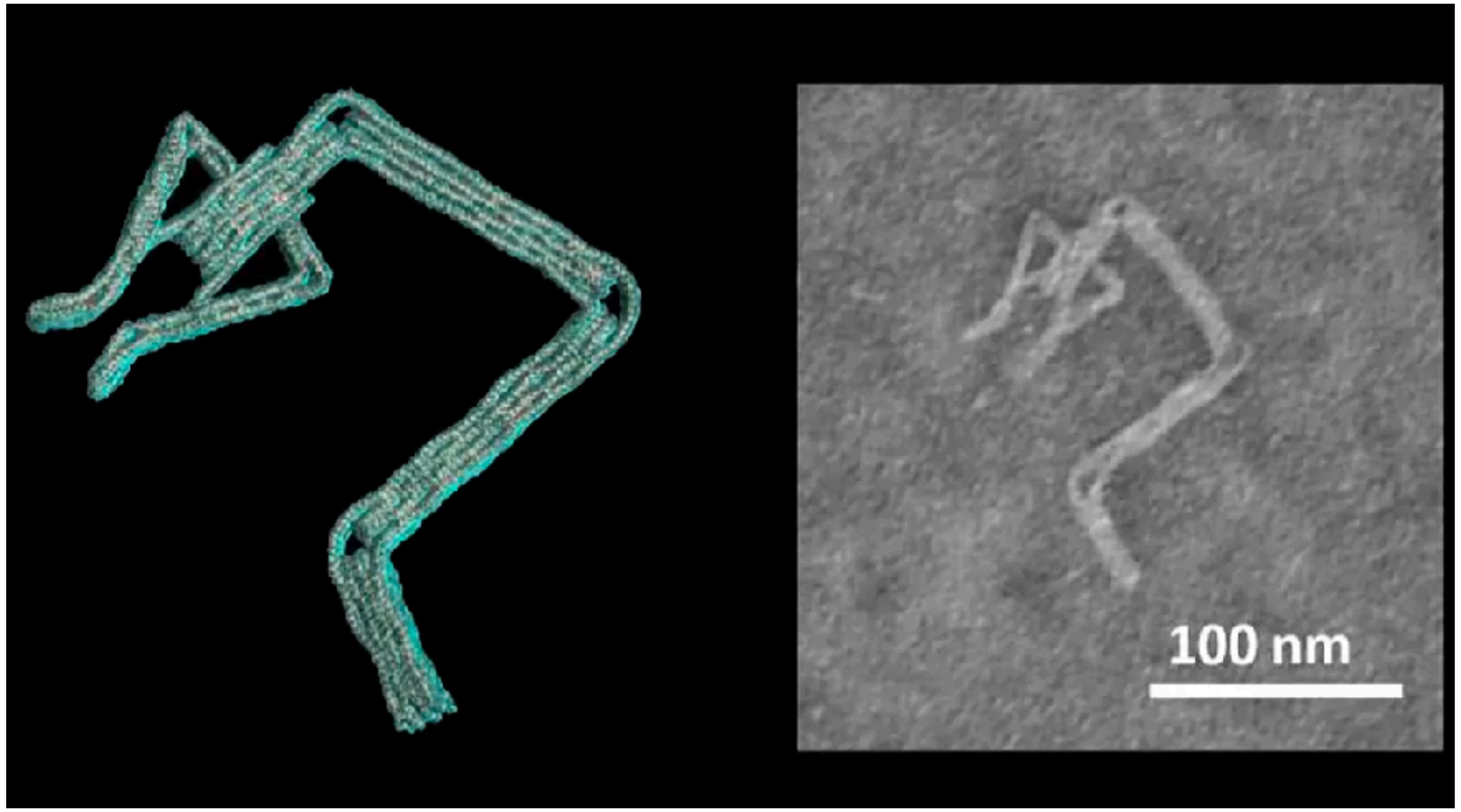
Introduction to Robot Intelligence
[Spring 2023]

Robot Transformations

February 16, 2023

Lerrel Pinto

Robots- a wide variety of sizes



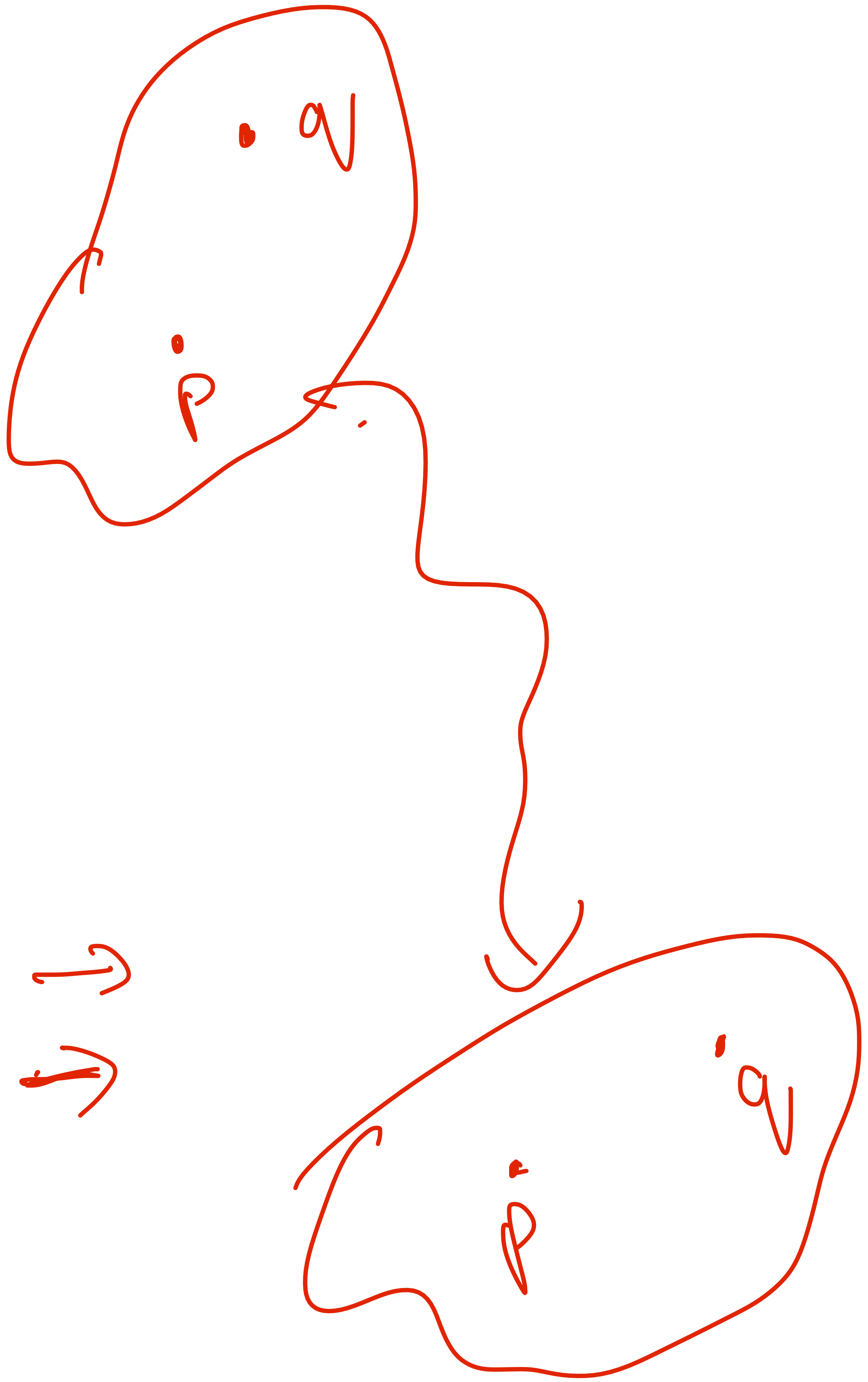
1. https://www.forconstructionpros.com/equipment/earthmoving/articulated-rigid-dump-trucks/press-release/21131477/caterpillar-cat-caterpillar-robot-trucks-haul-2-billion-tons-autonomously
2. https://www.advancedsciencenews.com/dna-robots-designed-in-minutes-instead-of-days/

