LAPLACE TRANSFORM REVIEW SOLUTIONS

1. Find the Laplace transform for the following functions. If an image is given, first write out the function and then take the transform.

   a) $e^{-t}\sinh(4t)$

   Solution:

   From #8 on the table:

   $$\mathscr{L}\{\sinh(4t)\} = \frac{4}{s^2-16}$$

   Using #12 on the table:

   $$\mathscr{L}\{e^{at}f(t)\} = Y(s-a) \implies \mathscr{L}\{e^{-t}\sinh(4t)\} = \frac{4}{(s+1)^2-16} \quad \blacksquare$$

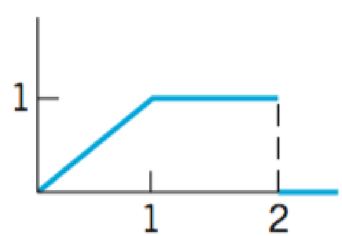
   b) $1.5\sin(3t-\pi/2)$

   Solution:

   $$\begin{aligned} f(t) &= 1.5\sin(3t-\pi/2) \\ &= 1.5\sin(3t)\cos(\pi/2) - 1.5\sin(\pi/2)\cos(3t) \\ &= -1.5\cos(3t) \\ \mathscr{L}\{f(t)\} &= -1.5\frac{s}{s^2+9} \quad \blacksquare \end{aligned}$$

   Note: we do not use #13 from the table (t-shift) because $f(t)$ is not multiplied by a Heaviside step function.

   c) Function given in the following figure:



Solution:

Whenever you're given an image that you need to find the transform of, it is first helpful to write it as a piecewise function, and then transform this piecewise function into a “single-line" form with Heaviside step functions we can take the Laplace transform of:

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 1 & 1 < t < 2 \\ 0 & t > 2 \end{cases}$$

Now we can multiply each part of the piecewise function with a boxcar function to “window" that part of the piecewise function to only its relevant interval:

$$f(t) = (1 - u(t-1))t + (u(t-1) - u(t-2))$$

Finally, we should rearrange this to be in a form such that we can apply #13 from the table (t-shift):

$$\begin{aligned}
f(t) &= (1-u(t-1))t + (u(t-1)-u(t-2)) \\
&= t - tu(t-1) + (u(t-1)-u(t-2)) \\
&= t - (t-1+1)u(t-1) + (u(t-1)-u(t-2)) \\
&= t - ((t-1)+1)u(t-1) + (u(t-1)-u(t-2)) \\
&= t - (t-1)u(t-1) - u(t-1) + u(t-1) - u(t-2) \\
&= t - (t-1)u(t-1) - u(t-2) \\
\mathscr{L}\{f(t)\} &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s} \quad \blacksquare
\end{aligned}$$

2. Solve the following initial value problems:
   a) $y'' + 9y = 10e^{-t}, \quad y(0) = y'(0) = 0$

   Solution:

$$\begin{aligned}
y'' + 9y &= 10e^{-t} \\
s^2Y(s) - sy(0) - y'(0) + 9Y(s) &= \frac{10}{s+1}
\end{aligned}$$

Apply initial conditions:

$$\begin{aligned}
s^2Y(s) + 9Y(s) &= \frac{10}{s+1} \\
Y(s)(s^2+9) &= \frac{10}{s+1} \\
Y(s) &= \frac{10}{(s+1)(s^2+9)}
\end{aligned}$$

Now we need to do a partial fraction decomposition:

$$\begin{aligned}
\frac{10}{(s+1)(s^2+9)} &= \frac{A}{s+1} + \frac{Bs+C}{s^2+9} \\
10 &= A(s^2+9) + Bs(s+1) + C(s+1) \\
&= As^2 + 9A + Bs^2 + Bs + Cs + C \\
&= s^2(A+B) + s(B+C) + 9A + C \\
A &= 1 \\
B &= -1 \\
C &= 1
\end{aligned}$$

