Functions that Preserve Manhattan Distance

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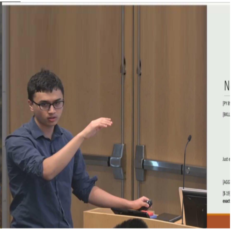
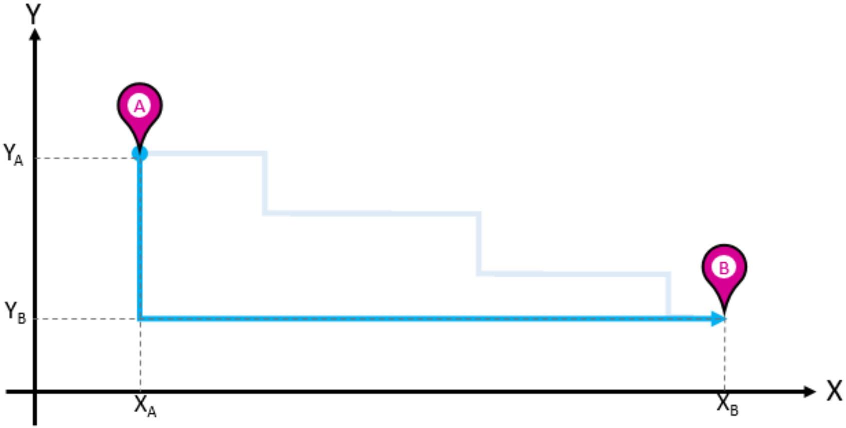
CMU MIT Stanford



Manhattan / Taxicab Distance



Taxicab Distance $= |X_A - X_B| + |Y_A - Y_B|$

A Motivating Puzzle

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- Raise each distance to the $2/3$ power.
- The result is always a Manhattan distance!

Why?

A Simple Example

- Given 3 points on a line at at 0, 1 and 2, compute Manhattan distances between each pair of points.

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- Given 3 points on a line at at 0, 1 and 2, compute Manhattan distances between each pair of points.
- Raise each distance to the $2/3$ power.
- Find 3 points whose Manhattan distances match these powered distances!

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- Powered distances: $1, 1, 2^{2/3}$.
- $$a = \left(\frac{1}{2} + \frac{2^{2/3}}{4}, \ \frac{1}{2} - \frac{2^{2/3}}{4}\right)$$
  $$b = (0, 0)$$
  $$c = \left(\frac{1}{2} - \frac{2^{2/3}}{4}, \ \frac{1}{2} + \frac{2^{2/3}}{4}\right).$$

Definition: The function $f(x) = x^{2/3}$ **sends** Manhattan distances to Manhattan distances.

The Core Mystery of our Talk

- Why does $x^{2/3}$ send Manhattan distances to Manhattan distances?

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  - This is new.
- Our talk focuses on the first point.

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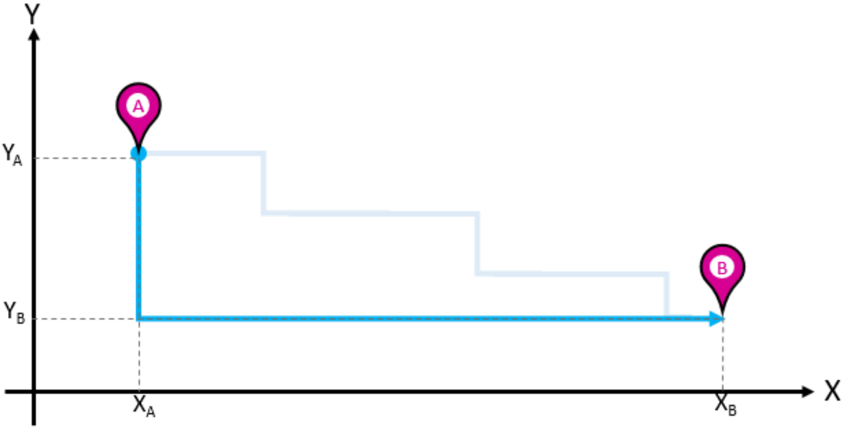
- Why does $x^{2/3}$ send Manhattan distances to Manhattan distances? A full proof.
- A deeper understanding of background tools:
    1. Manhattan distances
    2. How to determine whether a distance embeds into (squared) **Euclidean** distance.
    3. **Bonus**: A hidden use of Representation Theory of the group $\mathbb{Z}_2^n$.

Background Roadmap

1. Manhattan distances
2. How to determine whether a distance embeds into (squared) **Euclidean** distance.
3. **Bonus**: A hidden use of group Representation Theory.

