Stat 2911 Lecture Notes

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Proof of claim on expectation of a function of a RV, variance of Bernoulli, Geometric and Poisson RVs, computing the expectation of a function of a random vector, L^1 is a vector space and expectation is a linear operator on it (Rice 4.1-4.2)

Expectation

Def. The expected value, or mean of a discrete RV X with range $X(\Omega)=\{x_k\}$ is

$$E(X)=\sum_k{}' x_k p_X(x_k) \quad (*)$$

(*) provided the sum is well defined. This is the case if $X\geq 0$, or $X\leq 0$, or $X\in L^1$: $\sum_k{}'|x_k|p_X(x_k)<\infty$.

For $X\in L^1$ we can define $V(X)=E(X-\mu)^2$, where $\mu=E(X)$ and

$$V(X)=E(X^2)-E^2(X)$$

$$V(X)<\infty \iff E(X^2)=\sum{}' x_k^2 p_X(x_k)<\infty \iff X\in L^2$$

Claim. (expectation of a function of a RV)

If X is a discrete RV with $X(\Omega)=\{x_k\}$, $g:\mathbb{R}\to\mathbb{R}$ then:

(i) $Y=g(X)\in L^1$ iff $\sum_k{}'|g(x_k)|p_X(x_k)<\infty$.

(ii) $Y\in L^1 \Rightarrow E(Y)=\sum_k{}' g(x_k)p_X(x_k)$.

same if $Y\geq 0$ or $Y\leq 0$.

7.2

(i) $Y = g(\underline{X}) \in L^1$ iff $\sum_k' |g(x_k)| p_{\underline{X}}(x_k) < \infty$.

(ii) $Y \in L^1 \implies E(Y) = \sum_k' g(x_k) p_{\underline{X}}(x_k)$.

Proof. Let $\underbrace{\{y_n\}}_{\text{distinct}} = Y(\Omega) = \underbrace{\{g(x_k)\}}_{\text{not necessarily distinct}}$.

$$Y(\omega) = y \iff g(\underline{X}(\omega)) = y$$
$$\iff \underline{X}(\omega) \in g^{-1}(y) := \{x : g(x) = y\}$$

$$\Rightarrow P(Y = y) = P(\underline{X} \in g^{-1}(y))$$
$$= \sum_{k : g(x_k) = y}' p_X(x_k)$$

$$\Rightarrow \sum_n' |y_n| p_Y(y_n) = \sum_n' |y_n| \sum_{k : g(x_k) = y_n}' p_X(x_k)$$
$$= \sum_n' \sum_{k : g(x_k) = y_n}' |g(x_k)| p_X(x_k)$$
$$\overset{(*)}{=} \sum_k' |g(x_k)| p_X(x_k)$$

(*) holds because:

(i) each x_k appears exactly once in both expressions

(ii) the series is positive

Fubini's Theorem

(a) A positive series can be summed up in any order.

$\Rightarrow Y \in \mathcal{L}^1$ iff $\sum_k |g(x_k)| p_X(x_k) < \infty$.

(ii) If $Y \in \mathcal{L}^1$, then repeating the lines above without the $|\cdot|$ proves

$$E(Y) = \sum_k g(x_k) p_X(x_k).$$

The only difference is that in justifying (*), the series is no longer positive. However, the series converges absolutely therefore the terms can still be summed up in any order.

Fubini's Theorem (cont.)

A series $\sum a_n$ is absolutely convergent if $\sum |a_n| < \infty$.

(b) An absolutely convergent series can be summed up in any order.

