Stat 2911 Lecture Notes
Class 21, 2016
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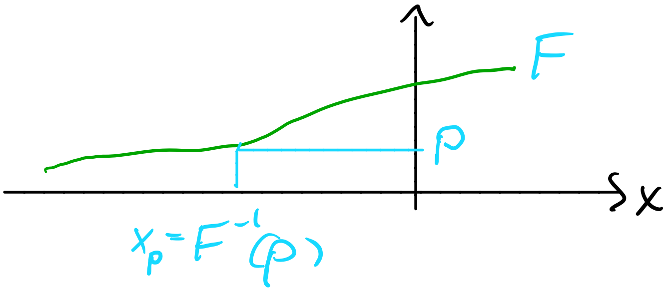
Quantiles (Rice 2.2), Functions of a RV, Sampling given a uniform sample (Rice 2.3)

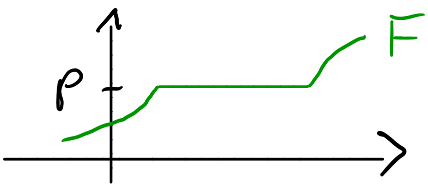
Quantiles

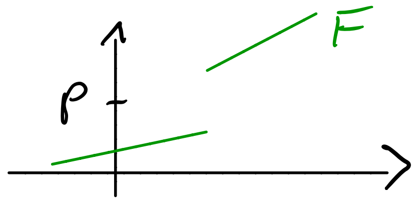
Suppose F is a continuous CDF which is strictly monotone. In this case, $F^{-1}:(0,1)\to\mathbb{R}$ is well-defined:

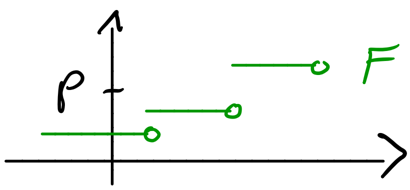


Def. The p^{th} quantile of F is $x_p = F^{-1}(p)$, or equivalently, $F(x_p)=p$.

What if F is not continuous and/or not strictly increasing?





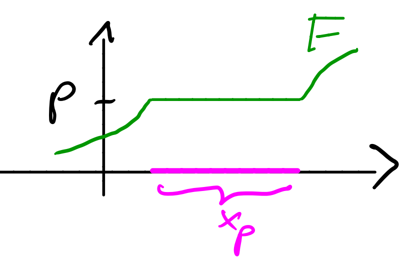


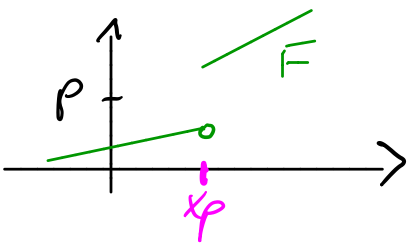
Def. For $p\in(0,1)$, $x_p\in\mathbb{R}$ is a p^{th} quantile of F if

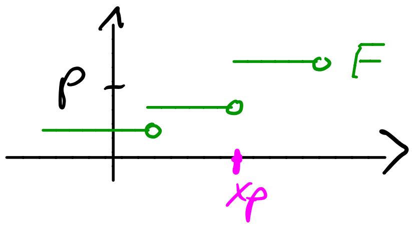
(i) $F(x_p)\geq p$

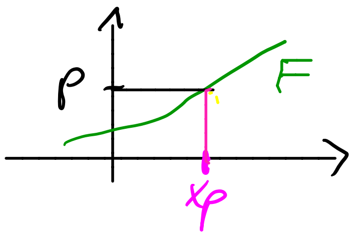
(ii) $F(x_p^-) := \lim\limits_{x\to x_p^-} F(x) \leq p$.

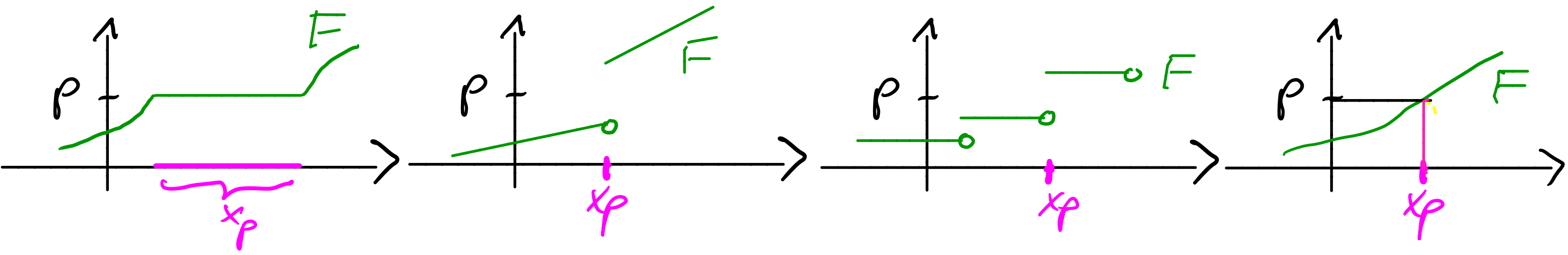
Cor. If $F(x_p)=p$ then x_p is a p^{th} quantile of F











