Stat 2911 Lecture Notes

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Examples of functions of RVs (Rice 2.3), Joint distribution function/density, Marginal density (Rice 3.1, 3.3)

Examples 1) $U \sim U(0,1)$ and $V = 1/U$, find f_V.

$V = g(U)$ where $g(u) = 1/u$ is smooth and strictly ↓ on $(0,1) = U(\Omega)$.

Therefore, by the theorem $f_V(v) = f_U(g^{-1}(v))\,|{g^{-1}}'(v)|$.

$v = 1/u \Rightarrow u = 1/v$ so $g^{-1}(v) = 1/v$ and

$$f_V(v) = f_U(1/v)\left|-1/v^2\right| = \begin{cases} 1/v^2 & v > 1 \\ 0 & v \le 1 \end{cases}.$$

Alternatively, for $v > 1$,

$$\begin{aligned} F_V(v) &= P(V \le v) \\ &= P(1/U \le v) \\ &= P(1/v \le U) \qquad (U, v > 0) \\ &= 1 - P(U < 1/v) \\ &= 1 - 1/v. \end{aligned}$$

$$\Rightarrow f_V(v) = \frac{d}{dv}(1 - 1/v) = 1/v^2$$

2) $Z \sim N(0,1)$, $Y = Z^2 = g(Z)$ where $g(z) = z^2$.

$\Rightarrow g^{-1}(y) = \sqrt{y}$

$$\Rightarrow \quad f_Y(y) \underset{y>0}{=} f_Z(\sqrt{y}) \left| \frac{1}{2\sqrt{y}} \right|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi y}} e^{-y/2}.$$

Sanity check:

$$\int_{-\infty}^{\infty} f_Y(y)\,dy = \frac{1}{2}\int_0^{\infty} \frac{1}{\sqrt{2\pi y}} e^{-y/2}\,dy \qquad \left[\begin{array}{c} \sqrt{y} = t \\ \frac{dy}{2\sqrt{y}} = dt \end{array} \right]$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$$

$$= 1/2 \ \ldots$$

What has gone wrong?!

$g(z) = z^2$ is not strictly monotone on $\mathbb{R} = Z(\Omega)$.

2) $Z \sim N(0,1)$ $\quad Y = Z^2$

Instead of using the formula we compute it directly.

For $y > 0$: $F_Y(y) = P(Y \le y)$

$$= P(Z^2 \le y)$$

$$= P(-\sqrt{y} \le Z \le \sqrt{y})$$

$$= P(-\sqrt{y} < Z \le \sqrt{y})$$

$$= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}),$$

where $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$.

$\Rightarrow F_Y$ is diff. and by chain rule, for $y > 0$

$$f_Y(y) = \Phi'(\sqrt{y}) \frac{1}{2\sqrt{y}} - \Phi'(-\sqrt{y})\left(-\frac{1}{2\sqrt{y}}\right)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{1}{2\sqrt{y}}$$

$$= \frac{(1/2)^{1/2}}{\sqrt{\pi}} y^{1/2 - 1} e^{-y/2}.$$

Recall the $\Gamma(\alpha, \lambda)$ density: $\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} 1_{x>0}$

$\Rightarrow Y = Z^2 \sim \Gamma(1/2, 1/2)$ aka χ^2_1.

$\Rightarrow \Gamma(1/2) = \sqrt{\pi}$.

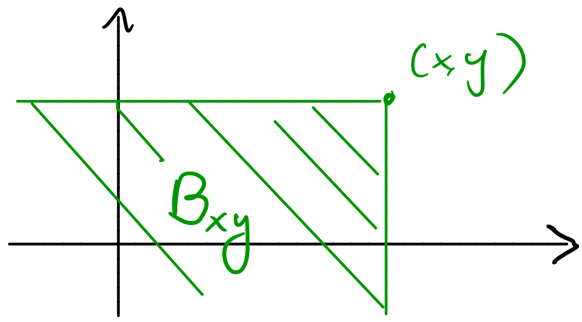
Joint Distributions

(discrete analogue: joint pmf)

