Stat 2911 Lecture Notes

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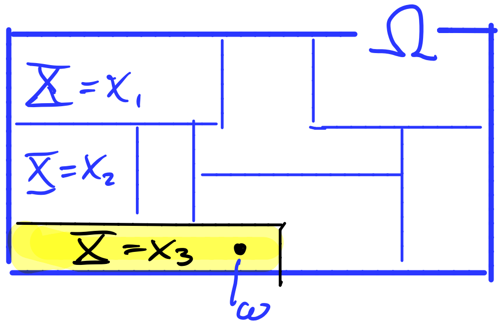
Conditional variance and the variance of the conditional expectation (Rice 4.4), Properties of CDF, Continuity of probability measure, Continuous RVs (Rice 2.2)

For $Y \in L^1$, $E(Y|X) = h(X)$, where $h(x) = E(Y|X=x)$.

For $Y \in L^2$, $V(Y|X) = g(X) - h^2(X)$, $g(x) = E(Y^2|X=x)$.



If $\omega \in \Omega$ satisfies $X(\omega) = x_3$, then

$$E(Y|X)(\omega) = E(Y|X=x_3)$$

$$V(Y|X)(\omega) = V(Y|X=x_3)$$

Thm. If $Y \in L^1$ then $E[E(Y|X)] = E(Y)$. (law of total expectation)

Random Sum: X_i are iid RVs and N is an ind. $\mathbb{N}$-valued RV. $T = \sum_1^N X_i$ is a random sum.

Claim. If $n \in N(\Omega)$, then the dist. of T given $N = n$ is the same as the dist. of $S_n = \sum_1^n X_i$.

Cor. If $X_i \in L^1$, $E(T|N=n) = n\,E(X_i)$, and if $X_i \in L^2$, $V(T|N=n) = n\,V(X_i)$

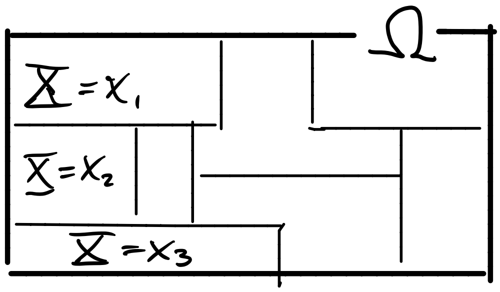
$$\Rightarrow E(T|N) \overset{X_i \in L^1}{=} N \cdot E(X_i), \quad V(T|N) \overset{X_i \in L^2}{=} N \cdot V(X_i)$$

If $X_i, N \in L^1$ then $E(T) = E[E(T|N)] = E(N)E(X_i)$.

Faulty monitor: $V(T) = \lambda p \underset{p \in (0,1)}{>} \lambda p^2 = V[E(T|N)]$

Theorem. If $y \in L^2$ then

$$V(y) = V[E(y|\underline{X})] + E[V(y|\underline{X})]$$



$V[E(y|\underline{X})]$ = the variance between the blocks/sets

$E[V(y|\underline{X})]$ = the average variance within the blocks

Proof.

$$E[V(y|\underline{X})] = E[g(\underline{X}) - h^2(\underline{X})]$$
$$= E[g(\underline{X})] - E[h^2(\underline{X})]$$

$$V[\underbrace{E(y|\underline{X})}_{h(\underline{X})}] = E[h^2(\underline{X})] - E^2[h(\underline{X})]$$

$$\Rightarrow E[V(y|\underline{X})] + V[E(y|\underline{X})]$$
$$= E[g(\underline{X})] - E^2[h(\underline{X})]$$
$$= E[E(y^2|\underline{X})] - \{E[E(y|\underline{X})]\}^2$$
$$\overset{?}{=} E(y^2) - [E(y)]^2$$
$$= V(y).$$

Back to the random sum $T = \sum_{1}^{N} X_i$, where X_i are iid L^2-RVs ind. of $N \in L^2$.