Manhattan / Taxicab Distance

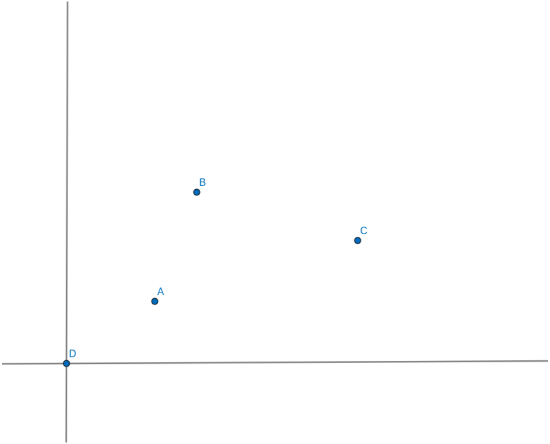


Taxicab Distance = $|X_A - X_B| + |Y_A - Y_B|$

Puzzles on Manhattan distance

- Show that any three point Manhattan metric is equivalent to a Manhattan metric on three corners of some high dimensional hyperbox.

Are Manhattan distances equivalent to Manhattan distances on a hyperbox?



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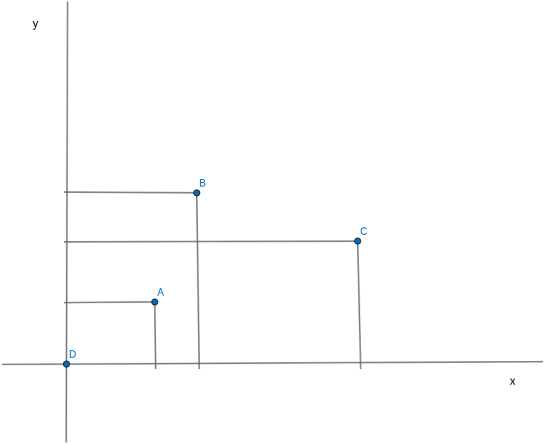

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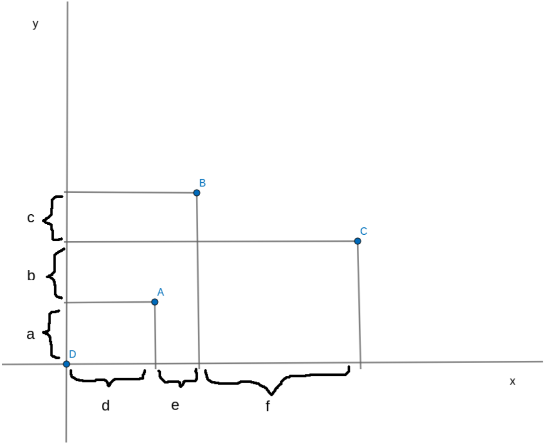

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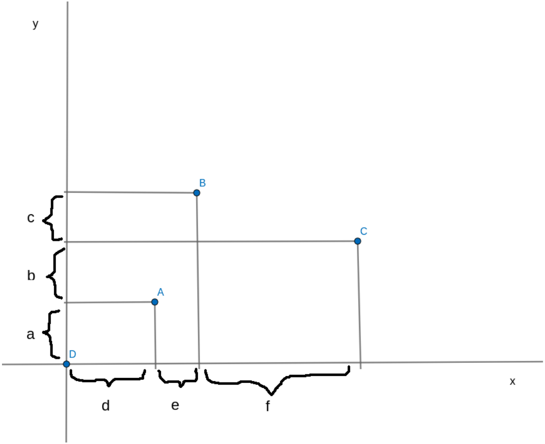

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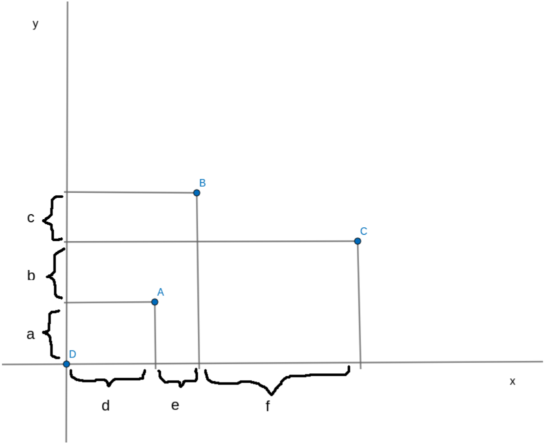

- Let A' be the point $(a, 0, 0, d, 0, 0)$.
- Let B' be the point $(a, b, c, d, e, 0)$.

