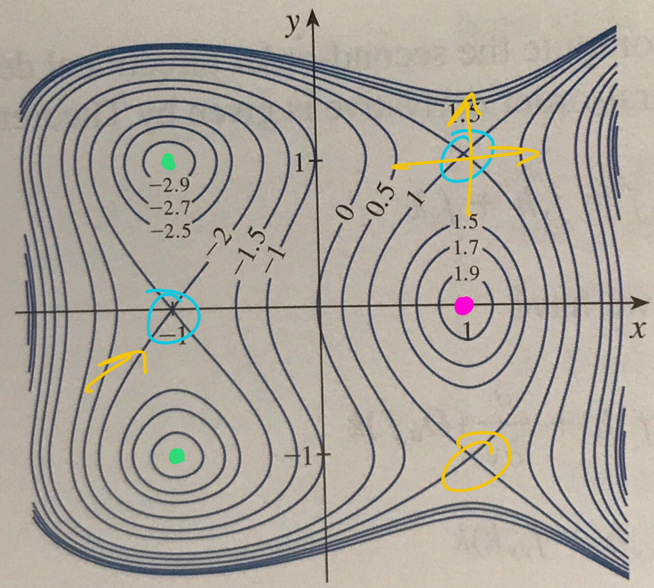
14.7 Max/min

(4)

4. $f(x, y) = 3x - x^3 - 2y^2 + y^4$



Option (1) Level curves

find max/min.

local min

$(-1,1), (-1,-1)$

local max

$(1,0)$.

Saddles at

$(1,1), (1,-1)$

& $(-1,0)$

Option 2: gradient.

$f(x, y) = 3x - x^3 - 2y^2 + y^4$

$$\vec{\nabla} f = \langle 3-3x^2, -4y+4y^3 \rangle = \vec{0} = \langle 0,0 \rangle$$

$3-3x^2=0$ $\qquad$ $-4y+4y^3=0$

$-y+y^3=0 \Rightarrow y(-1+y^2)=0$

$x = \pm 1$ $\qquad$ $y = 0, \pm 1$

Critical Points:

$(1,0)$, $(1,1)$, $(1,-1)$

$(-1,0)$, $(-1,1)$, $(-1,-1)$

Classify c.p.

use 2^{nd} derivatives test : $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$

$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$

$D > 0$, $f_{xx} > 0$ local min

$D > 0$ $f_{xx} < 0$ local max

$D < 0$ Saddle point

$D = 0$ Inconclusive - go look at level curves

$f(x, y) = 3x - x^3 - 2y^2 + y^4$

$f_x = 3 - 3x^2$ $f_{xx} = -6x$

$f_y = -4y + 4y^3$ $f_{yy} = -4 + 12y^2$

$f_{xy} = 0$

$D = (-6x)(-4 + 12y^2) - 0^2$

check:	$(1,0)$	$(1,1)$	$(1,-1)$	$(-1,0)$	$(-1,1)$	$(-1,-1)$
D	24	-48	-48	-24	48	48
$f_{xx} = -6x$	$-$	$\times$	$\times$	$\times$	$+$	$+$
Result	max	saddle	saddle	saddle	min	min

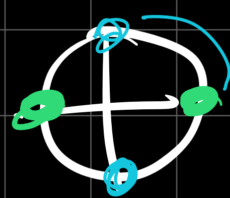
(19) $f(x,y) = y^2 - 2y\cos x \qquad -1 \le x \le 7.$

find local max/min.

$\vec{\nabla} f = \langle 2y\sin x,\ 2y - 2\cos x \rangle = \langle 0,0 \rangle$

$y\sin x = 0 \qquad\qquad y - \cos x = 0 \Rightarrow y = \cos x.$

$y = 0$ or $\sin x = 0$



if $y=0$ $\quad y = \cos x \Rightarrow \quad \cos x = 0$

$x = \frac{\pi}{2},\ \frac{3\pi}{2}$

if $\sin x = 0 \quad x = 0, \pi, 2\pi \qquad y = 1, -1$

critical points:

$(\frac{\pi}{2}, 0)$, $(\frac{3\pi}{2}, 0)$, $(0, 1)$, $(\pi, -1)$, $(2\pi, 1)$

$f_x = 2y\sin x$ $\quad f_y = 2y - 2\cos x$

Classify: $D = (2y\cos x)(2) - (2\sin x)^2$

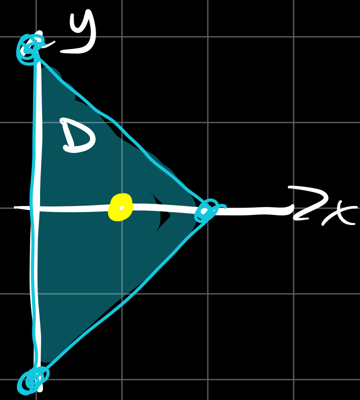
$= 4y\cos x - 4\sin^2 x$

now check all points.

