Introduction to Robot Intelligence [Spring 2023]

Control

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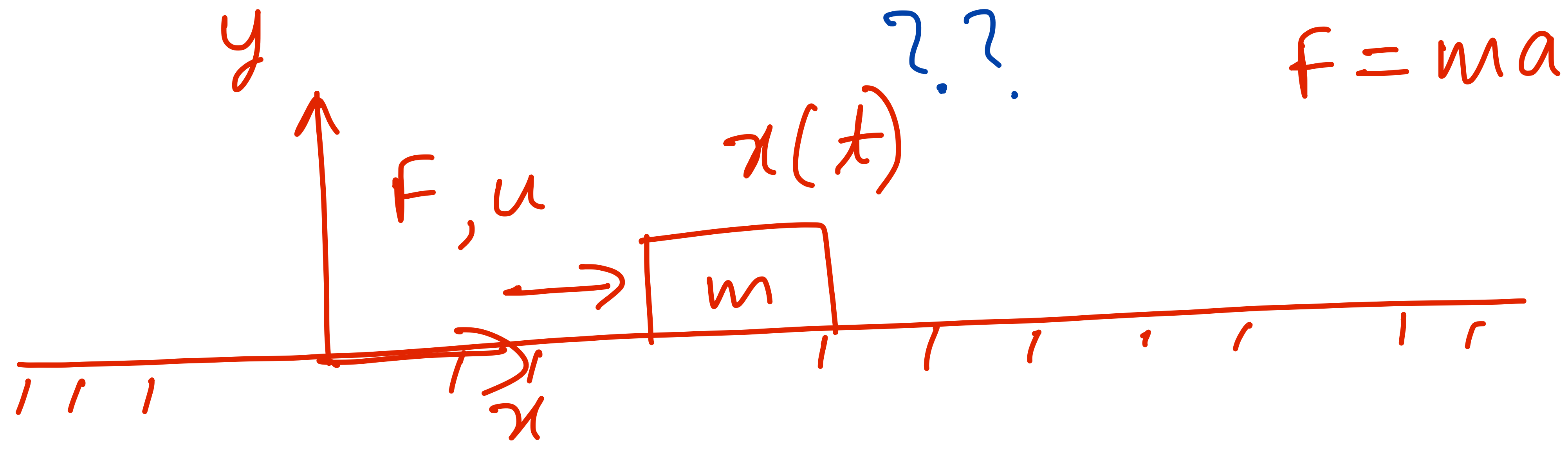
Robot Control

- Input: encoder position, (velocity, acceleration etc.)
- Output: motor toques, forces, (voltages etc.)
- One instance of the control problem:
    - Achieve desired position by controlling motor torques.

