Stat 2911 Lecture Notes

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Density of a function of a random point, Deriving the density of a sum from a 2-d transformation, General bi-variate normal, Rayleigh density, Box-Muller sampling, Quantile-Quantile Plot (Rice 3.6, 10.2.3)

Functions of jointly cont. RVs

Thm. (density of function of a random point)
Suppose $T : \underset{\text{open}}{D} \subset \mathbb{R}^2 \longrightarrow \underset{\text{open}}{R} \subset \mathbb{R}^2$ is diff and 1:1 on D with $J_T \neq 0$, and it is onto R.

Suppose (X, Y) is jointly cont. with density f_{XY} which vanishes outside of D, and let $(U,V) = T(X,Y)$.
Then (U,V) is jointly cont. and $\forall (u,v) \in R$

$$f_{UV}(u,v) = f_{XY}(T^{-1}(u,v)) \, |J_{T^{-1}}(u,v)| .$$

Examples.

1) $(u,v) = T(x,y) = (x, x+y)$ (linear map)

$(x,y) = T^{-1}(u,v) = (u, v-u)$

$$J_{T^{-1}} = \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \left| \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right| = 1 .$$

$$\Rightarrow f_{UV}(u,v) = f_{XY}(u, v-u) \cdot 1 .$$

In particular,

$$f_V(v) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,du$$

$$= \int_{-\infty}^{\infty} f_{XY}(u, v-u)\,du$$ (looks familiar?)

Since $V = X + Y$ we rediscovered the pdf of $X+Y$.

That's a general principle:

Suppose we want the pdf of $\varphi(X,Y)$, where $\varphi: \mathbb{R}^2 \to \mathbb{R}$.

Let $(u,v) = T(x,y) = (x, \varphi(x,y))$, so

$$T^{-1}(u,v) = (u, T_y^{-1}(u,v))$$ (assuming $\exists T^{-1}$)

$$\Rightarrow J_{T^{-1}} = \left| \begin{pmatrix} 1 & 0 \\ * & \frac{\partial T_y^{-1}}{\partial v} \end{pmatrix} \right| = \frac{\partial T_y^{-1}}{\partial v}$$ (assuming T^{-1} is diff)

$$\Rightarrow f_{\varphi(X,Y)}(v) = \int_{-\infty}^{\infty} f_{XY}(u, T_y^{-1}(u,v)) \left| \frac{\partial T_y^{-1}}{\partial v} \right| du .$$

2) (Z,W) is standard bivariate normal, $\rho\in(-1,1)$:

$$f_{ZW}(z,w)=\frac{1}{2\pi\sqrt{(1-\rho^2)}}\,e^{-\frac{1}{2(1-\rho^2)}(z^2-2\rho zw+w^2)}$$

Let $U=\sigma_1 Z+\mu_1 \qquad V=\sigma_2 W+\mu_2 \qquad \sigma_i>0,\ \mu_i\in\mathbb{R}$.

That is, with $T(z,w)=(\sigma_1 z+\mu_1,\sigma_2 w+\mu_2)$ (linear)

$$(U,V)=T(Z,W).$$

Finding T^{-1} is easy here:

$$T^{-1}(u,v)=\left(\frac{u-\mu_1}{\sigma_1},\frac{v-\mu_2}{\sigma_2}\right)$$

$$\Rightarrow J_{T^{-1}}=\left|\begin{pmatrix}1/\sigma_1 & 0\\ 0 & 1/\sigma_2\end{pmatrix}\right|$$

$$=\frac{1}{\sigma_1\sigma_2}$$

$\Rightarrow$

$$f_{UV}(u,v)=\frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma_1\sigma_2}\cdot e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{u-\mu_1}{\sigma_1}\right)^2+2\rho\frac{u-\mu_1}{\sigma_1}\frac{v-\mu_2}{\sigma_2}+\left(\frac{v-\mu_2}{\sigma_2}\right)^2\right]}$$

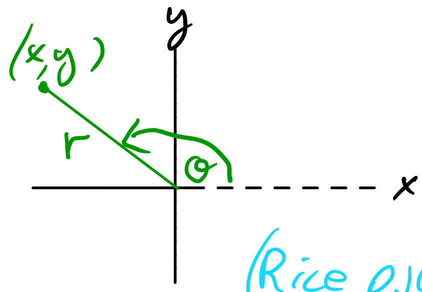
which is the <u>general</u> bivariate normal density.

Note that $U\sim N(\mu_1,\sigma_1^2)$ and $V\sim N(\mu_2,\sigma_2^2)$, (why?)

however the converse isn't true: if U and V are normally distributed it doesn't follow that they have a bivariate normal density!

3) X and Y are ind. $N(0,1)$ RVs and let (R, Θ) be the polar coord's of the Cartesian (X, Y).
More precisely, define $T: \mathbb{R}^2 \setminus \{(\mathbb{R}^+, 0)\} \to (0,\infty) \times (0, 2\pi)$ as

$$(r,\theta) = T(x,y) = \left(\sqrt{x^2+y^2},\ \tan^{-1}\left(\frac{y}{x}\right)\right)$$



(Rice p.100)

Then,
$$(R, \Theta) = T(X, Y)$$

$$T^{-1}(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

$$\Rightarrow J_{T^{-1}} = \left|\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\right| = r\cos^2\theta + r\sin^2\theta = r$$

$\Rightarrow$ For $r > 0$, $\theta \in (0, 2\pi)$

$$f_{R\Theta}(r,\theta) = f_{XY}(x,y)\, r$$

$x = x(r,\theta)$; $y = y(r,\theta)$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{2} - \frac{y^2}{2}} \cdot r$$

$$= \frac{1}{2\pi} e^{-r^2/2} \cdot r$$

$\Rightarrow$ For $\theta \in (0, 2\pi)$

$$f_\Theta(\theta) = \int_{r=0}^{\infty} \frac{1}{2\pi} e^{-r^2/2} \cdot r\, dr =$$

$\Rightarrow \Theta \sim U(0, 2\pi)$ (follows from the radial symmetry: the level sets of f_{XY} are concentric circles.)

