CME 102/ENGR 155A
Ordinary Differential Equations for Engineers
Winter 2017
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Review Problems for Midterm 2 Solutions

1. **Multiple-choice.** Circle the correct answer. Every question has exactly one correct answer. You do *not* need to justify your answer.

   (a) Consider the following numerical method

   $$y_{n+1} = y_n + hf(t_n + \frac{h}{2}, \frac{1}{2}(y_n + y_{n+1}))$$

   Which of the following statements is true about this method?

   (1) It is an explicit multistep method
   (2) It is an implicit multistep method
   (3) It is a Runge-Kutta type method
   (4) It is an implicit single-step method ✓

   (b) For which of the following ODEs can the general solution be written as a sum of the homogenous and particular solutions?

   (1) $yy' = -2xy^2$
   (2) $y'' + cosx = 2x^2y - 3y'$ ✓
   (3) $y' + e^y x = x^2$
   (4) All of the above

   (c) The method of undetermined co-efficients can be used for which of the following ODEs?

   (1) $x^2y'' - 4xy' + 4y = sinx$
   (2) $y'' + 4y' + 16 = x \ln x$
   (3) $xy'' - 2xy' + 4xy = x^4e^{2x}$ ✓
   (4) $y'' - 16y' + y^2 = 4$

   (d) Consider the second order ODE

   $$t^2y'' + 2ty' - 5e^{3t^2}y = 0$$

   Which of the following gives the Wronskian $W(t)$ as a function of time given $W(1) = 1$.

   (1) $W(t) = \frac{1}{t^2}$ ✓

(2) $W(t) = e^{1-t^2}$
(3) $W(t) = e^{-2t}$
(4) $W(t) = \frac{1}{t}$

(e) Below you can find the MATLAB code for the numerical method for solving an ODE given in the corresponding ODE function.

```
function [t,y] = my_numerical_method(f,tspan,y0,h)
    t(1) = tspan(1);
    y(1) = y0;
    n = 1;
    while t(n) < tspan(2)
        t(n+1) = t(n) + h;
        y(n+1) = y(n) + h*f(t(n),y(n));
        n = n+1;
    end
end

function yprime = my_ode(t,y)
r0 = 2;
yprime = -(r0^2)*y;
end
```

Which of the following is the maximum acceptable value of the time-step h to solve the above ODE using the given numerical method?

(1) $h = 0.1$
(2) $h = 0.5$ ✓
(3) $h = 4$
(4) $h = 0.99$

2. Match the ODE with its solution:

(a) $y' - 4 = 0$	(I) $y = C_1 e^{2x} + C_2 e^{-2x}$
(b) $y' - 4y = 0$	(II) $y = 4x + C$
(c) $y'' - 4y = 0$	(III) $y = C_1 \cos(2x) + C_2 \sin(2x)$
(d) $y'' + 4y = 0$	(IV) $y = C_1 e^{2x} + C_2 x e^{2x}$
(e) $y'' - 4y' + 4 = 0$	(V) $y = Ce^{4x}$

Answer:

(a) ______________ II

(b) ______________ V

(c) ______________ I

(d) ______________ III

(e) ______________ IV

3. Solve for the complete solution of the following ODE

$$x^2y'' - xy' + y = x\ln(\ln x)$$

Hint:

(a) Let $s = \ln x$ and transform the equation above to an ODE in y and s:

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{ds}$$
$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left[\frac{d^2y}{ds^2} - \frac{dy}{ds}\right]$$

(b) The following integral formulae maybe useful

$$\int \ln x dx = x\ln x - x; \qquad \int x\ln x dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}; \qquad \int x^2 \ln x dx = \frac{x^3}{3}\ln x - \frac{x^3}{9}$$

Solution:

Let $s = \ln x$ and transform the equation above into a second order linear ODE with constant coefficient:

$$\frac{d^2y}{ds^2} - 2\frac{dy}{ds} + y = e^s \ln s$$

Characteristic equation: $r^2 - 2r + 1 = 0 \quad \longrightarrow \quad r_1 = r_2 = 1.$

$$y_1 = e^s,\ y_2 = se^s \quad \longrightarrow \quad y_1' = e^s,\ y_2' = e^s(s+1) \quad \text{and} \quad W = e^{2s}.$$

$$y_p = -e^s \int s\ln s ds + se^s \int \ln s ds = -e^s\left[\frac{s^2}{2}\ln s - \frac{s^2}{4}\right] + se^s\left[s\ln s - s\right]$$

$$y = C_1 e^s + C_2 se^s + \frac{s^2}{4}e^s[2\ln s - 3]$$

Write y back as a function of x:

