Today: $\boxed{\text{Ken}}$

$\boxed{6.3}$ Disproofs & Algebraic Proofs

$\boxed{7.1}$ Functions defined on General Sets

Last time:

$\boxed{6.2}$ Properties of Sets

$\boxed{6.3}$ Disproofs & Algebraic Proofs

## Theorem 6.3.1

For every integer $n \geq 0$, if a set $X$ has $n$ elements, then $\mathcal{P}(X)$ has $2^n$ elements.

Proof:

$P(n)$:

If $N(X) = n$, then $N(\mathcal{P}(X)) = 2^n = 2^{N(X)}$.

① Base case, $P(0)$:

$N(\emptyset) = 0$

$\mathcal{P}(\emptyset) = \{\emptyset\} = \{\{\}\}$

$N(\mathcal{P}(\emptyset)) = 2^0 = 1$

② Suppose $P(k)$ for $k \geq 0$.

If $N(X) = k$, then $N(\mathcal{P}(X)) = 2^k$.

Let $S$ be a set such that $N(S) = k+1$. Since $k \geq 0$, $k+1 \geq 1$, so $S \neq \emptyset$ and let $z \in S$.

Consider $S = (S - \{z\}) \cup \{z\}$.

Since $N(S - \{z\}) = k+1-1 = k$,

$N(\mathcal{P}(S - \{z\})) = 2^k$ by induction hypothesis.

Define

$$\mathcal{U} = \{V \subset S : \forall W \in \mathcal{P}(S - \{z\})(V = W \cup \{z\})\}$$
$$= \{V \subset S : \forall W \subset (S - \{z\})(V = W \cup \{z\})\}$$

so that $\mathcal{P}(S) = \mathcal{U} \cup \mathcal{P}(S - \{z\})$.

Suppose $\mathcal{U} \cap \mathcal{P}(S - \{z\}) \neq \emptyset$.

Then, there exists a set

$Z \in \mathcal{U} \cap \mathcal{P}(S - \{z\})$ such that

$Z \in \mathcal{U}$ and $Z \in \mathcal{P}(S - \{z\})$.

Since $Z \in \mathcal{U}$, there exists $W \subset S - \{z\}$

such that $Z = W \cup \{z\}$ and $z \in Z$.

Since $Z \in \mathcal{P}(S - \{z\})$, $Z \subset S - \{z\}$,

$z \notin Z$. Thus $z \in Z$ and $z \notin Z$, a contradiction. Then $\mathcal{U} \cap \mathcal{P}(S-\{z\}) = \emptyset$. Since $\mathcal{U}$ defines a set for all $A \in \mathcal{P}(S-\{z\})$ and all $A \subset S-\{z\}$, $N(\mathcal{U}) = 2^k$. Also $N(\mathcal{P}(S-\{z\})) = 2^k$ and $N(\mathcal{U} \cap \mathcal{P}(S-\{z\})) = N(\emptyset) = 0$, so

$$
\begin{aligned}
N(\mathcal{P}(S)) &= N(\mathcal{U} \cup \mathcal{P}(S-\{z\})) \\
&= N(\mathcal{U}) + N(\mathcal{P}(S-\{z\})) \\
&= 2^k + 2^k = 2(2^k) \\
&= 2^{k+1}.
\end{aligned}
$$

$\square$

$S = \{1, 2\}$

$\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$S - \{2\} = \{1\}$

$\mathcal{P}(S - \{2\}) = \mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$ 2 elements

$\mathcal{U} = \{\emptyset \cup \{2\}, \{1\} \cup \{2\}\}$

$= \{\{2\}, \{1,2\}\}$ 2 elements

$N(\mathcal{U}) = 2$

$N(\mathcal{P}(S - \{2\})) = 2$

$\mathcal{U} \cap \mathcal{P}(S - \{2\}) = \emptyset$

$$N(\mathcal{U} \cup \mathcal{P}(S - \{2\})) = N(\mathcal{U}) + N(\mathcal{P}(S - \{2\}))$$

$$= 2^1 + 2^1$$

$$= 2(2^1) = 2^{1+1}$$

$$= N(\mathcal{P}(S))$$

The Cartesian product of two sets $A, B$ is defined as

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}.$$

If $N(A) = m$ and $N(B) = n$, then $N(A \times B) = mn$.

