Today:

$\boxed{3.1}$ Predicates & Quantifiers I

$\boxed{\text{Ken}}$

Last time:

$\boxed{2.3}$ Valid & Invalid Arguments

$\boxed{3.1}$ Predicates & Quantifiers I

The symbolic analysis of predicates and quantified statements is called the predicate calculus. The symbolic analysis of ordinary compound statements is called the propositional calculus (or statement calculus).

P : cares for patients at NYU Langone

Q : cares for patients at

P(x) : x cares for patients at NYU Langone.

Q(x,y) : x cares for patients at y.

Definition

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Definition

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}$$

$$\{x \in D : P(x)\}$$

e.g. ① Let $P_1(x)$ be the predicate "x is even"
with the domain $\mathbb{Z}^+ := \{1, 2, 3, \ldots\}$.

ⓐ Determine the truth values $P_1(1), P_1(2), P_1(5)$

ⓑ Find the truth set of $P_1(x)$.

ⓐ $P_1(1)$: 1 is even false

$P_1(2)$: 2 is even true

$P_1(5)$: 5 is even false

ⓑ domain $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

truth set $2\mathbb{Z}^+ := \{2, 4, 6, 8, \ldots\}$

② Let $P_2(x)$ be the predicate "$x \leq 1/x$"
with the domain $(0, \infty) = \{x \in \mathbb{R} : 0 < x < \infty\}$

ⓐ Determine the truth values $P_2(\pi), P_2(1), P_2(1/2)$

ⓑ Find the truth set of $P_2(x)$.

ⓐ $P_2(\pi)$: $\pi \leq 1/\pi$ false

$P_2(1)$: $1 \leq 1/1$ true

$P_2(1/2)$: $1/2 \leq 2$ true

ⓑ domain $(0, \infty)$

truth set $(0,1]$

(3) Let $P_3(x)$ be the predicate "$\sin(x) \geq 0$" with the domain $\mathbb{R}$.

(a) Determine the truth values

$P_3(0)$: $\sin(0) = 0 \geq 0$ true

$P_3(\pi/4)$: $\sin(\pi/4) = \sqrt{2}/2 \geq 0$ true

$P_3(\frac{5\pi}{4})$: $\sin(\frac{5\pi}{4}) = -\sqrt{2}/2 \geq 0$ false

(b) Find the truth set of $P_3(x)$.

$$\bigcup_{k=-\infty}^{\infty} [2k\pi, (2k+1)\pi]$$

$$= \cdots \cup [-2\pi, -\pi] \cup [0, \pi] \cup [2\pi, 3\pi] \cup [4\pi, 5\pi] \cup \cdots$$

Quantifiers

\forall

$\forall$ is called the universal quantifier, read "for any," "for all," "for every," "given any," or "for each."

\exists

$\exists$ is called the existential quantifier, read "there exists" or "there is."

Definition Universal Statement & Counterexample

Let $Q(x)$ be a predicate and D the domain of x. A universal statement is a statement of the form "$\forall x \in D (Q(x))$."
It's defined to be true if and only if $Q(x)$ is true for each individual x in D.
It's defined to be false if and only if $Q(x)$ is false for at least one x in D. A value x for which $Q(x)$ is false is called a counterexample to the universal statement.

Definition Existential Statement

Let $Q(x)$ be a predicate and D the domain of x. An existential statement is a statement of the form "$\exists x \in D(Q(x))$." It is defined to be true if and only if $Q(x)$ is true for at least one x in D. It is false if and only if $Q(x)$ is false for all x in D.

e.g.
- Truth & Falsity of Universal or Existential Statements
- Translating from Formal to Informal Language
- Statements with Multiple Quantifiers

Determine whether the quantified statements are true or false. Translate the statements from formal to informal language.

(a) (i) $\forall x \in \mathbb{Z}\left(\sqrt{x^2} = x\right)$ False

For any integer x, the square root of x squared equals x.

False, because $-1 \in \mathbb{Z}$ such that

$\sqrt{(-1)^2} = \sqrt{1} = 1 \neq -1$

(ii) $\forall x \in \mathbb{Z}^+ (\sqrt{x^2} = x)$ True

For any positive integer x, the square root of x squared equals x.

(b) $\exists x \in \mathbb{R} - \{0\} (x = 1/x)$ True

There exists a nonzero real number x such that x equals $1/x$.

True, consider $\{\pm 1\}$

(c) $\exists x \in \mathbb{Q} (x^2 = 2)$

There exists a rational number x such that $x^2 = 2$.

False

$\neg \exists x \in \mathbb{Q} (x^2 = 2) \equiv \forall x \in \mathbb{Q} (x^2 \neq 2)$ True

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(d) (i) $\exists x \in \mathbb{Q} (\forall y \in \mathbb{Q} (xy = yx = y))$ True

There exists a rational number x such that for any rational number y,

$xy = yx = y$.

there is $1 \in \mathbb{Q}$ such that for any $y \in \mathbb{Q}$, $1y = y1 = y$.

(ii) $\forall x \in \mathbb{Q} - \{0\} (\exists y \in \mathbb{Q} (xy = yx = 1))$

For any nonzero rational number x, there exists a rational number y such that $xy = yx = 1$.

True, for any $x \in \mathbb{Q} - \{0\}$, choose $y = 1/x$ such that $x(1/x) = (1/x)x = 1$.

(iii) $\exists x \in \mathbb{Z} (\forall y \in \mathbb{Z} (xy = yx = 1))$

There exists an integer x such that for any integer y, $xy = yx = 1$.

False

e.g. Trailing quantifiers and translating from informal to formal language

Rewrite each of the informal statements so that
(i) the quantifier trails the rest of the sentence
(ii) each statement is formal

(a) For any complex number z, the modulus (or absolute value) of z is nonnegative.

(ii) $\forall z \in \mathbb{C} \, (|z| \geq 0)$

(i) The modulus of z is nonnegative for any complex number z.

(b) There exists a complex number z such that $z^2 < -1$.

"$\exists$, $\ni$"

(ii) $\exists z \in \mathbb{C} \, (z^2 < -1)$

(i) $z^2 < -1$ for some complex number z.

true, choose $z = 6i$ such that $z^2 = -36 < -1$.

Universal Conditional Statements

A universal conditional statement takes form

$$\forall x\,(P(x) \to Q(x)).$$

e.g. Writing Universal Conditional Statements Informally

(a) $\forall x \in \mathbb{R}\,(|x| = 0 \to x = 0)$ true

(b) $\forall z \in \mathbb{C}\,(z = -\bar{z} \to z \notin \mathbb{R})$ false if $z = 0$

true for $\mathbb{C} - \{0\}$