Stat 2911 Lecture Notes
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Comparing the two HW estimators using the delta-method (Rice 4.6)

Hardy-Weinberg equilibrium

Model: $\underline{X} = (X_1, X_2, X_3) \sim \text{multinomial}(n; (1-\theta)^2, 2\theta(1-\theta), \theta^2)$ — AA, Aa, aa

Data: $n = 1029$ $\quad x_1 = 342 \quad x_2 = 500 \quad x_3 = 187$

Goal: estimate θ, the frequency of the minor allele.

Moment estimator: $\tilde{\theta} = \sqrt{x_3/n} \overset{\text{our case}}{=} \sqrt{187/1029} \simeq 0.4263$

MLE (also moment): $\hat{\theta} = \dfrac{x_2 + 2x_3}{2n} \overset{\text{our case}}{\simeq} 0.4247$

$E_\theta(\hat{\Theta}) = \theta \Rightarrow \hat{\Theta}$ is unbiased

$E(\tilde{\Theta}) = \sum_{k=0}^{n} \sqrt{k/n} \binom{n}{k} (\theta^2)^k (1-\theta^2)^{n-k} \le \theta$ $\quad (< \text{ if } \theta \notin \{0,1\})$

$MSE(\hat{\Theta}_n) = \frac{1}{2n}\theta(1-\theta) \xrightarrow[n]{} 0$ so $\hat{\Theta}$ is consistent

$MSE(\tilde{\Theta}) = 2\theta[\theta - E(\tilde{\Theta})]$

For any $\theta \in (0, 1/2)$ $\exists N = N(\theta)$ s.t. $\forall n \ge N$

(1) $\quad \dfrac{MSE(\tilde{\Theta}_n)}{MSE(\hat{\Theta}_n)} \ge 3/2 \quad$ ($\simeq 1.68$ in our case)

Cor. For any $\theta \in (0, 1/2)$ and any n sufficiently large

$$MSE(\hat{\Theta}_{2/3 n}) \le MSE(\tilde{\Theta}_n)$$

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Using the δ-method or, propagation of errors (Rice 4.6)
we will deduce the asymptotic behavior of
$E(\tilde{\Theta}_n) = E(\sqrt{X_3/n})$ (hard to compute) from
$E(\tilde{\Theta}_n^2) = E(X_3/n) =$ (easy: $X_3 \sim \text{binom}(\quad)$)

Let X be an L^2-RV, $g: \mathbb{R} \to \mathbb{R}$ smooth and let
$Y = g(X)$.

In our case, $X = \tilde{\Theta}^2 = X_3/n$ and $g(x) = \sqrt{x}$

$$Y = \sqrt{X} = \tilde{\Theta}$$

Expectation is additive, so approximate g with a simpler sum.
Consider the second order Taylor expansion of g
about $x_0 = \mu = E(X)$: ($= \theta^2$ in our case)

$$g(x) = g(\mu) + g'(\mu)(x-\mu) + \tfrac{1}{2}g''(\mu)(x-\mu)^2 + R_2(x),$$

where $R_2(x)/(x-\mu)^2 \to 0$ as $x \to \mu$.

$$\Rightarrow g(X) = g(\mu) + g'(\mu)(X-\mu) + \tfrac{1}{2}g''(\mu)(X-\mu)^2 + R_2(X).$$

(equation between RVs)

If $X \simeq \mu$ then

$$Y = g(X) \simeq g(\mu) + g'(\mu)(X-\mu) + \tfrac{1}{2}g''(\mu)(X-\mu)^2$$

and therefore

$$E(Y) \simeq g(\mu) + \tfrac{1}{2}g''(\mu)\sigma^2 \quad \text{where } \sigma^2 = V(X).$$

In our case, $g(x) = \sqrt{x}$

$$\Rightarrow \quad g'(x) = \tfrac{1}{2} x^{-1/2} \qquad g''(x) = -\tfrac{1}{4} x^{-3/2}$$

$$\mu = E(\bar{X}) = E(\tilde{\Theta}^2) = \theta^2$$

$$\sigma^2 = V(\tilde{\Theta}^2) = \frac{1}{n^2} V(X_3) = \frac{\theta^2(1-\theta^2)}{n}$$