Review from last class

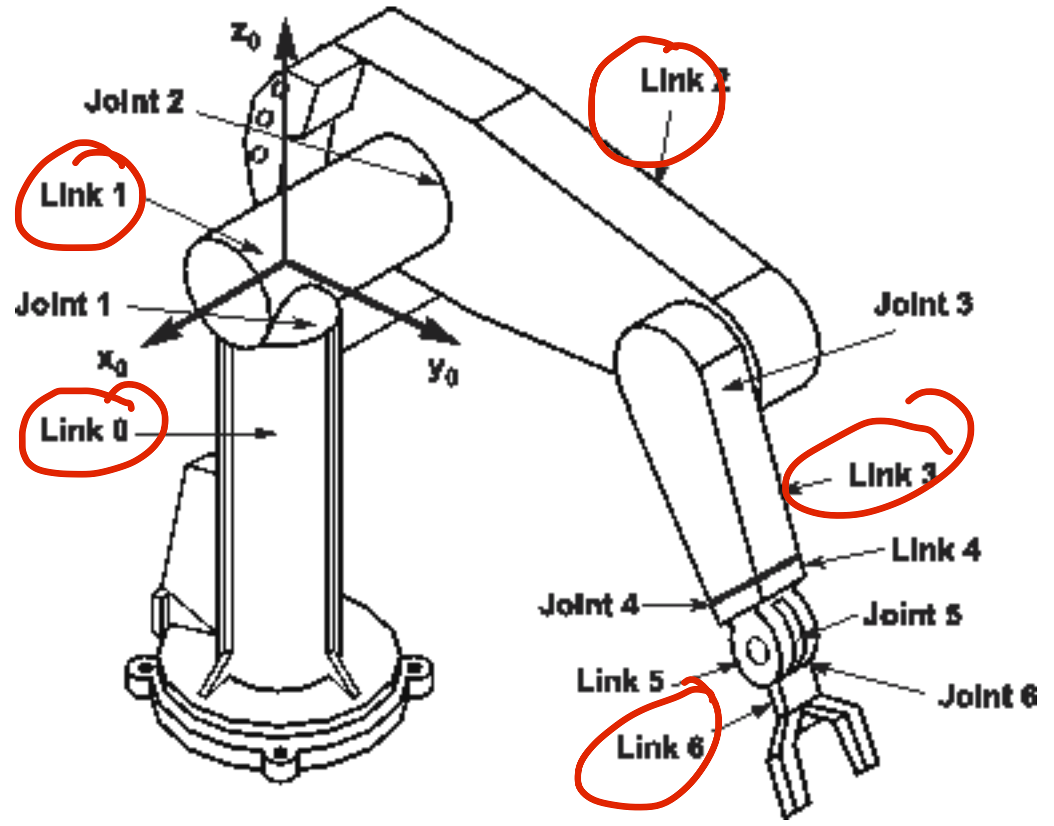
- Viewing a robot as a composition of rigid bodies.
- Fundamentals of rigid bodies

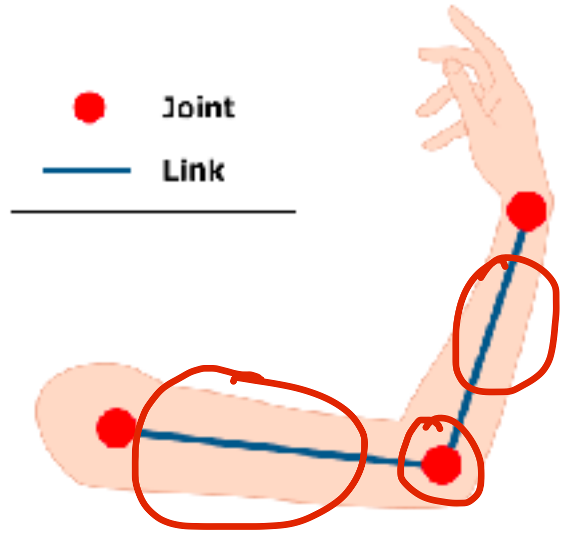
A mapping $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a *rigid body transformation* if it satisfies the following properties:

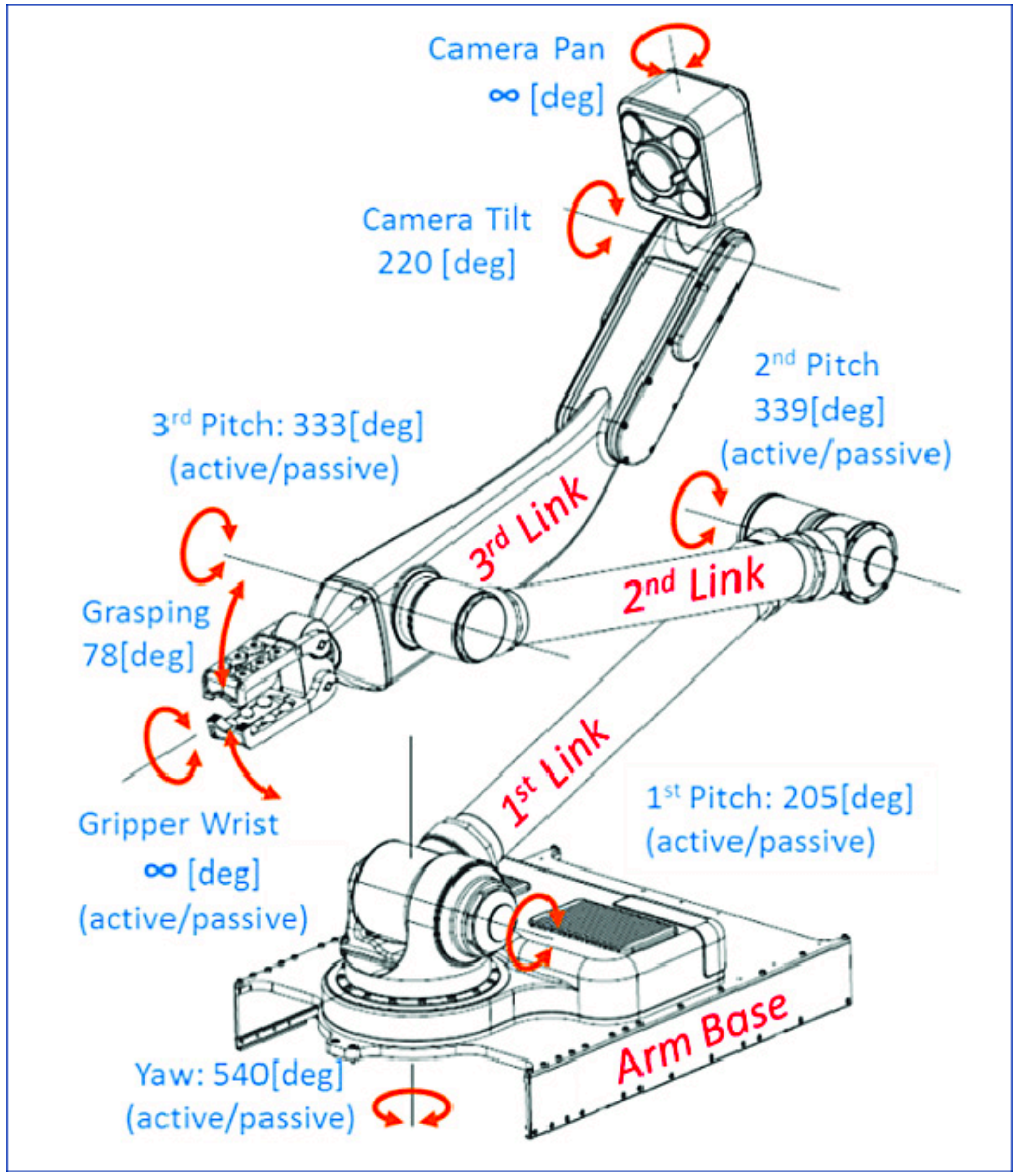
1. Length is preserved: $\|g(p) - g(q)\| = \|p - q\|$ for all points $p, q \in \mathbb{R}^3$.
2. The cross product is preserved: $g_*(v \times w) = g_*(v) \times g_*(w)$ for all vectors $v, w \in \mathbb{R}^3$.



