WEEK 9 SECTION SOLUTIONS

If not otherwise specified, solve the following problems. If initial conditions are given, solve for all constants of integration. It is okay to leave answers in implicit form or with unsolved integrals.

1. **Getting to know your Laplace Transform**
   Remember the mathematical definition of the Laplace transform:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

   Using this, derive the following properties of the Laplace transform. *Hint:* you will need to use substitution and integration by parts liberally.

   a) *s-Shift* $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$
   Solution:

$$\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-st+at}dt = \int_0^\infty f(t)e^{-t(s-a)}dt$$

$$\text{Substituting } s' = s-a\text{: } \int_0^\infty f(t)e^{-ts'}dt = F(s') = F(s-a)$$

   b) *t-Shift* $\mathscr{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$
   Solution:

$$\mathscr{L}\{u(t-a)f(t-a)\} = \int_0^\infty u(t-a)f(t-a)e^{-st}dt = \int_a^\infty f(t-a)e^{-st}dt$$

   Substituting $\tau = t-a$ and changing the bounds of integration:

$$\int_0^\infty f(\tau)e^{-s(\tau+a)}dt = \int_0^\infty f(\tau)e^{-s\tau}e^{-as}dt = e^{-as}\int_0^\infty f(\tau)e^{-s\tau}dt = e^{-as}F(s)$$

   c) *Derivative identity* For some function $f(t)$, $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$
   Solution:

$$\mathscr{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt$$

   Remember integration by parts: $\int u\,dv = uv - \int v\,du$. Take $u = e^{-st}$ and $v = f(t)$, then:

$$\int_0^\infty f'(t)e^{-st}dt = f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st})dt$$

$$= \lim_{t\to\infty} f(t)e^{-st} - f(0)e^{-s(0)} + s\int_0^\infty f(t)e^{-st}dt = sF(s) - f(0)$$

2. **Warm-up** Solve the following ODE using a Laplace transform.

$$y'' + 2y' + y = sin(t), \quad y(0) = 0, \quad y'(0) = 0$$

   Solution: Take the Laplace transform:

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) = \frac{1}{s^2+1}$$

$$\text{Applying initial condiitons: } s^2Y(s) + 2sY(s) + Y(s) = \frac{1}{s^2+1}$$

$$Y(s)(s^2+2s+1) = Y(s)(s+1)^2 = \frac{1}{s^2+1}$$
$$Y(s) = \frac{1}{(s^2+1)(s+1)^2} = \frac{-s}{2(s^2+1)} + \frac{1}{2(s+1)} + \frac{1}{2(s+1)^2}$$
$$y(t) = \frac{-1}{2}cos(t) + \frac{1}{2}e^{-t}t + \frac{1}{2}e^{-t}$$

Were you not to use a Laplace transform, which methods could you use instead to solve this ODE? Which of these methods (including Laplace transform) do you think is the easiest?

Solution: You could also solve this using *variation of parameters* or *undetermined coefficients.* All of these methods have their upsides and downsides, but Laplace transforms are probably the easiest method for solving a problem of this form. However, it comes down to personal preference.

3. **Laplace transform with impulse input** Suppose you have a mass-spring system governed by the following equation:
$$y'' + 4y' + 5y = \delta(t-3)$$
where $\delta(t-a)$ is a shifted Dirac delta function.

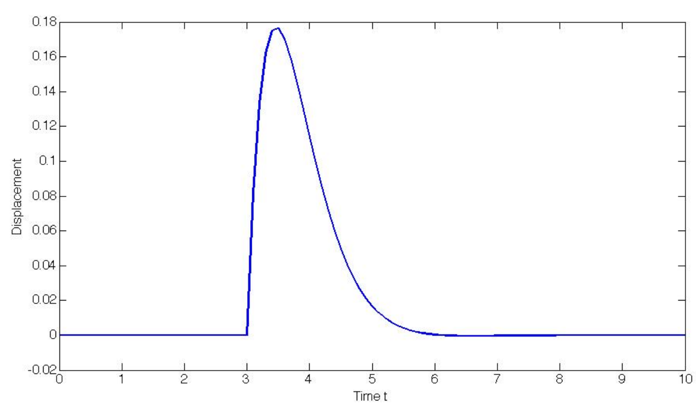
   a) Briefly describe intuitively what is going on in this system.

   Solution: A delta function represents an impulse. A good physical analogue of a delta would be slapping the mass spring system: it is a very brief impact from an outside source. For this system, it could be someone slapping the spring at time $t = 3$.

   b) Solve for $y(t)$ using the initial conditions $y(0) = 0$, $y'(0) = 0$.
   Solution: Since we are dealing with a discrete input, the only way to solve this problem is with Laplace transforms.
$$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 5Y(s) = e^{-3s}$$
   Applying initial conditions:
$$Y(s)(s^2+4s+5) = e^{-3s}$$
$$Y(s) = \frac{e^{-3s}}{s^2+4s+5} = \frac{e^{-3s}}{s^2+4s+4-4+5} = \frac{e^{-3s}}{(s+2)^2+1} = \frac{e^{-3s}}{((s+2)^2+1)}$$
$$y(t) = u(t-3)e^{-2(t-3)}sin(t-3)$$

   c) Qualitatively, what does this solution look like when plotted?

   Solution: The system is at rest at $y(t) = 0$ until time $t = 3$ when the system receives the input and begins oscillating as a decaying sinusoid. A plot is shown below:



4. **Laplace transform with discrete input**

Solve the following ODEs using Laplace transform. Also sketch a graph of the right-hand-side forcing function (i.e. $f(t)$ in each problem).

a)

$$y'' + 3y' + 2y = f(t), \qquad y(0) = y'(0) = 0$$

where $f(t) = 1$ for $0 < t < 1$ and $f(t) = 0$ for $t > 1$.

