Heat as a Model for Vertex Centrality

Timothy Chu
Carnegie Mellon University
tzchu@andrew.cmu.edu

Gary Miller
Carnegie Mellon University
glmiller@cs.cmu.edu

November 11, 2018

Abstract

In this note, we how heat on vertices of a graph cool when one fixed vertex is a heat sink. We propose using the long-term cooling on a graph with one vertex as a heat-sink, as a measure of that vertex's centrality in the graph. Intuitively, a sink at a central graph vertex may cause heat to rush out of the graph faster than a sink at a non-central vertex.

We can also compute a k-way partitioning of the graph, by picking the k vertices that drain heat the fastest out of a graph. Let these k vertices be designated as exemlars, and let each vertex in the graph be assigned to the exemplar that absorbed most its heat. This notion of dissipating heat quickly in a graph can be extended to edges, where we can find which edges dissipate heat quickly.

The hope is that this notion admits nice theorems, rather than just heuristical math results.

1 Modeling Heat in a Graph with Cold Sinks

We seek to model heat in a graph with cold sinks, as time goes to infinity. Suppose we have graph G with cold sink at vertex p. The heat h at time t given starting heats h_0 (at all points excluidng p) would be:

$$e^{tL_{-p}D^{-1}}h_0,$$

where L_{-p} denotes the Laplacian with the p^{th} row and column removed. Note that

$$h = D^{1/2}e^{tD^{-1/2}L_{-p}D^{-1/2}}D^{-1/2}h_0.$$

It appears to Tim Chu as if the behavior of this vector under any measure as $t \to \infty$, is determined by the smallest eigenvalue of

$$D^{-1/2}L_{-p}D^{-1/2}.$$

This is because the heat in the long term should be entirely governed by $e^{-\lambda_1 t}$, where λ_1 is the smallest eigenvalue of the normalized Laplacian.

This eigenvalue should govern the long-term heat as long as $D^{-1/2}h_0$ has some nontrivial portion of v_1 (the eigenvector corresponding to the smallest eigenvalue of the normalized Laplacian). The long-term heat in the graph under most measures is smallest when λ_1 is large. An identical problem: given a random walk, what vertex has the smallest long-term probability that a random walk avoids it?

2 Experiments

In this section, we build graphs and calculate the most central vertex. We provide copious visualizations and more.

We also build k-NN graphs from data, and calculate how fast heat flows on these graphs. Note that it is unclear how I want to set the vertex weights on these k-NN graphs.