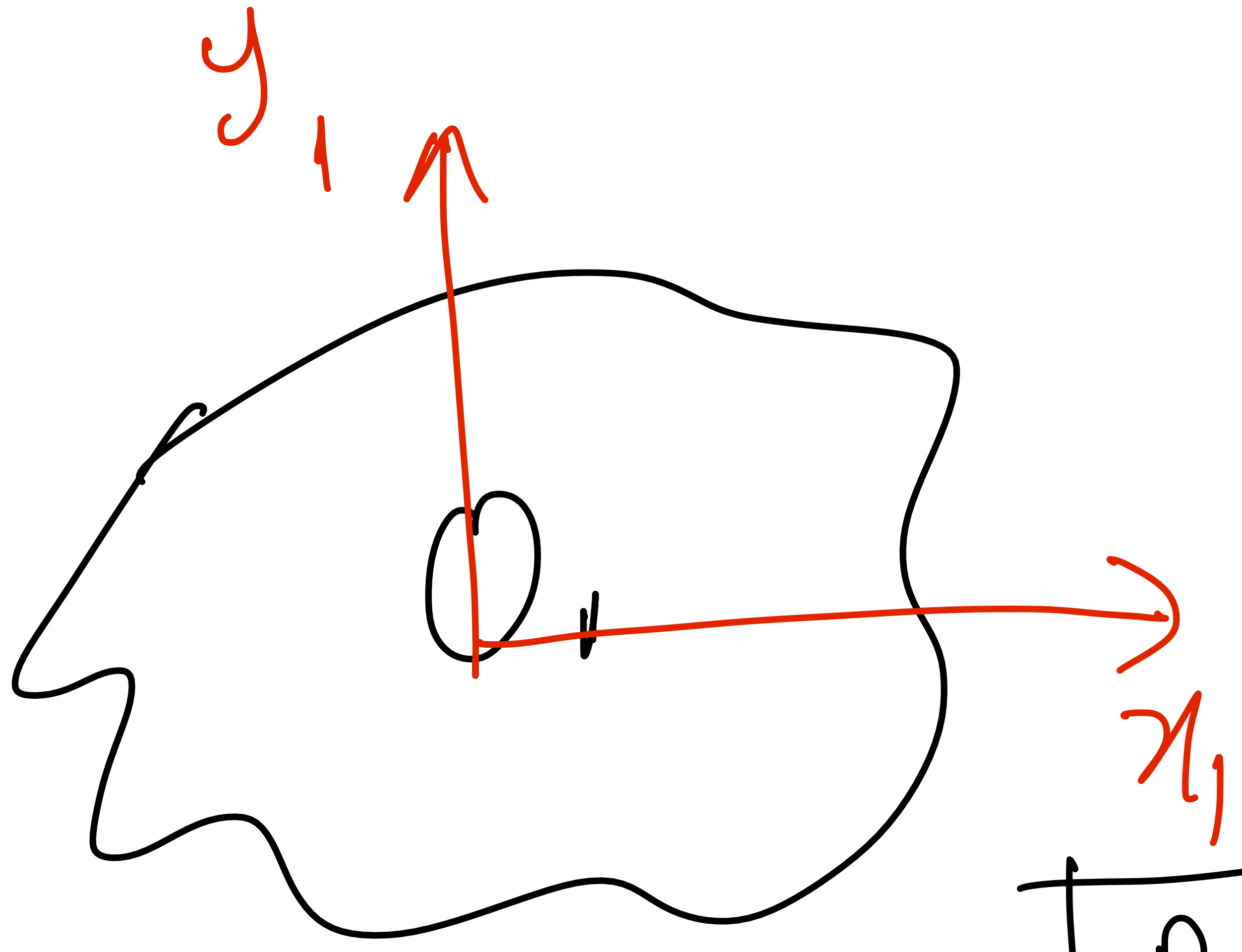
Introduction to Robot Intelligence
[Spring 2023]

Forward Kinematics

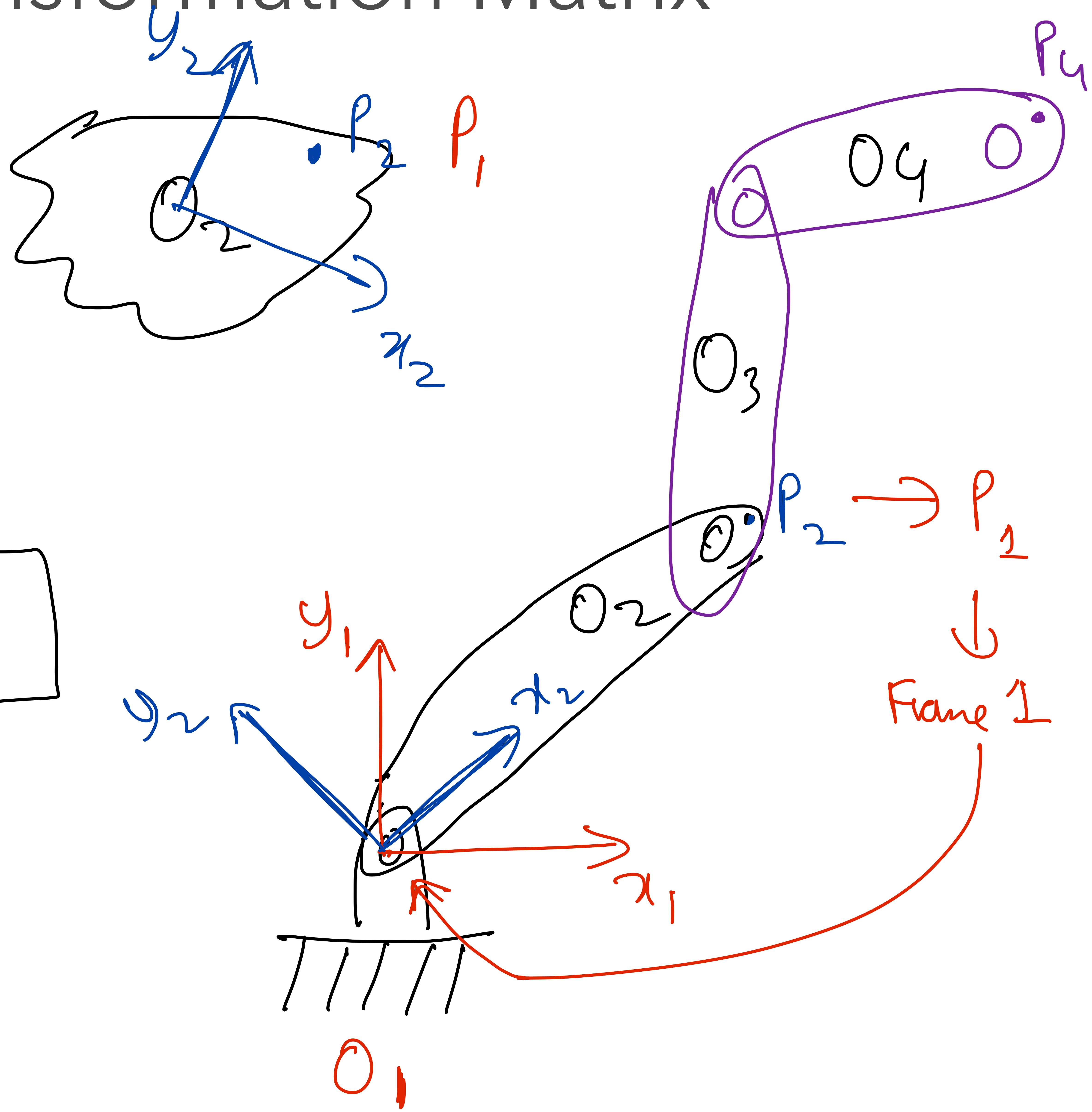
February 23, 2023

Lerrel Pinto

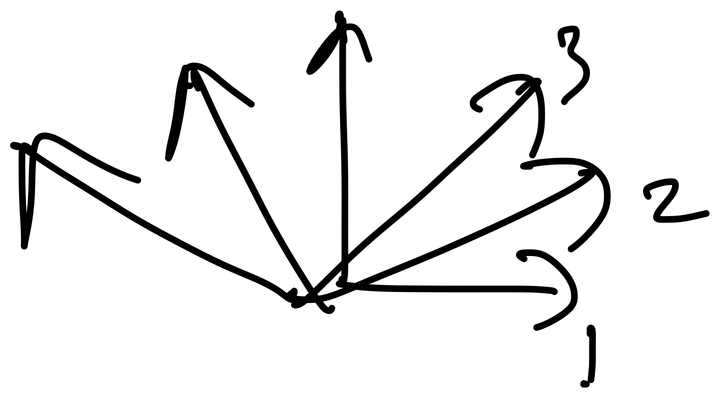
3D Homogenous Transformation Matrix



$$\boxed{P_1 = H\, P_2}$$



Properties of H



Inverse

$$\begin{bmatrix} - \\ - \\ - \\ 1 \end{bmatrix} H \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} R_{3\times3} & t_{3\times1} \\ 0 \; 0 \; 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

4×4

$$R^{-1} = R^T$$

$$R^{-1}$$

$$H^{-1} ?= \begin{bmatrix} R^T & -R^T t \\ 0 \; 0 \; 0 & 1 \end{bmatrix}$$

$$H^T$$

$$\begin{bmatrix} R^T & 0 \\ & 0 \\ & 0 \\ t^T & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$(t_x x + t_y y + t_z z + 1) \neq 1$$

$$H^{-1} \neq H^T$$

$$P_1 = H P_2$$

$$P_2 = H^{-1} P_1$$

Composition

$$R_3 = R_1 R_2$$

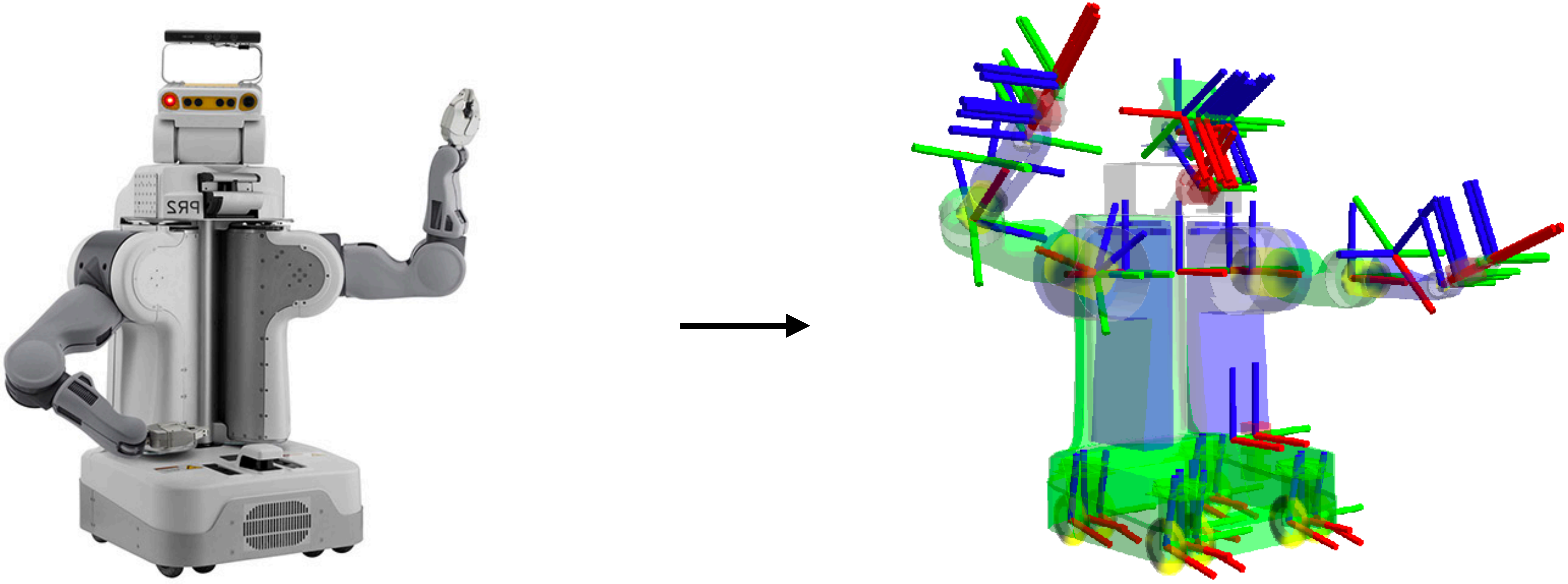
$$H_3 = H_1 H_2$$

$$H_3 = \begin{bmatrix} R_1 & t_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & t_2 \\ 0 & 1 \end{bmatrix}$$

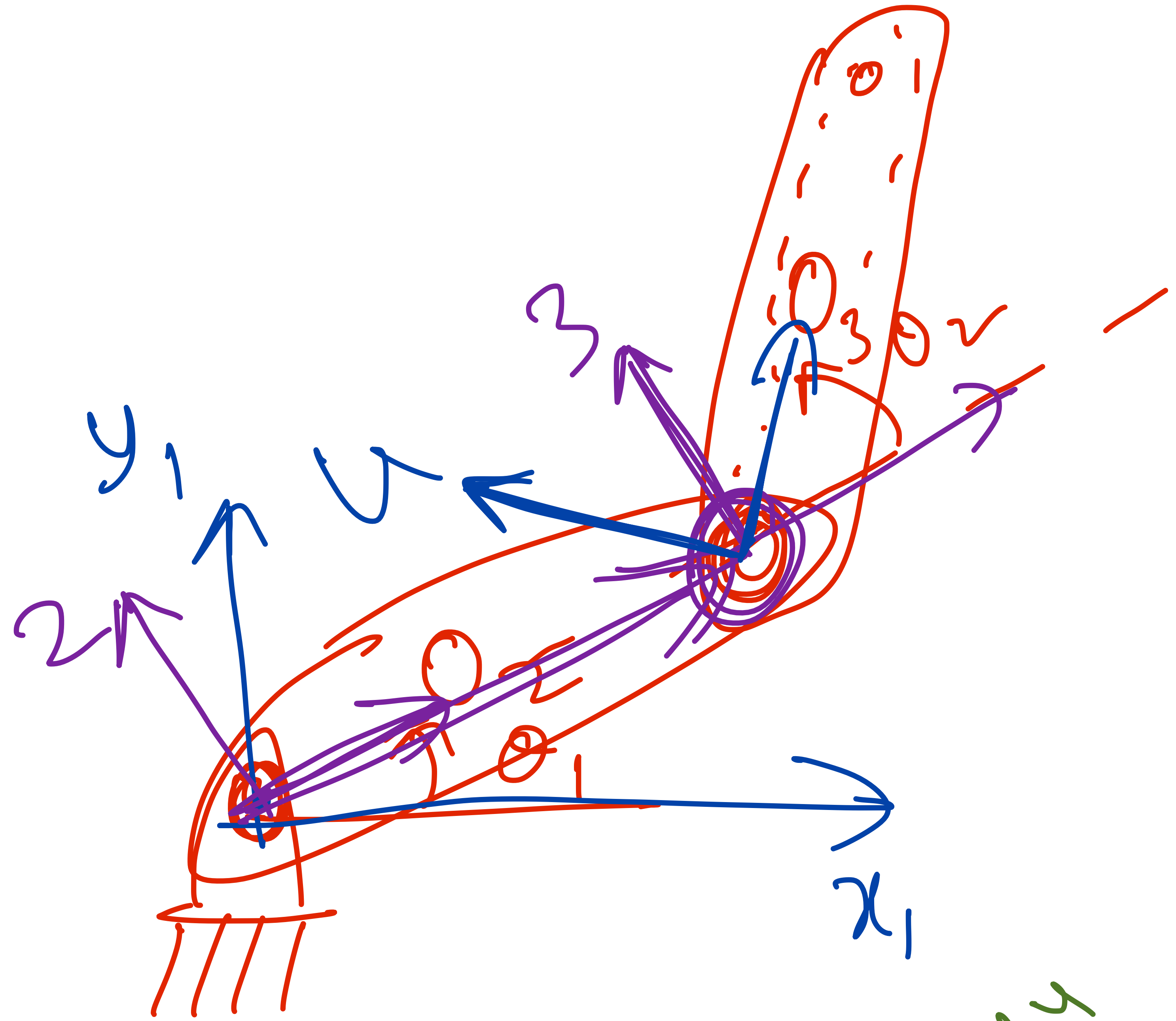
$$H_3 = \begin{bmatrix} R_1 R_2 & R_1 t_2 + t_1 \\ 0 & 1 \end{bmatrix}$$

$$R_3 = \boxed{R_1 R_2} \quad t_3 = R_1 t_2 + t_1$$

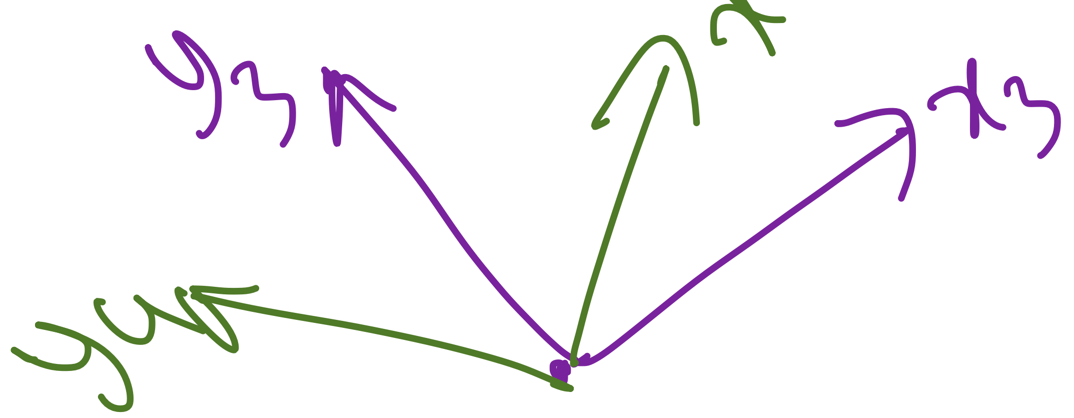
Robot as a collection of rigid bodies + transformations!



