CME 100 ACE – Final Exam Reference Sheet

Dot Product

$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = |\vec{v}|\,|\vec{w}|\cos\theta$

Projection vectors: the projection of $\vec{w}$ onto $\vec{v}$:

$$\text{proj}_{\vec{v}}\vec{w} = \frac{\vec{w}\cdot\vec{v}}{|\vec{v}|}\frac{\vec{v}}{|\vec{v}|}$$

Cross Product

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= (v_2 w_3 - w_2 v_3)\vec{i} - (v_1 w_3 - v_3 w_1)\vec{j} + (v_1 w_2 - w_2 v_1)\vec{k}$$

$|\vec{v}\times\vec{w}| = |\vec{v}|\,|\vec{w}|\sin\theta$

Area of parallelogram ABCD: $Area = |\vec{AB}\times\vec{AD}|$

Area of triangle ABD: $Area = \frac{1}{2}|\vec{AB}\times\vec{AC}|$

Lines and Planes

Parameterization of a line through P_0 parallel to vector $\vec{v}$:

$$\ell(t) = P_0 + t\vec{v}$$

Two vectors/lines are

- **Perpendicular (orthogonal)** if their *dot product* is 0
- **Parallel** if their *cross-product* is 0

The plan through point P_0 with normal $\vec{n}$:

$$(x - P_{0x}, y - P_{0y}, z - P_{0z})\cdot\vec{n} = 0$$

Vector-Valued functions

For a parameterized curve:

$$\vec{r} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

the velocity and acceleration are given by

$$\vec{v} = \frac{d\vec{r}}{dt}, \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2}$$

Arc length for a curve parameterized over $t_1 \le t \le t_2$ is given by:

$$L = \int_{t_1}^{t_2}\sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\,dt$$

TNB-frame, Curvature, and Torsion

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}, \qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\vec{B} = \vec{T}\times\vec{N}, \qquad \kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}$$

$$\tau = -\frac{d\vec{B}}{ds}\cdot\vec{N} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v}\times\vec{a}|^2}$$

Tangential and normal components of acceleration:

$$\vec{a} = a_T\vec{T} + a_N\vec{N}, \quad a_T = \frac{d}{dt}|\vec{v}|$$

$$a_N = \kappa|\vec{v}|^2 = \sqrt{|\vec{a}|^2 - a_T^2}$$

Matrix Operations

Vector: 1D array of values, taken to usually indicate a *column vector.*

Matrix: 2D array of values. $m\times n$ matrix indicates m rows and n columns. Entry a_{ij} is denoted by row number first, then column number.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

Note: it is easiest to think of all matrix/vector objects as matrices, and row/column vectors as matrices with size one along one dimension.

Transpose: flip the dimensions and indices of the entries. *Ex.* if $\mathbf{A}$ is $m\times n$ then $\mathbf{A}^T$ is $n\times m$ and $(A^T)_{ij} = a_{ji}$.

Symmetric matrix: a *square matrix* such that $\mathbf{A} = \mathbf{A}^T$.

Skew-symmetric matrix: a *square matrix* such that $\mathbf{A}^T = -\mathbf{A}$.

Diagonal matrix: special type of matrix such that only the diagonal elements are non-zero. *Note:* diagonal matrices are always square matrices.

$$\mathbf{D} = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & d_{nn} \end{bmatrix}$$

Identity matrix: diagonal matrix where the diagonal entries are all 1. Denoted as $\mathbf{I}$. Satisfies $\mathbf{A} = \mathbf{IA} = \mathbf{AI}$.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}$$

Matrix Multiplication

For a matrix-matrix product $\mathbf{C} = \mathbf{AB}$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i1}b_{1j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

- Size along the dimension being multiplied (the *inner dimension*) must match.
- Matrix product takes on the *outer dimensions. Ex.* if $\mathbf{A}$ is $m\times n$ and $\mathbf{B}$ is $n\times p$, then $\mathbf{C} = \mathbf{AB}$ is $m\times p$.
- Matrix multiplication is **not** commutative i.e. $\mathbf{AB} \neq \mathbf{BA}$.

Vector multiplication: special case of matrix multiplication where one matrix has size 1 along one of its outer dimensions.

