Ken

Today:

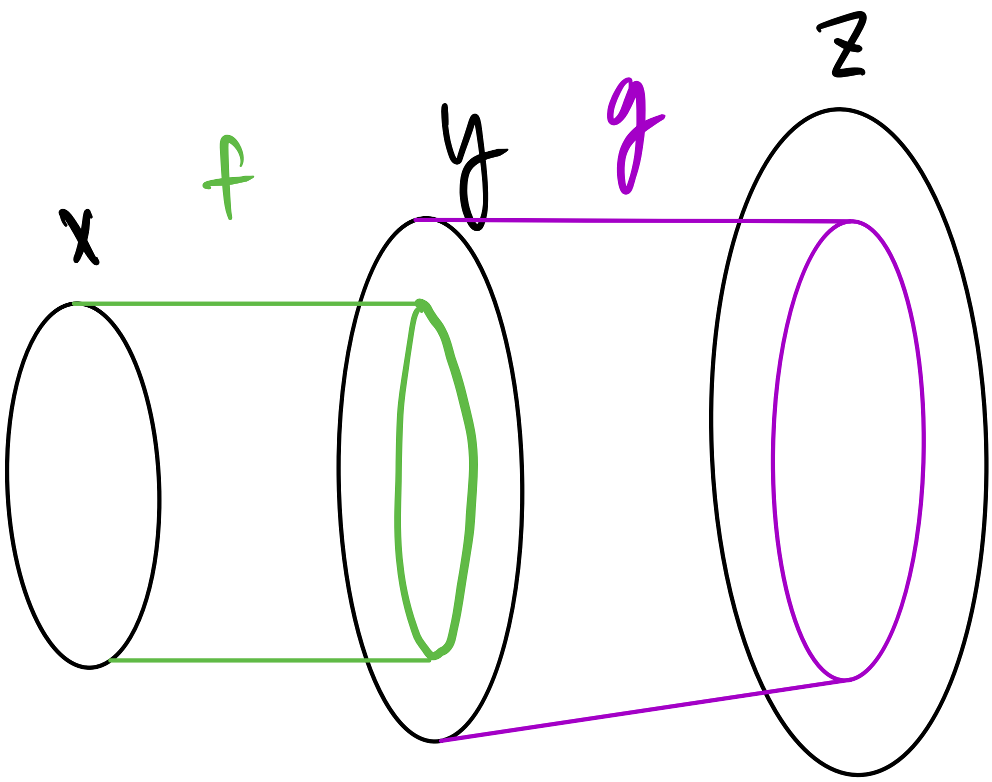
$\boxed{7.3}$ Composition of Functions

$\boxed{9.4}$ The Pigeonhole Principle

Last time:

$\boxed{7.3}$ Composition of Functions

$$f : X \hookrightarrow Y, \quad g : Y \hookrightarrow Z$$



Theorem 7.3.3

If $f: X \to Y$ and $g: Y \to Z$ are both injective functions, then $g \circ f$ is injective.

Proof? Last time.

Theorem 7.3.3

If $f: X \to Y$ and $g: Y \to Z$ are both surjective functions, then $g \circ f$ is surjective.

Proof?

Let $f: X \to Y$ and $g: Y \to Z$.

Suppose f, g are surjective.

$\forall y \in Y \, \exists x \in X \, (f(x) = y)$ f is surjective

$f(X) = Y$

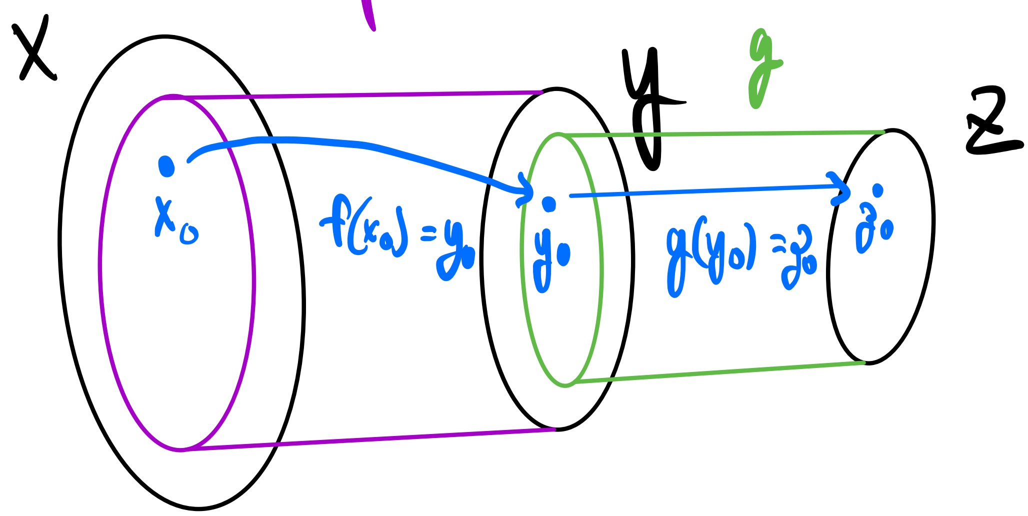
$\forall z \in Z \, \exists y \in Y \, (g(y) = z)$ g is surjective.