Using the δ-method, we get

$$E(\tilde{\Theta}_n) \simeq \sqrt{\theta^2} + \frac{1}{2}\left[-\frac{1}{4}(\theta^2)^{-3/2}\right] \frac{\theta^2(1-\theta^2)}{n}$$

$$= \theta - \frac{1}{8n} \frac{1-\theta^2}{\theta} \qquad \text{(need } \theta > 0\text{)}$$

Example. $\theta = 1/\sqrt{2} \simeq 0.7071$ with $n = 1000$

$E(\tilde{\Theta}) \simeq 0.7070$, compared with $\theta - \frac{1}{8n}\frac{1-\theta^2}{\theta} \simeq 0.7070$

$$\Rightarrow \quad E(\tilde{\Theta}_n) - \theta \simeq -\frac{1}{8n} \frac{1-\theta^2}{\theta} \xrightarrow[n]{} $$

so the negative bias vanishes as $n \to \infty$.

$$\Rightarrow \quad MSE(\tilde{\Theta}_n) = 2\theta[\theta - E(\tilde{\Theta})]$$

$$\simeq \frac{1-\theta^2}{4n} \xrightarrow[n]{}$$

So $\tilde{\Theta}$ is apparently also a estimator.

In addition,

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$$\frac{MSE(\tilde{\Theta}_n)}{MSE(\hat{\Theta}_n)} \simeq \frac{(1-\theta^2)/4n}{\theta(1-\theta)/2n}$$

$$= \frac{1+\theta}{2\theta}$$

$$= \frac{1}{2} + \frac{1}{2\theta}$$

$$> 3/2 \quad \text{if } \theta \in (0, 1/2)$$

Recall that $\theta < 1/2$ is expected since we are studying the minor allele.

WARNING: this is not rigorous analysis since "$\simeq$" is ambiguous.

Sanity check: the δ-method is based on a Taylor expansion of $g(x)$ ($\sqrt{x}$ in our case) about $x = E(X)$. That is, the analysis is predicated on $X \simeq E(X)$.

In our case $X = \hat{\Theta}_n^2 = X_3/n$ where $X_3 \sim \text{Binom}(n, \theta^2)$.

Is $X_3/n \simeq E(X) = \theta^2$ when n is large?

The LLN guarantees that $X_3/n \xrightarrow[n]{} \theta^2$, so $X_3/n \simeq E(X)$ for large n is plausible.

Still, a more rigorous analysis is generally preferable.

Claim. For $\Theta \in (0,1)$,

$$E(\tilde{\Theta}_n) = \underbrace{\Theta - \frac{1-\Theta^2}{8n\Theta}}_{\text{same as } \delta\text{-method}} + R_n,$$

where $nR_n \xrightarrow[n]{} 0$

The difference with the δ-method is that here we have control over the error, R_n.

Cor. 1 $MSE(\tilde{\Theta}_n) = \underbrace{\frac{1-\Theta^2}{4n}}_{\text{same as } \delta\text{-method}} + R_n'$ where $nR_n' \xrightarrow[n]{} 0$

Proof (of Cor 1)

$$\begin{aligned} MSE(\tilde{\Theta}_n) &= 2\Theta\left[\Theta - E(\tilde{\Theta}_n)\right] \\ &= 2\Theta\left[\frac{1-\Theta^2}{8n\Theta} - R_n\right] \\ &= \frac{1-\Theta^2}{4n} \underbrace{- 2\Theta R_n}_{R_n'} \end{aligned}$$

