Nonlinear progressive Rossby waves

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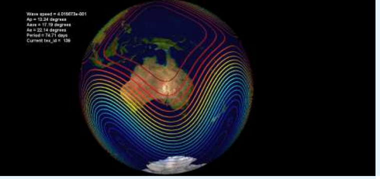
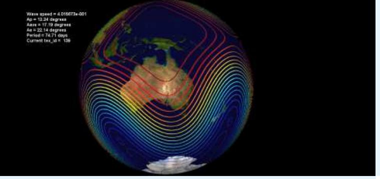
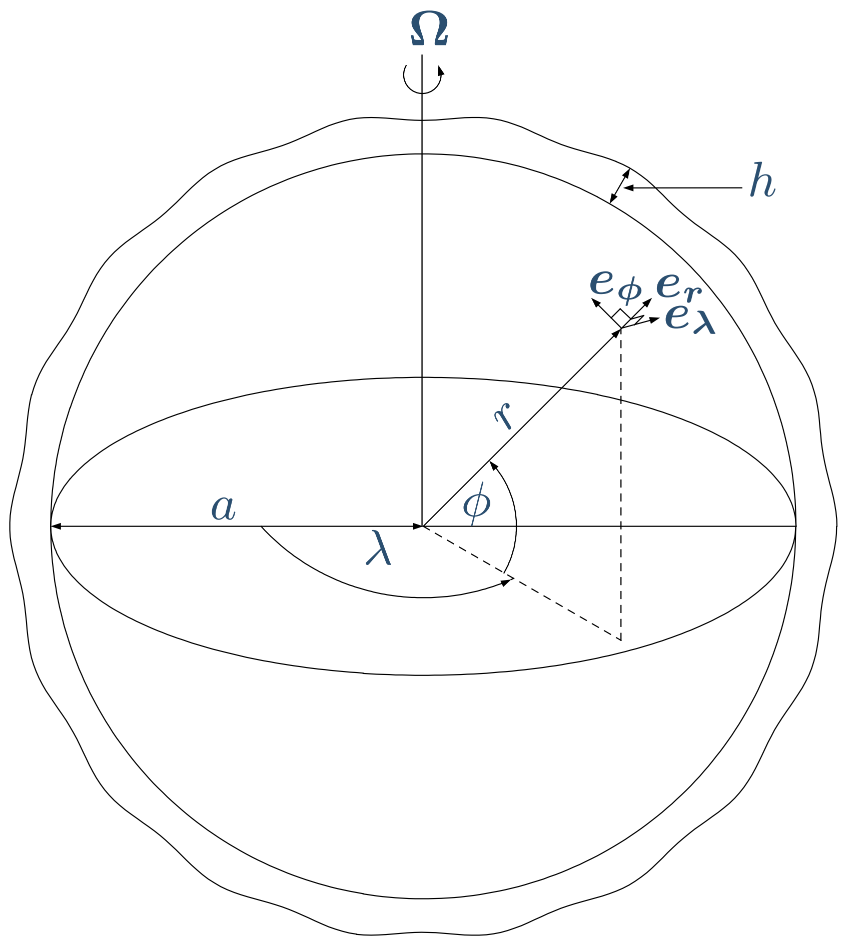


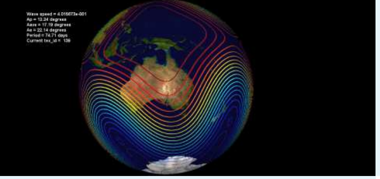
Preliminaries



Coordinate System

Variables:
Time, t
Radial coord, r
Longitude, λ
Latitude, ϕ
Density, ρ
Pressure, p
Gravity, $\boldsymbol{g} = -g\boldsymbol{e_r}$
Velocity,
$\boldsymbol{q} = u_r\boldsymbol{e_r} + u_\lambda\boldsymbol{e_\lambda} + u_\phi\boldsymbol{e_\phi}$
Free-surface depth,
$h(\lambda, \phi, t)$
Angular velocity, $\boldsymbol{\Omega}$





Equations of Motion

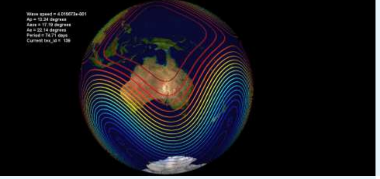
In a reference frame rotating with angular velocity $\boldsymbol{\Omega}$, conservation of mass for an incompressible inviscid fluid is expressed through the continuity equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{q} = 0 \tag{1}$$

and conservation of momentum requires the usual Euler equation

$$\frac{\mathrm{D}\boldsymbol{q}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{q} + \frac{1}{\rho}\boldsymbol{\nabla} p = \boldsymbol{f}, \tag{2}$$

where $\boldsymbol{f}$ is the combined effect of all body forces per unit mass.