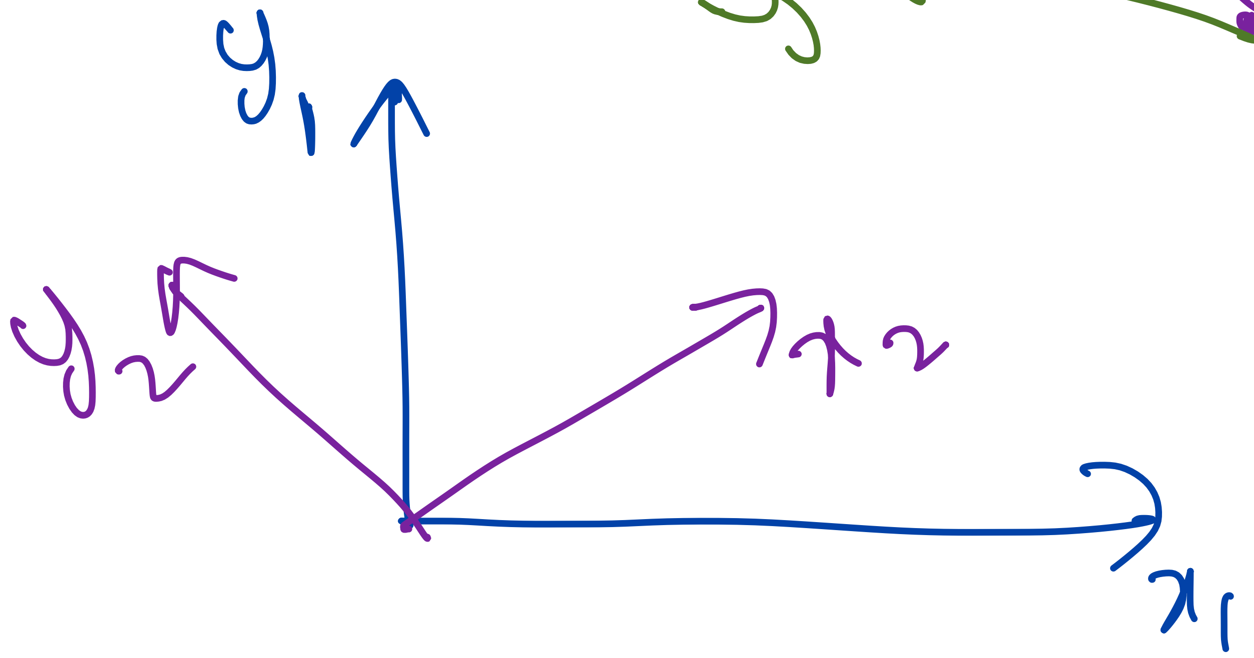
Modelling a robot



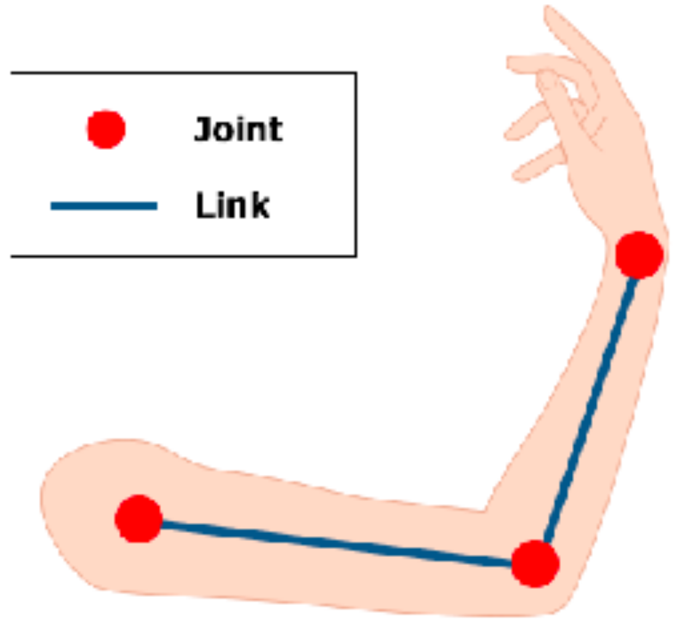
$$H = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

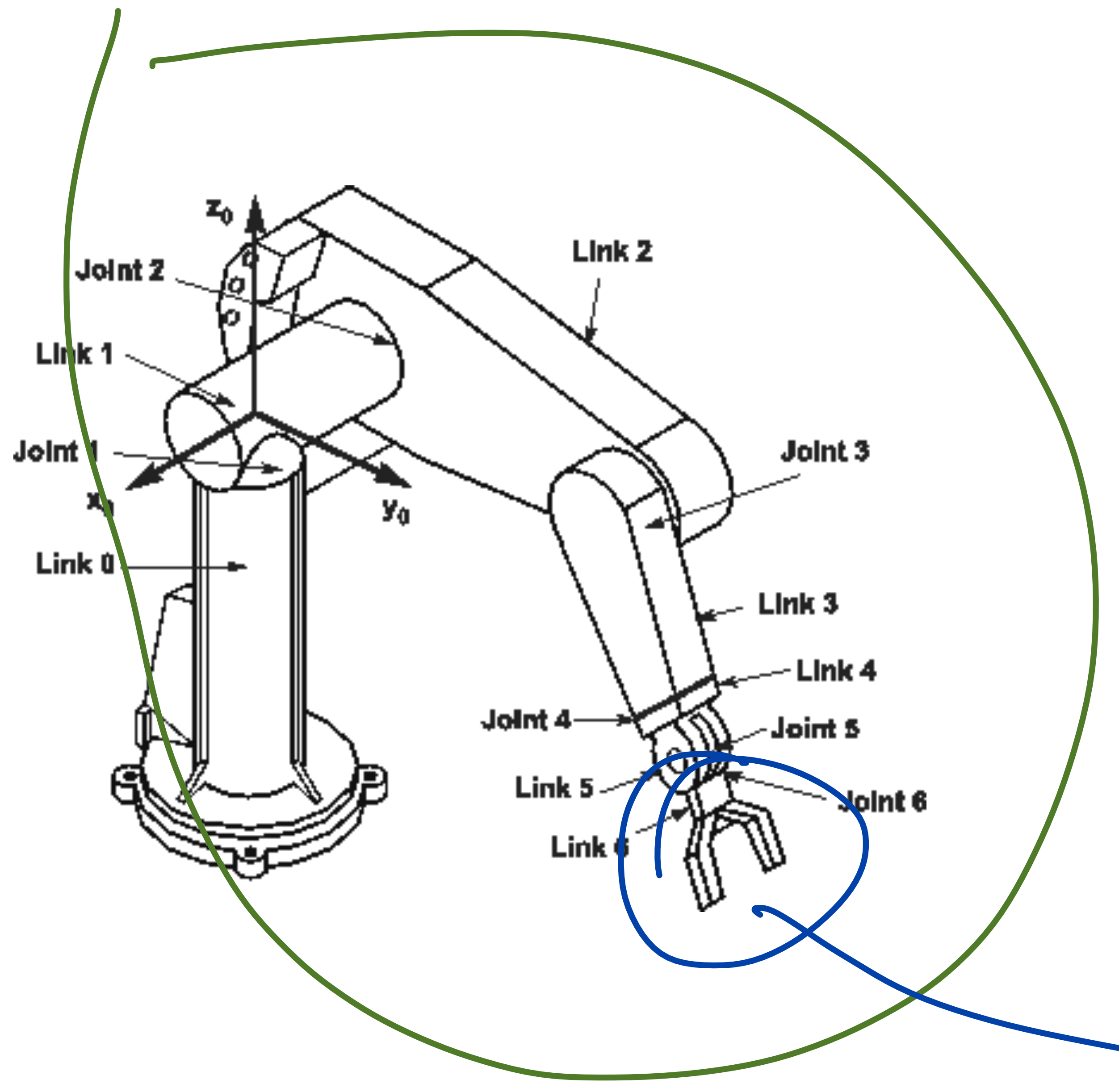


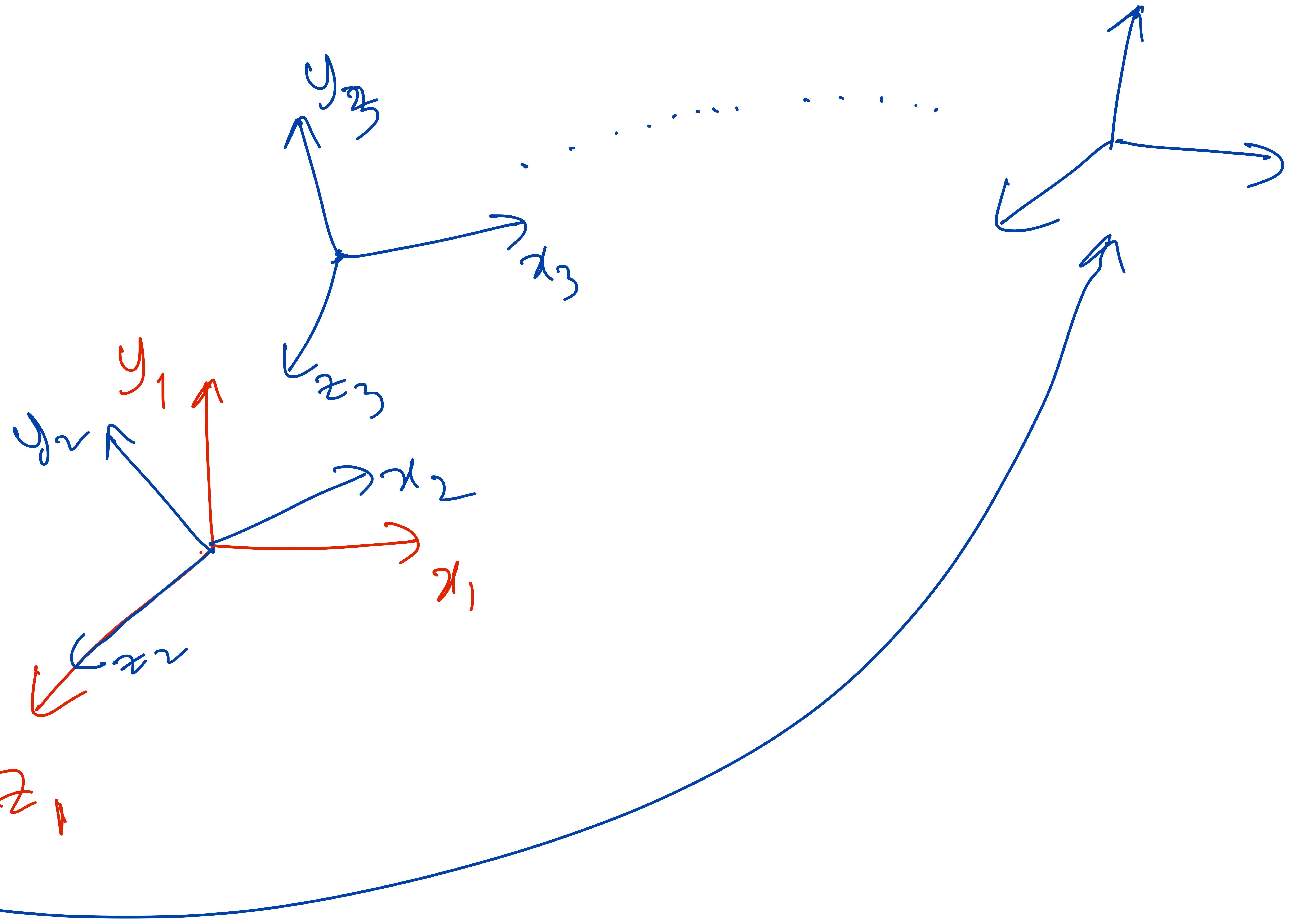
$$H = \begin{bmatrix} 1 & 0 & 0 & l_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