$g \circ f : X \to Z$

Let $z_0 \in Z$. Since $z_0 \in Z$ and g is surjective, there exists $y_0 \in Y$ such that $g(y_0) = z_0$. Since $y_0 \in Y$ and f is surjective, there exists $x_0 \in X$ such that $f(x_0) = y_0$. Then via substitution,

$$z_0 = g(y_0) = g(f(x_0)) = (g \circ f)(x_0).$$

Therefore, for any $z_0 \in Z$, there exists $x_0 \in X$ such that $(g \circ f)(x_0) = z_0$ and $g \circ f$ is surjective.



$f: X \twoheadrightarrow Y$

$g: Y \twoheadrightarrow Z$

$f: X \hookrightarrow\!\!\!\!\!\twoheadrightarrow Y$

A Set-theoretic Notation

Let Y^X denote the set of all functions from a set X into a set Y.

Y^X

# of distinct functions from X into Y is $N(Y)^{N(X)} = |Y|^{|X|}$

e.g. $A = \{1, 2, 3\}$

$B = \{b_1, b_2, b_3\}$

# of distinct functions $A \to B$ is $3^3 = 27$

$\mathbb{R}^{\mathbb{R}}$

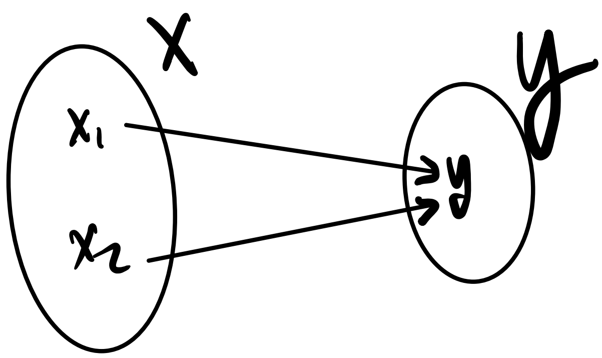
$N(\mathbb{R}^{\mathbb{R}}) > N(\mathbb{R}) > N(\mathbb{Z}) = N(\mathbb{Q}) = N(\mathbb{Z}^+)$

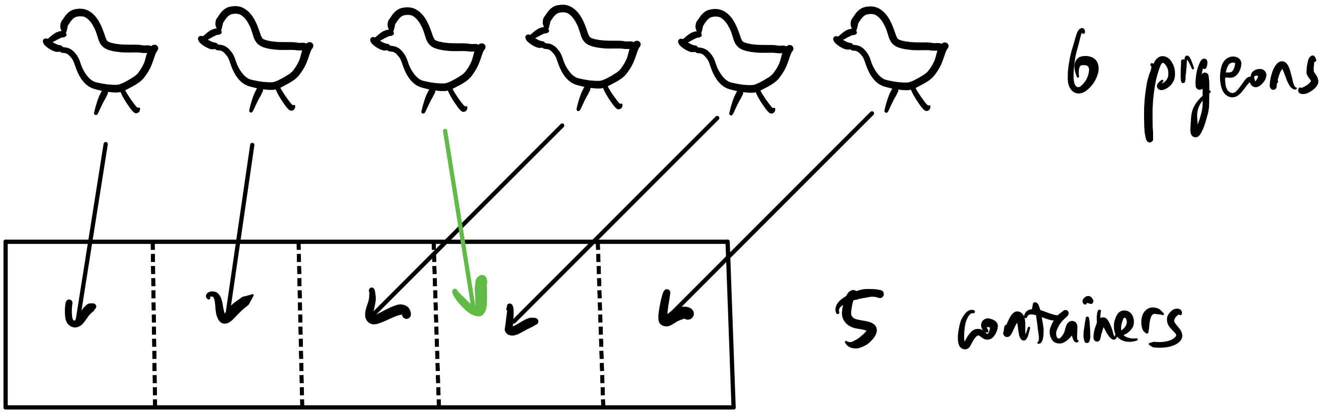
Cantor pairing function

(1,1)	(1,2)	(1,3)	(1,4)
(2,1)	(2,2)	(2,3)	
(3,1)	(3,2)	(3,3)	
(4,1)			

$\boxed{9.4}$ The Pigeonhole Principle

> **Pigeonhole Principle (PHP)**
>
> A function from one finite set to a smaller finite set cannot be injective. There must be at least two elements in the domain that have the same image in the co-domain.