Matrix Inverse

Matrix inverse: for a given matrix $\mathbf{A}$, the inverse is the matrix $\mathbf{M}$ such that $\mathbf{MA} = \mathbf{AM} = \mathbf{I}$.

- Denote the matrix inverse as $\mathbf{A}^{-1}$.
- To show that a given matrix B is the inverse of $\mathbf{A}$, show that $\mathbf{AB} = \mathbf{I}$.

Inverse of 2×2 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinants

Determinants only exist for square matrices.

2×2 **Determinant:**

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det\mathbf{A} = ad - cb$$

3×3 **Determinant:** "add the products of the down-diagonals, subtract the products of the up-diagonals"

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \det\mathbf{A} = aei + bfg + cdh - gec - hfa - idb$$

4×4 **Determinant:** multiply the first row element by the minor associated with that element, multiply by -1 if minor is formed by an even-numbered column. *Ex.:*

$$\det\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1\det\begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2\det\begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix}$$

$$+a_3\det\begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - a_4\det\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

Multi-variable differentiation

Gradient: direction of fastest increase in f & normal to level curve / level surface

$$\vec{\nabla}f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

Directional derivative: $\vec{\nabla}f\cdot\vec{n}$ (in direction of unit vector $\vec{n}$)

Chain rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \vec{\nabla}f\cdot\frac{d\vec{r}}{dt}$$

Linear approximation:
$f(x,y) \approx f(x_0,y_0) + (x - x_0)f_x(x_0,y_0) + (y - y_0)f_y(x_0,y_0)$,
$f(\vec{r}) \approx f(\vec{r}_0) + (\vec{r} - \vec{r}_0)\cdot\vec{\nabla}f(\vec{r}_0)$

Tangent line & plane to $f(x,y,z)$ at (x_0,y_0,z_0):

$$\vec{\nabla}f(\vec{r}_0)\cdot(\vec{r} - \vec{r}_0) = 0$$

Optimization

Unconstrained

Critical point ("first order condition"): $\vec{\nabla} f = 0$

Saddle point: $f_{xx}f_{yy} - f_{xy}^2 < 0$

Min ("second order condition"): $f_{xx}f_{yy} - f_{xy}^2 > 0, \quad f_{xx} > 0$

Max ("second order condition"): $f_{xx}f_{yy} - f_{xy}^2 > 0, \quad f_{xx} < 0$

If $f_{xx}f_{yy} - f_{xy}^2 = 0$, then the extremum is undefined, must use higher order derivative

Constrained

Problem: minimize or maximize $f(x,y,z)$ subject to $g(x,y,z) = 0$

Method 1 (Lagrange multipliers): solve the system

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$
$$g(x,y,z) = 0$$

Method 2: use $g(x,y,z) = 0$ as parametrized boundary & find critical points (plug in the constraint to the objective and optimize)

General coordinate transform

Formula:

$$\iiint f(x,y,z)dxdydz = \iiint f(u,v,w)|J(u,v,w)|dudvdw$$

Jacobian

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u & \partial z/\partial u \\ \partial x/\partial v & \partial y/\partial v & \partial z/\partial v \\ \partial x/\partial w & \partial y/\partial w & \partial z/\partial w \end{vmatrix} = \frac{1}{J(x,y,z)}$$

Polar coordinates: $x = r\cos\theta,\ y = r\sin\theta,\ J(r,\theta) = r$

Cylindrical: $x = r\cos\theta,\ y = r\sin\theta,\ z = z,\ J(r,\theta,z) = r$

Spherical:
$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi,\ J(\rho,\theta,\phi) = \rho^2\sin\phi$

Area, mass, centroid, moment of inertia

Volume, mass:

$$V = \iiint_D dxdydz; \quad M = \iiint_D \delta(x,y,z)dxdydz$$

Centroid (replace x by y or z for other components):

$$\overline{x} = \frac{1}{M}\iiint_D x\delta(x,y,z)dxdydz$$

Moment of inertia (rotate x, y and z for I_y and I_z):

$$I_x = \iiint_D (y^2+z^2)\delta(x,y,z)dxdydz$$

Line integral along path C

Work line integral: $W = \int_C \vec{F}\cdot d\vec{r} = \int_C Mdx + Ndy + Pdz$

Flux line integral: $\Phi = \int_C Mdy - Ndx$

Conservative $\vec{F}$ conditions:

$$M_y = N_x; \quad M_z = P_x; \quad N_z = P_y$$

Potential function $\vec{F} = \vec{\nabla} f; \quad M = f_x; \quad N = f_y; \quad P = f_z$

