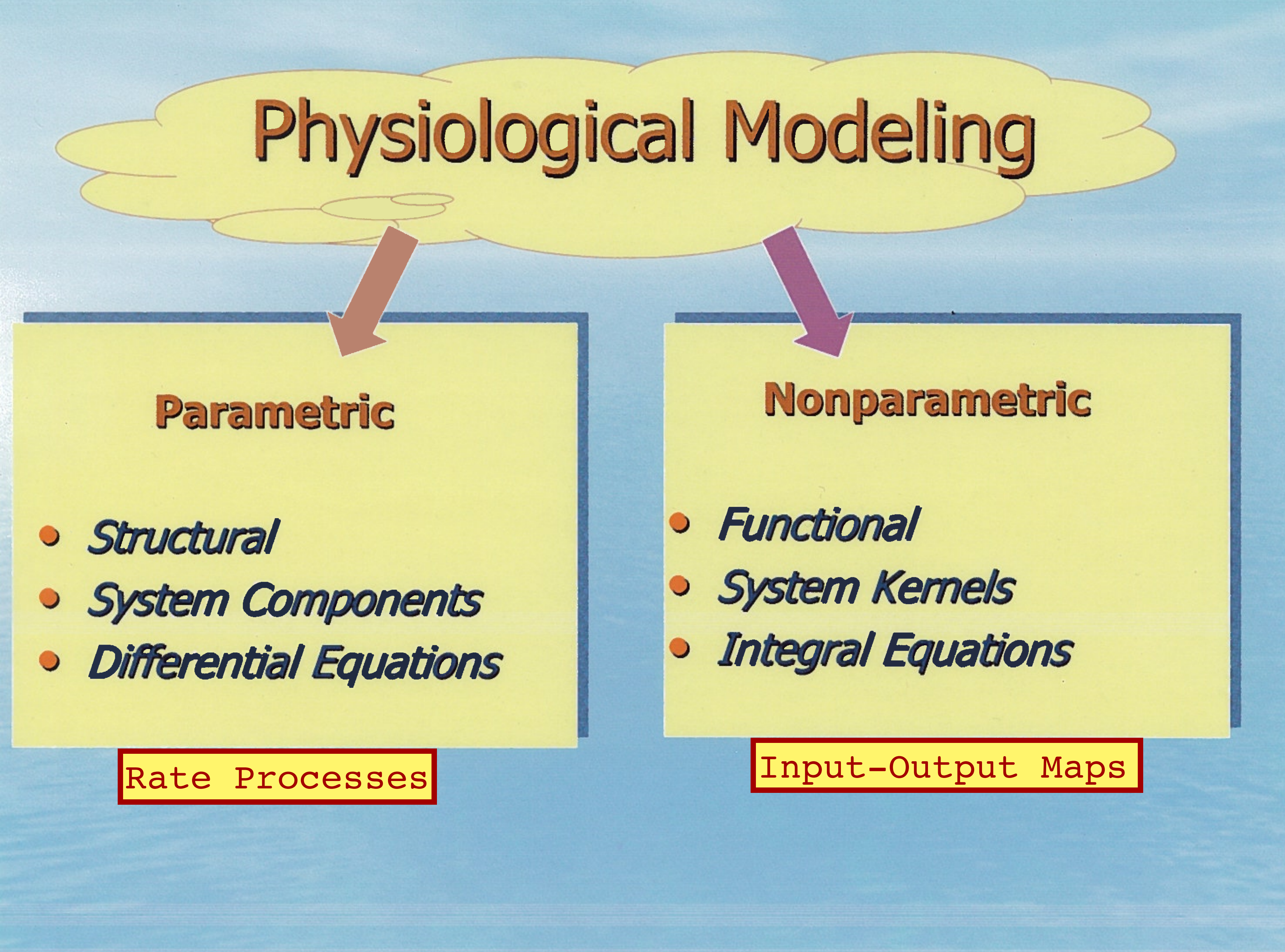
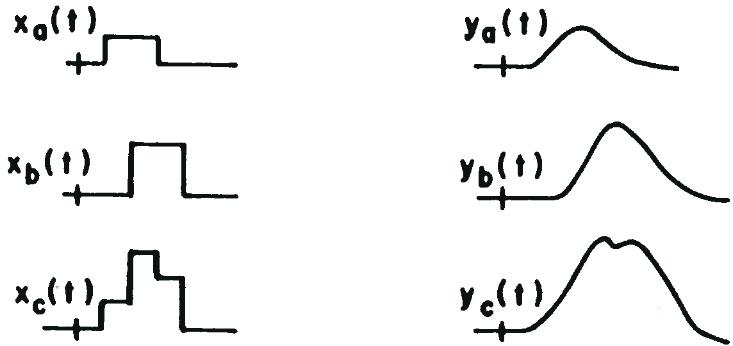
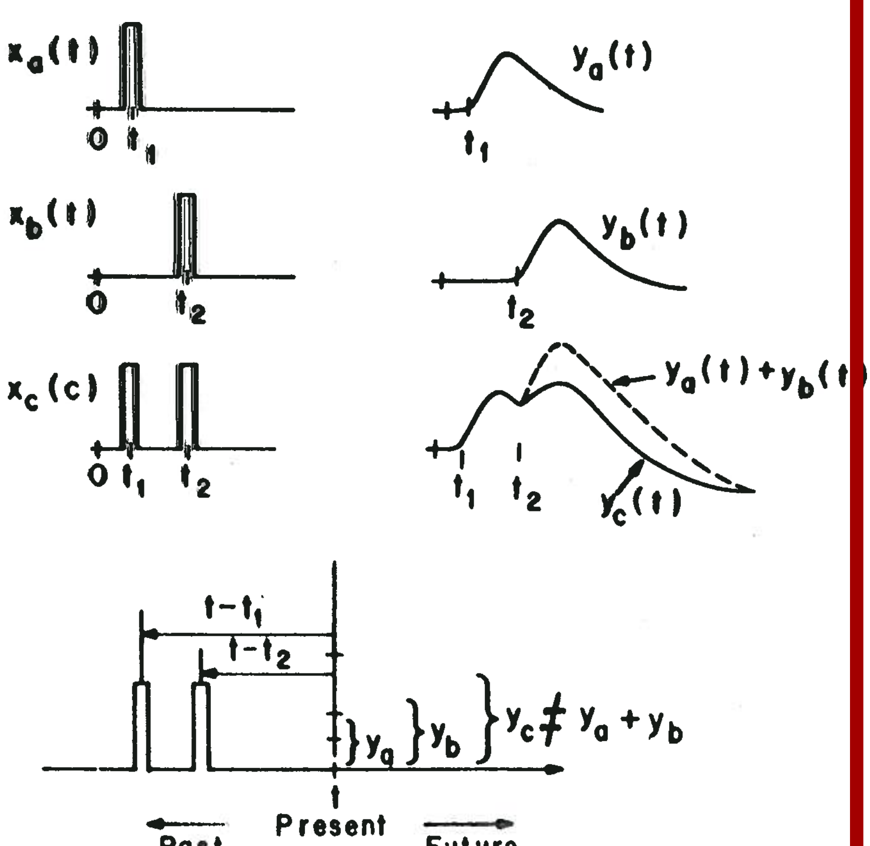

Physiological Modeling
Parametric
Structural
System Components
Differential Equations
Rate Processes
Nonparametric
Functional
System Kernels
Integral Equations
Input-Output Maps

Linear and Nonlinear Systems

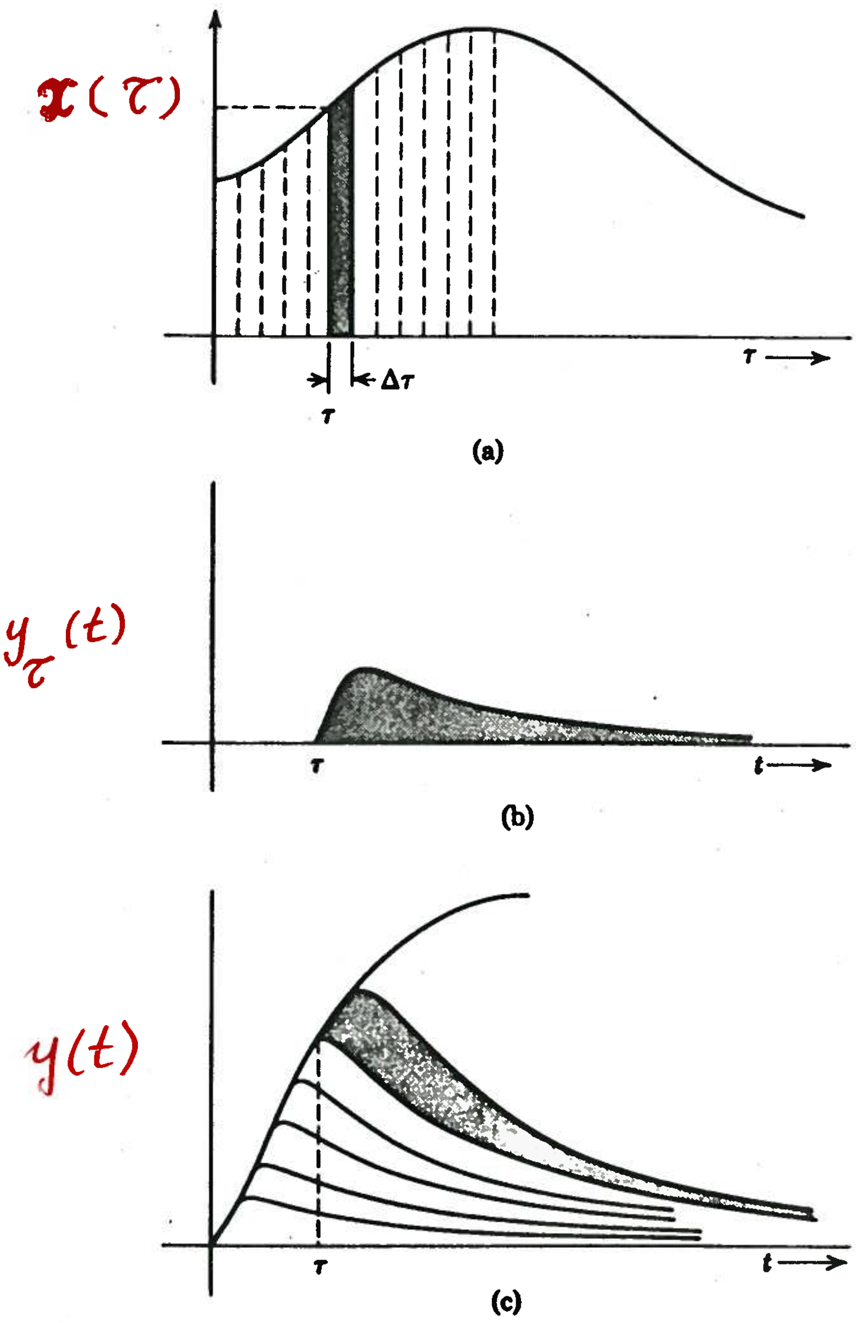
Linear



Non linear



Response of a Linear System



* $x(\tau)$ is represented as a sum of narrow pulses of width $\Delta\tau$ and height $x(\tau)$. As $\Delta\tau \to 0$ each narrow pulse becomes an impulse of strength $x(\tau)\Delta\tau$.

