Stat 2911 Lecture Notes

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Beta distribution, Density of a quotient, Distribution of a function of a random point (Rice 2.2, 3.6)

Let $Z = X + Y$ where X and Y have joint pdf f_{XY}.
Z is a cont. RV with density

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\,dx = \int_{-\infty}^{\infty} f_{XY}(z-y, y)\,dy.$$

ex: change of variable

If X and Y are ind. then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx, \text{ or}$$

$$f_Z = f_X * f_Y \quad .$$

Example. If $X \sim \Gamma(\alpha, \lambda)$ ind. of $Y \sim \Gamma(\beta, \lambda)$ then

$$X + Y \sim \Gamma(\alpha+\beta, \lambda)$$

Cor. 1

$$\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Cor. 2 If $X_1, \dots, X_n$ are iid $\exp(\lambda)$ RVs then
$\sum_1^n X_i \sim \Gamma(n, \lambda)$.
Hence, for large n $\Gamma(n, \lambda) \approx$ normal.

Def. The **Beta dist.** is a 2-parameters family of dists defined by the density

$$f_{ab}(u) = \frac{1}{\beta(a,b)} u^{a-1}(1-u)^{b-1} \, 1_{u\in(0,1)} ,$$

where $a,b>0$ are parameters.

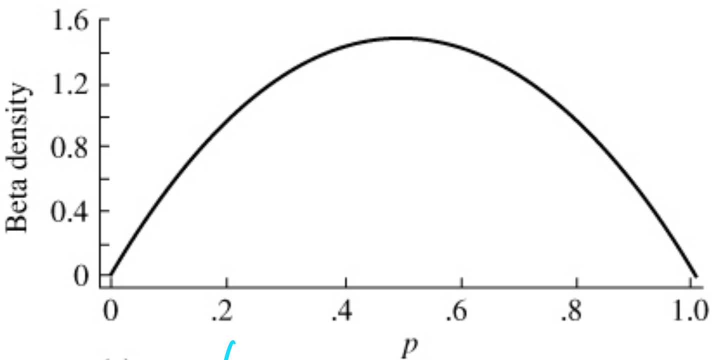
Note that $\int_0^1 f_{ab}(u)\,du = 1$ by def'n of $\beta(a,b)$.

Taking $a=b=1$,

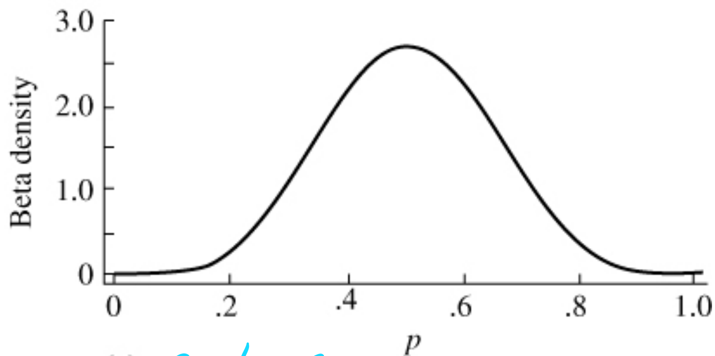
$$f(u) = c \cdot 1_{u\in(0,1)} = 1_{u\in(0,1)} ,$$

so the Beta(1,1) dist is a $U(0,1)$.

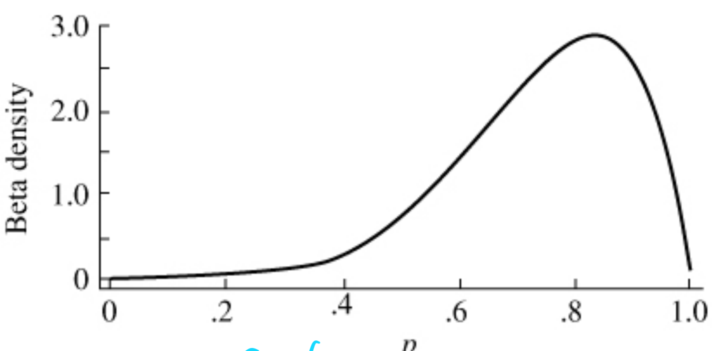
Examples of Beta densities:



