Title

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Abstract

There exist expanders without densifiers. In particular, there exist graphs G satisfying:

$$\frac{1}{\text{poly log}} K_n \leq G \leq K_n$$

that do not admit graphs with edge weights $\leq 10\text{poly log}/n$ that preserve all cuts up to a constant factor approximation. (Such a graph is called a densifier). Here, K_n is the complete graph on n vertices.

Construction: Construct a clique with vertices labeled 1 through a. Construct a second clique with vertices $1'$ through a'. Each clique has edge weights $\frac{1}{\log n}$. Now build a matching of weight a between vertices k and k' for all k. Such a graph has $2a$ vertices. This is a graph with a heavy matching between two cliques. Now set $a = n/2$ and let this graph be denoted as Q_n.

Basic lemmas about densifier:

Lemma 0.1. *1. If H is a densifier of G, then $T(H)$ is a densifier of G, where T is any graph isomorphism of G.*

2. If H_1 and H_2 are both densifiers of G, then any weighted average of H_1 and H_2 are densifiers of G.

3. If $\boldsymbol{T}$ is a family of isomorphisms of G and H is a densifier, then the weighted average

$$\sum_{T \in \boldsymbol{T}} T(H)$$

is also a densifier of G.

I think each item is straightforward – correct me if I'm wrong.

Theorem 0.2. *Q_n is an expander with poly log expansion*

Proof. Every cut has conductance between $O(n/logn)$ and $O(n)$ $\square$

Now our main theorem:

Theorem 0.3. *Q_n (defined in our construction) has no densifier*

Proof. If Q_n has a 2-densifier, then it has a densifier whose clique edge weights are all the same. This can be obtained by averaging over all graph isomorphisms that permute the clique vertices.

Moreover, the edge weights on the cliques are no more than twice the original edge weights, and no less than half the original edge weights. This is because the cut defined by vertices $1, 1', 2, 2', \dots a/2, a/2'$ has cut value of $a^2/2 \log n$, as it cuts no edges in the matching.

Thus, it follows that the cliques must have edge weights no more than double and no less than half of their original edge weights.

Now, by averaging over all permutations of vertices that fix the matching, you can similarly show that there is a densifier where edge weights on edge ij' are all the same, for $i \neq j$. These edge weights can't be too small, as the degree of each i must be preserved, weight on ii' cannot be large (thus the edge weight previously on ii' must be distributed across edges ij' for $j \neq i$).

This ensures that the cut $1, 1', 2, 2', \dots a/2, a/2'$ must have cut value of $a^2/2$, which is at least a $\log n$ distortion from the original cut value. (I skipped a couple steps in this deduction – check me on this.) $\square$