Assignment #2 (not mandatory)

feedback session for this assignment: 07-12-2020

General remark: This assignment deals with some first steps regarding (conditional) maximum likelihood. Since it is so short and there are many hints, I am not going to tape an extra intro video this time and instead post my additional remarks here:

- **In Task 1 of this assignment, you are first asked to simulate an AR(1) process. For this purpose, you can use your function from Assignment 1. After simulating 100 observations, we act as if we no longer knew the data-generating process and wanted to estimate the parameters using CML.**
- **In Task 2, you are asked to write a function that returns the conditional log-likelihood contributions (a vector). Two comments on this: (a) You see that Task 2 is really specific regarding (e.g.) the way in which data and parameters are supposed to be handed to the function. Please follow these instructions closely – getting used to this syntax will make your life easier later on. (b) Make sure to actually back-out the ε-vector in your function. Do not use your innovations from Task 1 (remember: You have forgotten that you simulated the process to be starting with).**
- **After evaluating the conditional log-likelihood function in Task 2.1, Task 3 now deals with estimating the parameters of interest. Please note that we are providing you with a mini-example for this purpose and that the code is heavily commented. So, please have a close look at the contents of mini_example.zip**

1. Simulate an AR(1) process

Use the AR simulation function from the first assignment. Simulate the AR(1) process:

$$y_t = 5 + 0.7y_{t-1} + \varepsilon_t,$$

where $\varepsilon_t \overset{iid}{\sim} \mathcal{N}(0,1)$ and $T = 100$. Set $y_0 = 17$ and plot the series. Make your graph look professional.

2. Compute the conditional log likelihood contributions of an AR(1) process

1. The log likelihood function of a Gaussian AR(1) process $y_t = c + \phi y_{t-1} + \varepsilon_t$, where $\varepsilon_t \overset{iid}{\sim} \mathcal{N}(0,\sigma^2)$, conditional on the first observation is given by:

$$\mathcal{L} = \sum_{t=2}^{T} \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{(y_t - c - \phi y_{t-1})^2}{-2\sigma^2}\right]\right) = \sum_{t=2}^{T}\left[\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{\varepsilon_t^2}{2\sigma^2}\right], \tag{1}$$

where $\varepsilon_t = y_t - c - \phi y_{t-1}$ and σ^2 denotes the variance of the Gaussian White Noise innovations ε_t.

Write a function that computes the conditional log likelihood contributions of a Gaussian AR(1) process. Input arguments should be a column vector of the parameters (c, ϕ, σ^2) and the time series $\mathbf{y} = (y_1, y_2, \cdots, y_T)'$. Please hand the input arguments to the function in this order: first the vector of parameters, then the series. The function should return a column vector of log likelihood contributions of the observations conditional on the first observation.

Hint: In the body of the function you have to read out the parameters from the parameter input vector, then calculate the sequence (vector) of $\varepsilon_t = y_t - c - \phi y_{t-1}$ and compute the conditional log likelihood contributions.

2. Evaluate your conditional log likelihood function from task 2.1 using the simulated series from the first task and the parameter combinations:

   a) $c = 5$, $\phi = 0.7$ and $\sigma^2 = 1$
   b) $c = -5$, $\phi = 0.1$ and $\sigma^2 = 0.25$

   Use your conditional log likelihood contributions function from task 2.1 and print the values of the conditional log likelihood function (not the vector of single contributions) for both parameter specifications to the command window. Interpret the result.

   Hint: Create a new MATLAB function that computes, first, the log likelihood contributions vector, and then sums it up to get the conditional log likelihood function.

3. Optimization Techniques

1. Estimate the parameters (c, ϕ, σ^2) of an AR(1) process via MATLAB's optimization functions. Print the parameter estimates to the command window. You should use the following three optimization functions for this task. Use as starting values the parameters in task 2.2 a and b. Interpret your results.

   Hint: Use MATLAB's excellent documentation to find (and understand!) the correct syntax for using the optimization functions. A code with an example of the correct use of `fminsearch` has been provided to you on Ilias (`main_mini.m` together with the function file `mini_ex.m`).

   (a) `fminunc`: is a gradient-based approach, that minimizes the unconstrained multivariate function.
   (b) `fminsearch`: is a derivative-free approach, that minimizes the unconstrained multivariate function.
   (c) `patternsearch`: is a global search for the minimum of a multivariate function.

   Hint: The three optimization functions calculate the minimum of a given function, while maximum likelihood estimation is concerned with finding the maximum of a given likelihood function. You should adjust your function from task 2.2, accordingly.

2. Generate a likelihood profile. In order to do so, use the estimates $\hat{c}$ and $\hat{\sigma}^2$ from task 3.1. and compute the value of the conditional log likelihood function for varying ϕ. Choose the values in the ϕ vector appropriately, such that it encompasses the values of both $\hat{\phi}$ from the previous task and $\phi_0 = 0.7$ (i.e. the true value of ϕ). Plot the resulting ϕ series against the relevant conditional log likelihood value and make your graph look professional. Interpret the graph.