CME 102 ACE – Midterm #2 Reference Sheet

Eigenvalues/Eigenvectors

For *system* of ODEs of the form $\vec{x}' = \mathbf{A}\vec{x}$:

1. Assume solution $\vec{x}(t) = C\vec{v}e^{\lambda t}$ and plug in to find $\mathbf{A}\vec{v} = \lambda\vec{v}$
2. Solve for eigenvalues λ of $\mathbf{A}$. For a 2x2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

3. Solve for eigenvectors of $\mathbf{A}$. Special cases for 2x2 matrix:
   Case 1: $c \neq 0$:

$$\vec{v}_1 = \begin{bmatrix} \lambda_1 - d \\ c \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} \lambda_2 - d \\ c \end{bmatrix}$$

   Case 2: $b \neq 0$:

$$\vec{v}_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} b \\ \lambda_2 - a \end{bmatrix}$$

   Case 3: b and c are zero:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

4. Assemble solution:
   (a) Two real, distinct eigenvalues:

$$\vec{x} = C_1\vec{v}_1 e^{\lambda_1 t} + C_2\vec{v}_2 e^{\lambda_2 t}$$

   (b) One real, repeated eigenvalue: find associated eigenvector $\vec{v}$, solve for $\vec{\rho}$ in

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{\rho} = \vec{v}$$

   then assemble solution:

$$\vec{x} = C_1\vec{v}e^{\lambda t} + C_2\vec{v}te^{\lambda t} + C_2\vec{\rho}e^{\lambda t}$$

   (c) Two complex conjugate eigenvalues $\lambda_{1,2} = \alpha + \beta i$:

$$\vec{x} = C_1 e^{\alpha t}\left(\vec{v}_R\cos(\beta t) - \vec{v}_I\sin(\beta t)\right) + C_2 e^{\alpha t}\left(\vec{v}_I\cos(\beta t) + \vec{v}_R\sin(\beta t)\right)$$

Second-Order Nonlinear

"Missing y" method

Second order nonlinear ODE that does not contain y:

$$F(y'', y', x) = 0$$

1. Make substitution $y' = u$ and $y'' = u'$ and rewrite ODE in terms of u and x
2. Solve new ODE for $u(x)$
3. Find y as $y = \int u(x)dx$

"Missing x" method

Second order nonlinear ODE that does not contain x:

$$F(y'', y', y) = 0$$

1. Make substitution $y' = u$ and $y'' = uu'$ and rewrite ODE in terms of u and y
2. Solve new ODE for $u(y)$
3. Substitute back $u = y'$ and solve for $y(x)$

Second-Order Linear Homogeneous

Homogeneous solution has two **basis functions** y_1 and y_2. Write homogeneous solution as linear combination of these:

$$y_h = C_1y_1 + C_2y_2$$

Variable Coefficients

ODE has form:

$$y'' + p(x)y' + q(x)y = 0$$

Solving using **reduction of order**. Given y_1, find y_2 as:

$$y_2 = y_1\int \frac{e^{-\int p dx}}{y_1^2}dx$$

Constant Coefficients

ODE has form:

$$ay'' + by' + cy = 0$$

with a, b, c constant. Solve using **characteristic equation** $y = e^{\lambda x}$. Three cases:

Case 1 $b^2 - 4ac > 0$ (two distinct real roots):

$$\lambda_1 = -\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}, \quad \lambda_2 = -\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$$
$$y_h = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$$

Case 2 $b^2 - 4ac = 0$ (double real root):

$$\lambda = -\frac{b}{2a}, \quad y_h = (C_1 + C_2x)e^{\lambda x}$$

Case 3 $b^2 - 4ac < 0$ (complex conjugate roots):

$$\alpha = -\frac{b}{2a}, \quad \beta = \frac{\sqrt{4ac-b^2}}{2a}, \quad \lambda_{1,2} = \alpha \pm i\beta$$
$$y_h = e^{\alpha x}\left(C_1\cos(\beta x) + C_2\sin(\beta x)\right)$$

Euler-Cauchy equation

ODE has form:

$$ax^2y'' + bxy' + cy = 0$$

with a, b, c constant. Solve using **characteristic equation** $y = e^{\lambda x}$. Three cases:

Case 1 $(b-a)^2 - 4ac > 0$:

$$m_{1,2} = -\frac{b-a}{2a} \pm \frac{\sqrt{(b-a)^2-4ac}}{2a}$$
$$y_h = C_1x^{m_1} + C_2x^{m_2}$$

Case 2 $(b-a)^2 - 4ac = 0$:

$$m = -\frac{b-a}{2a}, \quad y_h = (C_1 + C_2\ln|x|)x^m$$

Case 3 $(b-a)^2 - 4ac < 0$:

$$\alpha = -\frac{b-a}{2a}, \quad \beta = \frac{\sqrt{4ac-(b-a)^2}}{2a}, \quad m_{1,2} = \alpha \pm i\beta$$
$$y_h = x^{\alpha}\left(C_1\cos(\beta\ln|x|) + C_2\sin(\beta\ln|x|)\right)$$

Second-Order Linear Inhomogeneous

- You **must** always solve for the homogeneous solution first
- The final solution is $y = y_h + y_p$
- Apply initial conditions **after** finding the particular solution

Variation of Parameters

- Must use for variable coefficient second order linear ODE
- Integrals are hard, so only use if undetermined coefficients does not apply