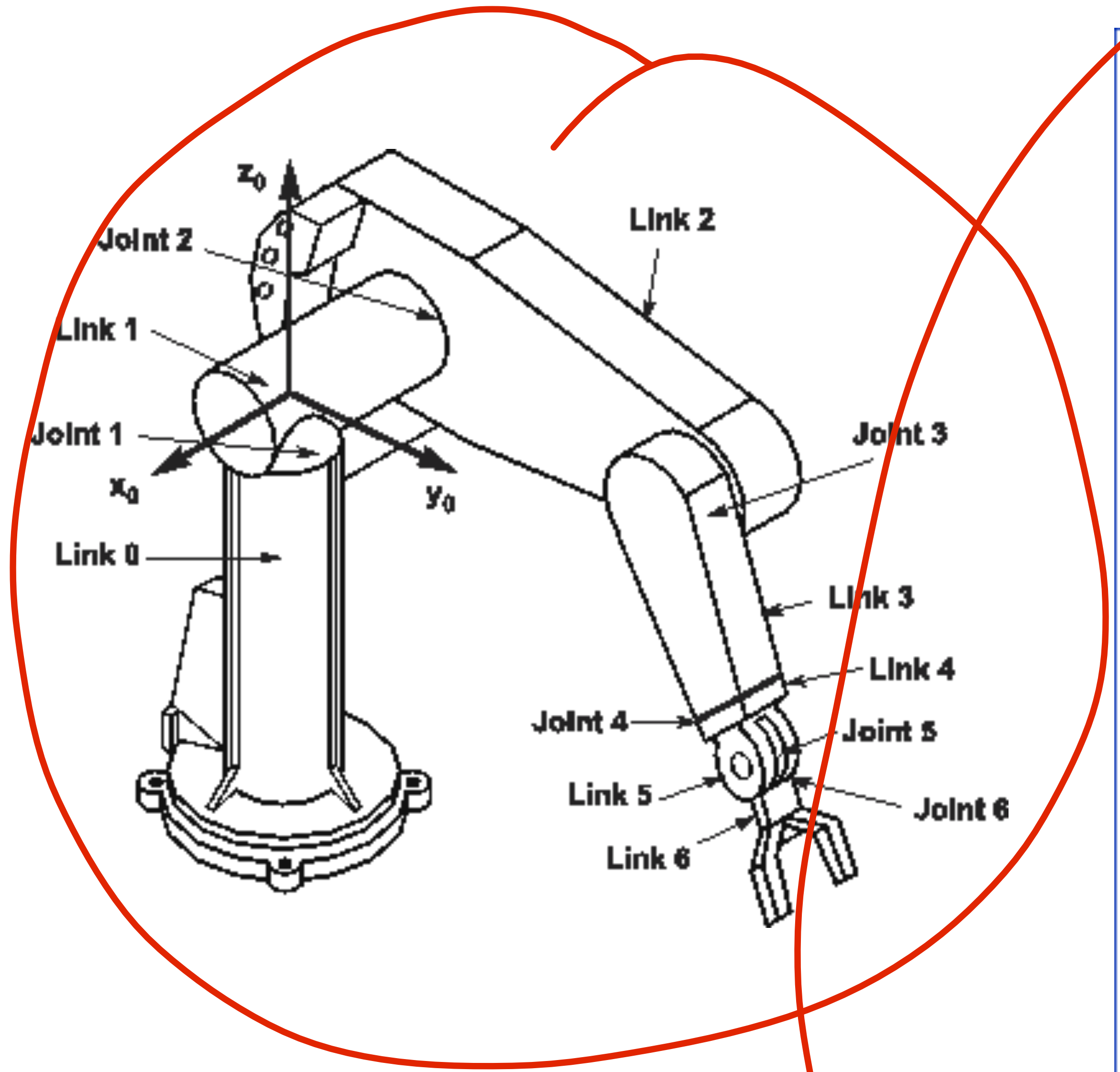
A single-joint robot

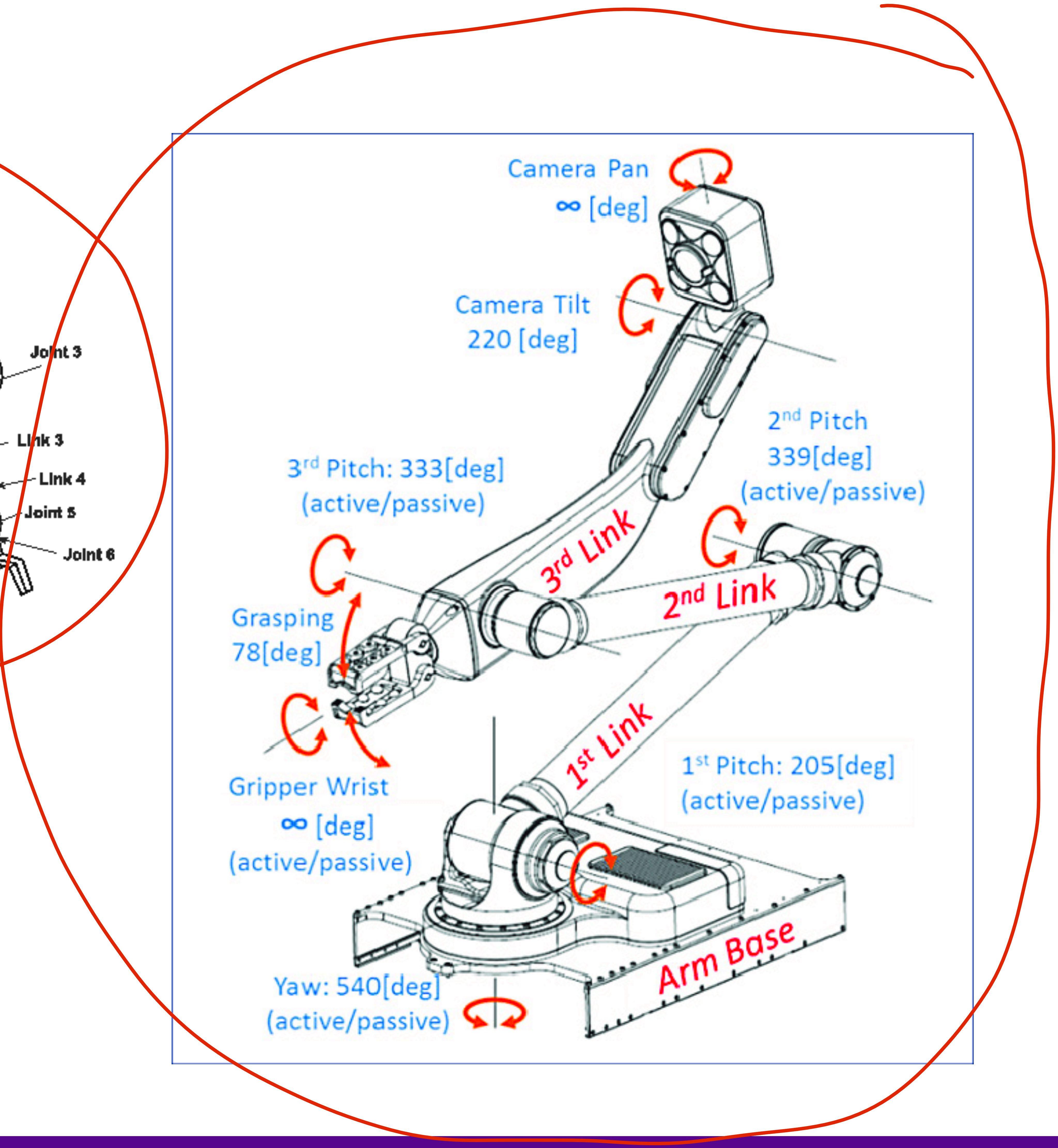


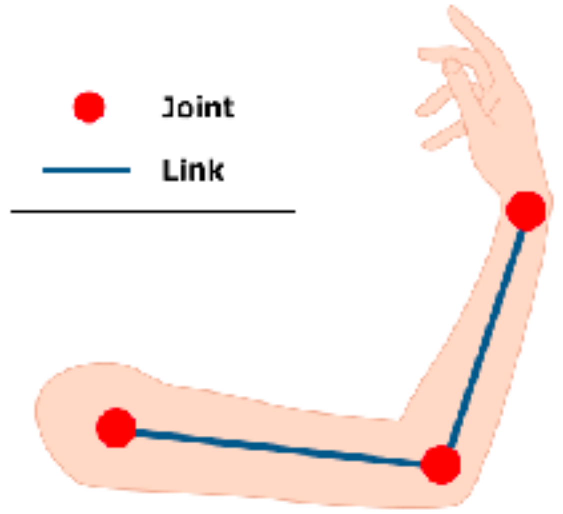


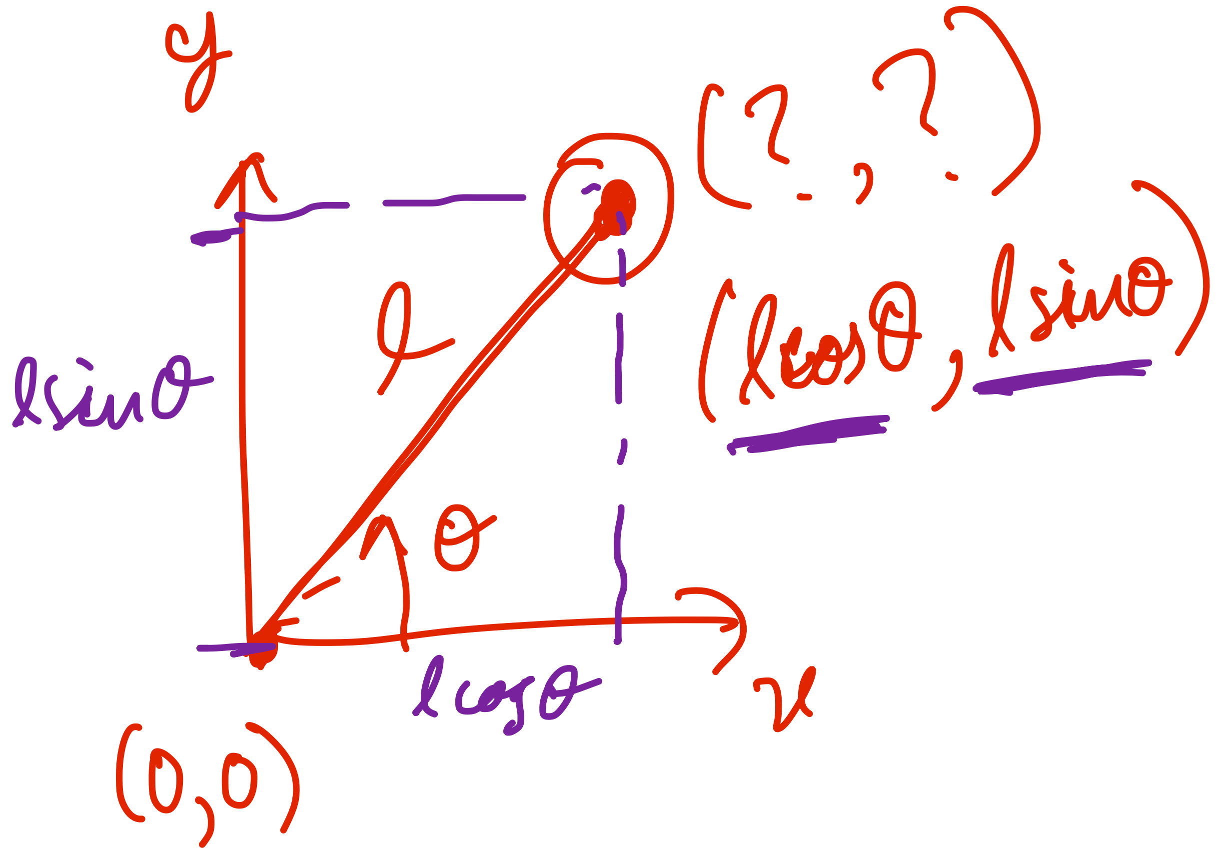


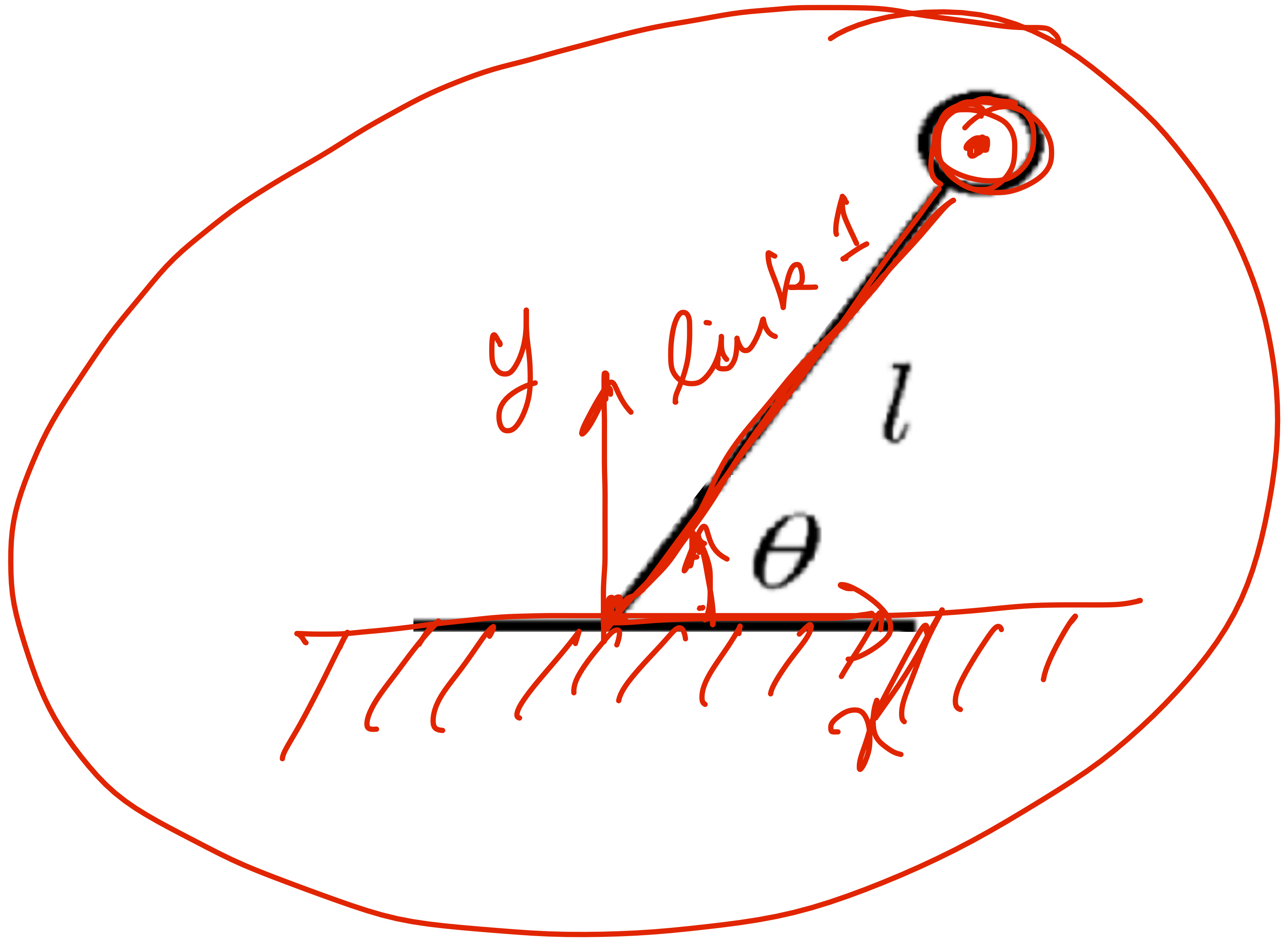
A single-joint robot



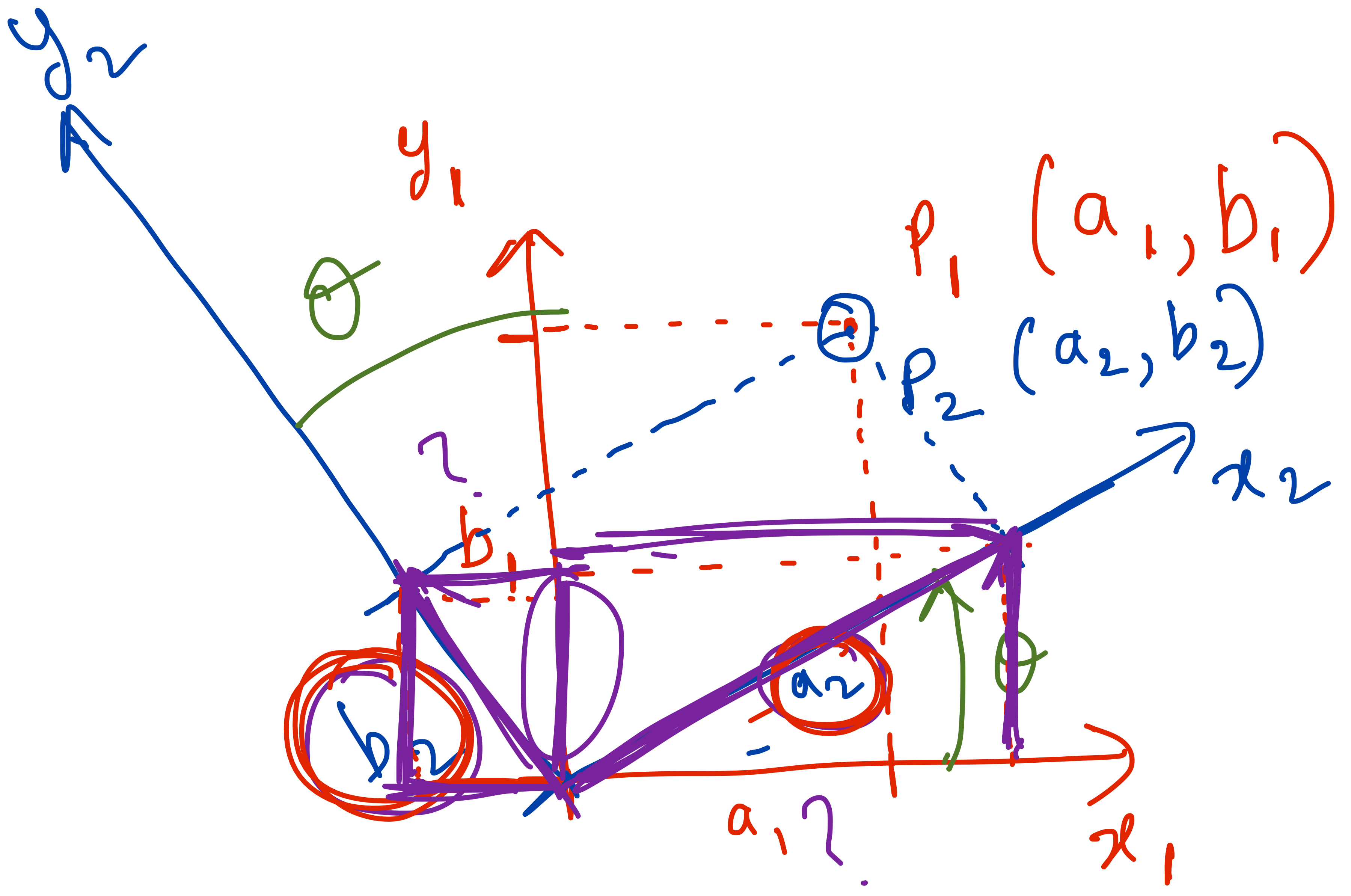


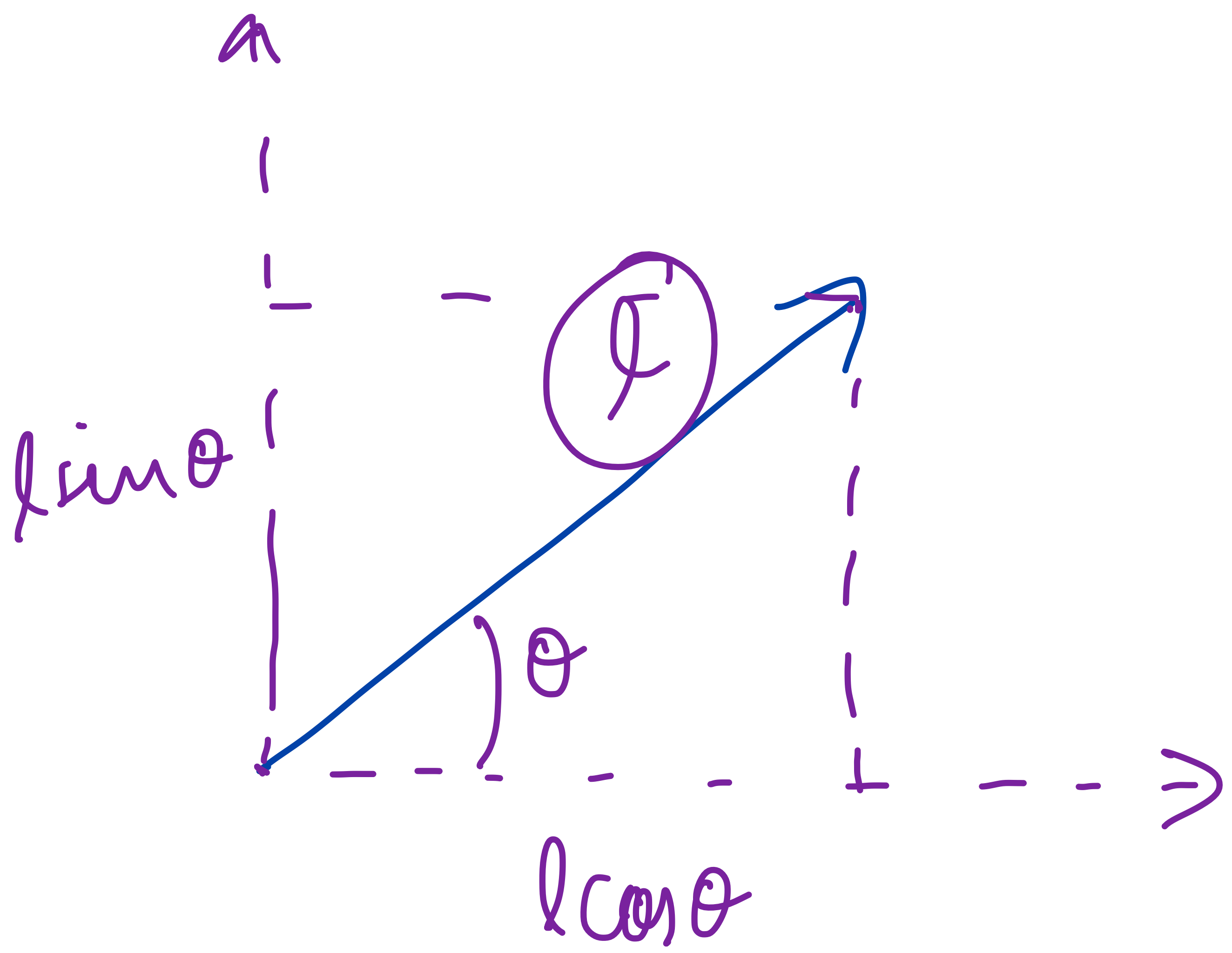






2D rotations





a_2, b_2

P_1 ?? P_2 ✓✓

$$a_1 = a_2 \cos\theta - b_2 \sin\theta \quad \text{Eq. 1}$$

$$b_1 = a_2 \sin\theta + b_2 \cos\theta \quad \text{Eq. 2}$$

