Today:

$\boxed{\text{Ken}}$

- $\boxed{3.3}$ Statements with Multiple Quantifiers
- $\boxed{3.4}$ Arguments with Quantified Statements
- $\boxed{4.1}$ Direct Proof and Counterexample I

Last time:

- $\boxed{3.2}$ Predicates & Quantifiers II
- $\boxed{3.3}$ Statements with Multiple Quantifiers

Note: Negation of a Biconditional

| $p$ | $q$ | $\neg q$ | $p \leftrightarrow q$ | $\neg(p \leftrightarrow q)$ | $p \leftrightarrow \neg q$ |
|---|---|---|---|---|---|
| T | T | F | T | F | F |
| T | F | T | F | T | T |
| F | T | F | F | T | T |
| F | F | T | T | F | F |

Note:
$\mathbb{Q}[x]$ is the set of all polynomials with rational coefficients, in a single variable $x$. E.g. if $p(x) = x^2 - 2$ then $p \in \mathbb{Q}[x]$.
Another example is $p(x) = \frac{2}{3}x^2 - x + \frac{1}{3}$.

Recall the example:

$\forall x \in \mathbb{R}$ ($x$ is transcendental) false

$\neg \forall x \in \mathbb{R}$ ($x$ is transcendental) $\equiv$ $\exists x \in \mathbb{R}$ ($x$ is not transcendental) true

Consider the two statements:

① For any real number $c$, there exists a polynomial $p$ with rational coefficients such that $c$ is algebraic if and only if $p(c) = 0$.

② There exists a real number $c$ such that, for any polynomial $p$ with rational coefficients, $c$ is transcendental if and only if $p(c) \neq 0$.

e.g. $\sqrt{2}$ is algebraic.

$p(x) = x^2 - 2$ and $p \in \mathbb{Q}[x]$

$p(\sqrt{2}) = (\sqrt{2})^2 - 2 = 0$

(a) Translate each statement from informal to formal language.

(b) Negate each statement and write it both formally and informally.

1 (a) $\forall c \in \mathbb{R}\ \exists p \in \mathbb{Q}[x]\ (c \text{ is algebraic} \longleftrightarrow p(c) = 0)$

true

(b) $\neg \forall c \in \mathbb{R}\ \exists p \in \mathbb{Q}[x]\ (c \text{ is algebraic} \longleftrightarrow p(c) = 0)$

$\equiv \exists c \in \mathbb{R}\ \forall p \in \mathbb{Q}[x]\ (\neg(c \text{ is algebraic} \longleftrightarrow p(c) = 0))$

$\equiv \exists c \in \mathbb{R}\ \forall p \in \mathbb{Q}[x]\ (c \text{ is algebraic} \longleftrightarrow p(c) \neq 0)$

false

2 (a) $\exists c \in \mathbb{R}\ \forall p \in \mathbb{Q}[x]\ (c \text{ is transcendental} \longleftrightarrow p(c) \neq 0)$