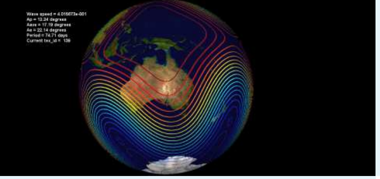
Approximations

- Shallow atmosphere, $h(\lambda, \phi, t) \ll a$.
- Mainly tangential motion, $u_r \ll u_\lambda$ and $u_r \ll u_\phi$.
- Radial coordinate approximated by $r = a$.
- Hydrostatic balance, $p(r, \lambda, \phi, t) = p_o + \rho g(a + h(\lambda, \phi, t) - r)$.
- Only allow progressive waves, define $\eta = \lambda - ct$, where $-ct$ term merely translates any initial wave structure.
- Nondimensionalize, reference scales v_{ref}, h_{ref} and c_{ref}, leading to dimensionless parameters

$$\mathrm{Sr} = \frac{a\, c_{ref}}{v_{ref}} \qquad \text{Strouhal number,}$$

$$\mathrm{Ro} = \frac{v_{ref}}{2\Omega a} \qquad \text{Rossby number,}$$

$$\mathrm{Fr} = \frac{v_{ref}}{\sqrt{g h_{ref}}} \qquad \text{Froude number.}$$



Shallow Atmosphere Equations

The conservation equations in spherical polar component form are given by

Mass

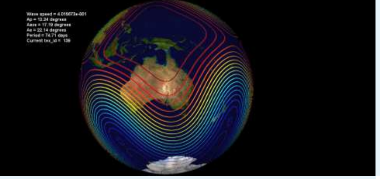
$$(u_\lambda - \mathrm{Sr}\, c\cos\phi)\frac{\partial h}{\partial \eta} + u_\phi \cos\phi \frac{\partial h}{\partial \phi} + h\left[\frac{\partial u_\lambda}{\partial \eta} + \cos\phi \frac{\partial u_\phi}{\partial \phi} - u_\phi \sin\phi\right] = 0,$$

λ momentum

$$(u_\lambda - \mathrm{Sr}\, c\cos\phi)\frac{\partial u_\lambda}{\partial \eta} + u_\phi \cos\phi \frac{\partial u_\lambda}{\partial \phi} - \left(\frac{\cos\phi}{\mathrm{Ro}} + u_\lambda\right) u_\phi \sin\phi + \frac{1}{\mathrm{Fr}^2}\frac{\partial h}{\partial \eta} = 0,$$

ϕ momentum

$$(u_\lambda - \mathrm{Sr}\, c\cos\phi)\frac{\partial u_\phi}{\partial \eta} + u_\phi \cos\phi \frac{\partial u_\phi}{\partial \phi} + \left(\frac{\cos\phi}{\mathrm{Ro}} + u_\lambda\right) u_\lambda \sin\phi + \frac{\cos\phi}{\mathrm{Fr}^2}\frac{\partial h}{\partial \phi} = 0.$$



Volume Specification

It is necessary to specify the total volume V_b of the atmosphere.

- Define exactly κ wavelengths around a latitude circle.

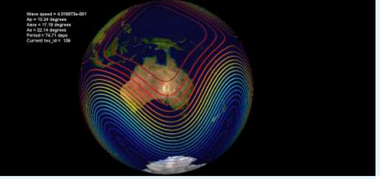
The total volume of fluid is

$$V = \frac{4\kappa}{3} \int_0^{\pi/\kappa} \int_0^{\pi/2} \left[h^3 + 3\hat{a}^2 h + 3\hat{a}h^2\right] \cos\phi \, \mathrm{d}\phi \mathrm{d}\eta.$$

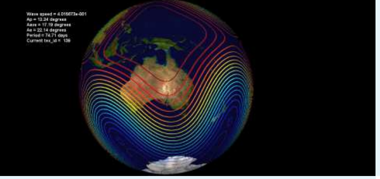
The volume specification condition is now written in the form

$$1 - \frac{V}{V_b} = 0.$$

The complete specification of a nonlinear atmospheric progressive Rossby wave in this model consists of solving all conservation equations subject to some condition defining the amplitude of the wave.



Linearized Theory



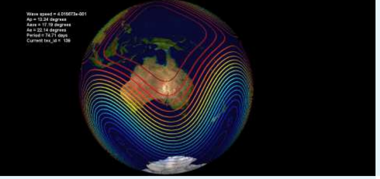
Zonal Flow and Perturbations

Consider small amplitude Rossby waves as perturbations to a base Westerly zonal flow. Have zonal flow of the form

$$
\begin{aligned}
u_{\lambda z} &= \omega \cos\phi, \\
u_{\phi z} &= 0, \\
h_z &= h_o + \frac{\omega \mathrm{Fr}^2}{2}\left(\frac{1}{\mathrm{Ro}} + \omega\right)\cos^2\phi,
\end{aligned}
$$

and then construct $O(\epsilon)$ perturbations

$$
\begin{aligned}
u_\lambda(\eta,\phi) &= u_{\lambda z} + \epsilon\cos(\kappa\eta)\,\Lambda(\phi) + O(\epsilon^2), \\
u_\phi(\eta,\phi) &= 0 \quad + \epsilon\sin(\kappa\eta)\,\Phi(\phi) + O(\epsilon^2), \\
h(\eta,\phi) &= h_z \ + \epsilon\cos(\kappa\eta)\,\mathcal{H}(\phi) + O(\epsilon^2).
\end{aligned}
$$