⋮

(31) $f(x,y) = x^2 + y^2 - 2x$ $\quad$ D is closed triangle w/ vertices $(2,0)$, $(0,2)$, $(0,-2)$

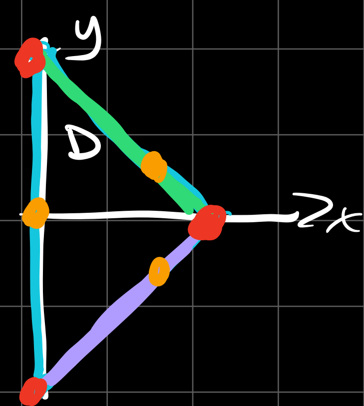
find abs max/min.



① find c.p.

$\vec{\nabla} f = \vec{0} \Rightarrow \langle 2x-2, 2y \rangle = \langle 0, 0 \rangle \Rightarrow x = 1, y = 0$

$f(1,0) = -1$



② find max/min on boundary.

$f(x,y) = x^2 + y^2 - 2x$

on • $x = 0$ $\quad -2 \le y \le 2$

$f(x,y)$ becomes: $y^2 = f(0,y)$
$= g(y)$

min: 0 at $(0,0)$

max: 4 at $(0,\pm 2)$

on $y = x-2 \quad 0 \le x \le 2$

$h(x) = f(x, x-2) = x^2 + (x-2)^2 - 2x$

$h'(x) = 2x + 2(x-2) - 2 = 4x - 6 = 0 \quad x = \frac{3}{2} \quad y = -\frac{1}{2}$

$(\frac{3}{2}, -\frac{1}{2})$

Endpoints: $(0,-2)$, $(2,0)$

x^2+y^2-2x

values: $f(0,-2) = 4$

$f(2,0) = 0$

$f(\frac{3}{2}, -\frac{1}{2}) = \frac{9}{4} + \frac{1}{4} - 3 = \frac{10}{4} - \frac{12}{4} = \frac{-2}{4} = -\frac{1}{2}$

on $y = -x+2 \quad 0 \le x \le 2$

$f(x, -x+2) = x^2 + (-x+2)^2 - 2x$

$\frac{d}{dx}\left(x^2 + (-x+2)^2 - 2x\right) = 0 \Rightarrow x = \frac{3}{2}, y = \frac{1}{2}$

$f(\frac{3}{2}, \frac{1}{2}) = (\frac{3}{2})^2 + (\frac{1}{2})^2 - 3(\frac{3}{2}) = -\frac{1}{2}$

endpoints already checked!

Points	Value
$(\frac{3}{2}, \frac{1}{2})$	$-\frac{1}{2}$
$(\frac{3}{2}, -\frac{1}{2})$	$-\frac{1}{2}$
$(2,0)$	0
$(0,-2)$	4
$(0,2)$	4
$(0,0)$	0
$(1,0)$	-1

— max value at $(0, \pm 2)$

— min value at $(1,0)$

(45) find three positive numbers that add to 100 and have a maximum product

$$0 < x < 100$$
$$0 < y < 100$$
$$0 < z < 100$$

$$x + y + z = 100 \qquad f(x,y,z) = xyz$$

$$z = 100 - x - y$$

$$g(x,y) = xy(100 - x - y) = 100xy - x^2y - xy^2$$

$$\vec{\nabla} g = \langle 100y - 2xy - y^2,\ 100x - x^2 - 2xy \rangle = \langle 0, 0 \rangle$$

$$\begin{cases} 100y - 2xy - y^2 = 0 \\ 100x - 2xy - x^2 = 0 \end{cases}$$

$$\begin{cases} y(100 - 2x - y) = 0 \\ x(100 - 2y - x) = 0 \end{cases} \qquad y, x \neq 0$$

$$\begin{array}{r} 2x + y = 100 \\ + \ (-2)(x + 2y) = (100) - 2 \\ \hline \end{array}$$

$-3y = -100 \Rightarrow y = \frac{100}{3} \Rightarrow x = \frac{100}{3}$

use $x + y + z = 100$ to find $z = \frac{100}{3}$

max product $\left(\frac{100}{3}\right)^3$

what if we let $f(x, y, z) = xyz$

found $\nabla f = \langle yz, xz, xy \rangle = \vec{0}$

$yz = 0 \qquad xz = 0 \qquad xy = 0$

lose context of our problem b/c we don't have our constraint.

Spoiler Alert!!

$f(x,y,z)$

constraint.

$g(x,y,z)=C$

$\underline{x+y+z=100}$

$\vec{\nabla} f$ $\qquad$ $\vec{\nabla} g$

find where are parallel.

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

scalar = Lagrange Multiplier.

$$\langle yz, xz, xy \rangle = \lambda \langle 1,1,1 \rangle$$

$x,y,z \neq 0$ bk of our problem.

$$\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \end{aligned} \quad \Rightarrow \quad \begin{aligned} y\cancel{z} &= x\cancel{z} \Rightarrow y = x \\ \cancel{x}z &= \cancel{x}y \qquad z = y \\ \cancel{y}z &= x\cancel{y} \qquad z = x \end{aligned}$$

$$x = y = z$$

<u>and</u> $\quad x+y+z=100$

$$x = y = z = \frac{100}{3}.$$