Quadratic Forms of Gaussian Random Variables

Tim Chen

1 Introduction

A quadratic form is a mathematical term that describes a polynomial equation of the second degree with multiple variables. It can be written as $\mathbf{Q}(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$, where $\mathbf{x}$ is a vector of variables, $\mathbf{A}$ is a square matrix, and $\mathbf{x}'$ is the transpose of $\mathbf{x}$. The purpose of quadratic forms is to analyze the properties of matrices and the behavior of systems that use quadratic functions. They have practical applications in diverse fields such as optimization, engineering, physics, and statistics. In statistics, quadratic forms are useful for determining the distance between data sets, computing the variance of linear combinations of variables, and testing statistical models. For instance, in linear regression, the error sum of squares (SSE) is a quadratic form that estimates the differences between the observed data and the predicted values from the regression model. Furthermore, quadratic forms are used in multivariate analysis to explore the interrelationships among several normally distributed variables, as we can see more in this document.

2 Fundamental Cases and Properties

Property 1. $\mathbf{Z} \sim \mathcal{N}_{\mathbf{n}}(\mathbf{0}, \mathbf{I}) \implies \mathbf{Z}'\mathbf{Z} \sim \chi^2_{\mathbf{n}}$

Property 2. $\mathbf{Z} \sim \mathcal{N}_{\mathbf{n}}(\mathbf{0}, \sigma^2\mathbf{I}) \implies \frac{\mathbf{Z}'\mathbf{Z}}{\sigma^2} \sim \chi^2_{\mathbf{n}}$

Property 3. $\mathbf{Z} \sim \mathcal{N}_{\mathbf{n}}(\mu, \sigma^2\mathbf{I}) \implies \frac{(\mathbf{Z}-\mu)'(\mathbf{Z}-\mu)}{\sigma^2} \sim \chi^2_{\mathbf{n}}$

Property 4. $\mathbf{Z} \sim \mathcal{N}_{\mathbf{n}}(\mu, \mathbf{\Sigma}) \implies (\mathbf{Z}-\mu)'\mathbf{\Sigma}^{-1}(\mathbf{Z}-\mu) \sim \chi^2_{\mathbf{n}}$

We start by introducing the simplest case, where $\mathbf{Z} \sim \mathcal{N}_{\mathbf{n}}(\mathbf{0}, \mathbf{I})$. It is a direct consequence that $Z_i \sim \mathcal{N}(0,1)\ \forall i \in [1,n]$, and therefore, $\mathbf{Z}'\mathbf{Z} = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{Z}_{\mathbf{i}}^2 \sim \chi^2_{\mathbf{n}}$. Notice, this property is THE most important of all when it comes to creating a central chi-squared distribution since all other properties, including 2 and 3 can be further transformed into 1.

Then, for $\mathbf{Z} \sim \mathcal{N}_{\mathbf{n}}(\mathbf{0}, \sigma^2\mathbf{I})$, we can see easily see that $\frac{\mathbf{Z}}{\sigma} \sim \mathcal{N}_{\mathbf{n}}(\mathbf{0}, \mathbf{I})$, and following above, we can derive $\frac{\mathbf{Z}'\mathbf{Z}}{\sigma^2} \sim \chi^2_{\mathbf{n}}$.

Now if $\mathbf{Z} \sim \mathcal{N}_{\mathbf{n}}(\mu, \sigma^2\mathbf{I})$, we can again get to the above stages by minus the mean and divide by the standard deviation and conclude that $\frac{(\mathbf{Z}-\mu)'(\mathbf{Z}-\mu)}{\sigma^2} \sim \chi^2_{\mathbf{n}}$.

