$\boxed{\text{Ken}}$

Today:

$\boxed{7.2}$ One-to-one, Onto, and Inverse Functions

Last time:

$\boxed{7.1}$ Functions defined on General Sets

$\boxed{7.2}$ One-to-one, Onto, and Inverse Functions

7.2 One-to-One, Onto, and Inverse Functions

Definition

A function $f: X \to Y$ is one-to-one or injective if and only if, for any $x_1, x_2 \in X$,

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2$$

or logically equivalently

$$x_1 \neq x_2 \text{ implies } f(x_1) \neq f(x_2).$$

Otherwise $f: X \to Y$ is not one-to-one if and only if there exist $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$.

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

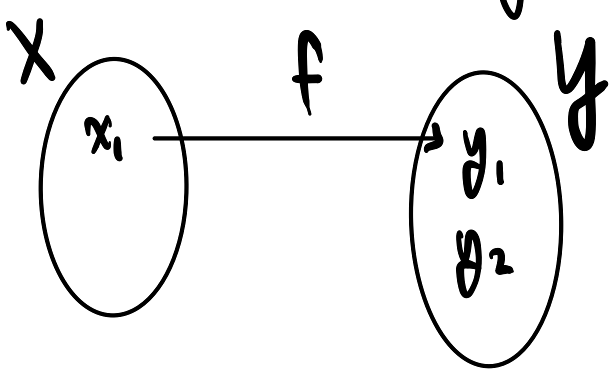
e.g. $2 \neq -2$ but $f(2) = 4 = f(-2)$

so f is not one-to-one/injective

Definition

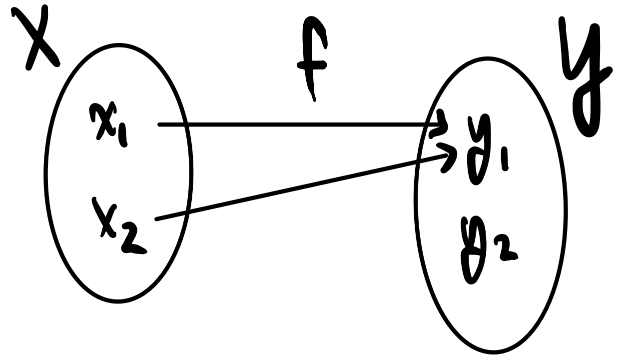
A function $f: X \to Y$ is onto or surjective if and only if, for any $y \in Y$, there exist(s) $x \in X$ such that $f(x) = y$. Equivalently, f is onto if and only if the image of domain X under f equals the codomain Y, i.e. $f(X) = Y$. $f(X) \subset Y$ & $Y \subset f(X)$

Otherwise $f: X \to Y$ is not onto if and only if there exists $y \in Y$ such that, for any $x \in X$, $f(x) \neq y$.



not onto

one-to-one



not onto
not one-to-one

e.g. $f : \mathbb{Z}^+ \to \mathbb{Z}^+$

$f(x) = 2x$ one-to-one

Let $x_1, x_2 \in \mathbb{Z}^+$.
Suppose $f(x_1) = f(x_2)$. Then
$$2x_1 = 2x_2$$
so $x_1 = x_2$. So f is one-to-one.

f is not onto because e.g.
$3 \in \mathbb{Z}^+$ but there is no $x \in \mathbb{Z}^+$
such that $3 = f(x)$.
Otherwise, suppose there exists
$c \in \mathbb{Z}^+$ such that $3 = f(c)$.
Then $3 = 2c$ so $c = 3/2 \notin \mathbb{Z}^+$.
$\Rightarrow\Leftarrow$

e.g. Let $f: \mathbb{R} \to \mathbb{R}$ such that, for any $x \in \mathbb{R}$, $f(x) = mx + b$ where $m, b \in \mathbb{R}$ and $m \neq 0$.

① f is injective.

Let $x_1, x_2 \in \mathbb{R}$.

Suppose $f(x_1) = f(x_2)$.

Then $$mx_1 + b = mx_2 + b$$
$$mx_1 = mx_2$$
$$x_1 = x_2.$$

So $x_1 = x_2$.

