Stat 2911 Lecture Notes

Class 23, 2017

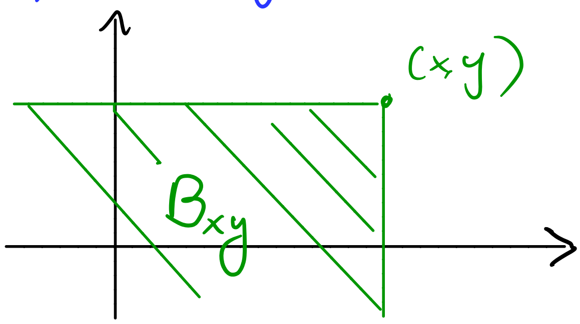
Uri Keich

2-d uniform distribution, Standard bivariate normal, Independent RVs, Bivariate normal are independent iff rho=0 (Rice 3.3., 3.4)

The joint dist. of the RVs X and Y, or the dist. of the random point $(X,Y)\in\mathbb{R}^2$ is determined by the joint dist. function:

$$F_{XY}(x,y)=P(X\le x, Y\le y)=P((X,Y)\in B_{xy}), \text{ where}$$

$$B_{xy}=\{(s,t): s\le x,\ t\le y\}$$



X and Y are jointly cont. if $\exists f_{XY}:\mathbb{R}^2\to\mathbb{R}^+$ s.t.

$$F_{XY}(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(s,t)\,dt\,ds=\iint_{B_{xy}} f_{XY}(s,t)\,ds\,dt \quad \forall (x,y)\in\mathbb{R}^2$$

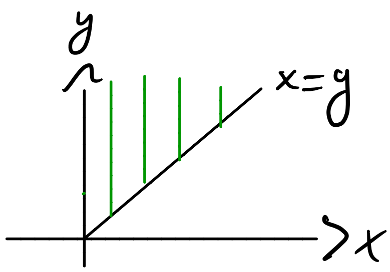
$\Rightarrow \forall B\subset\mathbb{R}^2$ (Borel)

$$P((X,Y)\in B)=\iint_B f_{XY}(x,y)\,dx\,dy$$

In this case,
$$f_X(x)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$
$$f_Y(y)=\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

<u>Examples</u>

1) $f_{XY}(x,y)=\lambda^2 e^{-\lambda y}$ $\quad y>x\ge 0$, $\lambda>0$



2) Recall that $U(a,b)$, the uniform dist. on $I=(a,b)$, has a density $f = c \cdot 1_I$ where $c = 1/|I|$. ($|I| = \text{length}(I) = b-a$)

Note that it models the random selection of a point in I (a random point), s.t. for any interval $A \subset I$

$$P(X \in A) = \int_A f\,dx = \int_A c\,dx = c\int_A dx = \frac{|A|}{|I|}.$$

We can generalize this to a uniform dist. over any "nice" region $R \subset \mathbb{R}^2$.

Def. A random point has a uniform dist. over $R \subset \mathbb{R}^2$ if it has a (joint) density

$$f(x,y) = c \cdot 1_R = \begin{cases} c & (x,y) \in R \\ 0 & \text{otherwise} \end{cases}$$

Note that $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = \iint_R c\,dx\,dy$

$$= c \cdot |R| \quad \Rightarrow \quad c = 1/|R|.$$

Again, $\forall A \subset R$ (measurable)

