CME 102 ACE – Midterm 1 Reference Sheet

First-Order ODE

Separation of Variables

For any nonlinear first order ODE, manipulate to be in form $f(y)dy = g(x)dx$ then integrate.

Two special cases for substitution:

- ODE of form $y' = f(y/x)$, use $u = y/x$
- ODE of form $y' = f(ay + bx + c)$, use $u = ay + bx + c$

Linear Inhomogeneous

ODEs of form $y' + p(x)y = r(x)$

Closed form solution:

$$y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx} r(x)dx + C\right]$$

Bernoulli Equation: $y' + p(x)y = q(x)y^n$

Solve by substituting $u = y^{1-n}$ to find ODE:

$$u' + (1-n)p(x)u = (1-n)q(x)$$

and solve for $u(x)$ to find $y(x)$

Equilibrium Solutions

System must be **autonomous** to have equilibria.

To find equilibrium solutions of $y' = f(y)$:

1. Find zeros of $f(y)$
2. Pick points in-between/outside of the zeros
3. Calculate y' at the test points
4. Classify based on sign of y' between points

Numerical Methods for IVP's

Accuracy

- Local error: error incurred over one step
- Global error: total error over the domain, one order of h less than local error, calculated as $\epsilon_{global} = N \times \epsilon_{local}$

Stability

- Derive amplification factor $\sigma(h)$ by starting with the model equation $y' = \lambda y$ and "stepping-through" the numerical method to derive relationship:

$$y_{n+1} = \sigma(h)y_n$$

- Stability condition is $|\sigma(h)| < 1$
- To find stable h, solve $\sigma(h) < 1$ and $\sigma(h) > -1$

Euler

Forward Euler: explicit, $\mathcal{O}(h)$ global accuracy:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Backward Euler: implicit, $\mathcal{O}(h)$ global accuracy:

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$$

When writing code for implicit scheme, need to first algebraically solve for y_{n+1} in terms of x_{n+1}, y_n, and h.

Trigonometric Identities

Regular trigonometric identities:

$$\sin^2 x + \cos^2 x = 1,\ \tan^2 x + 1 = \sec^2 x, 1 + \cot^2 x = \csc^2 x$$

$$\sin(2x) = 2\sin x\cos x,\ \tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

Hyperbolic trigonometric functions:

$$\sinh x = \frac{e^x - e^{-x}}{2},\ \cosh x = \frac{e^x + e^{-x}}{2},\quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1,\ \tanh^2 x + \text{sech}^2 x = 1,\ \coth^2 x - \text{csch}^2 x = 1$$

$$\sinh(2x) = 2\sinh x\cosh x,\quad \cosh(2x) = 2\cosh^2 x - 1$$

$$\tanh(2x) = \frac{2\tanh x}{1+\tanh^2 x}$$

Useful Integrals/Derivatives

Trigonometric function derivatives:

$$\frac{d}{dx}\sin x = \cos x,\ \frac{d}{dx}\cot x = -\csc^2 x,\ \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cot x = -\sin x,\ \frac{d}{dx}\sec x = \sec x\tan x,\ \frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan x = \sec^2 x,\ \frac{d}{dx}\csc x = -\csc x\cot x,\ \frac{d}{dx}\arctan x = \frac{1}{x^2+1}$$

Trigonometric and other integrals:

$$\int \tan x dx = -\log|\cos x| + C,\quad \int \cot x dx = \log|\sin x| + C$$

$$\int \csc x dx = -\log|\csc x + \cot x| + C$$

$$\int \sec x = \log|\sec x + \tan x| + C$$

$$\int \sin^2(x)dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

$$\int \cos^2(x)dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C,\ \int \ln x dx = x\ln x - x + C$$

$$\int e^{ax}\sin(bx)dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$$

$$\int e^{ax}\cos(bx)dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$$

Inverse trigonometric function integrals:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C,\ \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$$

Hyperbolic trig function derivatives:

$$\frac{d}{dx}\sinh(x) = \cosh(x),\quad \frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2(x),\quad \frac{d}{dx}\text{csch}(x) = -\coth(x)\,\text{csch}(x)$$

$$\frac{d}{dx}\text{sech}(x) = -\tanh x\,\text{sech}(x),\quad \frac{d}{dx}\coth x = 1 - \coth^2(x)$$