$$y = C_1 x + C_2 x\ln x + \frac{(\ln x)^2}{4}x[2\ln(\ln x) - 3]$$

4. (a) The figure show a resonance response of the mass m, thus $\omega = \gamma = 2$ and the particular solution is

$$x_p = -\frac{F_0}{2\omega} t\cos(\omega t) = -\frac{6}{2 \cdot 2} t\cos(2t)$$

$$x_p = -\frac{3}{2} t\cos(2t)$$

Since the system has equal spring on each side, the equivalent spring is

$$k_{\text{eq}} = 2k = \omega^2 m = 4 \cdot 1 \quad \longrightarrow \quad k = 2$$

(b) The complete solution is of the form

$$x(t) = C_1 \cos(2t) + C_2 \sin(2t) - \frac{3}{2} t\cos(2t)$$

Differenting:

$$x'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t) - \frac{3}{2}\cos(2t) + 3t\sin(2t)$$

Thus $C_1 = 1/2$ and $C_2 = 5/4$ from the initial conditions.
Total solution:

$$x(t) = \frac{1}{2}\cos(2t) + \frac{5}{4}\sin(2t) - \frac{3}{2} t\cos(2t)$$

5. Given the following 2nd-order Runge-Kutta algorithm

$$\begin{aligned} k_1 &= f(t_n, y_n) \\ k_2 &= f(t_n + \frac{2}{3}h, y_n + \frac{2}{3}hk_1) \\ y_{n+1} &= y_n + h\left(\frac{1}{4}k_1 + \frac{3}{4}k_2\right) \end{aligned}$$

(a) find the amplification factor σ in terms of λh when the algorithm is applied to the model equation $y' = \lambda y$

(b) Given that the amplification of the exact solution is

$$\sigma_{\text{exact}} = e^{\lambda h}$$

verify that the above algorithm is indeed 3rd-order accurate at each step (leading error term is $O(h^3)$). Helpful formula:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

Solution:

$$
\begin{aligned}
k_1 &= \lambda y_n \\
k_2 &= \lambda(y_n + \frac{2}{3}hk_1) \\
&= \lambda y_n + \frac{2}{3}h\lambda^2 y_n \\
y_{n+} &= y_n + h(\frac{1}{4}k_1 + \frac{3}{4}k_2) \\
&= y_n + \frac{1}{4}h\lambda y_n + \frac{3}{4}h\lambda y_n + \frac{3}{4}h(\frac{2}{3}h\lambda^2 y_n) \\
y_{n+1} &= y_n \underbrace{\left[1 + \lambda h + \frac{1}{2}\lambda^2 h^2\right]}_{\sigma_{RK2}}
\end{aligned}
$$

Given:

$$
\begin{aligned}
\sigma_{\text{exact}} &= e^{\lambda h} \\
&= 1 + \lambda h + \frac{1}{2}\lambda^2 h^2 + \frac{1}{6}\lambda^3 h^3 + \frac{1}{24}\lambda^4 h^4 + \cdots
\end{aligned}
$$

Thus, σ_{RK2} matches σ_{exact} up to h^2 term, and the leading error term is $\sim h^3$, which verifies that the method is indeed $O(h^3)$ at each time step.

6. For the following use the method of solving Cauchy-Euler ODE for the homogeneous part; then use Variation of Parameters for the particular solutions.

   (a) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2$
   Cauchy-Euler formula with $a = 1$, $b = -1$:

   $$m^2 + (a-1)m + b = m^2 - 1 = 0 \quad \rightarrow \quad m_1 = 1, \quad m_2 = -1$$

   Homogeneous solution:

   $$y_h = C_1 x^{m_1} + C_2 x^{m_2} = C_1 x + C_2 \frac{1}{x}$$

   ODE in standard form:

   $$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = 1 \quad \rightarrow \quad r(x) = 1$$

   and $W = y_1 y_2' - y_2 y_1' = -2/x$

   $$\begin{aligned} y_p &= -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx \\ &= -x \int \frac{1}{x}\left(-\frac{x}{2}\right) dx + \frac{1}{x}\int x \left(-\frac{x}{2}\right) dx \\ &= \frac{1}{3}x^2 \end{aligned}$$

   Finally,

   $$y = C_1 x + C_2 \frac{1}{x} + \frac{1}{x^2}$$

   (b) $x^2 \frac{d^2y}{dx^2} + 2y = 0$
   Cauchy-Euler formula with $a = 0$, $b = 2$:

   $$m^2 + (a-1)m + b = m^2 - m + 2 = 0 \quad \rightarrow \quad \text{complex roots} \quad m = \alpha \pm i\beta = \frac{1}{2} \pm \frac{\sqrt{7}}{2}$$