<u>Claim</u> For $p \in (0,1)$, $x_p \in \mathbb{R}$ the following are equivalent for a RV X with CDF F:

(I) x_p is a p^{th} quantile of F: (i) $F(x_p) \geq p$
(ii) $\lim_{x \to x_p^-} F(x) \leq p$

(II) $P(X \leq x_p) \geq p$ and $P(X < x_p) \leq p$

(III) Either $F(x_p) = p$, or $F(x_p) > p$ and $\forall \delta > 0$, $F(x_p - \delta) \leq p$.

<u>Proof.</u> Exercise

It would be useful to single out a unique quantile.

<u>Def.</u> The quantile function is defined as

$$Q_F(p) = \inf\{x : F(x) \geq p\} \quad p \in (0,1).$$

<u>Exercise.</u> $Q_F(p) = \min\{x : F(x) \geq p\}$ (F is right continuous)

<u>Claim</u> $Q_F(p)$ is a p^{th} quantile of F

<u>Proof.</u> $F(Q_F(p)) \geq p$ since $Q_F(p) \in \{x : F(x) \geq p\}$. why?

If $x < Q_F(p)$ then $F(x) < p$ (why?), therefore

$\lim_{x \to Q_F(p)^-} F(x) \leq p$. $\square$

<u>Claim</u>. $\forall p \in (0,1), x \in \mathbb{R}$: $x \geq Q_F(p)$ iff $F(x) \geq p$.

<u>Proof</u>.

If $x \geq Q_F(p)$ then $F(x) \underbrace{\geq}_{F\uparrow} F(Q_F(p))$

$$\underbrace{\geq}_{\text{last claim}} p.$$

Conversely, if $x < Q_F(p)$ then $F(x) < p$. (by def) $\square$

<u>Claim</u>. If F is cont. and strictly $\uparrow$ then $F^{-1} \equiv Q_F$

<u>Proof</u>. $\exists$ a unique $x_p \in \mathbb{R}$ s.t. $F(x_p) = p$, and therefore

$$F(x) \geq p \iff x \geq x_p,$$

(both $F, F^{-1} \uparrow$)

but by the previous claim

$$F(x) \geq p \iff x \geq Q_F(p).$$

Therefore,

$$Q_F(p) = x_p$$
$$= F^{-1}(p).$$

$\square$

Functions of RVs (Rice 2.3)

Special cases first.

1) Claim. Let X be a cont. RV with $F = F_X$ strictly $\uparrow$ on $X(\Omega)$ (so $\exists F^{-1}$ that is strictly $\uparrow$ on $(0,1)$). Then $F(X) \sim U(0,1)$.

Proof. $\forall x \in X(\Omega)$, $p \in (0,1)$:

$F(x) \le p \iff x \le F^{-1}(p)$ (Both F & F^{-1} are $\uparrow$)

Therefore, with $U = F(X)$ and $p \in (0,1)$

$$\begin{aligned} P(U \le p) &= P(F(X) \le p) \\ &= P(X \le F^{-1}(p)) \\ &= F(F^{-1}(p)) \\ &= p . \end{aligned}$$

Example. The p-value is used to quantify the level of surprise, or deviation from the null hypothesis.
For example, suppose T, the lifetime, in years, of a component is distributed according to an $\exp(\lambda)$ dist.

