Today: $\boxed{\text{Ken}}$

$\boxed{5.2}$ Mathematical Induction I

$\boxed{5.3}$ Mathematical Induction II

Last time:

$\boxed{5.1}$ Sequences

$\boxed{4.7}$ Indirect Argument: Contradiction & Contraposition

5.2 Mathematical Induction I: Proving Formulas

Empirical Problem of Induction

<u>Principle of Mathematical Induction</u>

Let $P(n)$ be a property that is defined for integers n, and let $a \in \mathbb{Z}$ be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For every integer $k \geq a$, if $P(k)$ is true, then $P(k+1)$ is true.

Then the statement

for every integer $n \geq a$, $P(n)$

is true.

Method of Proof by Induction

$\forall n \in \mathbb{Z}\,(n \geq a \to P(n))$

Consider a statement of the form,

"For every integer $n \geq a$, a property $P(n)$ is true."

To prove such a statement, perform the following two steps.

Step 1 (base case): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for every integer $k \geq a$, if $P(k)$ is true, then $P(k+1)$ is true. To perform this step,

Suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

(This assumption is called the inductive hypothesis.)

Then show that $P(k+1)$ is true.

Definition

If a sum with a variable number of terms is shown to equal an expression that does not contain either an ellipsis or a summation symbol, we say that sum is written in closed form.

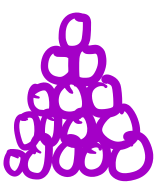
Theorem 5.2.1

The Sum of Consecutive Integers

For all $n \in \mathbb{Z}^+$,

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}.$$

Proof:



let $P(n)$ be the statement

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Step ① base case

$$P(1): \sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2}$$

Step ② Suppose for $n \geq 1$, $P(n)$ is true.

So $\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$ by assumption.

Then (to show $P(n+1)$),

$$\sum_{k=1}^{n+1} k = 1+2+3+\cdots+n+(n+1)$$

$$= \frac{n(n+1)}{2} + (n+1)$$ via induction hypothesis

$$= \frac{n(n+1)}{2} + \frac{(n+1)2}{2}$$

$$= \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

$$= \frac{(n+1)((n+1)+1)}{2}.$$

(So $P(n)$ implies $P(n+1)$)

Theorem 5.2.2

For any $r \in \mathbb{R}$ such that $r \neq 1$ and any $n \in \mathbb{Z}$ such that $n \geq 0$,

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r} = \frac{r^{n+1}-1}{r-1}$$

Proof:

Let $r \in \mathbb{R}$ such that $r \neq 1$. Define

$P(n)$ to be the statement

$$\sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}.$$

① base case, $P(0)$

$$\sum_{k=0}^{0} r^k = r^0 = 1 = \frac{1-r}{1-r} = \frac{1-r^{0+1}}{1-r}$$

② Inductive Step;

suppose, for $n \geq 0$, $P(n)$ is true,

meaning

$$\sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}.$$

Then, for $P(n+1)$

$$\sum_{k=0}^{n+1} r^k = 1 + r + \cdots + r^n + r^{n+1}$$

$$= \frac{1-r^{n+1}}{1-r} + r^{n+1}$$

$$= \frac{1-r^{n+1}}{1-r} + r^{n+1}\left(\frac{1-r}{1-r}\right)$$

$$= \frac{1-r^{n+1} + r^{n+1}(1-r)}{1-r}$$

$$= \frac{1-\cancel{r^{n+1}} + \cancel{r^{n+1}} - r^{n+1+1}}{1-r}$$

$$= \frac{1-r^{(n+1)+1}}{1-r}.$$

So $P(n)$ implies $P(n+1)$.

$$r^{n+1} r^1 = r^{n+1+1}$$

#34 ① Find a formula in $a, r \in \mathbb{R}$ and $m, n \in \mathbb{Z}$, $n \geq 0$ such that

$$ar^m + ar^{m+1} + ar^{m+2} + \cdots + ar^{m+n}$$

and prove the result via mathematical induction.

