CME 100 ACE – Midterm 1 Reference Sheet

Dot Product

$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = |\vec{v}|\,|\vec{w}| \cos\theta$

Projection vectors: the projection of $\vec{w}$ onto $\vec{v}$:

$$\text{proj}_{\vec{v}} \vec{w} = \frac{\vec{w} \cdot \vec{v}}{|\vec{v}|} \frac{\vec{v}}{|\vec{v}|}$$

Cross Product

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= (v_2 w_3 - w_2 v_3)\vec{i} - (v_1 w_3 - v_3 w_1)\vec{j} + (v_1 w_2 - w_2 v_1)\vec{k}$$

$|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \sin\theta$

Area of parallelogram ABCD: $Area = |\vec{AB} \times \vec{AD}|$

Area of triangle ABD: $Area = \frac{1}{2}|\vec{AB} \times \vec{AC}|$

Lines and Planes

Parameterization of a line through P_0 parallel to vector $\vec{v}$:

$$\ell(t) = P_0 + t\vec{v}$$

Two vectors/lines are

- **Perpendicular (orthogonal)** if their *dot product* is 0
- **Parallel** if their *cross-product* is 0

The plan through point P_0 with normal $\vec{n}$:

$$(x - P_{0x}, y - P_{0y}, z - P_{0z}) \cdot \vec{n} = 0$$

Vector-Valued functions

For a parameterized curve:

$$\vec{r} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

the velocity and acceleration are given by

$$\vec{v} = \frac{d\vec{r}}{dt}, \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2}$$

Arc length for a curve parameterized over $t_1 \le t \le t_2$ is given by:

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt$$

TNB-frame, Curvature, and Torsion

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}, \qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\vec{B} = \vec{T} \times \vec{N}, \qquad \kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

$$\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2}$$

Tangential and normal components of acceleration:

$$\vec{a} = a_T \vec{T} + a_N \vec{N}, \quad a_T = \frac{d}{dt}|\vec{v}|$$

$$a_N = \kappa |\vec{v}|^2 = \sqrt{|\vec{a}|^2 - a_T^2}$$

Matrix Operations

Vector: 1D array of values, taken to usually indicate a *column vector.*

Matrix: 2D array of values. $m \times n$ matrix indicates m rows and n columns. Entry a_{ij} is denoted by row number first, then column number.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

Note: it is easiest to think of all matrix/vector objects as matrices, and row/column vectors as matrices with size one along one dimension.

Transpose: flip the dimensions and indices of the entries. *Ex.* if $\mathbf{A}$ is $m \times n$ then $\mathbf{A}^T$ is $n \times m$ and $(A^T)_{ij} = a_{ji}$.

Symmetric matrix: a *square matrix* such that $\mathbf{A} = \mathbf{A}^T$.

Skew-symmetric matrix: a *square matrix* such that $\mathbf{A}^T = -\mathbf{A}$.

Diagonal matrix: special type of matrix such that only the diagonal elements are non-zero. *Note:* diagonal matrices are always square matrices.

$$\mathbf{D} = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & d_{nn} \end{bmatrix}$$

Identity matrix: diagonal matrix where the diagonal entries are all 1. Denoted as $\mathbf{I}$. Satisfies $\mathbf{A} = \mathbf{IA} = \mathbf{AI}$.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}$$

Matrix Multiplication

For a matrix-matrix product $\mathbf{C} = \mathbf{AB}$, then

$$c_{ij} = a_{i1} b_{1j} + a_{i1} b_{1j} + \cdots + a_{in} b_{nj} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- Size along the dimension being multiplied (the *inner dimension*) must match.
- Matrix product takes on the *outer dimensions. Ex.* if $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times p$, then $\mathbf{C} = \mathbf{AB}$ is $m \times p$.
- Matrix multiplication is **not** commutative i.e. $\mathbf{AB} \neq \mathbf{BA}$.

Vector multiplication: special case of matrix multiplication where one matrix has size 1 along one of its outer dimensions.

Matrix Inverse

Matrix inverse: for a given matrix $\mathbf{A}$, the inverse is the matrix $\mathbf{M}$ such that $\mathbf{MA} = \mathbf{AM} = \mathbf{I}$.

- Denote the matrix inverse as $\mathbf{A}^{-1}$.
- To show that a given matrix B is the inverse of $\mathbf{A}$, show that $\mathbf{AB} = \mathbf{I}$.

Inverse of 2×2 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinants

Determinants only exist for square matrices.

2×2 **Determinant:**

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det \mathbf{A} = ad - cb$$

3×3 **Determinant:** "add the products of the down-diagonals, subtract the products of the up-diagonals"

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \det \mathbf{A} = aei + bfg + cdh - gec - hfa - idb$$

4×4 **Determinant:** multiply the first row element by the minor associated with that element, multiply by -1 if minor is formed by an even-numbered column. *Ex.:*

$$\det \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1 \det \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2 \det \begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix}$$

$$+ a_3 \det \begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - a_4 \det \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

MATLAB

Matrix/vector multiplication: `A*B`

Dot product: `dot(u,v)`

Cross product: `cross(u,v)`

Vector magnitude: `norm(v)`

Absolute value: `abs(a)`

Determinant: `det(A)`

Trigonometric Identities

Regular trigonometric identities:

$$\sin^2 x + \cos^2 x = 1, \qquad \tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x, \sin(2x) = 2 \sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$$

Useful Integrals/Derivatives

Hyperbolic trigonometric functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\cosh^2 x - \sinh^2 x = 1, \quad \tanh^2 x + \mathrm{sech}^2 x = 1$$
$$\coth^2 x - \mathrm{csch}^2 x = 1$$
$$\sinh(2x) = 2\sinh x \cosh x, \quad \cosh(2x) = 2\cosh^2 x - 1$$
$$\tanh(2x) = \frac{2\tanh x}{1 + \tanh^2 x}$$

Trigonometric function derivatives:

$$\frac{d}{dx}\sin x = \cos x, \ \frac{d}{dx}\cot x = -\csc^2 x, \ \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cot x = -\sin x, \ \frac{d}{dx}\sec x = \sec x \tan x, \ \frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan x = \sec^2 x, \ \frac{d}{dx}\csc x = -\csc x \cot x, \ \frac{d}{dx}\arctan x = \frac{1}{x^2+1}$$

Trigonometric function integrals:

$$\int \csc x dx = -\log|\csc x + \cot x| + C$$
$$\int \sec x = \log|\sec x + \tan x| + C$$
$$\int \tan x dx = -\log|\cos x| + C, \quad \int \cot x dx = \log|\sin x| + C$$

Hyperbolic trig function derivatives:

$$\frac{d}{dx}\sinh x = \cosh x, \quad \frac{d}{dx}\cosh x = \sinh x$$
$$\frac{d}{dx}\tanh x = 1 - \tanh^2 x, \quad \frac{d}{dx}\mathrm{csch}\ x = -\coth x \mathrm{csch}\ x$$
$$\frac{d}{dx}\mathrm{sech}\ x = -\tanh x \mathrm{sech}\ x, \quad \frac{d}{dx}\coth x = 1 - \coth^2 x$$