Solution:
We first need to re-express the piece-wise function $f(t)$ in terms of Heaviside step functions:

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases} \implies f(t) = 1 - u(t-1)$$

So we can rewrite the ODE as $y'' + 3y' + 2y = 1 - u(t-1)$. Now, we can solve the ODE using a Laplace transform:

$$\begin{aligned}
y'' + 3y' + 2y &= 1 - u(t-1) \\
s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) + 2Y(s) &= \frac{1}{s} - \frac{e^{-s}}{s} \\
s^2Y(s) - s(0) - (0) + 3sY(s) - 3(0) + 2Y(s) &= \frac{1}{s} - \frac{e^{-s}}{s} \\
s^2Y(s) + 3sY(s) + 2Y(s) &= \frac{1}{s} - \frac{e^{-s}}{s} \\
Y(s)\left(s^2 + 3s + 2\right) &= \frac{1}{s} - \frac{e^{-s}}{s} \\
Y(s) &= \left(1 - e^{-s}\right)\left(\frac{1}{s(s^2+3s+2)}\right) \\
Y(s) &= \left(1 - e^{-s}\right)\left(\frac{1}{s(s+2)(s+1)}\right)
\end{aligned}$$

Now, we need to do a partial fraction decomposition:

$$\begin{aligned}
Y(s) &= \left(1 - e^{-s}\right)\left(\frac{1}{s(s+2)(s+1)}\right) \\
&= \left(1 - e^{-s}\right)\left(\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}\right) \\
1 &= A\left(s^2 + 3s + 2\right) + B\left(s^2 + s\right) + C\left(s^2 + 2s\right) \\
1 &= s^2(A + B + C) + s(3A + B + 2C) + 2A
\end{aligned}$$

$$\begin{aligned} 1 &= 2A \\ 0 &= 3A + B + 2C \\ 0 &= A + B + C \end{aligned} \implies \begin{aligned} A &= \tfrac{1}{2} \\ B &= \tfrac{1}{2} \\ C &= -1 \end{aligned}$$

$$\begin{aligned}
Y(s) &= \left(1 - e^{-s}\right)\left(\frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{s+1}\right) \\
y(t) &= \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - u(t-1)\left(\frac{1}{2} + \frac{1}{2}e^{-2(t-1)} - e^{-(t-1)}\right)
\end{aligned}$$

b)

$$y'' + y = f(t), \qquad y(0) = y'(0) = 0$$

where $f(t) = t$ for $0 < t < 1$ and $f(t) = 0$ for $t > 1$.

Solution:

Te-express $f(t)$ in terms of Heaviside step functions:

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 0 & t > 1 \end{cases} \implies f(t) = (1 - u(t-1))\,t$$

From here we can solve with Laplace transform:

$$\begin{aligned}
y'' + y &= (1 - u(t-1))\,t \\
&= t - u(t-1)(t-1+1) \\
&= t - u(t-1)(t-1) - u(t-1) \\
s^2 Y(s) - sy(0) - y'(0) + Y(s) &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \\
s^2 Y(s) + Y(s) &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \\
Y(s)\left(s^2 + 1\right) &= \frac{1}{s^2}\left(1 - e^{-s}\right) - \frac{e^{-s}}{s} \\
Y(s) &= \left(1 - e^{-s}\right)\frac{1}{s^2(s^2+1)} - \frac{e^{-s}}{s(s^2+1)}
\end{aligned}$$

We must do partial fractions on both of these fractions as follows:

$$\begin{aligned}
\frac{1}{s^2(s^2+1)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} \\
1 &= A(s^3+s) + B(s^2+1) + Cs^3 + Ds^2 \\
&= s^3(A+C) + s^2(B+D) + As + B
\end{aligned}$$

$$\begin{matrix} 1 = B \\ 0 = A \\ 0 = B + D \\ 0 = A + C \end{matrix} \implies \begin{matrix} A = 0 \\ B = 1 \\ C = 0 \\ D = -1 \end{matrix}$$

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1} \quad \blacksquare$$

$$\begin{aligned}
\frac{1}{s(s^2+1)} &= \frac{A}{s} + \frac{Bs+C}{s^2+1} \\
1 &= A(s^2+1) + Bs^2 + Cs \\
&= s^2(A+B) + Cs + A
\end{aligned}$$

$$\begin{matrix} 1 = A \\ 0 = C \\ 0 = A + B \end{matrix} \implies \begin{matrix} A = 1 \\ B = -1 \\ C = 0 \end{matrix}$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \quad \blacksquare$$

Finally, we can use these partial fractions to find our final solution:

$$
\begin{aligned}
Y(s) &= \left(1-e^{-s}\right)\frac{1}{s^2(s^2+1)} - \frac{e^{-s}}{s(s^2+1)} \\
&= \left(1-e^{-s}\right)\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) - e^{-s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) \\
&= \left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) - e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+1} + \frac{1}{s} - \frac{s}{s^2+1}\right) \\
y(t) &= t - \sin(t) - u(t-1)\left((t-1) - \sin(t-1) + 1 - \cos(t-1)\right) \\
&= t - \sin(t) - u(t-1)\left(t - \sin(t-1) - \cos(t-1)\right) \\
&= (1-u(t-1))\,t + u(t-1)(\sin(t-1) + \cos(t-1)) - \sin(t) \quad \blacksquare
\end{aligned}
$$