   Homogeneous solution:

   $$y_h = x^{\alpha}\left[C_1 \cos(\beta \ln(x)) + C_2 \sin(\beta \ln(x))\right]$$

   $$y = \sqrt{x}\left[C_1 \cos\left(\frac{\sqrt{7}}{2}\ln(x)\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}\ln(x)\right)\right]$$

7. The linear ODE with constant coefficients below

$$ay'' + by' + 10y = f(x)$$

has the following complete solution $y = y_h + y_p$, with

$$\begin{aligned} y_h &= e^x \left[\cos(3x) + \sin(3x)\right], \\ y_p &= e^{-2x}. \end{aligned}$$

Determine a, b and $f(x)$ of the ODE, and its initial conditions, $y(0)$ and $y'(0)$.

Solution

Roots of characteristic equation $ar^2 + br + c = 0$:

$$\begin{aligned} (r_1, r_2) &= \alpha \pm i\beta, \\ \alpha &= -\frac{b}{2a} = 1, \\ \beta &= \frac{\sqrt{40a - b^2}}{2a} = 3. \end{aligned}$$

Solving for a and b from this set of equations gives: $a = 1$, $b = -2$. (Alternatively, you can also write the characteristic equation in the form $(r - r_1)(r - r_2) = 0$; this will put the ODE in *standard form* right away in which $a = 1$).

The ODE becomes,

$$y'' - 2y' + 10y = f(x). \tag{1}$$

$y_p(x) = e^{-2x}$; thus $f(x)$ is of the form Ae^{-2x}. Substituting $f(x)$, y_p and its derivatives into Eq. (**??**) gives $A = 18$.

$$y'' - 2y' + 10y = 18e^{-2x}, \quad y(0) = 2, \; y'(0) = 2.$$

8. Certain areas in the US do not have logging access roads and helicopters are often used for carrying timber logs. The felled trees are slung beneath the helicopters on cables. Consider one such case where 2 logs are carried one below the other. Such a system can be modeled reasonably as a certain form of a double pendulum. The angles of oscillations of the 2 logs are governed by the following system of differential equations:

$$(m_1 + m_2)L_1\theta_1'' + m_2L_2\theta_2'' + (m_1 + m_2)g\theta_1 = 0 \tag{2}$$

$$L_1\theta_1'' + L_2\theta_2'' + g\theta_2 = 0 \tag{3}$$

where L_1, L_2, m_1, m_2 are cable lengths and masses, assumed to be constants for this problem. g denotes the acceleration due to gravity.

(a) Write the above system of second order ODEs as a system of first order ODEs.

Solution

We begin by setting $\theta_1' = u$ and $\theta_2' = v$. We then have

$$(m_1 + m_2)L_1 u' + m_2 L_2 v' + (m_1 + m_2) g\theta_1 = 0 \tag{4}$$

$$L_1 u' + L_2 v' + g\theta_2 = 0 \tag{5}$$

Multiply **??** by m_2 and subtract from **??** to get

$$m_1 L_1 u' = g(m_2\theta_2 - (m_1 + m_2)\theta_1)$$

$$L_2 v' = -g\theta_2 - \frac{g}{m_1}(m_2\theta_2 - (m_1 + m_2)\theta_1) = g(1 + \frac{m_2}{m_1})(\theta_1 - \theta_2)$$

Therefore the final system of ODEs can be written as

$$\begin{aligned}
\theta_1' &= u \\
\theta_2' &= v \\
u' &= \frac{g}{L_1}(\frac{m_2}{m_1}\theta_2 - (1 + \frac{m_2}{m_1})\theta_1) \\
v' &= \frac{g}{L_2}(1 + \frac{m_2}{m_1})(\theta_1 - \theta_2)
\end{aligned} \tag{6}$$

(b) Write a function for the system of first order ODEs obtained in (b) to be used by a solver method like RK4. The function should take t and y as inputs and output derivative vector *yprime*. Your ODE function also needs to accept the values of m_1, m_2, L_1, L_2 as variable parameters. You can define the value of g inside the function itself.

Solution

```
function yprime = logging(t,y,m1,m2,L1,L2)
    g = 9.81;
    yprime(1) = y(3);
    yprime(2) = y(4);
    yprime(3) = (g/L1)*(m2*y(2)/m1 - y(1) - m2*y(1)/m1);
    yprime(4) = (g/L2)*(1 + m2/m1)*(y(1) - y(2));
end
```