Week 10 Section Solutions

If not otherwise specified, solve the following problems. If initial conditions are given, solve for all constants of integration. It is okay to leave answers in implicit form or with unsolved integrals.

1. **Conceptual things:** For each of the following, give a short, snappy explanation or definition.

   a) Spanning vector/basis vector
   Solution Vectors that we can take a linear combination of to get any vector in a given vector space. We are usually concerned with 2-D space (a.k.a. x-y space), and we can form any vector/point in x-y space by taking a linear combination of the unit vectors in the x and y directions (the unit vectors lying along the axes).

   b) Basis function
   Solution Functions from which you can take a linear combination and form any other function in the space. In our purposes, we are usually concerned with functions in x-y space.

   c) Power series (a.k.a. Taylor series)
   Solution An expansion of a function into a linear combination of powers of x. This is possible for any function since the powers of x are a spanning basis for functions in x-y space.

   d) Series solution
   Solution A solution to an ODE that is given in the form of a linear combination of integer powers of x and, if possible, a recurrence relation between the coefficients.

2. **Example problems:** Solve the following using a power series solution.

   a) $y' + y = x$
   Solution
   Assume:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

   Plugging into the ODE:

$$\sum_{n=0}^{\infty} (n+1) a_n x^n + \sum_{n=0}^{\infty} a_n x^n = x$$

$$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} + a_n\right] x^n = x$$

   Matching coefficients we get:

$$a_1 = -a_0$$

$$a_2 = \frac{1 - a_1}{2}$$

   And we get the recurrence relation:

$$a_{n+1} = \frac{-a_n}{n+1}, \quad n > 2$$

   b) $y'' - xy = 0$
   This equation is known as the Airy Equation, and its solutions (known as Airy functions) have many important applications in optics and quantum mechanics.

Solution
Assume:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Plugging into the ODE:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Pulling out the 0th term in the first summation and reindexing the second:

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_{n-1}\right] x^n = 0$$

From which we can finally get the relations:

$$a_2 = 0$$

$$a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)}$$

c) $y'' - y' + x^2 y = 0$

Solution

$$y'' - y' + x^2 y = 0$$

Substituting in our power series expansions:

$$\left(\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n\right) - \left(\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n\right) + x^2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n+1)a_{n+1}\right] x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

For the first sum, let $n' = n - 2$. We can then reindex as:

$$\sum_{n'=-2}^{\infty} \left[(n'+4)(n'+3)a_{n'+4} - (n'+3)a_{n'+3}\right] x^{n'+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Now, pull the first two terms out of the first sum to make sure the bounds of our sum align:

$$[2a_2 - a_1] + [6a_3 - 2a_2]\,x + \sum_{n'=0}^{\infty} \left[(n'+4)(n'+3)a_{n'+4} - (n'+3)a_{n'+3}\right] x^{n'+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Combine the sums:

$$[2a_2 - a_1] + [6a_3 - 2a_2]\,x + \sum_{n'=0}^{\infty}\left[(n'+4)(n'+3)a_{n'+4} - (n'+3)a_{n'+3} + a_n\right]x^{n'+2} = 0$$

Matching coefficients then gives us the relationships:

$$\begin{aligned}
a_2 &= \frac{1}{2}a_1\\
a_3 &= \frac{2}{6}a_2 = \frac{1}{6}a_1\\
(n'+4)(n'+3)a_{n'+4} - (n'+3)a_{n'+3} + a_n &= 0\\
\implies a_{n+4} &= \frac{a_{n+3}}{n+4} - \frac{a_n}{(n+4)(n+3)}
\end{aligned}$$

Or we can reindex this as:

$$a_n = \frac{a_{n-1}}{n} - \frac{a_{n-4}}{n(n-1)}$$

which gives our recurrence relation.

Finally, we can write out our solution to 5th order as:

$$\begin{aligned}
y(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots\\
&= a_0 + a_1x + \frac{1}{2}a_1x^2 + \frac{1}{6}a_1x^3 + \left(\frac{a_3}{4} - \frac{a_0}{4(3)}\right)x^4 + \left(\frac{a_4}{5} - \frac{a_1}{5(4)}\right)x^5 + \cdots\\
&= a_0 + a_1x + \frac{1}{2}a_1x^2 + \frac{1}{6}a_1x^3 + \left(\frac{a_1}{24} - \frac{a_0}{12}\right)x^4 + \left(\frac{\frac{a_1}{24} - \frac{a_0}{12}}{5} - \frac{a_1}{20}\right)x^5 + \cdots\\
&= a_0 + a_1x + \frac{1}{2}a_1x^2 + \frac{1}{6}a_1x^3 + \left(\frac{a_1}{24} - \frac{a_0}{12}\right)x^4 + \left(\frac{a_1}{120} - \frac{a_0}{60} - \frac{a_1}{20}\right)x^5 + \cdots\\
&= a_0 + a_1x + \frac{1}{2}a_1x^2 + \frac{1}{6}a_1x^3 + \left(\frac{a_1}{24} - \frac{a_0}{12}\right)x^4 - \left(\frac{a_1}{24} + \frac{a_0}{60}\right)x^5 + \cdots \quad \blacksquare
\end{aligned}$$

3. **Power series in action** For the following ODE:

$$y'' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 2$$

   a) Solve this ODE using methods from earlier in the quarter. What method did you pick?
   Solution
   You could use several methods to solve this including direct integration or assuming a solution of the form $y(x) = e^{\lambda x}$ and plugging in or even Laplace transform. The solution is

$$y(x) = c_1 sin(x) + c_2 cos(x)$$

   Applying initial conditions gives $c_1 = 2 and c_2 = 1$, so

$$y(x) = 2sin(x) + cos(x)$$

b) Solve this ODE using the method of power series.
Solution
Assume:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Plugging into the ODE and reindexing:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

So the recurrence relation is:

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

But by definition we have:

$$a_0 = y(0) = 1 \qquad a_1 = y'(0) = 2$$

We can split these into relations for odd and even n. For any integer k, $(2k+1)$ is always odd and $2k$ is always even, so we can write:

$$y(x) = \sum_{n=0}^{\infty} \frac{2(-1)^n}{(2n+1)!} x^{2n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

c) Recall the Taylor expansions for sine and cosine:

$$sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Substitute these into your solution from part (a) and explain how this corresponds with your answer in part (b).
Solution
Pulling the constant out of the first summation in (b) gives:

$$y(x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

and substituting for the Taylor expansions given above we have:

$$y(x) = 2sin(x) + cos(x)$$

which is our original solution.

This problem shows that power series will always yield a correct solution if a solution does in fact exist, but the method is often more cumbersome and the solution less interpretable than with other methods.