The joint dist. of two RVs (a random point in $\mathbb{R}^2$) is determined by the joint dist. function:

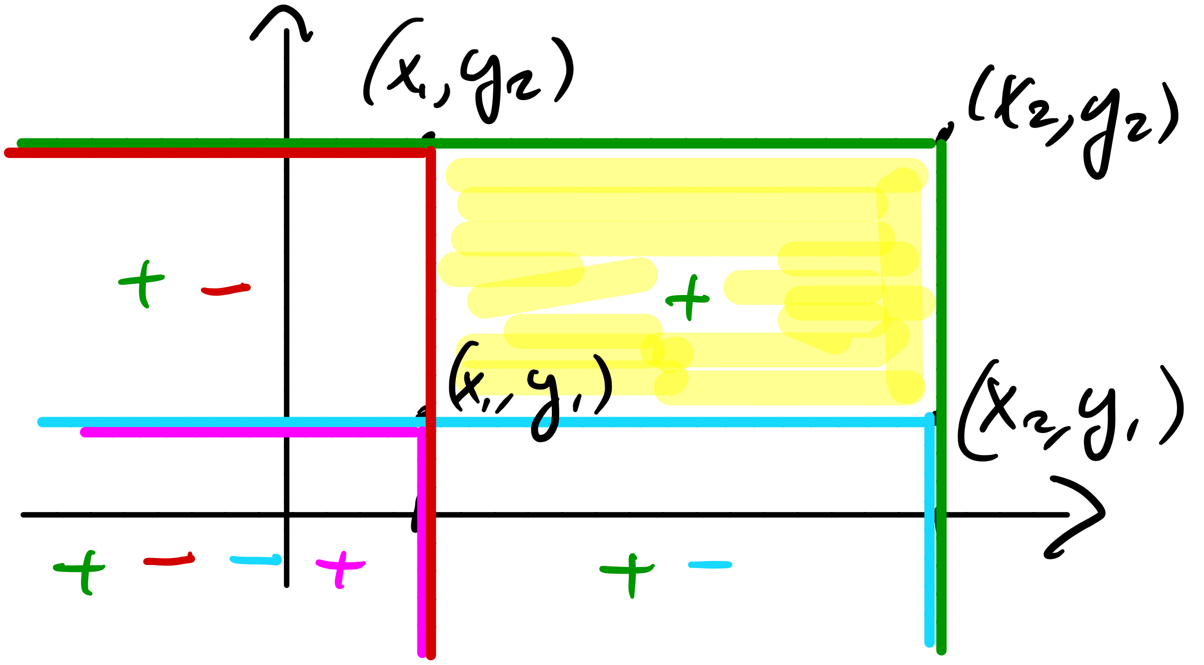
$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

$$B_{xy} = \{(s,t) : s \le x,\ t \le y\}$$



$$F_{XY}(x,y) = P((X,Y) \in B_{xy})$$

F_{XY} specifies the probability that $(X,Y) \in (x_1,x_2] \times (y_1,y_2]$:



$$P((X,Y) \in (x_1,x_2] \times (y_1,y_2]) = F(x_2,y_2) - F(x_1,y_2) - F(x_2,y_1) + F(x_1,y_1)$$

It follows (with effort) that for any Borel measurable $B \subset \mathbb{R}^2$, $P((X,Y) \in B)$ is specified by $F_{XY} = F$.

Def. X and Y have a joint density function if $\exists f: \mathbb{R}^2 \to \mathbb{R}^+$ s.t. $\forall (x,y) \in \mathbb{R}^2$:

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\,dt\,ds = \iint_{B_{xy}} f(s,t)\,ds\,dt$$

If X and Y have a joint density then they are jointly continuous.

Comments.

(i) If $f: \mathbb{R}^2 \to \mathbb{R}^+$ is integrable with $\iint_{\mathbb{R}^2} f = 1$, then $F(x,y) := \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\,dt\,ds$ is the joint CDF of jointly cont. RVs with the joint pdf f.

