CME 102 ACE – Final Exam Reference Sheet

First-Order ODE

Separation of Variables

For any nonlinear first order ODE, manipulate to be in form $f(y)dy = g(x)dx$ then integrate.

Two special cases:

- ODE of form $y' = f(y/x)$, use $u = y/x$
- ODE of form $y' = f(ay + bx + c)$, use $u = ay + bx + c$

Linear Inhomogeneous

ODEs of form $y' + p(x)y = r(x)$

Closed form solution:

$$y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx} r(x)dx + C\right]$$

Bernoulli Equation: $y' + p(x)y = q(x)y^n$

Substitute $u = y^{1-n}$, solve $u' + (1-n)p(x)u = (1-n)q(x)$

Eigenvalues/Eigenvectors

For *system* of ODEs of the form $\vec{x}' = \mathbf{A}\vec{x}$:

1. Assume solution $\vec{x}(t) = C\vec{v}e^{\lambda t}$ and plug in to find $\mathbf{A}\vec{v} = \lambda\vec{v}$
2. Solve for eigenvalues λ of $\mathbf{A}$. For a 2x2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

3. Solve for eigenvectors of $\mathbf{A}$. Special cases for 2x2 matrix:
   Case 1: $c \neq 0$:
   $$\vec{v}_1 = \begin{bmatrix} \lambda_1 - d \\ c \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} \lambda_2 - d \\ c \end{bmatrix}$$
   Case 2: $b \neq 0$:
   $$\vec{v}_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} b \\ \lambda_2 - a \end{bmatrix}$$
   Case 3: b and c are zero:
   $$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Second-Order Nonlinear

"Missing y" method

Second order nonlinear ODE that does not contain y:

$$F(y'', y', x) = 0$$

1. Make substitution $y' = u$ and $y'' = u'$ and rewrite ODE in terms of u and x
2. Solve new ODE for $u(x)$
3. Find y as $y = \int u(x)dx$

"Missing x" method

Second order nonlinear ODE that does not contain x:

$$F(y'', y', y) = 0$$

1. Make substitution $y' = u$ and $y'' = uu'$ and rewrite ODE in terms of u and y
2. Solve new ODE for $u(y)$
3. Substitute back $u = y'$ and solve for $y(x)$

Second-Order Linear Homogeneous

Homogeneous solution has two **basis functions** y_1 and y_2. Write homogeneous solution as linear combination of these:

$$y_h = C_1y_1 + C_2y_2$$

Variable Coefficients

ODE has form:

$$y'' + p(x)y' + q(x)y = 0$$

Solving using **reduction of order**. Given y_1, find y_2 as:

$$y_2 = y_1\int \frac{e^{-\int p dx}}{y_1^2}dx$$

Constant Coefficients

ODE has form:

$$ay'' + by' + cy = 0$$

with a, b, c constant. Solve using **characteristic equation** $y = e^{\lambda x}$. Three cases:

Case 1 $b^2 - 4ac > 0$ (two distinct real roots):

$$\lambda_1 = -\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}, \quad \lambda_2 = -\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$$
$$y_h = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$$

Case 2 $b^2 - 4ac = 0$ (double real root):

$$\lambda = -\frac{b}{2a}, \quad y_h = (C_1 + C_2x)e^{\lambda x}$$

Case 3 $b^2 - 4ac < 0$ (complex conjugate roots):

$$\alpha = -\frac{b}{2a}, \quad \beta = \frac{\sqrt{4ac-b^2}}{2a}, \quad \lambda_{1,2} = \alpha \pm i\beta$$
$$y_h = e^{\alpha x}\left(C_1\cos(\beta x) + C_2\sin(\beta x)\right)$$

Euler-Cauchy equation

ODE has form:

$$ax^2y'' + bxy' + cy = 0$$

with a, b, c constant. Solve using **characteristic equation** $y = x^m$. Three cases:

Case 1 $(b-a)^2 - 4ac > 0$:

$$m_{1,2} = -\frac{b-a}{2a} \pm \frac{\sqrt{(b-a)^2-4ac}}{2a}$$
$$y_h = C_1x^{m_1} + C_2x^{m_2}$$

Case 2 $(b-a)^2 - 4ac = 0$:

$$m = -\frac{b-a}{2a}, \quad y_h = (C_1 + C_2\ln|x|)x^m$$

Case 3 $(b-a)^2 - 4ac < 0$:

$$\alpha = -\frac{b-a}{2a}, \quad \beta = \frac{\sqrt{4ac-(b-a)^2}}{2a}, \quad m_{1,2} = \alpha \pm i\beta$$
$$y_h = x^{\alpha}\left(C_1\cos(\beta\ln|x|) + C_2\sin(\beta\ln|x|)\right)$$

