Interior point methods for LPs

Today: algorithm for solving LPs

Classic method: simplex algorithm

Works well in practice. Idea: walk along boundary of polytope, only taking steps which reduce cost function

Unfortunately, simplex alg. can be shown to have exponential complexity. This motivated the development of algorithms with polynomial complexity.

Recall standard form LP:

$$\begin{aligned} \text{minimize } & c^T x \\ \text{subject to } & Ax = b \\ & x \geq 0 \end{aligned}$$

and its dual:

$$\begin{aligned} \text{maximize } & b^T \lambda \\ \text{subject to } & A^T \lambda + s = c \\ & s \geq 0 \end{aligned}$$

The KKT conditions for both of these are the same:

$$\left.\begin{aligned} Ax^* &= b \\ x^* &\geq 0 \end{aligned}\right\} \text{primal feasibility}$$

$$\left.\begin{aligned} A^T\lambda^* + s^* &= c \\ s^* &\geq 0 \end{aligned}\right\} \text{dual feasibility}$$

$$x_i^* s_i^* = 0 \ \forall i \leftarrow \text{complementary slackness}$$

For primal feasible x and dual feasible (x, λ, s), we have:

$$A^T\lambda + s = c \implies x^TA^T\lambda + x^Ts = x^Tc$$

$$\iff b^T\lambda + x^Ts = x^Tc.$$

Furthermore, $x^Ts \geq 0$ with equality if and only if (x, λ, s) are optimal (satisfy the KKT conditions).

For this reason, we call:

$$x^Ts = x^Tc - b^T\lambda$$

the duality gap.

We will look at the primal-dual interior point method for solving an LP.

The idea is to solve the KKT conditions iteratively.

Actually, we will approximately solve a sequence of KKT conditions. We let $\mu > 0$, and try to find $x = x(\mu)$, $\lambda = \lambda(\mu)$, $s = s(\mu)$ which satisfy:

$$\left.\begin{array}{r} Ax = b \\ x \geq 0 \end{array}\right] \text{primal feasibility}$$

$$\left.\begin{array}{r} A^T\lambda + s = c \\ s \geq 0 \end{array}\right] \text{dual feasibility}$$

$$s_i x_i = \mu \ \forall i \leftarrow \text{relaxed complementary slackness.}$$

If $(x(\mu), \lambda(\mu), s(\mu))$ satisfy these relaxed KKT conditions, then $x(\mu)$ is primal feasible, $(\lambda(\mu), s(\mu))$ are dual feasible, and the duality gap is:

$$x^Tc - b^T\lambda = s^Tx = \sum_{i=1}^{n} s_i x_i = n\mu > 0.$$

The basic idea of this method is to construct a sequence $\{\mu_k\}$ such that $\mu_k \to 0$ as $k \to \infty$ and build an iterative scheme based on this idea.

First, how to solve the relaxed KKT conditions? They are a nonlinear system of equations because of the complementary slackness condition. We'll try to apply Newton's method. We need to write down a function F such that:

$$F(x, \lambda, s) = 0, \qquad x \geq 0, \quad s \geq 0,$$

if the relaxed KKT conditions are satisfied. We first rewrite the relaxed complementary slackness conditions as:

$$XS\mathbb{1} = \mu \mathbb{1}.$$

where:

$$X = \operatorname{diag}(x_1, \dots, x_n) = \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{bmatrix} \in \mathbb{R}^{n \times n},$$

$$S = \operatorname{diag}(s_1, \dots, s_n),$$

$$\mathbb{1} = (1, \dots, 1) \in \mathbb{R}^n.$$

Then:

$$x_i s_i = \mu, \quad i = 1, \dots, n \iff XS\mathbb{1} = \mu \mathbb{1}.$$

Then:

$$F(x,\lambda,s) = \begin{bmatrix} A^T\lambda + s - c \\ Ax - b \\ XS\mathbb{1} - \mu\mathbb{1} \end{bmatrix} \in \mathbb{R}^{m+2n}.$$

The Newton iteration is:

$$(x_{k+1}, \lambda_{k+1}, s_{k+1}) = (x_k, \lambda_k, s_k) - \alpha_k DF(x_k, \lambda_k, s_k)^{-1} F(x_k, \lambda_k, s_k).$$

The Jacobian of F is:

$$DF = \begin{bmatrix} D_x F_1 & D_\lambda F_1 & D_s F_1 \\ D_x F_2 & D_\lambda F_2 & D_s F_2 \\ D_x F_3 & D_\lambda F_3 & D_s F_3 \end{bmatrix} = \begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix}.$$

Hence, the Newton step satisfies:

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x_k \\ \Delta \lambda_k \\ \Delta s_k \end{bmatrix} = \begin{bmatrix} c - s - A^T\lambda \\ b - Ax \\ \mu_k\mathbb{1} - XS\mathbb{1} \end{bmatrix} \quad (*)$$

To start the algorithm, we pick some initial strictly primal-dual feasible $x_0 > 0$, $s_0 > 0$. Then, $\mu_0 = \frac{x_0^T s_0}{n}$.

At each step, we need to keep x_k, s_k strictly feasible (i.e. $x_k > 0$, $s_k > 0$). There are solutions to the KKT conditions which we avoid by doing this. Details here are out of scope, but this is the origin of the term interior point.

We have the following algorithm:

Given: (x_0, λ_0, s_0) st $x_0 > 0$, $s_0 > 0$

For $k = 0, 1, \dots$:

Solve $(*)$ for $\begin{bmatrix} \Delta x_k \\ \Delta \lambda_k \\ \Delta s_k \end{bmatrix}$ with $\mu_k = \frac{x_k^T s_k}{n}$.

Set $\begin{bmatrix} x_{k+1} \\ \lambda_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ \lambda_k \\ s_k \end{bmatrix} + \alpha_k \begin{bmatrix} \Delta x_k \\ \Delta \lambda_k \\ \Delta s_k \end{bmatrix}$, choosing

α_k such that $x_{k+1} > 0$ and s_{k+1}.