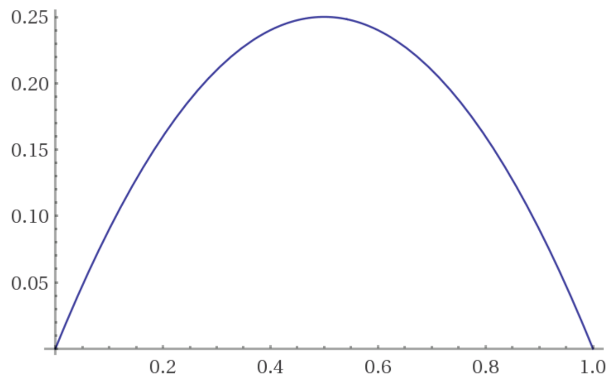
Examples (of variance)

1) $X \sim \text{Bernoulli}(p)$; $E(X) = p$

$$E(X^2) = 0^2 \cdot (1-p) + 1^2 \cdot p = p$$

$$\Rightarrow V(X) = E(X^2) - E^2(X) = p - p^2 = p(1-p)$$

$V(X)$



p

For $p \in \{0,1\}$ $V(X) = 0$

$V(X)$ is maximized for $p = \frac{1}{2}$

Binomial : later

2) $X \sim$ Geometric : $E(X) = 1/p$ $\quad (p \in (0,1))$

$$E(X^2) \overset{?}{=} \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p = \sum_{k=1}^{\infty} k^2 q^{k-1} p \quad (q = 1-p)$$

Need: $\sum_{k=1}^{\infty} k^2 q^{k-1} = ?$ $\qquad |q| < 1$

$k^2 = k(k-1) + k$ therefore,

$$\sum_{k=1}^{\infty} k^2 q^{k-1} \overset{?}{=} \sum_{k=1}^{\infty} k(k-1) q^{k-1} + \underbrace{\sum_{k=1}^{\infty} k q^{k-1}}_{= 1/(1-q)^2}$$

eventually ok for $|q| < 1$

$$\sum_{k=1}^{\infty} k(k-1) q^{k-1} = q \sum_{k=0}^{\infty} k(k-1) q^{k-2}$$

$$= q \sum_{k=0}^{\infty} \frac{d^2}{dq^2} (q^k)$$

$$\overset{?}{=} q \frac{d^2}{dq^2} \left(\sum_{k=0}^{\infty} q^k \right)$$

power series $|q| < 1$

$$= q \frac{d^2}{dq^2} \left(\frac{1}{1-q} \right) = \frac{2q}{(1-q)^3}$$

$$\Rightarrow E(X^2) = p \left(\frac{2q}{p^3} + \frac{1}{p^2} \right) = \frac{2-p}{p^2}$$

$$\Rightarrow V(X) = E(X^2) - E^2(X)$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

3) $\underline{X} \sim \text{Poisson}(\lambda)$; $\quad E(\underline{X}) = \lambda \quad (\lambda > 0)$

$$E(\underline{X}^2) = \sum_{k=0}^{\infty} k^2 e^{-\lambda} \lambda^k / k!$$

$$= \lambda \sum_{k=1}^{\infty} k\, e^{-\lambda} \lambda^{k-1} / (k-1)! \qquad k = (k-1)+1$$

$$\overset{?}{=} \lambda \left[\sum_{k=1}^{\infty} (k-1) e^{-\lambda} \lambda^{k-1} / (k-1)! + \sum_{k=1}^{\infty} e^{-\lambda} \lambda^{k-1} / (k-1)! \right]$$

$m = k-1$

$$= \lambda \left[\sum_{m=0}^{\infty} m e^{-\lambda} \lambda^m / m! + \sum_{m=0}^{\infty} e^{-\lambda} \lambda^m / m! \right]$$

$$= \lambda(\lambda+1)$$

$$\Rightarrow V(\underline{X}) = E(\underline{X}^2) - E^2(\underline{X})$$

$$= \lambda(\lambda+1) - \lambda^2$$

$$= \lambda$$

For $\text{Pois}(\lambda)$, $\quad E(\underline{X}) = V(\underline{X}) = \lambda$.

Goal: find $E(X+Y) = ?$

Does $X+Y$ have expectation?

