Heisenberg's Uncertainty Principle from Quantum Mechanics

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Abstract

In this note, we seek to prove Heisenberg's Uncertainty Principle from first principles. Our primary tool is an extension of Cauchy, which is known as the Hardy Inequality.

$$\int_{\mathbb{R}^3} |\nabla \psi(x)|^2 \geq \frac{1}{4} \int_{R^3} |\psi(x)|^2/|x|^2 \tag{1}$$

Remark 0.0.1. *Note that this is a 3 dimensional version of Hardy's inequality, and I'm not entirely sure whether the $\frac{1}{4}$ constant is tight. I also do not know how this follows from Hardy's inequality on one dimensions, or if an easy proof of the 3 dimensional case exists.*

Armed with this, we now consider the formulation of quantum kinetic energy:

$$KE := \frac{h^2}{2m} \int_{\mathbb{R}^3} |\nabla \psi(x)|^2. \tag{2}$$

Here, h represents the Planck Constant, and m is the mass of the particle. $\psi(x)^2$ is the probability distribution of the particle at point x. This formulation of Kinetic energy is an axiom.

Remark 0.0.2. *Is this formula the kinetic energy in the absence of some force? Otherwise I think it should have some kind of potential operator in it*

Now the standard deviation of the momentum of a particle is:

$$\sigma(p) := \sqrt{2m \cdot KE} \tag{3}$$

$$= h \left(\int_{\mathbb{R}^3} |\nabla \psi(x)|^2 \right)^{1/2}. \tag{4}$$

Remark 0.0.3. *Why is this the formula for momentum? And what is the momentum, without taking standard deviation?*

Thus: we can assert:

$$\left(h\int_{\mathbb{R}^3}|\nabla\psi(x)|^2\right)^{1/2}\left(\int_{\mathbb{R}^3}|x|^2|\psi(x)|^2\right)^{1/2} \tag{5}$$

$$\geq \frac{h}{2}\left(\int_{\mathbb{R}^3}|\psi(x)|^2/|x|^2\right)^{1/2}\left(\int_{\mathbb{R}^3}|x|^2|\psi(x)|^2\right)^{1/2} \tag{6}$$

$$\geq \frac{h}{2}\int_{\mathbb{R}^3}|\psi(x)|^2 \tag{7}$$

$$= h/2 \tag{8}$$

where the first inequality is from Hardy, the second is from Cauchy, and the final equality follows since $\psi(x)^2$ is a probability distribution, and thus has integral equal to 1 over the domain.

Remark 0.0.4. *Follow up question: When are the inequalities in both Hardy and Cauchy tight? Are they ever?*

Remark 0.0.5. *Momentum is usually a function of time, so where did that go in this equation? Schroedinger's equation promises an answer (but there, they have some imaginary number term that is difficult for me to comprehend). Furthermore, what kind of distribution is stable under Schroedinger's equation? I suspect that $|\psi(x)|$ is the complex norm of ψ, and is one reason they always deal with ψ^2 instead of ψ directly.*

Remark 0.0.6. *Are there any applications of this to graph theory? This proof suggests that the Laplacian of the grid, with mass-weights of $\frac{1}{|x|^2}$, has constant Hardy coefficient and thus constant eigenvalue (here x is the position). This would imply that this graph is an 'expander' of sorts, despite looking nothing like a classical expander.*