Geometry of Linear Programming (10/25)

So far, have seen optimization problems of the form:

$$\underset{x}{\text{minimize}} \; f(x), \quad x \in \mathbb{R}^n.$$

We assume that x can be any value in $\mathbb{R}^n$ — i.e., the minimization problem is unbounded. Eventually, we want to be able to solve constrained optimization problems of the form:

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && x \in \mathcal{X} \subseteq \mathbb{R}^n, \end{aligned}$$

where $\mathcal{X}$ is the domain of (feasible set) for the problem. Usually, we will encode these constraints as equality constraints and inequality constraints:

$$\begin{aligned} &\text{minimize} && f(x) && \\ &\text{subject to} && g_i(x) = 0, \quad i \in \mathcal{E} && \leftarrow \text{equality constraints} \\ & && h_i(x) \geq 0, \quad i \in \mathcal{I} && \leftarrow \text{inequality constraints} \end{aligned}$$

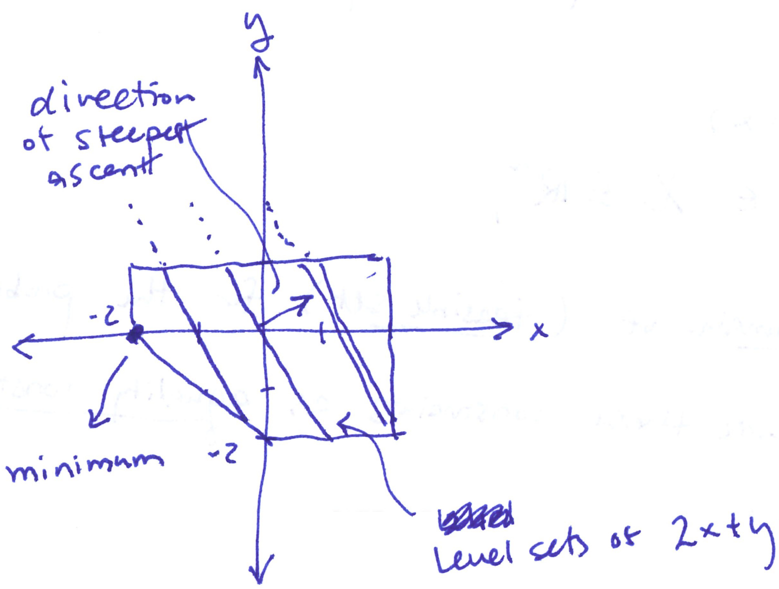
Depending on the properties of f, the g_i's, and the h_i's, we have different classes of optimization problems. If all of these functions are linear, we are working with the class of linear programs (LPs).

Example of an LP: Let's take a look at a simple LP in 2D before studying them in more detail:

$$\text{minimize} \quad 2x + y$$

$$\text{subject to} \quad x + y \geq -2$$
$$x \geq -2, \quad x \leq 2$$
$$y \geq -2, \quad y \leq 1$$

Let's sketch the constraint set:



In 2D, it is easy to visually inspect LPs to find their solution.

Note: (in 2D...)

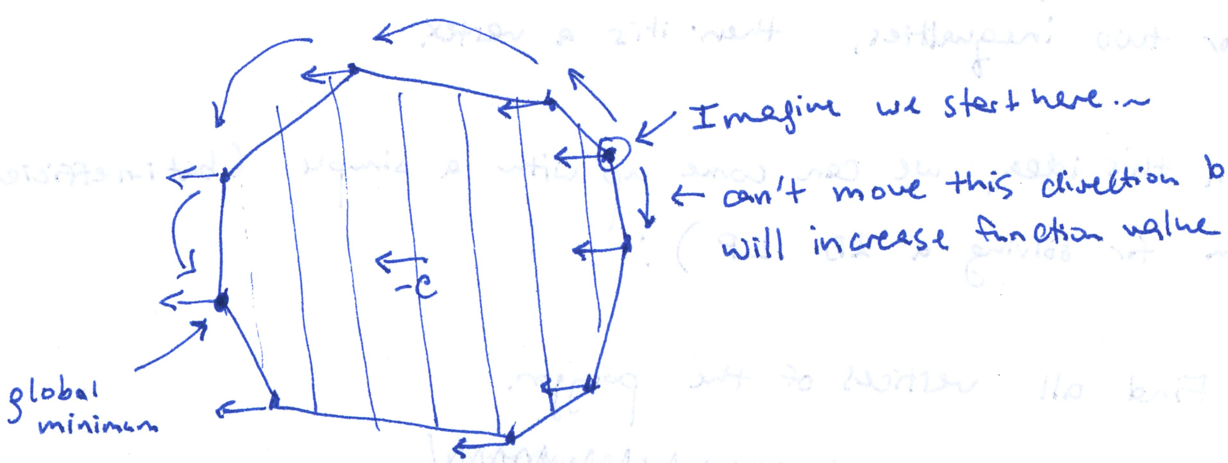
- the cost function is of the form $c^T x$, so that its gradient is c — then, direction of steepest descent is $-c$
- the inequalities taken together form a convex set — a polygon
- depending on direction, global min. on vertex of boundary, or edge