Example. $f(x,y) = \lambda^2 e^{-\lambda(x+y)} 1_{x>0} 1_{y>0}$ $(\lambda > 0)$.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^{\infty} \lambda e^{-\lambda x}\,dx \int_0^{\infty} \lambda e^{-\lambda y}\,dy$$

$$= 1$$

$\Rightarrow f$ is a joint pdf of RVs with joint CDF

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\,dt\,ds \qquad x,y > 0$$

$$= \int_0^{x} \lambda e^{-\lambda s}\,ds \int_0^{y} \lambda e^{-\lambda t}\,dt$$

$$= (1-e^{-\lambda x})(1-e^{-\lambda y})$$

(ii) By Tonelli's Theorem iterated/repeated integral of a non-negative function, is the same as the double integral:

$$\int_{-\infty}^{x}\int_{-\infty}^{y} f(s,t)\,dt\,ds = \iint_{B_{xy}} f(s,t)\,ds\,dt = \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt$$

Fubini's extension: if for any integrable f one of the above three integrals is finite, when f is replaced by |f|, then the equalités still hold.

(iii) From $P((X,Y)\in B_{xy}) = F_{XY}(x,y) = \iint_{B_{xy}} f\,ds\,dt \quad \forall (x,y)\in\mathbb{R}^2$

it follows that for any (Borel) set $B\subset\mathbb{R}^2$,

$$P((X,Y)\in B) = \iint_B f\,ds\,dt .$$

(iv) By the fundamental theorem of calculus, $\forall (x,y)\in\mathbb{R}^2$ where f is cont. at (x,y):

$$f(x,y) = \frac{\partial^2}{\partial x\,\partial y} F_{XY}\Big|_{(x,y)} = \frac{\partial^2}{\partial y\,\partial x} F_{XY}\Big|_{(x,y)}$$

(v) $\underline{X}$ and Y can be cont. RVs but not jointly cont.

(exercise)

(vi) If $\underline{X}$ is a cont. RV with pdf $f_{\underline{X}}$ then if $f_{\underline{X}}$ is cont. at x and $\delta > 0$ is small:

$$P\left(\underline{X} \in (x-\delta/2, x+\delta/2)\right) = \int_{x-\delta/2}^{x+\delta/2} f_{\underline{X}}(s)\,ds$$

$$\simeq f_{\underline{X}}(x)\cdot\delta \quad .$$

Similarly, if $(\underline{X}, \underline{Y})$ are are jointly cont. with pdf $f_{\underline{X}\underline{Y}}$, then if $f_{\underline{X}\underline{Y}}$ is cont. at (x,y) and $\delta, \varepsilon > 0$ are small, then

$$P\Big[(\underline{X},\underline{Y}) \in \underbrace{(x-\delta/2, x+\delta/2)\times(y-\varepsilon/2, y+\varepsilon/2)}_{R_{xy}}\Big] = \iint_{R_{xy}} f_{\underline{X}\underline{Y}}\,ds\,dt$$

$$\simeq f_{\underline{X}\underline{Y}}(x,y)\,|R_{xy}|$$

$$= f_{\underline{X}\underline{Y}}(x,y)\cdot\delta\cdot\varepsilon$$

Marginal distribution

Recall: $p_X(x) = \sum_y p_{XY}(x,y)$

Claim. If X and Y have a joint pdf f_{XY}, then X is a cont. RV and

$$f_X(x) = \int_{\mathbb{R}} f_{XY}(x,y)\,dy$$

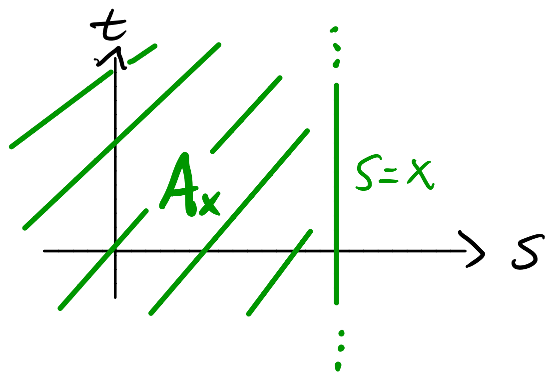
marginal density

(Same for f_Y)

