1 Abstract

We show that effective resistance satisfies, for any integer vector v whose coordinate sum is 1:

$$v^T Dv \leq 0.$$

In other words, we will show effective resistance is a Hypermetric.

This project is part of a larger goal: we'd like to determine where in the heirarchy of metrics Effective Resistance sits. The heirarchy in question is:

$$l_2 \subset Sphere \subset l_1 \subset HyperMetric \subset Neg - Type.$$

Note that we can show effective resistance is not contained in l_2 (trees are in ER), that l_2 is not contained in effective resistance (the Euclidean square cannot embed). It was known that effective resistances were negative type. We still do not know if effective resistances are in l_1.

2 Proof Overview

Suppose a graph G has positive conductances c_e and Laplacian L. Let R_e denote the effective resistance of edge e. For any weights $w_e \subset \mathbb{R}$:

$$\sum w_e R_e$$

can be reduced (or kept at the same quantity) by taking any edge e' and either raising $c_{e'}$ to be arbitrarily high, or zero. Note that $v^T Dv$ can be written as $\sum w_e R_e$ for weights $w_{ij} = v_i v_j$. (Proof omitted).

Consider any tree on G, where each tree edge has non-zero conductance. (There must exist such a tree if G is connected by resistance wires). For every non-tree edge, we either raise the conductance to infinity, or we zero the conductance: whichever one happens to increase $\sum w_e R_e$. If we zero an edge out, we reduce the vertex count. If we zero the conductance of an edge, we keep going until we've processed all the non-tree edges.

Now if we look at the conductances in the graph, we are either left with a tree (if we zero-ed out all non-tree edges), on which the hyper-metric property holds (trees are in l_1, which are hypermetrics), or we are left with an infinite-conductance edge. If we are left with a tree, we are finished: we raised $v^T Dv$ by zeroing out non-tree edges, and we were left with a quantity that was ≤ 0.

I claim that if we zero the conductance of an edge ij, we can finish by induction on the number of vertices. I omit the proof here, and will show it on the blackboard.

3 Filling in the Proofs for the Outline.

Let χ_{ij} be the vector that is 1 at i, and -1 at j, and zero elsewhere. Recall that $R_{ij} := \chi_{ij}^T L^\dagger \chi_{ij}$.

Theorem 3.1. *Let R_e denote the effective resistance of edge e in some graph G with non-negative conductacnes. For any real weights w_e' and any edge e, the expression $\sum_e w_e R_e$ is monotonic in $c_{e'}$,*

Proof. Sherman-Morrison for pseudoinverses tells us that:

$$(M + u^T u)^\dagger = M^\dagger - \frac{M^\dagger u u^T M^\dagger}{1 + u^T M^\dagger u}$$

Note that raising $c_{e'}$ changes the Laplacian by a rank one update. (Proof omitted). Let L' be the Laplacian L after raising $c_{e'}$ by scalar k, where k can be positive or negative. Now, u denotes the vector χ_e. Then:

$$(L')^\dagger = (L + k u^T u)^\dagger = L^\dagger - \frac{k \cdot L^\dagger u u^T L^\dagger}{1 + k \cdot u^T L^\dagger u}$$

Therefore the change in effective resistance (using the standard effective resistance formula) is:

$$\begin{aligned}
&\sum w_e (R_e - R_e') \\
&= \sum w_e \chi_e^T \left((L')^\dagger - L^\dagger \right) \chi_e \\
&= \sum -w_e \left(\chi_e^T \frac{k \cdot L^\dagger u u^T L^\dagger}{1 + k \cdot u^T L^\dagger u} \chi_e \right)
\end{aligned}$$

$$= \sum_e \frac{k}{1 + k \cdot u^T L^\dagger u} \left(-w_e \cdot \chi_e^T L^\dagger u u^T M^\dagger \chi_e \right). \tag{1}$$

Letting

$$C \stackrel{\text{def}}{=} u^T L^\dagger u$$
$$S_e \stackrel{\text{def}}{=} -w_e \cdot \chi_e^T L^\dagger u u^T L^\dagger \chi_e,$$

Equation 1 becomes:

$$\sum_e \frac{k \cdot S_e}{1 + k \cdot C} \tag{2}$$
$$= \frac{k \cdot \sum_e S_e}{1 + k \cdot C} \tag{3}$$
$$= \frac{k \cdot T}{1 + k \cdot C} \tag{4}$$

where $T := \sum S_e$.

This is a monotonic function in k (proof omitted). Therefore, the extremes occur when k is set to be as negative as possible (zero-ing out k), or by raising it to be as large as possible (setting it to be arbitrarily large).

(Note: I'm not actually sure Sherman Morrison is true for Pseudoinverses.... or what conditions need to be set on the rank one update. Perhaps it suffices that the rank one update has a nullspace inside the original matrix M's nullspace?) $\square$

This completes the proof of our main lemma. The rest of our proof follows the Proof Overview. (Detailed omitted.)