* The response of the system to a unit impulse at τ is $h_L(t-\tau)$. Hence, the response to an impulse of strength $x(\tau)\Delta\tau$ is

$$y_\tau(t) = \lim_{\Delta\tau \to 0} x(\tau)\Delta\tau\, h_L(t-\tau)$$

* The response $y(t)$ is the sum of such responses from $\tau=0$ to $\tau=t$

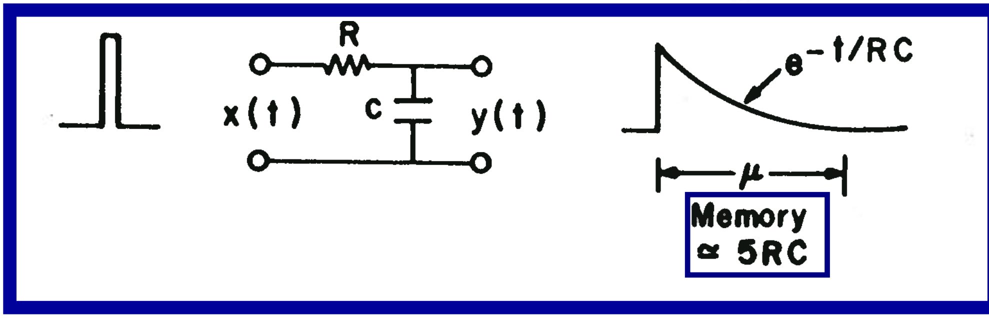
$$y(t) = \lim_{\Delta\tau \to 0} \sum_{\tau=0}^{t} x(\tau)\, h_L(t-\tau)\, \Delta\tau$$

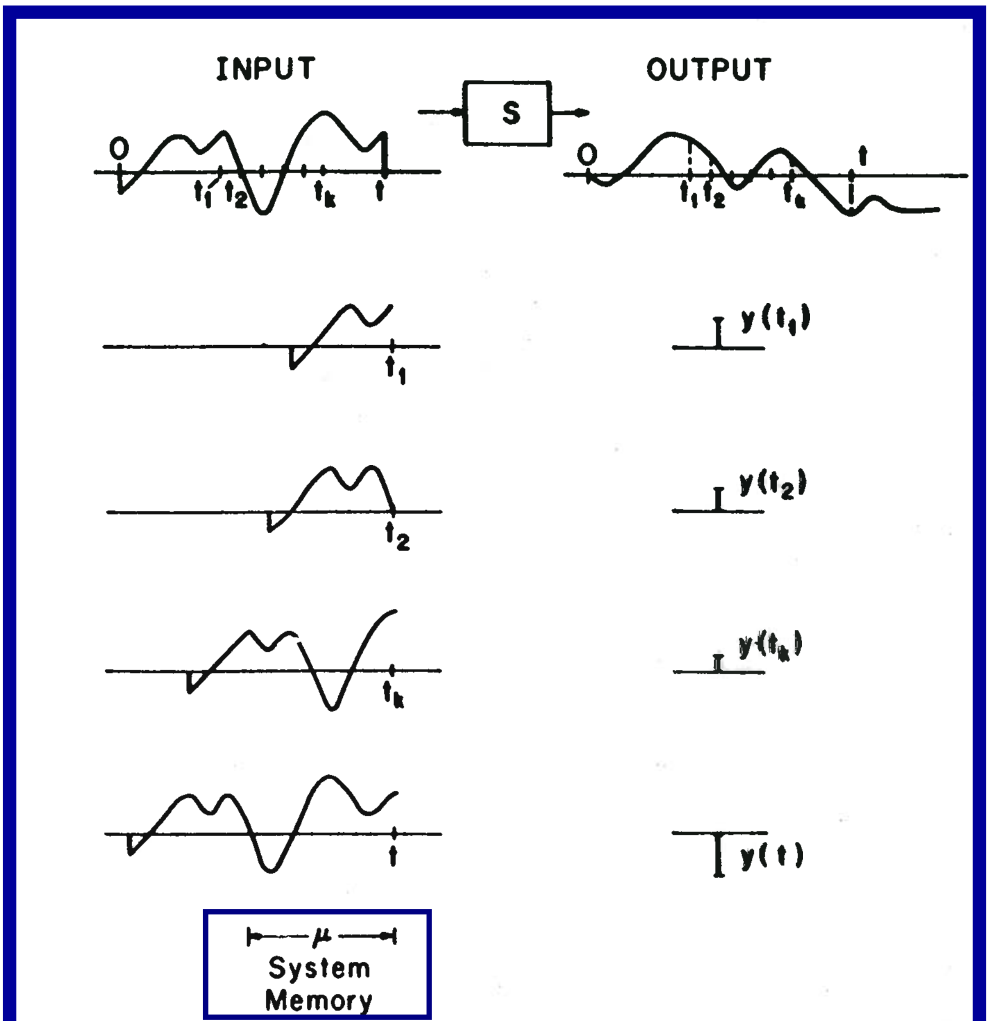
$$= \int_0^t x(\tau)\, h_L(t-\tau)\, d\tau$$

This is called the **Convolution Integral**

Output Dependence on the Iput Past

System Memory





Definition: A functional is a function of a function (i.e. a function whose argument is a function) and whose value is a number. We may write it as

$$\boxed{y = F[x(t)]}$$

where F is the functional, the function $x(t)$ is the argument, and y is the value of the functional.

Example:

The convolution integral

$$\boxed{y(t) = \int_0^t x(\tau)\, h_L(t-\tau)\, d\tau}$$

is a functional representation of a linear system.

Vito Volterra [1860 (Ancona) - 1940 (Roma)]

Vito Volterra was an **Italian mathematician and physicist**. He is known for his contributions to **mathematical biology and integral equations**. He joined the opposition to the Fascist regime (1922). Because of his political philosophy, he also refused to take a mandatory oath of loyalty (1931). He lived largely abroad, returning to Rome just before his death.

Royal Society (1910) - Royal Society of Edinburgh (1913)
A moon crater is named after him

FIRST-ORDER VOLTERRA SYSTEMS

$$y(t) = V_1[x(t)]$$

$$= \int_0^\infty k_1(\tau)\, x(t-\tau)\, d\tau$$

where

$V_1[x(t)] \triangleq$ First-order Volterra functional.

$k_1(t) \triangleq$ First-order Volterra kernel.

Example

Linear time-invariant systems

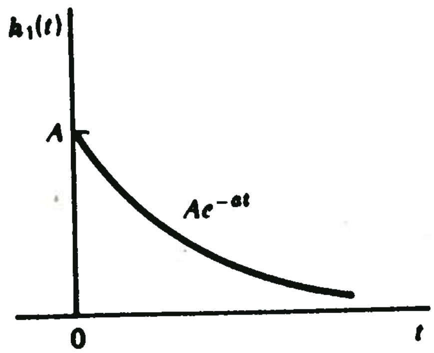
In this case $k_1(t) = h_L(t)$

where $h_L(t)$ is the impulse response of the system.

For example, $h_L(t) = A e^{-at}$ for $t \geq 0$ and $a > 0$.

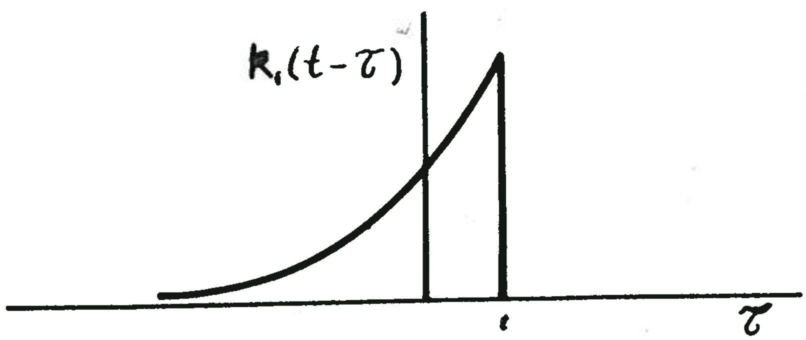
Linear Time-Invariant System

$$\boxed{k_1(t) = h_L(t)}$$



$k_1(t)$

Memory of System



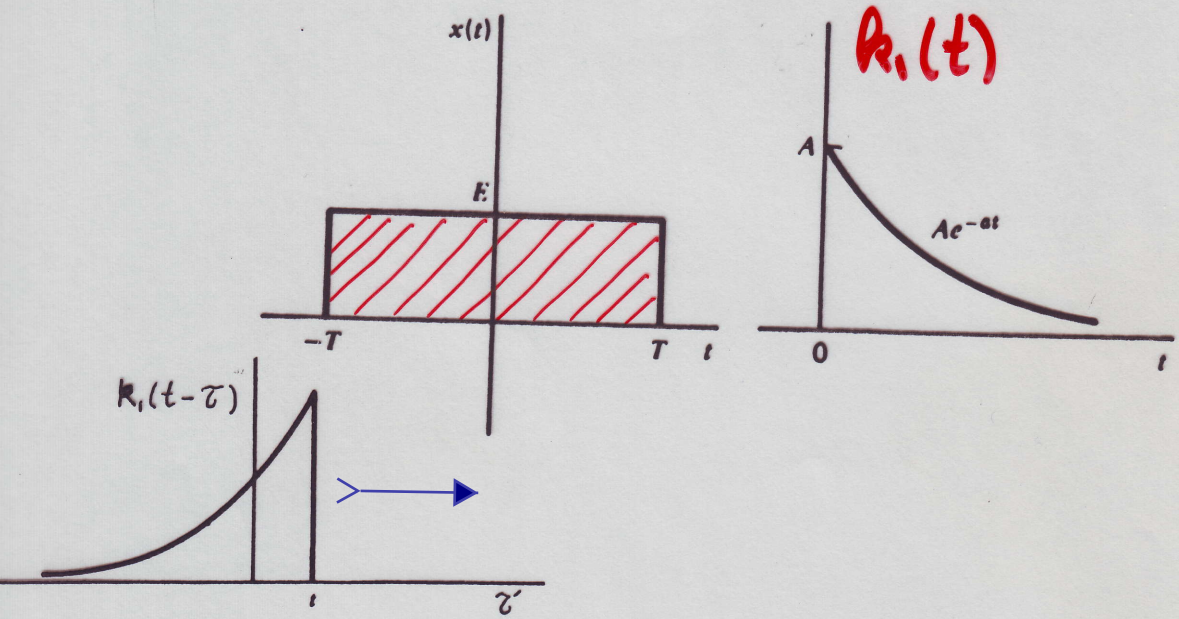
$k_1(t-\tau)$

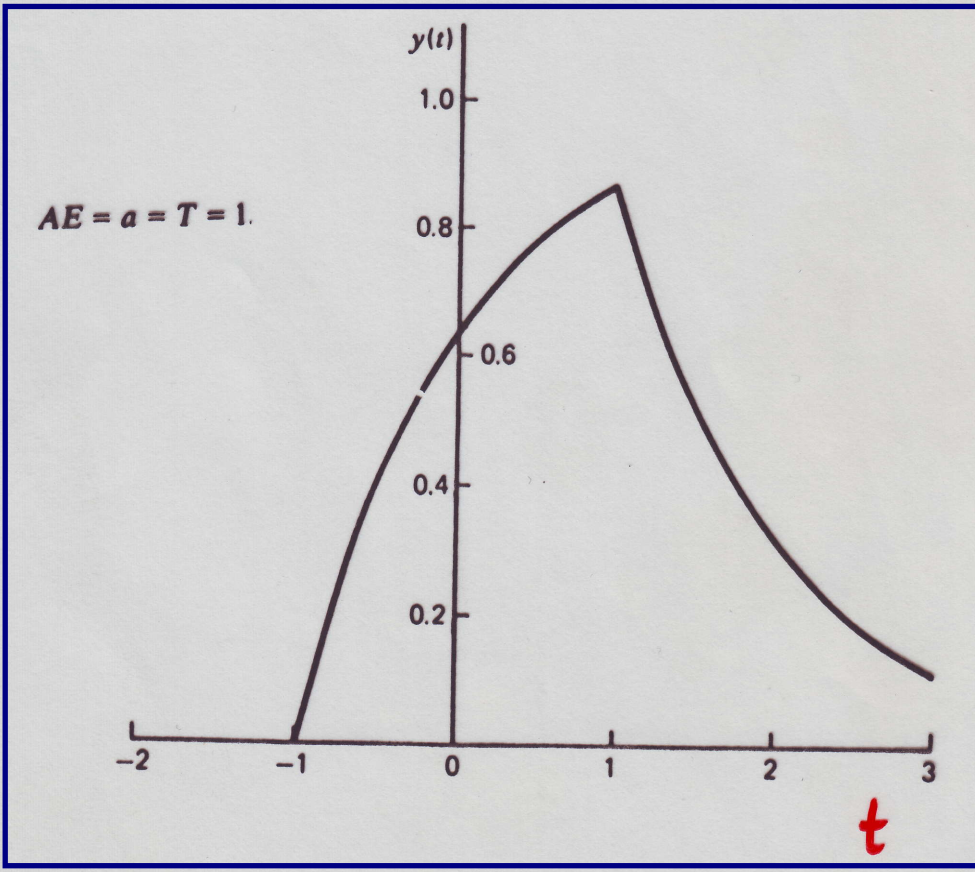
$$y(t) = V_1[x(t)] = \int_0^{\infty} x(\tau)\, k_1(t-\tau)\, d\tau$$

$$= \int_0^{\infty} k_1(\tau)\, x(t-\tau)\, d\tau$$

(The Convolution Integral)

For a rectangular input pulse x(t)





SECOND-ORDER VOLTERRA SYSTEMS

$$y(t) = V_2[x(t)]$$

$$= \int_0^\infty \int_0^\infty k_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2$$

where

$V_2[x(t)] \triangleq$ Second-order Volterra functional.

$k_2(\tau_1, \tau_2) \triangleq$ Second-order Volterra Kernel.