Exercise. $T \in L^2$

$$\Rightarrow \quad V(T) = V[E(T|N)] + E[V(T|N)]$$
$$= V[N \cdot E(X_i)] + E[N \cdot V(X_i)]$$
$$= E^2(X_i)\, V(N) + V(X_i)\, E(N)$$

Example. The faulty monitor: $X_i \sim \text{Bernoulli}(p)$, $N \sim \text{Poisson}(\lambda)$

$$V(T) = p^2 \cdot \lambda + p(1-p) \cdot \lambda$$
$$= \lambda p.$$

We knew that because $T \sim \text{Poisson}(\lambda p)$, but the theorem allows us to compute $V(T)$ without requiring the entire dist. of T (nor of X_i or N).

Continuous RVs

We are often interested in modeling random quantities that can take a continuous set of possible values, e.g., lifetime, tempratures, height of an individual.

The pmf cannot be used to specify such a distribution. Indeed, for such RVs we typically have $P(X=x)=0 \ \forall x$. For that we need the CDF (useful for any dist.).

Def. The cumulative distribution function (CDF) of a RV X is defined as:

$$F_X(x) = P(X \le x) .$$

Claim. If F is a CDF of a RV X with $P(X\in\mathbb{R})=1$, then:

(i) F is non-decreasing

(ii) F is right-continuous: $\lim_{x\to x_0^+} F(x) = F(x_0)$

(iii) $\lim_{x\to\infty} F(x)=1$ and $\lim_{x\to-\infty} F(x)=0$

Conversely, if F satisfies (i)-(iii) then F is the CDF of some RV X with $P(X\in\mathbb{R})=1$.

Proof. (i) $a\le b \Rightarrow \{X\le a\} \subset \{X \le b\}$

$$\Rightarrow F(a) = P(X\le a) \le P(X\le b) = F(b) .$$

To prove (ii) and (iii) we need the following lemma.

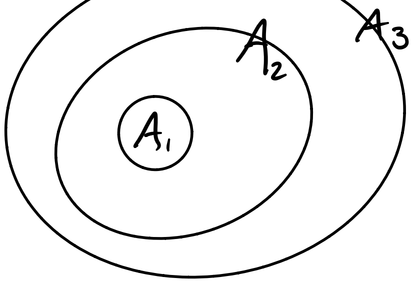
<u>Lemma</u>. (Continuity of the probability measure)

(I) If the events A_n are increasing ($A_n \uparrow$), i.e., if $A_n \subset A_{n+1}$, then $P(\bigcup_n A_n) = \lim_n P(A_n)$.

(II) If $A_n \downarrow$, i.e., if $A_n \supset A_{n+1}$, then $P(\bigcap_n A_n) = \lim_n P(A_n)$.

<u>Proof</u> (of lemma).

$A_n \uparrow$:



Define:

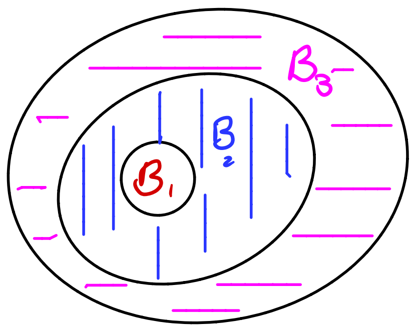
$B_1 = A_1$

$B_2 = A_2 \setminus A_1$

$B_3 = A_3 \setminus A_2$

$\vdots$

$B_n = A_n \setminus A_{n-1}$



B_n are disjoint events and $\bigcup_{k=1}^{n} B_k =$.

$$\Rightarrow \quad P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$= \sum_1^{\infty} P(B_n)$$

$$= \lim_{N \to \infty} \sum_1^{N} P(B_n)$$

$$= \lim_{N \to \infty} P\left(\bigcup_1^{N} B_n\right)$$

$$= \lim_{N \to \infty} P(A_N)$$

$\square$

(II) Exercise. Hint: If $A_n \downarrow$ then $A_n^c \uparrow$.

Why "continuity Lemma"?

Because $A_n \uparrow$, $A_n = \bigcup_1^n A_k$ and by the lemma:

$$\lim_n P(A_n) = P\left(\bigcup_1^\infty A_n\right) = P\left(\text{"}\lim_{n\to\infty} \bigcup_1^n A_k\text{"}\right) = P\left(\text{"}\lim_n A_n\text{"}\right).$$

Analogously to $\lim_n f(x_n) = f(\lim x_n)$ for a continuous f.