$$P((X,Y) \in A) = \iint_A f\,dx\,dy$$

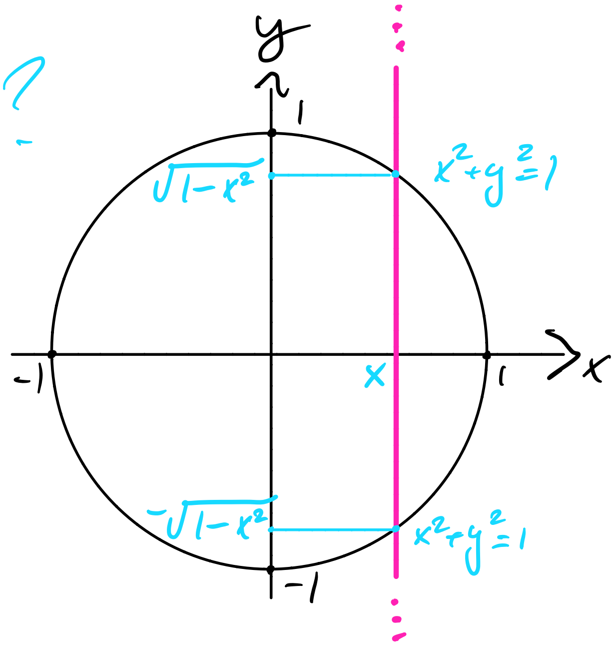
$$= \iint_A c\,dx\,dy$$

$$= c\iint_A dx\,dy = \frac{|A|}{|R|}.$$

Consider the uniform dist. on the unit disc

$$R = \{(x,y) \in \mathbb{R}^2 ; x^2 + y^2 \le 1\},$$

$|R| = \pi$, so the uniform dist. on R is given by $f_{XY}(x,y) = \frac{1}{\pi} \cdot \mathbb{1}_{x^2+y^2 \le 1}$.



What are the marginal dists?

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= 0 \quad \text{if } x \notin (-1,1)$$

If $x \in [-1,1]$,

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy = 2\sqrt{1-x^2} \cdot \frac{1}{\pi}.$$

- What is f_Y?
- Is X uniformly distributed on $[-1,1]$?

3) Bivariate Normal (special case)

The random point (Z,W) has a (standard) bivariate normal dist. if it has a density

$$f_{ZW}(z,w) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(z^2-2\rho z w+w^2)},$$

where $\rho \in (-1,1)$ is a parameter. (is it really a pdf?)

What are the marginal densities?

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)\,dw$$

Completing the square:

$$z^2-2\rho z w+w^2 = (w-\rho z)^2+(1-\rho^2)z^2$$

$$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \underbrace{\frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-(w-\rho z)^2/2(1-\rho^2)}}_{N(\mu=\rho z,\ \sigma^2=1-\rho^2) \text{ density}} dw$$

$$\Rightarrow f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$\Rightarrow Z \sim N(0,1)$ (verifying f_{ZW} is a pdf)

and by symmetry $W \sim N(0,1)$. (Z & W are std. normal)

Warning! Z, W can be $N(0,1)$ yet (Z,W) is not bivariate normal! Marginals generally can't specify the full density

Independent RVs

Recall: the discrete RVs $X_1, \dots, X_n$ are ind. if

$$p_{\underline{X}}(x_1, \dots, x_n) = p_{X_1}(x_1) \cdots p_{X_n}(x_n), \text{ where } \underline{X} = (X_1, \dots, X_n).$$

This generalizes to $(n=2)$:

$$F_{XY}(x,y) = P(X \leq x, Y \leq y) = \qquad \forall (x,y) \in \mathbb{R}^2$$

Def. The RVs $X_1, \dots, X_n$ are independent if $\forall \underline{x} \in \mathbb{R}^n$

$$F_{\underline{X}}(x_1, \dots, x_n) := P(X_1 \leq x_1, \dots, X_n \leq x_n) = F_{X_1}(x_1) \cdots F_{X_n}(x_n).$$

Suppose X and Y are jointly cont. RVs with pdf f_{XY}.
If they are ind., then $\forall (x,y)$ where f_{XY}, f_X & f_Y are cont.:

$$\begin{aligned} f_{XY}(x,y) &= \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) \\ &= \frac{\partial^2}{\partial x \partial y} F_X(x) F_Y(y) \\ &= \frac{\partial}{\partial x} F_X(x) f_Y(y) \\ &= f_X(x) f_Y(y) \end{aligned} \qquad (f_{XY} = f_X \otimes f_Y \text{ when cont.})$$

so the density factors. Conversely, if the density factors,

$$\begin{aligned} F_{XY}(x,y) &= \int_{-\infty}^{x} \int_{-\infty}^{y} f_X(s) f_Y(t) \, dt \, ds \\ &= \int_{-\infty}^{x} f_X(s) F_Y(y) \, ds \\ &= F_X(x) F_Y(y) . \end{aligned}$$

$\Rightarrow X$ and Y are ind.

Example. Bivariate standard normal with $\rho \in (-1,1)$:

$$f_{ZW}(z,w) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(z^2 - 2\rho z w + w^2)}$$

We saw: (i) $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ and $f_W(w) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2}$

(ii) Z, W are ind. iff $f_{ZW} = f_Z \otimes f_W$ (when cont.)

In particular, if Z and W are ind. then

$$\frac{1}{2\pi\sqrt{1-\rho^2}} = f_{ZW}(0,0) = f_Z(0) f_W(0) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi}$$

$\Rightarrow \rho = 0$.

Conversely, if $\rho = 0$ clearly $f_{ZW} = f_Z \otimes f_W$.