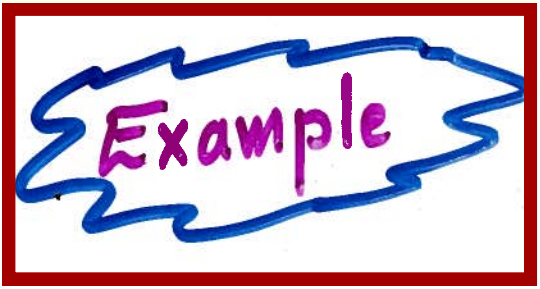
General Second-Order Volterra Systems

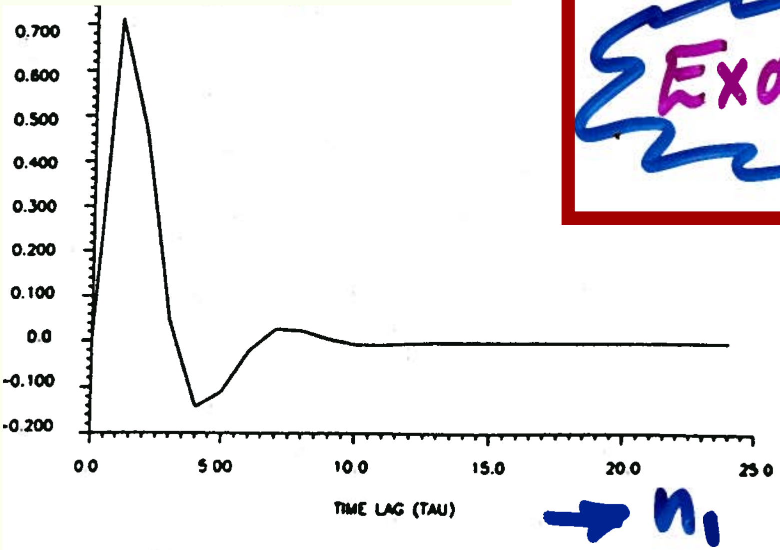
$$y(t) = V_0[x(t)] + V_1[x(t)] + V_2[x(t)]$$

$$= k_0 + \int_0^\infty k_1(\tau)\, x(t-\tau)\, d\tau$$

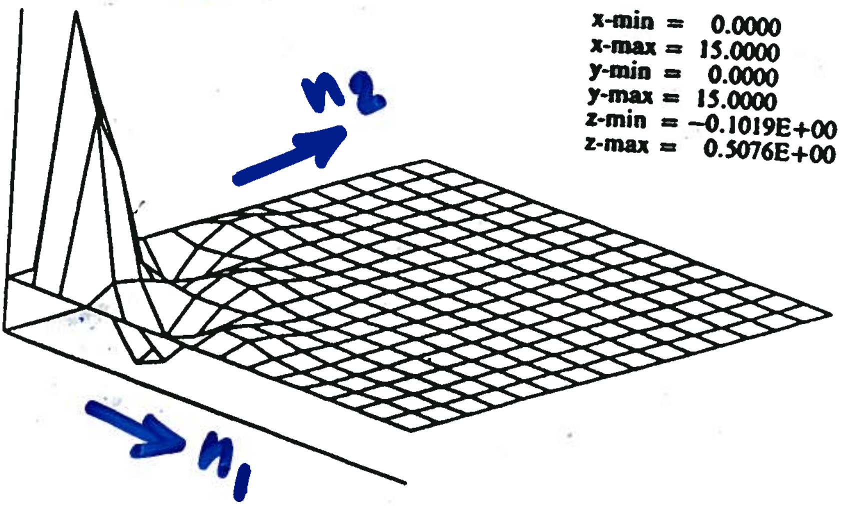
$$+ \int_0^\infty \int_0^\infty k_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2$$

First Order Kernel





Second Order Kernel



Volterra Series

$$x(t) \longrightarrow \boxed{} \longrightarrow y(t)$$

$$\begin{aligned} y(t) = \; & k_0 + \int_0^\infty k_1(\tau)\, x(t-\tau)\, d\tau \\ & + \int_0^\infty\!\!\int_0^\infty k_2(\tau_1,\tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2 \\ & + \int_0^\infty\!\!\int_0^\infty\!\!\int_0^\infty k_3(\tau_1,\tau_2,\tau_3)\, x(t-\tau_1)\, x(t-\tau_2)\, x(t-\tau_3)\, d\tau_1\, d\tau_2\, d\tau_3 \\ & + \cdots \end{aligned}$$

where $k_i(\cdot)$ is the $i\underline{th}$ Volterra Kernel.

The impulse response of a nonlinear system

$$x(t) = \delta(t)$$

$$y(t) = k_0 + k_1(t) + k_2(t,t) + k_3(t,t,t) + \cdots$$

GENERAL VOLTERRA SYSTEMS

$$y(t) = \sum_{n=0}^{\infty} V_n[x(t)]$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} k_n(\tau_1, \ldots, \tau_n)\, x(t-\tau_1) \cdots x(t-\tau_n)\, d\tau_1 \cdots d\tau_n$$

♫ This is a generalization of the convolution integral representation of a linear system.

Limitations of the Volterra Series :

1. Convergence : Same limitations as those encountered in Taylor series.
2. Measurement of Volterra Kernels.

Norbert Wiener : A famous child prodigy, he later became an early researcher in stochastic and noise processes, contributing work relevant to electronic engineering, electronic communication, and control systems.

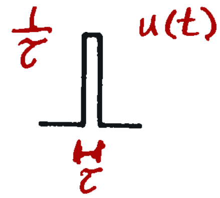
Wiener is considered the originator of ***cybernetics***, a formalization of the notion of feedback, with implications for engineering, systems control, computer science, biology, neuroscience, philosophy, and the organization of society.

Cybernetics: Or Control and Communication in the Animal and the Machine was written by Norbert Wiener and published in 1948. It is the first public usage of the term "cybernetics" to refer to ***self-regulating mechanisms***. The book laid the theoretical foundation for servomechanisms (whether electrical, mechanical or hydraulic), automatic navigation, analog computing, and reliable communications.

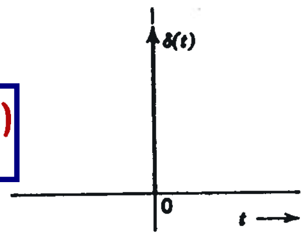
Wiener Filter : For signal processing. It's purpose is to reduce the amount of noise present in a signal by comparison with an estimation of the desired noiseless signal.

INPUT STIMULI

Pulse

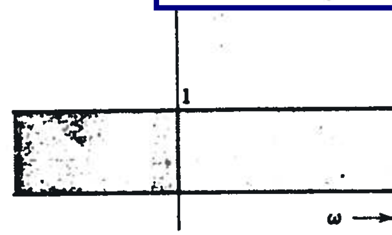


$\boxed{\lim_{\tau \to 0} u(t)}$



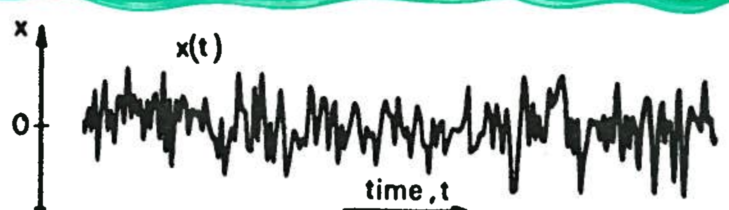
unit impulse

Fourier Transform

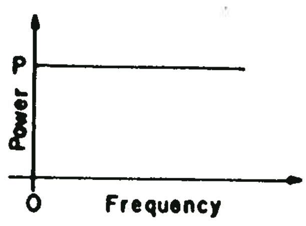


Frequency Spectrum

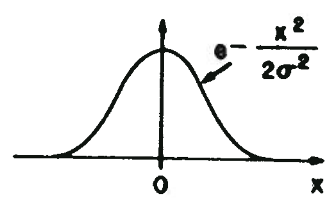
Gaussian White Noise



'zero mean'



Power Spectrum



Amplitude Distribution

- Any two samples are statistically independent.
- Stationary random process.

THE WIENER SERIES

$$y(t) = \sum_{n=0}^{\infty} G_n[h_n; x(t)]$$

The Wiener G-Functionals are orthogonal to each other with respect to a Gaussian White Noise (GWN) input $x(t)$.

Definition: The nth order Wiener G-functional is the sum of n+1 homogeneous Wiener functionals of decreasing order.

$$G_n[h_n; x(t)] = W_n[x(t)] + \sum_{i=0}^{n-1} W_{i(n)}[x(t)]$$

for $n \geq 1$

$$
\begin{aligned}
y(t) = {} & h_0 \\
& + \int_0^\infty h_1(\tau_1)\, x(t-\tau_1)\, d\tau_1 + h_{0(1)} \\
& + \int_0^\infty \int_0^\infty h_2(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2 \\
& + \int_0^\infty h_{1(2)}(\tau_1)\, x(t-\tau_1)\, d\tau_1 + h_{0(2)} \\
& + \int_0^\infty \int_0^\infty \int_0^\infty h_3(\tau_1, \tau_2, \tau_3)\, x(t-\tau_1)\, x(t-\tau_2)\, x(t-\tau_3) \\
& \qquad d\tau_1\, d\tau_2\, d\tau_3 \\
& + \int_0^\infty \int_0^\infty h_{2(3)}(\tau_1, \tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2 \\
& + \int_0^\infty h_{1(3)}(\tau_1)\, x(t-\tau_1)\, d\tau_1 \\
& + h_{0(3)} \\
& + \cdots\cdots
\end{aligned}
$$

$h_i(\cdot) \triangleq$ ith order Wiener Kernel

$h_{j(i)}(\cdot) \triangleq$ derived jth order Wiener Kernel associated with ith order Wiener kernel

Orthogonality Condition on G-functionals

$\Rightarrow \mathcal{E}\{G_i G_j\} = 0$ for GWN

$$h_{0(1)} = 0$$

$$h_{1(2)}(\tau_1) = 0$$

$$\boxed{h_{0(2)} = -P \int_0^\infty h_2(\tau_1, \tau_1)\, d\tau_1}$$

where $P \triangleq$ input power level

$$h_{2(3)}(\tau_1, \tau_2) = 0$$

$$\boxed{h_{1(3)}(\tau_1) = -3P \int_0^\infty h_3(\tau_1, \tau_2, \tau_2)\, d\tau_2}$$

$$h_{0(3)} = 0$$

G-functional	0^{th}-order kernels	1^{st}-order kernels	2^{nd}-order kernels	3^{rd}-order kernels	4^{th}-order kernels	5^{th}-order kernels
G_0	h_0					
G_1		h_1				
G_2	$h_{0(2)}$		h_2			
G_3		$h_{1(3)}$		h_3		
G_4	$h_{0(4)}$		$h_{2(4)}$		h_4	
G_5		$h_{1(5)}$		$h_{3(5)}$		h_5

The Relation Between Wiener & Volterra Kernels

$$y(t) = \sum_{n=0}^{\infty} V_n[x(t)] = \sum_{n=0}^{\infty} G_n[h_n; x(t)]$$

There is a unique analytical relation between the Wiener and Volterra kernels of a system, which represents the transformation of the functional expansion basis.

Example : For a fifth-order system

$$\sum_{n=0}^{5} V_n[x(t)] = \sum_{n=0}^{5} G_n[h_n ; x(t)]$$

Collect and Equate Similar terms

1. Constants

$$k_0 = h_0 + 0 + h_{0(2)} + 0 + h_{0(4)}$$

2. First-order Kernels

$$k_1 = 0 + h_1 + 0 + h_{1(3)} + 0 + h_{1(5)}$$

3. Second-order Kernels

$$k_2 = 0 + 0 + h_2 + 0 + h_{2(4)} + 0$$

4. Third-order Kernels

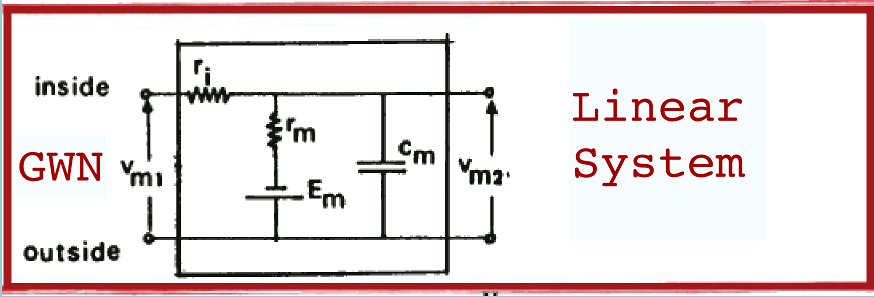
$$k_3 = 0 + 0 + 0 + h_3 + 0 + h_{3(5)}$$

