Week 6 Section Problems

If not otherwise specified, solve the following problems. If initial conditions are given, solve for all constants of integration. It is okay to leave answers in implicit form or with unsolved integrals.

1. For the following ODEs, if possible give the method we would use to solve them and why. *Note:* it is not necessary to solve these, simply give the steps you would hypothetically use to solve the ODE.

   a) $x^2 y'' + xy' + 2y = cos(x)$

      Solution: Variation of parameters (Cauchy-Euler equation to get the basis of the homogeneous solution).

   b) $(y'')^2 + y = cos(x)$

      Solution: Cannot be solved analytically, or at least with methods covered in this class. We move on.

   c) $y'' + 2y' + y = cos(x)$

      Solution: Undetermined coefficients. Variation of parameters would also work, but you should not use it since undetermined coefficients is much more efficient at solving this kind of equation.

   d) $y'' + 2y' + y = cos^2(x)$

      Solution: Variation of parameters or undetermined coefficients (if you split up the right hand side using trig identities).

2. $y'' - 4y' + 4y = e^{2x}$

   Solution
   For the homogeneous solution:

   $$\lambda^2 - 4\lambda + 4 = 0$$

   $\lambda = 2$, $y_h(x) = e^{2x}(c_1 + c_2 x)$
   We can then use undetermined coefficients to solve this ODE. However, note that the RHS term has the same exponent as the homogeneous solution, so we need to assume an answer of the form $y_p(x) = Ax^2 e^{2x}$:

   $$2Ae^{2x}(2x^2 + 4x + 1) - 2Ae^{2x}x(x+1) + 4Ax^2 e^{2x} = e^{2x}$$

   Matching coefficients, we find $A = \frac{1}{2}$ so the full final solution is:

   $$y(x) = e^{2x}(c_1 + c_2 x) + \frac{1}{2}x^2 e^{2x}$$

3. $y'' + 2y = sec(\sqrt{2}x)$

Solution
The homogeneous solution is giving by: $y(x) = c_1 sin(\sqrt{2}x) + c_2 cos(\sqrt{2}x)$. We then need to use variation of parameters to find the particular solution:

$$W = y_1 y_2' - y_2 y_1' = \sqrt{2}$$

$$y_p(x) = -\frac{\sqrt{2}}{2} cos(\sqrt{2}x) \int sec(\sqrt{2}x) sin(\sqrt{2}x) dx + \frac{\sqrt{2}}{2} sin(\sqrt{2}x) \int sec(\sqrt{2}x) cos(\sqrt{2}x) dx$$

$$y_p(x) = -\frac{\sqrt{2}}{2} cos(\sqrt{2}x) \int tan(\sqrt{2}x) dx + \frac{\sqrt{2}}{2} sin(\sqrt{2}x) \int dx$$

$$y_p(x) = \frac{1}{2} cos(\sqrt{2}x) ln(|cos(\sqrt{2}x)|) + \frac{\sqrt{2}}{2} x(sin(\sqrt{2}x))$$

$$y(x) = c_1 sin(\sqrt{2}x) + c_2 cos(\sqrt{2}x) + \frac{1}{2} cos(\sqrt{2}x) ln(|cos(\sqrt{2}x)|) + \frac{\sqrt{2}}{2} x(sin(\sqrt{2}x))$$

4. Improved Euler, also known as Heun's Method, has equations given by:

$$y' = f(t, y(t))$$

$$y_{i+1/2} = y_i + hf(t_i, y_i)$$

$$y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, y_{i+1/2})]$$

Derive the amplification factor for the general equation $y' = \lambda y$. Using this, could you give the local accuracy for this method?

Solution

$$y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, \tilde{y}_{i+1})] = y_i + \frac{h}{2}[\lambda y_i + \lambda y_i + h\lambda^2 y_i]$$

$$y_{i+1} = \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right) y_i$$

$$\sigma(h) = 1 + h\lambda + \frac{h^2\lambda^2}{2}$$

The method would be 3rd order accurate locally since the amplification factor matches the Taylor expansion of $y(x_i + h)$ up to the 3rd order term.