Finally, substitute back in the partial fraction and take the inverse transform:

$$\begin{aligned}
Y(s) &= \frac{10}{(s+1)(s^2+9)} \\
&= \frac{1}{s+1} + \frac{-s}{s^2+9} + \frac{1}{s^2+9}
\end{aligned}$$

$$= \frac{1}{s+1} + \frac{-s}{s^2+9} + \frac{1}{3}\frac{3}{s^2+9}$$
$$y(t) = e^{-t} - \cos(3t) + \frac{1}{3}\sin(3t) \quad \blacksquare$$

b) $y'' + 0.04y = 0.02t^2, \quad y(0) = -25, \quad y'(0) = 0$

Solution:

$$\begin{aligned}
y'' + 0.04y &= 0.02t^2 \\
s^2 Y(s) - sy(0) - y'(0) + 0.04Y(s) &= 0.02\frac{2}{s^3} \\
Y(s)(s^2+0.04) + 25s &= \frac{0.04}{s^3} \\
Y(s)(s^2+0.04) &= \frac{0.04}{s^3} - 25s \\
Y(s) &= \frac{0.04}{s^3(s^2+0.04)} - \frac{25s}{s^2+0.04}
\end{aligned}$$

Now, notice that we need to do a partial fraction decomposition on the first fraction but not on the second. We can already take the Laplace transform of this, so we do not need to decompose it (not that we could, anyways).

$$\begin{aligned}
\frac{0.04}{s^3(s^2+0.04)} &= \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{Ds+E}{s^2+0.04} \\
A &= 1 \\
B &= 0 \\
C &= -25 \\
D &= 25 \\
E &= 0 \\
\frac{0.04}{s^3(s^2+0.04)} &= \frac{1}{s^3} - \frac{25}{s} + \frac{25s}{s^2+0.04}
\end{aligned}$$

And now we can find the final solution:

$$\begin{aligned}
Y(s) &= \frac{0.04}{s^3(s^2+0.04)} - \frac{25s}{s^2+0.04} \\
&= \frac{1}{s^3} - \frac{25}{s} + \frac{25s}{s^2+0.04} - \frac{25s}{s^2+0.04} \\
&= \frac{1}{2}\frac{2}{s^3} - \frac{25}{s} + \frac{25s}{s^2+0.04} - \frac{25s}{s^2+0.04} \\
&= \frac{1}{2}\frac{2}{s^3} - \frac{25}{s} \\
y(t) &= \frac{1}{2}t^2 - 25 \quad \blacksquare
\end{aligned}$$

3. Find the inverse Laplace transform:

a) $Y(s) = \frac{2(e^{-s}-e^{-3s})}{s^2-4}$

Solution:

$$Y(s) = \frac{2(e^{-s}-e^{-3s})}{s^2-4}$$

$$= 2(e^{-s} - e^{-3s})\frac{1}{s^2-4}$$
$$= \frac{2e^{-s}}{s^2-4} - \frac{2e^{-3s}}{s^2-4}$$

Using #8 and #13 on the transform table:

$$y(t) = u(t-1)\sinh(2(t-1)) - u(t-3)\sinh(2(t-3)) \quad \blacksquare$$

b) $Y(s) = \frac{1+e^{2\pi(s+1)}(s+1)}{(s+1)^2+1}$

Solution:

First split up the fraction, since this will make it easier to handle the inverse transform:

$$\begin{aligned} Y(s) &= \frac{1+e^{2\pi(s+1)}(s+1)}{(s+1)^2+1} \\ &= \frac{1}{(s+1)^2+1} + \frac{e^{2\pi(s+1)}(s+1)}{(s+1)^2+1} \end{aligned}$$

Now, notice that we have both an s-shift and a t-shift. When applying the shifts, you first apply the shift in the domain you are in, then apply the shift for the domain you are going to. So in this case, we need to take case of the s-shift, *then* take care of the t-shift.

$$\begin{aligned} Y(s) &= \frac{1}{(s+1)^2+1} + \frac{e^{2\pi(s+1)}(s+1)}{(s+1)^2+1} \\ y(t) &= e^{-t}\sin(t) + u(t+2\pi)e^{-(t+2\pi)}\cos(t+2\pi) \quad \blacksquare \end{aligned}$$

4. Solve the initial value problem:

$$y'' + 9y = f(t), \quad y(0) = 0, \quad y'(0) = 4$$

where $f(t) = 8\sin(t)$ for $0 < t < \pi$ and 0 for $t > \pi$.

