Stat 2911 Lecture Notes
Class 16, 2016

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Asymptotic expression for the moment estimator from the HW problem, Parametric Bootstrap (Rice 8.4, 8.9)

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Claim. For $\theta \in (0,1)$, $E(\tilde{\Theta}_n) = \theta - \frac{1-\theta^2}{8n\theta} + R_n$,

where $nR_n \xrightarrow[n]{} 0$.

Proof.

The second order Taylor expansion of $g(x) = \sqrt{x}$ about $x_0 = \theta^2$:

$$\sqrt{x} = \underbrace{\theta + \frac{1}{2\theta}(x-\theta^2) - \frac{1}{8\theta^3}(x-\theta^2)^2}_{= T_2(x)} + R_2(x)$$

where $R_2(x) = \frac{g^{(3)}(\xi_x)}{6}(x-\theta^2)^3 = \frac{1}{16}\xi_x^{-5/2}(x-\theta^2)^3$

(ξ_x is between x and θ^2)

$x \to \tilde{\Theta}^2 = X_3/n$

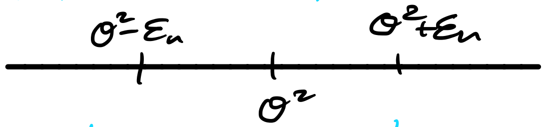
$$\Rightarrow \quad E(\tilde{\Theta}_n) = \underbrace{\theta - \frac{1-\theta^2}{8n\theta}}_{\text{same as } \delta\text{-method}} + \underbrace{E[R_2(X_3/n)]}_{R_n},$$

We need to show $nR_n \xrightarrow[n]{} 0$

$$R_n = E[R_2(X_3/n)]$$
$$= \sum_{k=0}^{n} R_2(k/n) P(X_3 = k)$$
$$= \underbrace{\sum_{k:|k/n-\theta^2|>\varepsilon_n} R_2(k/n)P(X_3=k)}_{V_n} + \underbrace{\sum_{k:|k/n-\theta^2|\le\varepsilon_n} R_2(k/n)P(X_3=k)}_{Q_n}$$

We will show that $nV_n \xrightarrow[n]{} 0$ and $nQ_n \xrightarrow[n]{} 0$, and hence $nR_n \xrightarrow[n]{} 0$.

We will argue that $V_n \ll 1$ because $X_3/n \xrightarrow[n]{} \theta^2$ (why?) and therefore $\sum'_{k : |k/n - \theta^2| > \varepsilon_n} P(X_3 = k) = P(|X_3/n - \theta^2| > \varepsilon_n) \ll 1$.



We will then argue that $Q_n \ll 1$ because when $x = k/n \approx x_0 = \theta^2$ the Taylor remainder, $R_2(x)$ is $\ll 1$.

In doing this, it is convenient to select $\varepsilon_n \longrightarrow 0$ at the "right rate", keeping in mind ε_n are auxiliary variables: $nR_n \xrightarrow[n]{} 0$ regardless of the choice of ε_n!

Recall that $R_2(x) = \sqrt{x} - T_2(x)$, which is a continuous function on $[0,1]$, so let $C_1 = \max_{x \in [0,1]} |R_2(x)| < \infty$.

$$|V_n| = \Big| \sum'_{k : |k/n - \theta^2| > \varepsilon_n,\ k \in \{0,1,\dots,n\}} R_2(k/n) P(X_3 = k) \Big|$$

$$\leq \sum'_{k : |k/n - \theta^2| > \varepsilon_n} |R_2(k/n)|\, P(X_3 = k)$$

$$\leq C_1 \sum'_{k : |k/n - \theta^2| > \varepsilon_n} P(X_3 = k)$$

$$= C_1 \cdot P(|X_3/n - \theta^2| > \varepsilon_n).$$

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$$|V_n| \le C_1 \cdot P\left(\left|\bar{X}_3/n - \theta^2\right| > \varepsilon_n\right).$$

<u>Lemma</u>. If $S_n \sim \text{Binom}(n,p)$ then $\forall \varepsilon > 0$

$$P\left(\left|S_n/n - p\right| > \varepsilon\right) \le C \cdot e^{-\sqrt{n}\,\varepsilon}$$

where $C \le 2e$.