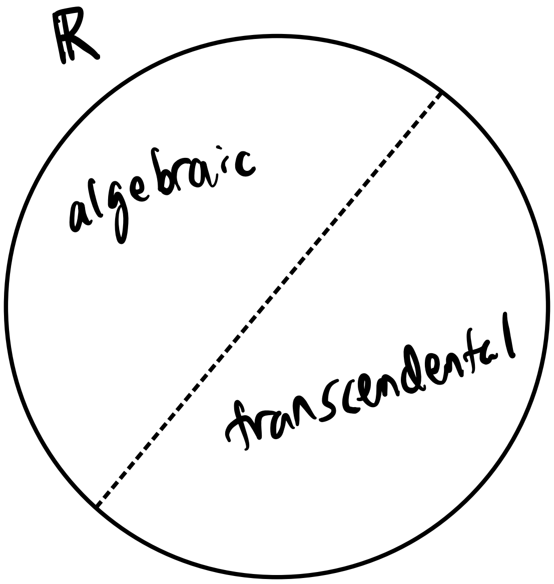
true

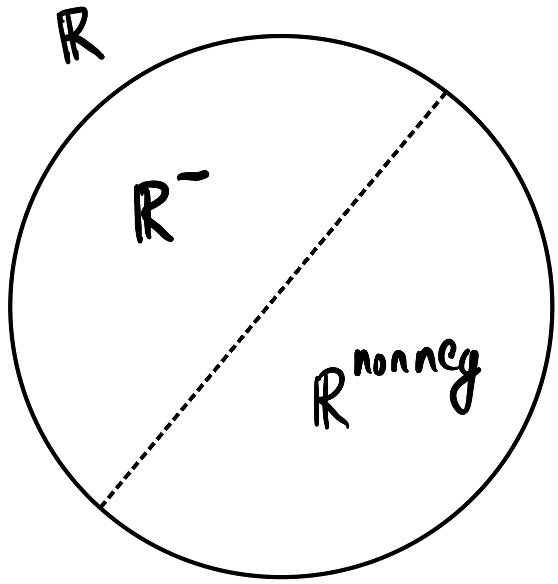
(b) $\neg \exists c \in \mathbb{R}\ \forall p \in \mathbb{Q}[x]\ (c \text{ is transcendental} \longleftrightarrow p(c) \neq 0)$

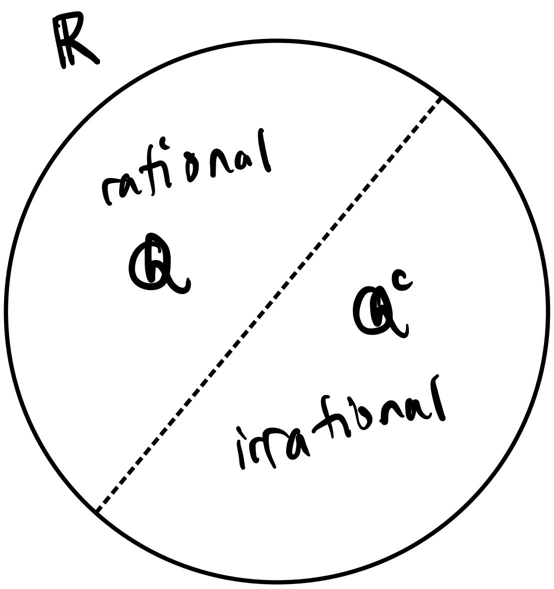
$\equiv \forall c \in \mathbb{R}\ \exists p \in \mathbb{Q}[x]\ (\neg(c \text{ is transcendental} \longleftrightarrow p(c) \neq 0))$

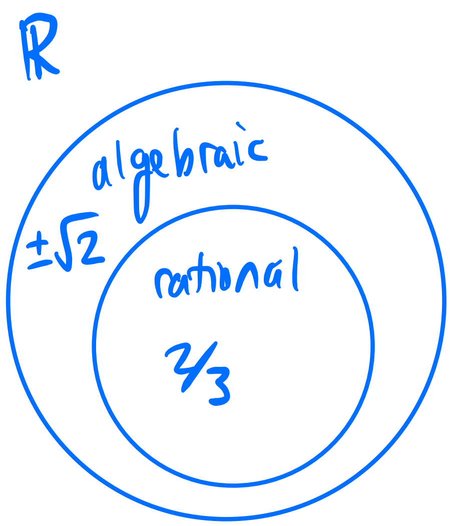
$\equiv \forall c \in \mathbb{R}\ \exists p \in \mathbb{Q}[x]\ (c \text{ is transcendental} \leftrightarrow p(c) = 0)$