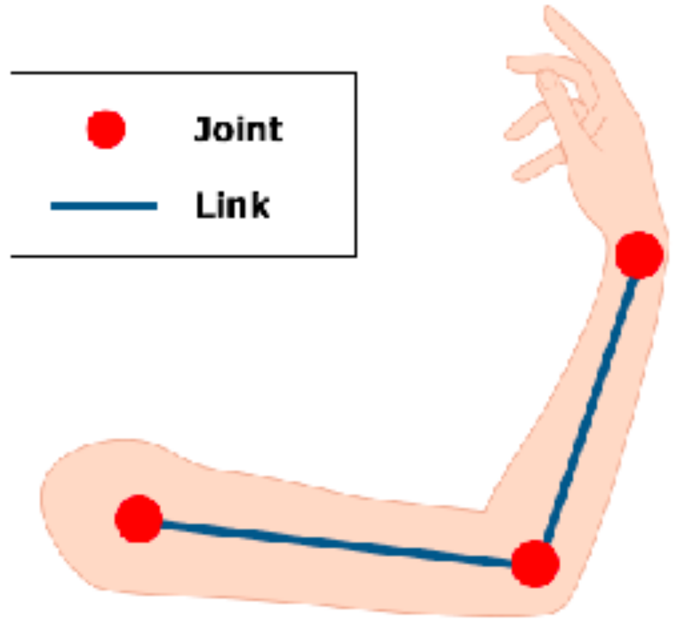
Modelling a robot

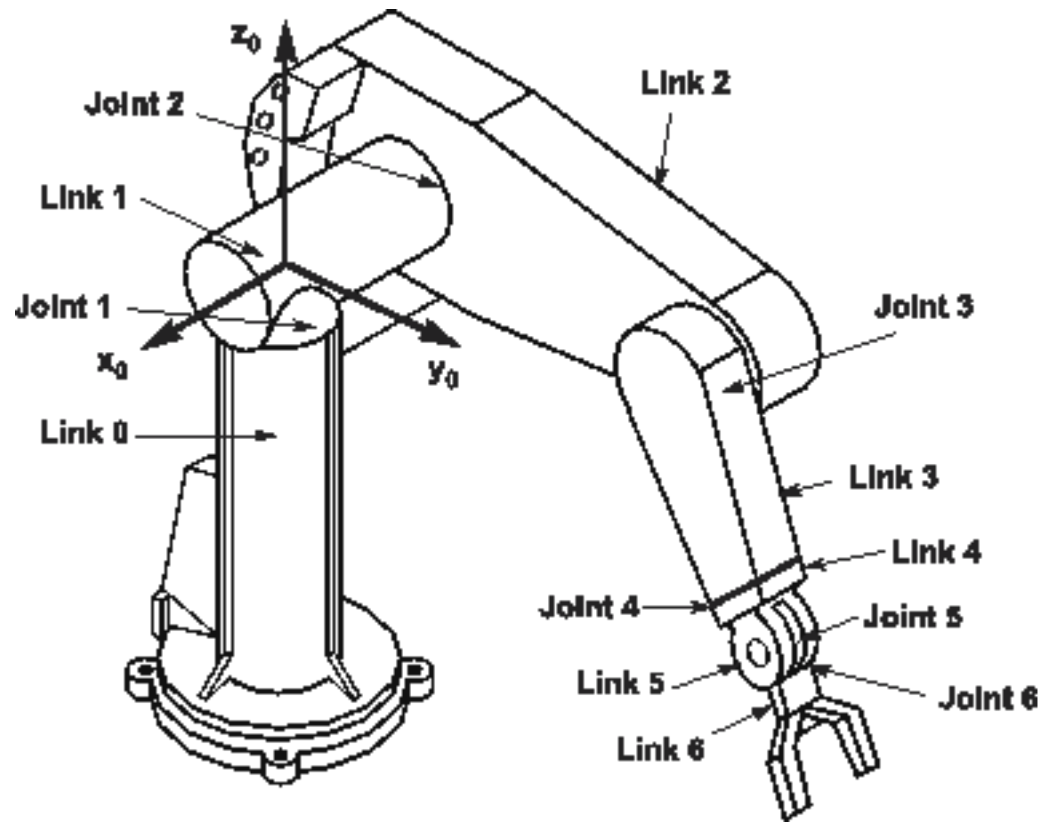






Modelling a robot

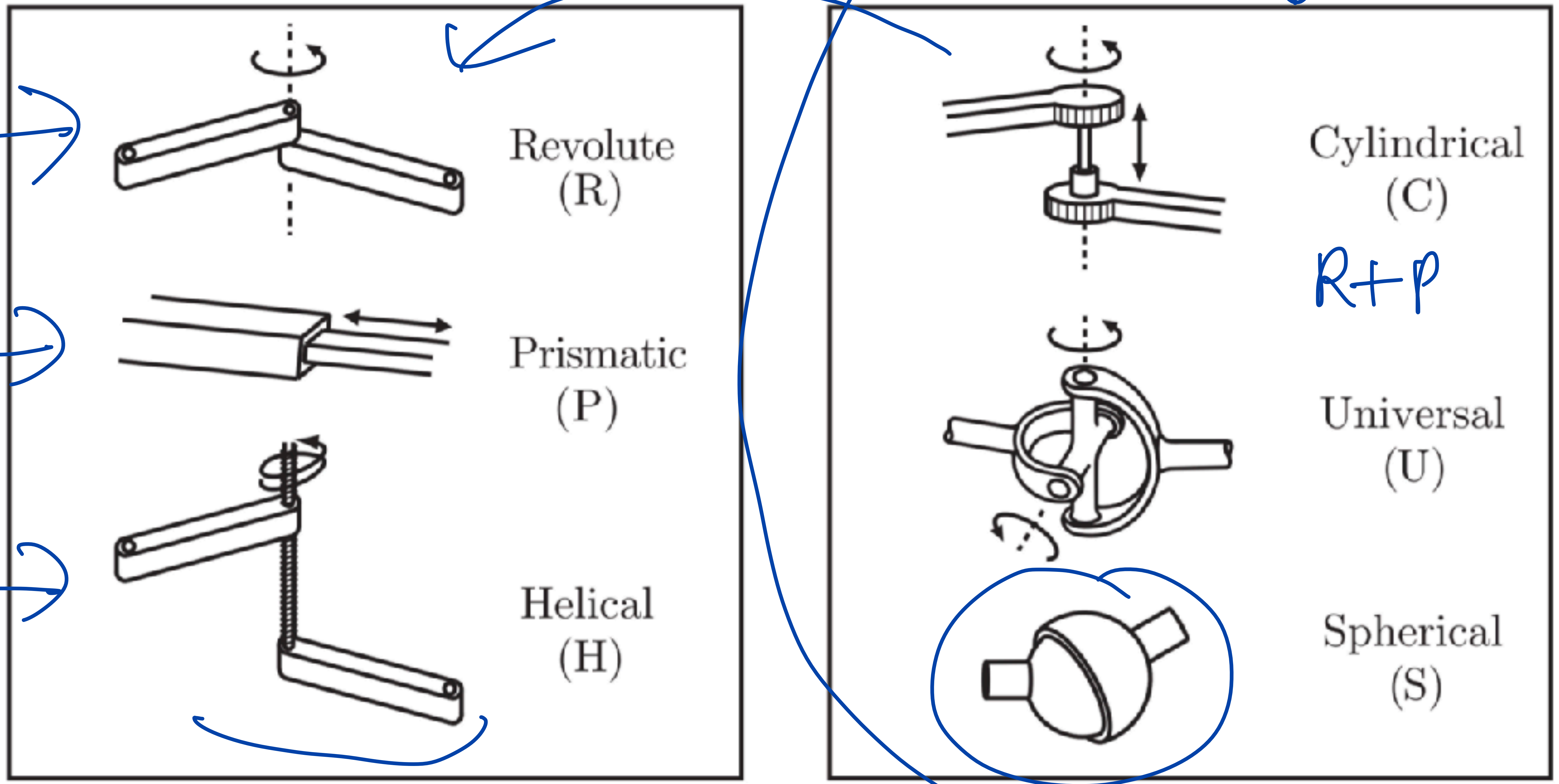




primitive

Types of joints

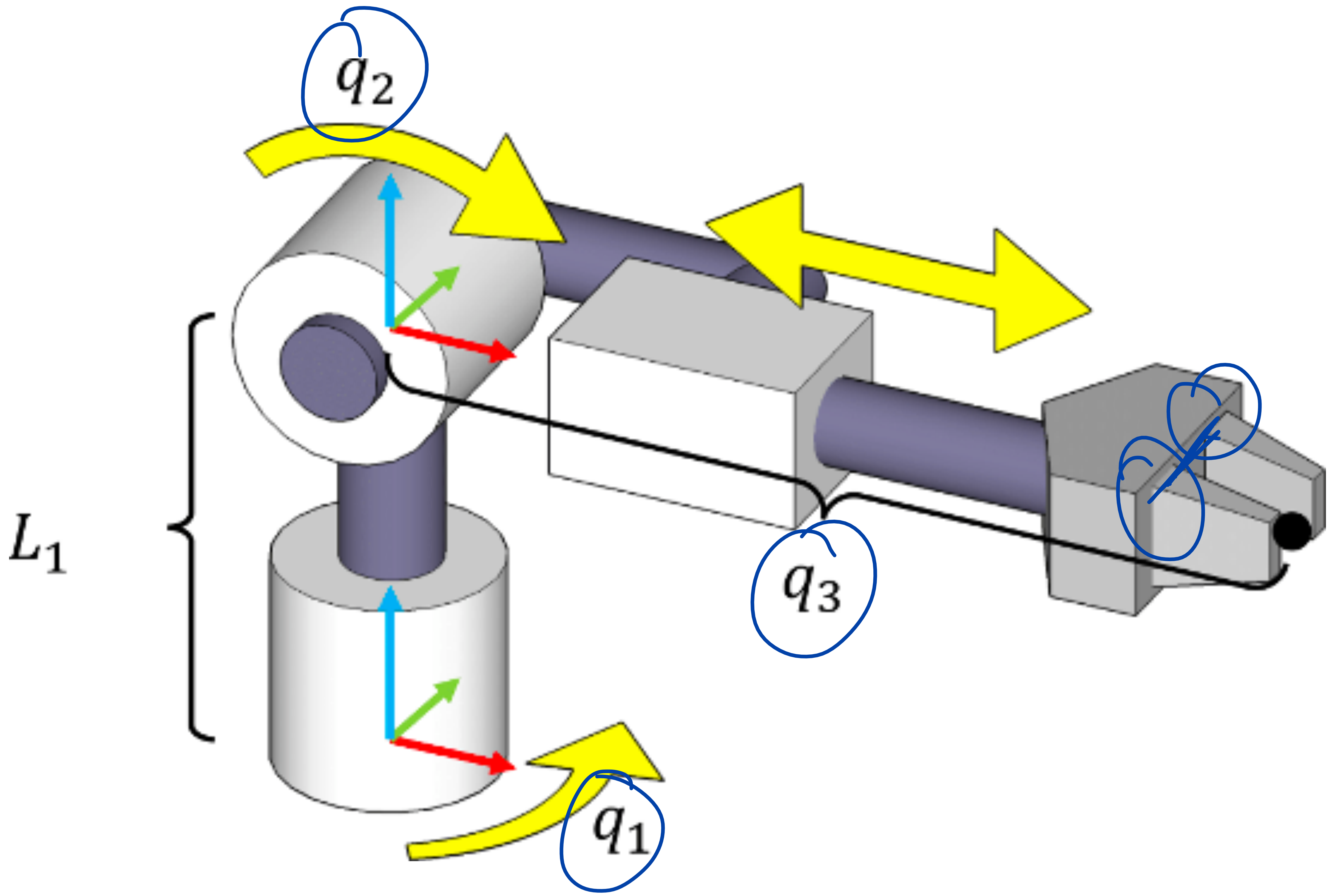
combination



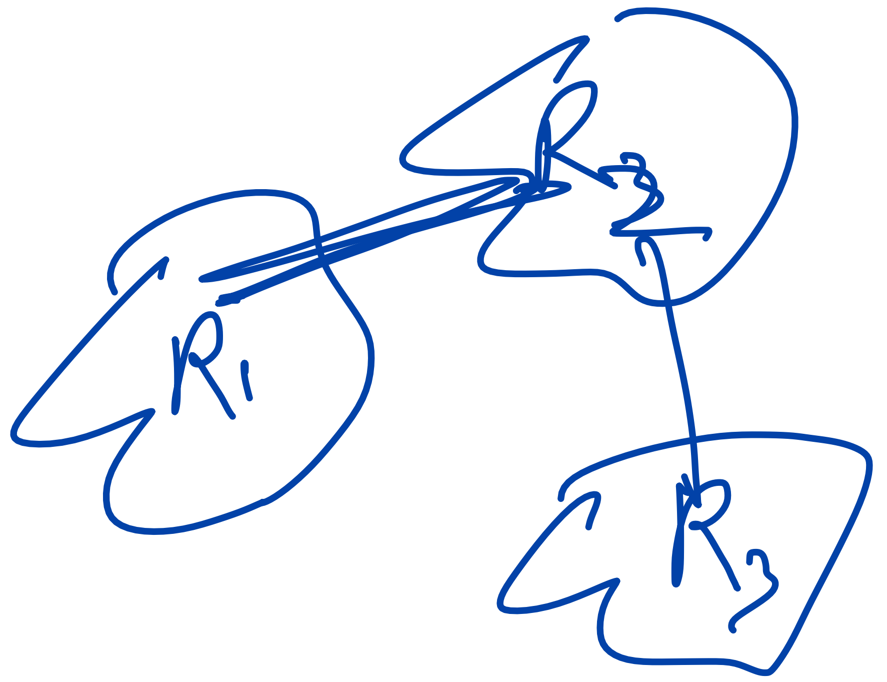
R+P

R+P

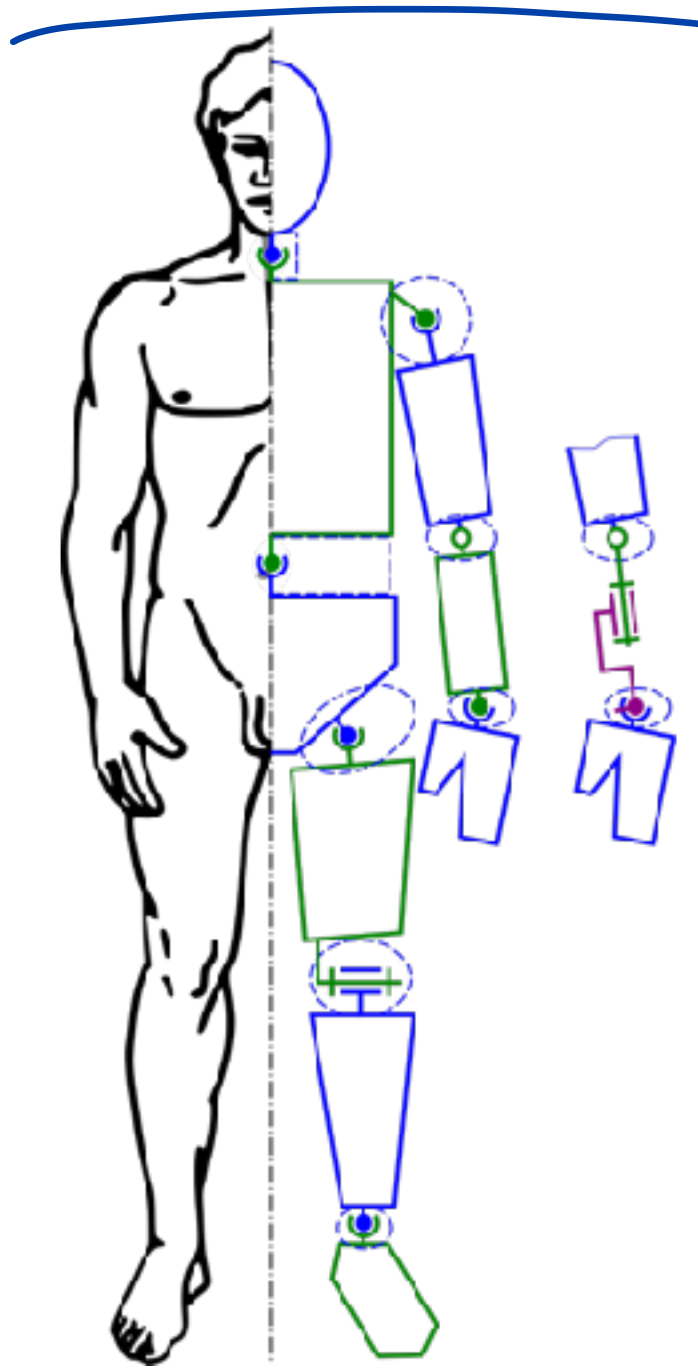
Modelling a robot

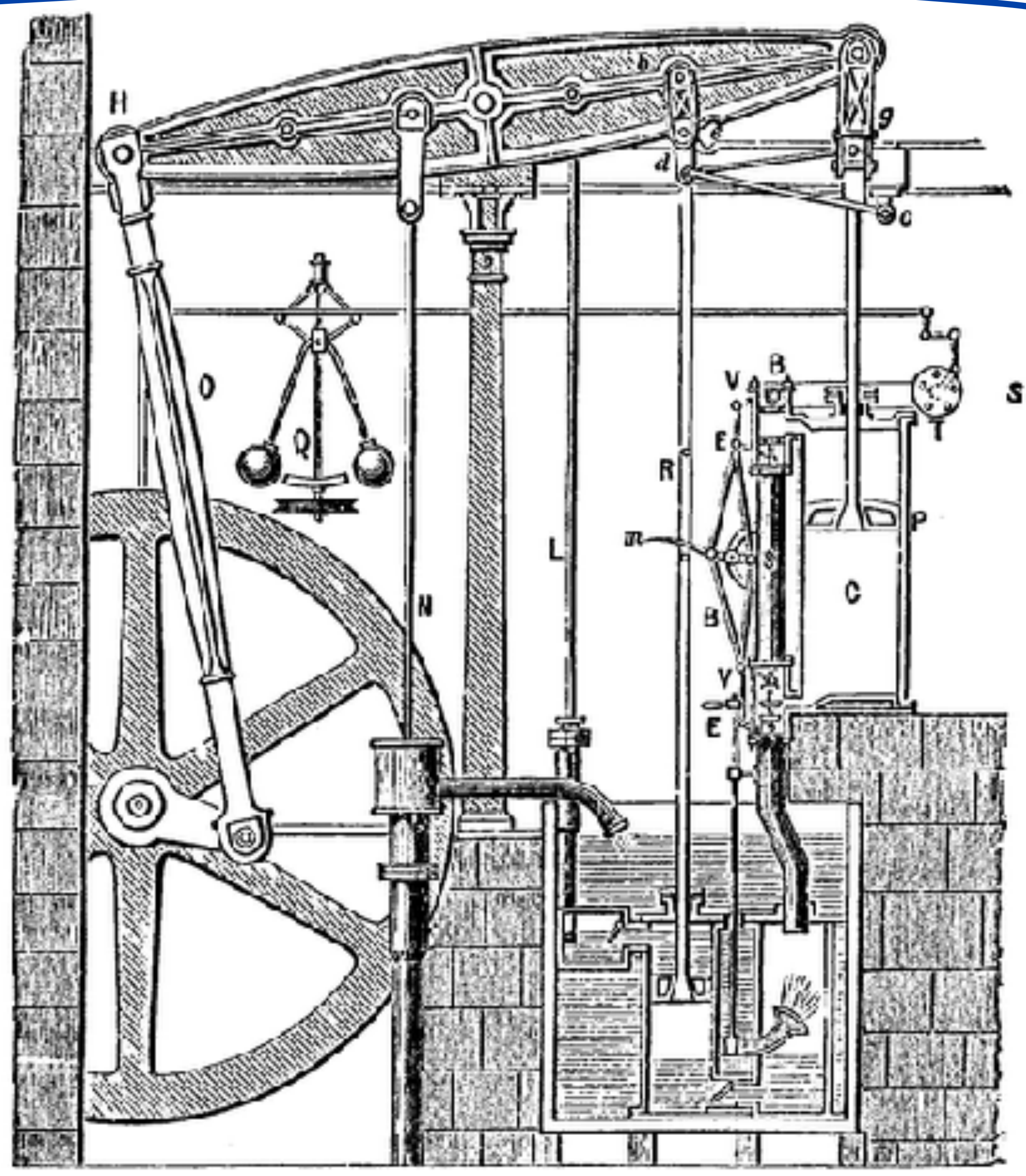


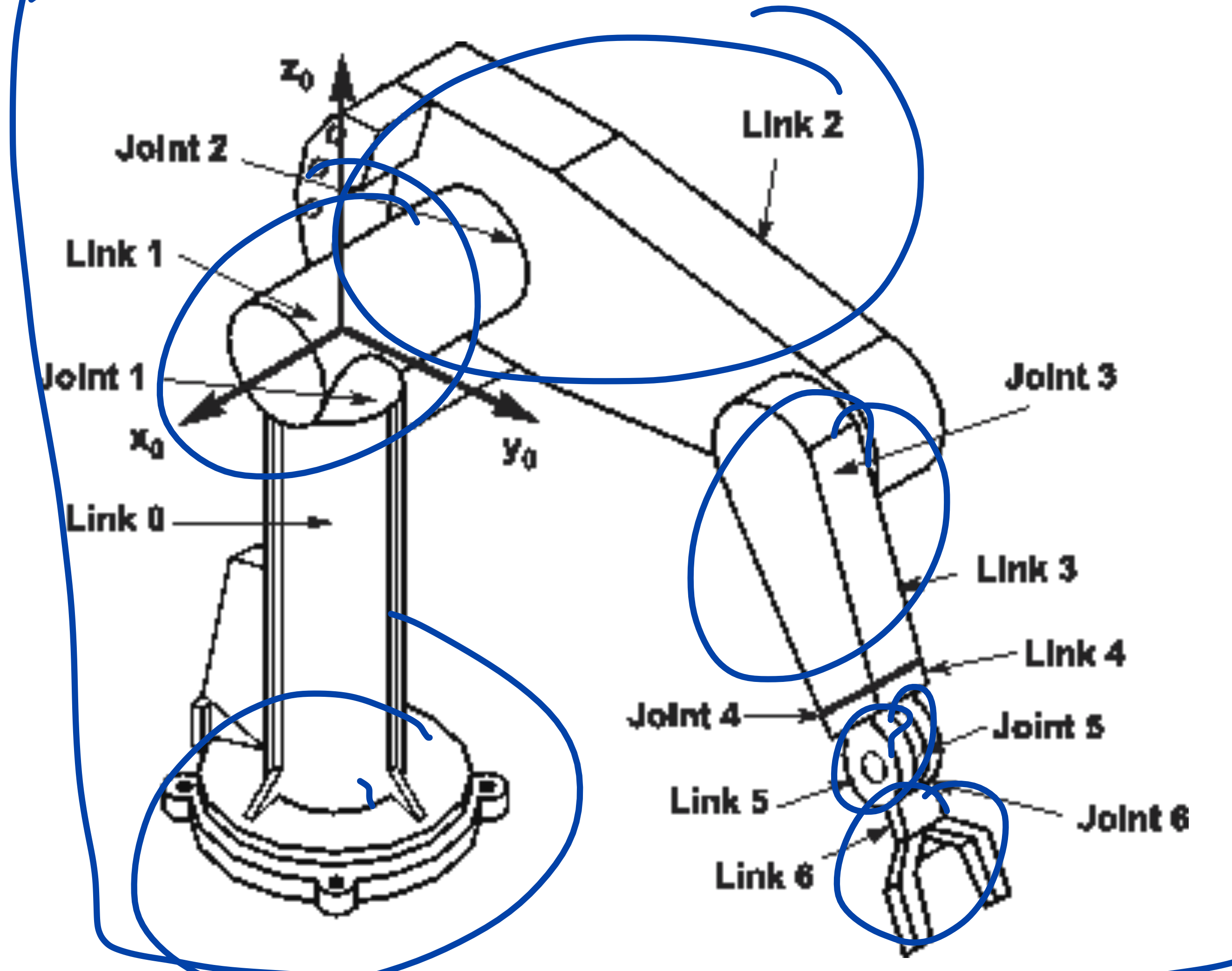
Today's class - Forward Kinematics



- Kinematic chain: A kinematic chain is an assembly of rigid bodies connected by joints to provide constrained (or desired) motion that is the mathematical model for a mechanical system.







Source: https://en.wikipedia.org/wiki/Kinematic_chain

Today's class - Forward Kinematics

