Heat as a Model for Vertex Centrality

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Abstract

In this note, we how heat on vertices of a graph cool when one fixed vertex is a heat sink. We propose using the long-term cooling on a graph with one vertex as a heat-sink, as a measure of that vertex's centrality in the graph. Intuitively, a sink at a central graph vertex may cause heat to rush out of the graph faster than a sink at a non-central vertex.

We can also compute a k-way partitioning of the graph, by picking the k vertices that drain heat the fastest out of a graph. Let these k vertices be designated as exemlars, and let each vertex in the graph be assigned to the exemplar that absorbed most its heat. This notion of dissipating heat quickly in a graph can be extended to edges, where we can find which edges dissipate heat quickly.

The hope is that this notion admits nice theorems, rather than just heuristical math results. Alternate measures of Vertex Centrality (degree centrailty, Katz Centrality, Eigenvector centrailty, etc.) might just be more useful, but this is another deuce in the pot. Note that Pagerank actually kind of sucks on Geometric graphs, because any d-regular graph has the same pagerank per vertex.

1 Modeling Heat in a Graph with Cold Sinks

We seek to model heat in a graph with cold sinks, as time goes to infinity. Suppose we have graph G with cold sink at vertex p. The heat h at time t given starting heats h_0 (at all points excluidng p) would be:

$$e^{tL_{-p}D^{-1}}h_0,$$

where L_{-p} denotes the Laplacian with the p^{th} row and column removed. Note that

$$h = D^{1/2}e^{tD^{-1/2}L_{-p}D^{-1/2}}D^{-1/2}h_0.$$

It appears to Tim Chu as if the behavior of this vector under any measure as $t \to \infty$, is determined by the smallest eigenvalue of

$$D^{-1/2}L_{-p}D^{-1/2}.$$

This is because the heat in the long term should be entirely governed by $e^{-\lambda_1 t}$, where λ_1 is the smallest eigenvalue of the normalized Laplacian.

This eigenvalue should govern the long-term heat as long as $D^{-1/2}h_0$ has some nontrivial portion of v_1 (the eigenvector corresponding to the smallest eigenvalue of the normalized Laplacian). The long-term heat in the graph under most measures is smallest when λ_1 is large. An identical problem: given a random walk, what vertex has the smallest long-term probability that a random walk avoids it?

2 Experiments

In this section, we build graphs and calculate the most central vertex. We provide copious visualizations and more.

We also build k-NN graphs from data, and calculate how fast heat flows on these graphs. Note that it is unclear how I want to set the vertex weights on these k-NN graphs.