Stat 2911 Lecture Notes

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Uri Keich

Continuous RVs, Uniform, exponential, Gamma and normal distributions, Quantiles (Rice 2.2)

Continuous RVs

The CDF of a (proper) RV X is $F_X(x) = P(X \le x)$.
F is non-decreasing, right-continuous, and $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.

X is a continuous RV if there exists a pdf $f \ge 0$, s.t. $F(x) = \int_{-\infty}^{x} f(t)\,dt$.

(ii) F is a cont. function, and for any x where f is cont. at x, $F'(x) = f(x)$.

(iii) If F_X, the CDF of a RV X, is cont. and it is differentiable everywhere except possibly at a finite number of points, then X is a cont. RV and $f = F_X'$ is its density.

(iv) For $b > a$, $\int_a^b f(t)\,dt = P(X \in (a,b])$.

$\Rightarrow$ For any (measurable) set $B \subset \mathbb{R}$, $P(X \in B) = \int_B f(t)\,dt$.

(v) If X is a cont. RV then $\forall x \in \mathbb{R}$, $P(X = x) = 0$.

(vi) The proof of (v) shows that for <u>any</u> RV X, $P(X = b) = F(b) - \lim_n F(b - 1/n)$. The corollary is that the CDF always determines the pmf.

For a discrete RV, the converse also holds:

$$F(x) = P(X \le x) = \sum_{k: x_k \le x} p(x_k)$$

$\{X \le x\} = \bigcup_{k: x_k \le x} \{X = x_k\}$

In general, however, the pmf does not specify the CDF. For example, for a cont. RV X, $p_X \equiv 0$.

(vii) If X is a cont. RV, then for $a<b$:

$$P(X \in [a,b]) = P(X \in (a,b)) = \int_a^b f(t)\,dt$$

(viii) <u>Claim.</u> If f is a pdf then: (I) $f \ge 0$, and (II) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Conversely, if f is integrable and f satisfies (I) & (II), then f is a pdf of a cont. RV X with the CDF $F(x) = \int_{-\infty}^{x} f(t)\,dt$.

<u>Proof.</u> If f is a pdf then (I) follows from the definition, whereas for (II):

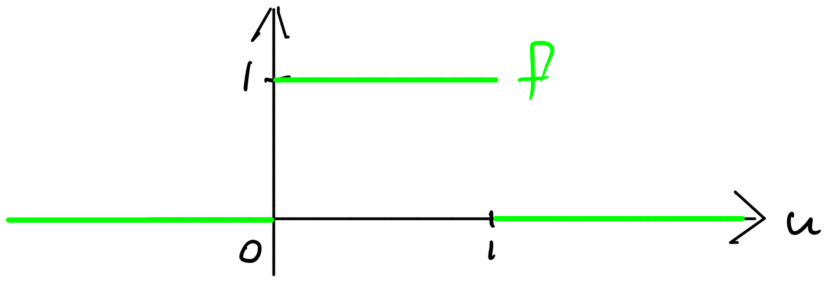
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{x\to\infty} \int_{-\infty}^{x} f(t)\,dt = \lim_{x\to\infty} F(\quad) = \quad .$$

To prove the converse, show that $F(x) \stackrel{d}{=} \int_{-\infty}^{x} f(t)\,dt$ is a CDF.

Examples. <u>Uniform distribution.</u>

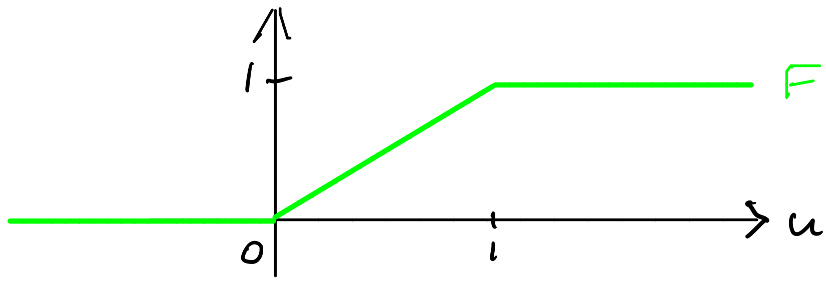
U has a uniform $(0,1)$ dist. if it has a density

$$f(u) = 1_{[0,1]}(u) = \begin{cases} 1 & u \in [0,1] \\ 0 & u \notin [0,1] \end{cases}$$



Clearly a valid density

$$\Rightarrow \quad F(u) = \int_{-\infty}^{u} f(t)\, dt = \begin{cases} & u \le 0 \\ & u \in (0,1) \\ & u \geq 1 \end{cases}$$



Note that $F'(x) = f(x)$ except $x \in \{0,1\}$.