- What is kinematics?

Today's class - Forward Kinematics

- What is kinematics? – Kinematics is **the study of motion without regard for the cause**. Dynamics: On the other hand, dynamics is the study of the causes of motion.

Degrees of Freedom

Definition: The number of independent ways by which a dynamic system can move, without violating any constraint imposed on it, is called number of degrees of freedom.

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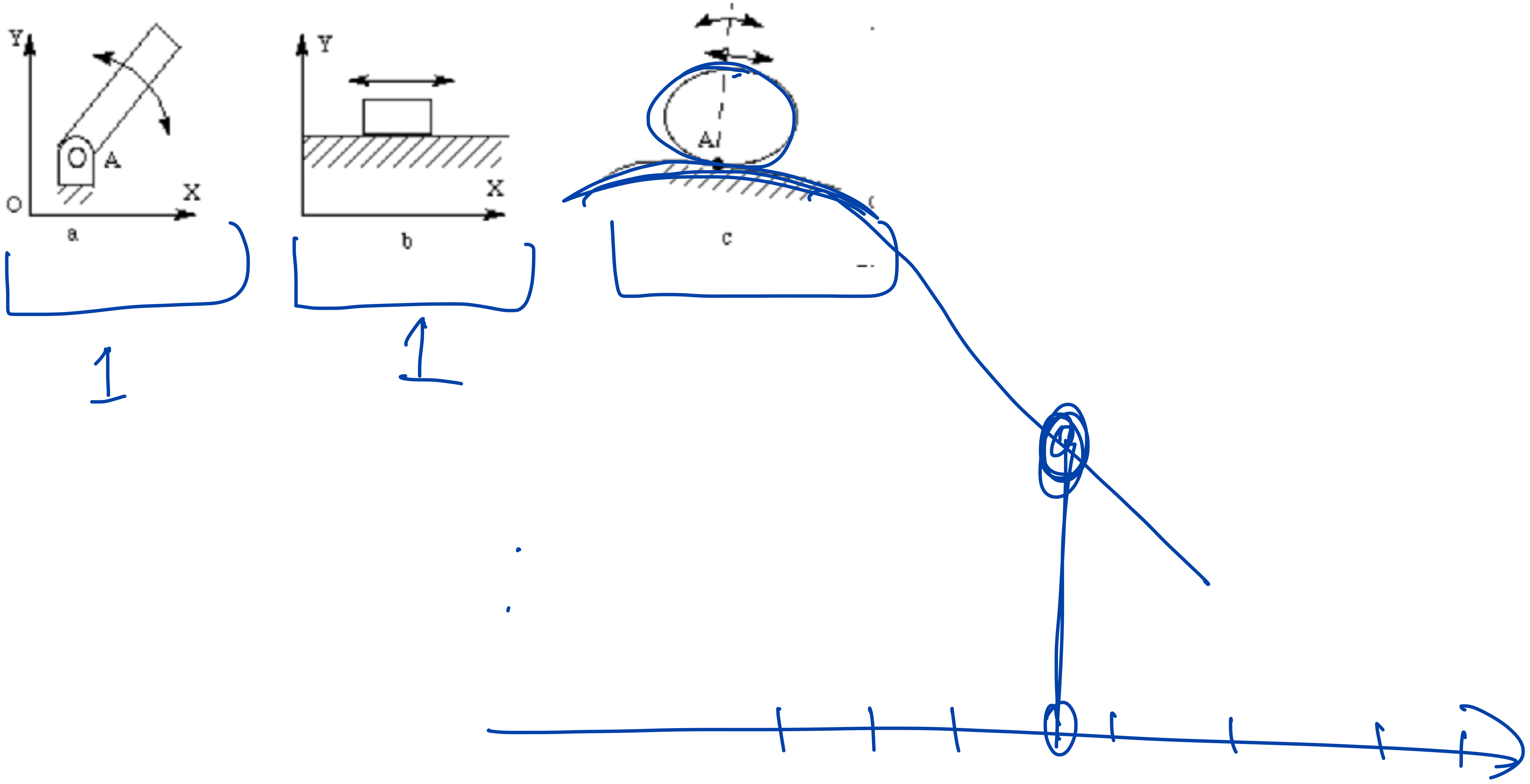

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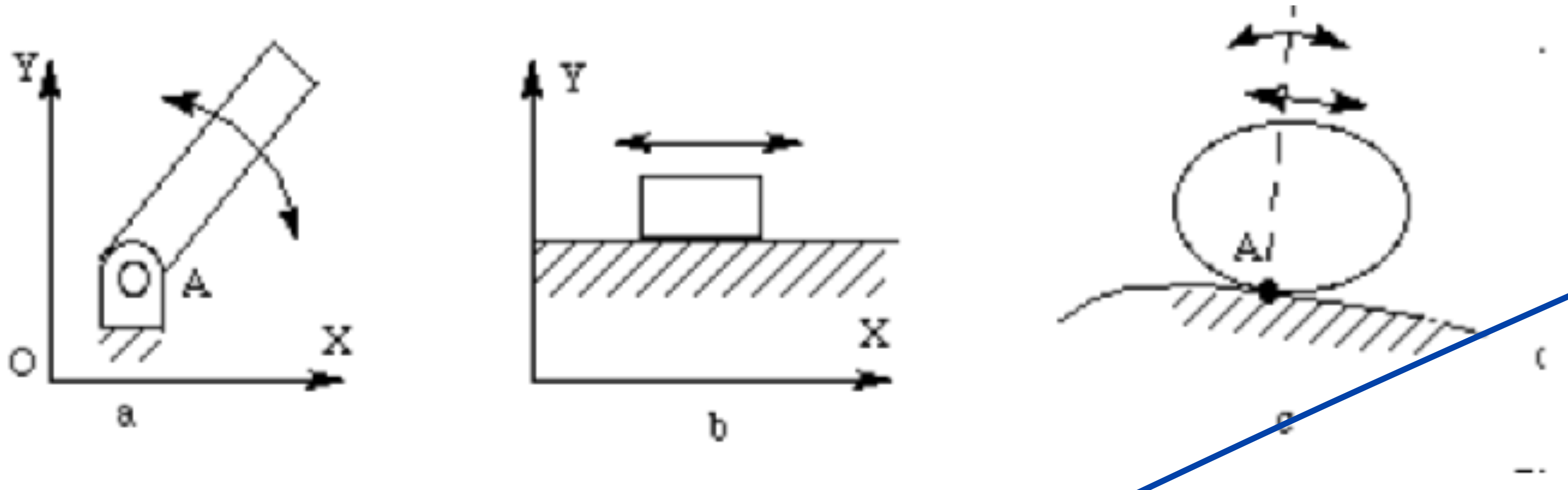

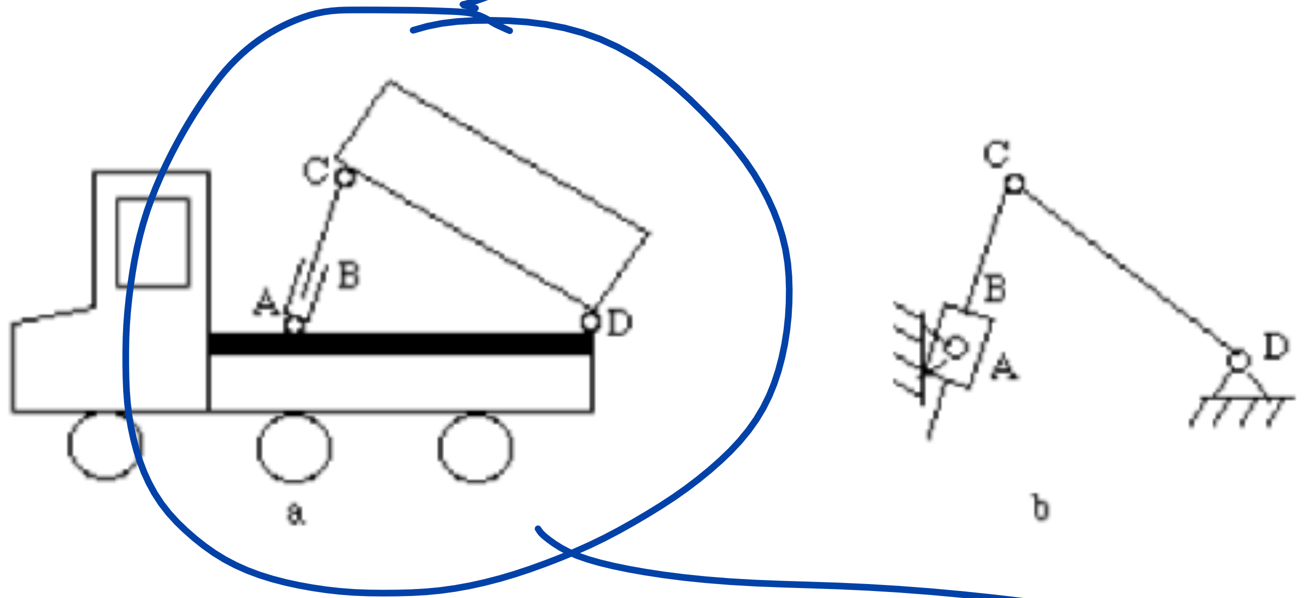

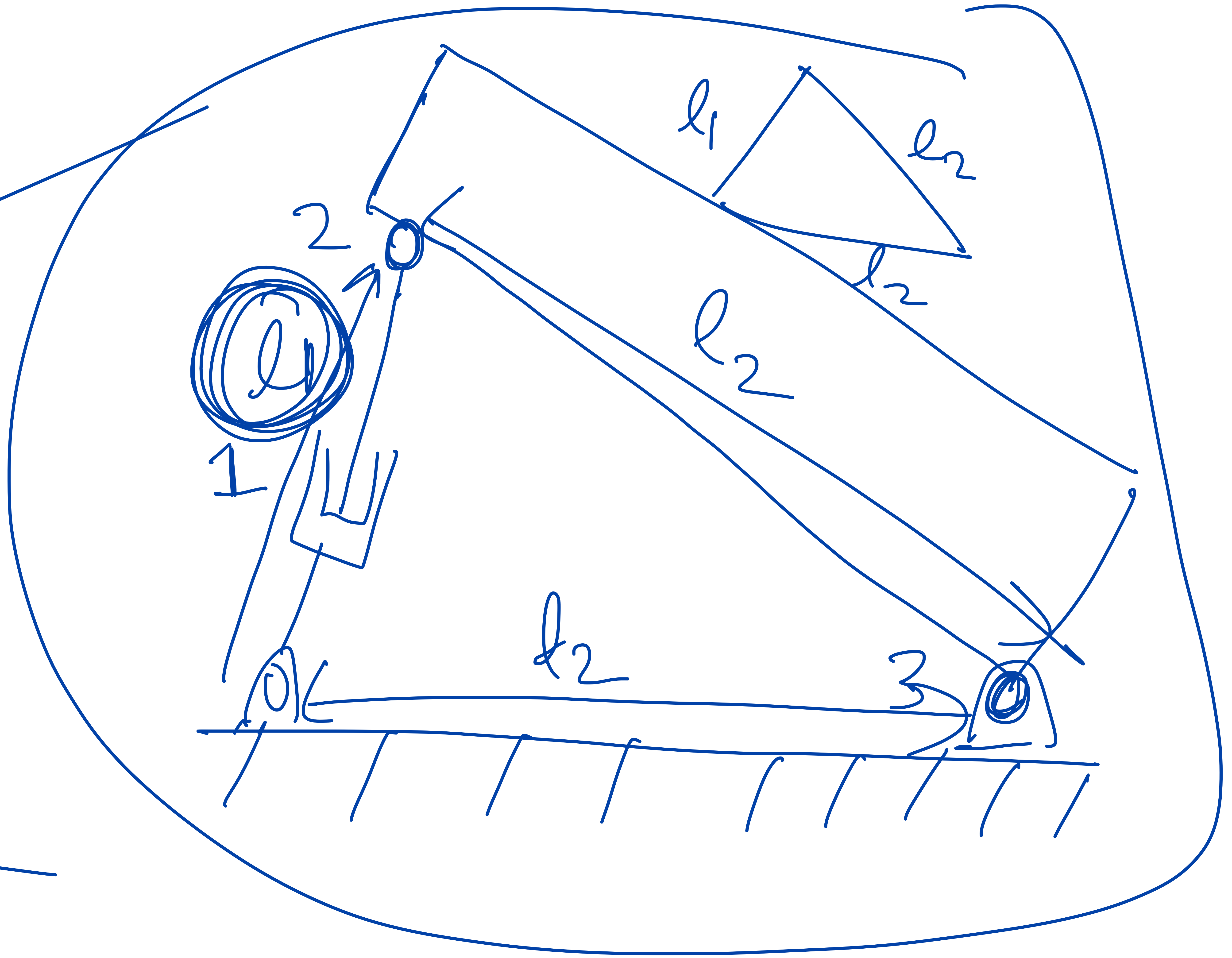

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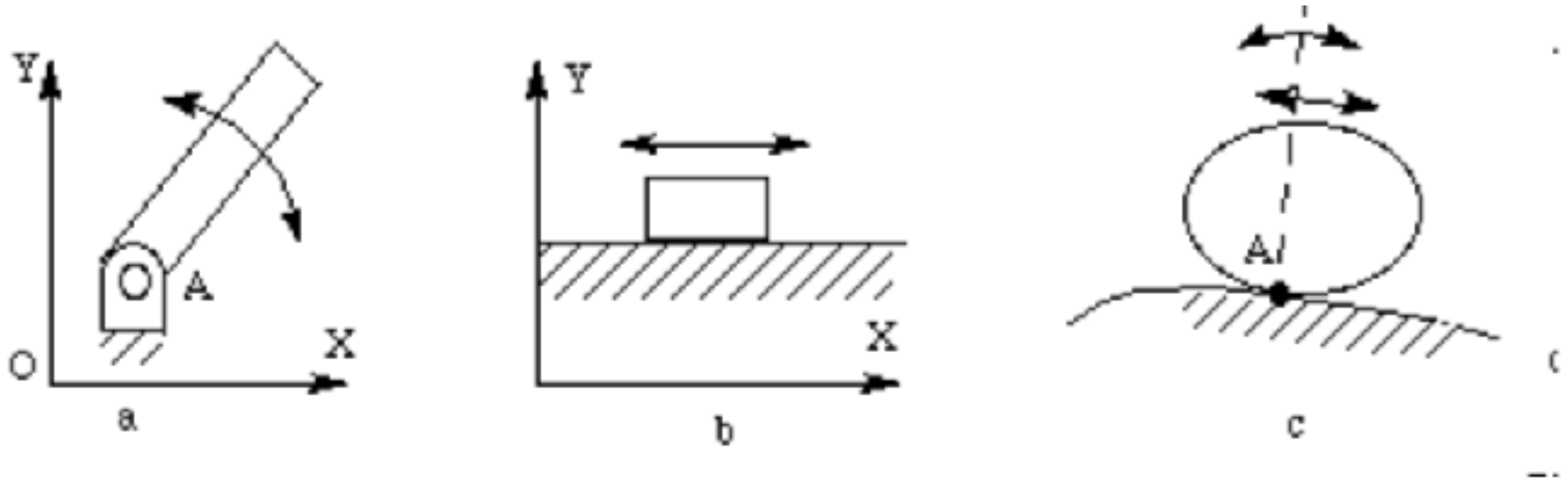