Cor. Z and W are ind. iff $\rho = 0$. (remember this)

Claim. Suppose $f_{XY}(x,y) = h(x) g(y)$, then $h = \frac{1}{c} \cdot f_X$, $g = c \cdot f_Y$ and it follows that X and Y are ind.

Proof.

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_{-\infty}^{\infty} h(x) g(y)\,dy = h(x) \underbrace{\int_{-\infty}^{\infty} g(y)\,dy}_{c}.$$

ex. finish the proof. (more than 1 line)

Suppose $X_1, \dots, X_n$ are ind. RVs then with $B_i = (-\infty, x_i]$, where $x_i \in \mathbb{R}$ we have $x \in B_i \iff x \le x_i$ so

$$P(X_1 \in B_1, \dots, X_n \in B_n) = F_{\underline{X}}(x_1, \dots, x_n)$$
$$= F_{X_1}(x_1) \dots F_{X_n}(x_n) = P(X_1 \in B_1) \dots P(X_n \in B_n)$$

This extends to any $B_i \subset \mathbb{R}$:

<u>Claim</u>. If the RVs $X_1, \dots, X_n$ are ind. then for any (Borel measurable) sets $B_i \subset \mathbb{R}$,

$$P(X_1 \in B_1, \dots, X_n \in B_n) = P(X_1 \in B_1) \dots P(X_n \in B_n).$$

<u>Proof</u>. For $n=2$ and jointly continuous X, Y with f_{XY}. Let $B_1 \times B_2 = \{(x,y) \in \mathbb{R}: x \in B_1, y \in B_2\}$, then

$$P(X \in B_1, Y \in B_2) = P((X,Y) \in B_1 \times B_2)$$
$$= \iint_{B_1 \times B_2} f_{XY}(x,y)\,dx\,dy$$
$$\overset{?}{=} \iint_{\mathbb{R}^2} 1_{B_1}(x) \cdot 1_{B_2}(y) f_X(x) f_Y(y)\,dx\,dy$$
$$\overset{?}{=} \int_{x=-\infty}^{\infty} 1_{B_1}(x) f_X(x) \int_{y=-\infty}^{\infty} 1_{B_2}(y) f_Y(y)\,dy\,dx$$
$$= \left(\int 1_{B_1}(x) f_X(x)\,dx\right)\left(\int 1_{B_2}(y) f_Y(y)\,dy\right)$$
$$= \left(\int_{B_1} f_X(x)\,dx\right)\left(\int_{B_2} f_Y(y)\,dy\right)$$
$$= P(X \in B_1) P(Y \in B_2).$$

Claim. If $X_1, \dots, X_n$ are ind. RVs and $h_i : \mathbb{R} \to \mathbb{R}$ (measurable) then with $Y_i := h_i(X_i)$, Y_i are ind. RVs.

Proof. If $h: \mathbb{R} \to \mathbb{R}$ and $A \subset \mathbb{R}$ then

$$h(x) \in A \underset{\text{tautology}}{\iff} x \in h^{-1}(A) := \{x : h(x) \in A\}$$

Therefore, with $y_i \in \mathbb{R}$,

$$\begin{aligned} P(Y_1 \le y_1, \dots, Y_n \le y_n) &= P\left(\bigcap_1^n \{h_i(X_i) \le y_i\}\right) \\ &= P\left(\bigcap_1^n \{h_i(X_i) \in (-\infty, y_i]\}\right) \\ &= P\left(\bigcap_1^n \{X_i \in \underbrace{h_i^{-1}(-\infty, y_i]}_{B_i}\}\right) \\ &= \prod_1^n P(X_i \in B_i) \\ &= \prod_1^n P(h_i(X_i) \in (-\infty, y_i]) \\ &= \prod_1^n P(Y_i \le y_i). \end{aligned}$$

$\Rightarrow$ Y_i are ind. RVs.

Cor. If U_i are iid $U(0,1)$ RVs then $Q_F(U_i)$ form an ind. sample of size n from the dist. determined by the CDF F.

Conditional Dist.

Recall: if $P(B)>0$ then $P(A|B)=\frac{P(A\cap B)}{P(B)}$.

If X and Y are discrete RVs with joint pmf p_{XY}, then for $y\in\mathbb{R}$ with $p_Y(y)>0$ we defined

$$p_{X|Y}(x|y)=\frac{p_{XY}(x,y)}{p_Y(y)}.$$

Suppose X and Y have a joint density f_{XY}.

We want to make sense of $F_{X|Y}(x|y)=P(X\le x\,|\,Y=y)$.

However, $P(Y=y)=0$ so we need to be creative.