$$\rightarrow \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation matrix}} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

$$\rightarrow \underline{P_1 = R\, P_2}$$

Properties of rotations

$R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|R| = ad - bc$

Determinant?

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

??

$$|R| = \cos\theta . \cos\theta - -\sin\theta \sin\theta$$
$$= \cos^2\theta + \sin^2\theta$$
$$= 1$$

$$\begin{array}{cc} R^T & R \end{array}$$
$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$\begin{bmatrix} c^2+s^2 & \cancel{-sc+cs} \, 0 \\ \cancel{sc-sc} & c^2+s^2 \end{bmatrix}$$

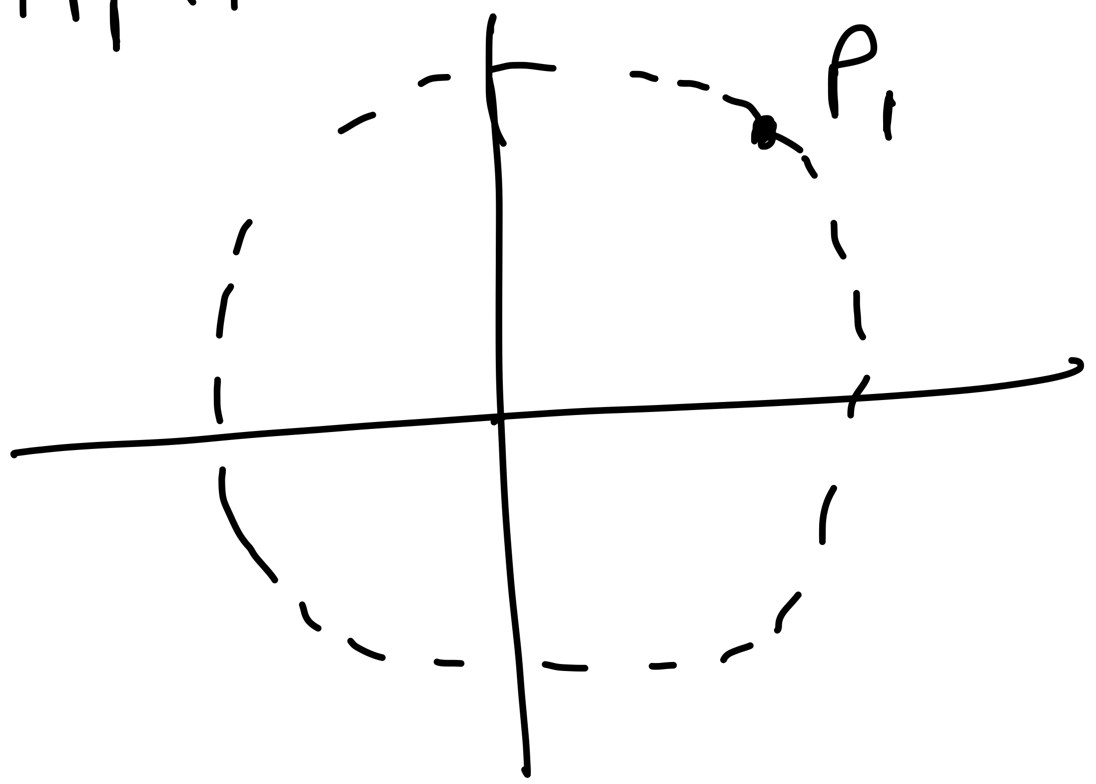
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_1 = R P_2$$

$$\|P_1\| = \|R P_2\| = \|P_2\|$$

$$\|P_1\|^2 = P_1^T P_1 = (RP_2)^T (RP_2) = P_2^T \underbrace{R^T R}_{I} P_2 = P_2^T P_2 .$$

$$= \|P_2\|^2$$



Properties of rotations

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad M^{-1} = \frac{1}{|M|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse?