<u>Proof</u> (of lemma): challenging exercise.

$$\Rightarrow |V_n| \le \underbrace{C_1 C}_{=C_2} \cdot e^{-\sqrt{n}\,\varepsilon_n}$$

For any choice of $\varepsilon_n = n^{-\alpha}$ with $\alpha \in (0, \frac{1}{2})$, $\beta = \frac{1}{2} - \alpha > 0$, therefore

$$n \cdot |V_n| \le n \cdot C_2 \cdot e^{-n^{1/2-\alpha}}$$

$$= C_2 \cdot n\, e^{-n^\beta} \xrightarrow[n]{} .$$

Can use L'Hôpital's rule to show

$$\frac{\log x}{x^\beta} \xrightarrow[x\to\infty]{} 0$$

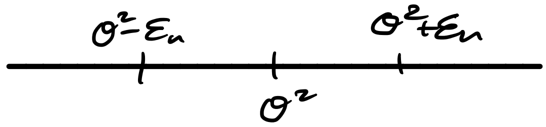
$$\Rightarrow \log x - x^\beta = x^\beta\left(\frac{\log x}{x^\beta} - 1\right) \xrightarrow[x\to\infty]{} -\infty$$

$$\Rightarrow e^{\log x - x^\beta} \xrightarrow[x\to\infty]{} 0$$

$$Q_n = \sum_{\substack{k: |k/n - \theta^2| \le \varepsilon_n,\ k \in \{0,1,\dots,n\}}}' R_2(k/n) P(X_3 = k)$$

If k satisfies $|k/n - \theta^2| \le \varepsilon_n$, then

$$|R_2(k/n)| \le \max_{|x-\theta^2| \le \varepsilon_n;\ x \in [0,1]} |R_2(x)|$$



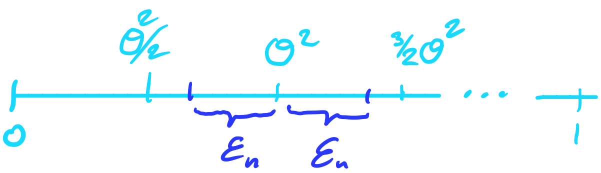
$$= \max_{\substack{|x-\theta^2| \le \varepsilon_n \\ x \in [0,1]}} \left| \frac{1}{16} \xi_x^{-5/2} (x - \theta^2)^3 \right| \quad \text{where } \xi_x \text{ is between } x \text{ and } \theta^2$$

$$\le \varepsilon_n^3 \max_{|x-\theta^2| \le \varepsilon_n;\ x \in [0,1]} \xi_x^{-5/2}$$

For a fixed ε_n, $|\xi_x^{-5/2}|$ can be arbitrarily large if $\theta^2 \to 0$. However, θ^2 is fixed and $\varepsilon_n = n^{-\alpha} \xrightarrow[n]{} 0$ with $\alpha \in (0, 1/2)$.

Choose $N = N(\alpha, \theta)$ s.t. $\varepsilon_N = N^{-\alpha} < \theta^2/2$. Then, $\forall n \ge N$

$$[\theta^2 - \varepsilon_n, \theta^2 + \varepsilon_n] \subset [\theta^2/2, 3/2\,\theta^2]:$$



$$\Rightarrow \quad |R_2(k/n)| \le \varepsilon_n^3 \underbrace{\max_{x \in [\theta^2/2,\, 3/2\,\theta^2]} \xi_x^{-5/2}}_{\le \left(\frac{\theta^2}{2}\right)^{-5/2}}$$

$\forall n \ge N$

$$\Rightarrow |R_2(k/n)| \le c_3 \varepsilon_n^3 \quad \text{where } c_3 = c_3(\theta) = \left(\frac{\theta^2}{2}\right)^{-5/2} < \infty.$$