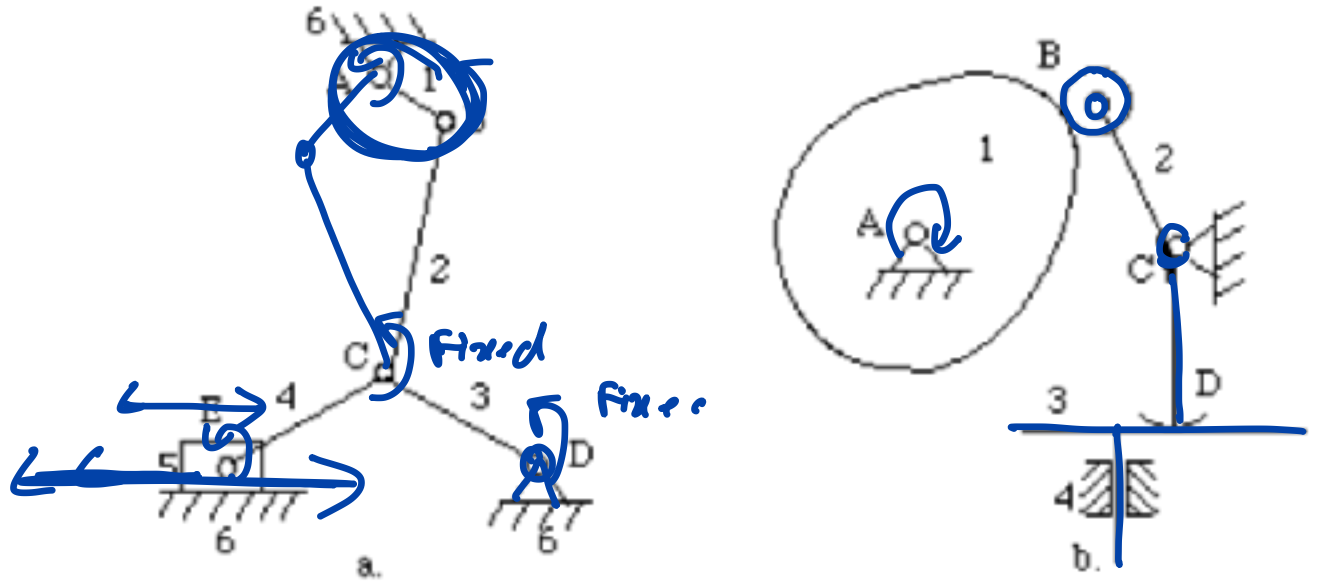

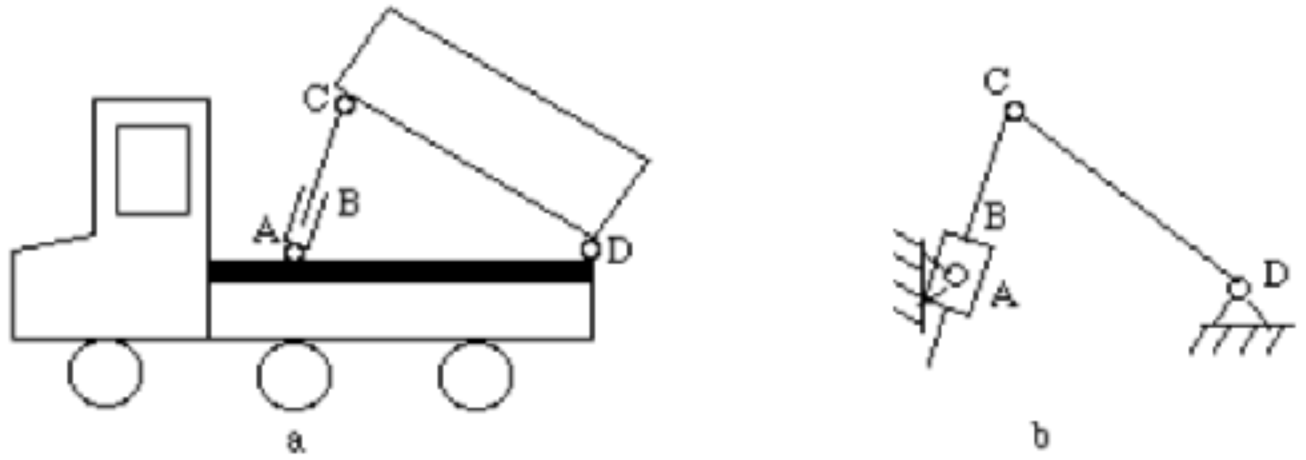

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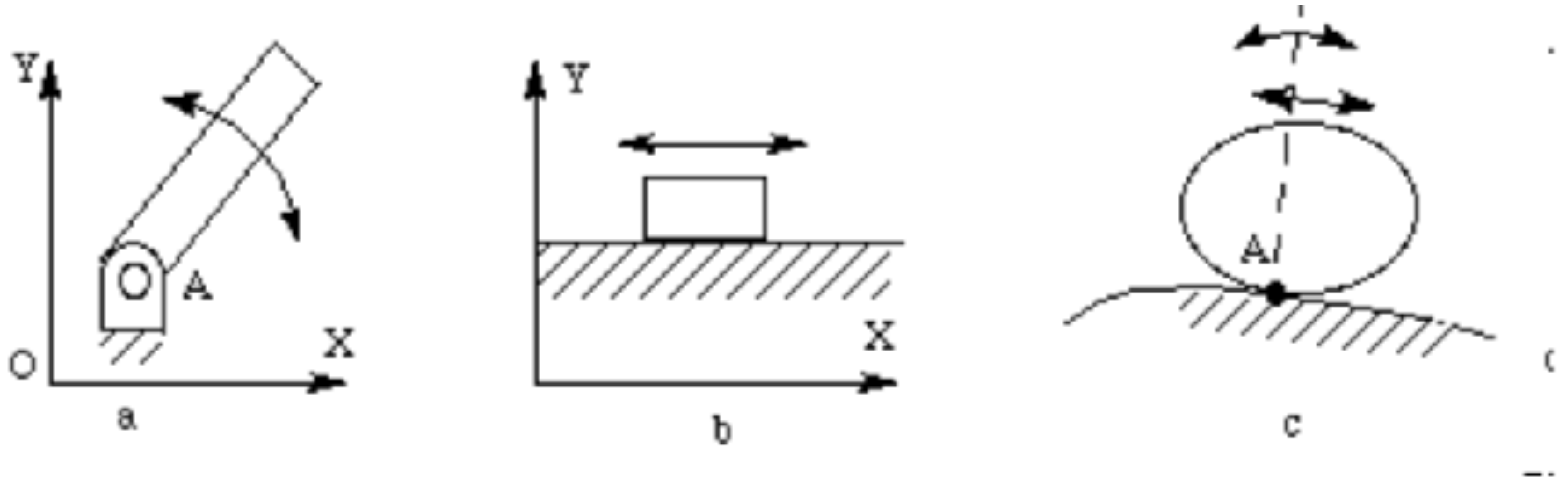
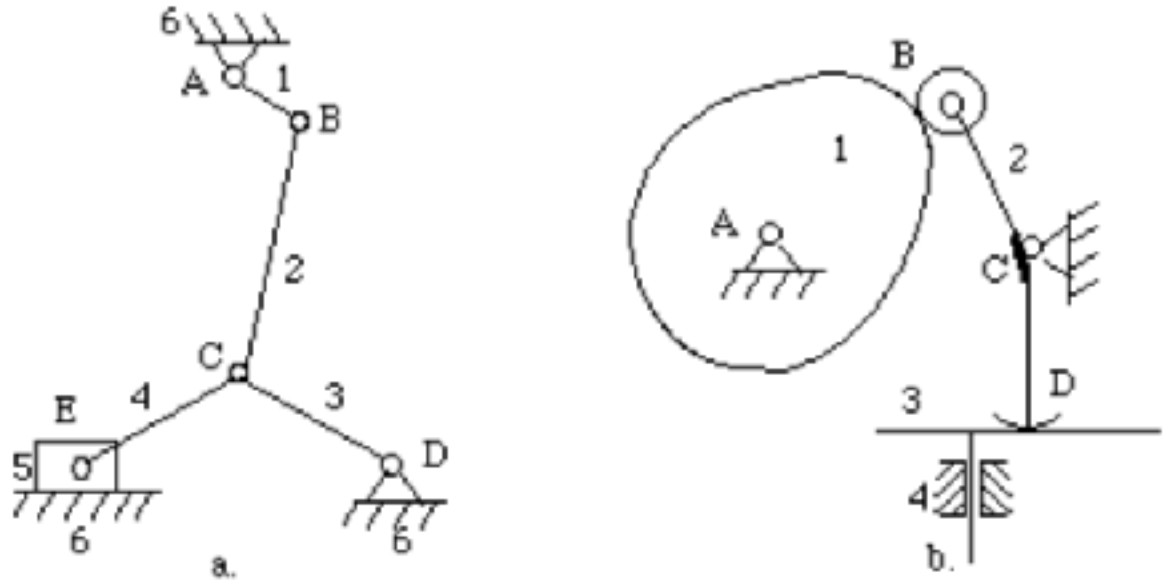
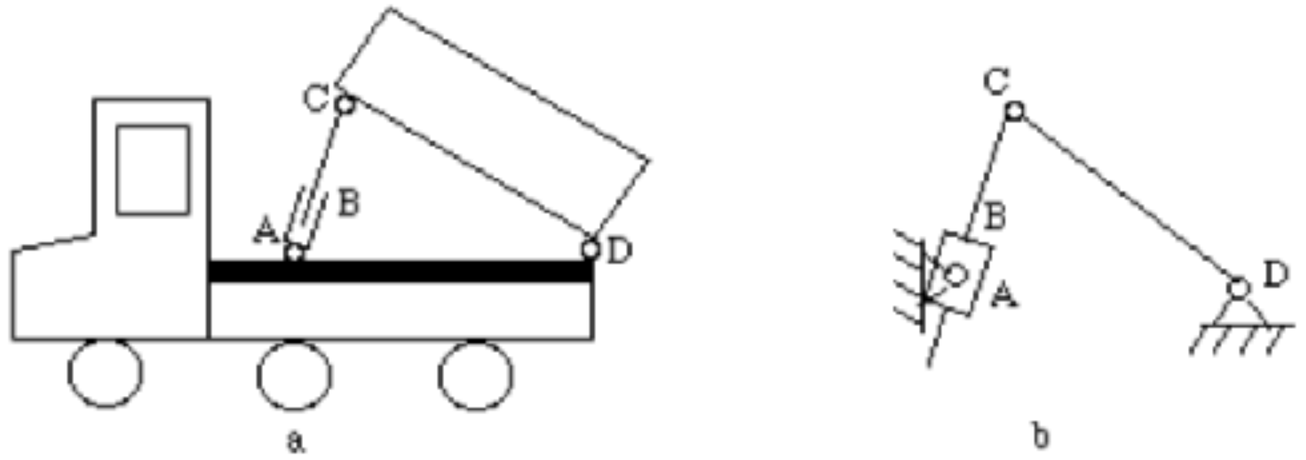
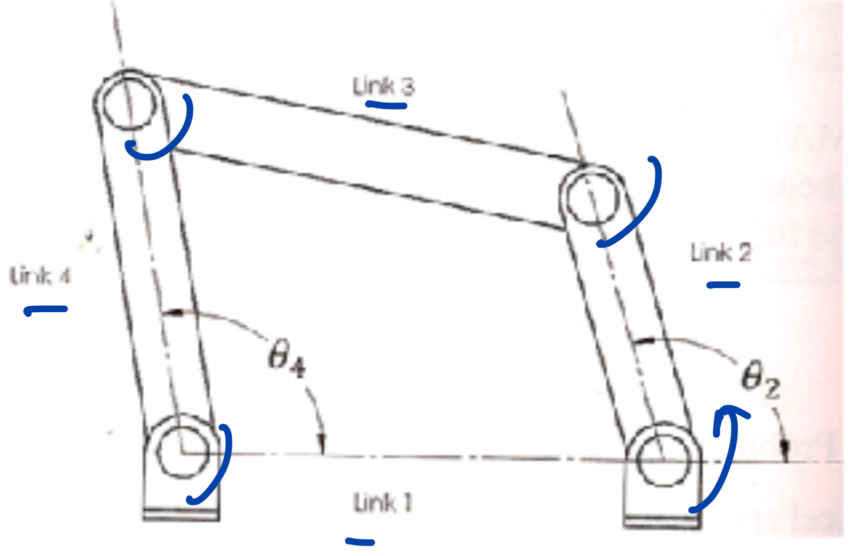
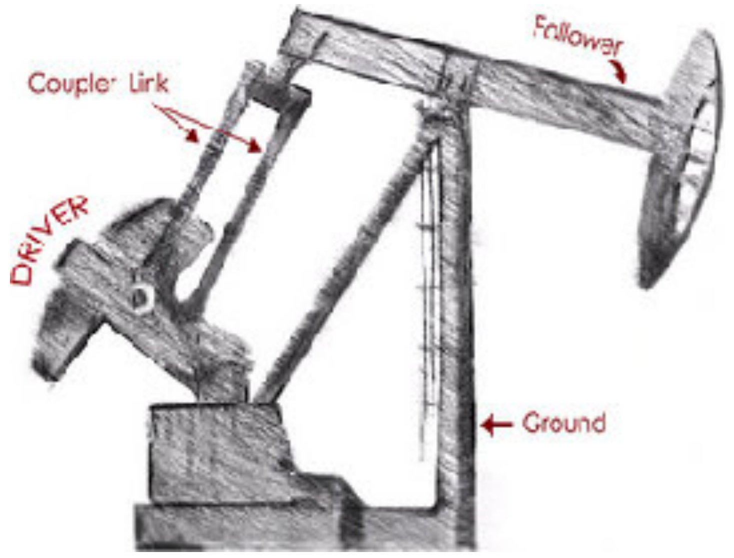