Solution:

We can use step functions to "window" the sine function to only be over $0 < t < \pi$. We can thus rewrite $f(t)$ with step functions, then solve the ODE:

$$\begin{aligned} f(t) &= (1-u(t-\pi))(8\sin(t)) \\ y'' + 9y &= (1-u(t-\pi))(8\sin(t)) \\ &= 8\sin(t) - 8u(t-\pi)\sin(t) \\ &= 8\sin(t) - 8u(t-\pi)\sin(t-\pi+\pi) \\ &= 8\sin(t) - 8u(t-\pi)(\sin(t-\pi)\cos(\pi) + \sin(\pi)\cos(t-\pi)) \\ &= 8\sin(t) + 8u(t-\pi)\sin(t-\pi) \\ s^2Y(s) - 4 + 9Y(s) &= \frac{8}{s^2+1} + \frac{8e^{-\pi s}}{s^2+1} \\ Y(s) &= \frac{8}{(s^2+1)(s^2+9)} + \frac{8e^{-\pi s}}{(s^2+1)(s^2+9)} + \frac{4}{s^2+9} \end{aligned}$$

Using partial fractions:

$$
\begin{aligned}
\frac{1}{(s^2+1)(s^2+9)} &= \frac{A}{s^2+1} + \frac{B}{s^2+9} \\
A &= \frac{1}{8} \\
B &= -\frac{1}{8} \\
\frac{1}{(s^2+1)(s^2+9)} &= \frac{1}{8(s^2+1)} - \frac{1}{8(s^2+9)}
\end{aligned}
$$

Resubstitute this fraction and take the inverse transform:

$$
\begin{aligned}
Y(s) &= 8\left(\frac{1}{8(s^2+1)} - \frac{1}{8(s^2+9)}\right) + 8e^{-\pi s}\left(\frac{1}{8(s^2+1)} - \frac{1}{8(s^2+9)}\right) + \frac{4}{(s^2+9)} \\
&= \left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right) + e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right) + \frac{4}{s^2+9} \\
&= \left(\frac{1}{s^2+1} - \frac{1}{3}\frac{3}{s^2+9}\right) + e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{3}\frac{3}{s^2+9}\right) + \frac{4}{3}\frac{3}{s^2+9} \\
y(t) &= \sin(t) - \frac{1}{3}\sin(3t) + u(t-\pi)\left(\sin(t-\pi) - \frac{1}{3}\sin(3(t-\pi)\right) + \frac{4}{3}\sin(3t) \\
&= \sin(t) + \sin(3t) + u(t-\pi)\left(\sin(t-\pi) - \frac{1}{3}\sin(3(t-\pi))\right) \quad \blacksquare
\end{aligned}
$$

5. Solve the initial value problem:

$$
y'' + 4y' + 5y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 3
$$

Solution:

We can solve this directly using the Laplace transform:

$$
\begin{aligned}
y'' + 4y' + 5y &= \delta(t-1) \\
s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 5Y(s) &= e^{-s} \\
s^2Y(s) - 3 + 4sY(s) + 5Y(s) &= e^{-s} \\
Y(s)\left(s^2+4s+5\right) &= e^{-s} + 3 \\
Y(s) &= \left(e^{-s}+3\right)\frac{1}{s^2+4s+5}
\end{aligned}
$$

We need to complete the square for the denominator, then we can take the inverse transform:

$$
\begin{aligned}
Y(s) &= \left(e^{-s}+3\right)\frac{1}{s^2+4s+5} \\
&= \left(e^{-s}+3\right)\frac{1}{s^2+4s+4+1} \\
&= \left(e^{-s}+3\right)\frac{1}{(s+2)^2+1} \\
&= \frac{e^{-s}}{(s+2)^2+1} + \frac{3}{(s+2)^2+1} \\
y(t) &= u(t-1)e^{-2(t-1)}sin(t-1) + 3e^{-2t}sin(t) \quad \blacksquare
\end{aligned}
$$

6. a) Find the Laplace transform: $f(t) = \frac{1}{2}te^{-3t}$

Solution:

