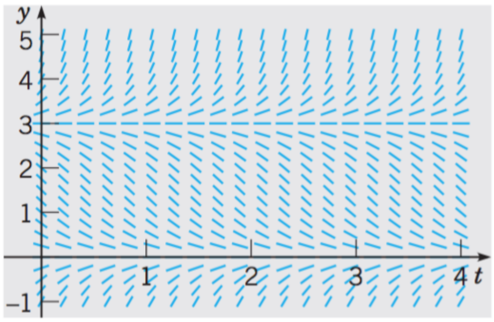
FINAL REVIEW PROBLEMS

If not otherwise specified, solve the following problems. If initial conditions are given, solve for all constants of integration. It is okay to leave answers in implicit form or with unsolved integrals.

1. **Direction fields and Equilibrium solutions:**

   Identify the equilibrium solutions and state their type.

   a) For the following figure:

   b) $y' = y(y-2)^2$

   c) $y' = y(y-3)(y-x)$

2. **Solution verification:** Verify the solution to the following ODEs.

   a) $x^2y'' + 2xy' + y = \ln(x) + 3x + 1, \quad y = \ln(x) + x$

   b) $(y'')^3 + (y')^2 - y - 3x^2 - = 0, \quad y = x^2$

   c) $y' = (x+y)^2, \quad y = \tan(x) - x$

3. **Separation of variables:** Solve the following ODEs using separation of variables.

   a) $y' + y^2\sin(x) = 0$

   b) $y' = 2 + 2x + 2y^2 + 2xy^2, \quad y(0) = 0$

   c) $y' = \frac{ty(4-y)}{1+t}$

4. **Existence and uniqueness:** Give the interval for existence and interval for uniqueness of the solution.

   a) $\sin(2x)dx + \cos(3y)dy = 0, \quad y(\pi/2) = \pi/3$

   b) $y^2(1-x^2)^{1/2}dy = \sin^{-1}(x)dx, \quad y(0) = 1$

5. **Linear first order ODEs:** Solve the following.

   a) $y' - 2y = 4 - t$

   b) $ty' + 2y = \sin(t), \quad y(\pi/2) = 0$

   c) $ty' + (t+1)y = t, \quad y(\ln(2)) = 1$

6. **Eigenvector solution to ODEs:** Solve the following using an eigenvalue/vector system.

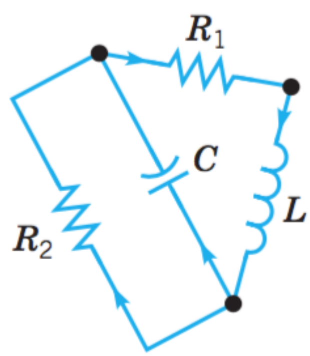
$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}$$

7. **Nonlinear second order ODEs:** Solve the following.
   a) $y'' = -2t(y')^2, \quad y(0) = 2, \quad y'(0) = -1$
   b) $y'' = 2yy'$

8. **Second order linear ODEs:** Solve the following. Clearly state the method you are using and why.
   a) $(x-1)y'' - xy' + y = 5, \quad x > 1, \quad y_1(x) = x$
      Hint 1: The solution of $y'' = \frac{x^2-2x+2}{x^2-x}y'$ is $y = c_1\frac{e^x}{x} + c_2$
      Hint 2: $\int \frac{x}{e^x(x-1)^2}dx = -\frac{e^{-x}}{x-1}$
   b) $y'' - 6y' + 9y = e^{3t} + 6$
   c) $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$
   d) $x^2y'' - 3xy' + 4y = x^2\ln(x)$
   e) $ty'' - (1+t)y' + y = t^2e^{2t}, \quad t > 0, \quad y_1 = t + 1$
      Hint: the solution of $y'' = \frac{x^2+1}{x^2+t}y'$ is $y = c_1\frac{e^t}{t+1} + c_2$

9. **Laplace transform:** Solve the following using a Laplace transform
   a) $y'' + 2y' + 2y = \cos(t) + \delta(t - \pi/2), \quad y(0) = y'(0) = 0$
   b) $y'' + 4y' + 4y = te^{-2t}, \quad y(0) = 0, \quad y'(0) = 1$

10. **Circuits** Derive the system of differential equations for this circuit.



    How would you solve the system for this circuit?

11. **Power series:** Solve the following using a power series and give the recurrence relation. Write out your solution up to fifth order. Clearly state why you cannot use another method to solve the system.
   a) $(2+x^2)y'' - xy' + 4y = 0, \quad y(0) = -1, \quad y'(0) = 3$
   b) $y'' + xy' + 2y = 0$

12. **Direct method:** Consider the ODE
$$y'' + x^2y = e^{-x}$$
    Write out the direct method system of equations for the following boundary conditions for this ODE with $N = 5$ nodes.
   a) $y(0) = 0, \quad y(4) = 5$

b) $y(0) = 0, \quad 4y'(2) + 3y(2) = 5$

13. **Numerical methods:** Write a short piece of MATLAB code to solve the following using backward Euler.

$$y' + 5y^2 = \sin(x), \quad y(0) = 10$$