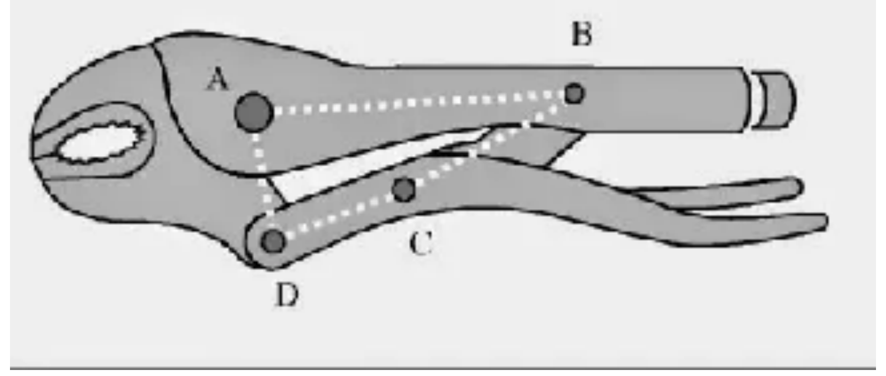






4 bar linkage





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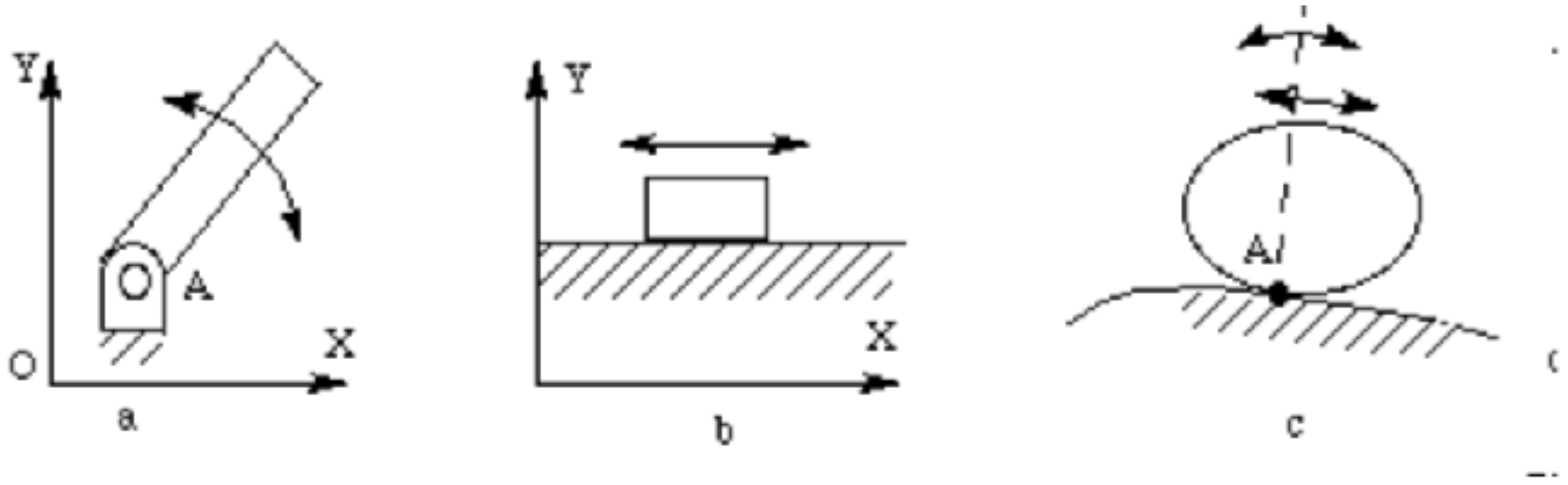
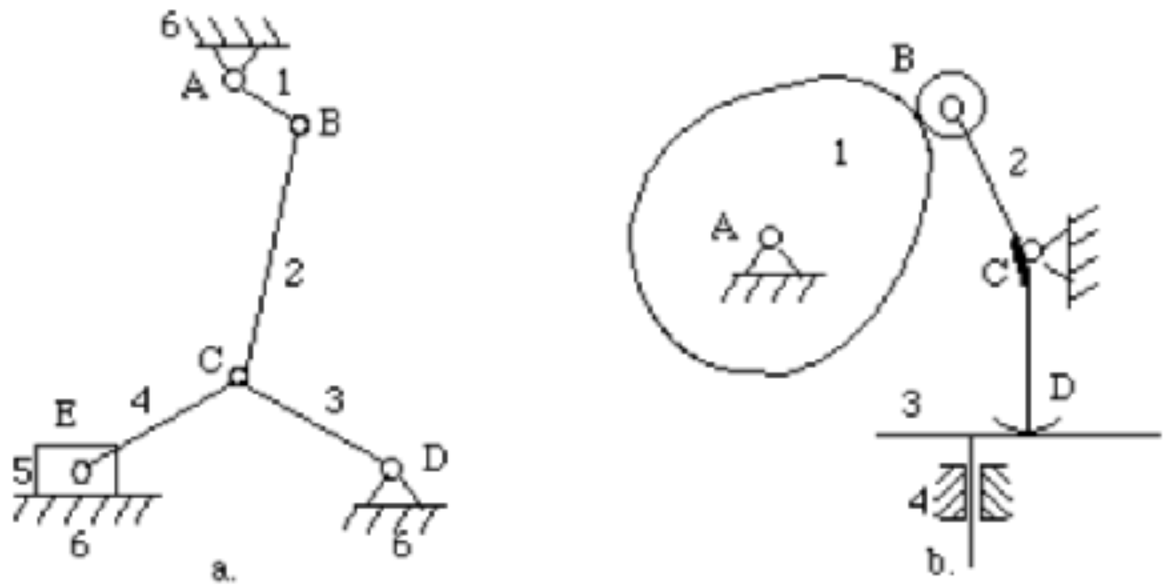
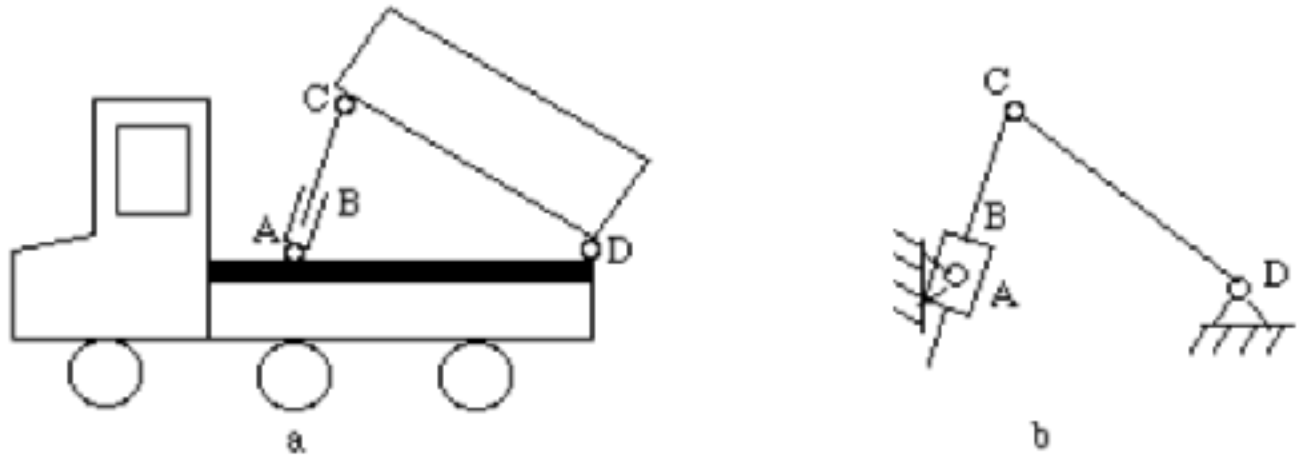
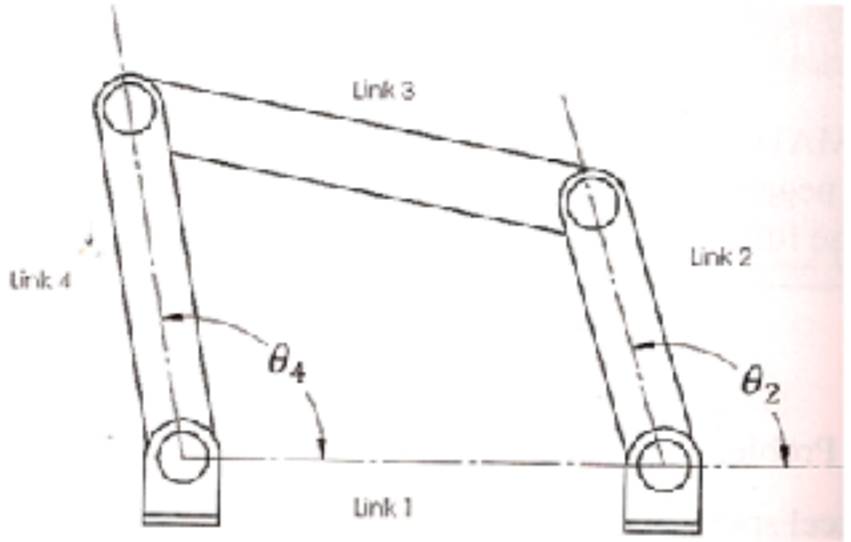
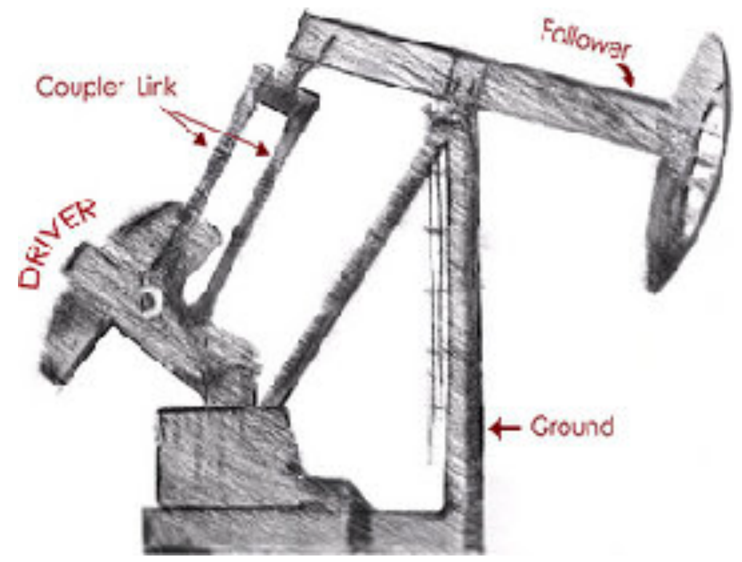
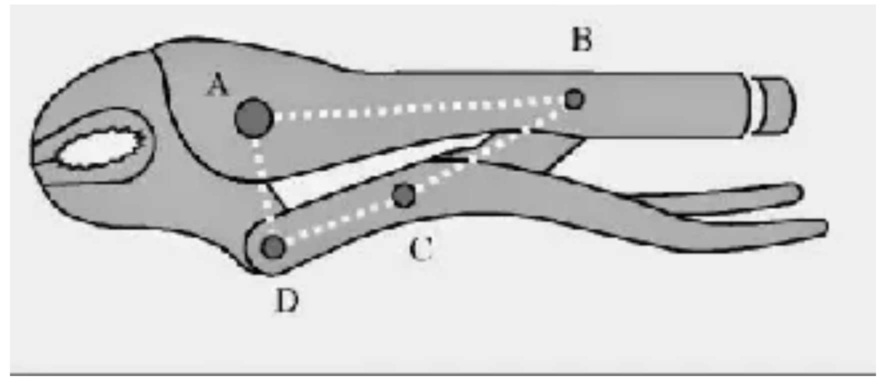






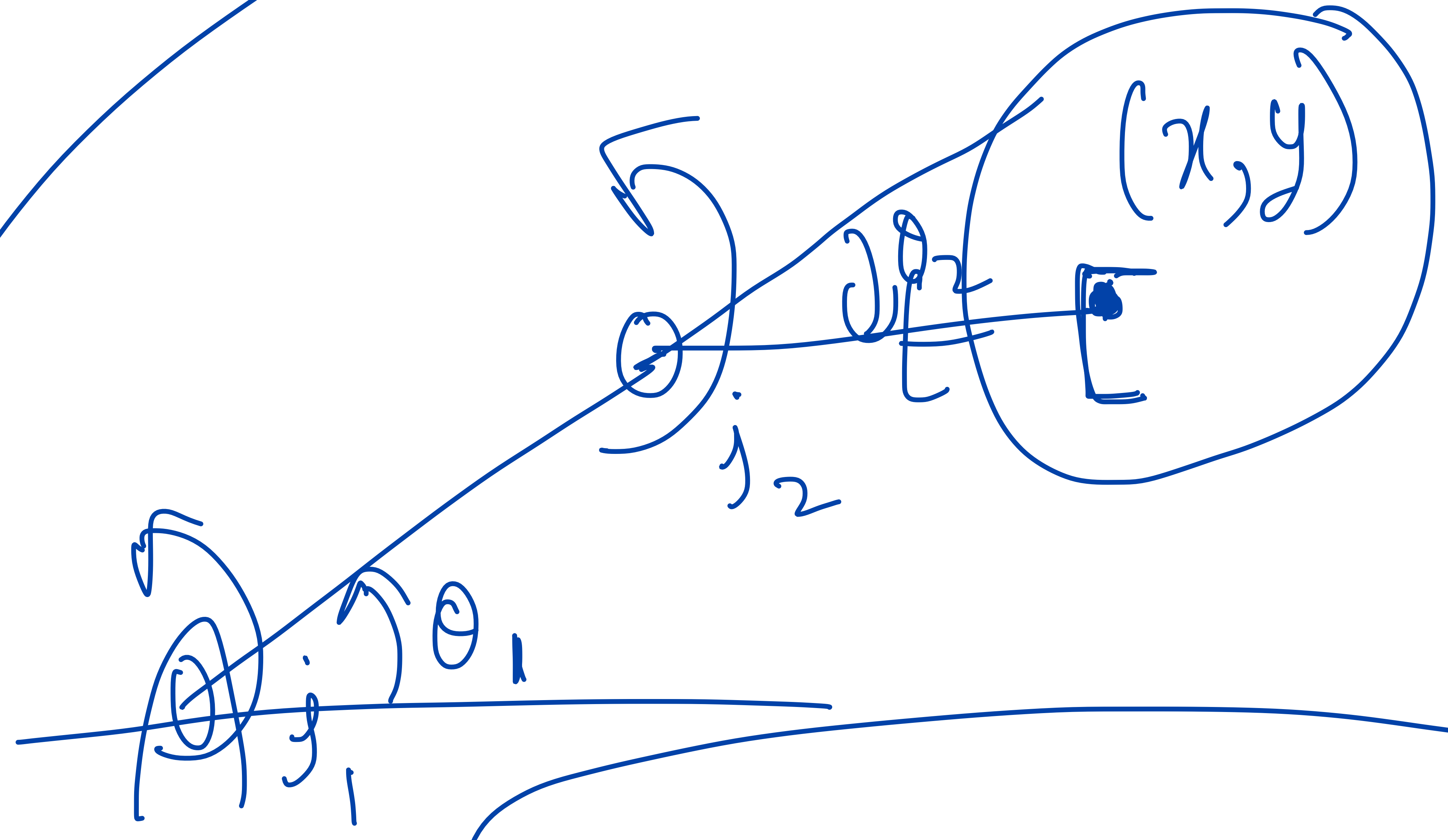






Want to learn more?
1. Gruebler's equation
2. https://www.cs.cmu.edu/~rapidproto/mechanisms/chpt4.html