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R^T$$

$$\boxed{\begin{aligned} P_1 &= R P_2 \\ P_2 &= R^T P_1 \end{aligned}}$$

$$R^{-1} P_1 = R^{-1} R \, P_2$$

? ✓ ✓ ?

$$P_2 = R^{-1} P_1$$

$$P_2 = R^T P_1$$

Properties of rotations

Multiplication of rotations?

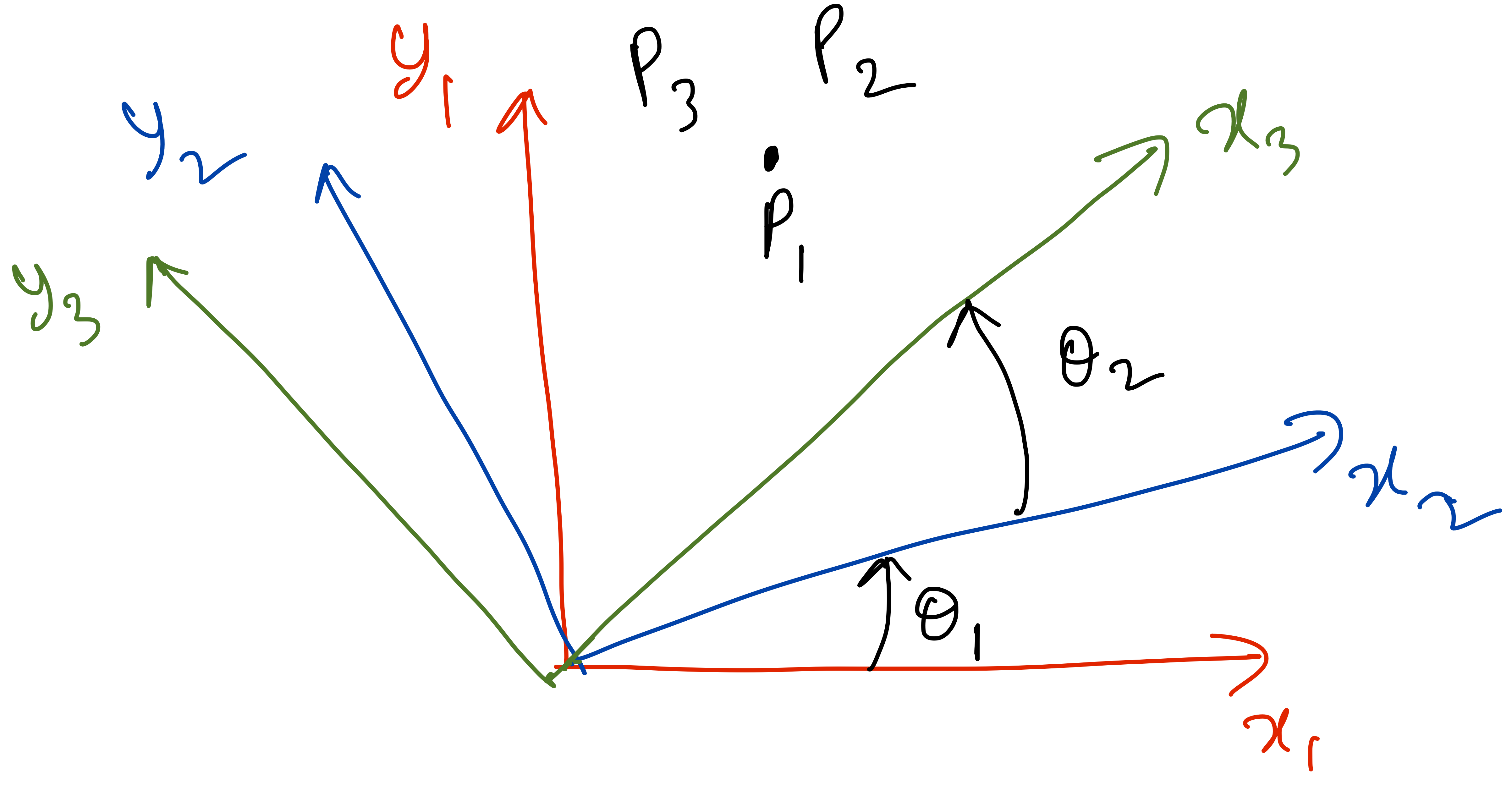
$$P_2 = R_2(\theta_2) P_3$$

$$= \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} P_3$$

$$P_1 = R_1(\theta_1) P_2$$

$$P_1 = \underbrace{R_1(\theta_1)}\,\underbrace{R_2(\theta_2)}\, P_3$$

$$= R(\theta_1 + \theta_2)$$



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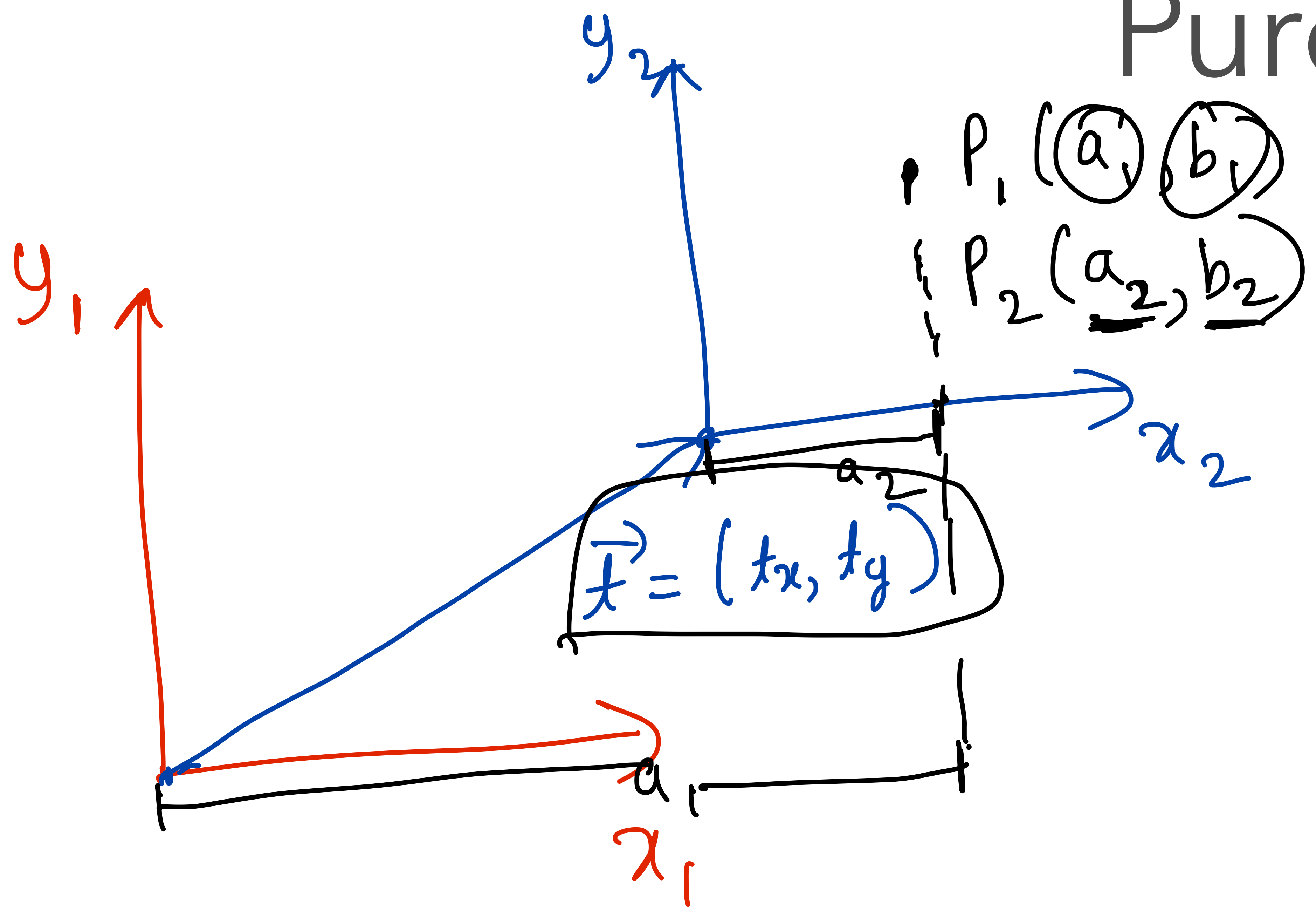
$$\begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

