Week 4 — Linear Independence, Span, and Basis

Linear Independence and Dependence

Definition: A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is **linearly independent** if the only solution to $c_1v_1 + c_2v_2 + \ldots + c_kv_k = 0$ is the trivial solution $c_1 = c_2 = \ldots = c_k = 0$.

Definition: A set is **linearly dependent** if there exists a **nontrivial** solution (not all $c_i = 0$) to $c_1v_1 + c_2v_2 + \ldots + c_kv_k = 0$.

Equivalent condition for dependence: At least one vector in the set can be written as a linear combination of the others.

> ***Important distinction (Week 4 Quiz Problem 1):***
>
> - *"If v_1 is a lin. combo of v_2 and v_3 → set is dependent" — TRUE (choice B)*
> - *"If v_1 is NOT a lin. combo of v_2 and v_3 → set is independent" — FALSE (choice C is false) Because v_2 could still be a linear combo of v_1 and v_3. Correct answer: ABD (not C).*

Quick Tests for Dependence (No Row Reduction Needed)

1. **Zero vector in set** → always dependent (c_1=1 for the zero vector, rest=0)
2. **Two vectors are scalar multiples** → dependent
3. **More vectors than the dimension** of the space → dependent (e.g., 4 vectors in R^3 are always dependent)

Week 4 WebWork Examples

Problem 5: $\{u = [4,-1,2]^T, v = [0,0,0]^T, w = [-8,4,-5]^T\}$ → $v = [0,0,0]^T$ (zero vector) → **linearly dependent** (answer E)

Problem 4: $[-16, 11, 6]^T = -3\cdot[-2,1,1]^T + 1\cdot[-2,2,1]^T + (-4)\cdot[5,-3,-2]^T$ → confirms the vector is a linear combination (coefficients: −3, 1, −4)

Subset and Superset Rules

Theorem: If $\{v_1, \ldots, v_k\}$ is **linearly independent**, then every **subset** is also linearly independent.

Theorem: If $\{v_1, \ldots, v_k\}$ is **linearly dependent**, then every **superset** containing it is also linearly dependent.

Week 4 WebWork Examples

Problem 8: $\{u_1, u_2, u_3, u_4\}$ is linearly independent. Is $\{u_1, u_2, u_3\}$ independent? → YES — always (answer A). It is a proper subset of an independent set.

Problem 9: $\{u_1, u_2, u_3\}$ is linearly dependent. Is $\{u_1, u_2, u_3, u_4\}$ independent? → NO — always dependent (answer D). Any superset of a dependent set is also dependent.

Week 4 Quiz Problem 3 (All True)

- Independent → every subset independent → **True**
- Spans R^3 → adding a 4th vector still spans R^3 → **True**
- v_1 = combo of v_2, v_3, v_4 → does v_4 = combo of v_1, v_2, v_3? → **True** (rearrange equation)

- Basis of R^3 → every vector in R^3 is a combo of basis vectors → **True**

Span

Definition: The **span** of $\{v_1, \ldots, v_k\}$ is the set of ALL linear combinations:

```
span{v₁, ..., vₖ} = {c₁v₁ + ... + cₖvₖ | c₁,...,cₖ ∈ ℝ}
```

Key fact: b is in $\text{span}\{v_1, \ldots, v_k\}$ **if and only if** the system $[v_1 \mid v_2 \mid \ldots \mid v_k \mid b]$ is **consistent** (has at least one solution).

Week 4 WebWork Problem 2

$b = [-2, -14, -10]^T$, $v_1 = [-1,-4,-2]^T$, $v_2 = [1,1,-1]^T$. Is b in $\text{span}\{v_1, v_2\}$? Form augmented matrix $[v_1 \mid v_2 \mid b]$ and row reduce → consistent with solution (4, 2). → $b = 4v_1 + 2v_2$, so **b is in $\text{span}\{v_1, v_2\}$**.

Basis

Definition: A set $\{v_1, \ldots, v_k\}$ is a **basis** for a subspace H if:

1. The set is **linearly independent**, AND
2. The set **spans** H.

Theorem: Every basis for R^n contains exactly **n** vectors.

Equivalence for n vectors in R^n: If you have exactly n vectors in R^n, the following are all equivalent:

- The vectors are linearly independent
- The vectors span R^n
- The vectors form a basis for R^n (Checking any one is sufficient to conclude all three.)