Work using path independence: $W = \int_a^b \vec{V}\cdot d\vec{r} = f(b) - f(a)$

Surface integral

Surface S described by $f(x,y,z) = 0$ (zero level set)

Parametrized S (e.g. spherical coordinates):

$$\iint_S g(x,y,z)d\sigma = \iint_R g(x(u,v),y(u,v),z(u,v))\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|dudv$$

where R is rectangular region in transformed space (u,v).

Explicit S ($z = f(x,y)$) (rotate partial derivatives if projecting into different plane):

$$\iint_S g(x,y,z)d\sigma = \iint_R g(x,y,z)\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}dxdy$$

Implicit S ($f(x,y,z) = 0$):

$$\iint_S g(x,y,z)d\sigma = \iint_R g(x,y,z)\,\frac{|\vec{\nabla} f|}{|\vec{\nabla} f\cdot\vec{k}|}\,dxdy$$

where R is projection of S on x-y plane.

Note: you do not introduce new variables for explicit/implicit surfaces, only in parameterized.

Green's theorems

Circulation-Curl form (for work/circulation):

$$\oint_C Mdx + Ndy = \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

Note: if conservative $\vec{F}$, then circulation $= 0$

Flux-Divergence form (for flux):

$$\oint_C Mdy - Ndx = \iint_R\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dxdy$$

Stokes' & Divergence theorems

Curl definition: curl $\vec{F} = \vec{\nabla}\times\vec{F}$

Stokes' theorem (for calculating circulation over a closed surface:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S(\vec{\nabla}\times\vec{F}\cdot\vec{n})\,d\sigma = \iint_R\frac{\vec{\nabla}\times\vec{F}\cdot\vec{\nabla} f}{|\vec{\nabla} f\cdot\vec{k}|}\,dxdy$$

where R is projection of S onto x-y plane (this transforms a circulation integral about the edge into a surface integral)

Divergence definition: div $\vec{F} = \vec{\nabla}\cdot\vec{F}$

Divergence theorem (for calculating flux over closed surface):

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D(\vec{\nabla}\cdot\vec{F})dV$$

where D is the volume enclosed by surface S and $\vec{n}$ is outward normal to surface S

Trigonometric Identities

Regular trigonometric identities:

$$\sin^2 x + \cos^2 x = 1, \qquad \tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x, \sin(2x) = 2\sin x\cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

Hyperbolic trigonometric functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\cosh^2 x - \sinh^2 x = 1, \quad \tanh^2 x + \text{sech}^2 x = 1$$
$$\coth^2 x - \text{csch}^2 x = 1$$
$$\sinh(2x) = 2\sinh x\cosh x, \quad \cosh(2x) = 2\cosh^2 x - 1$$
$$\tanh(2x) = \frac{2\tanh x}{1+\tanh^2 x}$$

Useful Integrals/Derivatives

Trigonometric function derivatives:

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cot x = -\csc^2 x, \quad \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cot x = -\sin x, \quad \frac{d}{dx}\sec x = \sec x\tan x, \quad \frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan x = \sec^2 x, \quad \frac{d}{dx}\csc x = -\csc x\cot x, \quad \frac{d}{dx}\arctan x = \frac{1}{x^2+1}$$

Trigonometric function integrals:

$$\int\csc x dx = -\log|\csc x + \cot x| + C$$
$$\int\sec x = \log|\sec x + \tan x| + C$$
$$\int\tan x dx = -\log|\cos x| + C, \quad \int\cot x dx = \log|\sin x| + C$$

Hyperbolic trig function derivatives:

$$\frac{d}{dx}\sinh(x) = \cosh(x), \quad \frac{d}{dx}\cosh(x) = \sinh(x)$$
$$\frac{d}{dx}\tanh x = 1 - \tanh^2(x), \quad \frac{d}{dx}\text{csch}(x) = -\coth(x)\ \text{csch}(x)$$
$$\frac{d}{dx}\text{sech}(x) = -\tanh x\ \text{sech}(x), \quad \frac{d}{dx}\coth x = 1 - \coth^2(x)$$