Second-Order Linear Inhomogeneous

- You **must** always solve for the homogeneous solution first
- The final solution is $y = y_h + y_p$
- Apply initial conditions **after** finding the particular solution

Variation of Parameters

- Must use for variable coefficient second order linear ODE
- Integrals are hard, so only use if undetermined coefficients does not apply

1. Find homogeneous basis solutions y_1 and y_2
2. Calculate Wronskian:
$$W = y_1y_2' - y_2y_1'$$
3. Put ODE into standard form:
$$y'' + p(x)y' + q(x)y = r(x)$$
4. Calculate the particular solution:
$$y_p = -y_1\int \frac{y_2 r}{W}dx + y_2\int \frac{y_1 r}{W}dx$$

Undetermined Coefficients

- Only applicable to constant coefficient equations
- If you need to solve for more than ~ 4 constants, variation of parameters probably faster

1. Find homogeneous basis solutions y_1 and y_2
2. Put ODE into standard form:
$$y'' + by' + cy = r(x)$$
3. Guess particular solution $y_p(x)$ based on table:

Form for $r(x)$:	Pick $y_p(x)$:
C	A
x^n	$A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0$
$e^{\gamma x}$	$Ae^{\gamma x}$
$\cos(\omega x)$ or $\sin(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$

   - Can have products of these functions (but only these functions) for $r(x)$ and the assumed solution
   - If any part of assumed solution also part of homogeneous basis, apply **modification rule** by multiplying assumed solution by x
   - Sometimes need to use trig identities to make $r(x)$ match something in the table
4. Take derivatives of assumed $y_p(x)$ and plug into ODE
5. Algebraically solve for the constants in assumed solution

Circuits

Kirchoff's voltage law: $\sum_i V_i = 0$ i.e. sum of all voltage drops across elements is zero (conservation of energy)

To form circuit equations:

1. Apply Kirchoff's voltage law around loop (or loops)
   - If there are multiple voltage loops i.e. you are forming a system of ODEs, make sure loops do not overlap
2. If there is a capacitor (and therefore $\int_0^t i(\tau)d\tau$ appears in the equation), take derivation through equation to form second order ODE
3. Solve second order linear ODE using Laplace transform or other method

Voltage drops for elements:

- Inductor: $v = Li'$
- Resistor: $v = Ri$ (Ohm's law)
- Capacitor: $v = \frac{Q}{C} = \frac{\int_0^t i(\tau)d\tau}{C}$ (charge over capacitance)

Numerical Methods for IVP's

Accuracy

- Local error: error incurred over one step
- Global error: total error over the domain, one order of h less than local error, calculated as $\epsilon_{global} = N \times \epsilon_{local}$

Stability

- Derive amplification factor $\sigma(h)$ by starting with the model equation $y' = \lambda y$ and "stepping-through" the numerical method to derive relationship:

$$y_{n+1} = \sigma(h)y_n$$

- Stability condition is $|\sigma(h)| < 1$
- To find stable h, solve $\sigma(h) < 1$ and $\sigma(h) > -1$

Euler

Forward Euler: explicit, $\mathcal{O}(h)$ global accuracy:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Backward Euler: implicit, $\mathcal{O}(h)$ global accuracy:

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$$

Runge-Kutta Methods

Trapezoidal: implicit, $\mathcal{O}(h^2)$ global accuracy, related to RK2, average of backward and forward Euler:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

Improved Euler/RK2: explicit, $\mathcal{O}(h^2)$ global accuracy:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\right]$$

RK4: explicit, $\mathcal{O}(h^4)$ global accuracy, basis of `ode45()`:

$$k_1 = f(t_n, y_n)$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$$

$$k_4 = f\left(t_n + h, y_n + hk_3\right)$$

$$y_{n+1} = y_n + h\left(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4\right)$$

Higher-Order Systems of ODEs

- ODEs (or systems of ODEs) that contain higher than first derivatives
- Need to convert to first order to apply numerical methods

Create table:

[Old] | [New] | [Derivative] | [New ODE]

Multi-Step Methods

- Uses information from multiple time steps to find next step.
- Methods are **not self-starting**