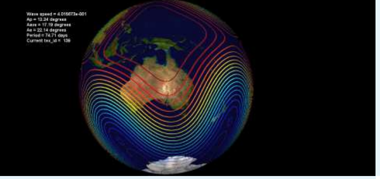
Series Solution

- Derive a linearized system of equations for the $O(\epsilon)$ corrections.
- Impose specific symmetry conditions.
- Solve using Fourier series of the form

$$\Lambda(\phi) = \sum_{n=1}^{N} P_{\kappa,n} \cos\big((2n-1)\phi\big),$$

$$\Phi(\phi) = \sum_{n=1}^{N} Q_{\kappa,n} \sin(2n\phi),$$

$$\mathcal{H}(\phi) = \sum_{n=1}^{N} H_{\kappa,n}(-1)^n \left[\cos(2n\phi) + \cos\big(2(n-1)\phi\big)\right].$$



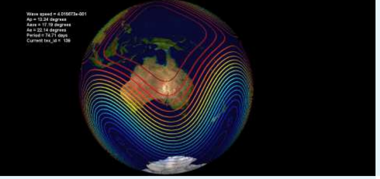
Generalized Eigenvalue Problem

- Use orthogonality to integrate the equations (Galerkin method).
- Derive a generalized eigenvalue problem of the form

$$A\boldsymbol{x} = cB\boldsymbol{x}.$$

- A and B are matrices corresponding to the left and right-hand sides of each of the algebraic equations obtained from orthogonality.
- The eigenvalue c is precisely the wavespeed for the progressive Rossby wave.
- Vector $\boldsymbol{x}$ is the eigenvector of unknown linearized coefficients, which is defined as

$$\boldsymbol{x} = \left[H_{\kappa,1}, \ldots, H_{\kappa,N}, P_{\kappa,1}, \ldots, P_{\kappa,N}, Q_{\kappa,1}, \ldots, Q_{\kappa,N}\right]^T.$$



Model Parameters

We use parameters that closely approximate those of the Earth.

$$a = 6.37122 \times 10^6 \text{ m}$$

$$\Omega = \frac{2\pi}{24 \times 3600} \approx 7.272 \times 10^{-5} \text{ s}^{-1}$$