Formalism for Dynamical Systems

- Recap Newtonian Physics



$F = ma \rightarrow a = \frac{F}{m} \Big| \frac{u}{m}$

$\ddot{x}(t) = \frac{u}{m}$

$\dot{x}(t) = \int \ddot{x}(t)$

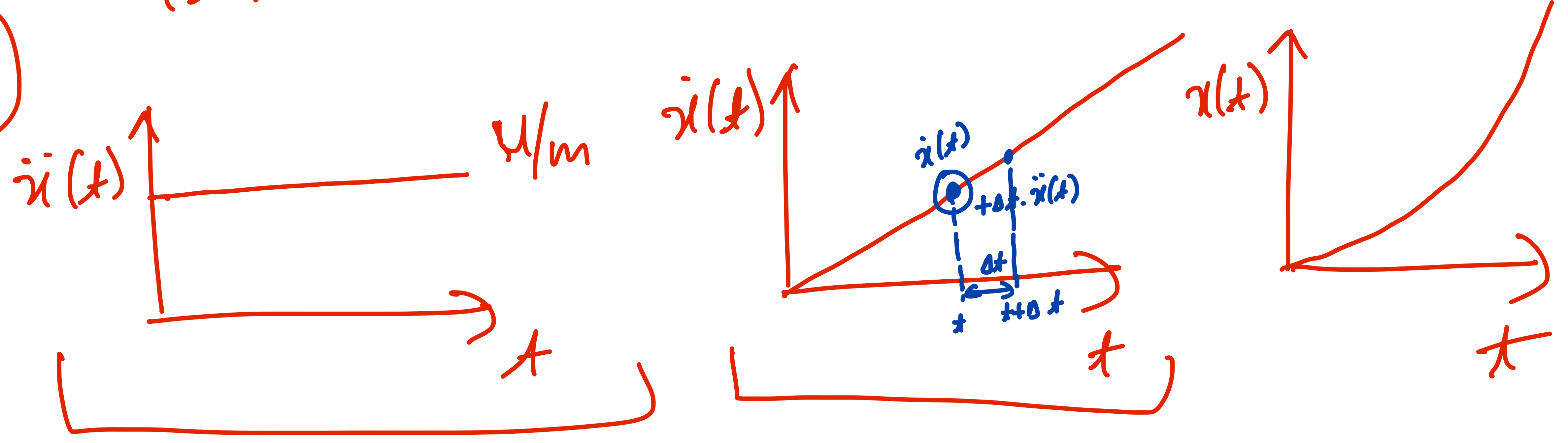
$x(t) = \int \dot{x}(t)$

$x(t) \rightarrow$ position
$\dot{x}(t) \rightarrow$ velocity
$\ddot{x}(t) \rightarrow$ acceleration.

at $t = 0$, $x(0) = 0$, $\dot{x}(0) = 0$, $\ddot{x}(0)$

$u = 1$, $m = 1$

at $t = 1$, $\dot{x}(1) = 1\ m/s$, $\ddot{x} = 1\ m/s^2$



$$\begin{bmatrix} x(t+1) \\ \dot{x}(t+1) \end{bmatrix} = \underline{f}\left(\begin{bmatrix} x(t) \\ \dot{x}(t+1) \end{bmatrix}, u \right)$$

$\rightarrow$ Previous state info

$\rightarrow$ Force / action info

$\Delta x(t) = \dot{x}(t) \cdot \Delta t + \frac{1}{2} \ddot{x} \Delta t^2$

Formalism for Dynamical Systems

$F = ma$

- Linearity of dynamical systems

$$\begin{bmatrix} x(t+\Delta t) \\ \dot{x}(t+\Delta t) \end{bmatrix} = f\left(\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, u \right)$$

$$\dot{x}(t+\Delta t) = \dot{x}(t) + \Delta t * \ddot{x}(t)$$

$$\dot{x}(t+\Delta t) = \dot{x}(t) + \Delta t \frac{u}{m} \quad \text{——— ①}$$

$$x(t+\Delta t) = x(t) + \Delta t \cdot \dot{x}(t) + \frac{1}{2}(\Delta t)^2 \ddot{x}(t)$$

$$x(t+\Delta t) = x(t) + \Delta t\, \dot{x}(t) + \frac{1}{2}(\Delta t)^2 \frac{u}{m} \quad \text{——— ②}$$