Joint space vs Task space

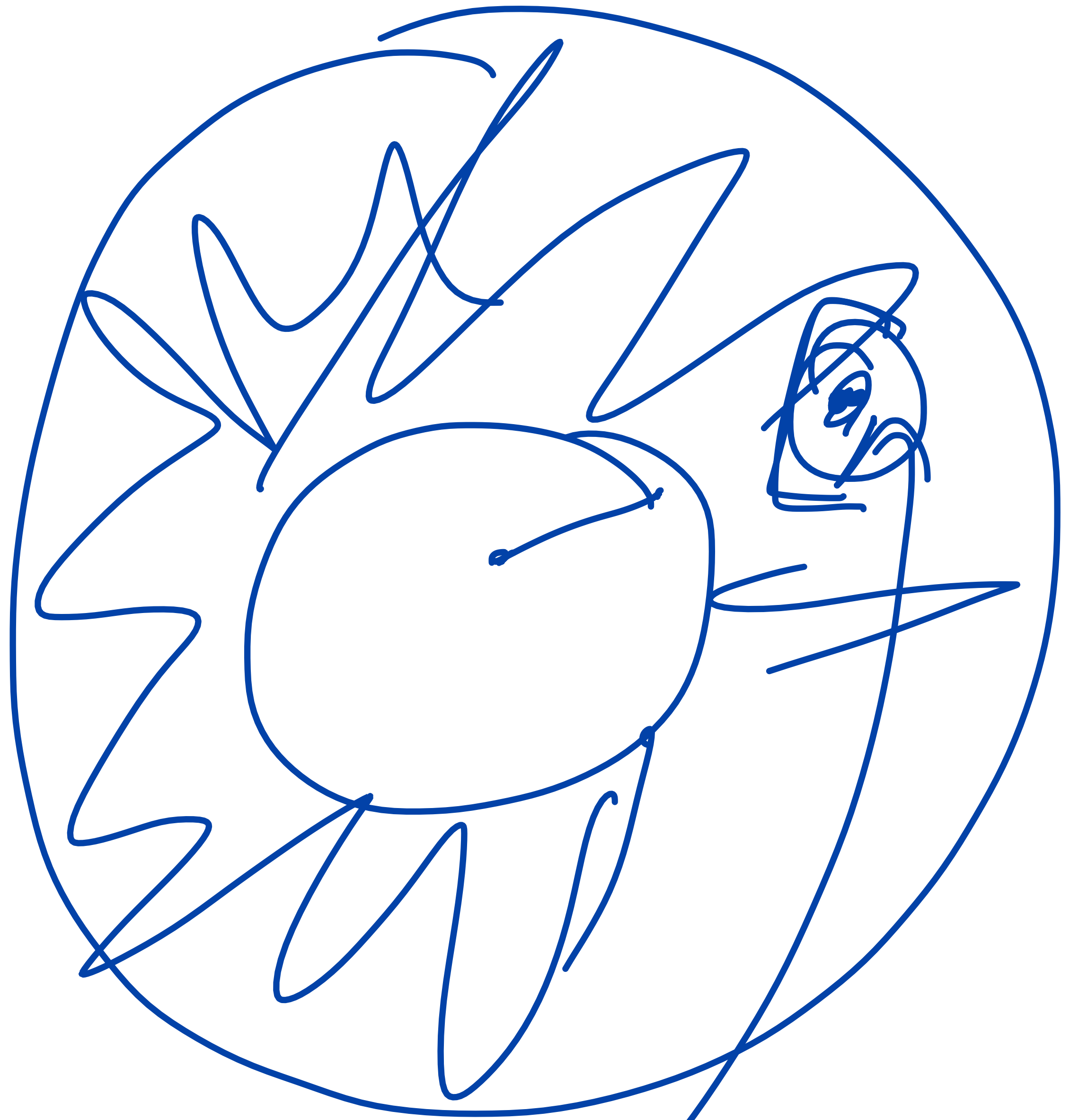


$j_1 > j_2$

$j_1 \to [0, 360^\circ]$

$j_2 \to [0, 360^\circ]$

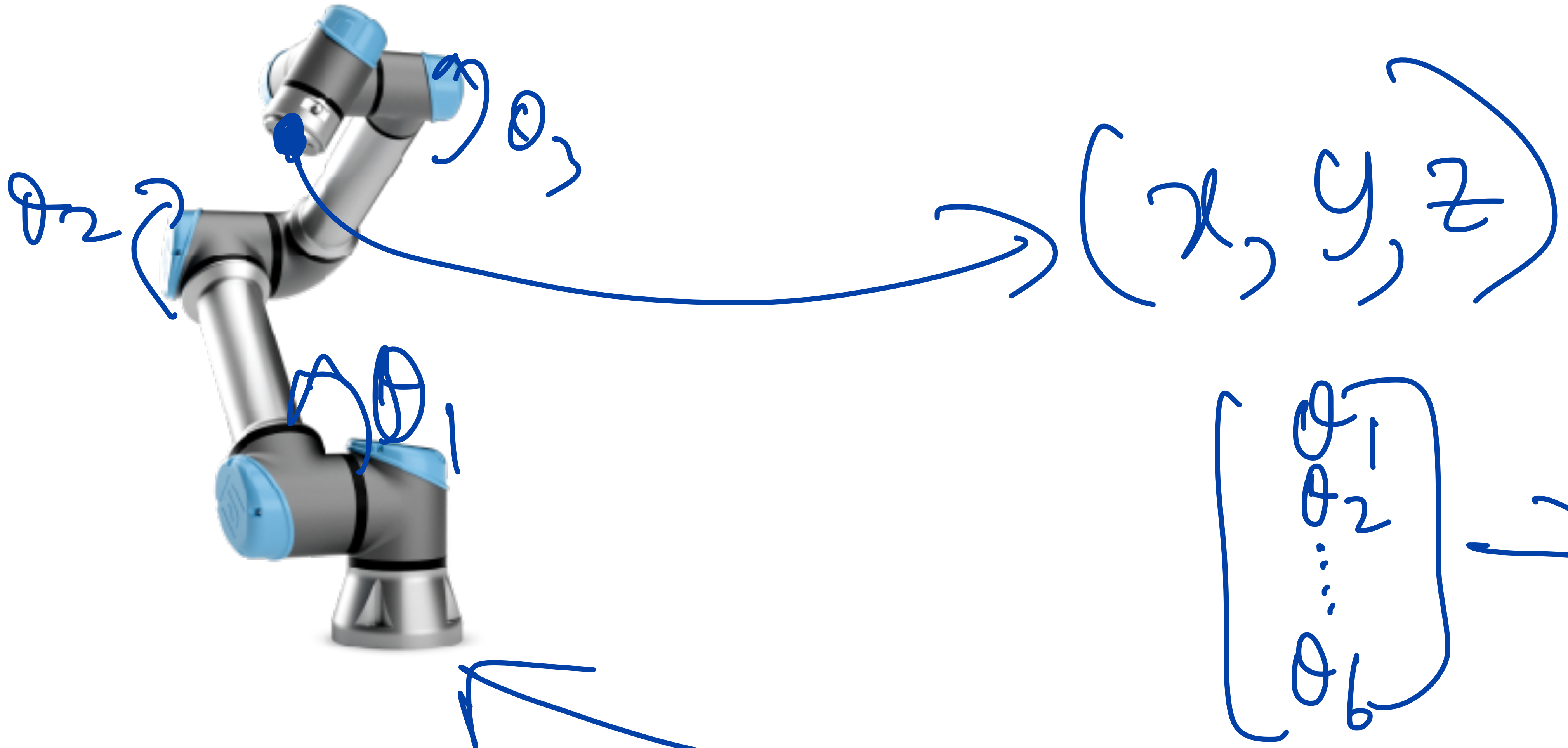
$j_3 \to [0, 1]$



θ_1, θ_2

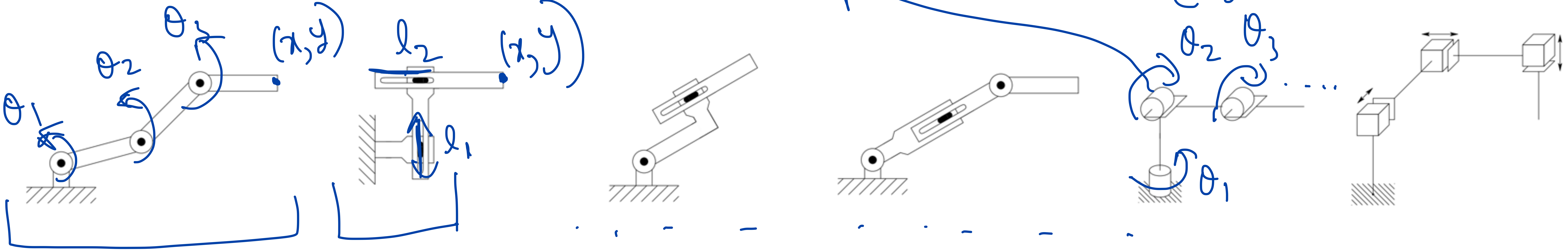
Forward Kinematics

- Forward kinematics: The use of the kinematic equations of a robot to compute the position of the end-effector from specified values for the joint parameters



$$\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} \longrightarrow \begin{bmatrix}x\\ y\end{bmatrix} \qquad \begin{bmatrix}l_1\\ l_2\end{bmatrix} \longrightarrow \begin{bmatrix}x\\ y\end{bmatrix}$$

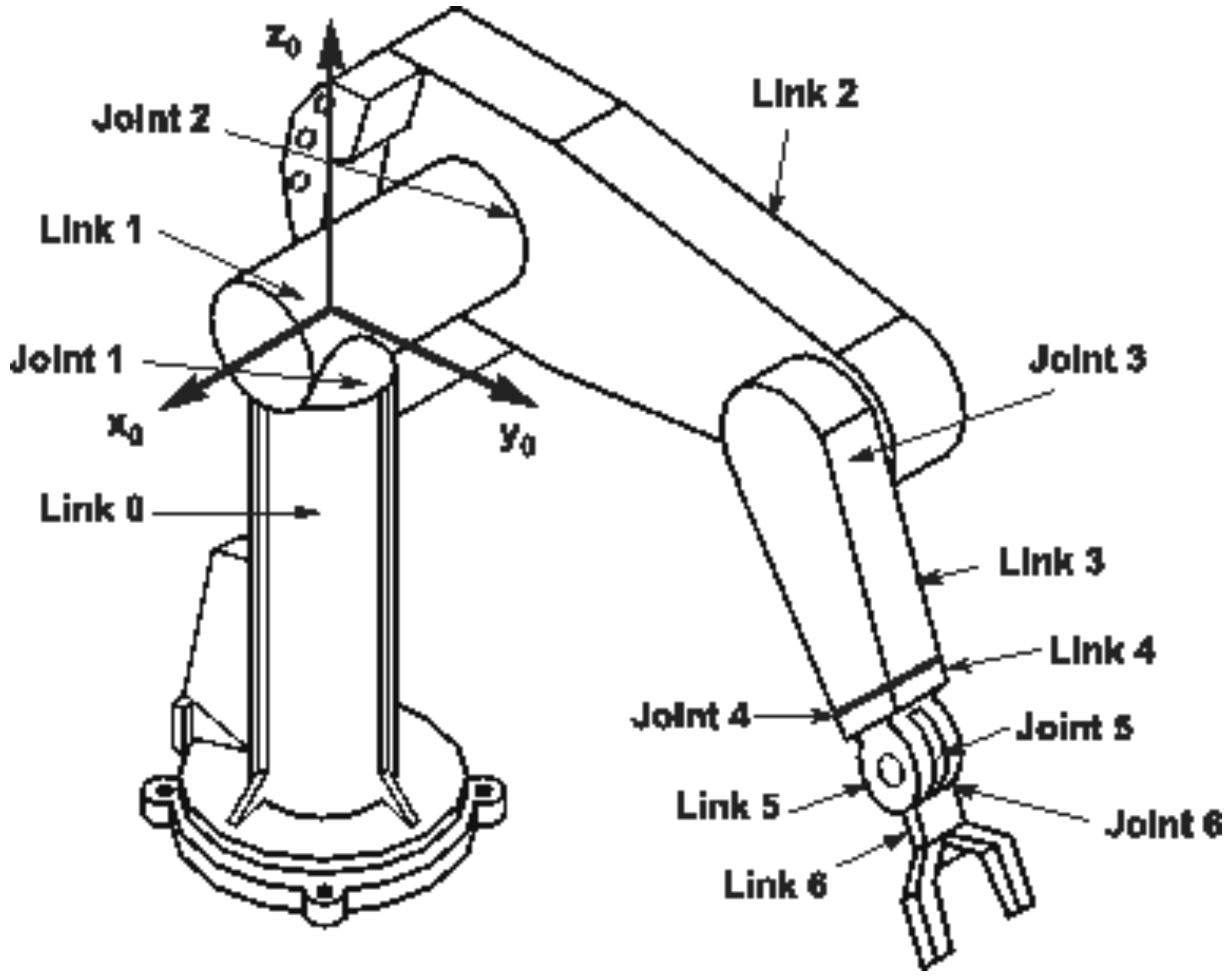
$$\begin{bmatrix}\theta_1\\ \theta_2\\ \vdots\\ \theta_6\end{bmatrix} \longrightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

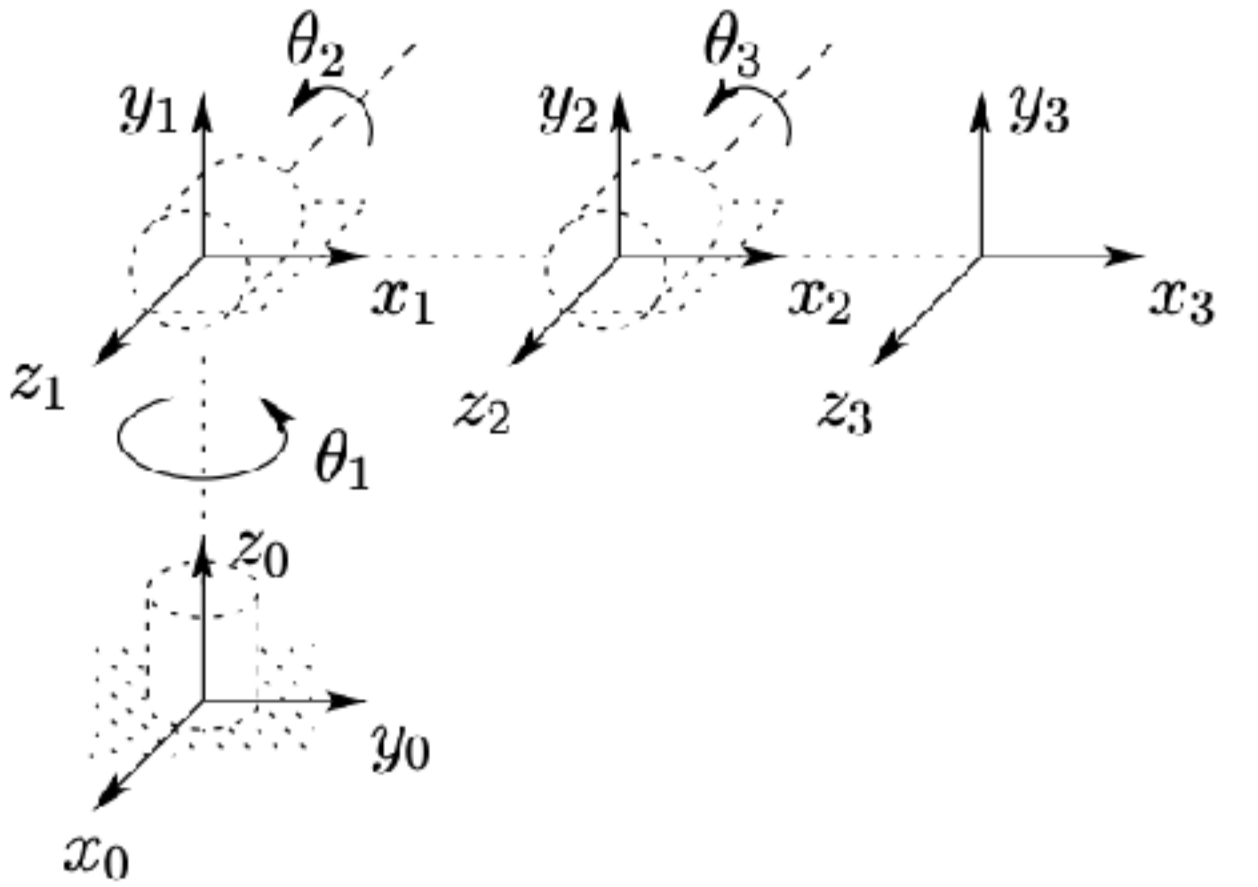


Forward Kinematics for Simple Mechanisms

Recall: Homogenous Transformation

A systematic approach for FK





https://users.cs.duke.edu/~brd/Teaching/Bio/asmb/current/Papers/chap3-forward-kinematics.pdf

A systematic approach for FK – Denavit Hartenberg Convention

We may summarize the above procedure based on the D-H convention in the following algorithm for deriving the forward kinematics for any manipulator.

Step l: Locate and label the joint axes $z_0, \ldots, z_{n-1}$.

Step 2: Establish the base frame. Set the origin anywhere on the z_0-axis. The x_0 and y_0 axes are chosen conveniently to form a right-hand frame.

For $i = 1, \ldots, n-1$, perform Steps 3 to 5.

Step 3: Locate the origin O_i where the common normal to z_i and z_{i-1} intersects z_i. If z_i intersects z_{i-1} locate O_i at this intersection. If z_i and z_{i-1} are parallel, locate O_i in any convenient position along z_i.

Step 4: Establish x_i along the common normal between z_{i-1} and z_i through O_i, or in the direction normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

Step 5: Establish y_i to complete a right-hand frame.

Step 6: Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the n-th joint is revolute, set $z_n = \boldsymbol{a}$ along the direction z_{n-1}. Establish the origin O_n conveniently along z_n, preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set $y_n = \boldsymbol{s}$ in the direction of the gripper closure and set $x_n = \boldsymbol{n}$ as $\boldsymbol{s} \times \boldsymbol{a}$. If the tool is not a simple gripper set x_n and y_n conveniently to form a right-hand frame.

Step 7: Create a table of link parameters a_i, d_i, α_i, θ_i.

$a_i =$ distance along x_i from O_i to the intersection of the x_i and z_{i-1} axes.

$d_i =$ distance along z_{i-1} from O_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

$\alpha_i =$ the angle between z_{i-1} and z_i measured about x_i (see Figure 3.3).

$\theta_i =$ the angle between x_{i-1} and x_i measured about z_{i-1} (see Figure 3.3). θ_i is variable if joint i is revolute.

Step 8: Form the homogeneous transformation matrices A_i by substituting the above parameters into (3.10).

Step 9: Form $T_n^0 = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

https://users.cs.duke.edu/~brd/Teaching/Bio/asmb/current/Papers/chap3-forward-kinematics.pdf

Example 1: Planar Elbow Manipulator

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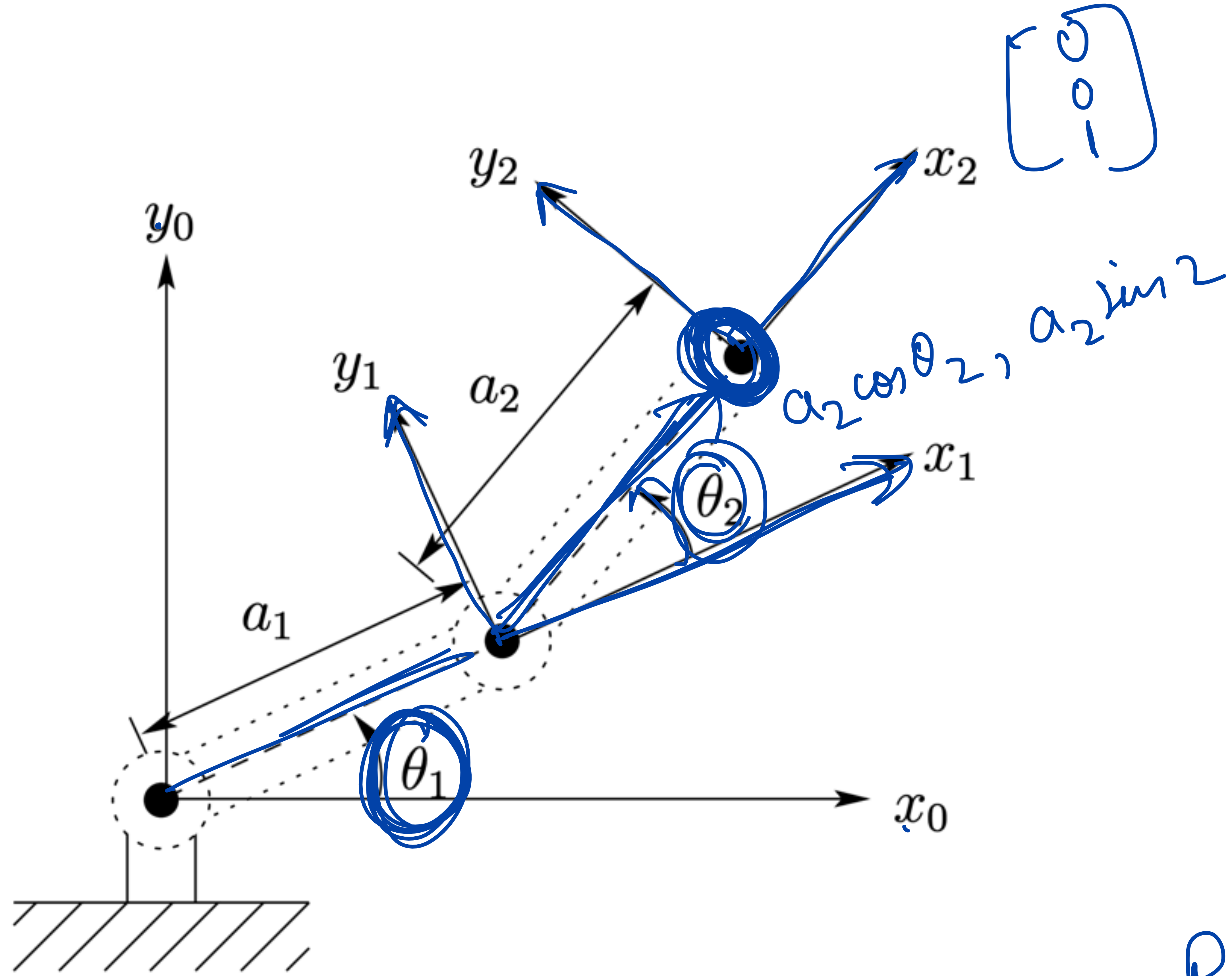


$$H_1 = \begin{bmatrix} R(\theta_2) & & \begin{matrix} a_2\cos\theta_2 \\ a_2\sin\theta_2 \end{matrix} \\ 0 & 0 & 1 \end{bmatrix}$$