In the last case $\mathbf{Z} \sim \mathcal{N}_{\mathbf{n}}(\mu, \mathbf{\Sigma})$, given the property of covariance matrix Σ being symmetric and invertible, we can look at $\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{Z}-\mu) \sim \mathcal{N}_{\mathbf{n}}(\mathbf{0}, \mathbf{I})$, then by applying property 1, we have, $(\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{Z}-\mu))'(\mathbf{\Sigma}^{-\frac{1}{2}}(\mathbf{Z}-\mu)) = (\mathbf{Z}-\mu)'\mathbf{\Sigma^{-1}}(\mathbf{Z}-\mu) \sim \chi^{\mathbf{2}}_{\mathbf{n}}$

2.1 Non-Central Chi-Squared Distribution

We know that if $\mathbf{Y} \sim \mathcal{N}_{\mathbf{n}}(\mu, \sigma^{\mathbf{2}}\mathbf{I})$, where $\mu = (\mu_1, \mu_2, ..., \mu_n)'$, we know from property 3 that $\frac{(\mathbf{Y}-\mu)'(\mathbf{Y}-\mu)}{\sigma^{\mathbf{2}}} \sim \chi^{\mathbf{2}}_{\mathbf{n}}$, and when $\mu_i = 0 \ \forall i$, $\frac{\mathbf{Y'Y}}{\sigma^{\mathbf{2}}} \sim \chi^{\mathbf{2}}_{\mathbf{n}}$.
Now We are interested in the case when for some $i, \mu_i \neq 0$. We first denote Q $= \frac{\mathbf{Y'Y}}{\sigma^{\mathbf{2}}} \sim \chi^{\mathbf{2}}_{\mathbf{n}}$, and now look at the MGF of Q.

$$M_Q(t) = E[exp(t\frac{\sum_{i=1}^n Y_i^2)}{\sigma^2}] = \prod_{i=1}^n E[exp(t\frac{Y_i^2}{\sigma^2}]$$

Also, for each

$$\begin{aligned}
E[exp(t\frac{Y_i^2}{\sigma^2})] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} exp(\frac{ty_i^2}{\sigma^2} - \frac{(y_i-\mu_i)^2}{2\sigma^2})dy_i \\
&= exp(\frac{t\mu_i^2}{(1-2t)\sigma^2}) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{1-2t}{2\sigma^2}(y_i - \frac{\mu_i}{1-2t})^2)dy_i \\
&= \frac{1}{\sqrt{1-2t}} exp(\frac{t\mu_i^2}{(1-2t)\sigma^2})
\end{aligned}$$

Then,

$$M_Q(t) = (1-2t)^{-\frac{n}{2}} exp(\frac{t\sum_i^n \mu_i^2}{(1-2t)\sigma^2})$$

. And thus, $Q = \frac{\mathbf{Y'Y}}{\sigma^{\mathbf{2}}} \sim \chi^{\mathbf{2}}_{\mathbf{n}}$ follows $\chi^2(n, \sum_i^n \mu_i^2)$

2.2 Applications in Linear Regression

2.2.1 Quadratic Form of Y

Consider the multiple regression model $\mathbf{Y} = \mathbf{X}\beta + \epsilon$ where $\epsilon \sim \mathcal{N}_{\mathbf{n}}(\mathbf{0}, \sigma^{\mathbf{2}}\mathbf{I})$.

Since $\mathbf{Y} \sim \mathcal{N}_{\mathbf{n}}(\mathbf{X}\beta, \sigma^{\mathbf{2}}\mathbf{I})$, we can apply property 3 and immediately get that $\frac{(\mathbf{Y}-\mathbf{X}\beta)'(\mathbf{Y}-\mathbf{X}\beta)}{\sigma^{\mathbf{2}}} \sim \chi^{\mathbf{2}}_{\mathbf{n}}$

2.2.2 Quadratic Form of $\hat{\beta}_{\mathbf{OLS}}$

In the same model as above, we can also take a closer look at the distribution of $\hat{\beta}_{\mathbf{OLS}}$, which is equal to $(\mathbf{X'X})^{\mathbf{-1}}\mathbf{X'Y}$. We have learnt before that

$$\mathbf{Y} \sim \mathcal{N}(\mu, \mathbf{\Sigma}) \implies \mathbf{AY} \sim \mathcal{N}_{\mathbf{m}}(\mathbf{A}\mu, \mathbf{A\Sigma A'}) \ \forall \mathbf{A} \in \mathbf{R^m}$$

So in this case we know

$$\hat{\beta}_{\mathbf{OLS}} \sim \mathcal{N}_{\mathbf{k+1}}((\mathbf{X'X})^{\mathbf{-1}}\mathbf{X'X}\beta, \sigma^{\mathbf{2}}((\mathbf{X'X})^{\mathbf{-1}}\mathbf{X'})((\mathbf{X'X})^{\mathbf{-1}}\mathbf{X'}))' = \mathcal{N}_{\mathbf{k+1}}(\beta, \sigma^{\mathbf{2}}(\mathbf{X'X})^{\mathbf{-1}})$$