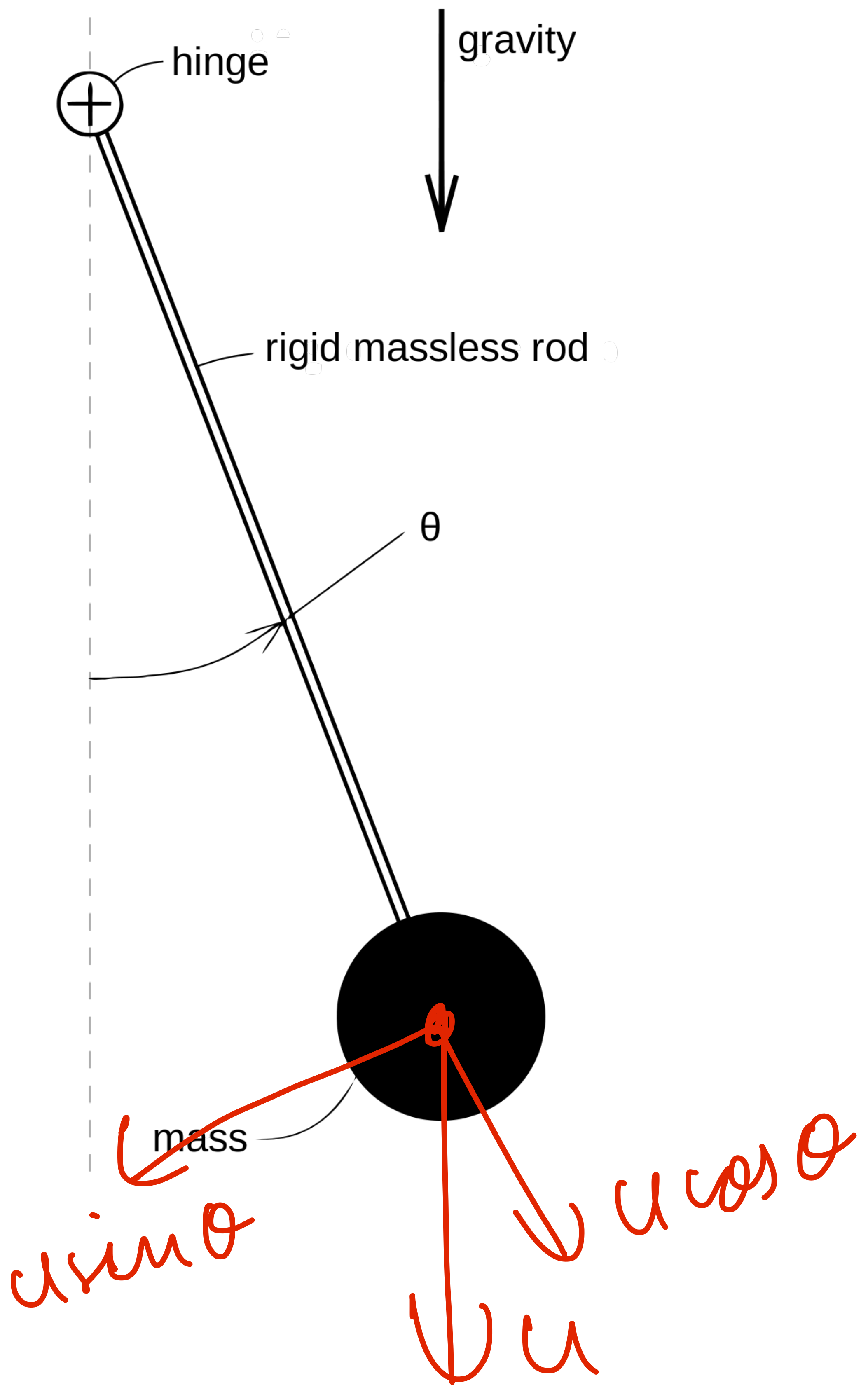
$$\begin{bmatrix} x(t+\Delta t) \\ \dot{x}(t+\Delta t) \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \begin{bmatrix} \frac{(\Delta t)^2}{2m} \\ \frac{\Delta t}{m} \end{bmatrix} \cdot u \quad \text{——— ③}$$

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix}_{t+\Delta t} = A \begin{bmatrix} x \\ \dot{x} \end{bmatrix}_{t} + B \cdot u$$

