₁₂] = [b₁]   [a₁₁ a₁₂ | b₁]
a₂₁x₁ + a₂₂x₂ = b₂      [a₂₁]      [a₂₂]   [b₂]   [a₂₁ a₂₂ | b₂]
```

In the vector equation form, the columns of A are the coefficient vectors.

Converting: Vector Equation → Linear System (Week 3 WebWork Problem 4)

```
x₁[3, -7]ᵀ + x₂[-9, 4]ᵀ = [1, 5]ᵀ
```

converts to:

```
3x₁ - 9x₂ = 1
-7x₁ + 4x₂ = 5
```

Converting: Linear System → Vector Equation (Week 3 Quiz Problem 4)

Each variable's coefficients form a column vector. The system

```
-5x₁ + 10x₃ - 8x₄ = 6
-9x₁ + x₂ + 3x₃ - 8x₄ = -8
...
```

becomes $x_1[-5,-9,-10,-1]^T + x_2[0,1,8,-10]^T + x_3[10,3,0,9]^T + x_4[-8,-8,-5,7]^T = [6,-8,7,-5]^T$

> ***Watch Out (Week 3 Quiz Problem 4):*** *The columns of the vector equation match the columns of the coefficient matrix — NOT the rows. The x_2 coefficient in equation 1 is 0 (the variable is missing), so the first component of the x_2 column vector is 0.*

Parametric Solution Form

When a system has infinitely many solutions, the general solution is written as:

```
x = p + s₁d₁ + s₂d₂ + ...
```

where:

- **p** is any particular solution (often read from the RREF)
- **d_1, d_2, ...** are direction vectors (one per free variable)
- **s_1, s_2, ...** are the free parameters

Example (Week 3 Quiz Problem 3)

```
x₁ = 15 + 16s + 14t
x₂ = t              ← free variable t
x₃ = -5 - 19s
x₄ = s              ← free variable s
x₅ = -27
```

Vector form:

```
[x₁]   [15]       [16]       [14]
[x₂]   [0 ]       [0 ]       [1 ]
[x₃] = [-5] + s * [-19] + t * [0 ]
[x₄]   [0 ]       [1 ]       [0 ]
[x₅]   [-27]      [0 ]       [0 ]
```

Note that the free-variable direction vectors have a 1 in the row corresponding to that free variable and 0 in the other free-variable row.

Zero Vector in Abstract Spaces

Definition: The **zero vector** in a vector space is the unique element 0 such that v + 0 = v for every vector v.

- In R^n: the zero vector is $[0, 0, ..., 0]^T$
- In P_4 (polynomials of degree ≤ 4): the zero vector is the **zero polynomial** 0 (the constant function that outputs 0 for every input)

> ***Watch Out (Week 3 Quiz Problem 5):*** *The zero polynomial in P_4 is written as 0, not x^4, and not the column vector $[0,0,0,0]^T$ (that would be in R^4). The elements of P_4 are polynomials, so the zero element is a polynomial.*

Subspaces

Definition: A subset H of a vector space V is a **subspace** of V if it satisfies all three:

1. **Contains zero:** The zero vector of V is in H.
2. **Closed under addition:** If $u \in H$ and $v \in H$, then $u + v \in H$.
3. **Closed under scalar multiplication:** If $v \in H$ and c is any scalar, then $cv \in H$.

If ANY condition fails, H is not a subspace.

> ***Note:*** *Conditions 2 and 3 can be combined: H is closed under linear combinations.*

Testing Subspaces — Worked Examples

Example 1 (Week 3 WebWork Problem 7, V_1): V_1 = all vectors in R^4 of the form $[a, 3a+b, a+2b, 4a-6b]^T$

Test condition 1: Set a=0, b=0 → $[0,0,0,0]^T \in V_1$. ✓ Test condition 2: Take two such vectors, add → still in form [a,3a+b,...]. ✓ Test condition 3: Multiply by scalar c → [ca, 3(ca)+cb,...] = same form. ✓ → **V_1 is a subspace of R^4.**

Example 2 (Week 3 WebWork Problem 7, V_2): V_2 = vectors $[a,b]^T$ in R^2 with $a \le b$.

Test condition 1: $[0,0]^T$, $0 \le 0$. ✓ Test condition 2: $[1,2]^T$ and $[3,4]^T$ → sum is $[4,6]^T$, $4 \le 6$. ✓ Test condition 3: Take $[1,2]^T$ with $c = -1$ → $[-1,-2]^T$, but $-1 \le -2$ is FALSE. × → **V_2 is NOT a subspace.** (Fails closure under scalar multiplication with negative scalars.)

> ***Watch Out — Classic Failure:*** *A set defined by an inequality ($a \le b$, $b_1 \le b_2$, etc.) always fails closure under negative scalar multiplication.*

Subspaces of R^3 (Week 3 WebWork Problem 8):

- $\{b_1=0\}$: passes all 3 → subspace ✓
- $\{b_2=2b_3\}$: passes all 3 → subspace ✓
- $\{b_1 \le b_2\}$: fails negative scalar → NOT a subspace ×
- $\{b_1=1\}$: does not contain zero → NOT a subspace × (zero vector has $b_1=0 \neq 1$)
- $\{b_3=b_1+b_2\}$: linear constraint, passes all 3 → subspace ✓
- $\{b_1=b_2\}$: linear constraint, passes all 3 → subspace ✓

> ***Pattern:*** *Any set defined by a* ***homogeneous linear constraint*** *(like $b_3=b_1+b_2$) is a subspace. Sets defined by inequalities or non-homogeneous equations ($b_1=1$) are not.*

Polynomial Subspaces (Week 3 WebWork Problem 9)

- Odd polynomials in P_3 ($ax^3 + bx$): subspace — closed under addition and scalar mult, zero is 0. ✓
- $f(1) \ge 0$ in P_3: NOT a subspace — fails scalar mult (multiply f by −1, then $f(1) \le 0$). ×
- $f(2) = 0$ in P_3: subspace — homogeneous linear condition on the coefficients. ✓