Theorem 9.4.1 (The Pigeonhole Principle)

Let X, Y be sets such that $N(X) = n \in \mathbb{Z}^+$ and $N(Y) = m \in \mathbb{Z}^+$. For any $f \in Y^X$, if $n > m$, then f is not injective, i.e. there exist $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$.

e.g. Suppose a cable has a USB type-A connector with two orientations, "up" and "down," that we attempt to connect to a compatible computer. How many attempts must we make to guarantee we connect the cable and computer? 2


#8 Let $T = \{1,2,3,4,5,6,7,8,9\}$. Suppose five integers are chosen from T. Must there be two integers whose sum is 10? No

e.g. $\{1,2,3,4,5\}$

$T = \{1,2,3,4,5,6,7,8,9\}$

$$\underbrace{\forall S \subset T \,\left(N(S)=5 \rightarrow \exists n_1, n_2 \in S \,(n_1+n_2=10)\right)}_{\text{false}}$$

#10, 11 Let $n \in \mathbb{Z}^+$ such that $S=\{1,2,3,\dots,2n\}$.

If we choose $n+1$ integers from S, must at least one of them be even? yes

If we choose $n+1$ integers from S, must at least one of them be odd? yes

Let $n \in \mathbb{Z}^+$. Define $S=\{1,2,3,\dots,2n\}$.

Then $S_E \subset S$ such that

$S_E = \{x \in S : \exists k \in \mathbb{Z}\ (x=2k)\}$

$N(S_E)=n$

$S_O = \{x \in S : \exists k \in \mathbb{Z}\,(x=2k-1)\}$

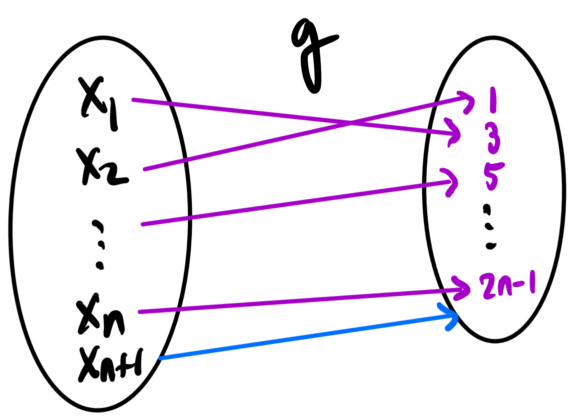
$N(S_O)=n$

$S = S_E \cup S_O$ and $S_E \cap S_O = \emptyset$

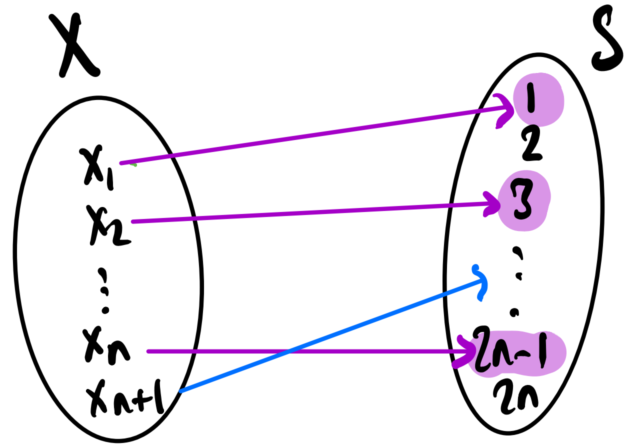
$n=3,\ n+1=4$

$S=\{1,2,3,4,5,6\}$

$X \subset S$ $\qquad S_0$



$N(X)=n+1 > N(S_0)=n$



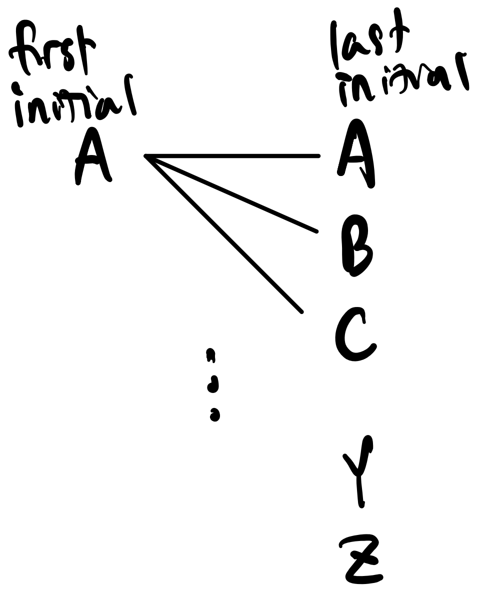
$S = S_E \cup S_0$

#19 How many integers from 100 through 999 must you pick in order to be sure that at least two of them have a digit in common?

$$\underbrace{111, 222, 333, 444, \ldots, 999}_{\text{nine three-digit \#'s}} \quad \underbrace{100}_{\text{1 more}}$$

answer: we must pick at least ten three digit numbers for at least one number to share digits or a digit

#4 In a group of 700 people, must there be 2 that have the same first and last initials?



\# of possible

first name initial 26

last name initial 26

$26^2 = 676 < 700$

#12 How many cards must you pick from a standard 52-card deck to be sure there's at least 1 red card? Why?

half the cards in a 52-card deck are red. the rest are black. So if we choose

$$\frac{52}{2} + 1 = 26 + 1 = 27$$

we are guaranteed to draw a red card.