Adams-Bashforth: $y_{n+1} = y_n + \frac{h}{2}[3f_n - f_{n-1}]$, second order accurate

Numerical Methods for BVP's

Finite differences

First derivatives

- Forward difference: $y_i' = \frac{y_{i+1} - y_i}{h}$, $\mathcal{O}(h)$ error
- Backward difference: $y_i' = \frac{y_i - y_{i-1}}{h}$, $\mathcal{O}(h)$ error
- Central difference: $y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$, $\mathcal{O}(h^2)$ error

Second derivative: use second order central difference:

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2},\ \mathcal{O}(h^2) \text{ error}$$

Boundary Conditions

- Dirichlet: value given (e.g. $y(x_L) = C$)
- Neuman: derivative given (e.g. $y'(x_L) = C$)
- Robin: combination of value and derivative (e.g. $y'(x_L) + \alpha y(x_L) = C$)

Setting Up Direct Method Equations

1. Replace derivatives in ODE with finite difference approximations, and group coefficients of y_{i-1}, y_i, and y_{i+1} to find *recursive equation*
2. Treat boundary conditions
   (a) **Dirichlet (value given):** plug boundary value (at y_1 or y_N) into recursive equation at node next to boundary (either $i = 2$ or $i = N - 1$)
   (b) **Robin or Neuman (includes derivative):** plug finite difference approximation for y' into the boundary condition, solve out for the ghost point, plug into recursive equation at the boundary node (either $i = 1$ or $i = N$)
3. Assemble system of equations in form $\mathbf{A}\vec{y} = \vec{f}$

Shooting Method

Main idea: treat BVP like IVP

1. Guessing the initial condition we don't know at left boundary
2. Solve ODE using method for IVP
3. Update guess for initial condition based on error at right boundary
4. Iterate until error is zero

MATLAB

`[t,y] = ode45(@(t,y) myODE(t,y), tspan, y0)`

If `myODE` has two arguments: `[t,y] = ode45(@myODE, tspan, y0)`

Anonymous function: `f = @(x) [expression]`

Trigonometric Identities

Regular trigonometric identities:

$$\sin^2 x + \cos^2 x = 1,\ \tan^2 x + 1 = \sec^2 x, 1 + \cot^2 x = \csc^2 x$$

$$\sin(2x) = 2\sin x\cos x,\ \tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

Hyperbolic trigonometric functions:

$$\sinh x = \frac{e^x - e^{-x}}{2},\ \cosh x = \frac{e^x + e^{-x}}{2},\quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1,\ \tanh^2 x + \operatorname{sech}^2 x = 1,\ \coth^2 x - \operatorname{csch}^2 x = 1$$

$$\sinh(2x) = 2\sinh x\cosh x,\quad \cosh(2x) = 2\cosh^2 x - 1$$

$$\tanh(2x) = \frac{2\tanh x}{1 + \tanh^2 x}$$

Useful Integrals/Derivatives

Trigonometric function derivatives:

$$\frac{d}{dx}\sin x = \cos x,\ \frac{d}{dx}\cot x = -\csc^2 x,\ \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\cot x = -\sin x,\ \frac{d}{dx}\sec x = \sec x\tan x,\ \frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\tan x = \sec^2 x,\ \frac{d}{dx}\csc x = -\csc x\cot x,\ \frac{d}{dx}\arctan x = \frac{1}{x^2 + 1}$$

Trigonometric and other integrals:

$$\int \tan x dx = -\log|\cos x| + C,\quad \int \cot x dx = \log|\sin x| + C$$

$$\int \csc x dx = -\log|\csc x + \cot x| + C$$

$$\int \sec x = \log|\sec x + \tan x| + C$$

$$\int \sin^2(x)dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

$$\int \cos^2(x)dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C,\ \int \ln x dx = x\ln x - x + C$$

$$\int e^{ax}\sin(bx)dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + C$$

$$\int e^{ax}\cos(bx)dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + C$$

Inverse trigonometric function integrals:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + C,\ \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C$$

Hyperbolic trig function derivatives:

$$\frac{d}{dx}\sinh(x) = \cosh(x),\quad \frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\frac{d}{dx}\tanh x = 1 - \tanh^2(x),\quad \frac{d}{dx}\operatorname{csch}(x) = -\coth(x)\operatorname{csch}(x)$$

$$\frac{d}{dx}\operatorname{sech}(x) = -\tanh x\operatorname{sech}(x),\quad \frac{d}{dx}\coth x = 1 - \coth^2(x)$$