Today: $\boxed{\text{Ken}}$

$\boxed{7.1}$ Functions defined on General Sets

$\boxed{7.2}$ One-to-one, Onto, and Inverse Functions

Last time:

$\boxed{6.3}$ Disproofs & Algebraic Proofs

$\boxed{7.1}$ Functions defined on General Sets

Definition

A function f from a set A to a set B is a set such that

① $\forall x \in A \; \exists y \in B \; ((x,y) \in f)$

where $f \subset A \times B$

② $\forall x \in A \; \forall y, z \in B \; ((x,y) \in f \wedge (x,z) \in f \Rightarrow y = z)$

① f is defined for all $x \in A$

② If $f(x) = y$ and $f(x) = z$ then $y = z$.

Definition

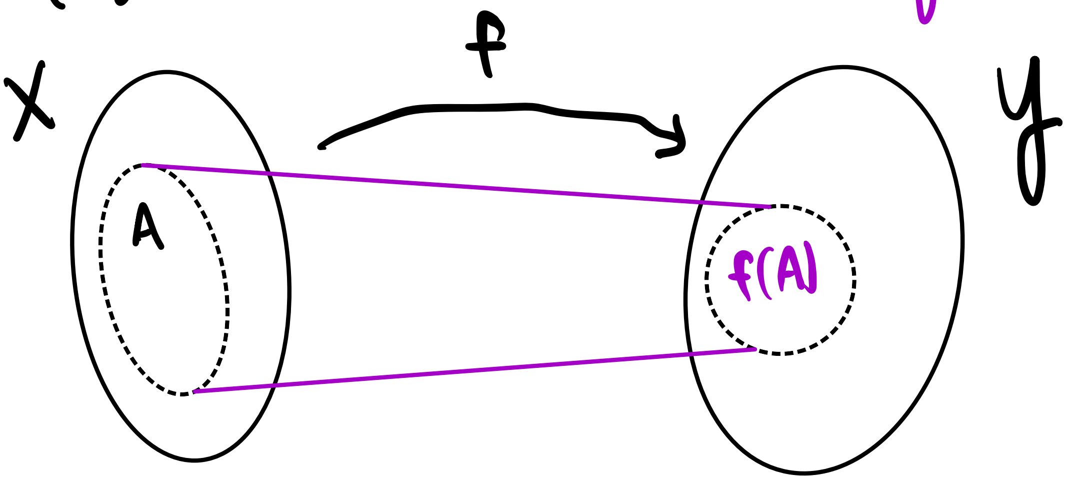
If $f: X \to Y$ is a function and $A \subset X$ and $B \subset Y$ then

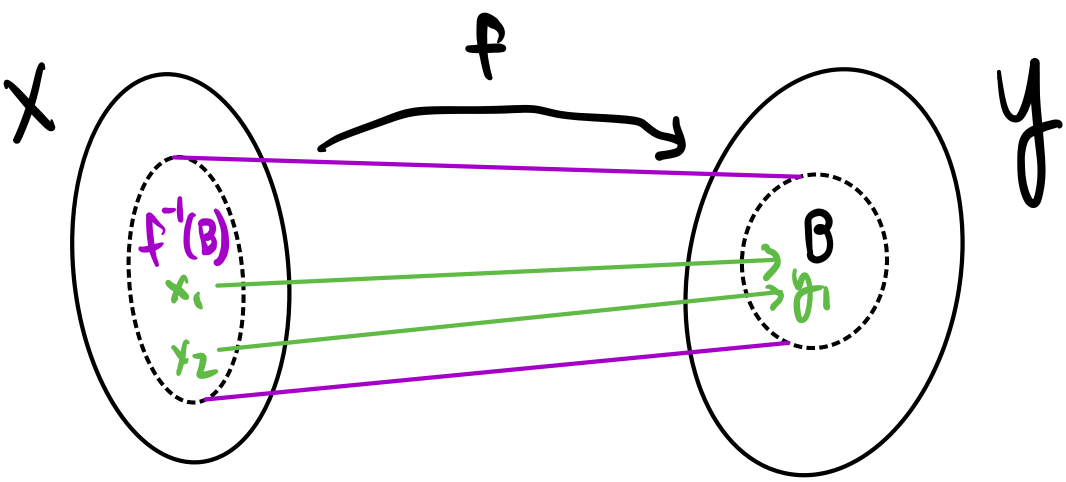
$$f(A) = \{y \in Y : \exists x \in A (y = f(x)\}$$

and

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

$f(A)$ is called the **image of A** and $f^{-1}(B)$ is called the **inverse image of B.**





