WEEK 9 SECTION PROBLEMS

If not otherwise specified, solve the following problems. If initial conditions are given, solve for all constants of integration. It is okay to leave answers in implicit form or with unsolved integrals.

1. **Getting to know your Laplace Transform**
   Remember the mathematical definition of the Laplace transform:

   $$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

   Using this, derive the following properties of the Laplace transform. *Hint:* you will need to use substitution and integration by parts liberally.

   a) *s-Shift* $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$
   b) *t-Shift* $\mathscr{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$
   c) *Derivative identity* For some function $f(t)$, $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$

2. **Warm-up**

   Solve the following ODE using a Laplace transform.

   $$y'' + 2y' + y = sin(t), \quad y(0) = 0, \quad y'(0) = 0$$

   Were you not to use a Laplace transform, which methods could you use instead to solve this ODE? Which of these methods (including Laplace transform) do you think is the easiest?

3. **Laplace transform with impulse input**

   Suppose you have a mass-spring system governed by the following equation:

   $$y'' + 4y' + 5y = \delta(t-3)$$

   where $\delta(t-a)$ is a shifted Dirac delta function.

   a) Briefly describe intuitively what is going on in this system.
   b) Solve for $y(t)$ using the initial conditions $y(0) = 0$, $y'(0) = 0$.
   c) Qualitatively, what does this solution look like when plotted?

4. **Laplace transform with discrete input**

   Solve the following ODEs using Laplace transform. Also sketch a graph of the right-hand-side forcing function (i.e. $f(t)$ in each problem).

   a)
   $$y'' + 3y' + 2y = f(t), \qquad y(0) = y'(0) = 0$$
   where $f(t) = 1$ for $0 < t < 1$ and $f(t) = 0$ for $t > 1$.

   b)
   $$y'' + y = f(t), \qquad y(0) = y'(0) = 0$$
   where $f(t) = t$ for $0 < t < 1$ and $f(t) = 0$ for $t > 1$.