More generally, for $a<b$, a uniform dist. on $[a,b]$ is given by the density

$$f(u) = \frac{1}{b-a} \cdot 1_{[a,b]}(u)$$

Exponential distribution

The exponential dist. is a 1-parameter family of distributions defined by the pdf

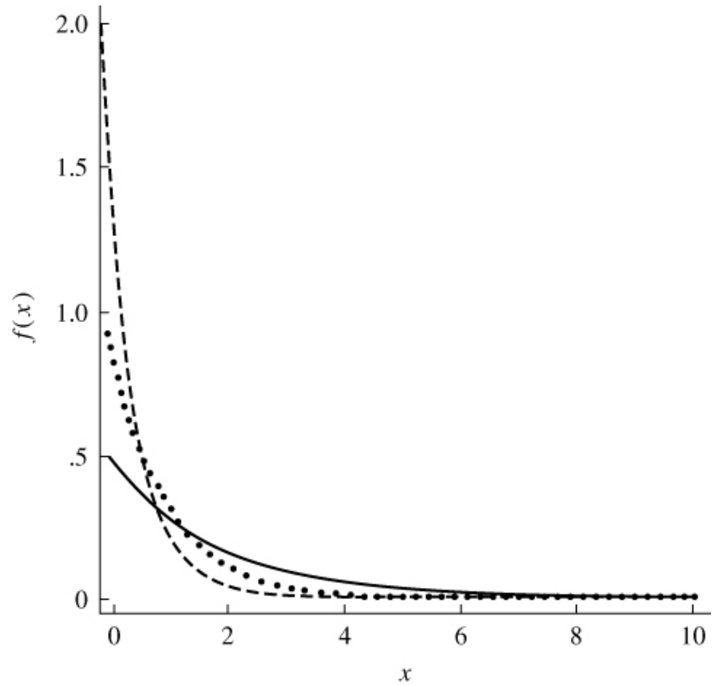
$$f_\lambda(x) = \lambda e^{-\lambda x} \cdot \mathbb{1}_{x \geq 0},$$

where $\lambda > 0$ is the rate parameter, alternatively,

$f_\beta(x) = \frac{1}{\beta} e^{-x/\beta} \cdot \mathbb{1}_{x \geq 0}$ where $\beta > 0$ is the scale parameter.

Examples: exponential densities with $\lambda \in \{1/2, 1, 2\}$.

Which is which?



As for the CDF, for $x \geq 0$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$
$$= \int_0^x \lambda e^{-\lambda t}\,dt$$
$$= -e^{-\lambda t}\Big]_0^x$$
$$= 1 - e^{-\lambda x}$$

$\Rightarrow F(x) = (1 - e^{-\lambda x})\mathbb{1}_{x \geq 0}$ and in particular,

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{x \to \infty} \int_{-\infty}^{x} f(t)\,dt = \lim_{x \to \infty} F(\ ) \overset{?}{=} \ .$$

(showing f is a density)

The exponential dist. is often used in modeling waiting times between events.

Suppose $T \sim \exp(\lambda)$, then for $t > 0$,

$$P(T > t) = 1 - P(T \leq t)$$
$$= 1 - (1 - e^{-\lambda t})$$
$$= e^{-\lambda t} \quad \text{(exponential decay)}$$

Therefore, for $t, s > 0$

$$P(T > t+s \mid T > s) = \frac{P(T > t+s, T > s)}{P(T > s)}$$
$$= \frac{P(T > t+s)}{P(T > s)}$$
$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$$
$$= e^{-\lambda t}$$
$$= P(T > t)$$

That is, the exponential dist. is memoryless (age is irrelevant).

discrete + memoryless $\Rightarrow$ geometric

continuous + memoryless $\Rightarrow$ exponential

Gamma distribution

Recall the Gamma function, defined for $\alpha > 0$ as:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

$\varphi(x) = x^{\alpha-1} e^{-x}$ has an integrable singularity at $x=0$ for $\alpha \in (0,1)$, and it is integrable at $x=+\infty$ for $\alpha \in \mathbb{R}$.