$f(x_1) = y_1$ $\quad$ $f^{-1}(\{y_1\}) = \{x_1, x_2\}$

$f(x_2) = y_1$

e.g. Let $f: S \to T$ and $U, V \subset S$.

Prove that $f(U \cap V) \subset f(U) \cap f(V)$.

Proof:

Let $a \in f(U \cap V)$.

$f(U \cap V) = \{y \in T : \exists x \in U \cap V\, (f(x) = y)\}$

So $a = f(b)$ for some $b \in U \cap V$.

By def. of intersection, $b \in U$ and $b \in V$.

$f(U) = \{y \in T : \exists x \in U\, (f(x) = y)\}$

Since $b \in U$ by specialization,

$f(b) = a \in f(U)$.

$f(V) = \{y \in T : \exists x \in V\, (f(x) = y)\}$

Since $b \in V$ by specialization,

$f(b) = a \in f(V)$.

So $a \in f(U) \cap f(V)$.

$f(U \cap V) \subset f(U) \cap f(V)$.

Prove that $f(U \cap V) \neq f(U) \cap f(V)$ in general.

A counterexample:

$S = \{1, 2, 3, 4\}$

$U = \{1, 2, 3\}$

$V = \{2, 3, 4\}$

$U \cap V = \{2, 3\}$

$T = \{a, b, c, d\}$

$f: S \to T$

$f(1) = a$

$f(2) = b$

$f(3) = c$

$f(4) = a$

$f(U) = \{a, b, c\}$

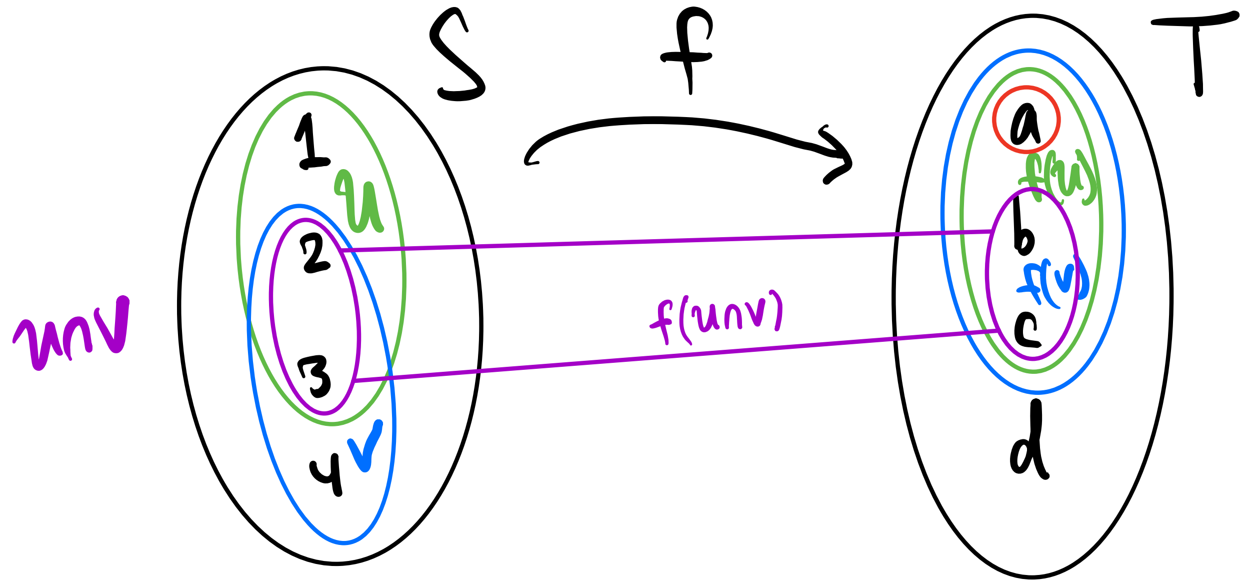
$f(V) = \{a, b, c\}$

$f(U \cap V) = \{b, c\}$

$f(U) \cap f(V) = \{a, b, c\}$

$f(U) \cap f(V) \not\subset f(U \cap V)$

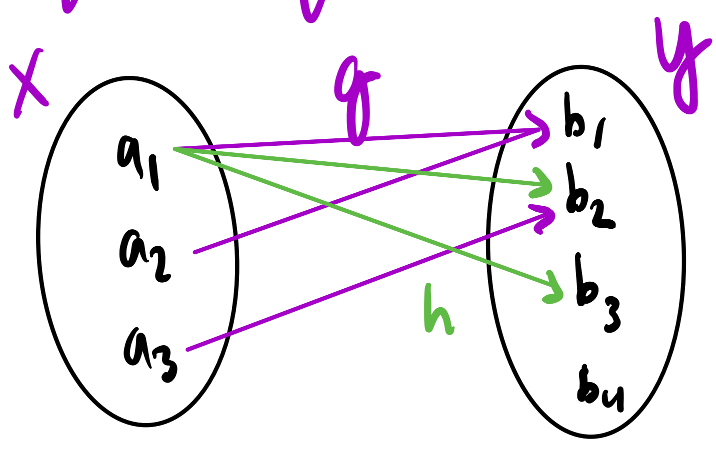
because $a \notin f(U \cap V)$



e.g. Arrow Diagrams

examples of functions and nonfunctions

$g : X \to Y$



$g = \{(a_1, b_1), (a_2, b_1), (a_3, b_2)\} \subset X \times Y$

is a function

$h = \{(a_1, b_2), (a_1, b_3)\} \subset X \times Y$

fails both properties ① & ②

