WEEK 7 SECTION PROBLEMS

If not otherwise specified, solve the following problems. If initial conditions are given, solve for all constants of integration. It is okay to leave answers in implicit form or with unsolved integrals.

1. **Solve** $y'' + 3y' + 2.25y = -10e^{-1.5x}$, $y(0) = 1$, $y'(0) = 0$

   Solution
   Characteristic polynomial:
   $$\lambda^2 + 3\lambda + 2.25 = 0$$
   $$(\lambda + 1.5)^2 = 0$$
   $$\lambda = -1.5$$
   so the homogeneous solution is
   $$y_h = e^{-1.5x}(c_1 + c_2 x)$$
   We can then use undetermined coefficients to find the particular solution. Assume a solution of the form $y_p = Ax^2e^{-1.5x}$:
   $$Ae^{-1.5x}(6x - 4.5x^2 + 2.25x^2 - 6x + 2) + Ax^2e^{-1.5x} = -10e^{-1.5x}$$
   $$A = -5$$
   $$y_p = -5x^2e^{-1.5x}$$
   $$y(x) = e^{-1.5x}(c_1 + c_2x) - 5x^2e^{-1.5x}$$
   $$c_1 = 1,\ c_2 = 1.5$$
   $$y(x) = e^{-1.5x}(1 + 1.5x - 5x^2)$$

2. **Solve** $y'' + 4y = sec(2x)$, $y(0) = 0$, $y'(0) = 1$

   Solution
   The homogeneous solution is giving by: $y(x) = c_1 sin(2x) + c_2 cos(2x)$. We then need to use variation of parameters to find the particular solution:
   $$W = y_1y_2' - y_2y_1' = 2$$
   $$y_p(x) = -\frac{1}{2}cos(2x)\int sec(2x)sin(2x)dx + \frac{1}{2}sin(2x)\int sec(2x)cos(2x)dx$$
   $$y_p(x) = -\frac{1}{2}cos(2x)\int tan(2x)dx + \frac{1}{2}sin(2x)\int dx$$
   $$y_p(x) = \frac{1}{4}cos(2x)ln(|cos(2x)|) + \frac{1}{2}x(sin(2x))$$
   $$y(x) = c_1 sin(2x) + c_2 cos(2x) + \frac{1}{2}cos(2x)ln(|cos(2x)|) + \frac{1}{2}x(sin(2x))$$
   $$c_1 = \tfrac{1}{2},\ c_2 = 0$$

$$y(x) = \frac{1}{2}sin(2x) + \frac{1}{2}cos(2x)ln(|cos(2x)|) + \frac{1}{2}x(sin(2x))$$

3. **Generalized RK2** The generalized formula for RK2 is given by:

$$k_1 = hf(t_n, y_n)$$

$$k_2 = hf(t_n + \alpha h, y_n + \alpha k_1)$$

$$y_{n+1} = y_n + (1 - \frac{1}{2\alpha})k_1 + \frac{1}{2\alpha}k_2$$

where α is a free parameter and $\alpha \in (0, 1]$. For the model equation $y' = \lambda y$, derive the amplification factor $\sigma(h, \alpha)$ and maximum step size $h_{max}(\alpha)$. For what value of α is h_{max} maximized? (*Hint*: there may not be a maximum or minimum to this function.)

Solution

$$y_{n+1} = y_n + (1 - \frac{1}{2\alpha})h\lambda y_n + \frac{h\lambda}{2\alpha}(y_n + \alpha h\lambda y_n)$$

$$y_{n+1} = y_n + h\lambda y_n + \frac{h^2\lambda^2}{2}y_n$$

We see that the amplification factor—and thus the maximum step size—is not dependent on α for the model problem. However, if were we dealing with a different problem, the α factor would be an important tuning parameter for minimizing the error given the dynamics of the problem.

4. **Direct Method** Consider the following boundary value problem:

$$y'' - 2y' + y = x^2, \qquad y(0) = 0, \qquad y(1) = 1$$

a) Solve the BVP analytically.
Solution: We solve identically to how we solve IVP's, except in this case we solve for the constants using boundary values instead of initial values.

$$y'' - 2y' + y = x^2$$

Using the characteristic equation to solve for the homogeneous solution:

$$\lambda^2 - 2\lambda + 1 = 0$$
$$\lambda = 1$$
$$y_h = (c_1 + c_2 x)e^{\lambda x}$$

and using undetermined coefficients to solve for the particular solution:

$$\begin{aligned}
y_p &= Ax^2 + Bx + C \\
y_p' &= 2Ax + B \\
y_p'' &= 2A \\
(2A) - 2(2Ax + B) + Ax^2 + Bx + C &= x^2 \\
Ax^2 + (-4A + B)x + (2A - 2B + C) &= x^2 \\
A &= 1 \\
-4A + B = 0 &\rightarrow B = 4 \\
2A - 2B + C = 0 &\rightarrow C = 6 \\
y_p &= x^2 + 4x + 6 \\
y(x) &= (c_1 + c_2 x)e^x + x^2 + 4x + 6 \\
y(0) = 0 &= c_1 + 6 \\
c_1 &= -6 \\
y(1) = 1 &= -6e^1 + c_2 e^1 + 1 + 4 + 6 \\
c_2 &= -10e^{-1} + 6 \\
y(x) &= (-10e^{-1} + 6)xe^x + x^2 + 4x + 6 \quad \blacksquare
\end{aligned}$$

b) Classify the boundary conditions
Solution: both boundary conditions are Type I (Dirichlet).

