Week 7 — Invertibility and ERO Matrices ⭐

Arithmetic of Linear Transformations

If S: $R^n \to R^m$ and T: $R^n \to R^m$ are linear transformations, and a, b are scalars, then:

```
(aS + bT)(v) = aS(v) + bT(v)
```

The combined transformation aS + bT is itself a linear transformation.

Week 7 Quiz Problem 1

S: $R^4 \to R^2$ and T: $R^4 \to R^2$ are linear with:

- $S([0,1,-4,9]^T) = [3,1]^T$
- $T([0,1,-4,9]^T) = [-6,4]^T$

Find $(-4S + 3T)([0,1,-4,9]^T)$:

```
= -4·S([0,1,-4,9]ᵀ) + 3·T([0,1,-4,9]ᵀ)
= -4·[3,1]ᵀ + 3·[-6,4]ᵀ
= [-12,-4]ᵀ + [-18,12]ᵀ
= [-30, 8]ᵀ   ← answer D
```

> ***Watch Out:*** *Both S and T act on the SAME input vector. You do not need to decompose the input — just apply the arithmetic of transformations formula directly.*

ERO Matrices: Construction Rules

Every elementary row operation can be represented as left multiplication by an **ERO matrix**. ERO matrices are derived from the identity matrix by performing the same ERO on the identity.

Type 1 — Row Swap ($R_i \leftrightarrow R_j$)

Construction: Take the identity matrix and swap rows i and j.

Example: $R_1 \leftrightarrow R_3$ on a 3×3 system:

```
Original I:   ERO matrix E:
[ 1  0  0 ]   [ 0  0  1 ]
[ 0  1  0 ]   [ 0  1  0 ]
[ 0  0  1 ]   [ 1  0  0 ]
```

Left-multiplying any 3×n matrix by E swaps its rows 1 and 3.

Week 7 Quiz Problem 2 (first part):

```
[ 0  0  1 ]
[ 0  1  0 ]   → This is I with rows 1 and 3 swapped → performs R₁ ⇔ R₃   Answer: C
[ 1  0  0 ]
```

Type 2 — Row Replacement ($cR_j + R_i \to R_i$)

Construction: Take the identity matrix and put c in position **(i, j)** — row i, column j. (i is the row being modified; j is the source row being scaled)

Example: $4R_3 + R_2 \to R_2$ on a 3×3 system (c=4, i=2, j=3):

```
Original I:   ERO matrix E:
[ 1  0  0 ]   [ 1  0  0 ]
[ 0  1  0 ]   [ 0  1  4 ]   ← 4 goes in position (2,3)
[ 0  0  1 ]   [ 0  0  1 ]
```

Week 7 Quiz Problem 2 (second part):

```
[ 1  0  0 ]
[ 0  1  4 ]   → 4 is in position (2,3) → performs 4R₃ + R₂ → R₂   Answer: C
[ 0  0  1 ]
```

> ***Watch Out:*** *The c goes in position (i, j), NOT (j, i). The MODIFIED row index is i (the ROW), and the SOURCE row index is j (the COLUMN). Getting this backwards is the most common error.*

> ***Watch Out:*** *The operation name is "$cR_j + R_i \to R_i$" — the source row is R_j, the destination is R_i. The 4 in position (row 2, col 3) means: 4 × (row 3) gets added to row 2.*

Type 3 — Row Scaling ($cR_i \to R_i$)

Construction: Take the identity matrix and replace the (i,i) entry with c.

Example: $3R_2 \to R_2$:

```
[ 1  0  0 ]
[ 0  3  0 ]   ← 3 in position (2,2)
[ 0  0  1 ]
```

Invertible Linear Transformations

Definition: A linear transformation $T: R^n \to R^n$ is **invertible** if there exists a linear transformation $T^{-1}: R^n \to R^n$ such that:

```
T(T⁻¹(v)) = v    for all v ∈ Rⁿ
T⁻¹(T(v)) = v    for all v ∈ Rⁿ
```

Key inference rule: If $T(v) = w$, then $T^{-1}(w) = v$ (and conversely).

Week 7 Quiz Problem 3

$T: R^2 \to R^2$ invertible with $T([5,1]^T) = [3,-3]^T$ and $T^{-1}([-5,-7]^T) = [-2,4]^T$.

Which statements must be true?

A: $T^{-1}([3,-3]^T) = [5,1]^T$ → We know $T([5,1]^T) = [3,-3]^T$. Apply T^{-1} to both sides: $T^{-1}(T([5,1]^T)) = T^{-1}([3,-3]^T) \to [5,1]^T = T^{-1}([3,-3]^T)$. **TRUE** ✓

B: $T(T^{-1}([-5,-7]^T)) = [-5,-7]^T$ → This is just the definition $T(T^{-1}(v)) = v$. **TRUE** ✓

C: $T^{-1}([-2,4]^T) = [-5,-7]^T$ → We know $T^{-1}([-5,-7]^T) = [-2,4]^T$. This says the INPUT to T^{-1} is $[-2,4]^T$. → Does $T^{-1}([-2,4]^T) = [-5,-7]^T$? This would require $T([-5,-7]^T) = [-2,4]^T$ — not given. **FALSE** ×

D: $T(T^{-1}([-5,-7]^T)) = [3,-3]^T$ → $T(T^{-1}(v)) = v$, not $[3,-3]^T$. **FALSE** ×

E: $T([-2,4]^T) = [-5,-7]^T$ → We know $T^{-1}([-5,-7]^T) = [-2,4]^T$. Apply T to both sides: $T(T^{-1}([-5,-7]^T)) = T([-2,4]^T)$ → $[-5,-7]^T = T([-2,4]^T)$. **TRUE** ✓

Correct answers: **ABE**

> ***Key rule:*** *$T^{-1}(w) = v \leftrightarrow T(v) = w$. These are exactly equivalent for invertible T. C is the trap: it reverses the direction incorrectly.*

Invertibility Equivalences

For a linear transformation $T: R^n \to R^n$ with matrix A (n×n), the following are all equivalent:

1. T is invertible
2. A is invertible ($\det(A) \neq 0$)
3. $RREF(A) = I_n$ (the n×n identity matrix)
4. rank(A) = n (full rank)
5. null(A) = {0} (trivial null space)
6. The columns of A form a basis for R^n
7. The system Ax = b has a unique solution for every $b \in R^n$

> ***Note:*** *If rank(A) < n, then T is NOT invertible, $RREF(A) \neq I$, null(A) has nonzero vectors, and some systems Ax = b have no solution while others have infinitely many.*

Quick Reference: ERO Matrices

ERO	ERO Matrix Form	Position of Off-Diagonal Entry
$R_i \leftrightarrow R_j$	I with rows i,j swapped	N/A (row swap)
$cR_i \to R_i$	I with (i,i) entry = c	On diagonal
$cR_j + R_i \to R_i$	I with (i,j) entry = c	Position **(i,j)**: row=destination, col=source

> ***Memory trick for row replacement:*** *The c in position (i,j) means "add c times row j to row i." Row of the c = row being changed. Column of the c = row being added.*