① fail: a_1 (domain element) maps to distinct values

② fails: h is not defined for all elements in X

h is NOT well-defined

$$g^{-1}(\{b_1\}) = \{a_1, a_2\}$$

$$g^{-1}(\{b_1, b_2\}) = \{a_1, a_2, a_3\} = X$$

$$g^{-1}(\{b_4\}) = \emptyset$$

Theorem 7.1.1

If $f: X \to Y$ and $g: X \to Y$ are functions, then $f = g$ if and only if $f(x) = g(x)$ for every $x \in X$.

$f = g$ if ① same domain
② same codomain
③ they agree on every element

Proof:

Let $f: X \to Y$ and $g: X \to Y$ be functions.

Suppose $f = g$ (as subsets of $X \times Y$).

$$f(x) = y \iff (x,y) \in f \underset{f=g}{\iff} (x,y) \in g \iff g(x) = y.$$

So $f(x) = g(x)$ for any $x \in X$.

Suppose $f(x) = g(x)$ for any $x \in X$.

$$(x, y) \in f \iff y = f(x) \underset{f(x)=g(x)}{\iff} y = g(x) \iff (x, y) \in g.$$

So $f = g$.

A "function" is **not well-defined** if it fails to satisfy at least one of the requirements to be a function.

#34 Suppose $f : \mathbb{Q} \to \mathbb{Q}$ such that, if $q \in \mathbb{Q}$ such that $q = m/n$ for some $m, n \in \mathbb{Z}$ where $n \neq 0$, $f(m/n) = \frac{m^2}{n}$. Is f well-defined?

$f(2/3) = 4/3$

$f(2/3) = f(4/6) = 16/6 = 8/3 \neq 4/3$

NOT WELL-DEFINED

e.g. let $\mathbb{Z}_5 = \{0,1,2,3,4\}$. Define

$f: \mathbb{Z}_5 \to \mathbb{Z}_5$ and $g: \mathbb{Z}_5 \to \mathbb{Z}_5$

such that

$$f(x) = (3x+1)^2 \bmod 5$$
$$g(x) = (4x^2+x+1) \bmod 5$$

Is $f = g$?