In fact, all these ideas generalize to multiple dimensions where LP is most important.

How do these ideas generalize? First point is clear — the gradient of the LP cost function is just c. Other two points:

- The inequalities taken together still form a polyhedral set. In 3D, set is a polyhedron. In ND, we call it a polytope.
- In multiple dimensions, say in $\mathbb{R}^n$, a polytope of maximum dimension will have boundary facets of dimensions 0 through $n-1$. (E.g. polyhedra have vertices (dim = 0), edges (dim = 1) and faces (dim = 2).

Let's look at a 2D LP again: The negative gradient gives a direction at each vertex:



In this case, each inequality will be of the form:

$$a_{i1}x_1 + a_{i2}x_2 \le b_i, \quad i = 1, \ldots, m,$$

where m is the number of constraints.

We can stack these constraints into a matrix:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ & \vdots \\ a_{m1} & a_{1m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\Updownarrow$$

$$\boxed{Ax \leq b}$$

to get a <u>linear matrix inequality</u>. Note that <u>LMIs</u> define polytopes, and we can use linear algebra to study some of the properties of the polytope.

In this case, for our 2D polygon, we can see that a point on the boundary if $a_{i1}x_1 + a_{i2}x_2 = b_i$ for some i. If this holds for two inequalities, then it's a vertex.

Based on this idea, we can come up with a simple (but inefficient algorithm for solving a 2D LP):

1) Find all vertices of the polygon.

2) Return the vertex with the minimum value $c^T x$.

For a very small LP in 2D, this may be OK. There are $\binom{m}{2} = \frac{m(m-1)}{2} = O(m^2)$ possible pairs of constraints to inspect. If m is small, may be OK. But for a higher dimensional ~~poly~~ polytope, this will scale like $O(m^n)$ — exponentially. We will need to come up with a better algorithm... more on this later.

Two more things... First, an example of an LP application.

Say we're a company with m factories making some product and n stores where ~~they~~ the product is sold. Assume the amount of the product ~~...~~ factory i can make is f_i, and the capacity of store j is c_j, and that it costs s_{ij} to ship a unit from factory i to store j. We want to ship everything while minimizing cost. This is the transportation problem, which can be expressed as an LP:

$$\text{minimize} \quad \sum_i \sum_j s_{ij} x_{ij}$$

amount of product shipped from factory i to store j

$$\begin{aligned} \text{subject to} \quad & \sum_j x_{ij} = f_i, \quad i = 1, \dots, m \\ & \sum_i x_{ij} = c_j, \quad j = 1, \dots, n \\ & x_{ij} \geq 0, \quad i = 1, \dots, m, \ j = 1, \dots, n. \end{aligned}$$

The first set of ~~inequalities~~ equalities requires all factories to ship all their products. The second requires all store capacities to be met. The inequalities require all shipping amounts to be nonnegative.

This is an example of an operations research problem.

Next, LPs are usually written in standard form:

$$\begin{aligned} &\text{minimize} \quad c^T x \\ &\text{subject to} \quad Ax = b \\ &\qquad\qquad\qquad x \geq 0. \end{aligned}$$

Here, A will be different from constraint matrix we saw earlier.

We can always put an LP in standard form using several basic operations:

1) Introduce slack variables: replace constraint "$Ax \leq b$" with "$Ax + z = b$" and "$z \geq 0$".

2) Convert maximization to minimization:

   "maximize $c^T x$" $\mapsto$ "minimize $-c^T x$"

3) Replace "$\geq$" w/ "$\leq$":

   "$x \leq 0$" $\mapsto$ "$z \geq 0$" and "$z = -x$"

4) "$x_1 \geq 5$" $\mapsto$ "$x_1' \geq 0$" and "$x_1' = x_1 - 5$"

5) free variable x $\mapsto$ "$x = y - z$, $y \geq 0$, $z \geq 0$"