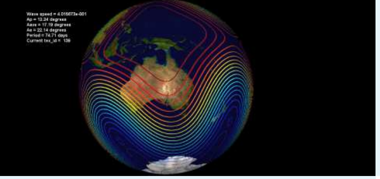
$$g = 9.80616 \text{ m s}^{-2}$$

$$v_{ref} = 40 \text{ m s}^{-1}$$

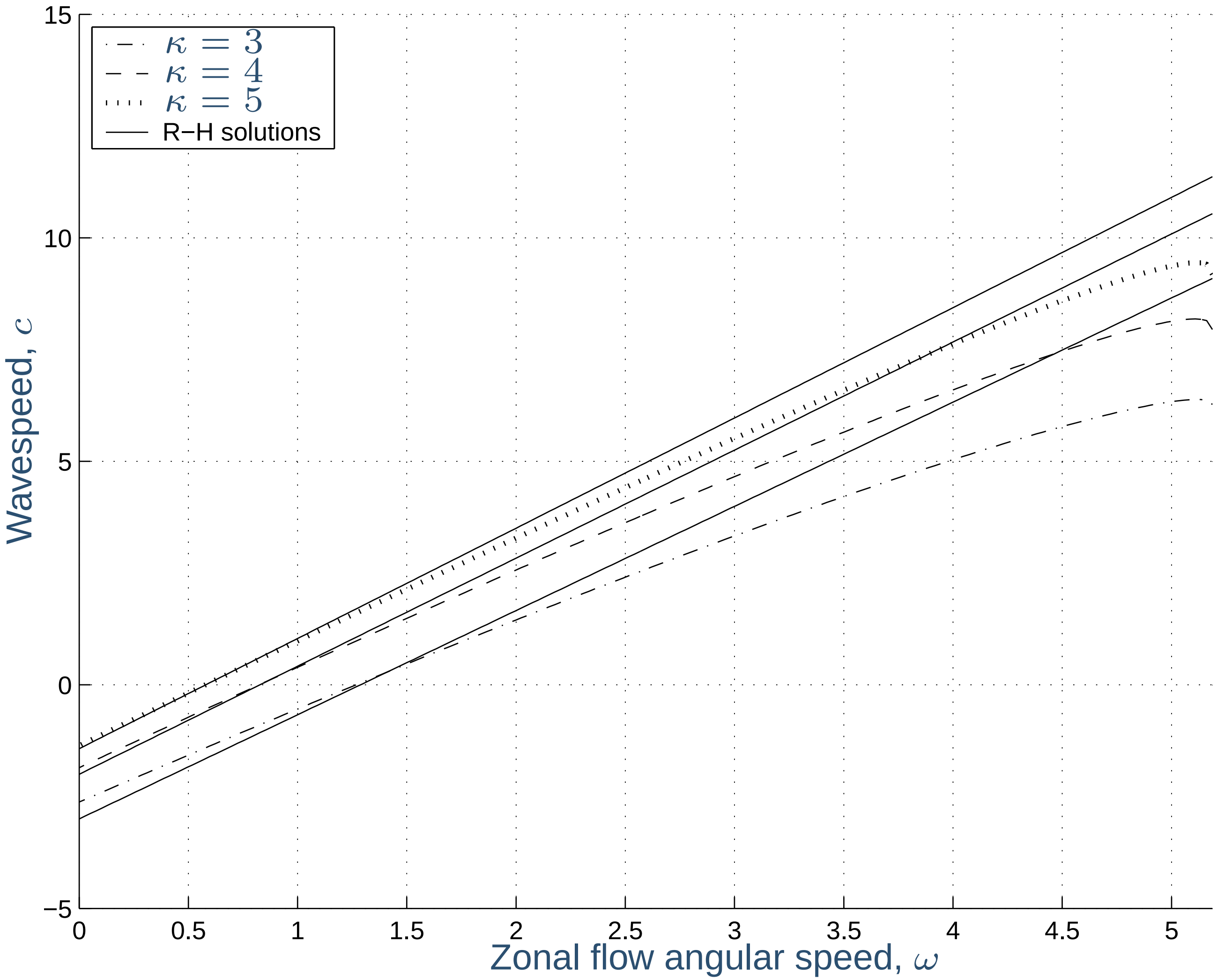
$$h_{ref} = 8.0 \times 10^3 \text{ m}$$

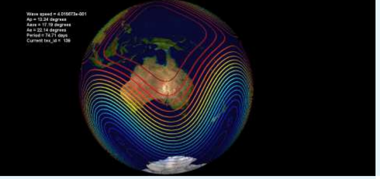
$$c_{ref} = \frac{\Omega}{30} \approx 2.4241 \times 10^{-6} \text{ s}^{-1}$$

Thus: $\mathrm{Sr} \approx 3.8611 \times 10^{-1}$, $\mathrm{Fr} \approx 1.4281 \times 10^{-1}$ and $\mathrm{Ro} \approx 4.3166 \times 10^{-2}$

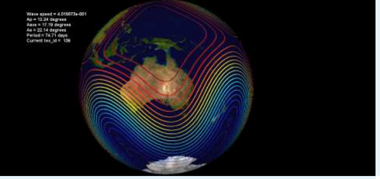


Comparison with R-H wavespeed





Nonlinear Theory



Series Solution

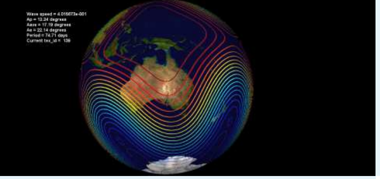
Seek solutions of the full nonlinear problem using Fourier series of the form

$$u_\lambda(\eta,\phi) = \omega\cos\phi + \sum_{m=1}^{M}\sum_{n=1}^{N} P_{m,n}\cos(\kappa m\eta)\cos\big((2n-1)\phi\big),$$

$$u_\phi(\eta,\phi) = \sum_{m=1}^{M}\sum_{n=1}^{N} Q_{m,n}\sin(\kappa m\eta)\sin(2n\phi)$$

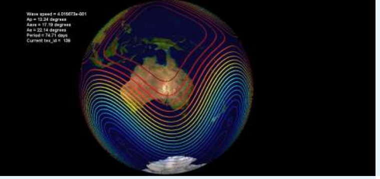
$$h(\eta,\phi) = \sum_{n=0}^{N} H_{0,n}\cos(2n\phi),$$

$$+\sum_{m=1}^{M-1}\sum_{n=1}^{N} H_{m,n}\cos(\kappa m\eta)(-1)^n\left[\cos(2n\phi)+\cos\big(2(n-1)\phi\big)\right].$$



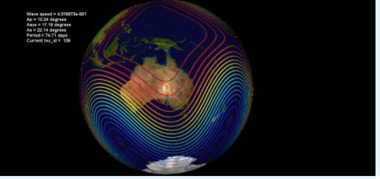
Amplitude Forcing and Measurement

- Force amplitude $\mathcal{A}$ through either $H_{1,1}$ or c.
- In this context, progressive Rossby waves are perturbations from a base Westerly zonal flow, for which the height contours are simply circles of constant ϕ. The unperturbed free-surface height contours at $\phi = \pm\pi/4$ are taken here as the base level, against which Rossby wave amplitudes are measured.
- Define the equatorial, polar and average amplitudes as $\mathcal{A}_e$, $\mathcal{A}_p$ and $\mathcal{A}_{ave}$ respectively.