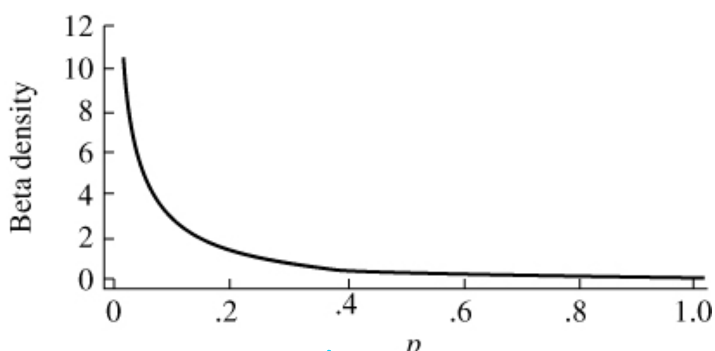
(a) $a=b=2$



(c) $a=b=6$



(b) $a=6,\ b=2$



(d) $a=0.5,\ b=4$

Quotients

Suppose X and Y have a joint density f_{XY} and let $Z = Y/X$. What is the dist. of Z? Is it ?
(since $P(X=0)=0$, Z is well defined)

Let $A_z = \{(x,y)\in\mathbb{R}^2 : y/x \le z\}$, then

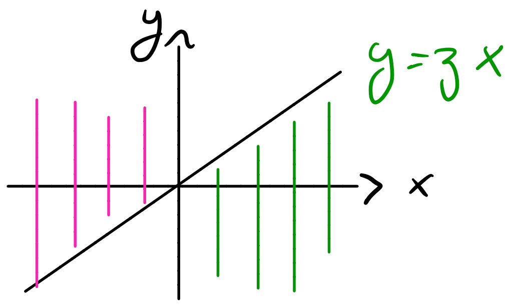
$$F_Z(z) = P(Y/X \le z) = \iint_{A_z} f_{XY}(x,y)\,dx\,dy.$$

How does A_z look like?

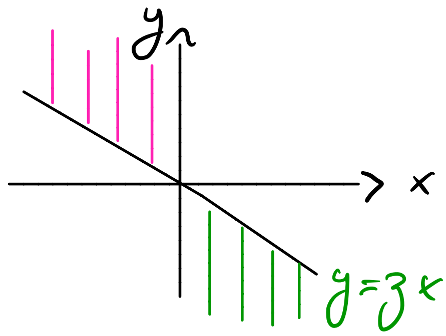
(i) $x>0$: $(x,y)\in A_z \iff y/x \le z \iff y \le z\cdot x$

(ii) $x<0$: $(x,y)\in A_z \iff y/x \le z \iff y \ge z\cdot x$

$z>0$:



$z<0$:



$$F_Z(z) = \int_{x=-\infty}^{0}\int_{y=zx}^{\infty} f_{XY}(x,y)\,dy\,dx + \int_{x=0}^{\infty}\int_{y=-\infty}^{zx} f_{XY}(x,y)\,dy\,dx$$

$x>0$: $$\int_{y=-\infty}^{zx} f_{XY}(x,y)\,dy \overset{v=y/x}{=} \int_{-\infty}^{z} f_{XY}(x, v\cdot x)\,\underset{|x|}{\underset{\shortparallel}{x}}\,dv$$

$x<0$: $\displaystyle\int_{y=zx}^{\infty} f_{XY}(x,y)\,dy \overset{v=y/x}{=} \int_{z}^{-\infty} f_{XY}(x,v\cdot x)\,x\,dv$

$$= \int_{-\infty}^{z} f_{XY}(x,v\cdot x)\underbrace{(-x)}_{=|x|}\,dv$$

Therefore

$$F_Z(z) = \int_{-\infty}^{0}\underbrace{\int_{-\infty}^{z} f_{XY}(x,vx)\,|x|\,dv}_{g(x)}\,dx + \int_{0}^{\infty}\underbrace{\int_{-\infty}^{z} f_{XY}(x,vx)\,|x|\,dv}_{g(x)}\,dx$$

$$= \int_{-\infty}^{\infty}\underbrace{\int_{-\infty}^{z} f_{XY}(x,vx)\,|x|\,dv}_{g(x)}\,dx$$

$$= \int_{-\infty}^{z}\underbrace{\left[\int_{-\infty}^{\infty} f_{XY}(x,vx)\,|x|\,dx\right]}_{h(v)}dv$$

$\Rightarrow$ $Z = Y/X$ is a cont. RV with

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x,\underbrace{zx}_{y})\,|x|\,dx \qquad y \Rightarrow y/x = z$$

Compare with

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_{XY}(x,\underbrace{z-x}_{y})\,dx \qquad y \Rightarrow x+y=z$$

Note the extra $|x|$ term for Y/X: density $\times$ length/area $=$ prob.