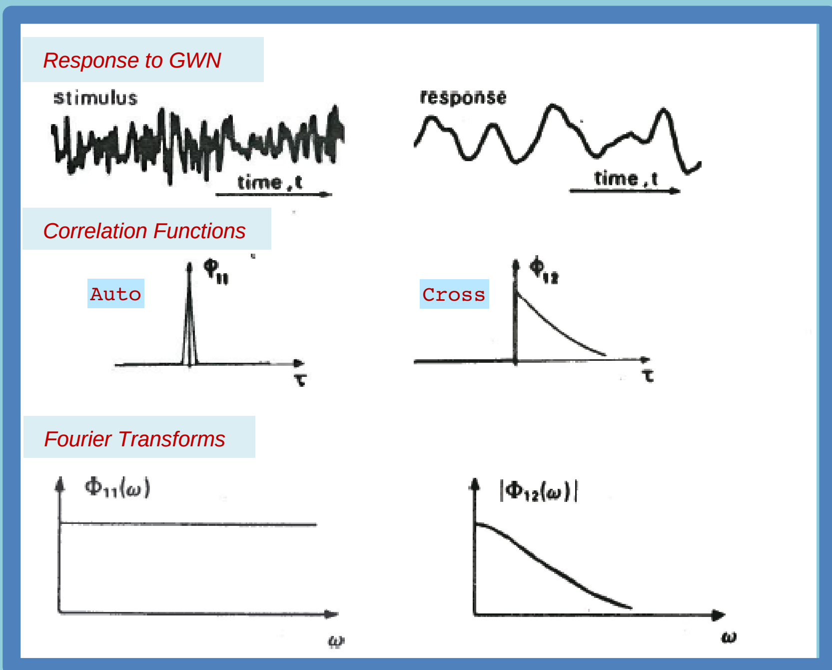
5. Fourth-order Kernels

$$\boxed{k_4 = 0 + 0 + 0 + 0 + h_4 + 0}$$

6. Fifth-order Kernels

$$\boxed{k_5 = 0 + 0 + 0 + 0 + 0 + h_5}$$


inside
outside
GWN
Linear System
r_i
r_m
c_m
E_m
V_{m1}
V_{m2}


Response to GWN
stimulus
time, t
response
time, t
Correlation Functions
Auto
ϕ_{11}
τ
Cross
ϕ_{12}
τ
Fourier Transforms
$\Phi_{11}(\omega)$
ω
$|\Phi_{12}(\omega)|$
ω

Auto correlation Function

$$\Phi_{xx}(\tau) = \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} x(t)\, x(t-\tau)\, dt$$

$$= \mathcal{E}\{x(t)\, x(t-\tau)\}$$

where $\mathcal{E}\{\cdot\}$ denotes Expected value.

Practically ϕ_{xx} is estimated as

$$\hat{\Phi}_{xx}(\tau) = \frac{1}{R-\tau} \int_{0}^{R} x(t)\, x(t-\tau)\, dt$$

* For white noise

$$\Phi_{xx}(\tau) = \mathcal{E}\{x(t)\, x(t-\tau)\}$$

$$= P\, \delta(\tau)$$

Power Level

Cross correlation Function

$$\phi_{yx}(\tau) = \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} y(t)\, x(t-\tau)\, dt$$

$$\hat{\phi}_{yx}(\tau) = \frac{1}{R-\tau} \int_{0}^{R} y(t)\, x(t-\tau)\, dt$$

$$\phi_{yx}(\tau) = \mathcal{E}\{ y(t)\, x(t-\tau) \}$$

$$= \mathcal{E}\{ y(t+\tau)\, x(t) \}$$

[redacted]

$*$ $\phi_{yx}(\tau) = \phi_{xy}(-\tau)$

[redacted]

For a Linear System: $\underset{\text{GWN}}{x(t)} \longrightarrow \boxed{h_L(\cdot)} \longrightarrow y(t)$

$$\Phi_{yx}(\tau) = \mathcal{E}\{x(t)\, y(t+\tau)\}$$

$$= \mathcal{E}\left\{x(t) \int_0^\infty h_L(\sigma)\, x(t+\tau-\sigma)\, d\sigma\right\}$$

$$= \int_0^\infty h_L(\sigma)\, \underbrace{\mathcal{E}\{x(t)\, x(t+\tau-\sigma)\}}\, d\sigma$$

$$\boxed{\begin{aligned} &= \Phi_{xx}(\sigma-\tau) \\ &= P\,\delta(\sigma-\tau) \\ &\text{for Gaussian White Noise} \end{aligned}}$$

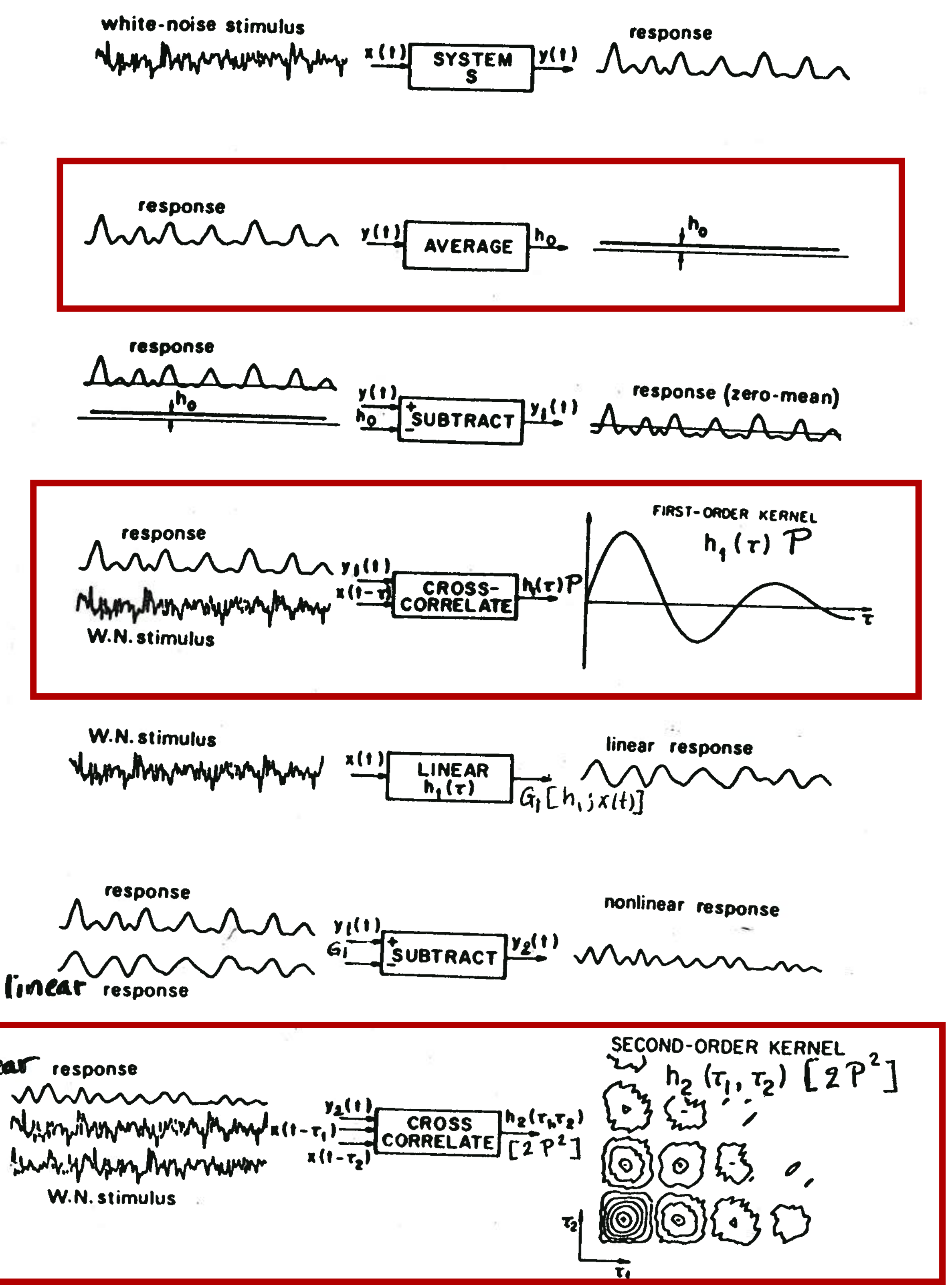
$$\boxed{\Phi_{yx}(\tau)} = P \int_0^\infty h_L(\sigma)\, \delta(\sigma-\tau)\, d\sigma$$

$$\boxed{= P\, h_L(\tau)}$$

The Lee-Schetzen Approach for Estimation of Wiener Kernels Using a GWN Input

Estimation of $\boxed{h_n(\tau_1, \ldots, \tau_n)}$ is through input-output crosscorrelation for $n \geq 1$

The Lee-Schetzen Approach



Limitations: (a) The requirement of a band-limited GWN stimulus that covers the entire system bandwidth.

(b) Long data-records are required to obtain estimates of satisfactory accuracy. Estimate converge to the true values at a rate proportional to $\sqrt{\text{record length}}$.

A Block-Structured Wiener Model

The Wiener-Bose Model

Consider a first-order Wiener System

$$y(t) = G_1[h_1; x(t)]$$

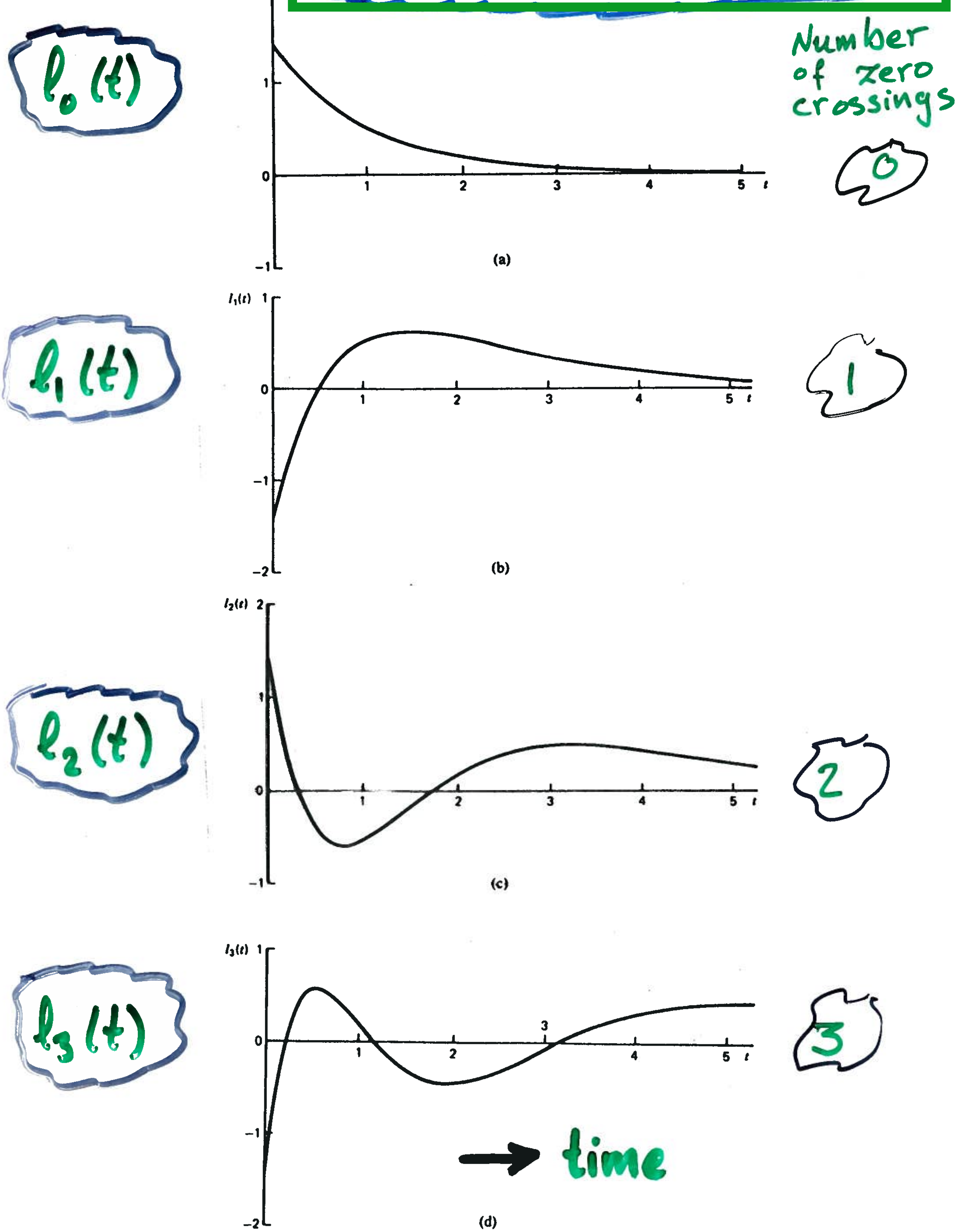
Expand $h_1(\tau)$ in terms of the orthogonal Laguerre functions $\ell_0(\tau), \ell_1(\tau), \ldots, \ell_n(\tau), \ldots$

