We prove Effective Resistance isometrically embeds into $L1$, completing our exploration on where in the Deza heirarchy of metrics that effective resistance fits into. The proof is an inductive argument using Sherman Morrison. Throughout this note, let χ_{ij} is defined as the vector with 1 at vertex i and -1 at vertex j and zero elsewhere.

Theorem 0.1. *Effective Resistance is in $L1$.*

We prove this using two theorems. Let R_e be the effective resistance of edge e, and c_e its conductance.

Theorem 0.2. *For any graph G, let $\mathcal{G}'$ be the set of graphs with one edge removed from G.*

$$L_G^\dagger = (m-n+1) \cdot \sum_{e \in \mathcal{E}(\mathcal{G})'} \lim_{c \to 1} (1 - c \cdot R_e c_e) \cdot L_{G-c \cdot e}^\dagger$$

How does this help us? Well, if G's minimum edge separator has a cardinality of more than 1, then the above equation is equivalent to:

$$L_G^\dagger = (m-n+1) \cdot \sum_{e \in \mathcal{E}(\mathcal{G})'} (1 - R_e c_e) \cdot L_{G-\cdot e}^\dagger$$

and thus:

$$\chi_{ij}^T L_G^\dagger \chi_{ij} = (m-n+1) \cdot \sum_{e \in \mathcal{E}(\mathcal{G})'} \chi_{ij}^T \left((1 - R_e c_e) \cdot L_{G-\cdot e}^\dagger \right) \chi_{ij}$$

and we are done by induction on the edges in G.

1 If e is an edge-separator

Note that if e is not an edge separator of G, then

$$\lim_{c \to 1} (1 - c \cdot R_e c_e) \cdot L_{G-c \cdot e}^\dagger$$

is equal to:

$$(1 - \cdot R_e c_e) \cdot L_{G-e}^\dagger.$$

Now we examine what happens if e is an edge-separator of G.

Theorem 1.1. *If e is an edge-separator of G that splits G into two clusters, then:*

$$\chi_{ij}^T \left(\lim_{c \to 1} (1 - c \cdot R_e c_e) \cdot L_{G-c \cdot e}^\dagger \right) \chi_i j$$

is equal to 0 if i and j are in the same cluster, and c_e otherwise.