Pure translation



$$a_1 = a_2 + t_x \quad \text{— Eq. 1}$$

$$b_1 = b_2 + t_y \quad \text{— Eq. 2}$$

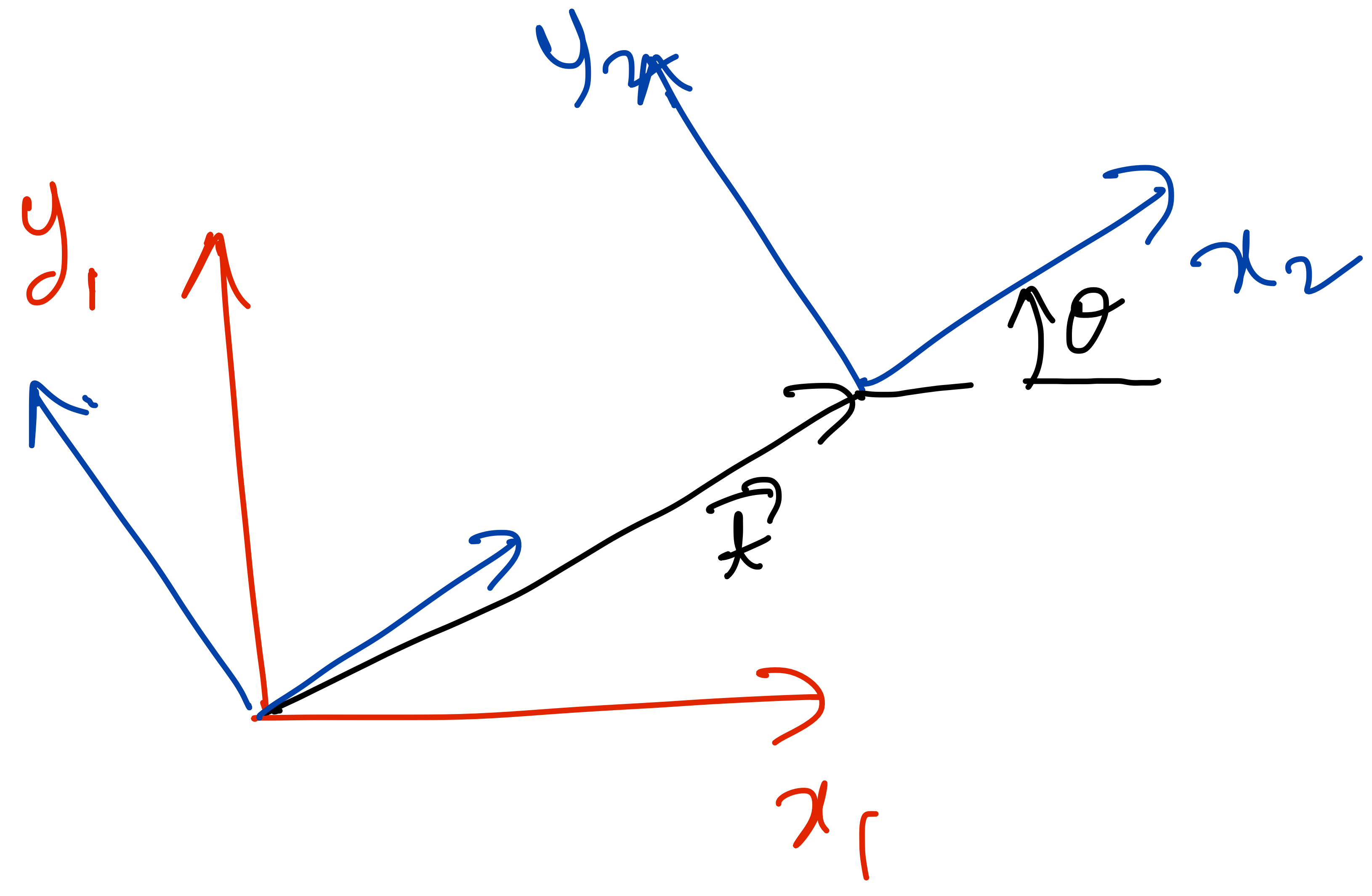
$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = M \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ b_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} t_x \\ t_y \end{matrix}} \\ \boxed{\begin{matrix} 0 & 0 \end{matrix}} & \boxed{1} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ 1 \end{bmatrix}$$

$$P_1 = M \; P_2$$

$$= \begin{bmatrix} I_{2\times2} & \vec{t} \\ 0_{1\times2} & 1 \end{bmatrix} P_2$$

2D rotations + translation



$$P_1 = M \; P_2$$

$$P_1 = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} P_2$$

$$P_1 = \underbrace{\begin{bmatrix} R(\theta) & \vec{t} \\ 0_{1\times 2} & 1 \end{bmatrix}}_{\text{Homogenous Transformation} \quad H} P_2$$

Summary of 2D Homogenous Transformation Matrix

Suppose a rotation by θ is performed, followed by a translation by x_t, y_t. This can be used to place the robot in any desired position and orientation. Note that translations and rotations do not commute! If the operations are applied successively, each $(x, y) \in \mathcal{A}$ is transformed to

$$\begin{pmatrix} x\cos\theta - y\sin\theta + x_t \\ x\sin\theta + y\cos\theta + y_t \end{pmatrix}. \tag{3.33}$$

The following matrix multiplication yields the same result for the first two vector components:

$$\begin{pmatrix} \cos\theta & -\sin\theta & x_t \\ \sin\theta & \cos\theta & y_t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta + x_t \\ x\sin\theta + y\cos\theta + y_t \\ 1 \end{pmatrix}. \tag{3.34}$$

This implies that the 3×3 matrix,

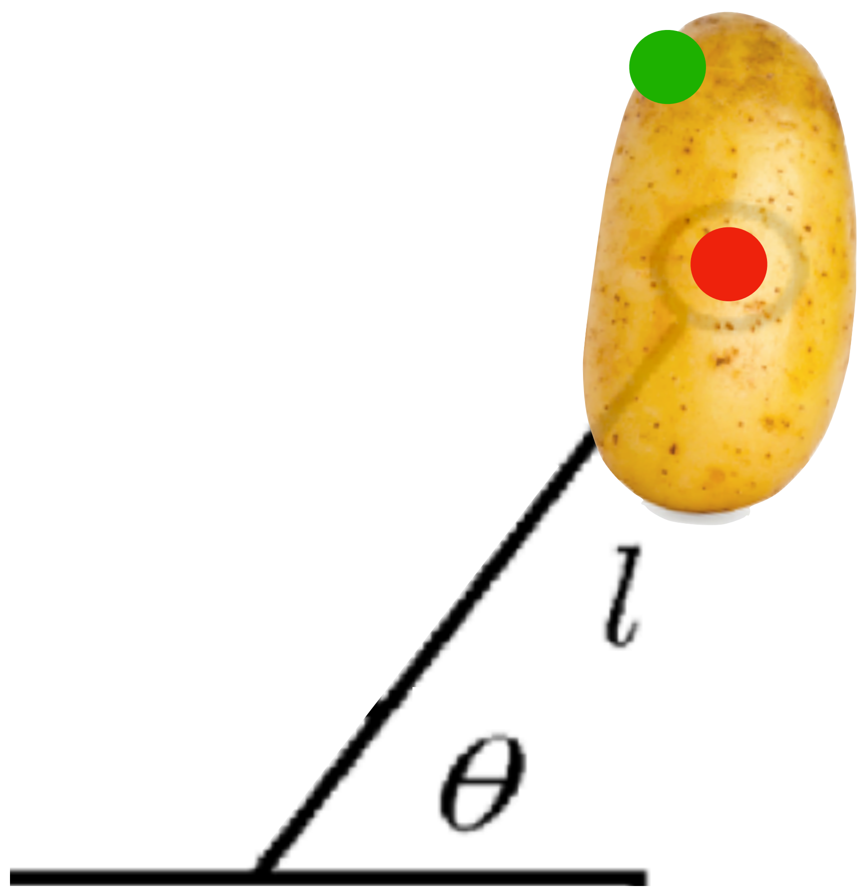
$$T = \begin{pmatrix} \cos\theta & -\sin\theta & x_t \\ \sin\theta & \cos\theta & y_t \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.35}$$

represents a rotation followed by a translation. The matrix T will be referred to as a *homogeneous transformation matrix*. It is important to remember that T represents a rotation *followed by* a translation (not the other way around). Each primitive can be transformed using the inverse of T, resulting in a transformed solid model of the robot. The transformed robot is denoted by $\mathcal{A}(x_t, y_t, \theta)$,