Back to proving (iii) of claim.

Let $A_n = \{X \leq n\}$, then $P(A_n) = F(\quad)$.

Clearly, $A_n \quad$ so by the lemma

$$P(\cup A_n) = \lim P(A_n) = \lim_n F(n)$$

Now,

$$\cup A_n = \bigcup_{n=1}^\infty \{X \leq n\} = \{X \qquad \}$$

$$\Rightarrow \lim_n F(n) = P(\cup A_n) = P(X < \infty) =$$

Note. $\varphi(x) = \sin(2\pi x)$ satisfies $\lim_n \varphi(n) = 0$ but $\lim_{x\to\infty} \varphi(x) \neq 0$

Finally, because F is monotone $\lim_{x\to\infty} F(x)$ exists, and since $\lim_{n\to\infty} F(n) = 1$, it follows that $\lim_{x\to\infty} F(x) = 1$.

As for the second half of (iii), let $A_n = \{ X \leq -n \}$.
A_n , so
$$P(\cap A_n) = \lim_n P(A_n) = \lim_n F(\quad)$$
But
$$\bigcap_1^\infty A_n = \bigcap_1^\infty \{ X \leq -n \} = \{ X \qquad \}$$
$$\Rightarrow \lim_{n\to\infty} F(-n) = P\left(\bigcap_1^\infty A_n\right) =$$

How would you finish this proof?

(ii) F is right-continuous at x_0.
Exercise. Hint: consider $A_n = \{ X \leq x_0 + 1/n \}$.

<u>Def</u>. A RV $X : \Omega \to \mathbb{R}$ with CDF F is called a continuous RV if there exists an integrable function $f \geq 0$ s.t.
$$F(x) = \int_{-\infty}^{x} f(t)\, dt .$$
In this case, f is called the density, aka, the pdf (probability density function) of X.

<u>Comments</u>. For a cont. RV X with CDF F and pdf f.

(i) Technically, X is an absolutely continuous RV.

(ii) F is a cont. function, and by the fundamental theorem of calculus, for any x where f is cont. at x, $F'(x) = f(x)$.

(iii) If F_X, the CDF of <u>some</u> RV X, is differentiable then X is a cont. RV and $f = F_X'$ is its density. The same holds if F_X is cont. and it is differentiable everywhere except on a finite set of points.

(iv) For $a<b$,

$$\int_a^b f(t)\,dt = \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$

$$= F(b) - F(a)$$

(always) $$= P(X \le b) - P(X \le a)$$

(always) $$\stackrel{?}{=} P(X \in (a,b]).$$ $(= F(b)-F(a))$ always holds!

with considerable effort $\Rightarrow$ For any (measurable) set $B \subset \mathbb{R}$,

$$P(X \in B) = \int_B f(t)\,dt. \qquad (*)$$

(v) <u>Claim</u>. If X is a cont. RV then for any $x \in \mathbb{R}$,

$$P(X = x) = 0.$$

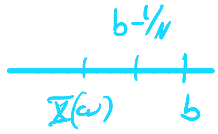
This leads to the following intriguing identity:

$$1 = P(X \in \mathbb{R}) = P\left(\bigcup_{x \in \mathbb{R}} \{X = x\}\right) = \sum_x P(X = x) = 0$$

<u>Proof</u>. You can take $B = \{x\}$ in (*) above.

Alternatively, we can prove this directly:

$$\{\omega : X(\omega) = b\} = \bigcap_n \underbrace{\{b - \tfrac{1}{n} < X \le b\}}_{B_n}$$



$$\Rightarrow P(X = b) = P\left(\bigcap_n B_n\right)$$

always holds

$$= \lim_n P(B_n)$$ since $B_n \downarrow$

$$= \lim_n P(X \in (b - 1/n, b])$$

$$\overset{?}{=} \lim_n \left[F(b) - F(b - \tfrac{1}{n})\right]$$

$$= 0$$ since F is cont. $\square$

(vii) The proof of (v) shows that for <u>any</u> RV X,

$P(X = b) = F(b) - \lim_n F(b - 1/n)$. The corollary is that the CDF always determines the pmf.