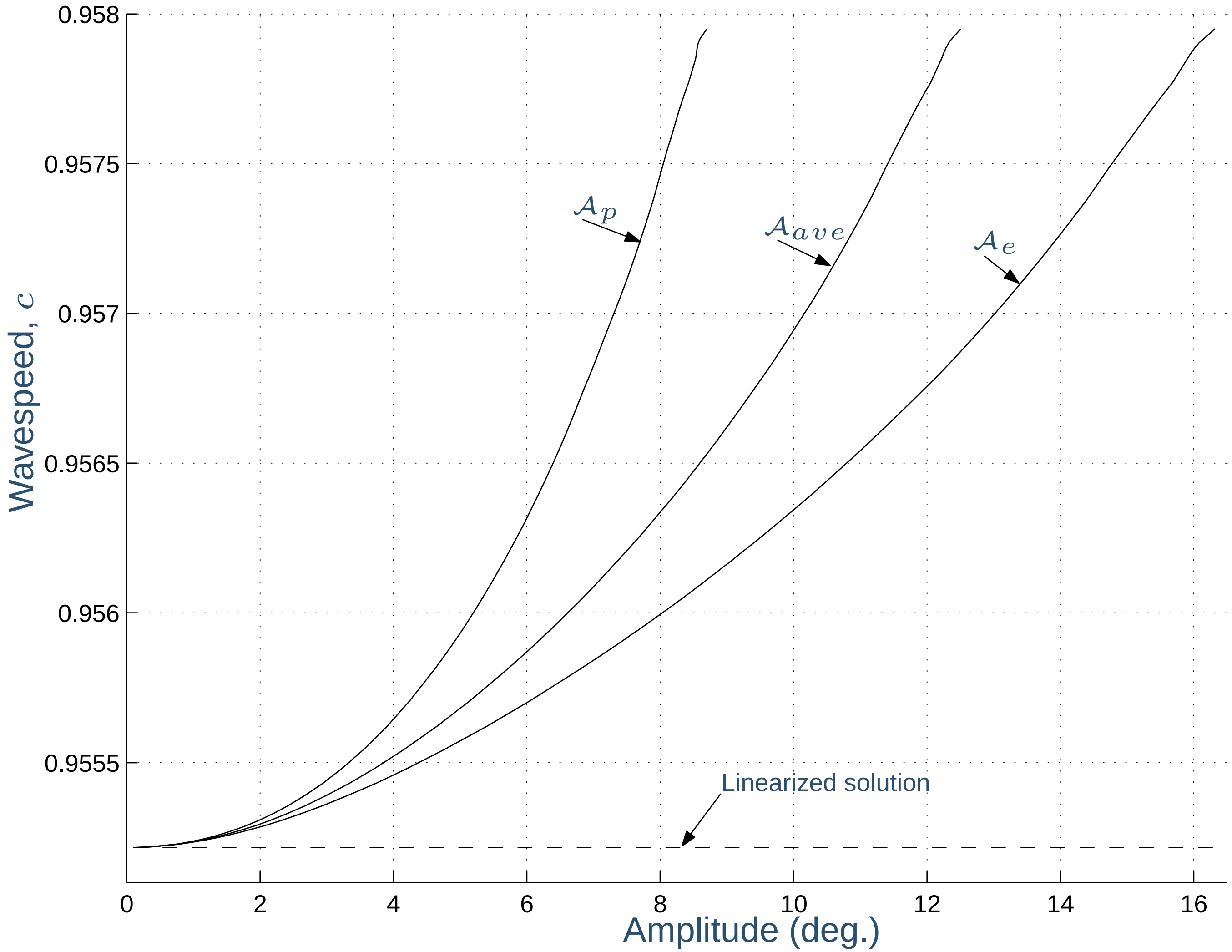


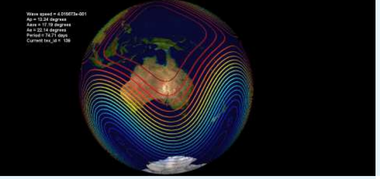
Solution Process

- Solve using Collocation. We evaluate the three governing dynamical equations at every point of the collocation mesh to give a vector of residuals. In addition, the volume specification equation is evaluated and appended to this vector.
- Residual equations are solved using a multi-dimensional Newton method.
- Linearized solutions are used to initialise the Newton method when the amplitude is small.
- Once a nonlinear solution has been found, the amplitude is slowly increased and bootstrapping is used to trace out the wavespeed versus amplitude curve incrementally.

