# Algebra Midterm Version A

1. How many $\sigma \in S_5$ are there such that $\sigma(1)=2$, $\sigma(2)=1$ and $\sigma$ is conjugate to $(1\,5)(2\,3)$?

$\sigma$ is conjugate to $(1\,5)(2\,3)$, so it has the same cycle type as $(1\,5)(2\,3)$.

$\sigma(1)=2$, $\sigma(2)=1$, so $\sigma=(1\,2)\tau$, where $\tau$ is a 2-cycle disjoint from $(1\,2)$.

The possibilities are $(1\,2)(3\,4)$, $(1\,2)(3\,5)$, $(1\,2)(4\,5)$

There are 3 possibilities for $\sigma$.

2. $G$ and $G'$ are groups. $f: G \to G'$ is a map. Define the set $H=\{(g,g')\in G\times G' \mid g'=f(g)\}$. Prove $H$ is a subgroup of $G\times G'$ if and only if $f$ is a homomorphism.

If ~~f~~ f is a homomorphism, for any $(g_1, f(g_1))\in H$ and $(g_2, f(g_2))\in H$.

$(g_1, f(g_1))^{-1}(g_2, f(g_2)) = (g_1^{-1}, f(g_1)^{-1})(g_2, f(g_2)) = (g_1^{-1}g_2, f(g_1^{-1}g_2))\in H$.

so $H$ is a subgroup of $G\times G'$.

Conversely, if $H$ is a subgroup of $G\times G'$, for any $g_1, g_2\in G$.

$(g_1, f(g_1))\cdot(g_2, f(g_2)) = (g_1g_2, f(g_1)f(g_2))\in H$

so $f(g_1)f(g_2)=f(g_1g_2)$, $f$ is a homomorphism.

3. How many subgroups of $\mathbb{Z}/14\mathbb{Z}$ are there?

$\mathbb{Z}/14\mathbb{Z}$ is cyclic, so any subgroup of it is cyclic.

$\langle \bar{0} \rangle = \{\bar{0}\}$.

$\langle \bar{1} \rangle = \langle \bar{3} \rangle = \langle \bar{5} \rangle = \langle \bar{9} \rangle = \langle \bar{11} \rangle = \langle \bar{13} \rangle = \mathbb{Z}/14\mathbb{Z}$. since these generators are units of $\mathbb{Z}/14\mathbb{Z}$.

$\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}, \bar{12}\}$ is a cyclic group of order 7.

so each non-identity element is a generator of this subgroup.

i.e., $\langle \bar{2} \rangle = \langle \bar{4} \rangle = \langle \bar{6} \rangle = \langle \bar{8} \rangle = \langle \bar{10} \rangle = \langle \bar{12} \rangle$

The only remaining case is

$\langle \bar{7} \rangle = \{\bar{0}, \bar{7}\}$.

In total there are 4 subgroups of $\mathbb{Z}/14\mathbb{Z}$.

4. What is the order of the group $\mathrm{Aut}(\mathrm{Aut}(\mathbb{Z}/10\mathbb{Z}))$?

$\mathrm{Aut}(\mathbb{Z}/10\mathbb{Z}) \cong (\mathbb{Z}/10\mathbb{Z})^\times = \{\bar{1}, \bar{3}, \bar{7}, \bar{9}\}$ is a cyclic group of order 4.

so $\mathrm{Aut}(\mathbb{Z}/10\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$

$\mathrm{Aut}(\mathrm{Aut}(\mathbb{Z}/10\mathbb{Z})) \cong \mathrm{Aut}(\mathbb{Z}/4\mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z})^\times = \{\bar{1}, \bar{3}\}$.

so $|\mathrm{Aut}(\mathrm{Aut}(\mathbb{Z}/10\mathbb{Z}))| = 2$.

5. $G$ is a group. If $|g|=2$ for any non-identity $g\in G$, prove that $G$ is abelian.

Since $\forall g\in G, g\neq 1, |g|=2$, we see $g^2=1$, $g=g^{-1}$.

Also, $1^{-1}=1$ so $g^{-1}=g$ for any $g\in G$.

For any $a,b\in G$, $ab=a^{-1}b^{-1}=(ba)^{-1}=ba$. so $G$ is abelian.

6. $\mathbb{R}$ is the group of real numbers with addition. $\mathbb{Q}$ is the subgroup of rational numbers. $r$ is a nonzero real number, denote $r\mathbb{Q}=\{rq\in\mathbb{R} \mid q\in\mathbb{Q}\}$.

Prove $\mathbb{R}/r\mathbb{Q}\cong\mathbb{R}/\mathbb{Q}$.

Define $f: \mathbb{R}\longrightarrow \mathbb{R}/r\mathbb{Q}$ by $f(x)=rx+r\mathbb{Q}$.

$f(x+y)=r(x+y)+r\mathbb{Q}=(rx+r\mathbb{Q})+(ry+r\mathbb{Q})=f(x)+f(y)$

so $f$ is a homomorphism

$f$ is surjective since for any $x+r\mathbb{Q}$, $x+r\mathbb{Q}=r\cdot\frac{x}{r}+r\mathbb{Q}=f(\frac{x}{r})$.

$\operatorname{Ker}(f)=\{x\in\mathbb{R} \mid f(x)=0+r\mathbb{Q}\}=\{x\in\mathbb{R} \mid rx\in r\mathbb{Q}\}=\{x\in\mathbb{R} \mid x\in\mathbb{Q}\}=\mathbb{Q}$.

By First Isomorphism Theorem, $\mathbb{R}/\mathbb{Q}\cong\mathbb{R}/r\mathbb{Q}$.

7. $f: G \to G'$ is a homomorphism. $K = \ker(f)$. $H$ is a subgroup of $G$, and $H' = \{f(h) \in G' \mid h \in H\}$, $M = \{g \in G \mid f(g) \in H'\}$. Prove that $M = HK$.

For any $h \in H$, $k \in K$, $f(hk) = f(h)f(k) = f(h) \in H'$. so $hk \in M$
we get $HK \subseteq M$

For any $m \in M$. $f(m) \in H'$, so $\exists\, h \in H$ such that $f(m) = f(h)$.
$f(h^{-1}m) = 1$. so $h^{-1}m \in K$, $m \in hK \subseteq HK$
we get $M \subseteq HK$