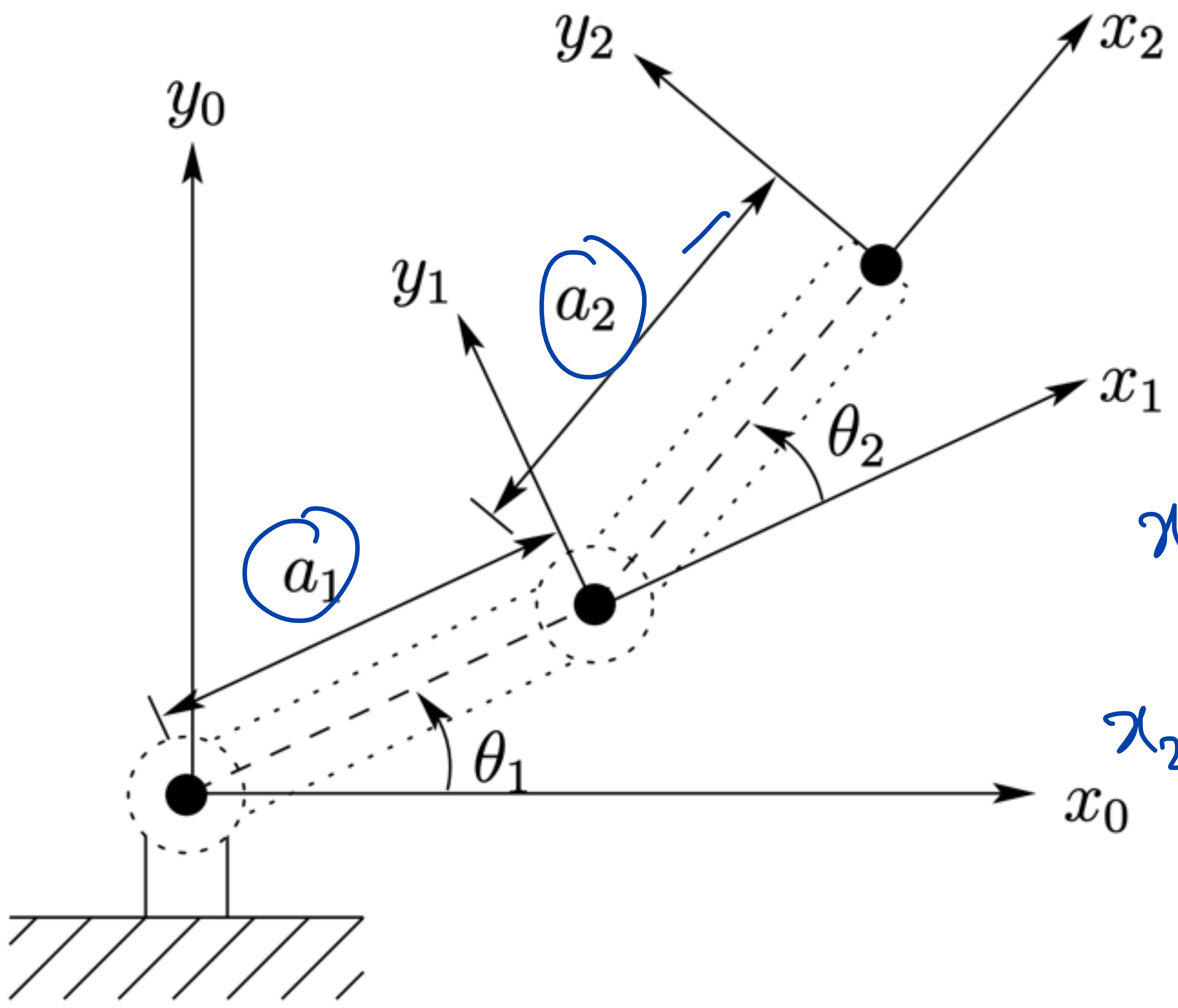
$$P_{x_1 y_1} = H_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} R(\theta_1) & & \begin{matrix} a_1\cos\theta_1 \\ a_1\sin\theta_1 \end{matrix} \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{x_0 y_0} = H_2 P_{x_1 y_1}$$

$$= H_2 H_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Example 1: Planar Elbow Manipulator



$\cos\theta_1$ $\sin\theta_1$

$x_1 y_1 \rightarrow x_0 y_0$

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

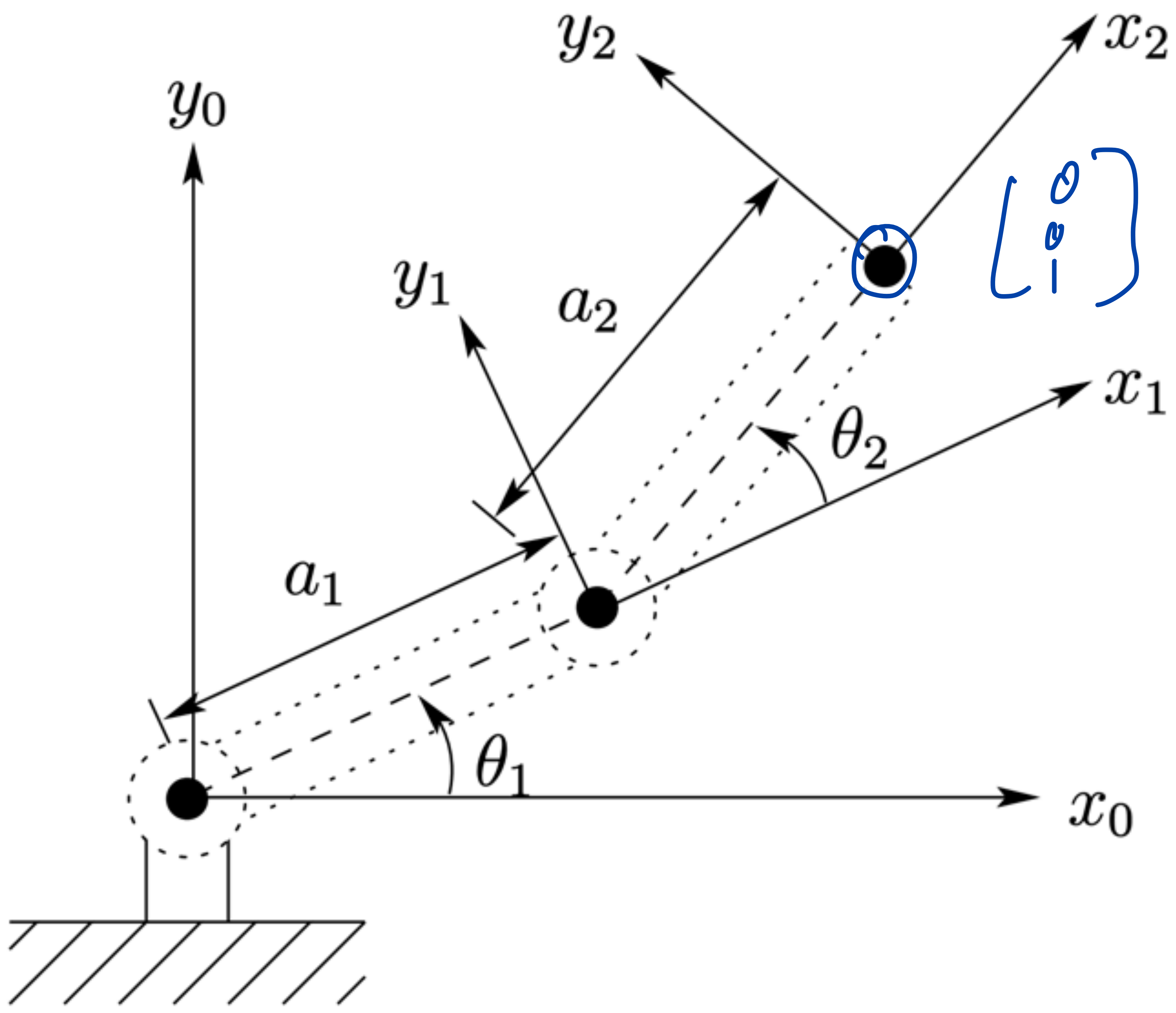
R

$x_2 y_2 \rightarrow x_1 y_1$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\cos\theta_2$ $\sin\theta_2$

Example 1: Planar Elbow Manipulator



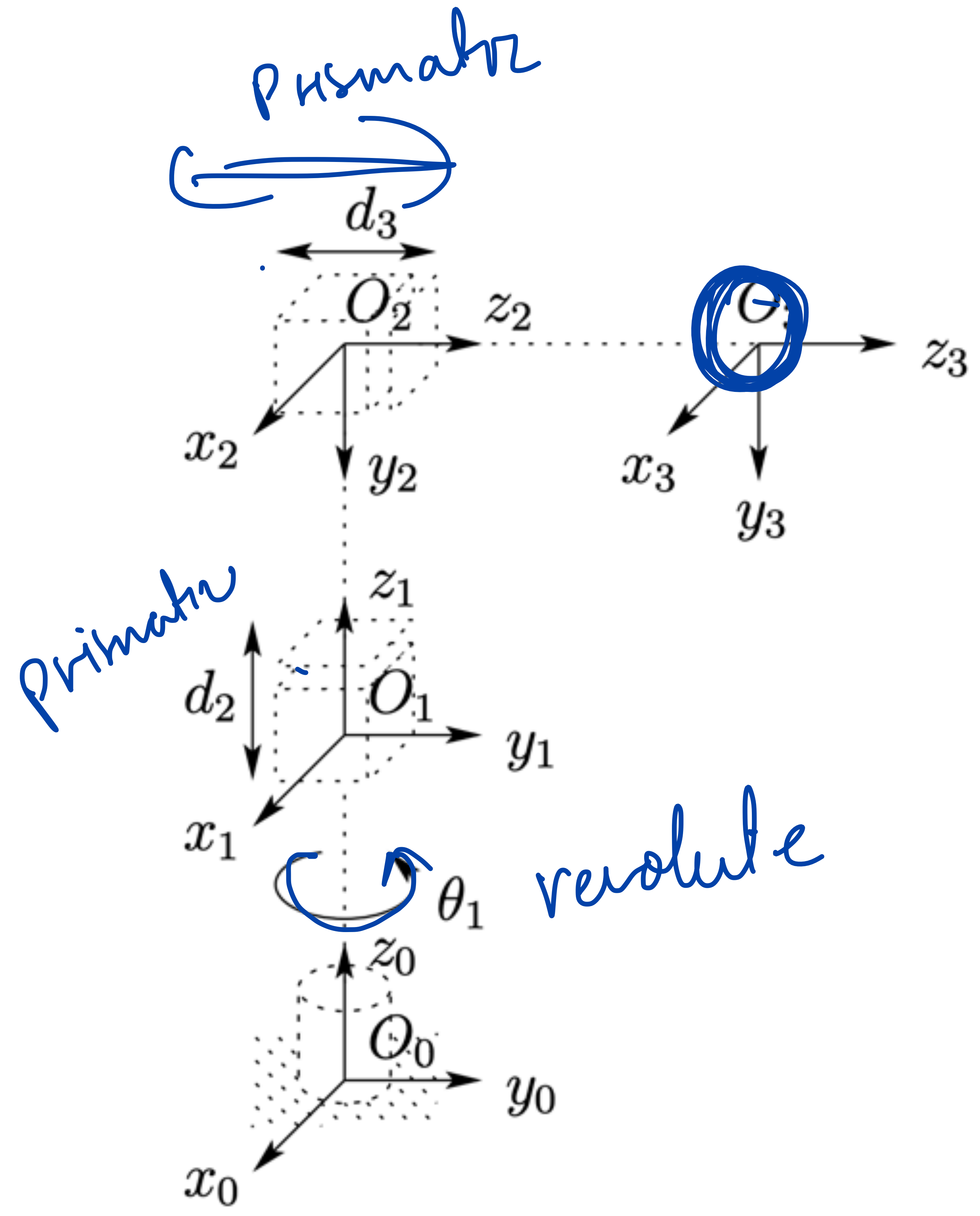
$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 = A_1A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 2: Three linked cylindrical robot



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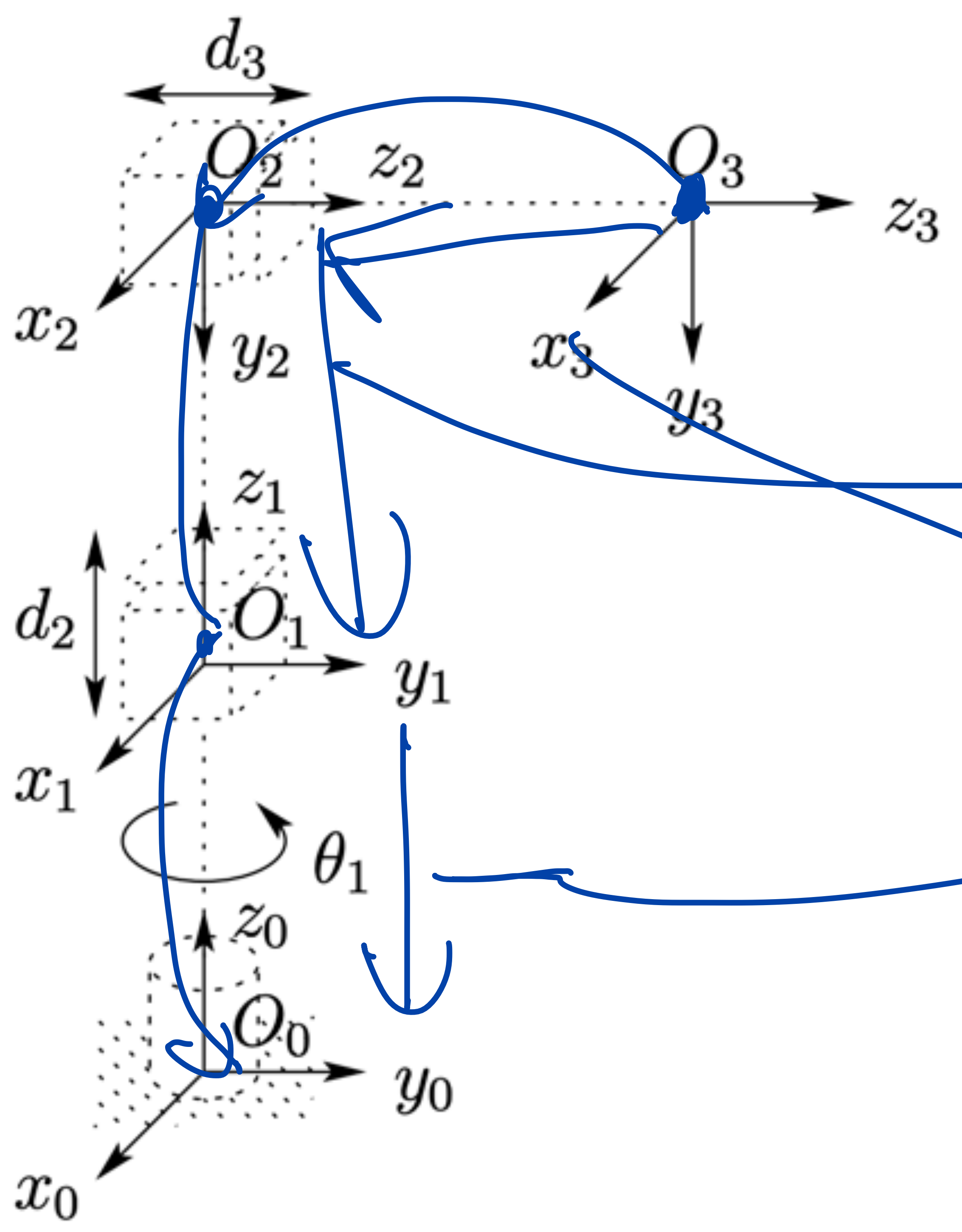

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 3: SCARA robot