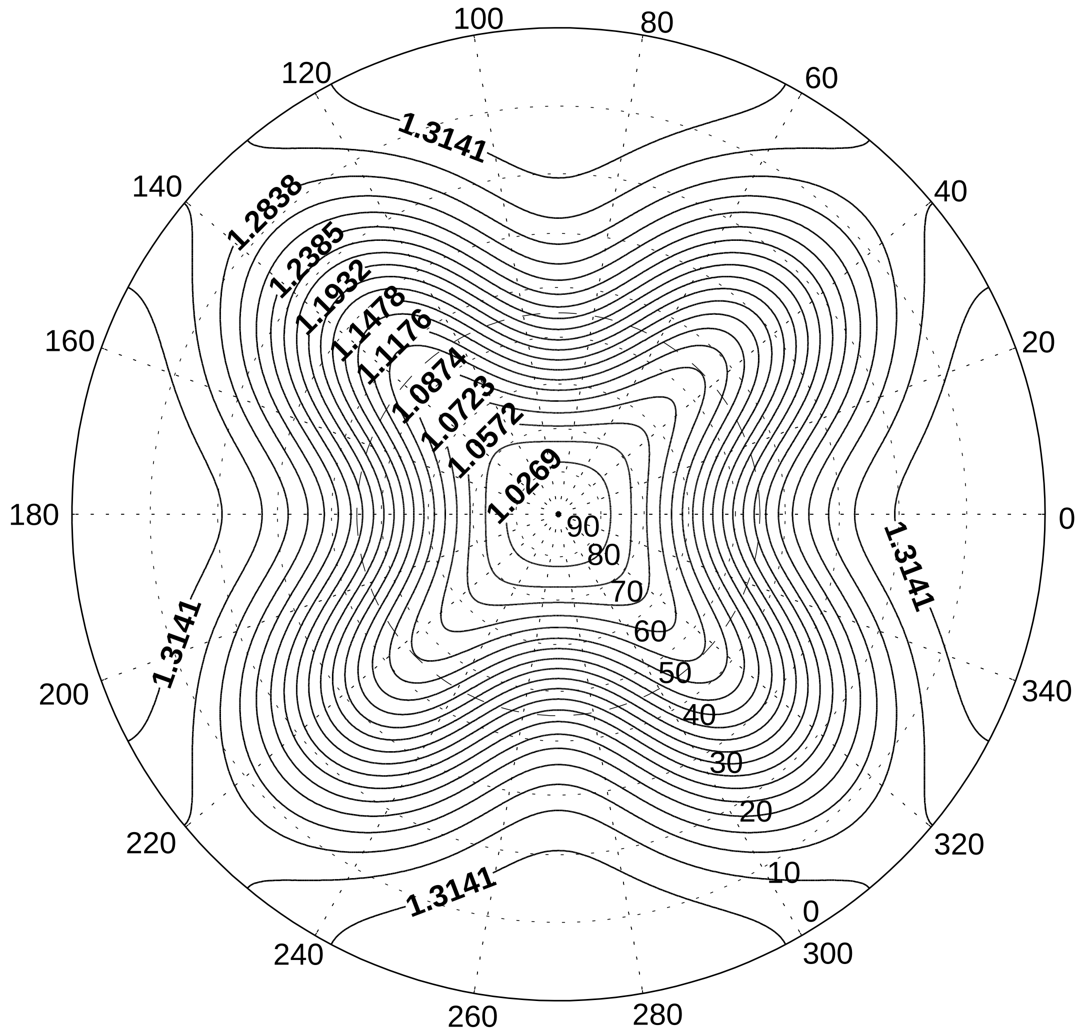


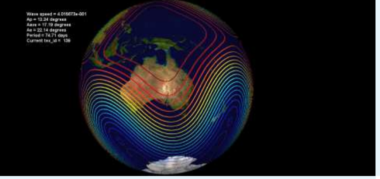
Results for $\kappa = 4$, $\omega = 1.25$