$$\boxed{h_1(\tau) = \sum_{n=0}^{\infty} c_n \ell_n(\tau)}$$

Wiener Kernel Expansion using Orthogonal Set of Basis Functions

Laguerre Functions

Number of zero crossings



A Block-Structured Wiener Model

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Consider a first-order Wiener System

$$y(t) = G_1[h_1; x(t)]$$

Expand $h_1(\tau)$ in terms of the orthogonal Laguerre functions $\ell_0(\tau), \ell_1(\tau), \ldots, \ell_n(\tau), \ldots$

$$\boxed{h_1(\tau) = \sum_{n=0}^{\infty} c_n \ell_n(\tau)}$$

$$G_1[h_1; x(t)] = G_1\left[\sum_{n=0}^{\infty} c_n \ell_n ; x(t)\right]$$

$$= \sum_{n=0}^{\infty} c_n \, G_1[\ell_n ; x(t)]$$

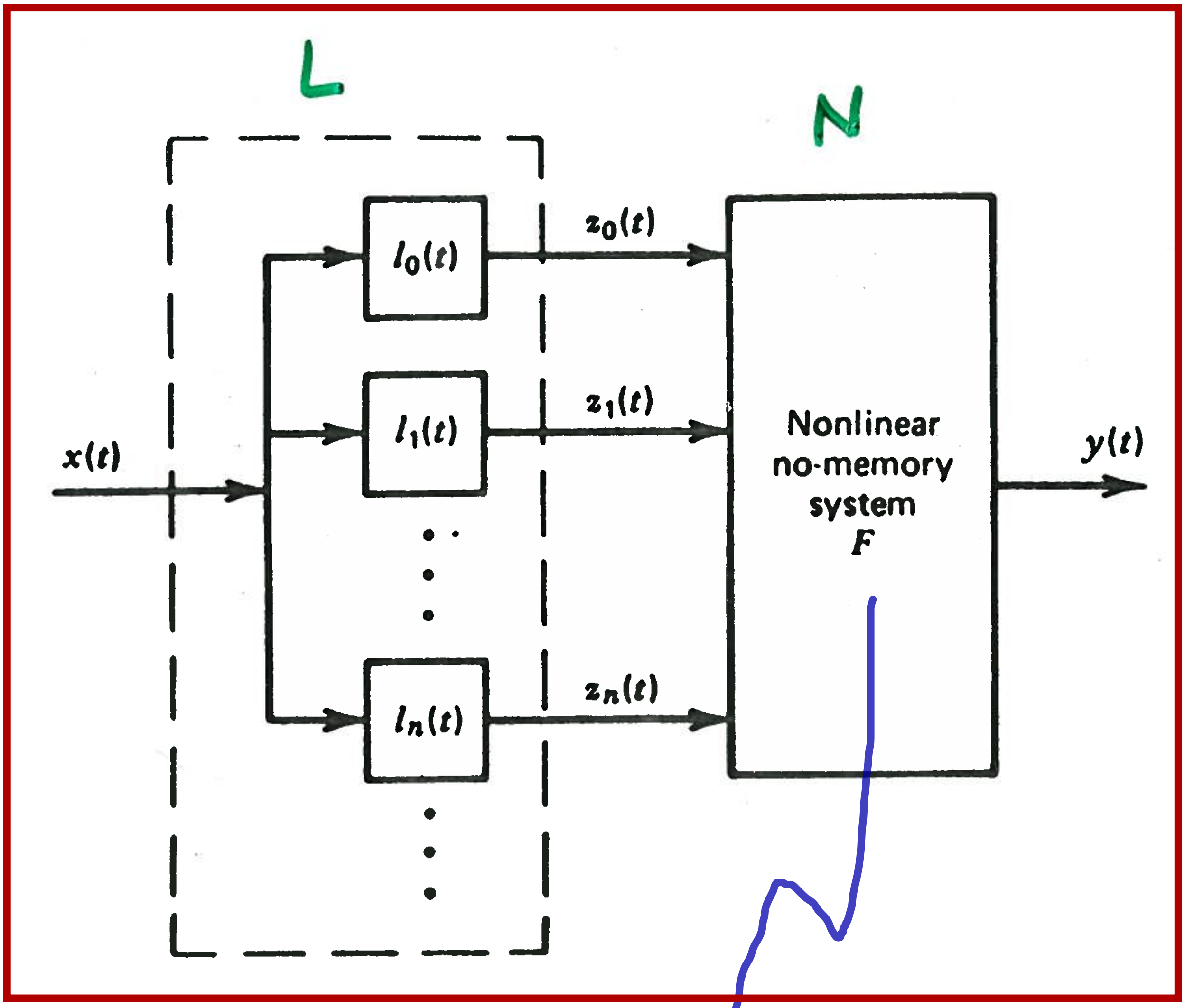
$$= \sum_{n=0}^{\infty} c_n \underbrace{\int_0^{\infty} \ell_n(\tau)\, x(t-\tau)\, d\tau}_{= z_n(t)}$$

$$y(t) = \sum_{n=0}^{\infty} c_n z_n(t)$$

♫ $\boxed{\text{In general} \quad y(t) = F(z_0(t), z_1(t), \ldots, z_n(t), \ldots)}$

and h_n is expanded in terms of $\{\ell_0, \ell_1, \ldots, \ell_n, \ldots\}$

The Wiener - Bose Model



♫ LN Cascade

♫ $y(t) = F(z_0(t), z_1(t), \ldots, z_n(t), \ldots)$

Let $F(z_0(t), z_1(t), \ldots)$ be analytic so that it can be expanded in a multidimensional power series

$$\boxed{\begin{aligned} y(t) = h_0 &+ \sum_{j=0}^{\infty} c_1(j)\, z_j(t) \\ &+ \sum_{j_1}\sum_{j_2} c_2(j_1, j_2)\, z_{j_1}(t)\, z_{j_2}(t) \\ &+ \sum_{j_1}\sum_{j_2}\sum_{j_3} c_3(j_1, j_2, j_3)\, z_{j_1}(t)\, z_{j_2}(t)\, z_{j_3}(t) \\ &+ \cdots \end{aligned}}$$

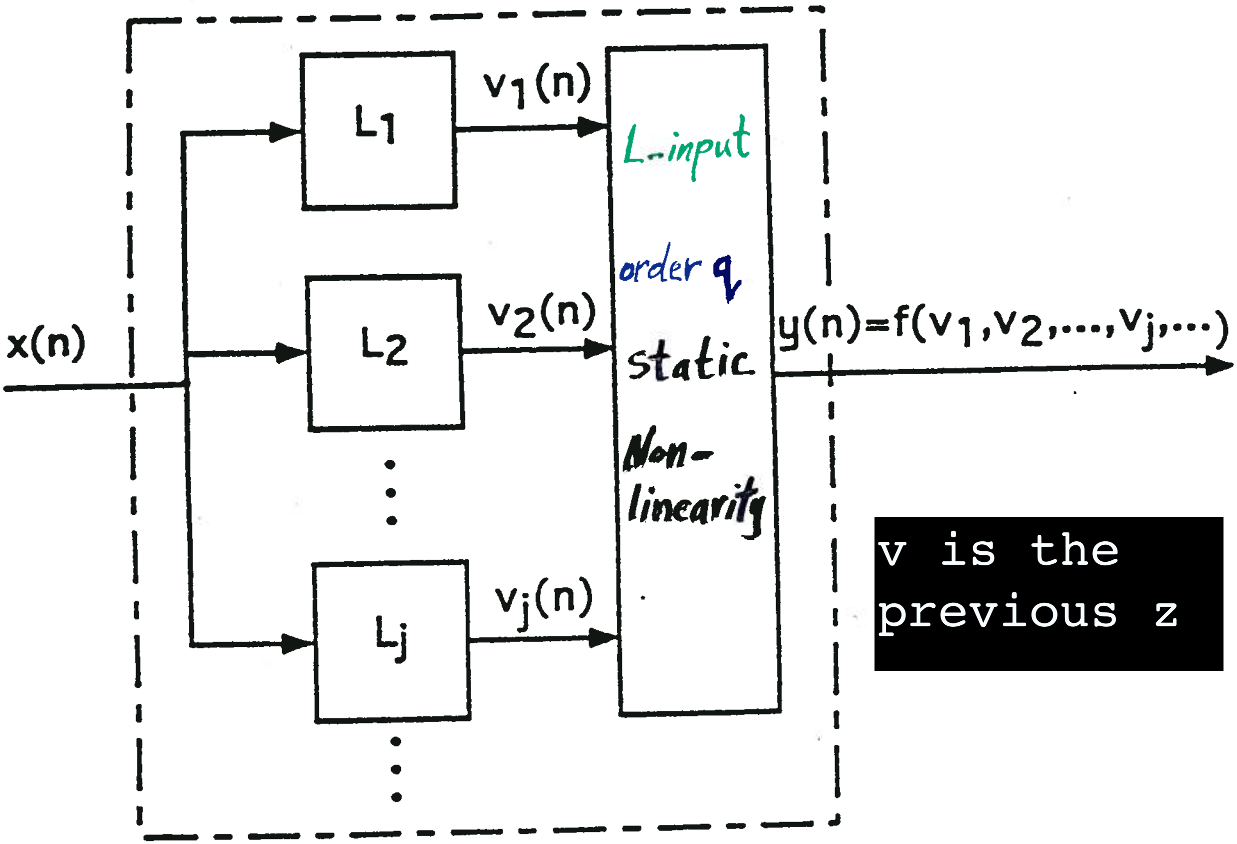
where

$$z_j = \int_0^{\mu} l_j(\tau)\, x(t-\tau)\, d\tau$$

$$\boxed{\mu \triangleq \text{System memory}}$$

The Wiener and Volterra Kernels are described by summations involving coefficients $c_n(\cdot)$ and Laguerre fns $l_n(t)$.

The Discrete-Time Wiener-Bose Model



v is the previous z

$L_j \triangleq$ Linear Filter whose impulse response $b_j(m)$ is the $(j-1)^{th}$ Laguerre function.

$h_m(n_1, \dots, n_m) \triangleq m^{th}$-order Wiener Kernel

$$\boxed{h_m(n_1,\dots,n_m) = \sum_{j_1} \cdots \sum_{j_m} c_m(j_1,\dots,j_m)\, b_{j_1}(n_1) \cdots b_{j_m}(n_m)}$$

The model parameters:

$$q\ ,\ L\ ,\ \{c_r(1,\dots,r);\ 0 \le r \le q\}$$

order of system

number of Laguerre functions

Coefficients of power series of f

$$y(n) = c_0 + \sum_j c_1(j)\, v_j(n) + \sum_{j_1}\sum_{j_2} c_2(j_1,j_2)\, v_{j_1}(n)\, v_{j_2}(n) + \dots$$

$$= f(v_1, v_2, \dots, v_j, \dots)$$

$$v_j(n) = \sum_m b_j(m)\, x(n-m)$$

- Apply $x(n) = \text{GWN}$ and measure $y(n)$
- $c_m(j_1, \ldots, j_m)$ can be estimated by least squares fitting using the known signals $y(n)$ and $\{v_j(n)\}$

How to compute v(n)

$$v_j(n) = \sum_m b_j(m)\, x(n-m)$$

For Laguerre basis functions, then

$$v_j(n) = \sqrt{\alpha}\left[v_j(n-1) + v_{j-1}(n)\right] - v_{j-1}(n-1)$$

$$v_1(n) = \sqrt{\alpha}\; v_1(n-1) + \sqrt{1-\alpha}\; x(n)$$

choice of α is based on the system memory and L.