If X and Y are ind. then

$$f_{Y/X}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z \cdot x) |x| \, dx \, .$$

Example. X, Y are iid $N(0,1)$ RVs, $Z = Y/X$

$$f_Z(z) = \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-(zx)^2/2} |x|}_{\text{even fn.}} \, dx$$

$$= 2 \cdot \int_0^{\infty} \frac{1}{2\pi} e^{-\frac{x^2}{2}(1+z^2)} \cdot x \, dx$$

$$= \frac{1}{\pi} \left[- \frac{e^{-x^2/2(1+z^2)}}{1+z^2} \right]_0^{\infty}$$

$$= \frac{1}{\pi} \frac{1}{1+z^2}$$

This is the density of the Cauchy dist.
Decays slowly (heavy tail).

Functions of jointly cont. RVs

Recall: if g is strictly monotone and diff. and if X is a cont. RV, then $Y=g(X)$ is a cont. RV and

$$f_Y(y)=f_X(g^{-1}(y))\,|{g^{-1}}'(y)| .$$

Suppose X and Y have a joint density f_{XY}, and $T:\mathbb{R}^2\to\mathbb{R}^2$ is 1:1, onto, and diff., and define the random point

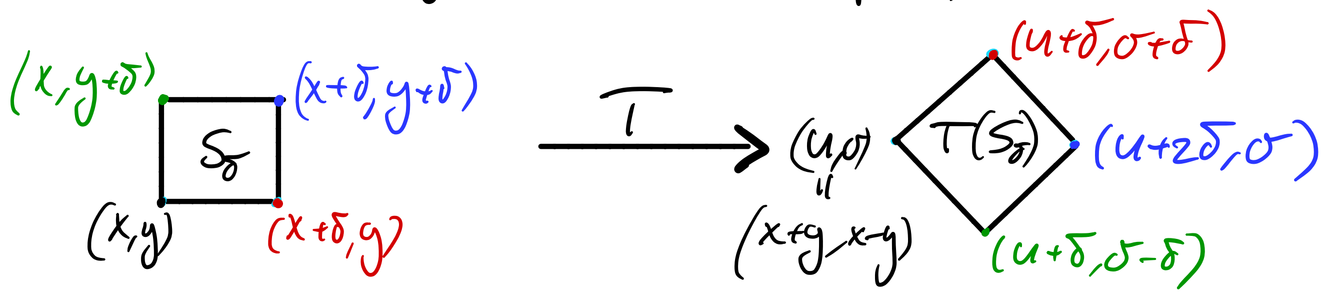
$$(U,V):=T(X,Y) .$$

For example,

$$T(x,y)=(x+y,\ x-y)$$

$$\Rightarrow \quad (U,V)=(X+Y,\ X-Y) .$$

Assume, for now, that (U,V) has a joint density f_{UV}, then with $(u,v)=T(x,y)$ in our example, and $\delta>0$



S_δ is a square with $|S_\delta|=\delta^2$.

$T(S_\delta)$ is a square with each side of length $\sqrt{\delta^2+\delta^2}=\sqrt{2}\,\delta$, so $|T(S_\delta)|=2\delta^2$.

Assume both f_{XY} and f_{UV} are cont. fns. It follows that

$$P((X,Y)\in S_\delta) = \iint_{S_\delta} f_{XY}(s,t)\,ds\,dt$$
$$= f_{XY}(x,y)|S_\delta| + R_\delta \cdot |S_\delta| ,$$

where

$$R_\delta = R_\delta(x,y) = \frac{1}{|S_\delta|}\iint_{S_\delta}\left(f_{XY}(s,t) - f_{XY}(x,y)\right)ds\,dt \xrightarrow[\delta\to 0]{\text{exercise}} 0.$$

Similarly,

$$P((U,V)\in T(S_\delta)) = \iint_{T(S_\delta)} f_{UV}(q,r)\,dq\,dr$$
$$= f_{UV}(u,v)|T(S_\delta)| + R'_\delta|T(S_\delta)|,$$

where $R'_\delta \xrightarrow[\delta\to 0]{} 0$.