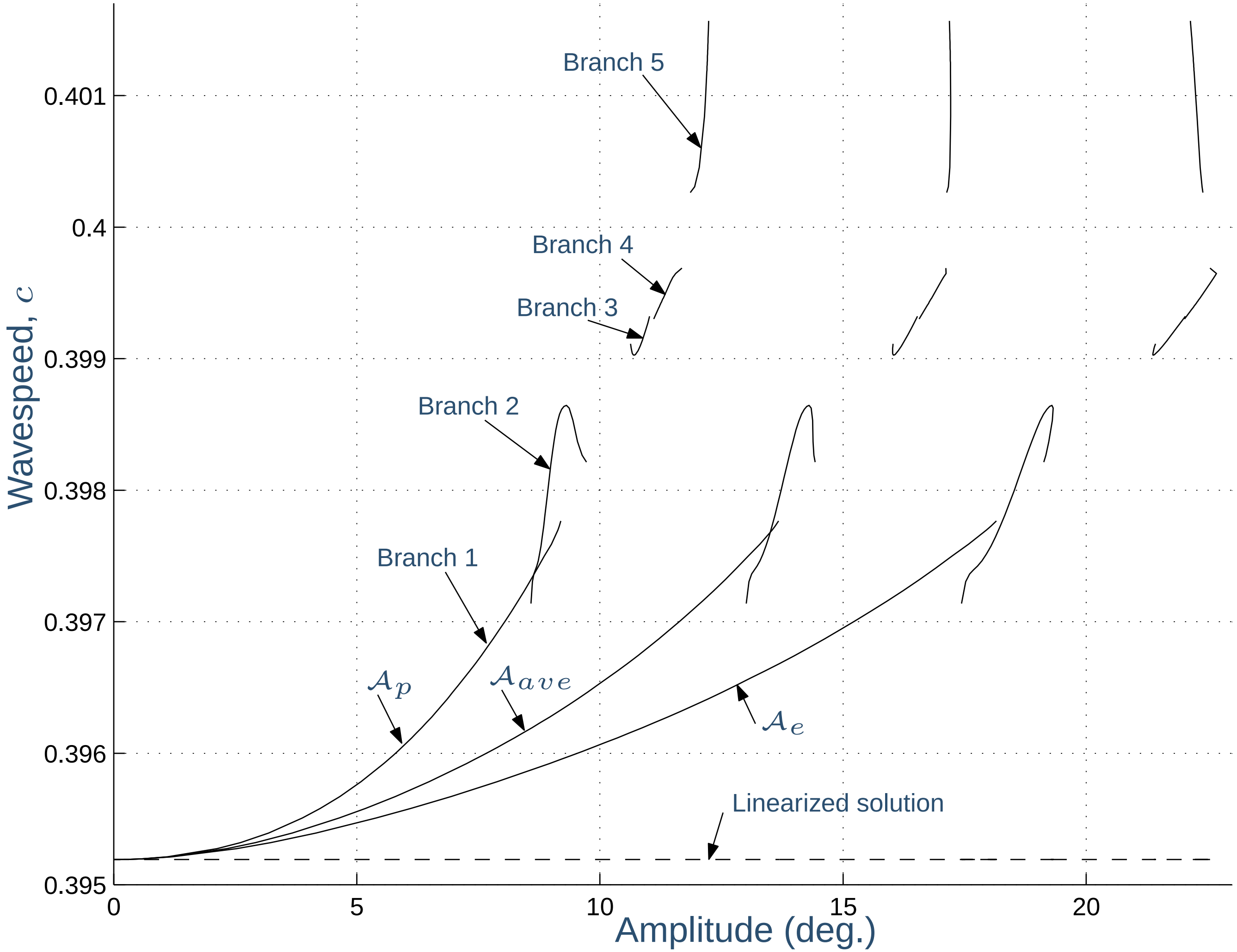


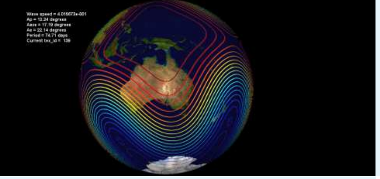
F-S contours, $\kappa = 4$, $\omega = 1.25$



