WEEK 3 SECTION SOLUTIONS

Solve the following problems. If initial conditions are given, solve for all constants of integration. It is okay to leave answers in implicit form or with unsolved integrals.

1. **Linear First Order ODEs:** Solve the following linear first order ODEs using either the formula derived in class or variation of parameters.

   a) $xy' + y = x$, $y(1) = 1$
   Solution
   Recall from class that an ODE of the form $y' + p(x)y = q(x)$ has solution:

   $$y(x) = e^{-\int p(x)dx}\Big(\int e^{\int p(x)dx} q(x)dx + C\Big)$$

   Applying this to these problems:

   $$y' + \frac{1}{x}y = 1$$

   $$y(x) = e^{-\int \frac{1}{x}dx}\Big(\int e^{\int \frac{1}{x}dx}(1)dx + C\Big)$$

   $$y(x) = \frac{1}{x}\Big(\int xdx + C\Big)$$

   $$y(x) = \frac{1}{x}\Big(\frac{x^2}{2} + C\Big)$$

   $$y(x) = \frac{x}{2} + \frac{C}{x}$$

   Applying the initial condition:

   $$y(1) = \frac{1}{2} + \frac{C}{(1)} = 1$$

   $$C = \frac{1}{2}$$

   $$y(x) = \frac{x}{2} + \frac{1}{2x}$$

   b) $xy' + y = sin(x)$, $y(1) = 0$
   Solution

   $$y' + \frac{1}{x}y = \frac{sin(x)}{x}$$

   $$y(x) = e^{-\int \frac{1}{x}dx}\Big(\int e^{\int \frac{1}{x}dx}\Big(\frac{sin(x)}{x}\Big)dx + C\Big)$$

$$y(x) = \frac{1}{x}\Big(\int sin(x)dx + C\Big)$$
$$y(x) = -\frac{cos(x)}{x} + \frac{C}{x}$$
$$y(1) = 0 = -\frac{cos(1)}{(1)} + \frac{C}{(1)}$$
$$C = cos(1)$$
$$y(x) = -\frac{cos(x)}{x} + \frac{cos(1)}{x}$$

c) $\frac{1}{2}y' + y = e^x$, $y(0) = 0$
Solution

$$y' + 2y = 2e^x$$
$$y(x) = e^{-\int 2dx}\Big(\int e^{\int 2dx}\big(2e^x\big)dx + C\Big)$$
$$y(x) = e^{-2x}\Big(\int 2e^{2x}e^x dx + C\Big)$$
$$y(x) = e^{-2x}\Big(\frac{2}{3}e^{3x} + C\Big)$$
$$y(x) = \frac{2}{3}e^x + Ce^{-2x}$$
$$y(0) = 0 = \frac{2}{3} + C$$
$$C = -\frac{2}{3}$$
$$y(x) = \frac{2}{3}e^x - \frac{2}{3}e^{-2x}$$

2. **Numerical Accuracy:** Show the following.

   a) Forward Euler is globally first-order accurate.
   Solution
   The Forward Euler method is defined by:

$$\frac{y_{n+1} - y_n}{h} = \tilde{y}'_n$$
$$\tilde{y}_{n+1} = y_n + hy'_n$$
$$y(x_{n+1}) = y(x_n + h) = y(x_n) + hy'(x_n) + \frac{1}{2}h^2 y''(x_n) + O(h^3)$$
$$y_{n+1} - \tilde{y}_{n+1} = h^2 y''_n + O(h^3)$$

   So, Forward Euler is $O(h^2)$ locally. To find global error, we multiply by n. Recall that $n = 1/h$, so $O(h^2) \times n = O(h^2)/h = O(h)$.

b) Backward Euler is locally second-order accurate.
Solution
The Backward Euler method is defined by:

$$\frac{y_{n+1} - y_n}{h} = y'_{n+1}$$

$$y_{n+1} = y_n + h\lambda y_{n+1}$$

$$y_{n+1} = \frac{y_n}{1 - h\lambda}$$

Using the taylor expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + O(x^4)$$

we have

$$y_{n+1} = y_n(1 + h\lambda + h^2\lambda^2 + h^3\lambda^3 + O(h^4))$$

For the actual solution $y(x)$, we can expand $y(x_{n+1})$ as:

$$y(x_{n+1}) = y(x_n + h) = y_n + hy'(x_n) + \frac{h^2}{2}y''(x_n) + O(h^3)$$

and applying the ODE we have:

$$y(x_{n+1}) = y(x_n + h) = y_n + h\lambda y_n + \frac{h^2\lambda^2}{2}y_n + O(h^3)$$

Then subtracting these, we have the final accuracy

$$y_{n+1} - y(x_{n+1}) = \frac{h^2\lambda^2}{2}y_n + O(h^3)$$

So, Backward Euler is $O(h^2)$ locally.

3. **Systems of Linear Equations:** Put the following systems of equations into matrix-vector form. State whether each has a unique solution.

a)

$$4x + 5y + 6z = 2$$

$$x + 7z = 5$$

$$8y + 2z = 0$$

Solution

We want to arrange equations in the form $A\vec{x} = b$, where A is a $m \times n$ matrix, $\vec{x}$ is a $n \times 1$ column vector, and b is a $m \times 1$ row vector. Arranging in this form, we have:

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 0 & 7 \\ 0 & 8 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$$

This will have a unique solution since we have three linearly independent equations and three unknowns. More mathematically, if the *rank* of the matrix is equation to the number of unknowns, then the solution will be unique.

b)

$$3y_1 + 2y_2 + 5y_3 = 2$$

$$y_3 + y_2 = 7$$

Solution

$$\begin{bmatrix} 3 & 2 & 5 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

There are only two equations for three unknowns, so the system is *underdetermined* and therefore has infinitely many solutions.

Remark: It is important to be able to characterize if a given system is unique to underdetermined. Specifically, if a system has infinitely many solutions, then we need to be aware of which solutions can be present, and how we can further constrain a system to produce to the solution we desire.

For ODEs, the order is the number of degrees of freedom and the initial/boundary conditions are what constrain or "pin down" our solution. By doing this, we can pick the desired solution by picking the appropriate constraints on the system. Remember that the constants of integration are determined by the initial conditions, and that these constants determine the solution. By picking the constants, we are able to pick the final solution.