In this note, we examine diffusion. We cover:

(a) Hitting Times.

(b) Commute Times.

(c) Diffusion, and thoughts on how to study that phenomenon in greater depth. This includes Gary's idea about the time it takes for the Gaussian heat kernel to hit the inflection point. Ideally, we may provide intuition for why hitting time is the same in the diffusive and in the standard model.

This follows a line of reasoning that: diffusion, or its generalizations, may be an interesting property to study within graph networks.

1 Hitting Times

In graph G, let h_{uv} denote the hitting time from x to y. For now, fix $u = 1$, and shorthand $h_v = h_{1v}$. h_v satisfies:

$$h_v = 1 + \sum_{x \in N(v)} \frac{c_{vx}}{c_v} h_x \tag{1}$$

or:

$$c_v h_v = c_v + \sum_{x \in N(v)} c_{vx} h_x. \tag{2}$$

This holds for all $v \neq (u = 1)$. Boundary condition: $h_u = 0$.

Then: this equation is equal to:

$$D \cdot h = \begin{pmatrix} c_1 - C \\ c_2 \\ \dots \\ c_n \end{pmatrix} + A \cdot h \tag{3}$$

or:

$$(D - A)h = \begin{pmatrix} c_1 - C \\ c_2 \\ \dots \\ c_n \end{pmatrix} \tag{4}$$

The $c_1 - C$ term comes from: $D - A$ has image orthogonal to the all ones vector. $c_1 - C$ is the unique possible entry in that vector. I do not know how to deduce $c_1 - C$ term otherwise.

$$h = L^{\dagger} \begin{pmatrix} c_1 - C \\ \dots \\ c_n \end{pmatrix} \tag{5}$$

plus a constant times the all ones vector. What the constant is, I do not know. (It is to ensure boundary condition is right.)

2 Commute Times

C_{xy} is commute time. $= h_{xy} + h_{yx} =$ (when $x = 1, y = n$):

$$\begin{pmatrix}-1, 0, 0, \dots, 1\end{pmatrix} L^\dagger \begin{pmatrix} c_1 - C \\ \dots \\ c_n \end{pmatrix} \tag{6}$$

$$+ \begin{pmatrix}c_1, 0, 0, \dots, c_n - C\end{pmatrix} L^\dagger \begin{pmatrix}1 & 0 & 0 & \dots - 1\end{pmatrix} \tag{7}$$

$$= \begin{pmatrix}-1, 0, 0, \dots, 1\end{pmatrix} L^\dagger \begin{pmatrix} c_1 - C \\ \dots \\ c_n \end{pmatrix} \tag{8}$$

$$+ \begin{pmatrix}-1, 0, 0, \dots 1\end{pmatrix} L^\dagger \begin{pmatrix}-c_1 & 0 & \dots & C - c_n\end{pmatrix} \tag{9}$$

$$= \begin{pmatrix}-1, 0, 0, \dots, -1\end{pmatrix} L^\dagger \begin{pmatrix} -C \\ \dots \\ C \end{pmatrix} \tag{10}$$

$$= C \cdot \chi_{1n} L^\dagger \chi_{1n} \tag{11}$$

3 Diffusion Equation

The diffusion equation: $e^{-LD^{-1}t}$.

$$= D^{1/2} e^{-D^{-1/2} L D^{-1/2} t} D^{-1/2} \tag{12}$$

So: If you want to find the discrete expected hitting time (from vertex 1), you would solve:

$$h_y = 1 + \sum_{x \in N_{df}(y)} \mathbf{df}_{xy} h_x \tag{13}$$

where $\mathbf{df}_x$ is the diffisuion matrix's value at xy for $y \neq 1$.

or:

$$(I - df)h = 1$$

or:

$$(I - D^{1/2} e^{-\tilde{L}t} D^{-1/2})h = 1$$

or:

$$D - (D^{1/2} e^{-\tilde{L}t} D^{1/2})h = \begin{pmatrix} c_1 - C \\ c_2 \\ \dots \\ c_n \end{pmatrix} \tag{14}$$

Here, c is the degree sequence, and C is the total sum of the degrees (or $2m$).

Commute time is:

$$2m\chi_{xy}(D - (D^{1/2} e^{-\tilde{L}t} D^{1/2})h)\chi_{xy} \tag{15}$$

I moved a little fast here, so confusions with my math may be errors with my math. Here, $\tilde{L} := D^{-1/2} L D^{-1/2}$.

Thus, it may be of interest to examine the eigenvectors of the discretized-time, continuous-walk matrix, or:

$$D - D^{1/2} e^{-\tilde{L}} D^{1/2} \tag{16}$$

However, this seems to always be a Laplacian: $I - df$ should have positive diagonals, as the random walk prescribed by df doesn't stay in the same place with more than probability 1. Right mulltiplying by D doesn't change the positivity of diagonals. That is ,we're just replacing one transition matrix (A) for another (the lazy random walk).

The above matrix also has all ones as eigenvector. So no new class of matrices is generated in this fashion. But maybe this matrix is still useful for finding distances between two points in a distribution....