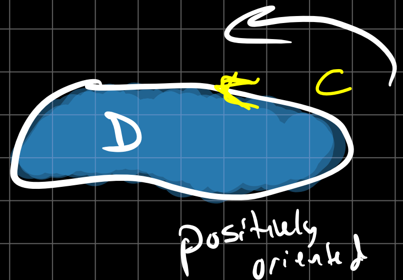
Green's Thm.



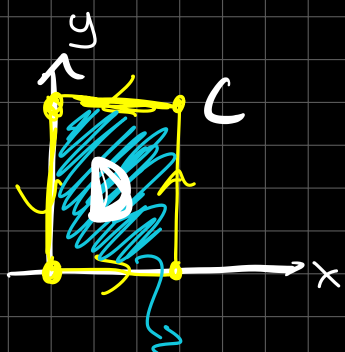
$$\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$

$$\text{work} = \int_C \vec{F}\cdot d\vec{r}$$

if $\vec{F} = \langle P, Q \rangle$

⑤ $\int_C ye^x\,dx + 2e^x\,dy$

C is rectangle with vertices $(0,0)$, $(3,0)$, $(3,4)$, $(0,4)$.
positively oriented.



$0 \le x \le 3$
$0 \le y \le 4$

$P = ye^x \qquad Q = 2e^x$

$P_y = e^x \qquad Q_x = 2e^x$

$Q_x - P_y = e^x$

$$\int_C ye^x\,dx + 2e^x\,dy = \int_0^4\int_0^3 e^x\,dx\,dy = 4(e^3 - 1)$$

⑨ $\boxed{\int_C y^3\,dx - x^3\,dy}$ C is circle $x^2+y^2=4$ positively oriented.

$P = y^3 \qquad Q = -x^3$

$P_y = 3y^2 \qquad Q_x = -3x^2$

$\int_C \vec{F}\cdot d\vec{r}$

$\vec{F} = \langle y^3, -x^3 \rangle$

$Q_x - P_y = -3x^2 - 3y^2$

too much!
So we use G.T.

D: $0 \le r \le 2$
$0 \le \theta \le 2\pi$

$Q_x - P_y = -3r^2$

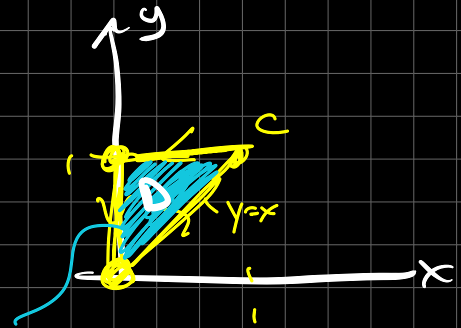
$$\int_C y^3\,dx - x^3\,dy = \int_0^{2\pi}\int_0^2 (-3r^2)\, r\,dr\,d\theta = -3\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta$$

$$= -3(2\pi)\left(\frac{r^4}{4}\right)\Bigg|_{r=0}^{r=2}$$

$$= -24\pi.$$

(14) $\vec{F}(x,y) = \langle \sqrt{x^2+1}, \tan^{-1}x \rangle$ C is triangle from $(0,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$

$$\int_C \vec{F}\cdot d\vec{r}$$



$P = \sqrt{x^2+1}$

$Q = \tan^{-1}(x)$

$P_y = 0$

$Q_x = \frac{1}{1+x^2}$

$0 \le y \le 1$
$0 \le x \le y$

or

$0 \le x \le 1$
$x \le y \le 1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1\int_0^y \frac{1}{1+x^2}\,dx\,dy = \int_0^1 \arctan(y)\,dy$$

I.B.P.
du = arctany du=dy

or switch!!