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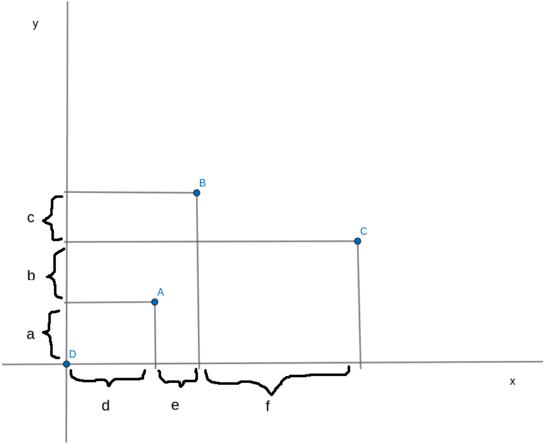

- Let A' be the point $(a, 0, 0, d, 0, 0)$.
- Let B' be the point $(a, b, c, d, e, 0)$.
- Let C' be the point $(a, b, 0, d, e, f)$.

Background Roadmap

1. Manhattan distances
2. How to determine whether a distance embeds into (squared) **Euclidean** distance.
3. **Bonus**: A hidden use of group Representation Theory.

What is a criterion for when points embed into **Squared** Euclidean Distance?

- A distance d on a set $\{x_1, \ldots x_n\}$ is a **squared** Euclidean distance if and only if the matrix D with:

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- But how can we test for this???

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A Sufficient Criterion for Negative Type Matrices

- Let D be a symmetric matrix.
- Suppose D has eigenvector $\overrightarrow{1}$, and all other eigenvectors have negative eigenvalue.
- Show that D is negative type: $x^T Dx \leqslant 0$ for all $x \perp 1$.

Example

- Let $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This has eigenvector $\overrightarrow{1}$, and all other eigenvectors $(1, -1)$ have negative eigenvalue.

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- Therefore, it of negative type, and the associated distance is a squared Euclidean distance.

Exercise

- Show that $D = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. is a negative type matrix!

A Sufficient Criterion for Showing a Distance is Squared Euclidean

Let d be a distance on set $x_1, \dots x_n$, such that the matrix D with $D_{ij} = d(x_i, x_j)$ satisfies:

- D has $\overrightarrow{1}$ as an eigenvector.
- All other eigenvalues of D are negative.

Then D_{ij} embeds into squared Euclidean distance!

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- Without loss of generality, let's assume the Manhattan distances are the full set of corners on some hyperbox.
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- Squared Euclidean distances on a hyperbox are Manhattan distances!

$x^{2/3}$ sends Manhattan distances to squared Euclidean distances

- We show $x^{2/3}$ sends Manhattan distances on a box to squared Euclidean distance.

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- We do ‘proof by example’, using a 2 dimensional box.

$x^{2/3}$ sends Manhattan-on-a-box to Squared Euclidean

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- The matrix D with $D_{ij} = d(x_i, x_j)$ is:

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- Claim: $\overrightarrow{1}$ is an eigenvector of D. (Why?)
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- Therefore: our Manhattan distances raised to the $2/3$ power are squared Euclidean distances.

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$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$
- The associated eigenvalues, we claim, are negative. We will show this by direct computation.

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- The eigenvalue λ corresponding to eigenvector $v = (1, -1, 1, -1)$ satisfies $D \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

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- This is $-f(a) + f(b) - f(a+b)$.

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- $= -\int_0^a f'(x)dx - \int_b^{a+b} f'(x)dx$
- Why is this eigenvalue negative, when $f = x^{2/3}$?

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$D_1 v = f(a) + f(b) - f(a+b)$

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- What property of $x^{2/3}$ did we use? What is the general property of functions f?

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- We show this by backing out the actual embedding.

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- The rows of P are on the corners of a box!
- The box corners are $(\pm\sqrt{-\lambda_2/2}, \pm\sqrt{-\lambda_3/2}, \pm\sqrt{-\lambda_4/2})$.

Proof Recap!

- Without loss of generality, we assumed the Manhattan distances are the full set of corners on some hyperbox.
- We showed that $x^{2/3}$ sends these distances to squared Euclidean distances **on a hyperbox.**
- Squared Euclidean distances on a hyperbox are Manhattan distances!
- Therefore: $x^{2/3}$ sends Manhattan distances to Manhattan distances!

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A **Bernstein function** is a (positive valued) function with positive derivative, negative second derivative, positive third derivative, negative fourth derivative. . . on $x \geqslant 0$.

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- Any positive linear combination of the above.