Today:

$\boxed{2.3}$ Valid & Invalid Arguments

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Last time:

$\boxed{2.1}$ Logical Form & Equivalence

$\boxed{2.2}$ Conditional Statements

$\boxed{2.3}$ Valid & Invalid Arguments

2.3

Definition

An argument is a sequence of statements, and an argument form is a sequence of statement forms. All statements in an argument and all statement forms in an argument form, except for the final one, are called premises (or assumptions or hypotheses). The final statement or statement form is called the conclusion. The symbol $\therefore$ meaning "therefore" is normally placed just before the conclusion.

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e.g. $\sqrt{2} < e$
$e < \pi$
$\therefore \sqrt{2} < e < \pi$ also abbreviated

To say that an argument form is valid means that no matter what particular statements are substituted for the statements variables in its premises, if the resulting premises are all true, then the conclusion is also true. To say an argument is valid means that its form is valid.

Testing an Argument Form for Validity

① Identify the premises and conclusion of the argument form.

② Construct a truth table showing the truth values of all the premises and the conclusion.

③ A row of the truth table in which all the premises are true is called a critical row. If there is a critical row in which the conclusion is false, then it is possible for an argument of the given form to have true premises and a false conclusion, and so the argument form is invalid. If the conclusion in every critical row is true, then the argument form is valid.

$\boxed{\text{Caution:}}$ If at least one premise of an argument is false, then we have no information about the conclusion: It might be true or it might be false.

e.g. multiple critical rows

$\neg p \vee q$ premise

$\therefore p \to q$ conclusion

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

3 critical rows
argument is valid

e.g. "proof by cases"

$2^3 = 8$

$p \vee q$
$p \to r$
$q \to r$
$\therefore\ r$

p	q	r	$p \vee q$	$p \to r$	$q \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	T	T	T
T	F	F	T	F	T
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	F	T	T
F	F	F	F	T	T

3 critical rows
valid

e.g. $\neg p \to q \vee r$

$q \vee r$

$\therefore p \vee r$

p	q	r	$\neg p$	$p \vee r$	$q \vee r$	$\neg p \to q \vee r$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	F	F	F

6 critical rows

not a valid argument

modus ponens

$p \to q$

p

$\therefore q$

modus tollens

$p \to q$

$\sim q$

$\therefore \sim p$

Additional Valid Argument Forms

Example 2.3.3 Generalization

$$\begin{array}{l} p \\ \therefore p \vee q \end{array} \qquad \begin{array}{l} q \\ \therefore p \vee q \end{array}$$

"$\vee$ intro"
"OR intro"

Example 2.3.4 Specialization

$$\begin{array}{l} p \wedge q \\ \therefore p \end{array} \qquad \begin{array}{l} p \wedge q \\ \therefore q \end{array}$$

"$\wedge$ elim"
"AND elim"

Example 2.3.5 Elimination

$$\begin{array}{l} p \vee q \\ \sim q \\ \therefore p \end{array} \qquad \begin{array}{l} p \vee q \\ \sim p \\ \therefore q \end{array}$$

Example 2.3.6 Transitivity

$$\begin{array}{l} p \to q \\ q \to r \\ \therefore p \to r \end{array}$$

Example 2.3.7 Proof by Division into Cases

$$\begin{array}{l} p \vee q \\ p \to r \\ q \to r \\ \therefore r \end{array}$$

"proof by cases"
see previous

Fallacies

A fallacy is an error in reasoning that results in an invalid argument. Three common fallacies are using ambiguous premises, and treating them as if they were unambiguous, circular reasoning (i.e. assuming what is to be proved without having derived it from the premises), and jumping to a conclusion (without adequate grounds).

Converse Error

fallacy of affirming the consequent

$p \to q$
q
$\therefore p$

e.g. If Jon is traveling to work, then Jon rides the subway.
Jon rides the subway.
Therefore Jon is traveling to work.

$p \to q \equiv \neg q \to \neg p$ (an aside)
$\neg q$

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e.g. If a function f is trigonometric, then f is periodic.

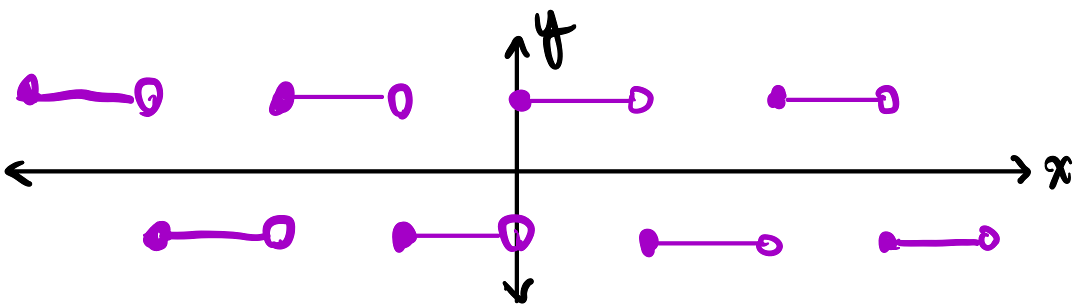
f is periodic.

Therefore f is trigonometric.

e.g. a counterexample

For any integer K, define the real function $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} -1 & 2K-1 \leq x < 2K \\ 1 & 2K \leq x < 2K+1 \end{cases}$$



Periodic function

$f: \mathbb{R} \to \mathbb{R}$ and $c > 0$, $c \in \mathbb{R}$,

for any $t \in \mathbb{R}$, $f(t+c) = f(t)$

Inverse Error

fallacy of denying the antecedent

$p \to q$

$\neg p$

$\therefore \neg q$

e.g. If Jon is traveling to work, then Jon rides the subway.
Jon is not traveling to work.
Therefore Jon does not ride the subway.

e.g. If A is an invertible matrix, then A is a square matrix.
A is not invertible.
Therefore A is not a square matrix.

e.g. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{R})$

$\det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0(0) - 0(1) = 0$

Validity vs Truth

Validity is a property of argument forms.
Truth is a property of statement forms.

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e.g. A Valid Argument with a False Premise
and a False Conclusion

If Canada is north of the United States, then the temperatures in Canada never rise above freezing. $p \to q$
Canada is north of the United States. p
∴ Temperatures in Canada never rise above freezing. $\therefore q$

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e.g. An Invalid Argument with True Premises
and a True Conclusion

If Boston is in New England, then Allston-Brighton is in New England. $p \to q$
Allston-Brighton is in New England. q
∴ Boston is in New England. $\therefore p$

Definition

An argument is called sound if and only if it is valid AND all its premises are true. An argument that is not sound is called unsound.

Proof by Contradiction (Contradiction Rule)

If we can show that the assumption p is false leads logically to a contradiction, then we can conclude that p is true.

$\neg p \to \bot$ $\qquad\qquad$ $\bot \equiv c$

$\therefore p$

p	$\neg p$	$\bot \equiv p \wedge \neg p$	$\neg p \to \bot$
T	F	F	T
F	T	F	F

Knights & Knaves

- Knights always tell the truth
- Knaves always lie

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(a) A says: both of us are knights
B says: A is a knave
What are A & B?

subproof

1	Suppose A is a knight.	assumption
2	A always speaks truth.	by def. of knight
3	A & B are both knights	what A said
4	B is a knight	specialization
5	A is a knave.	what B said
6	A is a knight and knave.	a contradiction via 1 5 conjunction
7	A is a knave	contradiction rule
8	B speaks truth.	via 7, what B said
9	B is a knight.	8 by def. of knight