Basic solutions & extreme points

Recall LP in standard form:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad \begin{aligned} & c, x \in \mathbb{R}^n \\ & b \in \mathbb{R}^m \\ & A \in \mathbb{R}^{m \times n} \end{aligned}$$

Assume A is full rank. This is fine. If A is not full rank, we have some inconsistent or redundant constraints.

Recall the rank-nullity theorem: For $A \in \mathbb{R}^{m \times n}$,

$$\underbrace{\text{rank}(A)}_{\substack{\text{dimension} \\ \text{of } \mathbb{R}^n \text{ mapped} \\ \text{under } A}} + \underbrace{\text{nullity}(A)}_{\substack{\text{dimension} \\ \text{of the kernel} \\ \text{(set of elements in} \\ \text{domain which} \\ \text{map to } \underline{0} \in \mathbb{R}^m)}} = n$$

We assume $m \leq n$. So, $\text{rank}(A) \leq m$, hence $\text{nullity}(A) \leq n-m$.

So, we further assume $\text{rank}(A) = m$ so that $\text{nullity}(A) = n-m$.

The nullity of A gives the affine dimension of the feasible set.

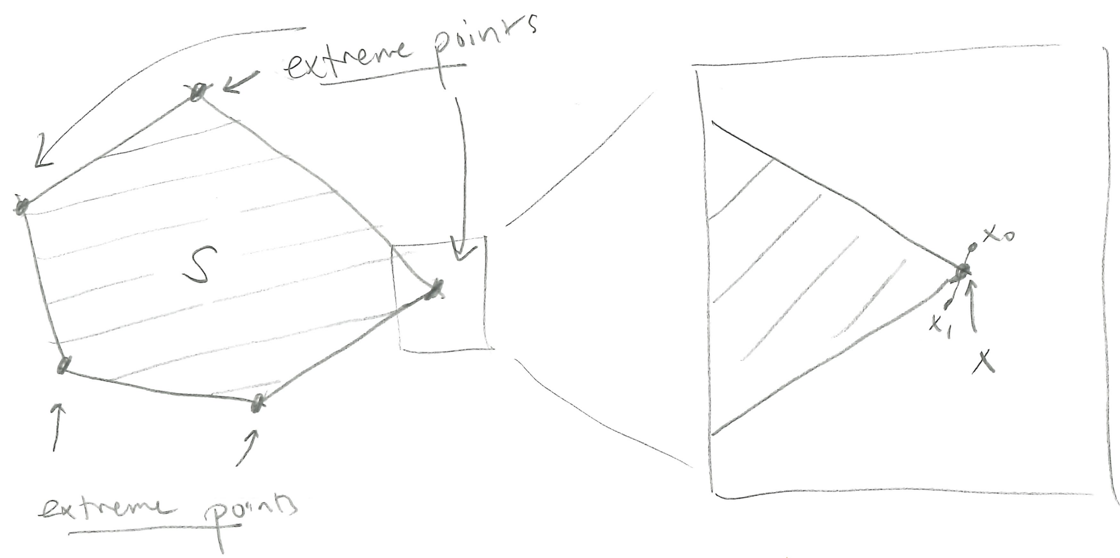
E.g. if $A \in \mathbb{R}^{1 \times 3}$, then $\text{nullity}(A) = 3-1=2$. We know from basic geometric considerations that in this case the feasible set is the subset of an affine plane.

Today, we want to relate a geometric notion of a solution to an LP and an algebraic one. These are the extreme points of an LP and an LP's basic feasible solutions.

An extreme point is actually a definition which applies to any convex set.

Def: Let S be a convex set. Then $x \in S$ is an extreme point or vertex of S if there do not exist points $x_0, x_1 \in S \setminus \{x\}$ and $\alpha \in (0,1)$ such that $x = (1-\alpha)x_0 + \alpha x_1$.

Eg:



Note: if a solution of an LP is extreme, that does not mean that it is unique.

How can we be sure the feasible set of an LP is convex?

First, recall that we can always put an LP in standard form. So, it suffices to check that a set of the form:

$$\mathcal{X} = \{ x \in \mathbb{R}^n : Ax = b \wedge x \geq 0 \}$$

is convex.

This is simple to check directly, but let's do it indirectly. We'll show: 1) A convex & B convex $\Rightarrow$ $A \cap B$ convex,

2) $Ax \leq b$ is convex.