Proof.

$$F_X(x) = P(X \le x)$$
$$= P(X \le x, Y < \infty)$$
$$= P((X,Y) \in A_x),$$

where $A_x = \{(s,t) \in \mathbb{R}^2 : s \le x\}$:



$$\Rightarrow F_X(x) = \iint_{A_x} f_{XY}(s,t)\,ds\,dt$$

(double integral)

$$= \int_{-\infty}^{x} \underbrace{\int_{-\infty}^{\infty} f_{XY}(s,t)\,dt}_{g(s)}\,ds$$

(Tonelli: repeated integral)

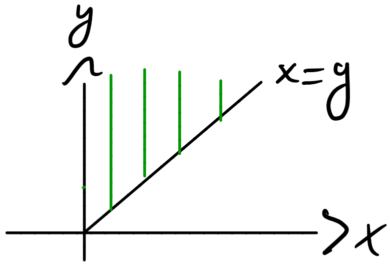
$$\Rightarrow F_X(x) = \int_{-\infty}^{x} g(s)\,ds$$

$\Rightarrow X$ is a cont. RV with pdf $f_X = g$.

$\square$

Examples

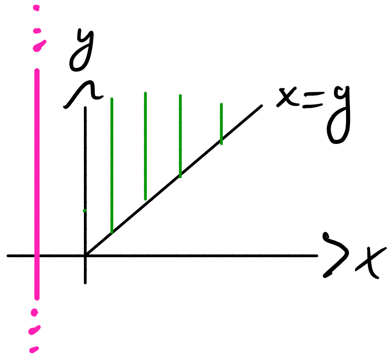
1) $f_{XY}(x,y) = \lambda^2 e^{-\lambda y}$ $\quad y > x \geq 0$, $\lambda > 0$



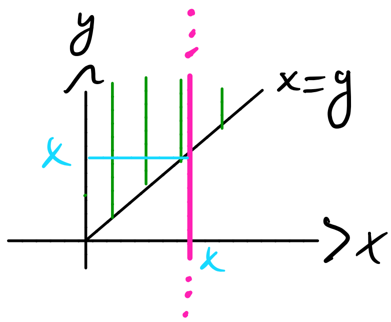
$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

Be particularly careful when plugging in the formula!

If $x < 0$, clearly $f_X(x) = 0$:



If $x \geq 0$ we have:



$$f_X(x) = \int_x^{\infty} \lambda^2 e^{-\lambda y}\,dy$$

$$= -\lambda e^{-\lambda y}\Big]_x^{\infty}$$

$$= \lambda e^{-\lambda x}.$$

$\Rightarrow X \sim \exp(\lambda)$ (not obvious from f_{XY}!)

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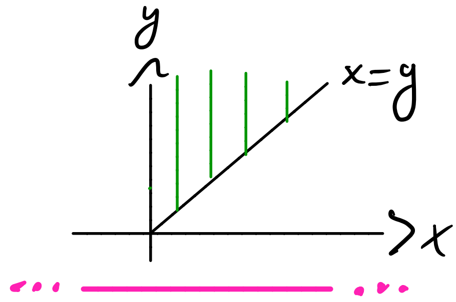
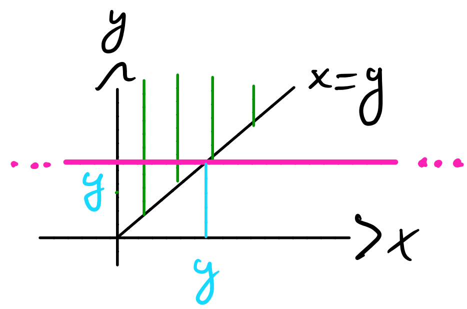
$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

If $y \le 0$, clearly $f_Y(y) = 0$:



If $y > 0$,



so
$$f_Y(y) = \int_0^y \lambda^2 e^{-\lambda y}\,dx$$
$$= y \cdot \lambda^2 e^{-\lambda y}$$

$\Rightarrow Y \sim \Gamma(\alpha = 2, \lambda)$