Suppose $X_1, \dots, X_n$ are jointly-distributed RVs, i.e., for each i, $X_i : \Omega \to \mathbb{R}$ is a discrete RV and $\exists\, p_{\underline{X}} : \mathbb{R}^n \to [0,1]$ s.t. $\forall\, \underline{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$

$$P(\underline{X} = \underline{x}) := P(X_1 = x_1, \dots, X_n = x_n) = p_{\underline{X}}(x_1, \dots, x_n) =: p_{\underline{X}}(\underline{x})$$

We say that $\underline{X} = (X_1, \dots, X_n)$ is an n-dimensional <u>random vector</u> with pmf $p_{\underline{X}}$. (we saw the case $n=2$)

Let $g : \mathbb{R}^n \to \mathbb{R}$ and let $Z = g(\underline{X})$

<u>Claim.</u>

(i) $Z \in L^1$ iff $\displaystyle\sum_{\substack{\underline{x} \in \mathbb{R}^n : \\ p_{\underline{X}}(\underline{x}) > 0}} |g(\underline{x})|\, p_{\underline{X}}(\underline{x}) < \infty$

(ii) If $Z \in L^1$ then $\displaystyle E(Z) = \sum_{\substack{\underline{x} \in \mathbb{R}^n : \\ p_{\underline{X}}(\underline{x}) > 0}} g(\underline{x})\, p_{\underline{X}}(\underline{x})$

(or $Z \geq 0$, or $Z \leq 0$)

<u>Proof.</u> Repeat the proof for $n=1$.

Cor. If X & Y are jointly distributed RVs with $X, Y \in L^1$ and $\alpha, \beta \in \mathbb{R}$ then

(i) $\alpha X + \beta Y \in L^1$

(i) plus $X \equiv 0 \in L^1$ shows L^1 is a vector space

(ii) $E(\alpha X + \beta Y) = \alpha E(X) + \beta E(Y)$

(expectation is linear regardless of dependency)

Note. If we take $Y \equiv 1$ (constant RV) then

$$E(\alpha X + \beta) = \alpha E(X) + \beta$$

scale   shift

Proof (of Cor.).

Let $g:\mathbb{R}^2 \to \mathbb{R}$ be $g(x,y) = \alpha x + \beta y$ and let

$Z = g(X,Y) = \alpha X + \beta Y$. Then,

$$\sum_{x,y}{}' |g(x,y)|\, p_{XY}(x,y) = \sum_{x,y}{}' |\alpha x + \beta y|\, p_{XY}(x,y) \qquad \left(\sum_{x_k, y_n}{}'\right)$$

$$\leq \sum_{x,y}{}' \left(|\alpha x| + |\beta y|\right) p_{XY}(x,y)$$

$$\overset{(*)}{=} \sum_{x,y}{}' |\alpha x|\, p_{XY}(x,y) + \sum_{x,y}{}' |\beta y|\, p_{XY}(x,y)$$

$$\overset{(*)}{=} \sum_{x}{}' |\alpha x| \sum_{y}{}' p_{XY}(x,y) + \sum_{y}{}' |\beta y| \sum_{x}{}' p_{XY}(x,y)$$

$$= |\alpha| \sum_{x}{}' |x|\, p_X(x) + |\beta| \sum_{y}{}' |y|\, p_Y(y) \overset{?}{<} \infty$$

$X, Y \in L^1$

$$\overset{?}{\Rightarrow} Z = X + Y \in L^1 \quad \text{and}$$

$$E(Z) = E(\alpha X + \beta Y)$$

$$\overset{?}{=} \sum_{x,y}{}' (\alpha x + \beta y)\, p_{XY}(x,y)$$

$= \ldots$ repeat above lines without $|\cdot|$: "$\leq$" $\leftrightarrow$ "$=$"

and $(*)$ are justified by absolute convergence

$$= \alpha E(X) + \beta E(Y)$$