<u>First</u>: for (1): let $x, y \in A \cap B$ and let $\lambda \in [0,1]$.

Since $x, y \in A \cap B$, we know $x, y \in A$, which is convex. Hence, $(1-\lambda)x + \lambda y \in A$. Likewise, $(1-\lambda)x + \lambda y \in B$. So, $(1-\lambda)x + \lambda y \in A \cap B$.

<u>Second</u>, for showing $\{Ax \leq b\}$ is convex. Let $x, y \in \{x \in \mathbb{R}^n : Ax \leq b\}$. So, $Ax \leq b$ and $Ay \leq b$. Let $\lambda \in [0,1]$.

Then:
$$A[(1-\lambda)x + \lambda y] = (1-\lambda)Ax + \lambda Ay$$
$$\leq (1-\lambda)b + \lambda b = (1-\lambda+\lambda)b = b.$$

Now to show that $\mathcal{X}$ is convex.

First, notice that $Ax = b$ iff $Ax \leq b$ and $Ax \geq b$. So, from (1) and (2), we can immediately see that $Ax = b$ is convex, Next, letting $A = -I$ and $b = 0$ in (2), we have that:

$$\{x \in \mathbb{R}^n : x \geq 0\} = \{x \in \mathbb{R}^n : Ax \leq b\}$$

is convex,

Finally, using (1), we have that:

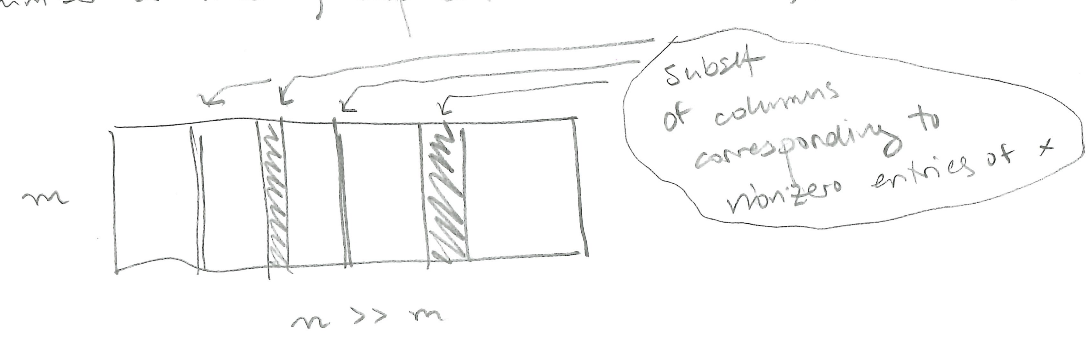
$$\mathcal{X} = \{x \in \mathbb{R}^n : Ax = b\} \cap \{x \in \mathbb{R}^n : x \geq 0\}$$

is convex.

Now let's consider the standard form LP algebraically.

Def: A point x is a basic solution of an LP if $Ax = b$ and the columns of A corresponding to the nonzero components of x are linearly independent.

Picture: A is a fat matrix since $m \leq n$. The maximum number of linearly dependent columns of A is m:

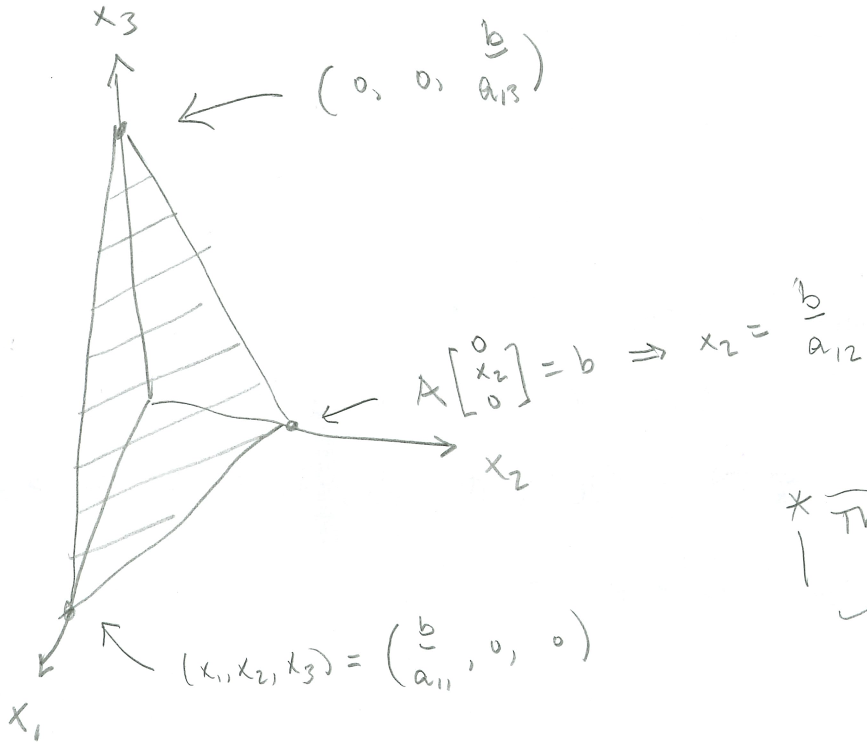


Refer to $n-m$ zero entries of x as nonbasic and the remaining variables as basic. If more than $n-m$ zeros, selection of columns can be nonunique.

Def: a point x is a basic feasible solution if it's a basic solution and $x \geq 0$.

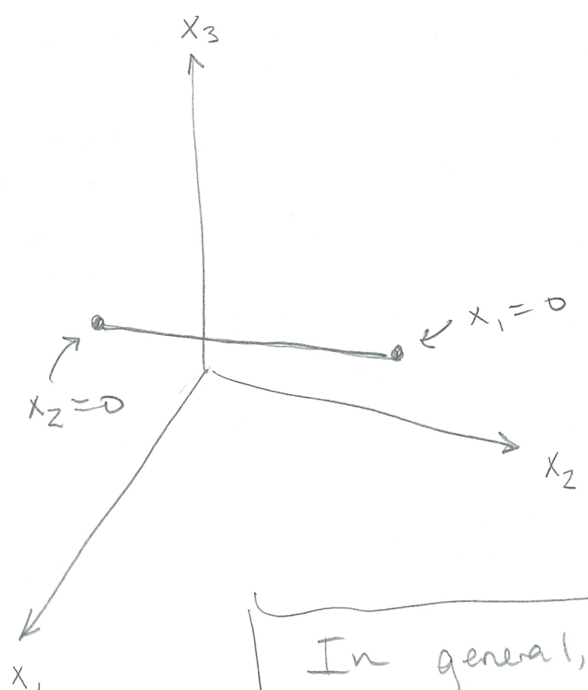
Eg: $m = 1$, $n = 3$. $A = [a_{11}\ a_{12}\ a_{13}]$.

If constraints are consistent, then this looks like:



* Three basic feasible solutions

Eg $m = 2$, $n = 3$: $\text{nullity}(A) = n - m = 1 \Rightarrow$ affine line:



* Two basic feasible solutions

In general, at most $\binom{n}{m} = \frac{n!}{(n-m)!\,m!}$ basic feasible solutions.

Theorem; a point $x \in \mathcal{X} = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ is an extreme point of $\mathcal{X}$ iff it is a basic feasible solution of the corresponding standard form LP.

Proof: ($\Leftarrow$). Let x be a basic feasible solution. Can always order x so that $x = (x_B\ x_N) = (x_B\ 0)$, where $x_B \in \mathbb{R}^m$. Let B be the corresponding subset of columns of A. We will show that x is an extreme point by contradiction.

Assume there are two other feasible points $y, z \in \mathcal{X}$ such that $y \neq x$, $z \neq x$, and $(1-\lambda)y + \lambda z = x$ for some λ st $0 < \lambda < 1$. Let's write $y = (y_B, y_N)$ and $z = (z_B, z_N)$. Note that $(1-\lambda)y_N + \lambda z_N = x_N = 0$. But because y and z are feasible, $y_N \geq 0$ and $z_N \geq 0$. Since $0 < \lambda < 1$, we have $\lambda > 0$ and $1-\lambda > 0$. Hence, the only way to get $(1-\lambda)y_N + \lambda z_N = 0$ is if $y_N = z_N = 0$. So we can actually write $y = \binom{y_B}{0}$ and $z = \binom{z_B}{0}$.

But also note that since y and z are feasible, they satisfy the equality constraints: $Ay = b$, $Az = b$. But since $y = \binom{y_B}{0}$ and $z = \binom{z_B}{0}$, this is equivalent to $By_B = Bz_B = Bx_B = b \Rightarrow x = y = z$.