WEEK 4 SECTION PROBLEMS

Solve the following problems. If initial conditions are given, solve for all constants of integration. It is okay to leave answers in implicit form or with unsolved integrals.

1. For the following ODEs, give its type (i.e. linear/nonlinear, order, homogeneous/inhomogeneous) and, if possible, solve the ODE.

   a) $\frac{1}{2}y'' + y' + y = 0$

      Solution

      This is a second order linear constantly coefficients homogeneous ODE.

      Solving the characteristic polynomial:

      $$\frac{1}{2}\lambda^2 + \lambda + 1 = 0$$

      $$\lambda = \frac{-1 \pm \sqrt{1 - 4 * 1/2}}{2 * 1/2} = -1 \pm \imath$$

      $$y(x) = e^{-x}(c_1 sin(x) + c_2 cos(x))$$

   b) $(y')^2 - y = 0$

      Solution

      First-order nonlinear homogeneous ODE.

      $$(y')^2 = y$$

      $$\frac{dy}{\sqrt{y}} = dx$$

      $$2\sqrt{y} = x + c$$

      $$y(x) = \frac{1}{4}(x+c)^2$$

   c) $xy' + y = x^2$

      Solution

      First order linear inhomogeneous ODE.

      $$y(x) = e^{-\int \frac{1}{x} dx}\Big(\int e^{\int \frac{1}{x} dx} x dx + c\Big)$$

      $$y(x) = \frac{x^2}{3} + \frac{C}{x}$$

2. **Existence and uniqueness:** For the following ODE, give the region R_1 where the solution exists, and the region R_2 where the solution is unique.

$$y' + \sqrt{y} ln(x) = 5$$

$$y(5) = 2$$

Solution
For existence, recall the first part of the existence and uniqueness theorem. It states that if there is a region R_1 containing y_0 within which $f(x, y)$ is continuous, then the solution is unique. For this problem, we have:

$$y' = f(x, y) = 5 - \sqrt{y} ln(|x|)$$

So, $f(x, y)$ is continuous for $x \in \mathcal{V} = \{x | x \in \mathbb{R}, x \neq 0\}$ and $y \in \mathcal{W} = \{y | y \geq 0\}$. We must have y_0 within our region R_1. $y_0 = (5, 2)$, so the region where the function is continuous is then:

$$R_1 = (x, y) \in \mathcal{Z} = \{x | x > 0\} \times \{y | y \geq 0\}$$

For uniqueness, we need $\partial f / \partial y$ continuous over some region $R_2 \subseteq R_1$:

$$\frac{\partial f}{\partial y} = \frac{-ln(|x|)}{2\sqrt{y}}$$

which is continuous for $x \neq 0$ and $y > 0$. So we can define

$$R_2 = (x, y) \in \mathcal{Z}' = \{x | x > 0\} \times \{y | y > 0\}$$

which is clearly a subset of R_1.

3. **Numerical Stability:** Give the amplification factor and maximum step size for the following numerical methods for the model ODE $y' = \lambda y$, assuming that $\lambda < 0$.

   a) Forward Euler

   Solution
   We can derive the amplification factor beginning with the definition of the numerical method:

$$y_{n+1} = y_n + h y_n'$$
$$y_n' = \lambda y_n$$
$$y_{n+1} = y_n + h\lambda y_n$$
$$y_{n+1} = (1 + h\lambda) y_n = (1 + h\lambda)^{n+1} y_0$$

   So we have the amplification factor $\sigma(h) = 1 + h\lambda$. For stability, we need $|\sigma(h)| \leq 1$:

$$\sigma(h) \geq -1$$

$$1 + h\lambda \geq -1$$
$$h_{max} = \frac{2}{|\lambda|}$$

b) Backward Euler

Solution

$$
\begin{aligned}
y_{n+1} &= y_n + h y'_{n+1} = y_n + h\lambda y_{n+1} \\
y_{n+1} &= \frac{1}{1-h\lambda} y_n \\
|\sigma(h)| &= \left|\frac{1}{1-h\lambda}\right| \leq 1
\end{aligned}
$$

which is valid for all h since $\lambda < 0$, so Backward Euler is *unconditionally stable.*

4. **ODEs and Eigenvalues:** Transform the following second order ODEs into a system of first order ODEs, solve for the eigenvalues of the resulting matrices, and relate these to the solution to the ODE.

a) $y'' - y = 0$

Solution

Substitute $v = y'$. This gives the system:

$$
\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}
$$

This matrix has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.

Real parts of the eigenvalues correspond to the exponential part of a solution. This ODE has solution $y(x) = c_1 e^x + c_2 e^{-x}$, so the exponential behavior of the system is apparent.

b) $y'' + y = 0$

Solution

Substitute $v = y'$.

$$
\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}
$$

This matrix has eigenvalues $\lambda_1 = \imath$ and $\lambda_2 = -\imath$.

Imaginary terms in the eigenvalue means there will be oscillatory terms in the solution. This follows from Euler's identity ($e^{\imath\pi} + 1 = 0$). This ODE has solution $y(x) = c_1 sin(x) + c_2 cos(x)$, which indeed has oscillatory terms.

Remark: Consider how this corresponds to stability of a dynamical system. The imaginary part has no effect on stability; only the real part affects stability since this determines whether the system is exponentially growing or decaying in time.

c) $y'' + 2y' + y = 0$

Solution

Substitute $v = y'$.

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}$$

This matrix has eigenvalue $\lambda = 1$ (a *repeated eigenvalue*). A repeated eigenvalue means that the solution takes the form $y(x) = e^{-x}(c_1 + c_2 x)$, which we could also verify by directly solving the homogeneous equation.