Stat 2911 Lecture Notes

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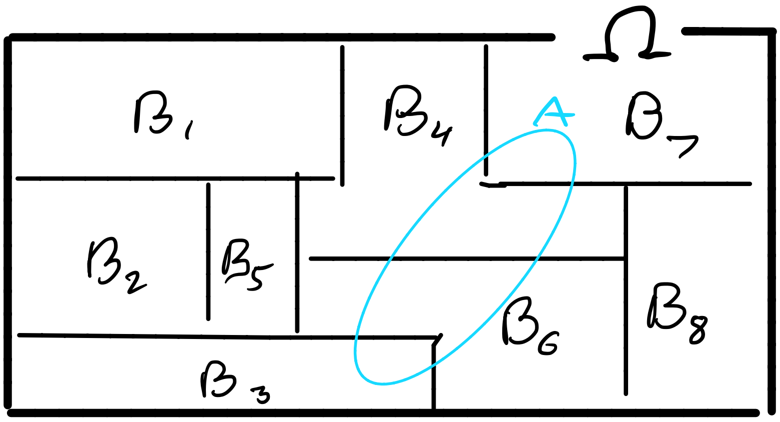
Bayes Rule, Random Variables, pmf, Bernoulli, Binomial, Geometric, Negative Binomial

Bayes' Rule

Diagnostic: which of the eventualities B_n triggered the event A?



$$P(B_n \mid A) = \frac{P(A \cap B_n)}{P(A)}$$

$$= \frac{P(A \mid B_n)\, P(B_n)}{\sum_k P(A \mid B_k)\, P(B_k)}$$

Random Variables

A random variable (RV) is a (measurable) function

$$X : \Omega \to \mathbb{R} \qquad (\text{sometimes } \mathbb{R} \cup \{-\infty, \infty\})$$

Why?! Same reason we use numbers!

A RV is discrete if its range, $X(\Omega) = \{X(\omega) : \omega \in \Omega\}$, is a countable set.

The prob. mass function (pmf) of X is defined as

$$p_X(x) = P(X = x) = P(\{\omega : X(\omega) = x\}) \quad \forall x \in \mathbb{R}$$

Distinguish between pmf and pdf.

How do we know $\{\omega : X(\omega) = x\} \in \mathcal{F}$? (trick question)

Claim. (i) $\forall x \in \mathbb{R} \setminus X(\Omega) \quad p_X(x) = 0$

(ii) With $\{x_i\} = X(\Omega) \quad \sum_i' p_X(x_i) = 1$

Proof. (ex)

The distribution (dist.) of a discrete RV is completely specified by its pmf. Indeed, $\forall A \subset \mathbb{R}$

$$P(X \in A) = \sum_{i : x_i \in A}' p_X(x_i)$$

We can thus specify the dist. of a RV X by specifying its pmf p_X.

Can any fn $p: \mathbb{R} \to [0,1]$ with a countable support $\{x : p(x) > 0\}^{(*)} = \{x_i\}$ s.t. $\sum'_{i} p(x_i) = 1$ be a pmf of <u>some</u> RV ?

(*) For a RV X, the support of its pmf can be smaller than its range $X(\Omega)$.

<u>Claim</u>. If $p : \mathbb{R} \to [0,1]$ has a countable support $\{x_i\}$ with $\sum'_i p(x_i) = 1$, then there exists a prob. space $(\Omega, \mathcal{F}, P)$ and a RV $X : \Omega \to \mathbb{R}$ s.t. $p_X = p$.

<u>Proof</u>. $\Omega = \{x : p(x) > 0\}$

$\mathcal{F} = 2^{\Omega}$

$P(A) = \sum'_{x \in A} p(x)$

$X(\omega) = \omega$

Ex: Complete the missing details.

The last claim allows us to define dists. by simply specifying a fn that has the properties of a pmf!

Examples

1) A Bernoulli(p) RV is defined by the pmf

$$p(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & x \notin \{0,1\} \end{cases}$$

"failure"
"success"

A 1-parameter family of distributions (p).

Example: $\Omega = \{H, T\}$ $\quad P(\{H\}) = p$

$$X(\omega) = \mathbb{1}_{\{H\}}(\omega) = \begin{cases} 1 & \omega = H \\ 0 & \omega \neq H \end{cases} \quad (\omega = T)$$

2) A <u>binomial</u> RV S_n models the number of successes in n independent and identically distributed (iid) Bernoulli(p) trials.

A 2-parameters family of distributions (n, p).

Let $X_i = \mathbb{1}_{\{\text{success at trial } i\}} = \begin{cases} 0 & \text{if failure at } i \\ 1 & \text{if success at } i \end{cases}$

Then $X_i \sim$ $\qquad$ and $S_n = \sum_{i=1}^{n} X_i$

Generally, for an event A, the RV $\mathbb{1}_A$ is the indicator of A:

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

which is a RV, with $p =$ $\qquad$.