1. Find homogeneous basis solutions y_1 and y_2
2. Calculate Wronskian:

$$W = y_1y_2' - y_2y_1'$$

3. Put ODE into standard form:

$$y'' + p(x)y' + q(x)y = r(x)$$

4. Calculate the particular solution:

$$y_p = -y_1\int \frac{y_2r}{W}dx + y_2\int \frac{y_1r}{W}dx$$

Undetermined Coefficients

- Only applicable to constant coefficient equations
- If you need to solve for more than ~ 4 constants, variation of parameters probably faster

1. Find homogeneous basis solutions y_1 and y_2
2. Put ODE into standard form:

$$y'' + by' + cy = r(x)$$

3. Guess particular solution $y_p(x)$ based on table:

Form for $r(x)$:	Pick $y_p(x)$:
C	A
x^n	$A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0$
$e^{\gamma x}$	$Ae^{\gamma x}$
$\cos(\omega x)$ or $\sin(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$

   - Can have products of these functions (but only these functions) for $r(x)$ and the assumed solution
   - If any part of assumed solution also part of homogeneous basis, apply **modification rule** by multiplying assumed solution by x
   - Sometimes need to use trig identities to make $r(x)$ match something in the table

4. Take derivatives of assumed $y_p(x)$ and plug into ODE
5. Algebraically solve for the constants in assumed solution

Numerical Methods

Accuracy

- Local error: error incurred over one step
- Global error: total error over the domain, one order of h less than local error, calculated as $\epsilon_{global} = N \times \epsilon_{local}$

Stability

- Derive amplification factor $\sigma(h)$ by starting with the model equation $y' = \lambda y$ and "stepping-through" the numerical method to derive relationship:

$$y_{n+1} = \sigma(h) y_n$$

- Stability condition is $|\sigma(h)| < 1$
- To find stable h, solve $\sigma(h) < 1$ and $\sigma(h) > -1$

Euler

Forward Euler: explicit, $\mathcal{O}(h)$ global accuracy:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Backward Euler: implicit, $\mathcal{O}(h)$ global accuracy:

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$$

Runge-Kutta Methods

Trapezoidal: implicit, $\mathcal{O}(h^2)$ global accuracy, related to RK2, average of backward and forward Euler:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

Improved Euler/RK2: explicit, $\mathcal{O}(h^2)$ global accuracy:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\right]$$

RK4: explicit, $\mathcal{O}(h^4)$ global accuracy, basis of `ode45()`:

$$k_1 = f(t_n, y_n)$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$$

$$k_4 = f\left(t_n + h, y_n + hk_3\right)$$

$$y_{n+1} = y_n + h\left(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4\right)$$

Higher-Order Systems of ODEs

- ODEs (or systems of ODEs) that contain higher than first derivatives
- Need to convert to first order to apply numerical methods

Create table:

[Old] | [New] | [Derivative] | [New ODE]

MATLAB

`[t,y] = ode45(@(t,y) myODE(t,y), tspan, y0)`

If `myODE` has two arguments: `[t,y] = ode45(@myODE, tspan, y0)`

Anonymous function: `f = @(x) [expression]`

Trigonometric Identities

Regular trigonometric identities:

$$\sin^2 x + \cos^2 x = 1,\ \tan^2 x + 1 = \sec^2 x, 1 + \cot^2 x = \csc^2 x$$

$$\sin(2x) = 2\sin x \cos x,\ \tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

Hyperbolic trigonometric functions:

$$\sinh x = \frac{e^x - e^{-x}}{2},\ \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1,\ \tanh^2 x + \text{sech}^2 x = 1,\ \coth^2 x - \text{csch}^2 x = 1$$

$$\sinh(2x) = 2\sinh x \cosh x, \quad \cosh(2x) = 2\cosh^2 x - 1$$

$$\tanh(2x) = \frac{2\tanh x}{1 + \tanh^2 x}$$

Useful Integrals/Derivatives

Trigonometric function derivatives:

$$\frac{d}{dx}\sin x = \cos x,\ \frac{d}{dx}\cot x = -\csc^2 x,\ \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos x = -\sin x,\ \frac{d}{dx}\sec x = \sec x \tan x,\ \frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan x = \sec^2 x,\ \frac{d}{dx}\csc x = -\csc x \cot x,\ \frac{d}{dx}\arctan x = \frac{1}{x^2+1}$$

Trigonometric and other integrals:

$$\int \tan x dx = -\log|\cos x| + C, \quad \int \cot x dx = \log|\sin x| + C$$

$$\int \csc x dx = -\log|\csc x + \cot x| + C$$

$$\int \sec x = \log|\sec x + \tan x| + C$$

$$\int \sin^2(x) dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

$$\int \cos^2(x) dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C,\ \int \ln x dx = x\ln x - x + C$$

$$\int e^{ax}\sin(bx)dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$$

$$\int e^{ax}\cos(bx)dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$$

Inverse trigonometric function integrals:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C,\ \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$$

Hyperbolic trig function derivatives:

$$\frac{d}{dx}\sinh(x) = \cosh(x), \quad \frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2(x), \quad \frac{d}{dx}\text{csch}(x) = -\coth(x)\,\text{csch}(x)$$

$$\frac{d}{dx}\text{sech}(x) = -\tanh x\,\text{sech}(x), \quad \frac{d}{dx}\coth x = 1 - \coth^2(x)$$