c) Set up the recursive equation for an interior node using second order central differencing schemes.

$$\begin{aligned}
y'' - 2y' + y &= x^2 \\
\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - 2\frac{y_{i+1} - y_{i-1}}{2h} + y_i &= x_i^2 \\
y_{i+1} - 2y_i + y_{i-1} - h(y_{i+1} - y_{i-1}) + h^2 y_i &= h^2 x_i^2 \\
(1-h)y_{i+1} + (h^2 - 2)y_i + (1+h)y_{i-1} &= h^2 x_i^2
\end{aligned}$$

And using that $h = 0.2$ for $N = 6$ nodes:

$$0.8y_{i+1} - 1.96y_i + 1.2y_{i-1} = 0.04x_i^2$$

d) Set up the matrix equation $A\vec{x} = b$ for $N = 6$ nodes.
Solution:

$$0.8y_{i+1} - 1.96y_i + 1.2y_{i-1} = 0.04x_i^2$$

Evaluating for each node:

$$\begin{aligned}
0.8y_3 - 1.96y_2 + 1.2y_1 &= 0.04x_2^2 \\
0.8y_4 - 1.96y_3 + 1.2y_2 &= 0.04x_3^2 \\
0.8y_5 - 1.96y_4 + 1.2y_3 &= 0.04x_4^2 \\
0.8y_6 - 1.96y_5 + 1.2y_4 &= 0.04x_5^2
\end{aligned}$$

Plugging in for the right hand side values:

$$\begin{aligned}
0.8y_3 - 1.96y_2 + 1.2y_1 &= 0.04(0.2)^2 = 0.0016 \\
0.8y_4 - 1.96y_3 + 1.2y_2 &= 0.04(0.4)^2 = 0.0064 \\
0.8y_5 - 1.96y_4 + 1.2y_3 &= 0.04(0.6)^2 = 0.0144 \\
0.8y_6 - 1.96y_5 + 1.2y_4 &= 0.04(0.8)^2 = 0.0256
\end{aligned}$$

Then plugging in for the boundary conditions:

$$\begin{aligned}
0.8y_3 - 1.96y_2 + 1.2(0) &= 0.0016 \\
0.8y_3 - 1.96y_2 &= 0.0016 \\
0.8(1) - 1.96y_5 + 1.2y_4 &= 0.0256 \\
-1.96y_5 + 1.2y_4 &= -0.7744
\end{aligned}$$

And finally putting into a matrix-vector form:

$$\begin{bmatrix} -1.96 & 0.8 & 0 & 0 \\ 1.2 & -1.96 & 0.8 & 0 \\ 0 & 1.2 & -1.96 & 0.8 \\ 0 & 0 & 1.2 & -1.96 \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0.0016 \\ 0.0064 \\ 0.0144 \\ 0.7744 \end{bmatrix}$$

e) Complete the previous parts, except with the right boundary condition now changed to $y'(1) = 1$. (*Hint:* very little will change for this part.)
Solution:

For the analytical solution:

$$\begin{aligned}
y(x) &= (c_1 + c_2 x)e^x + x^2 + 4x + 6 \\
y(0) = 0 &= c_1 + 6 \\
c_2 &= -6 \\
y'(x) &= -6e^x + c_2 e^x + c_2 x e^x + 2x + 4 \\
y'(1) = 1 &= -6e^1 2c_2 e^1 + 2 + 4 \\
c_2 &= \frac{-5}{2}e^{-1} + 6 \\
y(x) &= \left(-\frac{5}{2}e^{-1} + 6\right)xe^x + x^2 + 4x + 6 \quad \blacksquare
\end{aligned}$$

Left boundary is a Type I (Dirichlet) boundary condition, right boundary is a Type II (Neumann) boundary condition.

Recursive equation for interior nodes remains the same:

$$0.8y_{i+1} - 1.96y_i + 1.2y_{i-1} = 0.04x_i^2$$

The matrix remains largely unchanged, except the node at $i = 6$ is now a degree of freedom, so there is an additional equation in the matrix. The right boundary equation is given by:

$$0.8y_7 - 1.96y_6 + 1.2y_5 = 0.04(1)^2$$

We must reformulate the Neumann BC, then plug this into our recursive to eliminate the ghost point:

$$\begin{aligned}
y'(1) &= 1 \\
\frac{y_7 - y_5}{2h} &= 1 \\
y_7 &= 2h + y_5 \\
0.8y_7 - 1.96y_6 + 1.2y_5 &= 0.04 \\
0.8(2h + y_5) - 1.96y_6 + 1.2y_5 &= 0.04 \\
0.8(0.2 + y_5) - 1.96y_6 + 1.2y_5 &= 0.04 \\
0.16 + 0.8y_5 - 1.96y_6 + 1.2y_5 &= 0.04 \\
-1.96y_6 + 2y_5 &= -0.12
\end{aligned}$$

This gives the final matrix equation as:

$$\begin{bmatrix} -1.96 & 0.8 & 0 & 0 & 0 \\ 1.2 & -1.96 & 0.8 & 0 & 0 \\ 0 & 1.2 & -1.96 & 0.8 & 0 \\ 0 & 0 & 1.2 & -1.96 & 0.8 \\ 0 & 0 & 0 & 2 & -1.96 \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0.0016 \\ 0.0064 \\ 0.0144 \\ 0.0256 \\ -0.12 \end{bmatrix}$$