From here we can use property 3 and get

$$\frac{(\hat{\beta}_{\mathbf{OLS}} - \beta)'\mathbf{X}'\mathbf{X}(\hat{\beta}_{\mathbf{OLS}} - \beta)}{\sigma^{\mathbf{2}}} \sim \chi^{\mathbf{2}}_{\mathbf{k+1}}$$

3 All About Sample Variance and $\frac{(\mathbf{n-k-1})\mathbf{S_e^2}}{\sigma^{\mathbf{2}}}$

To find the distribution of this USEFUL equation, we first need to familiarize with some lemmas and theorems.

3.1 Lemmas and Theorems

Lemma 3.1. *Let* $\mathbf{A}$ *be a symmetric and idempotent matrix, then the eigenvalues of* $\mathbf{A}$ *equal to either one or zero, and* $\mathbf{tr}(\mathbf{A})$ *gives the number of eigenvalues equal to 1.*

Proof: To find all the eigenvalues of $A \in R^{n\times n}$, we are trying to find all λ such that $\forall x \in R^n,\ Ax = \lambda x$. Since A is idempotent, $Ax = AAx = \lambda^2 x \implies \forall x \in R^n,\ \lambda^2 x = \lambda x \implies \lambda^2 - \lambda = 0 \implies \lambda = 0$ or $\lambda = 1$. Since A is symmetric and idempotent, we can use spectral decomposition to write $A = UDU^{-1}$ where D is the diagonal matrix with eigenvalues on the diagonal, and by cyclical property of trace, we have $tr(A) = tr(UDU^{-1}) = tr(DU^{-1}U) = tr(D)$ is clearly a count of how many 1's are in the diagonal, which gives the number of eigenvalues equal to 1.

Theorem 3.2. *Suppose* $\mathbf{B}$ *is an* $n\times n$ *symmetric and idempotent matrix. Then there exists an* $n \times p$ *matrix* $\mathbf{E}$ *such that* $\mathbf{B} = \mathbf{EE'}$ *and* $\mathbf{E'E} = \mathbf{I_p}$ *where* $\mathbf{p} = \mathbf{trace}(\mathbf{B})$.

Proof: Since B is symmetric and idempotent, $B = B', BB' = B = B'B \implies B$ is normal, and thus the eigendecomposition $B = PDP^{-1}$composes of unitary matrix P, i.e. $P' = P^{-1}$, and thus $B = PDP'$. We can split P into $E \in R^{n\times p}$ and $E^* \in R^{n\times(n-p)}$, and from Lemma3.1, we can split D into corresponding parts as follows:

$$P = \begin{bmatrix} E & E^* \end{bmatrix}, \qquad D = \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix}, \qquad P' = \begin{bmatrix} E' \\ E^{*'} \end{bmatrix}$$

$$B = PDP' = ED_rE' = EI_pE' = EE'$$

also, Since $I_n = P^{-1}P = P'P$, We have

$$I_n = \begin{bmatrix} E' \\ E^{*'} \end{bmatrix} \begin{bmatrix} E & E^* \end{bmatrix} = \begin{bmatrix} E'E & E'E^* \\ E^{*'}E & E^{*'}E^* \end{bmatrix}$$

and from here we can get $E'E = I_p$, where $p = trace(B)$.

Theorem 3.3. *Let* $\mathbf{Y} \sim \mathcal{N}_{\mathbf{n}}(\mu, \sigma^{\mathbf{2}}\mathbf{I})$ *and let* $\mathbf{B}$ *be an* $n\times n$ *symmetric and idempotent matrix such that* $\mathbf{tr}(\mathbf{B}) = \mathbf{p}$ *and* $\mu'\mathbf{B}\mu = \mathbf{0}$*, then* $\frac{\mathbf{Y'BY}}{\sigma^{\mathbf{2}}} \sim \chi^{\mathbf{2}}_{\mathbf{p}}$.