$$\Gamma(\alpha+1) = \int_0^\infty \underbrace{x^\alpha}_{v} \underbrace{e^{-x} dx}_{du} \qquad u = -e^{-x}$$

$$= -e^{-x} x^\alpha \Big]_0^\infty + \int_0^\infty e^{-x} \cdot \alpha x^{\alpha-1} dx$$

$$= 0 + \alpha \int_0^\infty e^{-x} x^{\alpha-1} dx$$

$$= \alpha \Gamma(\alpha).$$

Since $\Gamma(1) = \int_0^\infty e^{-x} dx =$, we have $\forall n \in \mathbb{N}$

$$\Gamma(n+1) = n \cdot \Gamma(n)$$

$$= n(n-1)\Gamma(n-1)$$

$$\cdots \quad \text{induction}$$

$$= n!\,\Gamma(1)$$

$$= n!$$

$\Rightarrow$ Stirling's formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

The Gamma distribution is a 2-parameter family of distributions defined by the pdf

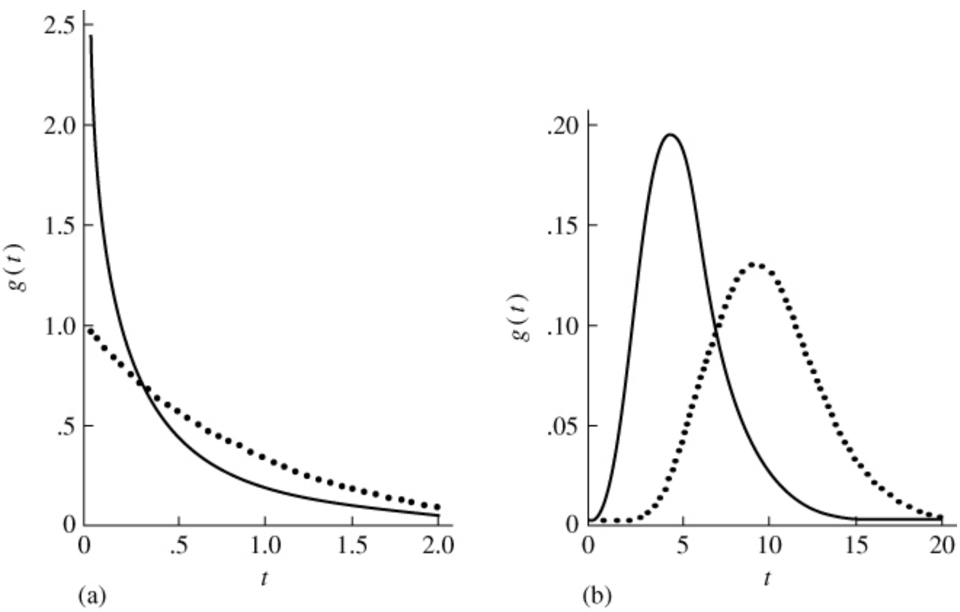
$$f_{\lambda,\alpha}(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \cdot 1_{x\geq 0},$$

where $\alpha > 0$ is the <u>shape</u> parameter, and $\lambda > 0$ is the <u>rate</u> parameter:

$$f(x)dx = \frac{(\lambda x)^{\alpha-1} e^{-(\lambda x)}}{\Gamma(\alpha)} \cdot 1_{\lambda x \geq 0}\, d(\lambda x) \quad \text{ex.} \Rightarrow \int_{-\infty}^{\infty} f_{\lambda,\alpha}(x)\, dx = 1.$$

For $\alpha = 1$, $f_{\lambda,1}(x) = \lambda e^{-\lambda x} 1_{x\geq 0}$ which is an exponential pdf, so the Gamma dist. generalizes the exponential dist.

Examples:
Gamma densities with $\lambda = 1$ and $\alpha \in \{1/2, 1\}$; $\alpha \in \{5, 10\}$



The Gamma dist. often models the time to the nth event with exponentially distributed waiting times between events. Similar to the relation between the geometric and negative binomial dists.