$$= \int_0^1\int_x^1 \frac{1}{1+x^2}\,dy\,dx = \int_0^1 \left(\frac{1}{1+x^2}\right)\underbrace{\int_x^1 dy}_{1-x}\,dx = \int_0^1 \frac{1}{1+x^2} - \frac{x}{1+x^2}\,dx$$

$$= \arctan(1) - \arctan(0) - \int_0^1 \frac{x}{1+x^2}\,dx$$

$u = 1+x^2$

$du = 2x\,dx \quad \frac{1}{2}du = x\,dx$

$$= \frac{\pi}{4} - \frac{1}{2}\int_1^2 \frac{1}{u}\,du$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln(2) + \frac{1}{2}\ln(1) = \frac{\pi}{4} - \frac{1}{2}\ln(2)$$

Curl

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k}$$ ← vector field

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

divergence

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = P_x + Q_y + R_z$$ ← Scalar field.

Could be in $\mathbb{R}^n$ too.

(*) $\vec{F}$ conservative $\Rightarrow$ $\operatorname{curl}\vec{F} = \operatorname{curl}\vec{\nabla}f = \vec{0}$

(**) if $\vec{F}$ defined on $\mathbb{R}^3$ and components have continuous 2^{nd} p.d. AND $\operatorname{curl}\vec{F} = \vec{0}$ then $\vec{F}$ is conservative.

(***) $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$ if $\operatorname{div}\vec{G} = 0$ we say $\vec{G}$ is incompressible

$\vec{G} = \operatorname{curl}\vec{F} \Rightarrow \vec{G}$ is a curl field

16.5 (2) $\vec{F}(x,y,z) = x^3yz^2\hat{j} + y^4z^3\hat{k}$ $P = 0$, $Q = x^3yz^2$, $R = y^4z^3$

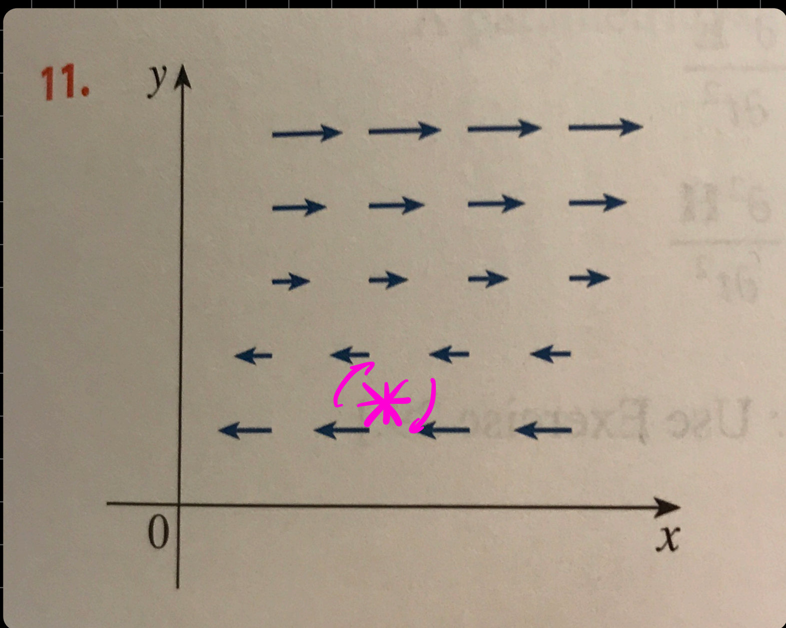
find curl $\vec{F}$ find div $\vec{F}$

$$\vec{\nabla} \times \vec{F} = (R_y - Q_z)\hat{i} + (P_z - R_x)\hat{j} + (Q_x - P_y)\hat{k}$$

$$= (4y^3z^3 - 2x^3yz)\hat{i} + 0\hat{j} + (3x^2yz^2 - 0)\hat{k}$$

$$= (4y^3z^3 - 2x^3yz)\hat{i} + 3x^2yz^2\hat{k}$$

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = P_x + Q_y + R_z = 0 + x^3z^2 + 3y^4z^2$$



z - component = 0.

is div $\vec{F}$ pos, neg, zero.

curl $\vec{F} = \vec{0}$? NO!