#36 ② Prove the theorem:

> <u>Theorem</u>
>
> For any $n \in \mathbb{Z}^+$,
>
> $$1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Definition

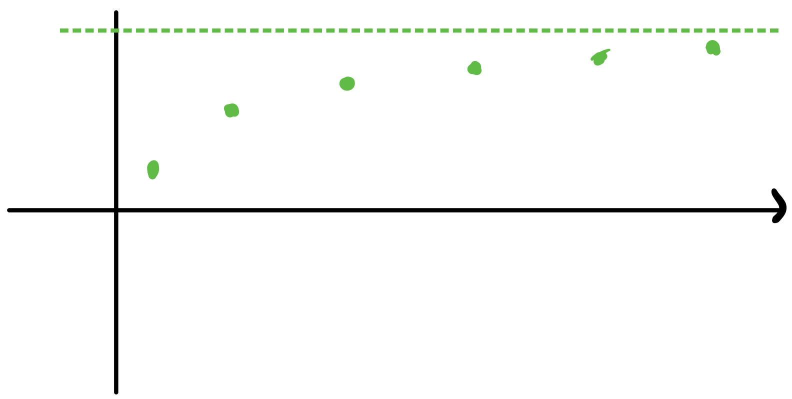
1. $\{a_n\}$ is monotone if $\{a_n\}$ is increasing or decreasing.
2. $\{a_n\}$ is increasing if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.
3. $\{a_n\}$ is decreasing if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.
4. $\{a_n\}$ is bounded above if there exists $M_1 \in \mathbb{R}$ such that $a_n \leq M_1$ for all $n \in \mathbb{N}$.
5. $\{a_n\}$ is bounded below if there exists $M_2 \in \mathbb{R}$ such that $a_n \geq M_2$ for all $n \in \mathbb{N}$.
6. A sequence $\{a_n\}$ is bounded if $\{a_n\}$ is bounded below and bounded above.

Theorem

A bounded monotone sequence converges.

e.g. Let $a_0 = \sqrt{3}$ and $a_k = \sqrt{3a_{k-1}}$ for all $k \in \mathbb{Z}$ such that $k \geq 1$. Prove that $\{a_k\}_{k=0}^{\infty}$ converges. What does a_k approach as k approaches infinity?

$\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \ldots$



5.3 Mathematical Induction I: Proving Formulas

Proposition 5.3.2

For every $n \in \mathbb{Z}$ such that $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

Proof?

Let $P(n)$ be the statement

$3 \mid 2^{2n} - 1$.

① base case; $P(0)$:

$$2^{2(0)} - 1 = 2^0 - 1 = 1 - 1 = 0$$

$$= 0(3)$$

so there exists $0 \in \mathbb{Z}$ such that

$3(0) = 2^{2(0)} - 1$ and $3 \mid 2^{2n} - 1$.

② inductive step; suppose $P(n)$ is true for $n \geq 0$. $P(n)$ states

$3 \mid 2^{2n}-1$ so there exists

$k \in \mathbb{Z}$ such that $2^{2n}-1 = 3k$.

Then

$$2^{2(n+1)}-1 = 2^{2n+2}-1 = 2^{2n}(2^2)-1$$

$$= 2^{2n}(4)-1 = 2^{2n}(3+1)-1$$

$$= 3(2^{2n}) + \underbrace{2^{2n}-1}_{\text{induction hypothesis}}$$

$$= 3(2^{2n}) + 3k$$

$$= 3(2^{2n}+k)$$

where $2^{2n}+k =: \ell$ such that

$2^{2(n+1)}-1 = 3\ell$ for $\ell \in \mathbb{Z}$. So

$3 \mid 2^{2(n+1)}-1$.

Proposition 5.3.3

For all $n \in \mathbb{Z}$ such that $n \geq 3$,

$2n+1 < 2^n$.

Proof?

Let $P(n)$ be the statement, $2n+1 < 2^n$ (where $n \in \mathbb{Z}$).

① Base case, $P(3)$:

$$2(3)+1 = 7 < 2^3 = 8.$$

② Inductive step:

Suppose $P(n)$ for $n \geq 3$, namely

$2n+1 < 2^n$.

Then, for $P(n+1)$,

$$\begin{aligned} 2(n+1)+1 &= 2n+2+1 \\ &= (2n+1)+2 \\ &< 2^n + 2^n = 2^n(1+1) = 2^n 2^1 = 2^{n+1}. \end{aligned}$$

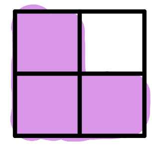
via induction hypothesis

$\square$

$n \geq 3$

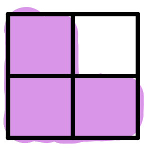
$2^n \geq 2^3 = 8 > 2$

Theorem 5.3.4

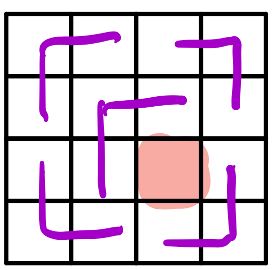


Covering a Board with Trominoes

For any $n \in \mathbb{Z}^+$, if one square is removed from a $2^n \times 2^n$ checkerboard, the remaining squares can be completely covered by L-shaped trominoes.



$2^1 \times 2^1$



$2^2 \times 2^2 = 4 \times 4$