Theorem 0.1. *Let D be a negative type matrix. Let $LRO_k(S)$ denote the optimal low-rank approximation of S in the Froebenius norm. Then in $O(npoly\frac{k}{\varepsilon})$ time, we can compute matrix D' of rank k such that:*

$$||D - D'||_F \leq (1+\varepsilon)||D - LRO_{k-2}(D)||_F \tag{1}$$

To do this, we will use Musco and Woodruff and a careful use of Cauchy's Interlacing Theorem.

Theorem 0.2. *(Musco and Woodruff) Let M be a positive semi-definite matrix. Then we can compute a matrix M' of rank k in $O(npoly\frac{k}{\varepsilon}$ time such that:*

$$||M - M'||_F \leq (1+\varepsilon)||M - LRO_k(M)||_F \tag{2}$$

Cauchy's Interlacing theorem, as stated in lemma 3.4 of https://arxiv.org/pdf/1408.4421v1.pdf:

Theorem 0.3. *(Optimal Low Rank Approximation of Matrices): Let $\mu_1, \ldots \mu_n$ be eigenvectors of symmetric matrix S, where $|\mu_1| \leq \ldots \leq |\mu_n|$. Then:*

$$S - LRO_k(S) = l_2(\mu_{k+1}, \ldots, \mu_n)$$

We state the above theorem without proof. Here, l_2 represents the l_2 norm.

Theorem 0.4. *(Cauchy's Interlacing Theorem) Let λ_i^X denote the i^{th} largest eigenvalues of X for any matrix X. If A is a symmetric matrix and v is a vector, then the eigenvalues of $A + vv^T$ satisfy:*

$$\lambda_i^A \leq \lambda_i^{A+vv^T} \leq \lambda_{i+1}^A \tag{3}$$

$$\lambda_i^{A+vv^T} \leq \lambda_{i+1}^A \leq \lambda_{i+1}^{A+vv^T} \tag{4}$$

Corollary 0.4.1.

$$\lambda_{i-1}^A \leq \lambda_i^{A+vv^T-ww^T} \leq \lambda_{i+1}^A \tag{5}$$

$$\lambda_{i-1}^{A+vv^T-ww^T} \leq \lambda_i^A \leq \lambda_{i+1}^{A+vv^T-ww^T} \tag{6}$$

This follows from Theorem 0.4.

Definition 0.5. *Let $M := [d_{ij} - d_{0i} - d_{0j}]$, where d_{ij} is the ij entry of D. Let $A_1 = [d_{0i}]$, and $A_2 = [d_{0j}]$, where A1 and A2 are n by n matrices. By construction, $D = M + A_1 + A_2$, and sampling from M can be done quickly by taking three samples from D.*

Let v be the vector with entries $(1 + d_{0i})/2$, and w be the vector with entires $(1 - d_{0i})/2$.

Lemma 0.6. $A1 + A2 = vv^T - ww^T$

(Proof omitted. This is straightforward computation I think.)

Lemma 0.7. $D = M + vv^T - ww^T$

Proof. This follows from lemma 0.6. □

Corollary 0.7.1. *Using D and M as defined in 0.5:*

$$\lambda_{i-1}^M \leq \lambda_i^D \leq \lambda_{i+1}^M \tag{7}$$

$$\lambda_{i-1}^D \leq \lambda_i^M \leq \lambda_{i+1}^D \tag{8}$$

Proof. This follows from corollary 0.4.1 and Lemma 0.6 □

Now, we list basic facts about negative type distances D.

Theorem 0.8.

$$\lambda_1^D \leq \ldots \lambda_{n-1}^D \leq 0 < \lambda_n^D$$
$$\lambda_n^D \geq |\lambda_1^D| \geq \ldots |\lambda_{n-1}^D| \geq 0$$

The second chain of inequalities holds since D has at most one positive eigenvalue (known, I can show you how to prove this offline). Also, the trace of D is 0, which means the positive eigenvalue is equal to the negative sum of the negative eigenvalues.

Theorem 0.9. *M is negative-semidefinite, and so:*

$$\lambda_1^M \leq \ldots \leq \lambda_n^M = 0$$

$$|\lambda_1^M| \geq \ldots |\lambda_n^M| \geq 0$$

That M is negative-semidefinite, is well established in the literature.

Theorem 0.10. *For all $i < n$:*

$$|\lambda_{i-1}^M| \geq |\lambda_i^D| \geq |\lambda_{i+1}^M| \tag{9}$$

For all $2 \leq i < n - 1$:

$$|\lambda_{i-1}^D| \geq |\lambda_i^M| \geq |\lambda_{i+1}^D| \tag{10}$$

Proof. This follows from Corollary 0.7.1. □

Theorem 0.11. *Let M' be the output of Woodruff and Musco, applied to M, where M' is rank $k-2$. Then let $D' = M' + vv^T - ww^T$. Recall that $D = M + vv^T - ww^T$. Then*

$$||D - D'||_F = ||M - M'|| \leq (1+\varepsilon)||M - LRO_{k-2}(M)||$$
$$= (1+\varepsilon) \cdot l_2(\lambda^M_{k-1}, \ldots \lambda^M_n)$$

Theorem 0.12. *For $j \geq 1$,*

$$||D - LRO_j(D)||_F = l_2(\lambda^D_j, \ldots \lambda^D_{n-1})$$

Theorem 0.13.

$$l_2(\lambda^M_{k-1}, \ldots \lambda^M_n) \leq l_2(\lambda^D_{k-2}, \ldots \lambda^D_{n-1}) = ||D - LRO_{k-2}(D)||_F$$

Proof. The first inequality follows from Theorem 0.10. □

Theorem 0.14.

$$||D - D'||_F \leq (1+\varepsilon)||D - LRO_{k-2}(D)||_F$$

As desired. Note that I think we can actually bound $||D-D'||_F$ with $||D-LRO_{k-1}(D)||_F$ with more careful eigenvalue bounding, but whatevs. Note that D' can be computed quickly (M' can be obtained quickly by sampling from D, and v and w can be computed in $O(n)$ time.)