$$nR_n' = -2\Theta n R_n \xrightarrow[n]{} .$$

□

Note: this rigorously establishes the consistency of $\tilde{\Theta}$,

Cor. 2 For $\theta \in (0,1)$,

$$\lim_{n \to \infty} \frac{MSE(\tilde{\theta}_n)}{MSE(\hat{\theta}_n)} = \underbrace{\frac{1}{2} + \frac{1}{2\theta}}_{\delta\text{-method}}$$

$$> 3/2 \text{ if } \theta \in (0, 1/2)$$

Proof (of Cor 2)

$$\frac{MSE(\tilde{\theta}_n)}{MSE(\hat{\theta}_n)} = \frac{(1-\theta^2)/4n + R_n^1}{\theta(1-\theta)/2n}$$

$$= \frac{1+\theta}{2\theta} + \underbrace{\frac{2}{\theta(1-\theta)} \cdot n R_n^1}_{\xrightarrow[n]{}}$$

$\square$

Cor. 3 For $\theta \in (0, 1/2)$ $\exists N_\theta \in \mathbb{N}$ s.t. $\forall n \geq N_\theta$

$$\frac{MSE(\tilde{\theta}_n)}{MSE(\hat{\theta}_n)} > 3/2$$

Proof (of Cor 3)

$$\lim_{n \to \infty} \frac{MSE(\tilde{\theta}_n)}{MSE(\hat{\theta}_n)} > 3/2.$$

$\square$

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<u>Claim</u>. For $\theta \in (0,1)$, $E(\tilde{\Theta}_n) = \theta - \frac{1-\theta^2}{8n\theta} + R_n$,
where $nR_n \xrightarrow[n]{} 0$.

<u>Proof</u>.

Recall the second order Taylor expansion of g about x_0 that we used in the δ-method:

$$g(x) = \underbrace{g(x_0) + g'(x_0)(x-x_0) + \frac{1}{2} g''(x_0)(x-x_0)^2}_{T_2(x) = T_2(g;x_0)(x)} + \underbrace{R_2(x)}_{R_2(g;x_0)(x)}$$

$$R_2(x) = g(x) - T_2(x) = \frac{g^{(3)}(\xi_x)}{3!}(x-x_0)^3,$$

$\boxed{R_2 \text{ was ignored in the } \delta\text{-method!}}$

where ξ_x is between x and x_0 (Lagrange Remainder).

In our case, $g(x) = \sqrt{x}$ and $x_0 = \theta^2 > 0$, so
$g'(x) = \frac{1}{2}x^{-1/2}$, $g''(x) = -\frac{1}{4}x^{-3/2}$, $g^{(3)}(x) = \frac{3}{8}x^{-5/2}$ and

$$\sqrt{x} = \underbrace{\theta + \frac{1}{2\theta}(x-\theta^2) - \frac{1}{8\theta^3}(x-\theta^2)^2}_{T_2(x)} + R_2(x)$$

Plugging in $x = \tilde{\Theta}_n^2 = X_3/n$: (like we did in the δ-method)

$$\tilde{\Theta}_n = \theta + \frac{1}{2\theta}\left(X_3/n - \theta^2\right) - \frac{1}{8\theta^3}\left(X_3/n - \theta^2\right)^2 + R_2\left(X_3/n\right)$$

Using $X_3 \sim \text{Binom}(n, \theta^2)$ and taking expectations of both sides of this RVs equation, we get

$$E(\tilde{\Theta}_n) \simeq \theta - \frac{1}{8\theta^3}\frac{\theta^2(1-\theta^2)}{n} + E\left[R_2\left(\bar{X}_3/n\right)\right]$$

$$= \underbrace{\theta - \frac{1-\theta^2}{8n\theta}}_{\text{same as } \delta\text{-method}} + \underbrace{E\left[R_2\left(\bar{X}_3/n\right)\right]}_{R_n},$$

where $R_2(x) = \frac{g^{(3)}(\xi_x)}{6}(x-\theta^2)^3$ (ξ_x is between x and θ^2)

$$= \frac{1}{16}\xi_x^{-5/2}(x-\theta^2)^3$$

We need to show $nR_n \xrightarrow[n]{} 0$

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