false



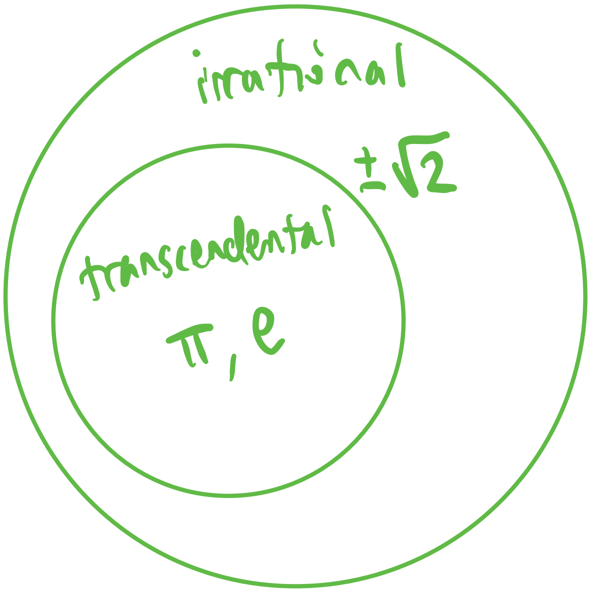

$2/3$

$p(x) = 3x - 2$

$p(2/3) = 3(2/3) - 2 = 2 - 2 = 0$

$a/b \quad b \neq 0;\ a, b \in \mathbb{Z}$

$p(x) = bx - a$



## 3.4 Arguments with Quantified Statements

**Universal Instantiation**

If a property is true of everything in a set, then it is true of any particular thing in the set.

e.g. All bachelors are unmarried.
Jon is a bachelor.
$\therefore$ Jon is unmarried.

**Universal Modus Ponens**

Formal Version

$\forall x(P(x) \rightarrow Q(x))$

$P(a)$ for some particular $a$.

$\therefore Q(a)$

Informal Version

If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true.

$a$ makes $P(x)$ true.

$\therefore$ $a$ makes $Q(x)$ true.

## Definition

$q$ is a rational number if and only if $q = m/n$ for some integers $m$ and $n$ such that $n \neq 0$.

e.g. If $x$ is a rational number, then $x$ is a real number.

$2/3$ is a rational number.

$\therefore$ $2/3$ is a real number.

$$\forall x\,(x \in \mathbb{Q} \rightarrow x \in \mathbb{R})$$

$$2/3 \in \mathbb{Q}$$

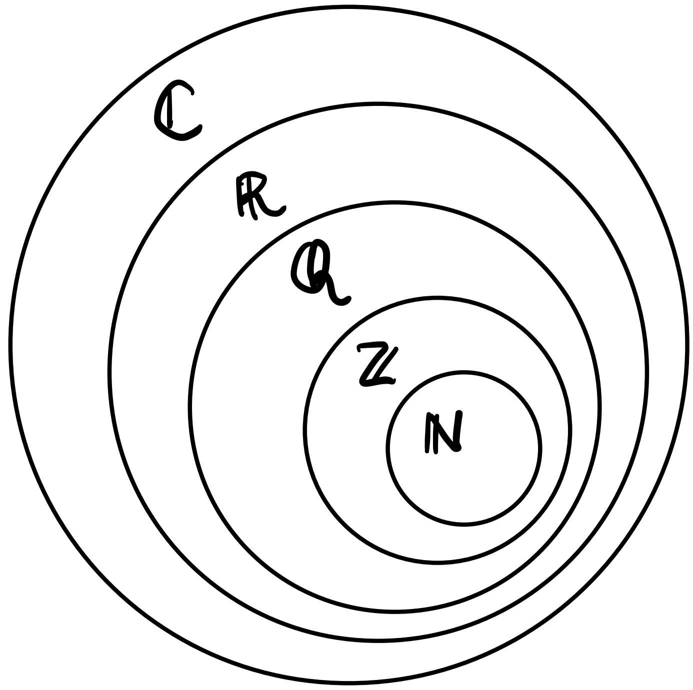
$$\therefore\ 2/3 \in \mathbb{R}$$

e.g. If $x$ is a rational number, then $x$ is a real number.

$$\forall x\,(x \in \mathbb{Q} \rightarrow x \in \mathbb{R})$$

$\mathbb{Q} \subset \mathbb{R}$ so

$$\forall x \in \mathbb{Q}\,(x \in \mathbb{R})$$




## Universal Modus Tollens

**Formal Version**

$\forall x(P(x) \to Q(x))$

$\neg Q(a)$ for some particular $a$.