♫ The discrete-time Laguerre function is

$$l_j(m) = \alpha^{\frac{m-j}{2}} (1-\alpha)^{\frac{1}{2}} \sum_{k=0}^{j} (-1)^k \binom{m}{k} \binom{j}{k} \cdot \left[\alpha^{(j-k)} (1-\alpha)^k \right]$$

for $m \geq 0$, $j = 0, 1, \ldots, L-1$,

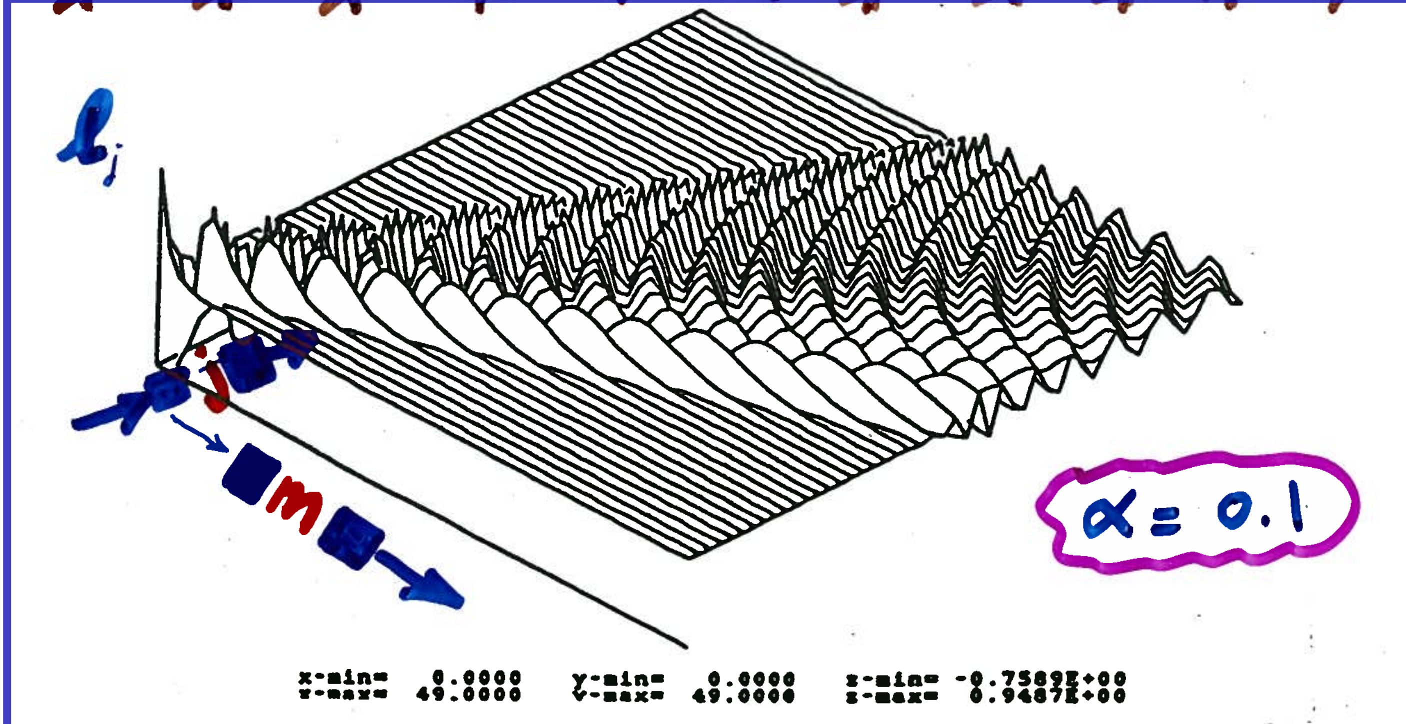
$0 < \alpha < 1$

discrete-time Laguerre parameter

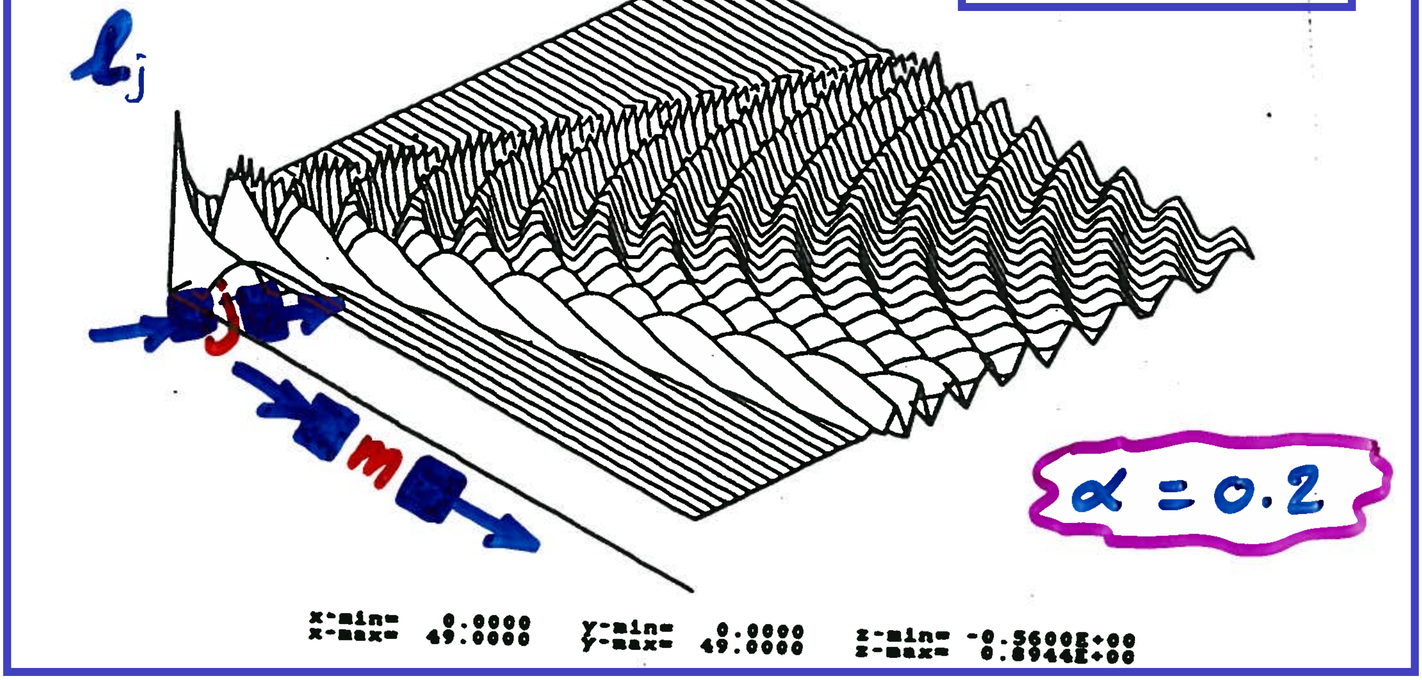
♫ $\binom{j}{k} = \dfrac{j!}{(j-k)!\, k!}$ for $k \leq j$

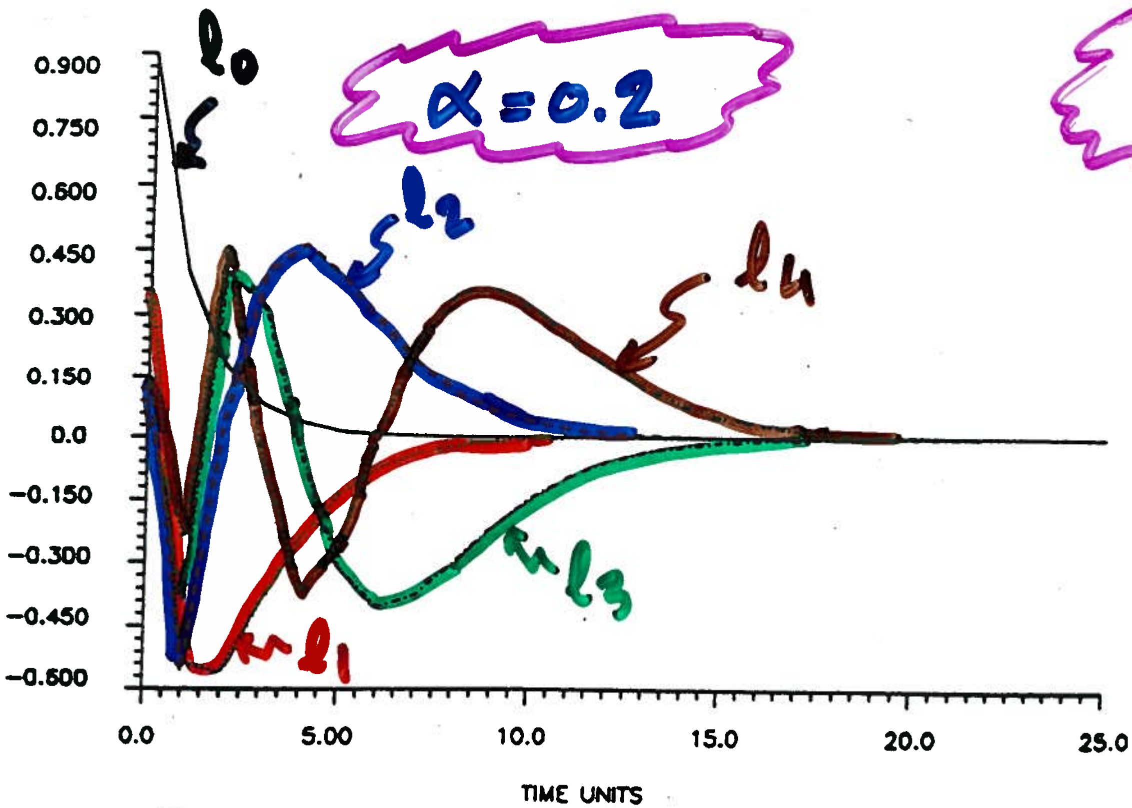
$$\boxed{b_j(m) = l_{j-1}(m) \quad ; \; j = 1, \ldots L}$$

The first 50 Laguerre Functions, plotted as a square matrix over 50 time units (0-49 lag)

