This note contains an observation about characters in representation theory, and conjectures a link between this, QM, and the theory of graph laplacians or PSD matrices in general.

Section 2 contains a summary of Fulton Harris chapters 1 and 2, according to Tim Chu (Orthogonality of Characters via dimension of center of Tensors, which occurs via Schur's lemma and the idea of G-linear maps).

0.1 Character Tables as Eigenvectors?

Character tables are orthogonal, and it turns out that $(P^*DP)_{ij}$ (D diagonal with elements a_i, corresponding to characters of representation A) is equal to:

(a) the size of $(V_i \otimes A \otimes V_j^*)_G$, AKA the size of the center of $(V_i \otimes A \otimes V_j^*)$

(b) the V_i term in the irreducible decomposition of $(A \otimes V_j)$.

Here, P is the unitary matrix with elements $\chi_{V_j}(C_i)$, as element ij, where V_j is the j^{th} irreducible representation of G and C_i is the i^{th} conjugacy class for the group g. The enumeration of C and V is arbitrary.

Here, P^*DP is a matrix with eigenvectors equal to the columns of P, with eigenvalues D (at least in the case of reals. What happens for complex numbers?)

1 Reason For Considering This

The reason I bring this up is that:

(a) Magically you get unitary matrices from representation characters, just like you get unitary matrices when doing an eigen-decomposition of any PSD matrix.

(b) Fourier series can be interpreted as projecting onto the eigenspectrum of the discrete Laplacian, and Fourier transforms can be interpreted(-ish) as projections onto the continuous circle (a compact set, which may be important). However, imaginary numbers don't necessarily come up here – you can get away with decompositions into sine and cosine. (This is sorta why group representations may be fundamentally different than Laplacian eigenvectors – most Laplacians don't have complex eigenvectors.)

(c) Fourier series can also be interpreted as projecting onto the character of the cyclic group.

Some possible other throwaway reasons for considering this include:

(d) Magically, there is a unique Hermetian inner product for any irreducible representation V, such that the representation of all group elements (in V) are unitary matrices under this inner product (Last exercise of Chp 1 in Fulton Harris). This is something I haven't looked into yet.

(e) Unitary matrices appear in Quantum Mechanics, and orthonormal matrices appear when decomposing any positive semi-definite matrix. (Unitary appears when decomposing a Hermitian matrix.)

(f) There exists an extension of Fourier analysis to non-abelian groups, via character theory. Probably there is some 'Fourier analysis of semi-simple Abelian groups' too or something (used in Terry Tao's proof of the Froebenius Kernel Theorem, some theorem in which characters are essential but I don't know what it says).

(g) PSD matrices appear all over TCS, and appear in the study of negative type distances.

2 Orthogonality of Characters Note

For proof on the orthogonality of characters, refer to Fulton-Harris. The main idea is that inner products in characters of V and W is equal to: the dimension of the fixed points of $V^* \otimes W$ under G, aka the **center** of G in $V^* \otimes W$ (for any representations V and W). (Why is inner product equal to the center's dimension? Why does the center's dimension imply orthogonality?)

A note on characters, and proof that the center is trivial for $V^* \otimes W$ on irreducible non-iso V and W

Let g_{V^*} have eigenvalues $\alpha_1 \dots \alpha_m$, and g_W have eigenvalues $\beta_1 \dots \beta_n$. Then $g_{V^* \otimes W}$ has eigenvalues $\alpha_i \beta_j$, whose sum is $(\sum_i \alpha_i)(\sum_j \beta_j)$. This implies that

$$\chi_{V^* \otimes W}(g) = \chi_{V^*}(g)\chi_W(g).$$

However, the center of $V^* \otimes W$ is equal to the dimension of the space of G-linear maps (maps $\psi : V \to W$ such that $\psi(g_V \_) = g_W \psi(\_)$) from V to W, which by Schur's lemma is 0 if V and W are irreducible and non-isomorhpic, and 1 otherwise (up to some scaling, which is important but which I'm too lazy to think about).

Schur's Lemma:

Schur's lemma holds since the kernel of ψ is a sub-representation of V, and the image of ψ is a sub-representation of W. (This follows from the definition of a G-linear map, and in fact illustrates why the concept of a G-linear map is so important.) So if V and W are irreducible (have no sub-representation), then either ψ is an isomorphism, or its the zero map.

(Schur part 2) Isomorphisms must be scalar copies of the identity:

For any G-linear map ψ (why do we care about linearity again? I see why we care about the G-preservation part), it has an eigenvalue λ. Then $\psi - \lambda I$ is also a G-linear map, since G-linear maps are closed under linear combinations (why?). But then $\psi - \lambda I$ must be 0, by the first part of Schur's lemma.

Note that V and W are treated as vector spaces over the complex numbers, which critically are algebraically complete (otherwise, ψ might not have an eigenvalue). I don't really understand this part, and it would be worth going into detail about this at some point.

2.1 Tensors Blah

Recall that the linear transform g operating on $V^* \otimes W$, has eigenvectors $EigVec(g_v) \otimes EigVec(g_w)$ with eigenvalues $EigVal(g_v) \cdot EigVal(g_w)$. (This is crappy notation.)

As a reminder, recall that

$$g_{V\otimes W}(\sum_i v_i \otimes w_i) = \sum_i (g_V \cdot v_i) \otimes (g_W \cdot w_i)$$

2.2 Miscellaneous Notes

As a final note, it's still unclear to me how characters are really magical. I suspect they can't really be that strong – the character orthogonality is just disguising Schur's lemma, and some properties of tensors. Somehow they don't *quite* seem like the right fundamental structure.....

3 Contributors

Observation about P^*DP thanks to Ben Gunby.