Rotates clockwise.

curl $\vec{F}$ in negative z or $\hat{k}$ direction.

if curl $\vec{F} = \vec{0}$ everywhere, $\vec{F}$ is Irrotational.

(12) Let f be a scalar field and **F** a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.

(a) curl f ✗
(b) grad f ✓
(c) div **F** ✓
(d) curl(grad f) ✓
(e) grad **F** ✗
(f) grad(div **F**) ✓
(g) div(grad f) ✓
(h) grad(div f) ✗
(i) curl(curl **F**) ✓
(j) div(div **F**) ✗
(k) (grad f) × (div **F**) ✗
(l) div(curl(grad f)) ✓

(g) $\operatorname{div}(\operatorname{grad} f) = \vec{\nabla}\cdot(\vec{\nabla} f) = \vec{\nabla}\cdot\langle f_x, f_y, f_z\rangle$

$$= \frac{\partial}{\partial x} f_x + \frac{\partial}{\partial y} f_y + \frac{\partial}{\partial z} f_z$$

$$= f_{xx} + f_{yy} + f_{zz}$$

$\vec{\nabla}\cdot(\vec{\nabla} f) = 0$

Laplace's equation $\nabla^2 f = 0$

or

$\Delta f = 0$

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(13) $\vec{F}(x,y,z) = y^2z^3\hat{\imath} + 2xyz^3\hat{\jmath} + 3xy^2z^2\hat{k}$

is $\vec{F}$ conservative? P, Q, R, are ∞ diff'able.

now find $\operatorname{curl}\vec{F} = \vec{0}$

$\vec{F}$ conservative. $\vec{F} = \nabla f$

↳ scalar potential!

(14) is it possible for $\vec{G}$ to satisfy NO!

$\operatorname{curl}\vec{G} = \langle x\sin y, \cos y, z - xy\rangle$?

vector field has no vector potential.

If so, $\operatorname{div}\operatorname{curl}\vec{G} = 0$

$\vec{\nabla}\cdot\langle x\sin y, \cos y, z - xy\rangle$

$$= \frac{\partial}{\partial x} x\sin y + \frac{\partial}{\partial y}\cos y + \frac{\partial}{\partial z}(z - xy) = 1$$

(32) $\vec{F} = \dfrac{\vec{r}}{r^p}$ $\quad r = \|\vec{r}\|$ $\quad \vec{r} = \langle x, y, z \rangle$

find $\operatorname{div} \vec{F} = \dfrac{\partial}{\partial x}(\,?\,) + \dfrac{\partial}{\partial y}(\,?\,) + \dfrac{\partial}{\partial z}(\,?\,)$

$$\vec{F} = \frac{\langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}^{\,p}}$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{p/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{p/2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{p/2}}\right)$$

$$= \frac{(x^2+y^2+z^2)^{p/2}(1) - (x)\left(\frac{p}{\cancel{2}}\right)(x^2+y^2+z^2)^{\frac{p}{2}-1}(\cancel{2}x)}{(x^2+y^2+z^2)^p} + \cdots$$

$$= \frac{\cancel{(x^2+y^2+z^2)^{p/2}}\left(1 - x^2 p (x^2+y^2+z^2)^{-1}\right)}{(x^2+y^2+z^2)^{p/2}} + \cdots$$

$$= \frac{1 - x^2 p r^{-2}}{r^p} + \frac{1 - y^2 p r^{-2}}{r^p} + \frac{1 - z^2 p r^{-2}}{r^p}$$

$$= \frac{3 - p\,\boxed{r^{-2}(\underbrace{x^2+y^2+z^2}_{r^2})}}{r^p} = \frac{3-p}{r^p}$$

(*) when $p=3$

$\operatorname{div}\vec{F} = 0$ everywhere.

look at 33, 34