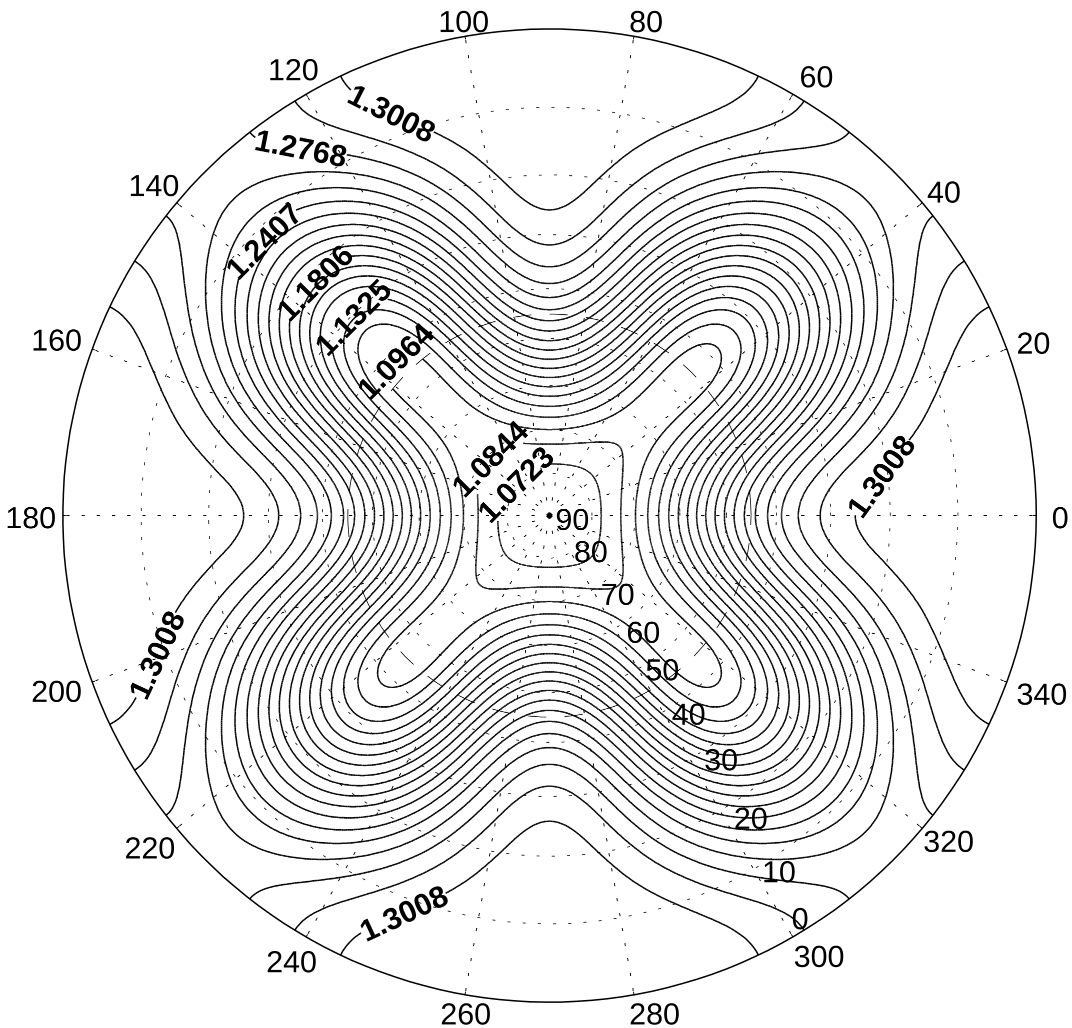


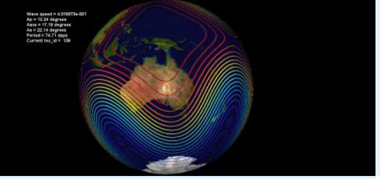
Results for $\kappa = 4$, $\omega = 1.0$



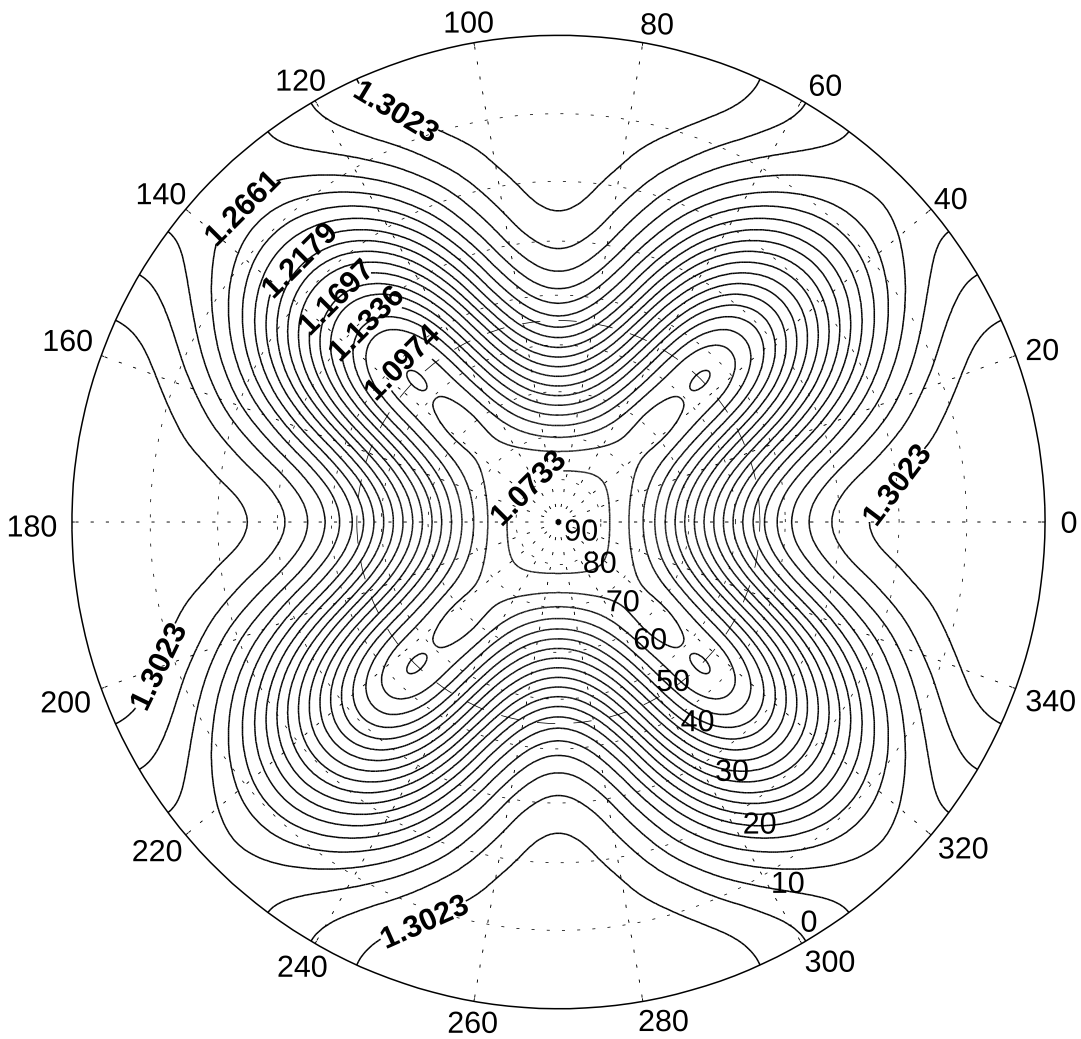


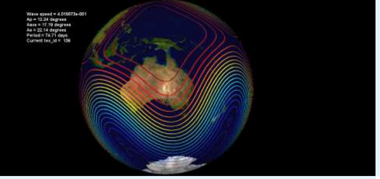
F-S contours (end Branch 4), $\kappa = 4$, $\omega = 1.0$





F-S contours (end Branch 5), $\kappa = 4$, $\omega = 1.0$





Velocity (end Branch 5), $\kappa = 4$, $\omega = 1.0$