Using #9 on the Laplace transform table:

$$\mathscr{L}\{t^n g(t)\} = (-1)^n G^{(n)}(s) \implies \mathscr{L}\{tg(t)\} = -\frac{d}{ds}(G(s))$$

$$\begin{aligned} g(t) &= \frac{1}{2}e^{-3t} \\ f(t) &= tg(t) \\ F(s) &= -\frac{d}{ds}\mathscr{L}\{g(t)\} \\ &= -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s+3}\right) \\ &= \frac{1}{2(s+3)^2} \quad \blacksquare \end{aligned}$$

b) Find the inverse Laplace transform: $F(s) = \cot^{-1}\left(\frac{s}{\pi}\right)$. *Hint:* $\frac{d}{dx}\left(\cot^{-1}(x)\right) = \frac{-1}{1+x^2}$.

Solution:

We have to use #9, but in the other direction this time:

$$\begin{aligned} F(s) &= \cot^{-1}\left(\frac{s}{\pi}\right) \\ F'(s) &= \frac{1}{\pi}\frac{-1}{1+(s/\pi)^2} \\ &= \frac{-\pi}{\pi^2+s^2} \\ -F'(s) &= \frac{\pi}{\pi^2+s^2} \end{aligned}$$

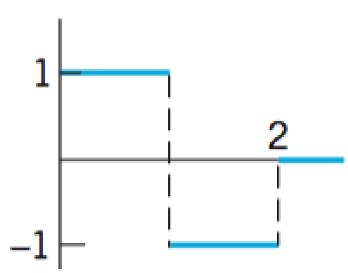
Using #6 on the transform table:

$$\mathscr{L}^{-1}\{-F'(s)\} = \sin(\pi t)$$

and then using #9 from the table:

$$\begin{aligned} tf(t) &= \sin(\pi t) \\ f(t) &= \frac{\sin(\pi t)}{t} \quad \blacksquare \end{aligned}$$

7. Solve $y'' + 4y = f(t)$, $\quad y(0) = y'(0) = 0$ with $f(t)$ defined by the following figure:



Solution:

You first need to express $f(t)$ as a piecewise function, then express it using Heaviside step functions:

$$\begin{aligned}
f(t) &= \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \\ 0 & t > 2 \end{cases} \\
\implies f(t) &= (1 - u(t-1)) - (u(t-1) - u(t-2)) \\
&= 1 - 2u(t-1) + u(t-2)
\end{aligned}$$

Now, we can plug this into our ODE and solve via Laplace transform.

$$\begin{aligned}
y'' + 4y &= f(t) \\
&= 1 - 2u(t-1) + u(t-2) \\
s^2 Y(s) - sy(0) - y'(0) + 4Y(s) &= \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} \\
s^2 Y(s) + 4Y(s) &= \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} \\
Y(s)(s^2+4) &= \left(1 - 2e^{-s} + e^{-2s}\right)\frac{1}{s} \\
Y(s) &= \left(1 - 2e^{-s} + e^{-2s}\right)\frac{1}{s(s^2+4)} \\
\frac{1}{s(s^2+4)} &= \frac{A}{s} + \frac{Bs+C}{s^2+4} \\
1 &= As^2 + 4A + Bs^2 + Cs \\
1 &= 4A, \quad 0 = C, \quad 0 = A + B \\
A &= \frac{1}{4} \\
B &= -\frac{1}{4} \\
\frac{1}{s(s^2+4)} &= \frac{1}{4s} - \frac{s}{4(s^2+4)} \\
Y(s) &= \left(1 - 2e^{-s} + e^{-2s}\right)\left(\frac{1}{4s} - \frac{s}{4(s^2+4)}\right) \\
&= \frac{1}{4s} - \frac{s}{4(s^2+4)} - 2e^{-s}\left(\frac{1}{4s} - \frac{s}{4(s^2+4)}\right) + e^{-2s}\left(\frac{1}{4s} - \frac{s}{4(s^2+4)}\right) \\
y(t) &= \frac{1}{4} - \frac{1}{4}\cos(2t) - 2u(t-1)\left(\frac{1}{4} - \frac{1}{4}\cos(2(t-1))\right) + \\
&\quad u(t-2)\left(\frac{1}{4} - \frac{1}{4}\cos(2(t-2))\right) \quad \blacksquare
\end{aligned}$$