WEEK 8 SECTION SOLUTIONS

If not otherwise specified, solve the following problems. If initial conditions are given, solve for all constants of integration. It is okay to leave answers in implicit form or with unsolved integrals.

1. For the following, give the natural frequency ω_0. State whether or not there is resonance or beats, and give a reason for why.

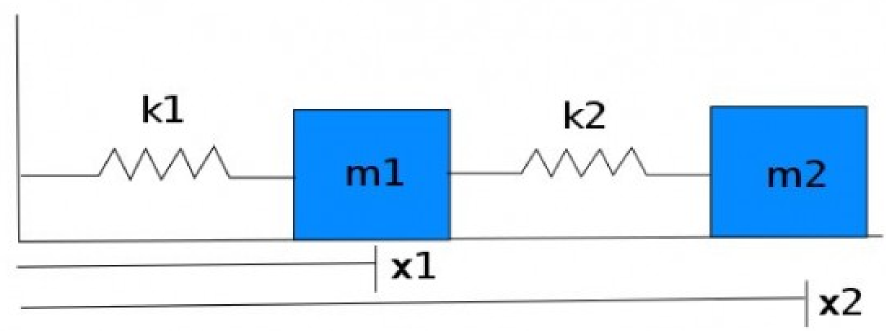
   a) $y'' + 4y = sin(2t)$
   Solution: The homogeneous equation is $y'' + 4y = 0$ or $y'' = -4y$. Then the natural frequency is $\omega_0 = 2$ since our ODE is of the form $y'' = -\omega^2 y$. This matches the frequency of the forcing term (the right hand side), so our particular solution is of the form $y_p(t) = Atsin(2t) + Btcos(2t)$ which grows in time. So, we have <u>resonance</u>.

   b) $y'' + 2y' + 5y = cos(2t)$
   Solution: The homogeneous equation is $y'' + 5y' + 2y = 0$ which has roots $\lambda = -1 \pm 2i$, so the natural frequency is $\omega_0 = 2$. This matches the frequency of the forcing term (the right hand side). If this were an undamped system, we would have resonance. However, the system is damped and thus will not blow up, so we have neither beats nor resonance.

   c) $y'' + y' + 2y = 3$
   Solution: he homogeneous equation is $y'' + y' + 2y = 0$ which has roots $\lambda = -1/2 \pm i\sqrt{7}/2$, so the natural frequency is $\omega_0 = \sqrt{7}/2$. The forcing term is not periodic, so we do not have beats or resonance.

   d) $y'' + 4y = sin(2.05t)$
   Solution: The homogeneous problem is the same as in 1(a), and the natural frequency is $\omega_0 = 2$. The forcing term has frequency $\omega = 2.05$, so $\omega_0 \approx \omega$ and we have <u>beats</u>.

2. For the system described by the image below, derive the system of ODEs governing the mass-spring system. Treat the masses as point masses and assume no damping or friction.



   Derive the system of ODEs governing the mass-spring system. Treat the masses as point masses and assume no damping or friction.
   Solution

$$m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2$$

$$m_2 x_2'' = k_2 x_1 - k_2 x_2$$

   Which in matrix form is: $\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

3. Perform the following integrals:

a) $\int_0^\infty e^{-sx} dx$
Solution

$$\int_0^\infty e^{-sx} dx = \frac{-1}{s} e^{-sx}\Big|_0^\infty = 0 - (\frac{-1}{s}) = \frac{1}{s}$$

b) $\int_0^\infty sin(x) e^{-sx} dx$
Solution
With integration by parts:

$$u = sin(x)\ ,\ dv = e^{-sx}$$
$$du = cos(x) \text{ and } v = \frac{-1}{s} e^{-sx}$$
$$\int_0^\infty sin(x) e^{-sx} dx = sin(x) e^{-sx} - \int_0^\infty \frac{-1}{s} e^{-sx} cos(x) dx$$

Doing this again for the second integral:

$$u = cosx\ ,\ dv = e^{-sx}$$
$$du = -sinx \text{ and } v = \frac{-1}{s} e^{-sx}$$
$$\int_0^\infty cos(x) e^{-sx} dx = cos(x) e^{-sx} - \int_0^\infty \frac{-1}{s} e^{-sx} cos(x) dx$$

And then finally:

$$\int_0^\infty sin(x) e^{-sx} dx = \frac{e^{-sx}((s)sin(x) - cos(x))}{s^2+1}\Big|_0^\infty = \frac{1}{s^2+1}$$

c) $\int_0^\infty u(x-2) sin(x-2) e^{-sx} dx$
Solution
Use the substitution $\tau = x - 2$, then the integral becomes:

$$\int_0^\infty u(x-2) sin(x-2) e^{-sx} dx = \int_0^\infty u(\tau) sin(\tau) e^{-s(\tau+2)} dx = \int_0^\infty sin(\tau) e^{-s(\tau+2)} dx =$$
$$\int_0^\infty sin(\tau) e^{-s\tau} e^{-2s} d\tau = e^{-2s} \int_0^\infty sin(\tau) e^{-s\tau} d\tau$$

The integral is the same as in 3(b), so the answer is then:

$$\int_0^\infty sin(x-2) e^{-sx} dx = \frac{e^{-2s}}{s^2+1}$$

4. Solve the following using a Laplace transform:

$$y'' - 2y' + y = 2\ ; \qquad y(0) = 0 \qquad y'(0) = 0$$

Solution
Taking the Laplace Transform of each side:

$$s^2 Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + Y(s) = \frac{2}{s}$$

Applying initial conditions:

$$s^2 Y(s) - 2sY(s) + Y(s) = \frac{2}{s}$$
$$Y(s) = \frac{2}{s(s^2-2s+1)} = \frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2}$$
$$y(t) = 1 - e^t + te^t$$