Stat 2911 Lecture Notes

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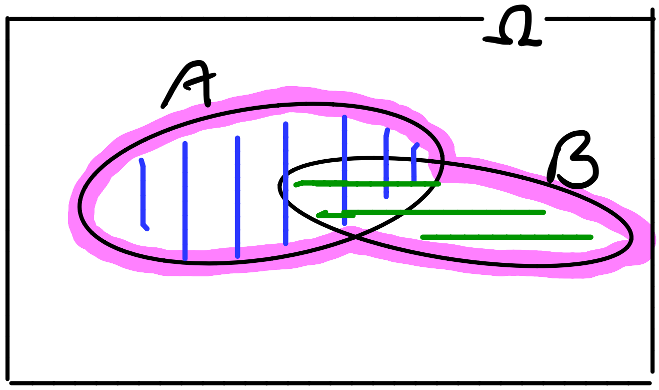
P(A U B), equiprobable spaces, birthday questions, conditional probability, independence, total probability law

What if A and B are not disjoint?
You can use a Venn diagram to get an idea:



prob. ~ area

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

How do we know that $A \cap B \in \mathcal{F}$?

Ex. Using de Morgan laws: $(A \cup B)^c = A^c \cap B^c$
$(A \cap B)^c = A^c \cup B^c$

show that if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

Claim. $\forall A, B \in \mathcal{F}$ $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof. $A \cup B = A \,\overset{\text{disjoint}}{\dot{\cup}}\, (B \setminus A)$ where $B \setminus A \overset{\text{set minus}}{:=} B \cap A^c \overset{\text{why?}}{\in} \mathcal{F}$

$\overset{(*)}{\Rightarrow} P(A \cup B) = P(A) + P(B \setminus A)$

$B = (B \setminus A) \,\dot{\cup}\, (B \cap A)$

$\Rightarrow P(B) = P(B \setminus A) + P(B \cap A)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Equiprobable Spaces

An equiprobable space consists of a <u>finite</u> sample space Ω s.t. $\forall \omega \in \Omega \quad P(\{\omega\}) = c > 0$ $\quad (\mathcal{F}_e = 2^{\Omega})$

Note. $\omega \in \Omega$ is a sample point (possible outcome)

$\{\omega\} \in \mathcal{F}$ is the event which includes only the sample point ω.

"$P(\omega)$" $:= P(\{\omega\})$

Why is Ω finite ?

Otherwise, $\exists \{\omega_n\}_1^\infty \subset \Omega$ with $\omega_i \neq \omega_j$ for $i \neq j$ so

$$P\left(\bigcup_n \{\omega_n\}\right) = \sum_n{}' P(\omega_n)$$
$$= \sum_n{}' c = +\infty$$

But $\forall A \in \mathcal{F} \quad P(A) \in [0,1]$ (why?!)

Exercise. If $A, B \in \mathcal{F}$ and $A \subset B$ then $P(A) \leq P(B)$.

Claim. $\forall A \in \mathcal{F}_e \qquad P(A) = \frac{|A|}{|\Omega|}$ $\qquad |A| =$ cardinality of A = # of elements in A

Proof. $A = \bigcup_{\omega \in A} \{\omega\}$

$$\Rightarrow P(A) = P\left(\bigcup_{\omega \in A} \{\omega\}\right) \overset{?}{=} \sum_{\omega \in A}{}' P(\omega) =$$

be careful with this notation

Take $A = \Omega \Rightarrow 1 = P(\Omega) = |\Omega| \cdot c$

$$\Rightarrow c = 1/|\Omega|$$

$$\Rightarrow P(A) = |A| / |\Omega| \qquad \square$$

$\Rightarrow$ combinatorics!

Examples. 1) A fair die is rolled: for $A \subset \{1, \dots, 6\}$

$$P(A) = |A|/6$$

2) A group of n people meets at a party. What is the prob. at least two of them share their birthday?

$\Omega = \{(i_1, i_2, \dots, i_n) : i_k \in \{1, \dots, 365\}\}$

$\forall \omega \in \Omega \qquad P(\omega) = 1/|\Omega|$

$|\Omega| =$

$\Rightarrow \quad P(\omega) = 365^{-n}$

Modeling questions: (which we already made in this equiprobable context)

- leap years?
- seasonal effects?
- annual party of the octuplets association?

Our model assumes each of the 365 days is equally likely, and no dependence between guests.

$A = \{(i_1, i_2, \dots, i_n) \in \Omega : \exists j \neq k \qquad\qquad \}$

To compute $P(A)$ we first compute $P(A^c)$.

Exercise. For any event $E \in \mathcal{F}$ $P(E) + P(E^c) = 1$.

