Exercise 1 - (6%)

State the converse, contrapositive, and inverse of the conditional statement.

A positive integer is a prime only if it has no divisors other than 1 and itself.

Con : A positive integer has no divisors other than 1 and itself only if it is a prime ✓

Contra: A positive integer has divisors other than one and itself only if it is not a prime ✓

Inverse A positive integer is not a prime only if it has divisors other than one and itself ✓

Exercise 2 - (5%)

Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent.

p	q	$\neg p \leftrightarrow q$	$p \leftrightarrow \neg q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

✓

Exercise 3 - Translation (12%)

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

$P(x)$ = x is perfect

(a) No one is perfect. $\forall x \neg P(x) = \neg \exists x P(x)$ ✓ $F(x)$ = x is my friend

(b) Not everyone is perfect. $\neg \forall x P(x)$ ✓ $= \exists x \neg P(x)$

(c) All your friends are perfect. $\forall x (F(x) \rightarrow P(x))$ ✓

(d) At least one of your friends is perfect. $\exists x (F(x) \rightarrow P(x))$

(e) Everyone is your friend and is perfect. $\forall x (F(x) \wedge P(x))$

(f) Not everybody is your friend or someone is not perfect. $\neg \forall x (F(x)) \vee \exists x \neg P(x)$

Exercise 4 - Proof (8%)

Show that $\sqrt{2}+\sqrt{3}$ is an irrational number or a rational number?

let $x = \sqrt{2}+\sqrt{3}$

$x^2 = 2 + 2\sqrt{6} + 3$

$= 5 + 2\sqrt{6}$

x^2 is irrational b/c $\sqrt{6}$ is irrational

thus x is irrational because $(\frac{p}{q})^2 = \frac{p^2}{q^2}$, which must be rational

Proving $\sqrt{6}$ is irrational:

assume $\sqrt{6}$ is rational, $\sqrt{6} = \frac{p}{q}$, $p \in \mathbb{Z}$, $q \in \mathbb{Z}$, $q \neq 0$, p & q in lowest term,

$6 = \frac{p^2}{q^2} \Rightarrow 6q^2 = p^2 \Rightarrow p$ is divisible by 6, so $p = 6K$, $K \in \mathbb{Z}$

$6q^2 = 36k^2$

$q^2 = 6k^2 \Rightarrow q$ is divisible by 6, so $q = 6q$, $q \in \mathbb{Z}$

so $\frac{p}{q} = \frac{6K}{6q}$, which contradicts assumption.

$\therefore \sqrt{6}$ is irrational

Exercise 5 - Sets (9%)

Can you conclude that $A = B$ if A, B, and C are sets such that

(a) $A \cup C = B \cup C$? Yes No

(b) $A \cap C = B \cap C$? Yes No

(c) $A \cup C = B \cup C$ and $A \cap C = B \cap C$? Yes ?

Exercise 6 - Sets (8%)

Suppose that g is a function from A to B and f is a function from B to C.

(a) Show that if both f and g are injective functions, then $f \circ g$ is also injective.

(b) Show that if both f and g are surjective functions, then $f \circ g$ is also surjective.

a) f is injective $\Rightarrow \forall a,b\ ((f(a)=f(b)) \to a=b))$

g is injective $\Rightarrow \forall a,b\ ((g(a)=g(b)) \to a=b))$

assuming there is $f(g(a)) = f(g(b))$. Since f is injective, $g(a) = g(b)$. Since g is injective, $a=b$.

b) f is surjective $\Rightarrow \forall y \in C\ \exists x \in B\ (f(x)=y)$

g is surjective $\Rightarrow \forall y \in B\ \exists x \in A\ (g(x)=y)$

So there exists an x such that $f(y)=x$

there also exists a y such that $g(z)=y$

thus $f(g(z))=x$, and since f and g are onto,

every $z \in A$ maps to $x \in C$, $\Rightarrow f \circ g$ is onto.

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Exercise 7 - Power set (6%)

Suppose that A and B are sets such that the power set of A is a subset of the power set of B. Does it follow that A is a subset of B? Prove your answer.

If $x \in A$, $\{x\} \in P(A)$

Since $P(A) \subseteq P(B)$, $\{x\} \in P(B) \Rightarrow x \in B$, thus $A \subseteq B$

Exercise 8 - Matrices (6%)

Let **A** be the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

where $ad - bc \neq 0$, show its inverse.

$AA^{-1} = I$ let $A^{-1} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$ax_1 + bx_3 = 1$
$ax_2 + bx_4 = 0$
$cx_1 + dx_3 = 0$
$cx_2 + dx_4 = 1$

$x_1 = \frac{1-bx_3}{a} = \frac{1-b(-\frac{cx_1}{d})}{a} = \frac{1+\frac{bcx_1}{d}}{a} = \frac{d+bcx_1}{ad}$

$x_3 = \frac{-cx_1}{d}$

$adx_1 = d + bcx_1$
$(ad-bc)x_1 = d$
$x_1 = \frac{d}{ad-bc}$

and so on...

