Today: $\boxed{\text{Ken}}$

$\boxed{4.3}$ Direct Proof and Counterexample III

$\boxed{4.4}$ Direct Proof and Counterexample IV

Last time:

$\boxed{4.1}$ Direct Proof and Counterexample I

$\boxed{4.2}$ Direct Proof and Counterexample II

Exam 1 10/7/2021 8:00am in-person

Chapter 2 & 3

$\boxed{4.3}$ Direct Proof and Counterexample III

Recall:

<u>Definition</u>

q is a rational number if and only if $q = m/n$ for some integers m and n such that $n \neq 0$.

$$q \in \mathbb{Q} \iff \exists m \in \mathbb{Z}\ \exists n \in \mathbb{Z} - \{0\}\ (q = m/n)$$

e.g. Which of the following are rational numbers?

1. $3/4$ yes
2. 3.1415 yes
3. $0.\overline{3} = 0.333\ldots$ yes
4. $-22/7$ yes
5. $1/0$ no
6. Provided $m, n \in \mathbb{Z} - \{0\}$, $\frac{m+n}{mn}$ yes

   proof left as exercise

Let X be either $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.

Zero Product Property

If $x_1 \in X$ and $x_2 \in X$ are nonzero, then their product $x_1 x_2$ is nonzero.

$$\forall x_1 \in X\ \forall x_2 \in X\ (x_1 \neq 0 \land x_2 \neq 0 \rightarrow x_1 x_2 \neq 0)$$

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are the integers, rational numbers, real numbers, and complex numbers respectively. $\land$

Theorem 4.3.1

Every integer is a rational number.

$$\forall n \in \mathbb{Z}\ (n \in \mathbb{Q})$$

Proof:

Let $n \in \mathbb{Z}$. Consider $1 \in \mathbb{Z}$ such that

$n = n/1 \in \mathbb{Q}$.

Theorem 4.3.2

The sum of any two rational numbers is a rational number.

$\forall p \in \mathbb{Q}\ \forall q \in \mathbb{Q}\ (p+q \in \mathbb{Q})$

Proof:

Let $p, q \in \mathbb{Q}$. So by definition of rational number $p = m_1/n_1$ and $q = m_2/n_2$ where $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ and $n_1 \neq 0, n_2 \neq 0$. Consider

$$p+q = \frac{m_1}{n_1} + \frac{m_2}{n_2} = \left(\frac{n_2}{n_2}\right)\frac{m_1}{n_1} + \frac{m_2}{n_2}\left(\frac{n_1}{n_1}\right)$$

$$= \frac{m_1 n_2 + m_2 n_1}{n_2 n_1} \in \mathbb{Q}$$

because $m_1 n_2 + m_2 n_1 \in \mathbb{Z}$ by closure of $\mathbb{Z}$ under addition & multiplication and $n_1 n_2 \in \mathbb{Z}$ (by closure of multiplication again) where $n_1 n_2 \neq 0$ via zero product property.

Deriving Additional Results about Even and Odd Integers

1. The sum, product, and difference of any two even integers are even.
2. The sum and difference of any two odd integers are even.
3. The product of any two odd integers is odd.
4. The product of any even integer and any odd integer is even.
5. The sum of any odd integer minus any even integer is odd.
6. The difference of any odd integer minus any even integer is odd.
7. The difference of any even integer minus any odd integer is odd.

Prove that if m is any even integer and n is any odd integer, then $\frac{m^2+n^2+1}{2}$ is an integer.

Suppose $m \in 2\mathbb{Z}$ and $n \in \mathbb{Z} - 2\mathbb{Z}$.
Then $m = 2K$ for some $K \in \mathbb{Z}$ and $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$ via definition of even and odd respectively.

$$\frac{m^2+n^2+1}{2} = \frac{(2K)^2+(2\ell+1)^2+1}{2}$$ by substitution

$$= \frac{4K^2+4\ell^2+4\ell+1+1}{2}$$

$$= \frac{4K^2+4\ell^2+4\ell+2}{2} = 2K^2+2\ell^2+2\ell+1$$

$$= 2(K^2+\ell^2+\ell)+1$$

by basic algebra

where $K^2+\ell^2+\ell \in \mathbb{Z}$ via closure of products and sums so

$$\frac{m^2+n^2+1}{2} \in \mathbb{Z} - 2\mathbb{Z} \subset \mathbb{Z}.$$

Corollary 4.3.3

The double of a rational number is rational.