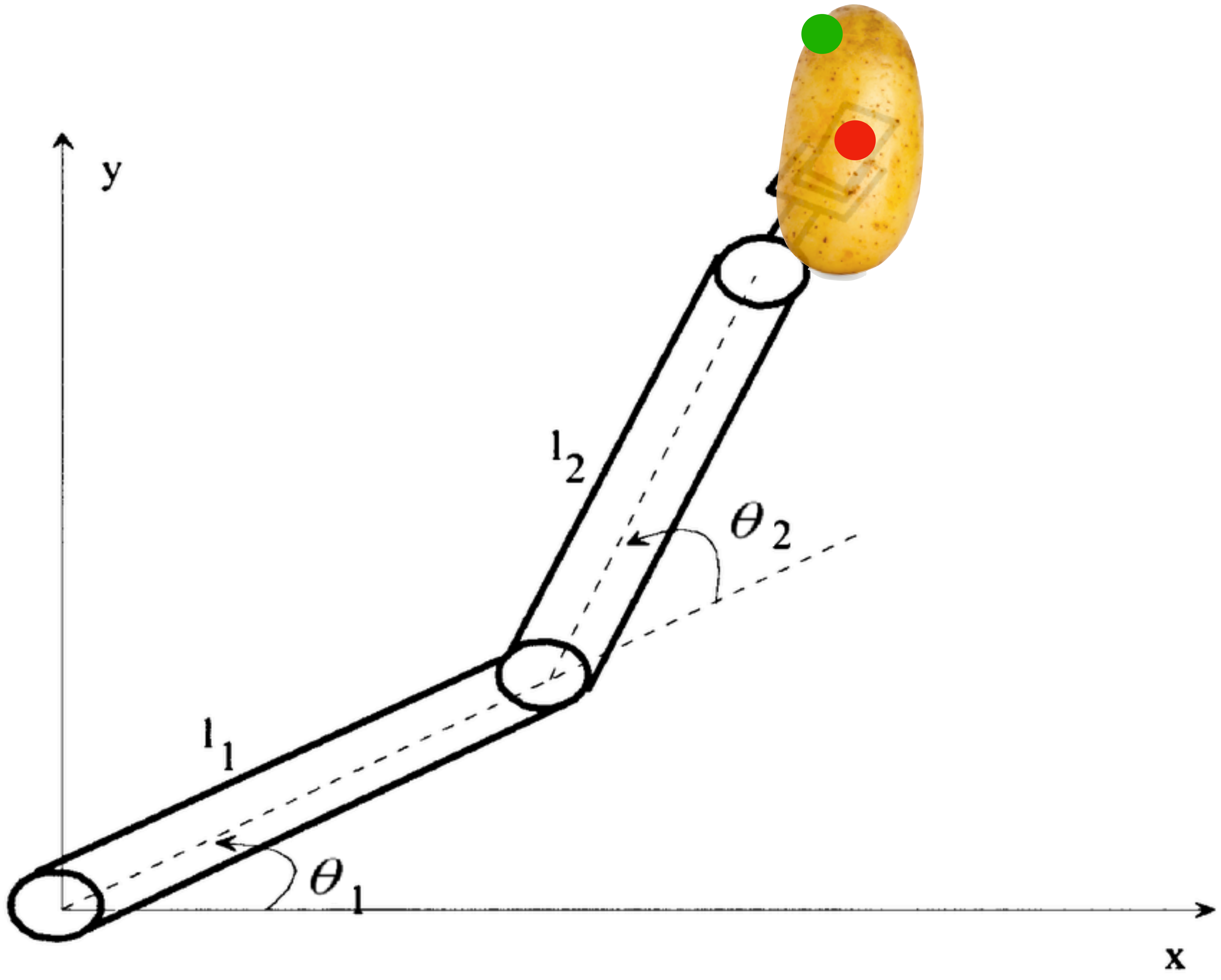
and in this case there are three degrees of freedom. The homogeneous transformation matrix is a convenient representation of the combined transformations; therefore, it is frequently used in robotics, mechanics, computer graphics, and elsewhere. It is called homogeneous because over $\mathbb{R}^3$ it is just a linear transformation without any translation. The trick of increasing the dimension by one to absorb the translational part is common in projective geometry [804].

http://planning.cs.uiuc.edu/node99.html

A localization exercise

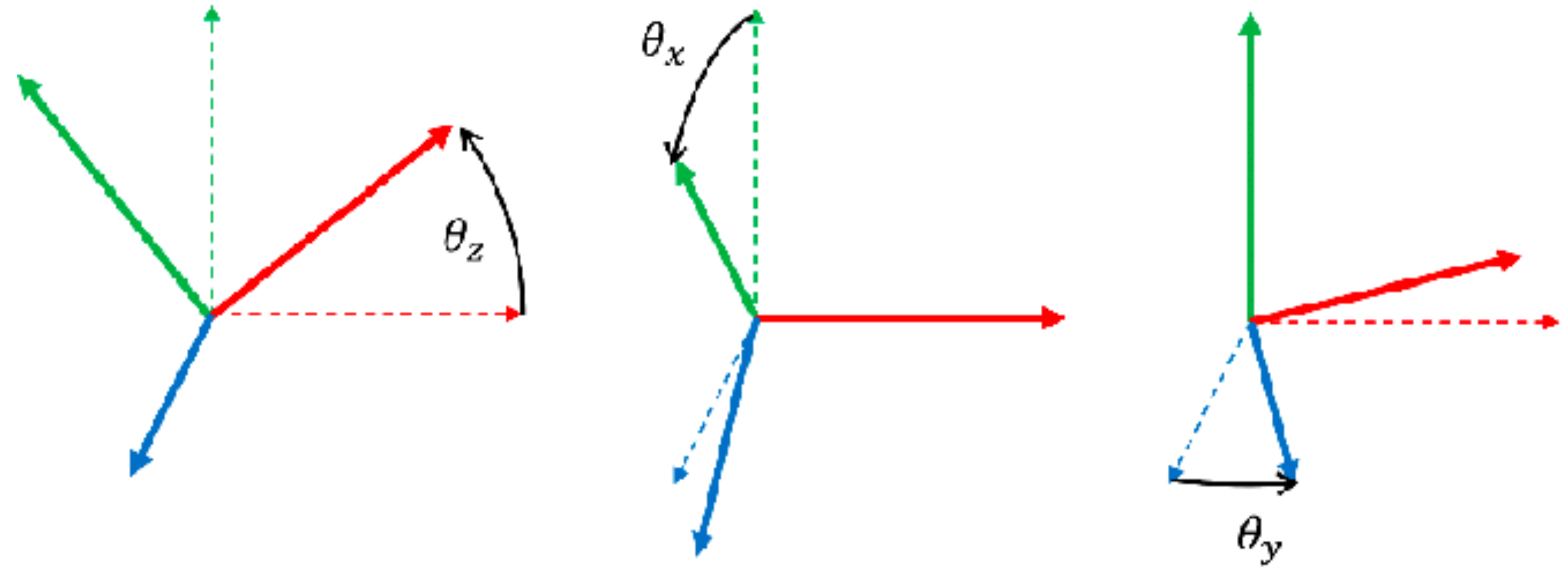


A slightly more complicated localization exercise



3D rotations

Summary of 3D rotations



First, the matrix for rotation about the Z axis contains a 2D rotation matrix in its upper corner:

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

This can be interpreted by imagining the Z axis pointing out of the page, and the X and Y axes marking the axes of a standard graph on the page. The rotation about θ is a CCW rotation in the plane of the page. Notice that the Z coordinate of any point is preserved by this operation, a property maintained by the third row $(0, 0, 1)$, nor does it affect the X and Y coordinates, a property maintained by the first two 0 entries of the third column. In the (X, Y) plane, the upper 2×2 matrix is identical to a 2D rotation matrix.

The rotation about the X axis is similar, with a 2D rotation matrix appearing along the Y and Z axes:

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}. \tag{6}$$

Here the X coordinate is preserved while the Y and Z entries perform a 2D rotation.

Finally, the rotation about the Y axis is similar, but with a sign switch of the $\sin$ terms:

$$R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}. \tag{7}$$

The reason why the $-\sin\theta$ term switches to below the diagonal is that if one were to orient a frame so the Y axis points out of the page, and align the X axis to point rightward, the Z axis would point downward instead of upward. Instead, the matrix can be derived by aligning the Z axis to the rightward direction and the X axis to the upward direction, so that the $-\sin\theta$ term arrives in the Z, X spot. A mnemonic to help remember the sign switch on rotations about Y is that the order of the two coordinate directions defining the orthogonal plane is derived from a *cyclic* ordering of the axes: Y follows X, Z follows Y, and X follows Z. So, the plane orthogonal to X is (Y, Z), the plane orthogonal to Y is (Z, X), and the plane orthogonal to Z is (X, Y).

1. http://scipp.ucsc.edu/~haber/ph216/rotation_12.pdf
2. http://motion.pratt.duke.edu/RoboticSystems/3DRotations.html

3D rotations + translations

http://scipp.ucsc.edu/~haber/ph216/rotation_12.pdf

3D Homogenous Transformation Matrix

Robot as a collection of rigid bodies + transformations!