Proof: By Theorem 3.2, we know that $\exists E \in R^{n\times p}$ such that $EE' = B$ and $E'E = I_p$, where $p = tr(B)$. Now we replace B with EE', and get the following:

$$\mu' B\mu = 0 \implies (E'\mu)'(E'\mu) = 0 \implies (E'\mu) = 0$$

We get the last equation from the fact that in n-d Real Space: squares are are always greater or equal to 0, so sum of squares equals 0 implies all elements are 0. Then applying the above, we can see that

$$E'Y \sim \mathcal{N}_p(E'\mu, E'\sigma^2 E) = \mathcal{N}_p(0, \sigma^2 I)$$

Then, we transform $\frac{\mathbf{Y'BY}}{\sigma^2} = \frac{\mathbf{(E'Y)'E'Y}}{\sigma^2}$, and from there we can apply property 2, and get that $\frac{\mathbf{Y'BY}}{\sigma^2} \sim \chi^2_{\mathbf{p}}$.

Lemma 3.4. *Let $Y \sim \mathcal{N}_n(0, I)$. If $\mathbf{P}$ is orthogonal matrix (i.e. $\mathbf{P'P} = \mathbf{I}$) then $\mathbf{Z} = \mathbf{P'Y} \sim \mathcal{N}_{\mathbf{n}}(\mathbf{0}, \mathbf{I})$.*

Proof: The proof is a one-liner. Since Linear Transformation of a multivariate normal distribution is still a multivariate normal distribution, proved by the uniqueness of MGF. Therefore, $P'Y \sim \mathcal{N}_n(0, P'IP) = \mathcal{N}_n(0, I)$.

Theorem 3.5. *Let $\mathbf{Y} \sim \mathcal{N}_{\mathbf{n}}(\mathbf{0}, \mathbf{I})$, and let $\mathbf{A}$ be a symmetric and idempotent matrix. Then $\mathbf{Y'AY} \sim \chi^2_{\mathbf{r}}$, where $\mathbf{r}$ is the number of eigenvalues of $\mathbf{A}$ equal to 1. The other $\mathbf{n} - \mathbf{r}$ eigenvalues are equal to zero.*

Proof: We use Spectral decomposition: $A = P\Lambda P^{-1}$, which in this case, $P^{-1} = P'$ since A is symmetric. Then $Y'AY = (P'Y)'\Lambda(P'Y)$. Denote $Z = (P'Y)$. Then using lemma 3.1 we know Λ is composed of r counts of 1s and $n - r$ 0s in the diagonal, we can sort them during spectral decomposition so that all 1's are are the left top corner, and 0's the rest, Denote top left matrix as Λ_r. From there, it is intuitive to split Z into $Z_1 \in R^{r\times n}$ and $Z_2 \in R^{(n-r)\times n}$.

$$Z' = \begin{bmatrix} Z_1' & Z_2' \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \begin{matrix} 1 & 0 & 0 & \dots & \\ 0 & 1 & 0 & \dots & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{matrix} & \\ & \begin{matrix} 0 & 0 & 0 & \dots & \\ 0 & 0 & 0 & \dots & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{matrix} \end{bmatrix} = \begin{bmatrix} \Lambda_r & 0 \\ 0 & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

Then $Z'\Lambda Z = Z_1'\Lambda_r Z_1 = Z_1'Z_1$. From here, we can apply Lemma 3.4 to get that $Z_i \sim \chi^2_1$ since $Z = P'Y$ and P is orthogonal matrix. Thus, $Z_1'Z_1 \sim \chi^2_r$ where r is the number of eigenvalues of A equal to 1.

3.2 Applying to Linear Regression

3.2.1 Sample Variance S_e^2

Method 1. Use **Theorem 3.3** to prove that in Multiple Regression $\mathbf{\frac{(n-k-1)S_e^2}{\sigma^2}} \sim \chi^2_{\mathbf{n-k-1}}$.