$$f = u \cdot m$$

Formalism for Dynamical Systems

- Examples of non linear systems

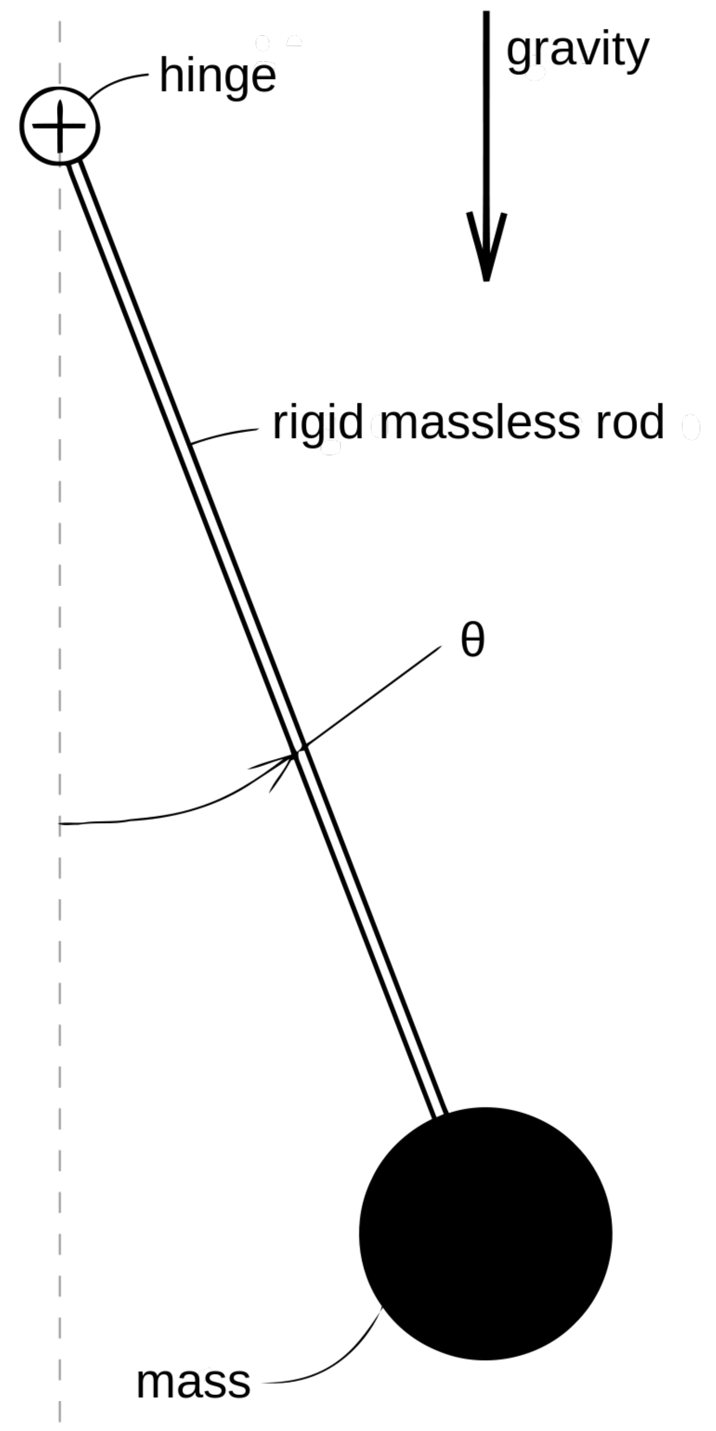