$\therefore \neg P(a)$

**Informal Version**

If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true.

$a$ does not make $Q(x)$ true.

$\therefore$ $a$ does not make $P(x)$ true

e.g. All rational numbers are real numbers.

$i = \sqrt{-1}$ is not a real number.

$\therefore i = \sqrt{-1}$ is not a rational number.

$\forall x (x \in \mathbb{Q} \rightarrow x \in \mathbb{R})$

$i = \sqrt{-1} \notin \mathbb{R}$

$\therefore\ i = \sqrt{-1} \notin \mathbb{Q}$

## Definition

To say that an argument form is valid means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true. An argument is called valid if and only if its form is valid. It is called sound if and only if its form is valid and its premises are true.

e.g. All rational numbers are algebraic.

$\forall x \in \mathbb{Q}$ ($x$ is algebraic)

For any real number $x$, if $x$ is a rational number, then $x$ is algebraic.

$\forall x \in \mathbb{R}\,(x \in \mathbb{Q} \rightarrow x \text{ is algebraic})$

e.g. All transcendental numbers are irrational.

$\forall x \in \mathbb{R}\,(x \text{ is not algebraic} \longrightarrow x \notin \mathbb{Q})$

$\forall x \in \mathbb{R}\,(x \text{ is transcendental} \longrightarrow x \notin \mathbb{Q})$

## Converse Error (Quantified Form)

**Formal Version**

$\forall x\,(P(x) \longrightarrow Q(x))$

$Q(a)$ for some particular $a$.

$\therefore P(a)$

**Informal Version**

If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true.

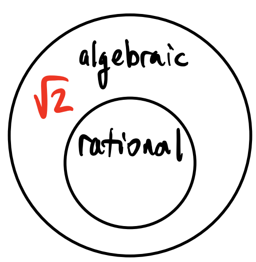
$a$ makes $Q(x)$ true.

$\therefore$ $a$ makes $P(x)$ true.

e.g. All rational numbers are algebraic.

$c$ is algebraic.

$\therefore$ $c$ is a rational number.



## Inverse Error (Quantified Form)

**Formal Version**

$\forall x (P(x) \rightarrow Q(x))$

$\neg P(a)$ for some particular $a$.

$\therefore \neg Q(a)$

**Informal Version**

If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true.

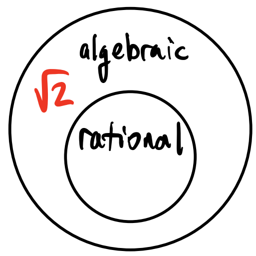
$a$ does not make $P(x)$ true

$\therefore$ $a$ does not make $Q(x)$ true

e.g. All rational numbers are algebraic.

$c$ is irrational.

$\therefore$ $c$ is not algebraic.




## Universal Transitivity

**Formal Version**

$\forall x (P(x) \rightarrow Q(x))$

$\forall x (Q(x) \rightarrow R(x))$

$\therefore \forall x (P(x) \rightarrow R(x))$

**Informal Version**

If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true.

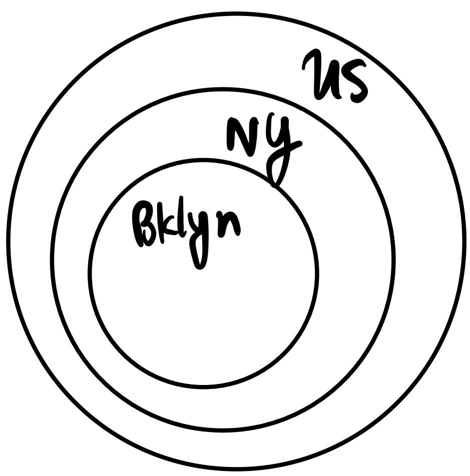
If $x$ makes $Q(x)$ true, then $x$ makes $R(x)$ true.

$\therefore$ If $x$ makes $P(x)$ true, then $x$ makes $R(x)$ true.

e.g. Apply Universal Transitivity to draw a conclusion. Can you write the argument in formal language? Is the argument valid? Is the argument sound? Draw a diagram to support your answer.