$f(0) = 1 \bmod 5 = 1 = g(0) = 1 \bmod 5$

$f(1) = 16 \bmod 5 = 1 = g(1) = 6 \bmod 5$

$f(2) = 49 \bmod 5 = 4 = g(2) = 19 \bmod 5$

$f(3) = 100 \bmod 5 = 0 = g(3) = 40 \bmod 5$

$f(4) = 169 \bmod 5 = 4 = g(4) = 69 \bmod 5$

$f = g$

$$f(x) = (9x^2+6x+1) \bmod 5$$

$$g(x) = (4x^2+x+1) \bmod 5$$

e.g. For any set X, define $id_X : X \longrightarrow X$ such that $\forall x \in X\,(id_X(x) = x)$. Then id_X is called the identity function on X.

e.g. Suppose $X = \mathbb{Z}$.

(a) Is it true that

$$\forall x, y \in \mathbb{Z}\,(id_{\mathbb{Z}}(x+y) = id_{\mathbb{Z}}(x) + id_{\mathbb{Z}}(y))?$$

Let $x, y \in \mathbb{Z}$.

$id_{\mathbb{Z}}(x+y) = x+y = id_{\mathbb{Z}}(x) + id_{\mathbb{Z}}(y)$

(b) Is it true that

$$\forall x, y \in \mathbb{Z}\,(id_{\mathbb{Z}}(xy) = id_{\mathbb{Z}}(x)\, id_{\mathbb{Z}}(y))?$$

Let $x, y \in \mathbb{Z}$.

$id_{\mathbb{Z}}(xy) = xy = id_{\mathbb{Z}}(x)\, id_{\mathbb{Z}}(y)$

(c) Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is any function with the property $\forall x, y \in \mathbb{Z}\, (f(x+y) = f(x) + f(y))$.

Why is $f(0) = 0$?

$$f(0) = f(0+0) = f(0) + f(0)$$

$$f(0) = f(0) + f(0)$$

$$-f(0) \qquad -f(0)$$

$$0 = f(0)$$

e.g. Suppose $A = \{a_1, a_2, a_3\}$. Define $f : \mathcal{P}(A) \to \mathbb{Z}^{\text{nonneg}}$ such that

$f(X) = N(X)$. What's the arrow diagram for f?

Definition

The formal definition of a sequence of real numbers or $\{a_n\}_{n=1}^{\infty}$ is $f : \mathbb{Z}^+ \to \mathbb{R}$ such that, for any

$n \in \mathbb{Z}^+$, $f(n) = a_n$.

"OEIS"

Definition

The logarithmic function base a is defined $\log_a : \mathbb{R}^+ \to \mathbb{R}$ such that ...

(review)

Definition

Given a set S, a string over S can be regarded as a finite sequence of elements of S. The number of characters in a string is called the length of the string and the null string over S is the string with no characters, denoted λ, with length 0.

Definition

An (n-place) Boolean function f is a function whose domain is the set of all ordered n-tuples of 0's and 1's whose codomain is the set $\{0,1\}$. I.e.

$f: \{0,1\}^n \to \{0,1\}$ where $\{0,1\}^n$ is the Cartesian product

$\underbrace{\{0,1\} \times \cdots \times \{0,1\}}_{n \text{ times}}$.

e.g. Consider $f: \{0,1\}^3 \to \{0,1\}$ such that $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3) \bmod 2$. What's the table of all possible input and output values of f?

$f(0,0,0) = 0$

$f(1,0,0) = 1$

$f(0,1,0) = 1$

$f(0,0,1) = 1$

$f(1,1,0) = 0$

$f(0,1,1) = 0$

$f(1,0,1) = 0$

$f(1,1,1) = 1$

$\boxed{7.2}$ One-to-One, Onto, and Inverse Functions

Definition

A function $f: X \to Y$ is **one-to-one** or **injective** if and only if, for any $x_1, x_2 \in X$,

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2$$

or logically equivalently

$$x_1 \neq x_2 \text{ implies } f(x_1) \neq f(x_2).$$

Otherwise $f: X \to Y$ is not one-to-one if and only if there exist $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$.

$\sin: \mathbb{R} \to [-1,1]$

$\sin(\pi) = 0 = \sin(0)$ but $\pi \neq 0$

So sine is not one-to-one on $\mathbb{R}$.