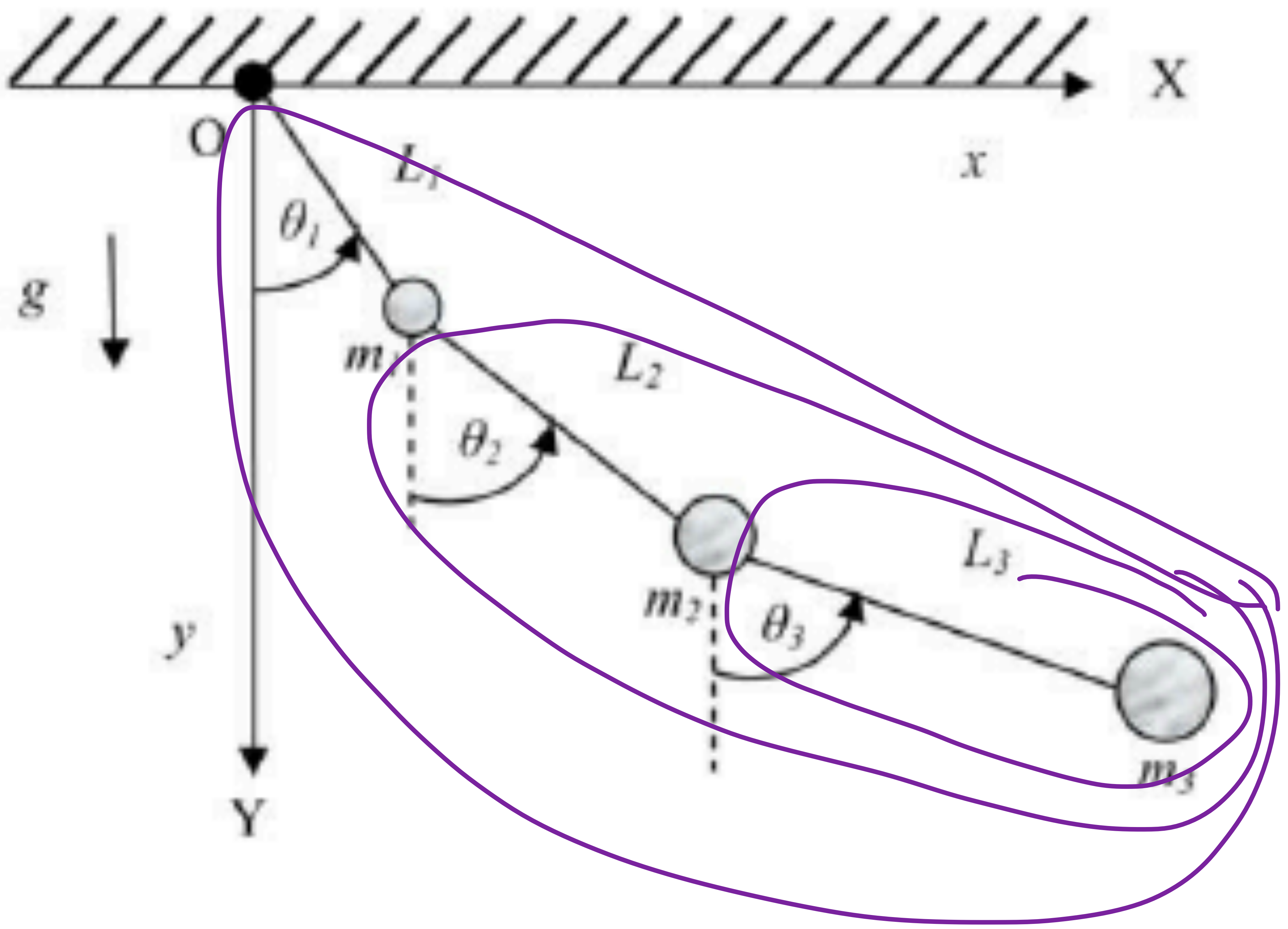
$$u \sin\theta \cdot l = \tau = \underbrace{I\ddot{\theta}}$$
$$= m l^2 \ddot{\theta}$$