Exercise 9 - Induction (8%)

The harmonic numbers are defined by $H_j = \sum_{k=1}^{j} \frac{1}{k}$. Show that for all $n \geqslant 0$, $H_{2^n} \geqslant 1 + \frac{n}{2}$.

Exercise 10 - Probability (8%)

Suppose that one person in 100,000 has a particular rare disease for which there is a fairly acc diagnostic test. This test is correct 99.0% of the time when given to a person selected at ra who has the disease; it is correct 99.5% of the time when given to a person selected at randor does not have the disease. Given this information can we find

(a) the probability that a person who tests positive for the disease has the disease?

(b) the probability that a person who tests negative for the disease does not have the dis

Reminder: No need to give the exact number for the results. You can stop at the formulat

Exercise 11 - Combinations (9%)

How many solutions are there to the equation
$x + y + z + r = 17$,
where x, y, z, and r are non-negative integers such that

(a) $x \geqslant 1$?

(b) $x \geqslant 1$, $y \geqslant 1$, $z \geqslant 2$, and $r \geqslant 2$?

(c) $0 \leqslant r \leqslant 11$?

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Reminder: No need to give the exact number for the results. You can stop at the formulation.

D = has disease

$P(D) = 0.00001 \Rightarrow P(\bar{D}) = 0.99999$

T = test positive for D

$P(T|D) = 0.99 \Rightarrow P(\bar{T}|D) = 0.01$

$P(\bar{T}|\bar{D}) = 0.995 \Rightarrow P(T|\bar{D}) = 0.005$

a) $P(D|T) = \frac{P(T/D) \cdot P(D)}{P(T/D) \cdot P(D) + P(T/\bar{D}) \cdot P(\bar{D})} = \frac{0.99 \times 0.00001}{0.99 \times 0.00001 + 0.005 \times 0.99999} = 0.00198$

b) $P(\bar{D}|\bar{T}) = \frac{P(\bar{T}/\bar{D}) \cdot P(\bar{D})}{P(\bar{T}/\bar{D}) \cdot P(\bar{D}) + P(\bar{T}/D) \cdot P(D)} = \frac{0.995 \times 0.99999}{0.995 \times 0.99999 + 0.01 \times 0.00001} = 0.999999$

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a) let $x' = x - 1$

$x' + y + z + r = 16$

$C(16+4-1, 3)$

b) $y' = y-1$, $z' = z-2$, $r' = r-2$

$x' + y' + z' + r' = 11$

$C(11+4-1, 3)$

c) let $r' = r-12$

if $r' \geq 0$, $x+y+z+r' = 5$

$C(5+4-1, 3)$

total $= C(17+4-1, 3)$

ans $= C(17+4-1, 3) - C(5+4-1, 3)$

Exercise 12 - Binomial coefficient (5%)

What is the coefficient of x^7 in $(3-2x)^{13}$?

$$(3-2x)^{13} = \sum_{i=0}^{13} \binom{13}{i} (3)^i (-2x)^{13-i}$$

on x^7, $i=6$

$$\text{coeff} = \binom{13}{6} (3)^6 (-2)^7$$

$$= -160,123,392$$

Exercise 13 - Expected Value and Variance (10%)

Find the expectation and standard deviation of the random variables X whose value when two fair dice are rolled is $X((i,j)) = i - j$, where i is the number appearing on the first die and j is the number appearing on the second die.

$$E(X) = \frac{1}{36} \sum_{i=1}^{6} \sum_{j=1}^{6} (i-j)$$

$$= 0$$

$$V(X) = E(X^2) - E^2(X) = \cancel{17.5} \quad \frac{210}{36}$$

$$\sigma(X) = \sqrt{V(X)} = \sqrt{\cancel{17.5}} \quad \sqrt{\frac{210}{36}}$$

1	1
2	2
3	3
4	4
5	5
6	6

1	2	3	4	5	6
$1-1=0$	$2-1=1$	$3-1$ 2	3	4	5
$1-2=-1$	$2-2=0$	$3-2$ 1	2	3	4
$1-3=-2$	$2-3=-1$	$3-3=$ 0	1	2	3
$1-4=-3$	$2-4=-2$	$3-4$ -1	0	1	2
$1-5=-4$	$2-5=-3$	$3-5$ -2	-1	0	1
$1-6=-5$	$2-6=-4$	$3-6$ -3	-2	-1	0

$$\text{Sum} = \cancel{-15} + \cancel{(-9)} + \cancel{(-3)} + \cancel{3} + \cancel{9} + \cancel{15} = 0$$

$$\text{Sum}^2 = \left[6(0)^2 + 5(-1)^2 + 4(-2)^2 + 3(-3)^2 + 2(-4)^2 + 1(-5)^2\right] \times 2$$

$$= 210$$