The null hypothesis is: $\lambda = 1$ (H_0).
The alternative hypothesis is: the component is a fake with $\lambda > 1$ (H_1).
We measured a lifetime of $t_0 = \frac{1}{2}$ year. How suspicious should we be?
The p-value of t_0 is the probability of observing a lifetime of $t_0 (= \frac{1}{2})$ or shorter, assuming an exp(1) dist.
That is, with T denoting the (random) lifetime of the component, then

$$\text{p-value}(t_0) = P(T \leq t_0)$$
$$= F_T(t_0) \simeq 0.39 \quad (\text{so we won't reject } H_0).$$

What is the probability that, assuming H_0 is correct, $\underbrace{\text{the p-value of } T}_{F_T(T)}$ is ≤ 0.05?

$$P(F_T(T) \leq 0.05) = ?$$

By the claim $F_T(T) \sim U(0,1)$ therefore

$$P(F_T(T) \leq 0.05) = 0.05.$$

More generally, for any test statistic T, with a cont. and strictly $\uparrow$ F_T on $T(\Omega)$, the p-value $\sim U(0,1)$ under H_0.

2) <u>Claim</u>. Suppose the CDF F has an inverse F^{-1} on $(0,1)$ (that is, F is cont. and strictly $\uparrow$ on $I = F^{-1}(0,1)$), and let $U \sim U(0,1)$.

Define $X = F^{-1}(U)$, then $F_X \equiv F$.

<u>Proof</u>. Take x with $F(x) \in (0,1)$, then

$$\begin{aligned} F_X(x) &= P(X \le x) \\ &= P(F^{-1}(U) \le x) \\ &= P(U \le F(x)) \qquad (F, F^{-1} \ldots) \\ &= F(x) \quad . \qquad (\text{exercise: } F_X \equiv F) \end{aligned}$$

$\square$

What is the significance of this claim?

Sampling!

If we know how to sample from a $U(0,1)$ dist. then we can sample from any dist. with a cont. and strictly $\uparrow$ F: all we need is to find F^{-1}.

What about sampling from other dists?

Recall that $F^{-1} \equiv Q_F$ for any cont. strictly $\uparrow$ F.

<u>Claim</u>. Let F be a CDF and $U \sim U(0,1)$ RV. Then

$$Q_F(U) \sim F.$$

<u>Proof</u>. We showed that $\forall p \in (0,1)$, $x \in \mathbb{R}$:

$$p \le F(x) \iff Q_F(p) \le x .$$

Therefore,

$$\{\omega : U(\omega) \le F(x)\} = \{\omega : Q_F(U(\omega)) \le x\}$$

$$\Rightarrow \quad P(Q_F(U) \le x) = P(U \le F(x))$$

$$= F(x) .$$

<u>Cor</u>. If $U_1, \dots, U_n$ is a sample from the $U(0,1)$ dist., then $Q_F(U_1), \dots, Q_F(U_n)$ is a sample from the dist. determined by the CDF F.

To complete the proof of the corollary we need the intuitively obvious fact that $Q_F(U_1), \dots, Q_F(U_n)$ are ind. RVs.

<u>Theorem</u>. Suppose X is a cont. RV with a pdf f_X, and g is a continuously differentiable and strictly monotone function on some interval $I \supset X(\Omega)$.
Then $Y := g(X)$ is a cont. RV with pdf

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = f_X(g^{-1}(y)) \left| \frac{1}{g'(g^{-1}(y))} \right|.$$

<u>Proof</u>. Since g is strictly monotone, g^{-1} exists and is also strictly monotone.
Moreover, since g is cont. diff., so is g^{-1} and by the chain rule,

$$\begin{aligned} 1 &= \frac{d}{dy} y \\ &= \frac{d}{dy} g(g^{-1}(y)) \\ &= g'(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) \end{aligned}$$

$$\Rightarrow \quad \frac{d}{dy}(g^{-1}(y)) = \frac{1}{g'(g^{-1}(y))} .$$

To simplify the argument below we assume F_X is diff. everywhere.

$$F_Y(y) = P(Y \le y)$$
$$= P(g(\underline{X}) \le y).$$

(i) Assume $g \uparrow$, then $g^{-1} \uparrow$ and therefore

$$F_Y(y) = P(\underline{X} \le g^{-1}(y))$$
$$= F_{\underline{X}}(g^{-1}(y)).$$

By chain rule F_Y is diff. and

$$f_Y(y) = F_{\underline{X}}'(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$
$$= f_{\underline{X}}(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| .$$ (why?)

(ii) Assume $g \downarrow$, then $g^{-1} \downarrow$ and therefore

$$F_Y(y) = P(\underline{X} \geq g^{-1}(y))$$
$$= 1 - P(\underline{X} < g^{-1}(y))$$
$$= 1 - F_{\underline{X}}(g^{-1}(y))$$ why?

So, again by chain rule, F_Y is diff. with

$$f_Y(y) = -f_{\underline{X}}(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$
$$= f_{\underline{X}}(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| .$$ why?