Similarly, for $r>0$,

$$f_R(r) = \int_0^{2\pi} \frac{1}{2\pi} e^{-r^2/2} r \, d\theta = r \cdot e^{-r^2/2}$$

Rayleigh density.

Note: $f_{R\Theta}(r,\theta) = \frac{1}{2\pi} r e^{-r^2/2} \mathbb{1}_{r>0} \mathbb{1}_{\theta\in(0,2\pi)}$

$$= \left(\frac{1}{2\pi} \cdot \mathbb{1}_{\theta\in(0,2\pi)}\right)\left(r e^{-r^2/2} \mathbb{1}_{r>0}\right)$$

$$= f_\Theta(\theta) f_R(r) .$$

$\Rightarrow$ R and Θ are ind. RVs.

Not always the case but true here due to the radial symmetry.

It follows that $S := R^2 = X^2 + Y^2$ and Θ are also ind. RVs (why?), and note that for $s>0$

$$f_S(s) = f_R(\sqrt{s}) \left| \frac{d}{ds} \sqrt{s} \right|$$

$$= \sqrt{s}\, e^{-s/2} \cdot \frac{1}{2\sqrt{s}}$$

$$= \frac{1}{2} e^{-s/2} .$$

$\Rightarrow S \sim \exp(\lambda = 1/2)$.

To summarize: if X and Y are iid $N(0,1)$ RVs then $S' = X^2 + Y^2$ and $\Theta = \tan^{-1}(Y/X)$ are ind. $\exp(1/2)$ and $U(0,2\pi)$ RVs.

Conversely, if (S', Θ) are as above then $(R' = \sqrt{S'}, \Theta)$ are ind. Rayleigh and $U(0,2\pi)$ RVs and defining

$$(X', Y') = \Upsilon(R', \Theta) = T^{-1}(R', \Theta) = (R'\cos\Theta, R'\sin\Theta)$$

(X', Y') are ind $N(0,1)$ RVs. Intuitively this is clear, but we can formalize this as follows.

<u>Claim.</u> Suppose $X_1, \ldots, X_n \sim F_{\underline{X}}$ and $\underline{Y} = T(\underline{X})$ where T is a 1:1 map from $D \subset \mathbb{R}^n$ onto $R \subset \mathbb{R}^n$.

Suppose $\underline{Y} \sim G_{\underline{Y}}$ then for any $\underline{Z} \sim G_{\underline{Y}}$, $T^{-1}\underline{Z} \sim F_{\underline{X}}$.

<u>Proof.</u>

Fix $\underline{x} \in \mathbb{R}^n$, and let $B_{\underline{x}} = \{\underline{t} \in \mathbb{R}^n : t_1 \le x_1, \ldots, t_n \le x_n\}$.

$$F_{\underline{X}}(\underline{x}) = P(X_1 \le x_1, \ldots, X_n \le x_n) = P(\underline{X} \in B_{\underline{x}}) = P(\underline{Y} \in T B_{\underline{x}})$$

$$\Rightarrow P(T^{-1}\underline{Z} \in B_{\underline{x}}) \overset{?}{=} P(\underline{Z} \in T B_{\underline{x}}) \overset{?}{=} P(\underline{Y} \in T B_{\underline{x}}) = F_{\underline{X}}(\underline{x})$$

$$\Rightarrow T^{-1}\underline{Z} \sim F_{\underline{X}}.$$

Box-Muller Sampling

Suppose we want to sample from the $N(0,1)$ dist.

If we can compute $\Phi^{-1}(p)$ where $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$,

then with $U \sim U(0,1)$, $\Phi^{-1}(U) \sim N(0,1)$.

Computing Φ^{-1} is relatively costly, certainly compared with computing $F_{S'}^{-1}$ for $S' \sim \exp(\lambda)$!

$$p = 1 - e^{-\lambda s} \Rightarrow F_{S'}^{-1}(p) = -\frac{1}{\lambda} \log(1-p).$$

The Box-Muller method relies on this observation and the previous analysis to efficiently sample a pair of ind. $N(0,1)$ values:

(i) sample $\Theta \sim U(0, 2\pi)$ and $s \sim \exp(1/2)$

(ii) report $x = \sqrt{s} \cdot \cos(\Theta)$

$y = \sqrt{s} \sin(\Theta)$

Visual comparison of distributions

Since we want to compare arbitrary dists. we use CDFs rather than pmfs or pdfs.

We can compare $F_X(t)$ and $F_Y(t)$ by plotting one against the other, as we vary $t \in \mathbb{R}$.

Alternatively, if F_X^{-1} and F_Y^{-1} exist we can plot them as we vary $p \in (0,1)$.

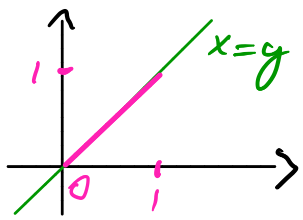
The **Q-Q (quantile-quantile) plot** of F_X and F_Y is defined as the graph

$$\{(F_X^{-1}(p), F_Y^{-1}(p)) : p \in (0,1)\}$$

Examples. 1) $F_X \equiv F_Y$. In this case $F_X^{-1} \equiv F_Y^{-1}$ so the Q-Q plot is the diagonal $x=y$:

e.g. $F_X \equiv F_Y \sim U(0,1)$

a section of



$F_X \equiv F_Y \sim N(0,1)$