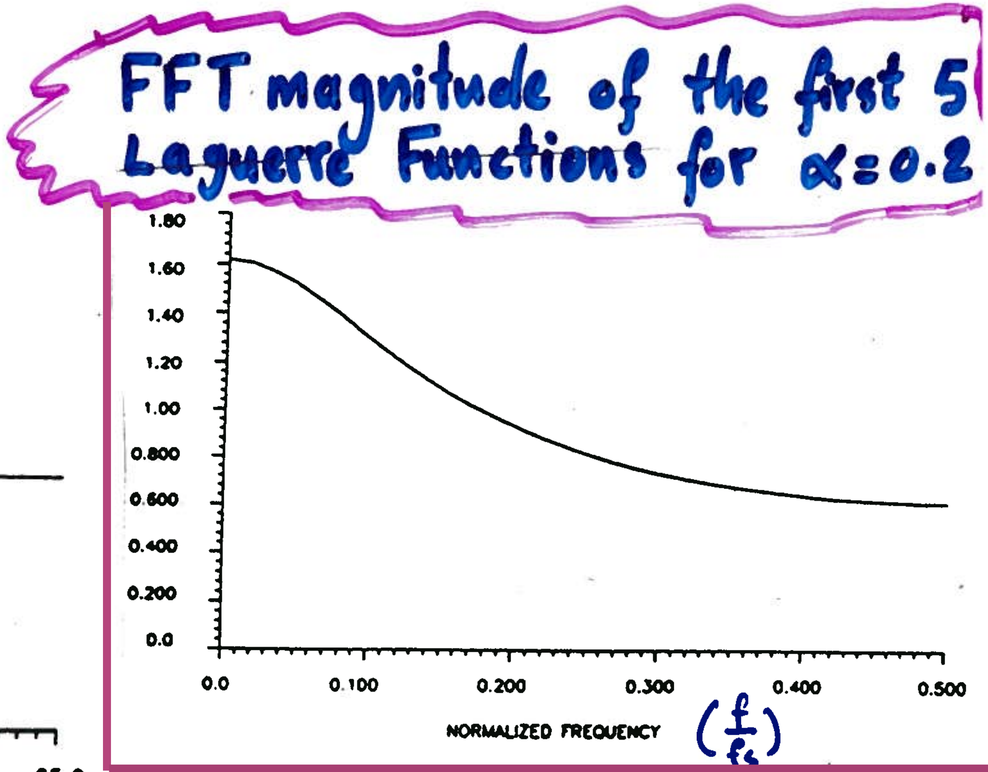


j ≡ order
m ≡ time

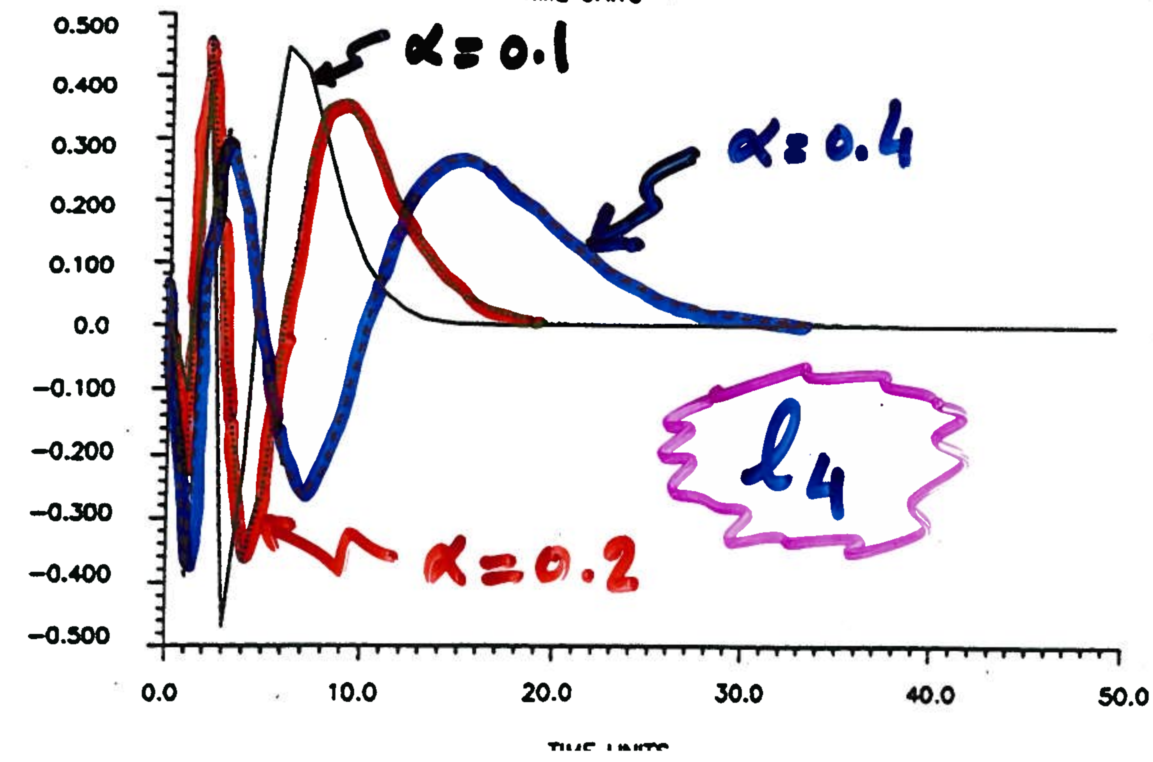




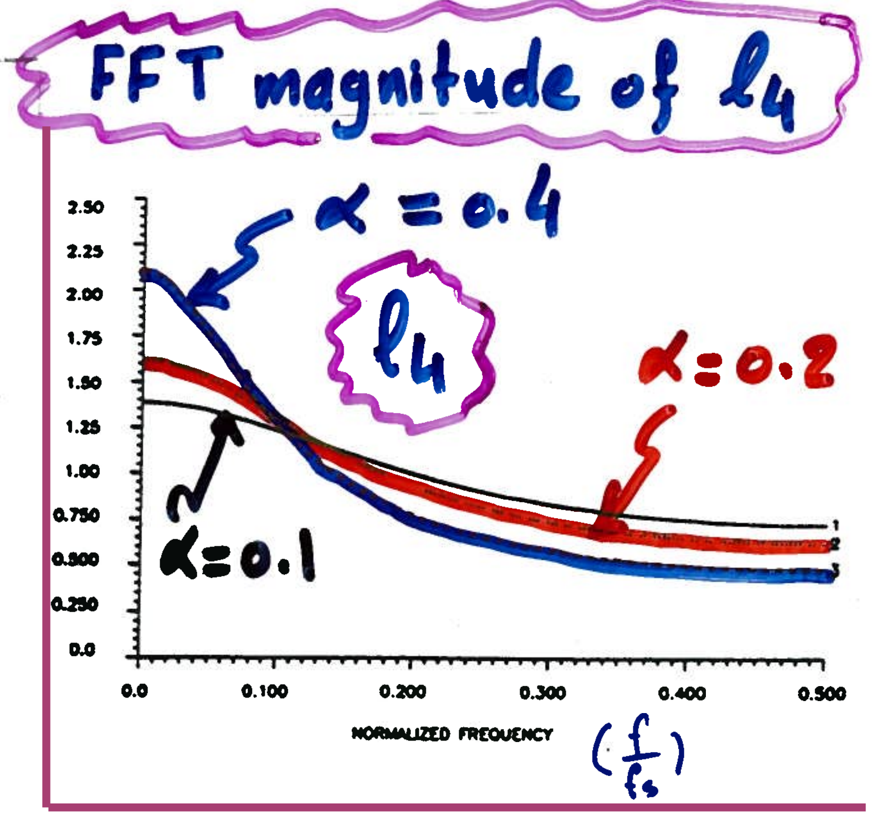
α=0.2
l0
l1
l2
l3
l4
0.900
0.750
0.600
0.450
0.300
0.150
0.0
−0.150
−0.300
−0.450
−0.600
0.0
5.00
10.0
15.0
20.0
25.0
TIME UNITS



FFT magnitude of the first 5 Laguerre Functions for α=0.2
1.80
1.60
1.40
1.20
1.00
0.800
0.600
0.400
0.200
0.0
0.0
0.100
0.200
0.300
0.400
0.500
NORMALIZED FREQUENCY
(f/fs)



α=0.1
α=0.4
α=0.2
l4
0.500
0.400
0.300
0.200
0.100
0.0
−0.100
−0.200
−0.300
−0.400
−0.500
0.0
10.0
20.0
30.0
40.0
50.0



FFT magnitude of l4
α=0.4
l4
α=0.2
α=0.1
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.750
0.500
0.250
0.0
0.0
0.100
0.200
0.300
0.400
0.500
NORMALIZED FREQUENCY
(f/fs)

Choice of L

- The number of Laguerre functions L must be much smaller than the memory-bandwidth product M of the system.

$$\boxed{L \ll M}$$

- e.g., $M = 100 \Rightarrow L = 10$

Practical Hints for LET

1. Inject $x(t) = \text{GWN}$ and estimate response $y(t)$.

2. $\text{FFT}\{y(t)\} \Rightarrow$ System Bandwidth

3. Estimate first order kernel $k_1(\tau)$ by:
   a) Cross correlation of $x(t)$ & $y(t)$.

   or b) $\mathcal{F}^{-1}\left\{\dfrac{\mathcal{F}[y(t)]}{\mathcal{F}[x(t)]}\right\}$ $\boxed{\mathcal{F} \triangleq \text{FFT}}$

4. Duration of $k_1(\tau) \Rightarrow$ System Memory

5. Estimate number of Laguerre Functions
   $L \ll$ Memory-Bandwidth product

6. Estimate α to cover system memory.

LYSIS

LYSIS (the Greek word for "solution") is an interactive software of a set of modular programs (each performing a specific task) that provide an integrated computing environment for data analysis and system modeling. Unique capabilities of LYSIS include input-output nonlinear system modeling and the novel methodology of "Principal Dynamic Modes" (PDMs). This package has evolved over time to incorporate emerging methodologies developed by the Biomedical Simulations Resource (BMSR) at USC, under the supervision of Prof. V.Z. Marmarelis.

LYSIS 7.2 – Matlab version is the current version and is focused on the unique capabilities of input-output nonlinear system modeling using the Volterra-Wiener approach and its efficient variants that have been developed and tested by the BMSR over the years, utilizing Laguerre expansions of the kernels and the novel concept/tool of Principal Dynamic Modes (PDMs). The source code of modular Matlab programs are made available that can be incorporated in the Matlab computational environment of the user.

https://bmsr.usc.edu/software/lysis/

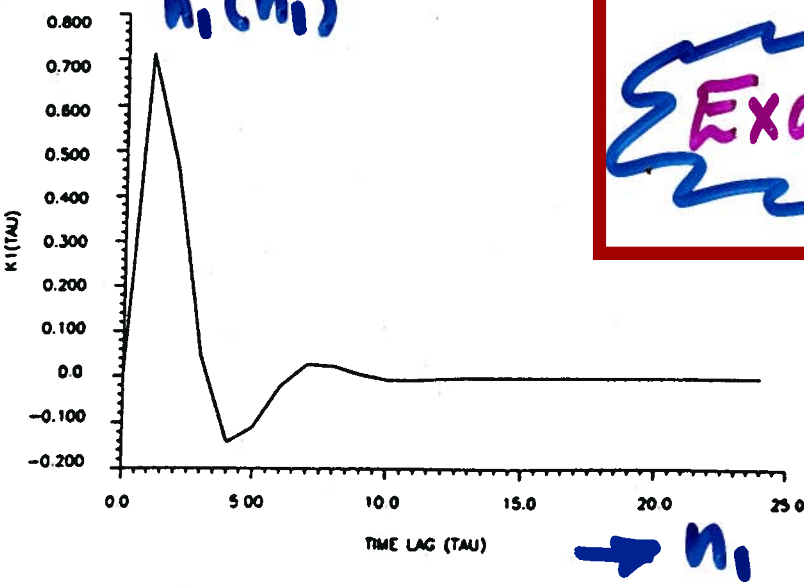
The development of LYSIS is supported by the Biomedical Simulations Resource (BMSR) in the Department of Biomedical Engineering at the University of Southern California, under support from the National Institute for Biomedical Imaging and Bioengineering (P41-EB001978) and the National Center for Research Resources (P41-RR01861) of the National Institutes of Health. LYSIS is made available to the biomedical community at large free of charge in order to promote research in this area and enhance the computational capabilities of biomedical investigators nationwide.