Proof: Since $\mathbf{\frac{(n-k-1)S_e^2}{\sigma^2}} = \mathbf{\frac{e'e}{\sigma^2}} = \mathbf{\frac{Y'(I-H)Y}{\sigma^2}}$, it is now enough to show that $(I-H)$ is symmetric, idempotent, and $tr(I-H) = n-k-1$, $(X\beta)'(I-H)X\beta = 0$. Clearly, $I-H$ is symmetric since I and H are both symmetric, it is idempotent, because $(I_n - H)(I_n - H) = I_n - 2H + HH = I_n - 2H + H = 1 - H$. Then, $tr(I_n - H) = tr(I_n) - tr(X(X'X)^{-1}X') = tr(I_n) - tr((X'X)^{-1}X'X) = tr(I_n) - tr(I_{k+1}) = n-k-1$. Lastly, from $(I-H)X = X - X = 0$ we can easily get $(X\beta)'(I-H)X\beta = 0$. Therefore, the proof is complete.

Method 2. Use **Theorem 3.5** to prove that in Multiple Regression $\mathbf{\frac{(n-k-1)S_e^2}{\sigma^2}} \sim \chi^2_{\mathbf{n-k-1}}$.

Proof: To use Theorem 3.5, we need to first transform

$$\frac{(n-k-1)S_e^2}{\sigma^2} = \frac{e'e}{\sigma^2} = \frac{(X\beta+\epsilon)'(I-H)(X\beta+\epsilon)}{\sigma^2} = \frac{\epsilon'(I-H)\epsilon}{\sigma^2}$$

Then, from Gauss-Markov Condition, we know that $\epsilon \sim \mathcal{N}_n(0, I)$ and combining with the proof earlier that $I-H$ is symmetric and idempotent with $tr(I-H) = n-k-1$, we can conclude by Theorem 3.5 that $\frac{(n-k-1)S_e^2}{\sigma^2} \sim \chi^2_{n-k-1}$.

Case with Minor Modification: Suppose $\mathbf{Y} \sim \mathcal{N}_\mathbf{n}(\mu\mathbf{1}, \sigma^\mathbf{2}\mathbf{I})$. Show that $\mathbf{\frac{(n-1)S_y^2}{\sigma^2}} \sim \chi^2_{\mathbf{n-1}}$, where S_y^2 is the sample variance of $\mathbf{Y} = (Y_1, ..., Y_n)'$.

Proof: $S_y^2 = \frac{Y'(I-\frac{1}{n}11')Y}{n-1}$, in this case $B = (I - \frac{1}{n}11')$, which is symmetric because both terms are symmetric and idempotent because

$$(I - \frac{1}{n}11')(I - \frac{1}{n}11') = I - 2\frac{1}{n}11' + \frac{1}{n^2}n11' = I - \frac{1}{n}11'$$

Also, $tr(B) = tr(I) - \frac{1}{n}tr(11') = n-1$, and $(\mu 1)'B\mu 1 = (\mu 1)'(I - \frac{1}{n}11')\mu 1 = 0$ because μ is a fixed constant and $\frac{1}{n}11'\mu 1 = \mu 1$ Then we can apply Theorem 3.3 to reach the conclusion that $\frac{(n-1)S_y^2}{\sigma^2} = \frac{Y'(I-\frac{1}{n}11')Y}{\sigma^2} \sim \chi^2_{n-1}$

4 Fun Notes

4.1 Sufficient Condition on Theorem 3.3

Let $\mathbf{Y} = (Y_1, ..., Y_n)'$, where $Y_i \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma)\ \forall i \in [1, n]$. Let $\mathbf{A}$ be a real symmetric matrix.
\- *Note: If* $\mathbf{A} = \mathbf{I}$ *then we have seen that* $\mathbf{\frac{Y'Y}{\sigma^2}} \sim \chi^2_\mathbf{n}$.
We can use the moment generating function of the quadratic form $\mathbf{\frac{Y'AY}{\sigma^2}}$ and investigate the conditions on $\mathbf{A}$ so that $\mathbf{\frac{Y'AY}{\sigma^2}}$ follows a chi square distribution.