---

$A = \{1, 2, 3\}$

$B = \{x, y\}$

$N(A) = 3$
$N(B) = 2$
$N(A \times B) = 3(2) = 6$

$A \times B = \{(1,x), (2,x), (3,x), (1,y), (2,y), (3,y)\}$

---

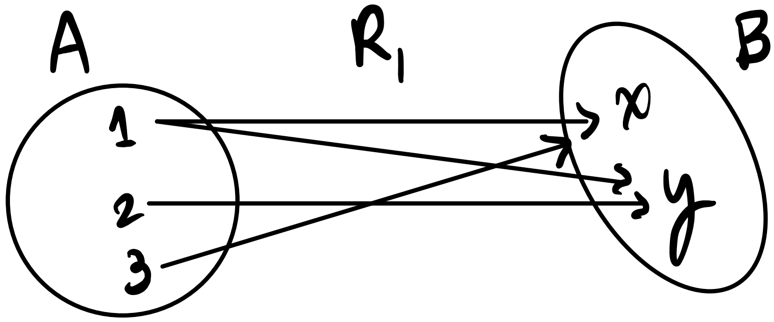
Given sets $A, B$, a relation $R$ is a subset of $A \times B$, i.e. $R \subset A \times B$.

---

Define $R_1 \subset A \times B$ such that

$R_1 = \{(1,x), (3,x), (1,y), (2,y)\}$.

We can draw the correspondence




We also write $1 R_1 x$ to indicate $(1, x) \in R_1$ and $2 \not{R_1} x$ because $(2, x) \notin R_1$. $R_1$ is not a function.

## Definition

Let $A, B$ be sets. A relation $R$ from $A$ to $B$ is a subset of $A \times B$. Given an ordered pair $(x,y) \in A \times B$, $x$ is related to $y$ by $R$, written $x R y$ if and only if $(x,y) \in R$. The set $A$ is called the domain of $R$ and the set $B$ is called the codomain of $R$.

$$x R y \text{ means } (x,y) \in R$$

$$x \not{R} y \text{ means } (x,y) \notin R$$

Another word for $R$ as described here is a binary relation.

## 7.1 Functions defined on General Sets

**Definition**

A function $f$ from a set $X$ to a set $Y$ written $f: X \to Y$ is a relation from $X$, the domain of $f$, to $Y$, the codomain of $f$, that satisfies two properties:

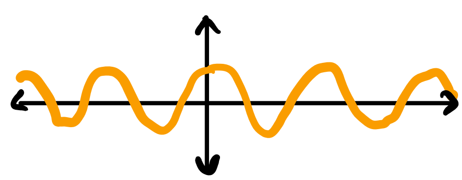
1. every element in $X$ is related to some element in $Y$
2. no single element in $X$ is related to more than one element in $Y$.

So, for any $x \in X$, there is a unique element $y \in Y$ that is related to $x$ via $f$.

We say that "$f$ sends $x$ to $y$" or "$f$ maps $x$ to $y$" and write $x \xrightarrow{f} y$ or $f: x \to y$. If the function $f$ is unspecified in name, we might also write $x \mapsto y$ to mean "$x$ maps to $y$" (by a function). The unique element to which $f$ sends $x$ is denoted $f(x)$ and is called

1. $f$ of $x$
2. the output of $f$ for input $x$
3. the value of $f$ at $x$
4. the image of $x$ under $f$.

$\cos : \mathbb{R} \to \mathbb{R}$



$\cos(\mathbb{R}) = [-1, 1] \subsetneq \mathbb{R}$ image of $\mathbb{R}$ under cos

The set of all values of $f$ altogether is called the range of $f$ or the image of $X$ under $f$ and is

range of $f$ = image of $X$ under $f$

$$= \{ y \in \mathcal{Y} : y = f(x) \text{ for some } x \in X \}.$$

Some mathematicians denote this set $R(f)$, $\mathrm{im}(f)$, or $f(X)$, the last notation not to be confused with the notation $f(x)$.

Given an element $y \in \mathcal{Y}$, there may exist many elements in $X$ with $y$ as their image. When $x \in X$ such that $f(x) = y$, then $x$ is called a preimage of $y$ or an inverse image of $y$. The set of all inverse images of $y$ is called the inverse image of $y$,

denoted

the inverse image of $y = \{x \in X : f(x) = y\}$.

---

<u>Definition</u>

A function $f$ from a set $A$ to a set $B$ is a set such that

① $\forall x \in A \; \exists y \in B \; ((x,y) \in f)$

where $f \subset A \times B$

② $\forall x \in A \; \forall y, z \in B \; ((x,y) \in f \wedge (x,z) \in f \Rightarrow y = z)$

---

e.g. $f : \mathbb{R} \to \mathbb{R}$

$x \longmapsto x^2$

$f(\mathbb{R}) = [0, \infty)$

## Definition

If $f : X \to Y$ is a function and $A \subset X$ and $B \subset Y$ then

$$f(A) = \{y \in Y : \exists x \in A \, (y = f(x)\}$$

and

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

$f(A)$ is called the image of $A$ and $f^{-1}(B)$ is called the inverse image of $B$.