② f is surjective.

$f(\mathbb{R}) \subset \mathbb{R}$ because f is well-defined.

Let $y \in \mathbb{R}$. Define $c \in \mathbb{R}$ to be $c = \frac{y-b}{m}$.
$$f(c) = m\left(\frac{y-b}{m}\right) + b$$
$$= y - b + b = y.$$

So $\mathbb{R} \subset f(\mathbb{R})$ and $f(\mathbb{R}) = \mathbb{R}$.

(3) Since f is one-to-one and onto, f is a one-to-one correspondence or bijection.

$3 = e^{\ln(3)}$; $c = \ln(3)$

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e.g. let $f: \mathbb{R} \to \mathbb{R}$ such that, for any $x \in \mathbb{R}$, $f(x) = x^2$.

f is not onto because, for any $y < 0$, there is no $x \in \mathbb{R}$ such that $f(x) = y$. E.g. $-561 \neq f(x)$ for any $x \in \mathbb{R}$.

$f: [0, \infty) \to [0, \infty)$; $f(x) = x^2$

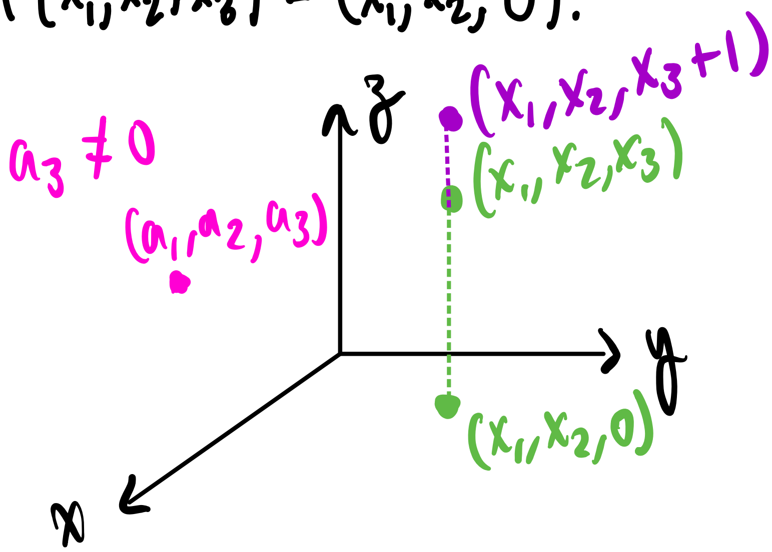
(with restricted domain and codomain)

f is a bijection.

$f^{-1}(x) = \sqrt{x}$.

e.g. let $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ (meaning $f: \mathbb{R}\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{R}\times\mathbb{R}$) such that, for any $(x_1, x_2, x_3) \in \mathbb{R}^3$,

$f(x_1, x_2, x_3) = (x_1, x_2, 0)$.



(a projection)

f is not one-to-one because e.g.

$f(1,2,3) = (1,2,0) = f(1,2,4)$

but $(1,2,3) \neq (1,2,4)$.

f is not onto. Why?

Suppose f is onto.

Then e.g. $(0,0,-5) = f(c_1, c_2, c_3)$

for some $(c_1, c_2, c_3) \in \mathbb{R}^3$.

So $(c_1, c_2, 0) = f(c_1, c_2, c_3) = (0,0,-5)$

and $0 = -5$ which is a contradiction.

Definition

A **one-to-one correspondence** or **bijection** is a function $f: X \to Y$ such that f is both one-to-one (injective) and onto (surjective).

example 7.2.10 Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ (also written $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$) such that for any $(x,y) \in \mathbb{R}^2$, $f(x,y) = (x+y, x-y)$.

Show that f is a bijection.

#29 Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ such that,

for any $(x_1, x_2) \in \mathbb{R}^2$, $f(x_1, x_2) = (x_1+1, 2-3x_2)$

Claim: f is a bijection.

① Claim: f is injective.