Proof:

$$
\begin{aligned}
M_{\frac{Y'AY}{\sigma^2}} &= E(exp(t\frac{Y'AY}{\sigma^2})) \\
&= \int \cdots \int_Y exp(t\frac{Y'AY}{\sigma^2})(2\pi)^{-n/2} \left|\sigma^2 I\right|^{-\frac{1}{2}} exp(-\frac{1}{2\sigma^2}Y'Y)\, dy_1 \ldots dy_n \\
&= \int \cdots \int_Y (2\pi)^{-n/2} \left|\sigma^2 I\right|^{-\frac{1}{2}} exp(-\frac{1}{2}Y'(\sigma^2(I-2tA)^{-1})^{-1}Y)\, dy_1 \ldots dy_n \\
&= \int \cdots \int_Y (2\pi)^{-n/2} \left|\sigma^2 (I-2tA)^{-1}(I-2tA)\right|^{-\frac{1}{2}} exp(-\frac{1}{2}Y'(\sigma^2(I-2tA)^{-1})^{-1}Y)\, dy_1 \ldots dy_n \\
&= \left|I-2tA\right|^{-\frac{1}{2}} \int \cdots \int_Y (2\pi)^{-n/2} \left|\sigma^2 (I-2tA)^{-1}\right|^{-\frac{1}{2}} exp(-\frac{1}{2}Y'(\sigma^2(I-2tA)^{-1})^{-1}Y)\, dy_1 \ldots dy_n \\
&= \left|I-2tA\right|^{-\frac{1}{2}}
\end{aligned}
$$

We know that A is symmetric, so spectral decomposition leads to $A = P\Lambda P'$, and $P'AP = \Lambda$. Since $PP' = I_n$, $1 = \left|I_n\right| = \left|PP'\right| = \left|P\right| \left|P'\right|$. Since all determinant are scalar values, we can easily see $\left|I-2tA\right| = \left|P'\right| \left|I-2tA\right| \left|P\right| = \left|P'(I-2tA)P\right|$,

$$
P'(I-2tA)P = I - 2t\Lambda = \begin{bmatrix} 1-2ta_1 & 0 & \ldots & 0 \\ 0 & 1-2ta_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1-2ta_n \end{bmatrix}
$$

where a_i is the i-th eigenvalue of A.

$$
M_{\frac{Y'AY}{\sigma^2}} = \left|(I-2tA)\right|^{-1/2} = \left|P'(I-2tA)P\right|^{-1/2} = (\prod_{i=1}^{n} 1-2ta_n)^{-1/2}
$$

Now by comparing the MGF above with MGF of χ^2_n (Special Gamma Distribution with $\alpha = n/2$ and $\beta = 2$): $(1-2t)^{-n/2}$, we can see that the condition $\forall i \in [1,n]\ a_i = 1$ has to be met in order for the two equations to align. That is to say $\Lambda = I_n$, and following that we have $AA = (P\Lambda P')(P\Lambda P') = PI_n(P'P)I_nP' = PI_nP' = A$. By definition, it means that A is idempotent.

From this proof, we can write out the following Lemmas:

Lemma 4.1. *Let* $\mathbf{Y} \sim \mathcal{N}_\mathbf{n}(\mathbf{0}, \sigma^\mathbf{2}\mathbf{I})$.
If $\mathbf{A}$ *is a real and symmetric, MGF of* $\frac{\mathbf{Y'AY}}{\sigma^\mathbf{2}}$ *is* $\mathbf{M}_{\frac{\mathbf{Y'AY}}{\sigma^\mathbf{2}}} = |\mathbf{I_n} - \mathbf{2tA}|^{-\frac{1}{2}}$

Lemma 4.2. *Let* $\mathbf{Y} \sim \mathcal{N}_\mathbf{n}(\mathbf{0}, \sigma^\mathbf{2}\mathbf{I})$. *If* $\mathbf{A}$ *is a real and symmetric,* $\frac{\mathbf{Y'AY}}{\sigma^\mathbf{2}} \sim \chi^2_n \implies \mathbf{A}$ *is idempotent.*

4.2 Independence between Quadratic Forms

Theorem 4.3. *Let* $\mathbf{Y} = (Y_1, ..., Y_n)'$*, where* $Y_i \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma)\ \forall i \in [1, n]$*. Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two real symmetric matrices.* $\mathbf{Q_1} = \mathbf{Y'AY}$ *and* $\mathbf{Q_2} = \mathbf{Y'BY}$*.* $\mathbf{Q_1}$ *and* $\mathbf{Q_2}$ *are independent if and only if* $\mathbf{AB} = \mathbf{0}$*.*