Since T is 1:1,

$$(X,Y)\in S_\delta \iff (U,V) = T(X,Y) \in T(S_\delta)$$

$$\Rightarrow \quad P((X,Y)\in S_\delta) = P((U,V)\in T(S_\delta))$$

$$\Rightarrow \quad (f_{XY}(x,y) + R_\delta)\,\delta^2 = (f_{UV}(u,v) + R'_\delta)\cdot 2\delta^2$$

Divide by δ^2 and let $\delta\to 0$ to get:

$$f_{XY}(x,y) = 2f_{UV}(u,v), \quad \text{or}$$
$$f_{UV}(u,v) = \frac{1}{2} f_{XY}(T^{-1}(u,v))$$

The unit area in the x-y coordinates is 2× larger when mapped to the u-v coord's. Therefore, the density in the u-v coord's is 2× smaller.

The factor of 2 is the "blow up" factor:

$$|T(S_\delta)| / |S_\delta| = 2$$

By how much does a small unit square in the x-y coord's grow, when it is mapped under T.

It is given by the Jacobian determinant, or the determinant of the Jacobian/Derivative matrix:

$$J_T = |D_T| = \left| \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \right| ,$$

which in our case with $u = x+y$, $v = x-y$ is

$$J_T = \left| \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right| = -2 ,$$

constant! A feature of linear transformation.

and $|J_T| = 2$.

$$\Rightarrow f_{XY}(x,y) = f_{UV}(T(x,y)) \, |J_T| \quad \text{or equivalently,}$$

$$f_{UV}(u,v) = f_{XY}(T^{-1}(u,v)) \, |J_T|^{-1}$$

$$= f_{XY}(T^{-1}(u,v)) \, |J_{T^{-1}}(u,v)| .$$

Note that if $T: D \subset \mathbb{R}^n \to \mathbb{R}^n$ where D is an open set and T is diff., then if $J_T \neq 0$ at $\underline{x} \in \mathbb{R}^n$, then T is invertible in the neighborhood of $T(\underline{x})$ and by the chain rule $J_T^{-1} = J_{T^{-1}}$.

Our result generalizes as follows.

<u>Thm.</u> (density of function of a random point)

Suppose $T: \underset{\text{open}}{D} \subset \mathbb{R}^2 \to \underset{\text{open}}{R} \subset \mathbb{R}^2$ is diff and 1:1 on D with $J_T \neq 0$, and it is onto R.

Suppose (X, Y) is jointly cont. with density f_{XY} which vanishes outside of D, and let $(U,V) = T(X,Y)$.

Then (U,V) is jointly cont. and $\forall (u,v) \in R$

$$f_{UV}(u,v) = f_{XY}(T^{-1}(u,v))\, |J_{T^{-1}}(u,v)|.$$

The theorem can be proved with the help of the following general theorem in calculus.

<u>Thm.</u> If T is as above then for any open $A \subset D$, $B = T(A)$ is an open subset of R, and for any integrable $\psi: A \to \mathbb{R}$ we have

$$\iint_A \psi(x,y)\,dx\,dy = \iint_B \psi(T^{-1}(u,v))\,|J_{T^{-1}}(u,v)|\,du\,dv$$

The first (density) then follows from the second (change of variable):

Take $\psi = f_{XY}$, then for any open $B \subset \mathbb{R}$

$$P((U,V) \in B) = P((X,Y) \in A = T^{-1}(B)) \quad (T \text{ is } 1:1)$$

$$= \iint_A f_{XY}(x,y)\,dx\,dy$$

$$= \iint_{B=T(A)} f_{XY}(T^{-1}(u,v))\,|J_{T^{-1}}(u,v)|\,du\,dv$$

Since this holds for all open $B \subset \mathbb{R}$, (U,V) has a density which is again:

$$f_{UV}(u,v) = f_{XY}(T^{-1}(u,v))\,|J_{T^{-1}}(u,v)|\,.$$