Exact Kernels

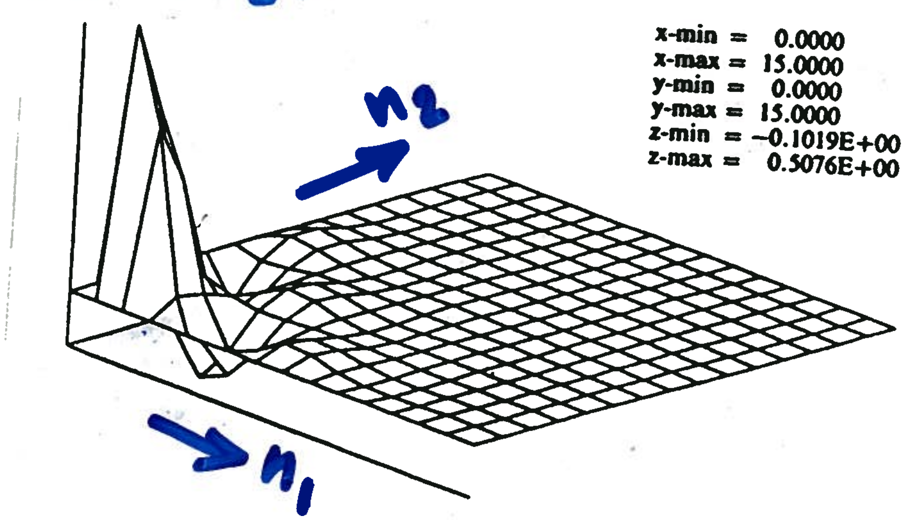
$q = 2$

Example

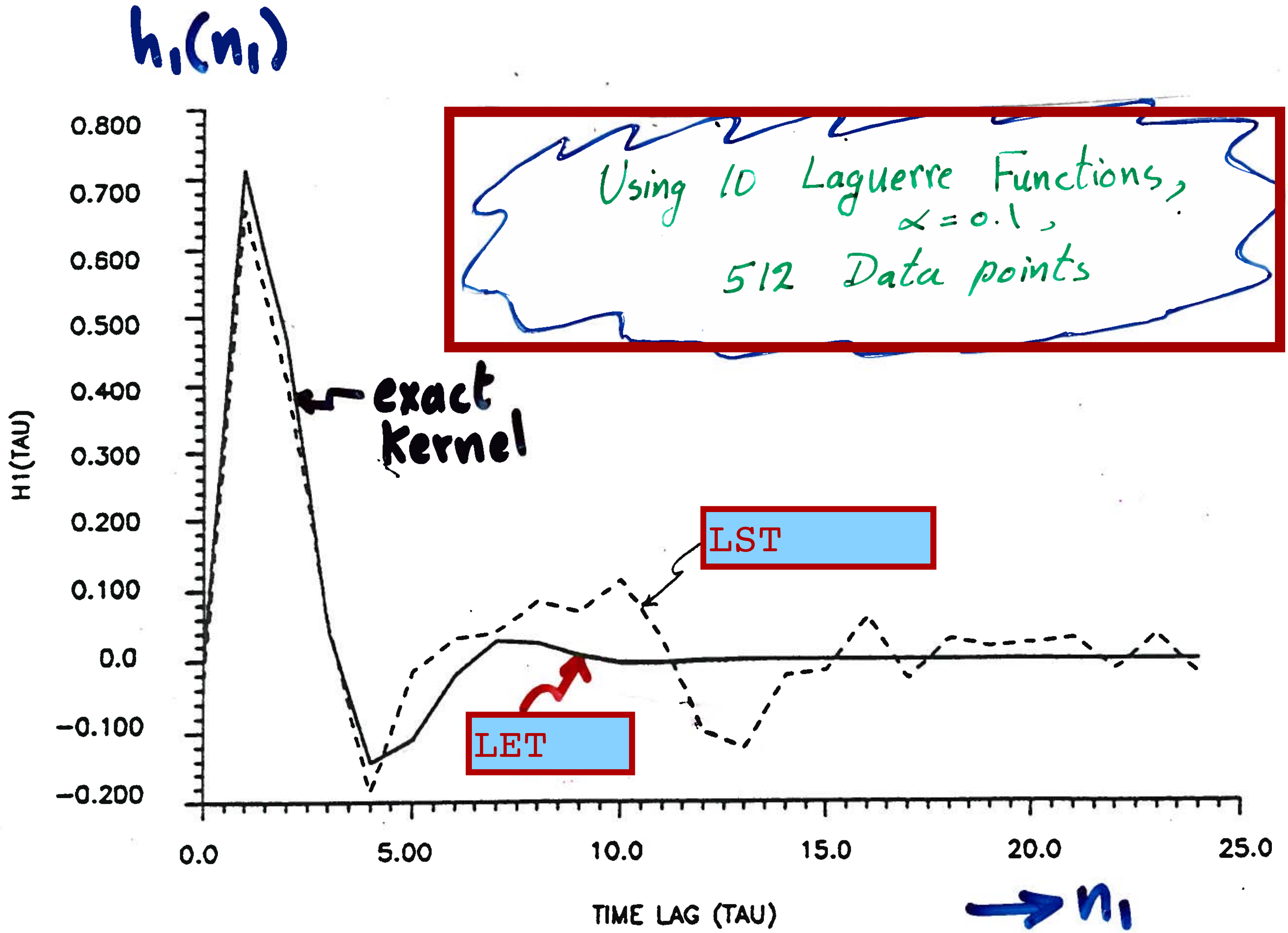
$h_1(n_1)$



$h_2(n_1, n_2)$



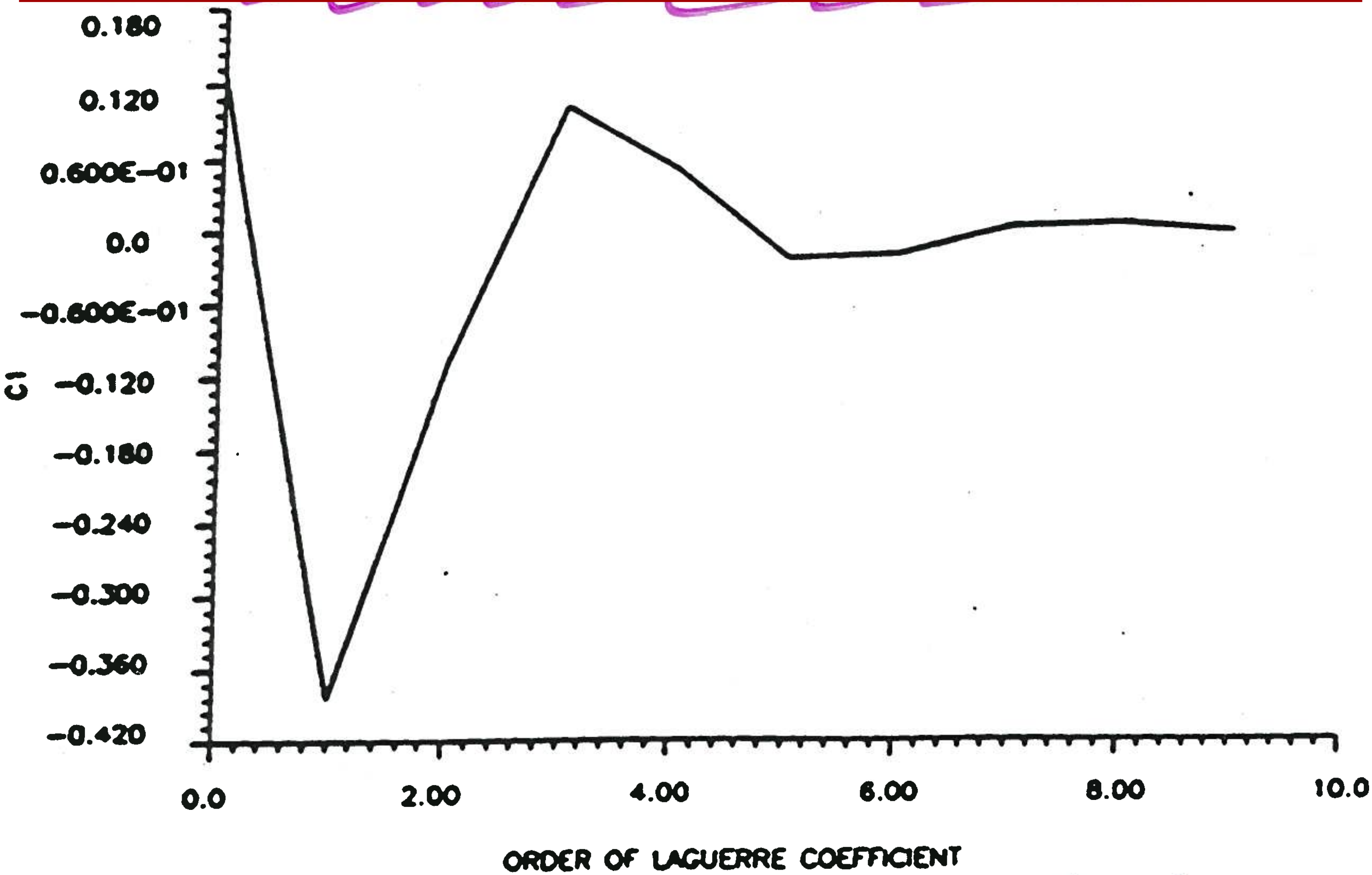
The model output is produced by convolution of GWN with the system kernels



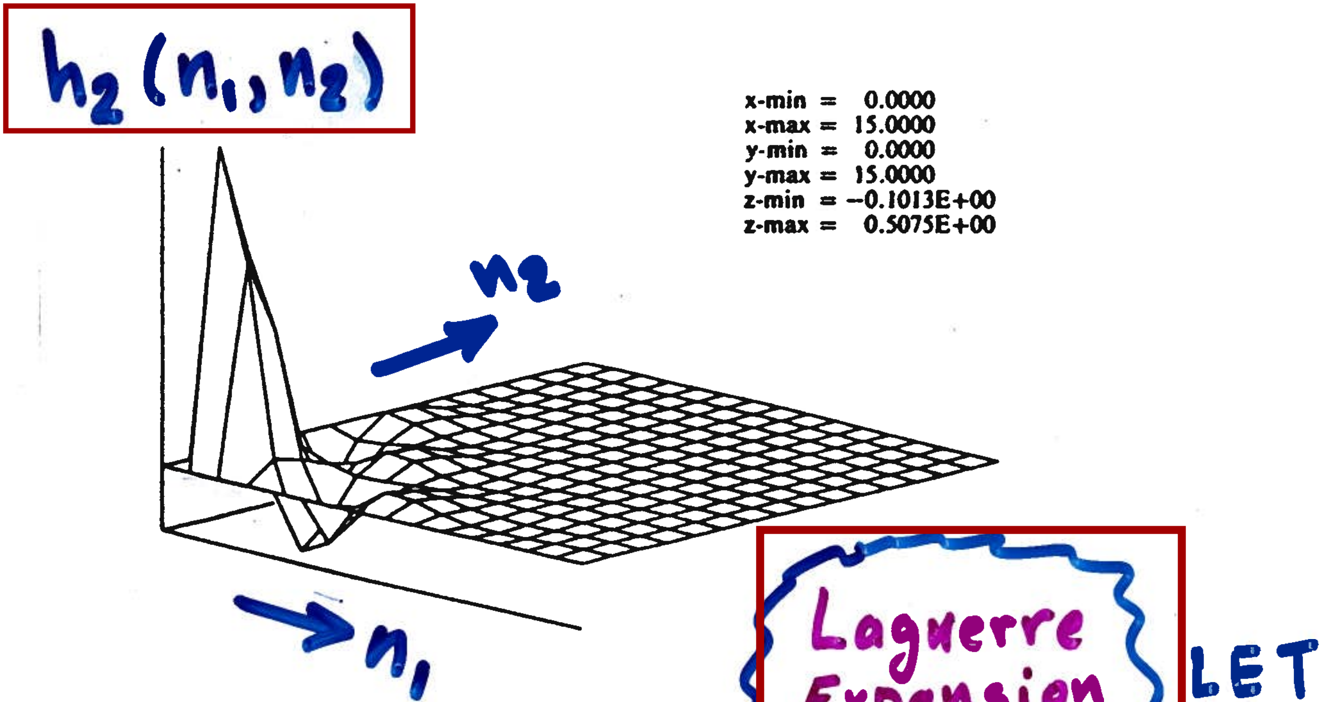
LST $\triangleq$ Lee-Schetzen Technique

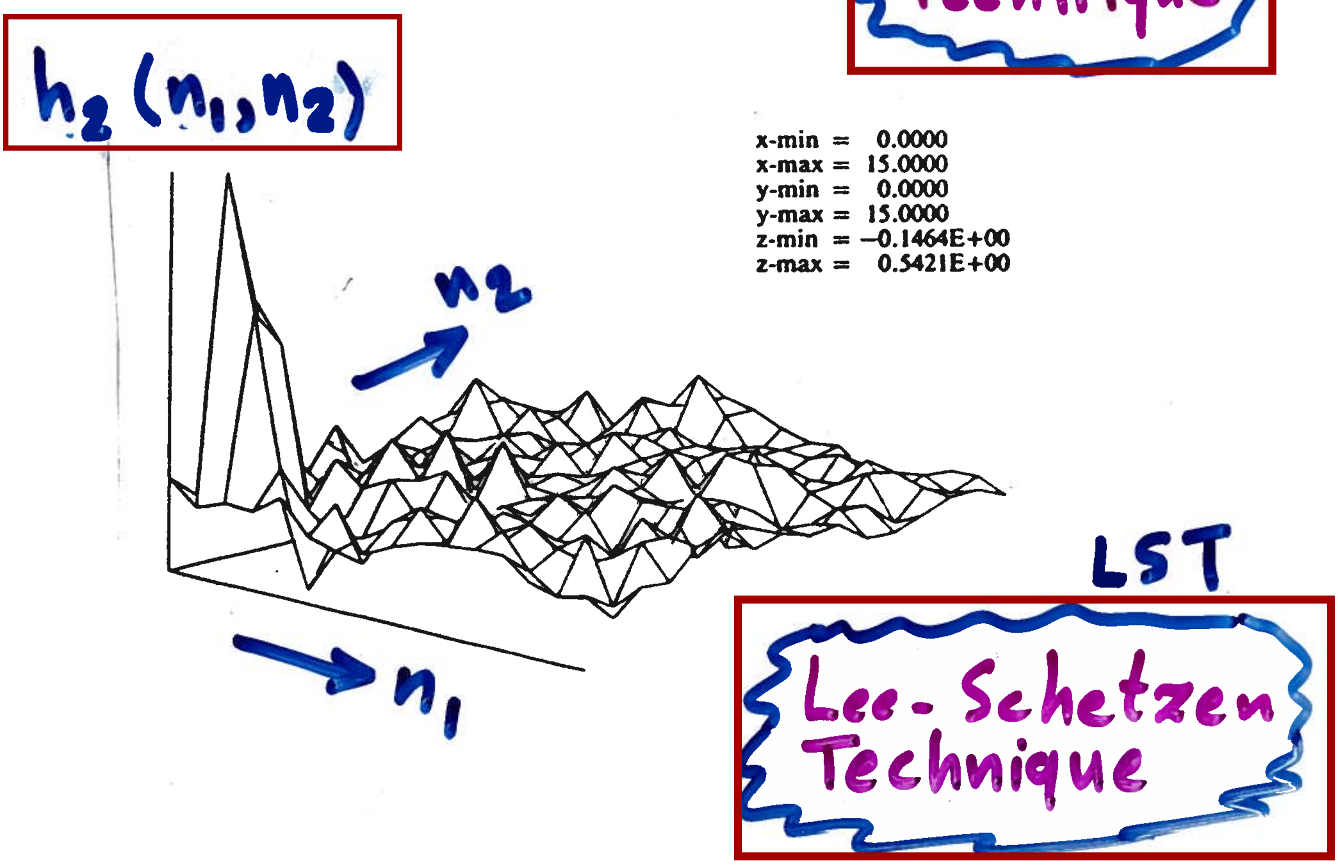
LET $\triangleq$ Laguerre Expansion Technique

Estimates of the 10 Laguerre Expansion Coefficients for the first-order Kernel



The required L can be determined by estimating the first-order kernel for a large L, then inspect coeff^t estimates to select minimum number for signifiant values.


h2 (n1, n2)
x-min = 0.0000
x-max = 15.0000
y-min = 0.0000
y-max = 15.0000
z-min = −0.1013E+00
z-max = 0.5075E+00
n2
n1
Laguerre Expansion Technique
LET


h2 (n1, n2)
x-min = 0.0000
x-max = 15.0000
y-min = 0.0000
y-max = 15.0000
z-min = −0.1464E+00
z-max = 0.5421E+00
n2
n1
LST
Lee-Schetzen Technique