MATLAB

Matrix/vector multiplication: `A*B`

Dot product: `dot(u,v)`

Cross product: `cross(u,v)`

Vector magnitude: `norm(v)`

Absolute value: `abs(a)`

Determinant: `det(A)`

MATLAB examples

Example 12-5

As an example, we evaluate the following integral

$$A = \int_0^2 \sqrt{x}dx$$

The Matlab script below uses $\Delta x = 0.01$ (200 intervals) which gives $A = 1.8783$, 1.8857, and 1.8925 from left, middle and right Riemann sums, respectively; whereas the exact solution is 1.8856. It is clear that, for the same number of intervals, the middle Riemann sum is the most accurate while the other two either overestimate or underestimate the exact integral.

```
    % integral of f(x) = sqrt(x) from 0 to 2
    % exact answer:  1.8856
clear
close all
```



```
a = 0; b = 2;
n = 200;
dx = (b-a)/n;
x = a:dx:b;
area_l = 0.;
area_m = 0.;
area_r = 0.;
for i=1:length(x)-1
    area_l = area_l + sqrt(x(i))*dx;
    area_m = area_m + sqrt(x(i)+dx/2)*dx;
    area_r = area_r + sqrt(x(i)+dx)*dx;
end
fprintf('\n number of intervals = %i \n',length(x)-1)
fprintf(' area (Riemann left sum) = %8.5f \n',area_l)
fprintf(' area (Riemann middle sum) = %8.5f \n',area_m)
fprintf(' area (Riemann right sum) = %8.5f \n',area_r)
```

Matlab results

```
number of intervals = 200
area (Riemann left sum) =  1.87834
area (Riemann middle sum) =  1.88568
area (Riemann right sum) =  1.89248
```

Matlab script

```
    % double integral of f(x,y) = 4x+2
    % boundaries: from y = 2x to y = x^2
    % Riemann sum:  middle sum
    % exact answer:  8.0
clear
close all
N = 200; M = 400;            % number of intervals in x & y directions
a = 0; b = 2; c = 0; d = 4;
x = linspace(a,b,N+1);       % array containing x points (N+1)
y = linspace(c,d,M+1);       % array containing y points (M+1)
dx = x(2)-x(1);              % x spacing
dy = y(2)-y(1);              % y spacing
    % start the integral loops
volume = 0.;
    % y integral is the outer loop
```



```
for j=1:length(y)-1
        % find lower x limit at this y location
    x_lower = (y(j)+dy/2)/2;     % x = y/2, Riemann middle sum
        % find starting element of x at lower limit
    k1 = length(find(x_lower >= x));
        % find upper x limit at this y location
    x_upper = sqrt(y(j)+dy/2);   % x = sqrt(y), Rieamann middle sum
        % find ending element of x at upper limit
    k2 = length(find(x_upper >= x));
        % x integral is the inner loop, from i=k1 to i=k2-1
    for i=k1:k2-1
        volume = volume + (4*(x(i)+dx/2)+2)*dx*dy;
    end
end
fprintf('\n volume = %8.5f \n',volume)
```

Matlab result

```
volume =  7.97852
```

Command	What it does
`a = B(:,1)`	transfer all elements of column 1 of B to a
`sum(a)`	sum all elements in vector a
`length(a)`	get the length of vector a (longest dimension)
`x = A\b`	matrix operation: $x = A^{-1}b$

Solution

Matlab script

```
fileID = fopen('buildingstories.txt','r');
formatSpec = '%f %f %f';
sizeB = [3 Inf];
%  Read building data into array B and reorder as 3 columns.
B = fscanf(fileID,formatSpec,sizeB);
B = B';
%  Close the file
fclose(fileID);
%
xi = B(:,3);
yi = B(:,2);
N = length(xi);
A = [sum(xi.^2) sum(xi); sum(xi) N];
b = [sum(xi.*yi) ; sum(yi)];
```

```
mb = A\b
%
figure(1)
plot(xi,yi,'ro')
hold on
plot(xi,mb(1)*xi+mb(2),'k')
xlabel('Number of Stories')
ylabel('Building height')
title('Relationship between building height and stories')
```