$\forall p \in \mathbb{Q}\,(2p \in \mathbb{Q})$

Proof?

Let $p \in \mathbb{Q}$. Via Theorem 4.3.2, $p+p \in \mathbb{Q}$ so $2p \in \mathbb{Q}$.

#19 For all real numbers a and b, if $a<b$ then $a < \frac{a+b}{2} < b$.

Written formally?

Proof?

#20 Given any two rational numbers r and s with $r<s$, there exists another rational number between r and s.

#28 Suppose $a,b,c,d \in \mathbb{Z}$ and $a \neq c$. Suppose also that $x \in \mathbb{R}$ satisfying the equation

$$\frac{ax+b}{cx+d} = 1.$$

Must x be rational? If so, express x as a ratio of integers.

$\boxed{4.4}$ Direct Proof and Counterexample IV

Definition

If n and d are integers then

n is divisible by d if and only if
n equals d times some integer and $d \neq 0$.

Instead of "n is divisible by d," we can say that

$$d \mid n \Leftrightarrow \exists k \in \mathbb{Z}\,(n = dk \wedge d \neq 0)$$

d divides n

d is a divisor of n

d is a factor of n

n is a multiple of d

$$(2,\infty) = \{x \in \mathbb{R} : x > 2\}$$
$$= \{x \in \mathbb{R} \mid x > 2\}$$

The notation $d \mid n$ reads "d divides n."

Symbolically, if n and d are integers:

$$d \mid n \Leftrightarrow \exists k \in \mathbb{Z}\,(n = dk \wedge d \neq 0)$$

The notation $d \nmid n$ reads "d does not divide n."

For all integers n and d

$$d \nmid n \Leftrightarrow \forall k \in \mathbb{Z}\,(n \neq dk \vee d = 0)$$
$$d \nmid n \Leftrightarrow \forall k \in \mathbb{Z}\,(n/d \neq k \vee d = 0)$$

i.e. n/d is not an integer.

example 4.4.2 Divisors of Zero

If k is any nonzero integer, does k divide 0?

Yes because $0 = k(0)$.

$\frac{0}{k} = 0$

Theorem 4.4.1

For all integers m and n, if m and n are positive and m divides n, then $m \le n$.

$\forall m \in \mathbb{Z} \, \forall n \in \mathbb{Z} \, (m > 0 \wedge n > 0 \wedge m \mid n \rightarrow m \le n)$

Property T20 of $\mathbb{R}$:

If $a < b$ and $c > 0$, then $ac < bc$.

$\forall a \in \mathbb{R} \, \forall b \in \mathbb{R} \, \forall c \in \mathbb{R} \, (a < b \wedge c > 0 \rightarrow ac < bc)$

Property T25 of $\mathbb{R}$:

If $ab > 0$ then both a and b are positive or both are negative.

$\forall a \in \mathbb{R} \, \forall b \in \mathbb{R} \, (ab > 0 \rightarrow (a > 0 \wedge b > 0) \vee (a < 0 \wedge b < 0))$

$\forall m \in \mathbb{Z} \, \forall n \in \mathbb{Z} \, (m > 0 \wedge n > 0 \wedge m \mid n \rightarrow m \le n)$

Proof:

Let $m, n \in \mathbb{Z}$. Suppose $m > 0$, $n > 0$, and $m | n$. By definition of divides, there exists $k \in \mathbb{Z}$ such that $n = km$. Via property T25 $km = n > 0$ so $k > 0$ and $m > 0$. Since $k \in \mathbb{Z}$ and $k > 0$, $k \geq 1$. Via property T20, since $m \geq 0$ and $k \geq 1$,

$$mk \geq m(1) = m$$

but $mk = n$ so $n = mk \geq m$.

Theorem 4.4.2

The only divisors of 1 are 1 and -1.

Property T12:

Rule for Multiplication with Negative Signs

$$(-a)b = a(-b) = -(ab)$$

$$(-a)(-b) = ab$$

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}.$$

$$\forall k \in \mathbb{Z}\ (k \mid 1 \rightarrow (k = 1 \lor k = -1))$$

next time