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Therefore, $\forall n \geq N$

$$|Q_n| = \left| \sum_{k: |k/n - \theta^2| \leq \varepsilon_n}' R_2(k/n) P(X_3 = k) \right|$$

$$\leq \sum_{k: |k/n - \theta^2| \leq \varepsilon_n}' |R_2(k/n)| P(X_3 = k)$$

$$\leq \sum_{k: |k/n - \theta^2| \leq \varepsilon_n}' C_3 \varepsilon_n^3 P(X_3 = k)$$

$$\leq C_3 \varepsilon_n^3 \sum_{k=0}^{n}' P(X_3 = k)$$

$$= C_3 n^{-3\alpha} \xrightarrow[n]{} 0 \text{ but we need } nQ_n \to 0$$

Thus, if $\alpha > 1/3$, $1 - 3\alpha < 0$ and therefore

$$n|Q_n| \leq C_3 n^{1-3\alpha} \xrightarrow[n]{}$$

It follows that for any choice of $\alpha \in (1/3, 1/2)$, with $\varepsilon_n = n^{-\alpha}$ we have

$$n|R_n| \leq n|V_n| + n|Q_n| \longrightarrow$$

Regardless, (the choice of ε_n was just means to an end)

$$nR_n \longrightarrow 0$$

$\square$

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Parametric Bootstrap

Example. Binomial$(\underline{\theta})$, $\underline{\theta}=(m,p)$ are unknown

Moments estimators:

$\tilde{m} = \bar{X}^2/(\bar{X}-\hat{\sigma}^2)$; $\tilde{p} = \bar{X}/\tilde{m} = (\bar{X}-\hat{\sigma}^2)/\bar{X}$ $(\tilde{m}\tilde{p}=\bar{X})$

MLE:

$\hat{m} = \max_m \sum_{k=1}^{n} \log\left[\binom{m}{X_k}\left(\frac{\bar{X}}{m}\right)^{X_k}\left(1-\frac{\bar{X}}{m}\right)^{m-X_k}\right]$; $\hat{p} = \bar{X}/\hat{m}$.

Which estimator is better? (for fixed n)

Difficult to answer analytically, but we can try numerically.

How can we estimate the bias/var/MSE of $(\hat{m}_n, \hat{p}_n)$?

Still a hard problem but what if we knew (m,p)?

- Draw N samples of size n from a Binom(m,p) dist:

  $\underline{x}^i = (x_1^i, \dots, x_n^i)$ $\quad i=1,\dots,N$

- For each, compute $\hat{m}^i = \hat{m}(\underline{x})$ and $\hat{p}^i = \hat{p}(\underline{x})$
- Empirically estimate the bias/var/MSE from the sample $(\hat{m}^i, \hat{p}^i)$ $i=1,2,\dots,N$.

For example, $\widehat{\text{bias}}(\hat{m}) = \overline{\hat{m}^i} - m$ where $\overline{\hat{m}^i} = \frac{1}{N}\sum_{1}^{N}\hat{m}^i$

$$\widehat{MSE}(\hat{m}) = \sum_{1}^{N}(\hat{m}^i - m)^2 \cdot \frac{1}{N}$$

Of course, we don't typically know $\underset{\sim}{\theta}=(m,p)$, so how can we still estimate, say, the MSE?

Recall that when estimating Bernoulli(θ) with $\hat{\theta}=\bar{X}$

$$MSE(\hat{\Theta}_n) = \frac{\theta(1-\theta)}{n}$$

θ is unknown here as well: how did we get around this?

To estimate $MSE(\hat{\Theta}_n)$ you plug in $\hat{\theta}=\bar{x}$:

$$\widehat{MSE(\hat{\Theta}_n)} = \frac{\hat{\theta}(1-\hat{\theta})}{n} = \frac{\bar{x}(1-\bar{x})}{n}$$

Going back to estimating the $MSE(\hat{m})$ from Binom($\underset{\sim}{\theta}$):

Parametric Bootstrap: execute the MC (Monte Carlo) simulations above, with $\hat{\underset{\sim}{\theta}}=(\hat{m},\hat{p})$ instead of (m,p).

Example. Hardy-Weinberg equilibrium.