Week 4 Quiz Problem 4 — Bases for R^2

Set	Basis?	Why
$\{[1,0]^T, [0,1]^T\}$	YES	Standard basis
$\{[-7,-5]^T, [-21,-15]^T\}$	NO	Second = 3× first (dependent)
$\{[-4,-2]^T, [4,2]^T\}$	NO	Second = −1× first (dependent)
$\{[7,1]^T, [-4,-1]^T\}$	YES	Not scalar multiples, 2 vectors in R^2
$\{[8,-6]^T, [16,-13]^T\}$	YES	RREF has two pivots

Pivot Column Method: Testing Independence, Span, Basis

Procedure: Let B be the matrix whose columns are your vectors.

1. Row reduce B to RREF.
2. Count pivot columns = **rank(B)**.
3. **Independence:** No free variables ↔ all columns are pivot columns.
4. **Span R^n:** Pivots in every row of B ↔ B has m pivot rows (where B is m×k).

5. **Basis:** Both conditions hold (rank = m = k, i.e., square matrix with full rank).

Example (Week 4 WebWork Problems 6 and 7)

B (2×4 matrix):

```
[ 5  5  5  5 ]      RREF: [ 1  0 -1 -2 ]
[ -1 -2 -3 -4]            [ 0  1  2  3 ]
```

Pivot columns: 1 and 2. Rank = 2.

- Spans R^2? Yes (pivots in both rows). ✓
- Independent? No (free columns 3 and 4 exist). ×
- Basis? No (not independent). ×

B (3×3 matrix):

```
[ 2  5  7  ]      RREF: [ 1  0  0 ]
[ 12 9  20 ]            [ 0  1  0 ]
[ -7 -5 -11]            [ 0  0  1 ]
```

RREF = I → rank 3, all columns pivot.

- Independent: YES | Spans R^3: YES | Basis: YES

Representing a Vector w.r.t. a Basis

Definition: If $B = \{b_1, \ldots, b_k\}$ is a basis for subspace H, and $v \in H$, then the **representation of v w.r.t. B** is the unique vector $[c_1, \ldots, c_k]$ such that:

```
v = c₁b₁ + c₂b₂ + ... + cₖbₖ
```

How to find it: Solve the system $[b_1 \mid b_2 \mid \ldots \mid b_k \mid v]$ via row reduction.

Example (Week 4 WebWork Problem 14)

Basis $B = \{[1,1,1]^T, [-1,1,3]^T\}$ for the solution set of $x - 2y + z = 0$. Find Rep_B(v) for $v = [5, -1, -7]^T$.

Solve $[1\ {-1} \mid 5;\ 1\ 1 \mid -1;\ 1\ 3 \mid -7]$ → $c_1 = 2$, $c_2 = -3$. Check: $2[1,1,1]^T + (-3)[-1,1,3]^T = [2+3, 2-3, 2-9]^T = [5, -1, -7]^T$ ✓

Rep_B(v) = $[2, -3]^T$.

Example (Week 4 WebWork Problem 10 — Polynomial Basis)

Vectors in P_2 represented w.r.t. $B = \{1, x, x^2\}$: just read off coefficients.

- Rep_B($-15 - 15x - 12x^2$) = $[-15, -15, -12]^T$

Then form matrix of these representations and row-reduce to test independence/span.

When Exactly n Vectors in R^n Form a Basis (Week 4 Quiz Problem 5)

Given: $\{[-4,4,-12]^T, [-3,8,1]^T, [-2,-1,-8]^T\}$ is a basis of R^3. Then (without computation):

- **A: Adding a 4th vector → set is linearly dependent** ✓ (4 vectors in R^3 always dependent)
- **B: Any vector in R^3 is in the span** ✓ (basis spans R^3)
- **C: Any vector can be written as a linear combination** ✓ (same as B)
- **D: First vector in span of other two** × (independent set, no redundancy)
- **E: Two vectors span R^3** × (need at least 3 vectors to span R^3)

> ***Exam Pattern:*** *Given a basis of R^n, you should immediately know: (1) adding any vector creates dependence, (2) every vector in R^n is in the span, (3) any vector can be written as a linear combination, (4) removing any vector destroys the spanning property.*