1 Introduction

Suppose D is a matrix with elements d_{ij}. Is it the effective resistance in an underlying graph G? (Note that this paper does not account for 0 resistance edges, as those break the pseudoinverse propety.)

2 Definitions

1. Let χ_v denote the vector with 1 at vertex v and 0 everywhere else.
2. Let χ_{uv} denote the vector $\chi_u - \chi_v$.
3. Let $P := I - \frac{1}{n}J$ where J is the all ones matrix.
4. L is the Laplacian of a graph G.

Definition 2.1. *Effective resistance between vertice u and v is defined to be:*

$$\boldsymbol{r}_{uv} \stackrel{\text{def}}{=} \chi_{uv}^T L^\dagger \chi_{uv}.$$

where L_{ij} is the **conductance** of the unique wire connecting i and j in underlying graph G. Note that this is the classic definition of effective resistances, except that it only works when the resistances are non-zero. (Conductance cannot be infinite).

3 Is a distance ER? If so....

$$d_{uv} = \chi_{uv}^T L^\dagger \chi_{uv}. \tag{1}$$

$$d_{uv} = L^\dagger_{uu} + L^\dagger_{vv} - 2 \cdot L^\dagger_{uv}. \tag{2}$$

$$D = [L^\dagger_{uu}] + [L^\dagger_{vv}] - 2L^\dagger. \tag{3}$$

$$\frac{-P^T D P}{2} = L^\dagger. \tag{4}$$

$$\left(\frac{-P^T D P}{2}\right)^\dagger = L. \tag{5}$$

Or alternately,

$$\left(-P^T DP\right)^\dagger = 2L. \tag{6}$$

Now we would like to check if $-P^T DP$ being Laplacian AND RANK N-1 is enough to guarantee that D is an effective resistance matrix. Here, I claim that $-P^T DP$ is the inverse Laplacian that generates distance matrix D.

Proof: We claim

$$d_{uv} = \chi_{uv}^T(-P^T DP/2)\chi_{uv}.$$

Because $P \cdot \chi_{uv} = \chi_{uv}$ since $J \cdot \chi_u v = 0$ then this claim is equivalent to:

$$d_{uv} = \chi_{uv}^T(-D/2)\chi_{uv}.$$

or

$$d_{uv} = -\frac{1}{2}(d_{uu} + d_{vv} - 2d_{uv}).$$

Since $d_{uu} = d_{vv} = 0$, then this expression is d_{uv} as desired.

Note that the $P \cdot \chi_{uv} = \chi_{uv}$ for all χ_{uv}, and P squashing the all ones vector were the only essential requirements on P. Our choice of P satisfies both of these.