Proof:
Using Lemma 4.1, We know that $M_{\frac{Q_1}{\sigma^2}}(t_1) = M_{\frac{Y'AY}{\sigma^2}} = |I_n - 2t_1A|^{-\frac{1}{2}}$ and $M_{\frac{Q_2}{\sigma^2}}(t_2) = M_{\frac{Y'BY}{\sigma^2}} = |I_n - 2t_2B|^{-\frac{1}{2}}$. Since we know the definition of independece, it is know enough to make sure the joint moment generating function of Q_1 and Q_2 equals to the product of the marginal MGF, which is $|I_n - 2t_1A - 2t_2B + 4t_1t_2AB|^{-\frac{1}{2}}$.
On the other hand,

$$
\begin{aligned}
M_{\frac{Q_1}{\sigma^2}, \frac{Q_2}{\sigma^2}} &= E(exp(t_1\frac{Y'AY}{\sigma^2} + t_2\frac{Y'BY}{\sigma^2})) \\
&= \int \cdots \int_Y exp(\frac{Y'(t_1A + t2_B)Y}{\sigma^2})(2\pi)^{-n/2} \left|\sigma^2 I\right|^{-\frac{1}{2}} exp(-\frac{1}{2\sigma^2}Y'Y)\, dy_1 \dots dy_n \\
&= \int \cdots \int_Y (2\pi)^{-n/2} \left|\sigma^2 I(I_n - 2t_1A - 2t_2B)^{-1}(I_n - 2t_1A - 2t_2B)\right|^{-\frac{1}{2}} \\
&\quad exp(-\frac{1}{2\sigma^2}Y'(I_n - 2t_1A - 2t_2B)Y)\, dy_1 \dots dy_n \\
&= |I_n - 2t_1A - 2t_2B|^{-\frac{1}{2}} \int \cdots \int_Y (2\pi)^{-n/2} \left|\sigma^2 I(I_n - 2t_1A - 2t_2B)^{-1}\right|^{-\frac{1}{2}} \\
&\quad exp(-\frac{1}{2\sigma^2}Y'(I_n - 2t_1A - 2t_2B)Y)\, dy_1 \dots dy_n \\
&= |I_n - 2t_1A - 2t_2B|^{-\frac{1}{2}}
\end{aligned}
$$

Because of the Uniqueness of MGF and the independence theorem of MGF, we can conclude that $Q1$ and Q_2 are independent if and only if $AB = 0$.

4.3 Sum of Squares Formula

SST=SSR+SSE

SST: **T**otal Sum of Squares

$$
\begin{aligned}
&= \sum_{i=1}^{n}(y_i - \bar{y})^2 \\
&= \mathbf{Y'(I - \frac{1}{n}11')Y}
\end{aligned}
\tag{1}
$$

SSR: **R**egression Sum of Squares

$$
\begin{aligned}
&= \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 \\
&= \mathbf{Y}'(\mathbf{H} - \frac{\mathbf{1}}{\mathbf{n}}\mathbf{1}\mathbf{1}')\mathbf{Y}
\end{aligned} \tag{2}
$$

SSE: **E**rror Sum of Squares

$$
\begin{aligned}
&= \sum_{i=1}^{n} (\hat{y}_i - yi)^2 \\
&= \mathbf{e}'\mathbf{e} = \mathbf{Y}'(\mathbf{I} - \mathbf{H})\mathbf{Y} = \epsilon'(\mathbf{I} - \mathbf{H})\epsilon
\end{aligned} \tag{3}
$$

SSR and SSE are always confusing because SSR also means **E**xplained Sum of Squares, and SSE means **R**esidual Sum of Squares, which has the exact opposite starting letter.
Clearly Sum of Squares Formula makes sense in the matrix notation since

$$
Y'(I - \frac{1}{n}11')Y = Y'(H - \frac{1}{n}11')Y + Y'(I - H)Y
$$

References

[1] Christou, N. (2023). *Distribution of quadratic forms of normally distributed random variables* [Class handout]. UCLA Department of Statistics.
https://bruinlearn.ucla.edu/courses/157535/files/12488260?module_item_id=5874831

[2] Christou, N. (2023). *Distributions related to normal distribution* [Class handout]. UCLA Department of Statistics.
https://bruinlearn.ucla.edu/courses/157535/files/12558432?module_item_id=5882672