The range of S_n is $S_n(\Omega) = \{ \qquad \}$.
Fix $k \in \{0, 1, \dots, n\}$ and consider the following configuration of Bernoulli outcomes for which $S_n = k$:

$$\underbrace{ss\dots s}_{k}\ \underbrace{f\dots f}_{n-k}$$

Its probability is:

$$p^k (1-p)^{n-k}$$

All other configurations of k successes and $n-k$ failures have the same prob.
How many different such configs are there?

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\Rightarrow \quad P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k \in S_n(\Omega)$$

It is easy to verify that $\sum_{k=0}^{n} P(S_n = k) = 1$
(binomial theorem)

3) A geometric RV X models the # of iid Bernoulli(p) trials it takes till the first success.
A 1-parameter family (p).

$$X(\Omega) = \{1, 2, 3, \dots \} \cup \{\infty\}$$

For $k \in \mathbb{N}_1 = \{1, 2, \ldots\}$, $X = k$ iff the outcome of the Bernoulli sequence of trials is $\underbrace{ff\ldots f}_{k-1}s$. Therefore,

$$P(X=k) = (1-p)^{k-1}p.$$

Ex. If $p > 0$ then $\sum_{k=1}^{\infty} P(X=k) = 1$.

The geometric dist. is often used to model the lifetime of an electronic component. why?

$X > k \iff \underbrace{f \ldots f}_{k}$ so $P(X > k) =$,

$$P(X > n+k \mid X > n) = \frac{P(X > n+k, X > n)}{P(X > n)}$$ note the ","

$$= \frac{P(X > n+k)}{P(X > n)}$$

$$= \frac{(1-p)^{n+k}}{(1-p)^n}$$

$$= (1-p)^k = P(\qquad)$$

The geometric dist. is <u>memoryless</u>: the lifetime of a component is ind. of its age!

If $X_i = 1_{\{\text{success at trial } i\}}$ again then

$$X = \min\{i : X_i = 1\}$$

4) A negative binomial RV X^r models the number of iid Bernoulli(p) trials till the r^{th} success, where $r \in \mathbb{N}$, is fixed.

A 2-parameters family (r,p).

Note that X^1 is a   RV.

$X^r(\Omega) = \{r, r+1, \dots \} \cup \{\infty\}$,

Fix $k \in \{r, r+1, \dots\}$, and consider the following config. of Bernoulli trials for which $X^r = k$:

$$\underbrace{f \dots f}_{k-r}\ \underbrace{ss \dots s}_{r}$$

Its probability is:

$$(1-p)^{k-r} p^r$$

All other configurations of r successes and $k-r$ failures have the same prob.

How many such configs are there for which $X^r = k$?

$$\underbrace{\overset{f/s}{\circ}\ \overset{f/s}{\circ}\ \dots\ \overset{f/s}{\circ}}_{k-1}\ \ ⓢ \quad \Rightarrow$$

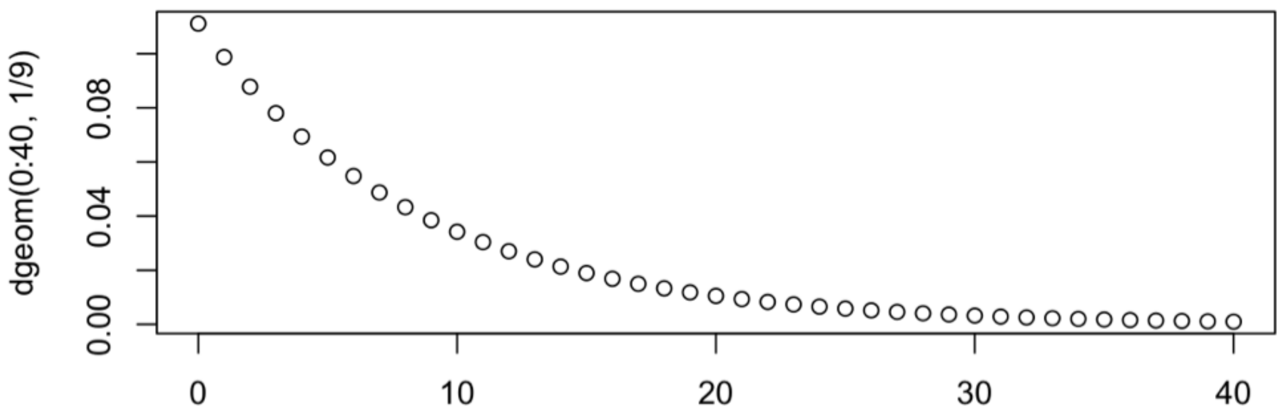
$$\Rightarrow \qquad P(X^r = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r} \qquad k \in \{r, r+1, \dots\}$$

Ex.(*) If $p > 0$ then $\sum_{k=r}^{\infty} P(X^r = k) = 1$

With $X_i = 1_{\{\text{success at trial } i\}}$ again

$$X = \min\{m : \sum_{1}^{m} X_i = r\}$$

pmf of a geometric RV with $p = 1/9$



pmf of a negative binomial RV with $r=2$, $p=1/9$