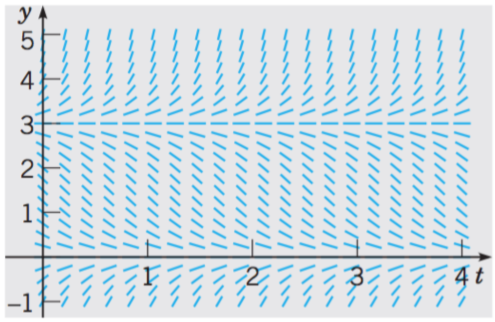
FINAL REVIEW PROBLEMS

If not otherwise specified, solve the following problems. If initial conditions are given, solve for all constants of integration. It is okay to leave answers in implicit form or with unsolved integrals.

1. **Direction fields and Equilibrium solutions:**

   Identify the equilibrium solutions and state their type.

   a) For the following figure:

   b) $y' = y(y-2)^2$

   c) $y' = y(y-3)(y-x)$

2. **Solution verification:** Verify the solution to the following ODEs.

   a) $x^2y'' + 2xy' + y = \ln(x) + 3x + 1, \quad y = \ln(x) + x$

   b) $(y'')^3 + (y')^2 - y - 3x^2 - = 0, \quad y = x^2$

   c) $y' = (x+y)^2, \quad y = \tan(x) - x$

3. **Separation of variables:** Solve the following ODEs using separation of variables.

   a) $y' + y^2 \sin(x) = 0$

   b) $y' = 2 + 2x + 2y^2 + 2xy^2, \quad y(0) = 0$

   c) $y' = \frac{ty(4-y)}{1+t}$

4. **Existence and uniqueness:** Give the interval for existence and interval for uniqueness of the solution.

   a) $\sin(2x)dx + \cos(3y)dy = 0, \quad y(\pi/2) = \pi/3$

   b) $y^2(1-x^2)^{1/2}dy = \sin^{-1}(x)dx, \quad y(0) = 1$

5. **Linear first order ODEs:** Solve the following.

   a) $y' - 2y = 4 - t$

   b) $ty' + 2y = \sin(t), \quad y(\pi/2) = 0$

   c) $ty' + (t+1)y = t, \quad y(\ln(2)) = 1$

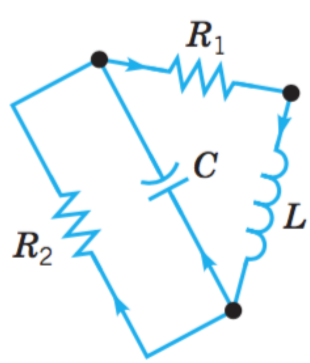
6. **Eigenvector solution to ODEs:** Solve the following using an eigenvalue/vector system.

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}$$

7. **Nonlinear second order ODEs:** Solve the following.
   a) $y'' = -2t(y')^2, \quad y(0) = 2, \quad y'(0) = -1$
   b) $y'' = 2yy'$
8. **Second order linear ODEs:** Solve the following. Clearly state the method you are using and why.
   a) $(x-1)y'' - xy' + y = 5, \quad x > 1, \quad y_1(x) = x$
      Hint 1: The solution of $y'' = \frac{x^2-2x+2}{x^2-x} y'$ is $y = c_1 \frac{e^x}{x} + c_2$
      Hint 2: $\int \frac{x}{e^x(x-1)^2} dx = -\frac{e^{-x}}{x-1}$
   b) $y'' - 6y' + 9y = e^{3t} + 6$
   c) $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$
   d) $x^2 y'' - 3xy' + 4y = x^2 \ln(x)$
   e) $ty'' - (1+t)y' + y = t^2 e^{2t}, \quad t > 0, \quad y_1 = t + 1$
      Hint: the solution of $y'' = \frac{x^2+1}{x^2+t} y'$ is $y = c_1 \frac{e^t}{t+1} + c_2$
9. **Mass-spring system:** Consider the equation of motion for a mass-spring system:

$$mx'' + \beta x' + kx = f(t)$$

   For the following values of m, β, and k and form of $f(t)$, state if the homogeneous solution is over-damped, critically damped, If the motion is forced, state whether we will have beats, resonance, or neither, and why.
   a) $m = 1, \beta = 5, k = 2, f(x) = 0$
   b) $m = 2, \beta = 0, k = \frac{1}{2}, f(t) = \sin(0.49t)$
   c) $m = 2, \beta = 8, k = 8, f(x) = 0$
   d) $m = 3, \beta = 0, k = \frac{1}{3}, f(t) = \sin(t/3)$
   e) $m = 5, \beta = 2, k = 1, f(t) = \sin(2t/5)$
10. **Laplace transform:** Solve the following using a Laplace transform
   a) $y'' + 2y' + 2y = \cos(t) + \delta(t - \pi/2), \quad y(0) = y'(0) = 0$
   b) $y'' + 4y' + 4y = te^{-2t}, \quad y(0) = 0, \quad y'(0) = 1$
11. **Circuits** Derive the system of differential equations for this circuit.



   How would you solve the system for this circuit?

12. **Direct method:** Consider the ODE

$$y'' + x^2 y = e^{-x}$$

Write out the direct method system of equations for the following boundary conditions for this ODE with $N = 5$ nodes.

a) $y(0) = 0, \quad y(4) = 5$

b) $y(0) = 0, \quad 4y'(2) + 3y(2) = 5$

13. **Numerical methods:** Write a short piece of MATLAB code to solve the following using backward Euler for $1 \le t \le 10$ with $h = 0.01$.

$$y' + 5ty = \sin(t), \quad y(1) = 10$$