Alternative Symmetrizations of Hitting Times in Graphs

Timothy Chu
Carnegie Mellon University
tzchu@andrew.cmu.edu

Gary Miller
Carnegie Mellon University
glmiller@cs.cmu.edu

November 24, 2018

Abstract

Finite point sets equipped with a given negative type distance can be embedded as points in Euclidean space, where the distance squared between points matches the given metric. It is known that l_1, l_2, effective resistance, hypermetrics, spherical metrics, and more are of negative type.

For points in Euclidean space, we can calculate the covariance matrix. It is known that if the distance in question is an effective resistance distance, then the covariance matrix is a pseudo-inverse Laplacian. It is also known that any power $0 < p < 1$ of the pseudoinverse Laplacian gives rise to a new pseudoinverse Laplacian.

Therefore, one may ask the question: given an l_1 covariance matrix, what powers of it are still l_1 covariance matrices? Likewise for l_2 covariance matrices, and negative type covariance matrices in general.

Experimental evidence indicates that: taking the p power of any covariance matrix (l_1, l_2) fixes the class, for $0 < p < 1$. We aim to prove it, and hope that our results have applications in... something.

First, we prove that $L^{1/2}$ is a Laplacian.

$$L = (D - A)$$
$$(D - A)^{1/2} = D^{1/4}(1 - \hat{A})^{1/2} D^{1/4}$$

Now we sub $\hat{A} = A$ and ignore the D terms.

$$(I - A)^{1/2} = I - A/2 - A^2/8 - A^3/16 - \dots$$

where $p(A)$ is an degree-weighted adjacency matrix for any positive polynomial p. It's clear that $L^{1/2}$ has the same nullspace as L, and half powers are possible given the matrix is PSD. Thus, the result has negativ off-diagonals, positive diagonals (why?), and has the right null-space: so it must be Laplacian.

The core insight is the following:

$$D^{-1/2} A D^{-1/2}$$

is an adjacency matrix, and so is any power thereof.

We seek to emulate this strategy for general matrices. For instance, suppose M is an l_1 covariance matrix. Now we consider:

$$\widehat{A} = D_M^{-1/2} A_M D_M^{-1/2}$$

where A_M is the off-diagonal portion and D_M is the on-diagonal portion. Particularly, we would like to test whether:

$$D_M^{1/2}(I - \widehat{A}^p) D_M^{1/2}$$

is an l_1 covariance matrix.