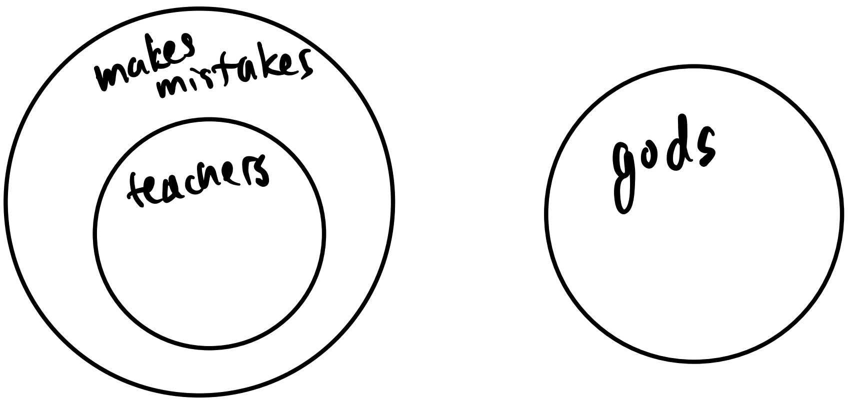
Any person born in Brooklyn was born in New York. Any person born in New York was born in the United States.

∴ Any person born in Brooklyn was born in the United States.



e.g. Indicate whether the argument is valid or invalid. Support your answer by drawing a diagram.

#23 All teachers occasionally make mistakes.
No gods ever make mistakes.
∴ No teachers are gods.




$$\forall x\,(P(x) \to Q(x))$$

$$\forall x\,(R(x) \to \neg Q(x)) \equiv \forall x\,(\neg\neg Q(x) \to \neg R(x))$$

$$\equiv \forall x\,(Q(x) \to \neg R(x))$$

$\therefore \forall x\,(P(x) \to \neg R(x))$ via universal transitivity

$\boxed{4.1}$ Direct Proof and Counterexample I

## Assumptions

① We assume a familiarity with the laws of basic algebra listed in Appendix A of the textbook.

② The Equivalence Relation "equals" or "="

For all objects $A, B, C$

(i) Reflexivity: $A = A$

(ii) Symmetry: $A = B$ if and only if $B = A$

(iii) Transitivity: if $A = B$ and $B = C$, then $A = C$

③ We use the principle of substitution:

For all objects $A$ and $B$, if $A = B$, then we may substitute $B$ wherever we have $A$.

④ (i) $\neg \exists x \in \mathbb{Z} \, (0 < x < 1)$

We assume there is no integer between 0 and 1.

(ii) $\forall x \in \mathbb{Z} \, \forall y \in \mathbb{Z} \, (x + y \in \mathbb{Z})$

(iii) $\forall x \in \mathbb{Z} \, \forall y \in \mathbb{Z} \, (x - y \in \mathbb{Z})$

(iv) $\forall x \in \mathbb{Z} \, \forall y \in \mathbb{Z} \, (xy \in \mathbb{Z})$

We assume the set of integers are closed under (i) addition (ii) subtraction (iii) multiplication

## Even, Odd, Prime, and Composite Integers

An integer n is **even** if and only if n equals twice some integer. An integer n is **odd** if and only if n equals twice some integer plus 1.

I.e.

(i) n is **even** $\iff n = 2K$ for some integer K

(ii) n is **odd** $\iff n = 2K+1$ for some integer K

We may denote the set of even integers

$$2\mathbb{Z} = \{0, \pm 2, \pm 4, \pm 6, \dots\}$$

and odd integers

$$\mathbb{Z} - 2\mathbb{Z} = \{\pm 1, \pm 3, \pm 5, \pm 7, \dots\}$$

We may formalize (i) & (ii):

(i) $n \in 2\mathbb{Z} \iff \exists K \in \mathbb{Z}\,(n = 2K)$

(ii) $n \in \mathbb{Z} - 2\mathbb{Z} \iff \exists K \in \mathbb{Z}\,(n = 2K+1)$