$A^c = \{(i_1, \dots, i_n) \in \Omega : \forall j \neq k \qquad \}$

$|A^c| =$

$\Rightarrow P(A^c) = \prod_{1}^{n} \frac{365-(i-1)}{365}$ $\qquad \left(\prod_{i=1}^{n} a_i = a_1 a_2 \cdots a_n\right)$

$= \prod_{1}^{n} \left(1 - \frac{i-1}{365}\right) \underset{n=23}{\approx} 0.5 \Rightarrow P(A) \approx 0.5$ for $n=23$

Compare that with the prob. of the event

$B = \{$ one of the guests shares <u>your</u> birthday $\}$

In particular, what is the number of guests, n, for which $P(B) \approx 1/2$? $\approx 365/2$???

$$B = \{(i_1, i_2, \dots, i_n) \in \Omega : \exists j \qquad \},$$

where $x \in \{1, \dots, 365\}$ is your birthday.

Again, it is easier to deal with B^c:

$$B^c = \{(i_1, i_2, \dots, i_n) \in \Omega : \forall j \qquad \}$$

$|B^c| =$

$\Rightarrow P(B^c) = \frac{364^n}{365^n}$

$= \left(1 - \frac{1}{365}\right)^n \simeq e^{-n/365}$

For $n = 253$ $\quad P(B^c) \simeq 0.5$

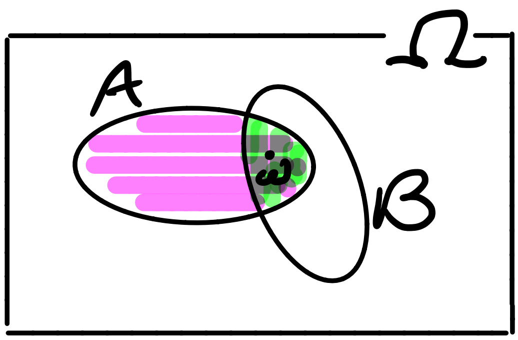
$\Rightarrow P(B) \simeq 0.5$

Note: $n > 365/2$, why? Related to the first part!

Conditional Probability

If $A, B \in \mathcal{F}$ and $P(A) > 0$ we can define

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$



Example. $\Omega = \{1, 2, \dots, 6\}$

$A = \{1, 2\}$ $\quad B = \{1, 3, 5\}$

Independence

$A \in \mathcal{F}$ is independent (ind.) of $B \in \mathcal{F}$ if knowing only whether or not A occurred, does not give us any information on whether or not B occurred:

$$P(B|A) = P(B)$$

$$\Rightarrow \quad \frac{P(A \cap B)}{P(A)} = P(B)$$

$$\Rightarrow \quad \boxed{P(A \cap B) = P(A)\,P(B)}$$

(definition of ind events)

Opaque but more robust! - symmetric in A & B

- no need to assume $P(A) > 0$
- easier to generalize to multiple events

Warning. Do not confuse ind. and disjoint events!

Specifically, if $A, B \in \mathcal{F}$ are disjoint $P(A \cap B) = 0$ so they are not ind. unless $P(A) = 0$ or $P(B) = 0$.

More generally $A_1, \dots, A_n \in \mathcal{F}$ are (mutually) ind. if

$$\forall\ 1 \le i_1 < \dots < i_k \le n \qquad P(A_{i_1} \cap \dots \cap A_{i_k}) = \prod_{j=1}^{k} P(A_{i_j})$$

These events are pairwise ind. if $\forall\ i \neq j$ A_i and A_j are ind. $P(A_i \cap A_j) = P(A_i)\,P(A_j)$

Clearly mutual ind. implies pairwise ind. but the reverse implication generally fails. (tutorial ex)

The total prob. law

A prob. of an event can be computed by summing up all eventualities.

Example

3 machines	A	B	C
production rate	0.5	0.2	0.3
failure rate	0.01	0.02	0.005

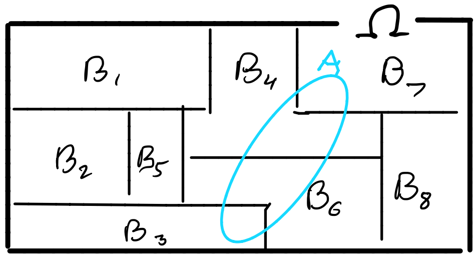
P(failed product)

$$= 0.5 \times 0.01 + 0.2 \times 0.02 + 0.3 \times 0.005 = 0.0105$$

We simplify the problem by conditioning on the producing machine: the failure rate of each machine is explicitly given

More generally, suppose the events $\{B_n\}$ form a countable (finite or infinite) partition of Ω:
$\Omega = \bigcup_n B_n$ and $\{B_n\}$ are disjoint events.

Example (pictorial):



Claim. $\forall A \in \mathcal{F}$ $\quad P(A) = \sum_n' P(A|B_n) P(B_n)$

In practice we define the partition $\{B_n\}$ so that $P(A|B_n)$ and $P(B_n)$ are easy to compute.

Proof.
$$P(A) = P(A \cap \Omega)$$
$$= P[A \cap (\bigcup_n B_n)]$$

Distributive laws:
$$a \times (b+c) = a \times b + a \times c$$
$$a + (b \times c) \neq (a+b) \times (a+c)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$\Rightarrow P(A) \overset{?}{=} P[\bigcup_n (A \cap B_n)]$$
$$\overset{?}{=} \sum_n' P(A \cap B_n)$$
$$= \sum_n' P(A|B_n) P(B_n) \qquad \square$$