Normal distribution

The standard normal dist. is defined by the density

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}. \qquad \left(\text{ex. } \int_{-\infty}^{\infty} f(z)\,dz = 1\right)$$

Let Z be a standard normal RV and for $\sigma > 0$ and $\mu \in \mathbb{R}$ define $X = \sigma \cdot Z + \mu$.

$$\Rightarrow F_X(x) = P(X \le x)$$
$$= P(\sigma Z + \mu \le x)$$
$$\overset{\sigma>0}{=} P\left(Z \le \frac{x-\mu}{\sigma}\right)$$
$$= F_Z\left(\frac{x-\mu}{\sigma}\right)$$

By chain rule F_X' exists, so X is a cont. RV with a pdf

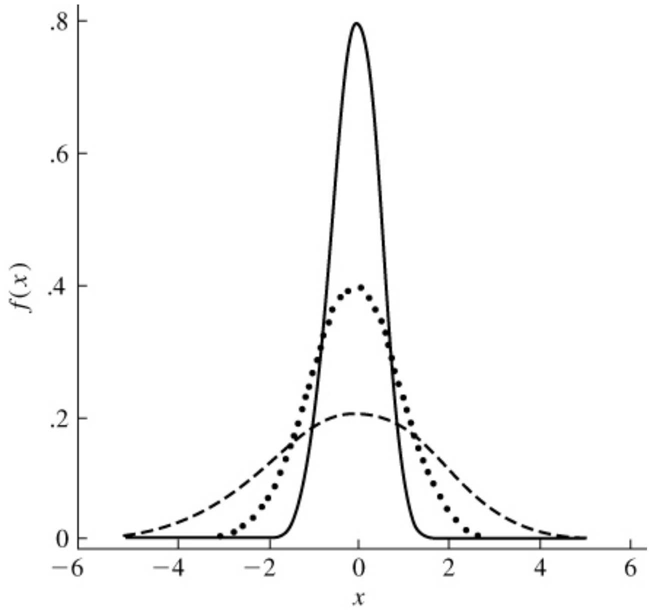
$$F_X'(x) = F_Z'\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot \frac{1}{\sigma}$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

Def. If a RV X has the above density then X has a normal $N(\mu, \sigma^2)$ distribution.

The standard normal dist. is $N(0,1)$.

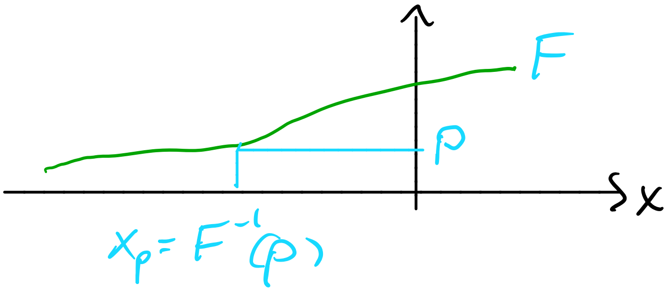
By construction, μ is a location parameter and $\sigma > 0$ is a scale parameter.

Examples.
Normal densities with $\mu = 0$
and $\sigma \in \{1/2, 1, 2\}$.
which is which?



Quantiles

Suppose F is a continuous CDF which is strictly monotone. In this case, $F^{-1}:(0,1)\to\mathbb{R}$ is well-defined:



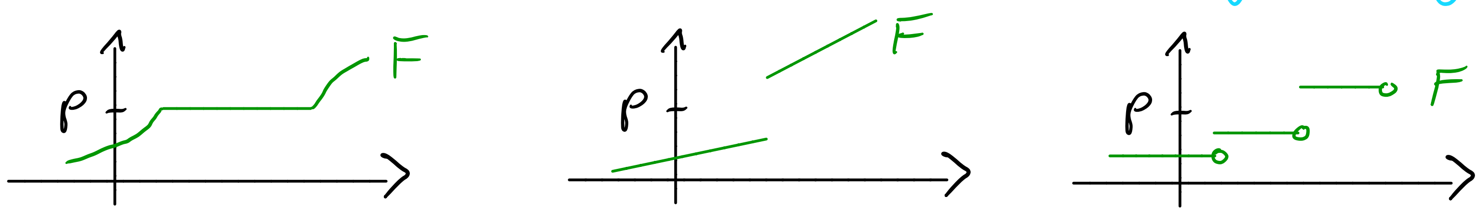
Def. The p^{th} quantile of F is $x_p = F^{-1}(p)$, or equivalently, $F(x_p) = p$.

x_p is aka the $(100p)^{th}$ percentile.

Examples

p	1/2 (50%)	1/4 (25%)	3/4 (75%)
x_p	median	lower quartile	upper quartile

What if F is not continuous and/or not strictly increasing?



Def. For $p\in(0,1)$, $x_p\in\mathbb{R}$ is a p^{th} quantile of F if

(i) $F(x_p) \geq p$

(ii) $F(x_p^-) := \lim_{x\to x_p^-} F(x) \leq p$.