Proof: Let $a = (a_1, a_2) \in \mathbb{R}^2$

and $b = (b_1, b_2) \in \mathbb{R}^2$.

Suppose $f(a_1, a_2) = f(b_1, b_2)$.

Can you show $a = b$?

I.e $a_1 = b_1$ & $a_2 = b_2$.

② Claim: f is surjective.

Proof: Let $y = (y_1, y_2) \in \mathbb{R}^2$.

Can you find $c = (c_1, c_2) \in \mathbb{R}^2$

such that $f(c_1, c_2) = (y_1, y_2)$?

Hint: $c_1 = y_1 - 1$ and $c_2 = \frac{2-y_2}{3}$.

A hash function is a function defined from a larger, possibly infinite, set of data to a smaller fixed-size set of integers.

e.g. $h: \mathbb{Z}^+ \longrightarrow \{0,1\}$ such that, for any $k \in \mathbb{Z}^+$,

$$h(k) = \begin{cases} 0 & k \notin 2\mathbb{Z} \\ 1 & k \in 2\mathbb{Z} \end{cases}$$

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e.g. Dirichlet function $f: \mathbb{R} \rightarrow \{0,1\}$ such that, for any $x \in \mathbb{R}$,

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} - \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases}$$

Theorem 7.2.2

Suppose $f: X \to Y$ is a one-to-one correspondence (a bijection). Then there is a function $f^{-1}: Y \to X$ such that

$$f^{-1}(y) = x \iff y = f(x).$$

f^{-1} is called the inverse function of the function f.

SEE TEXTBOOK

Theorem 7.2.3

If X and Y are sets and $f: X \to Y$ is one-to-one and onto, then $f^{-1}: Y \to X$ is also one-to-one and onto.

Suppose $f: X \to Y$ is one-to-one and onto. So there is an inverse function $f^{-1}: Y \to X$.

① Claim: f^{-1} is one-to-one.

Let $y_1, y_2 \in Y$.

$f^{-1}(y_1) = x_1$ for some $x_1 \in X$.

$f^{-1}(y_2) = x_2$ for some $x_2 \in X$.

Suppose $f^{-1}(y_1) = f^{-1}(y_2)$.

So $x_1 = x_2$. Apply f

to obtain $f(x_1)=f(x_2)$

but $f^{-1}(y_1)=x_1 \iff f(x_1)=y_1$

and $f^{-1}(y_2)=x_2 \iff f(x_2)=y_2$.

So $y_1=y_2$.

2 Claim: $f^{-1}(Y)=X$.

Namely, show

$f^{-1}(Y)\subset X$ and $X\subset f^{-1}(Y)$.

Hint: let $x\in X$.

$f(x)=y \iff f^{-1}(y)=x$.

Exponential and Logarithmic Functions

Define the exponential function base $b \in \mathbb{R}^+$ $(b \neq 1)$ $\exp_b : \mathbb{R} \to \mathbb{R}^+$ such that, for any $x \in \mathbb{R}$, $\exp_b(x) = b^x$.

Likewise, define the logarithmic function base $b \in \mathbb{R}^+$ $(b \neq 1)$ $\log_b : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\log_b(x) = y \iff x = b^y.$$

Laws of Exponents

For any $a, b \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$,

$$b^x b^y = b^{x+y}$$

$$(b^x)^y = b^{xy}$$

$$\frac{b^x}{b^y} = b^{x-y}$$

$$(ab)^x = a^x b^x .$$

Laws of Logarithms

For any $a, b \in \mathbb{R}^+ - \{1\}$, $x, y \in \mathbb{R}^+$, $z \in \mathbb{R}$,

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$\log_a(x/y) = \log_a(x) - \log_a(y)$$

$$\log_a(x^z) = z\log_a(x)$$

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}.$$

$\boxed{7.3}$ Composition of Functions

Definition

Let $f : X \to Y_1$ and $g : Y_2 \to Z$ such that $f(X) \subset Y_2$. Define $g \circ f : X \to Z$ such that, for any $x \in X$, $(g \circ f)(x) = g(f(x))$.

The function $g \circ f$ is called the composition of f and g.