$$\ddot{\theta} = \frac{u \boxed{\sin\theta}}{m l}$$

Formalism for Dynamical Systems

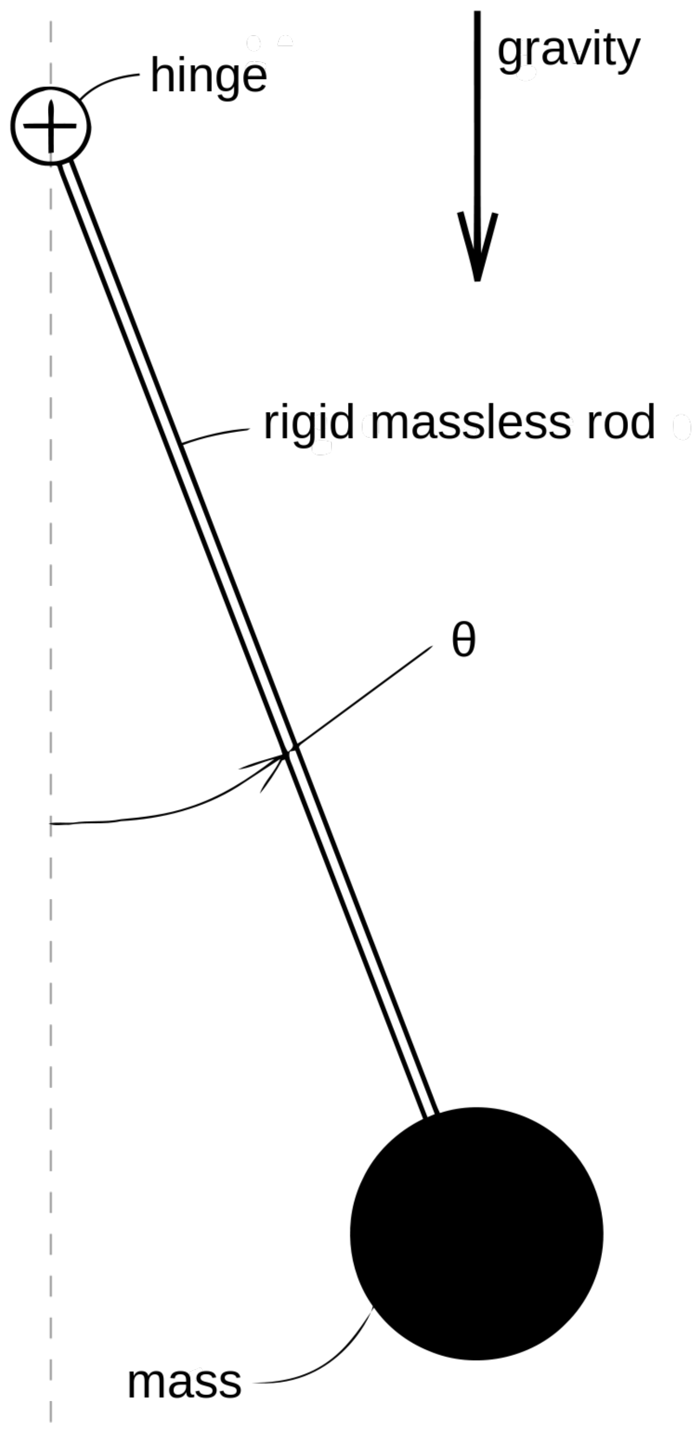
- Examples of non linear systems

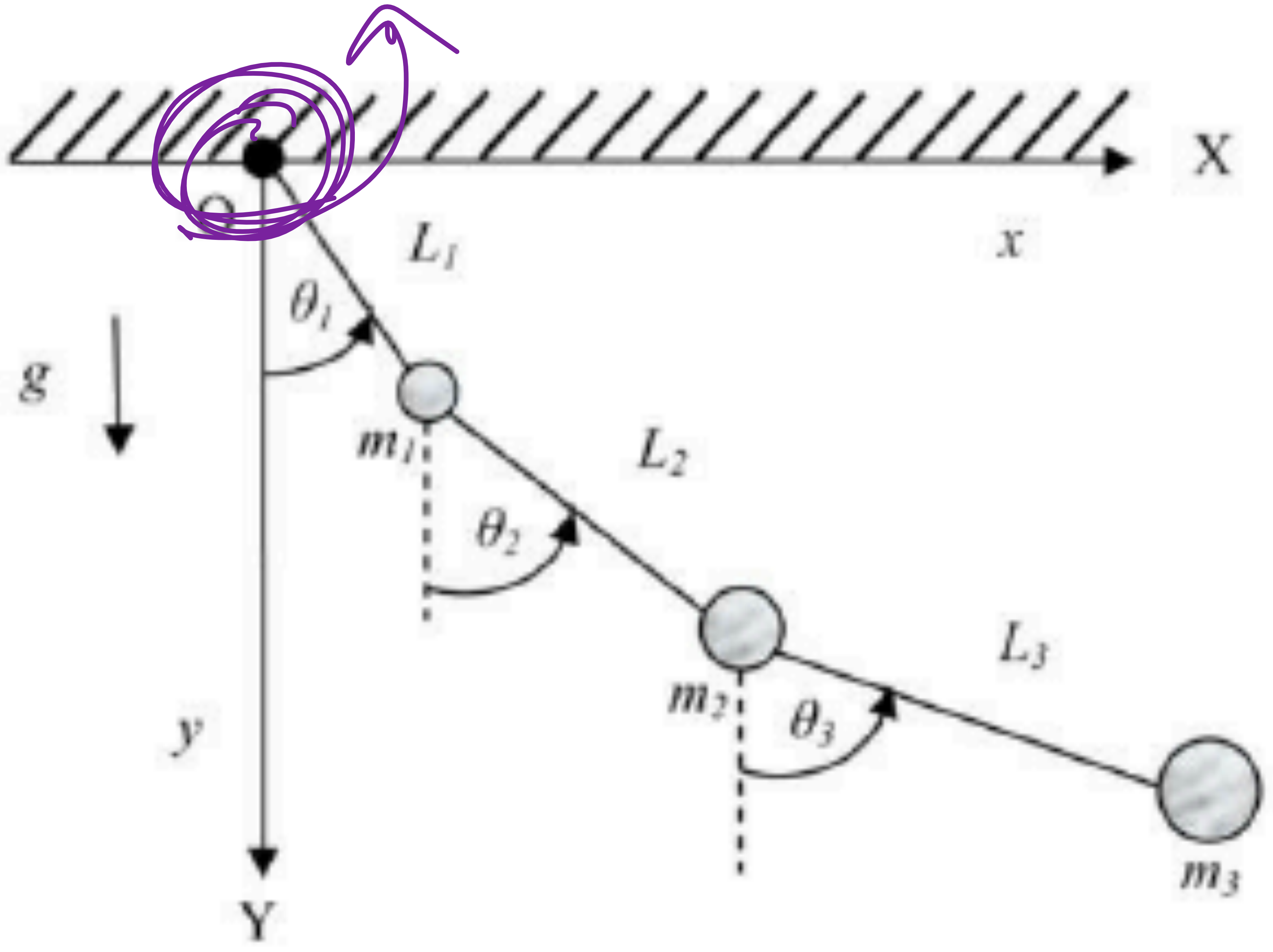


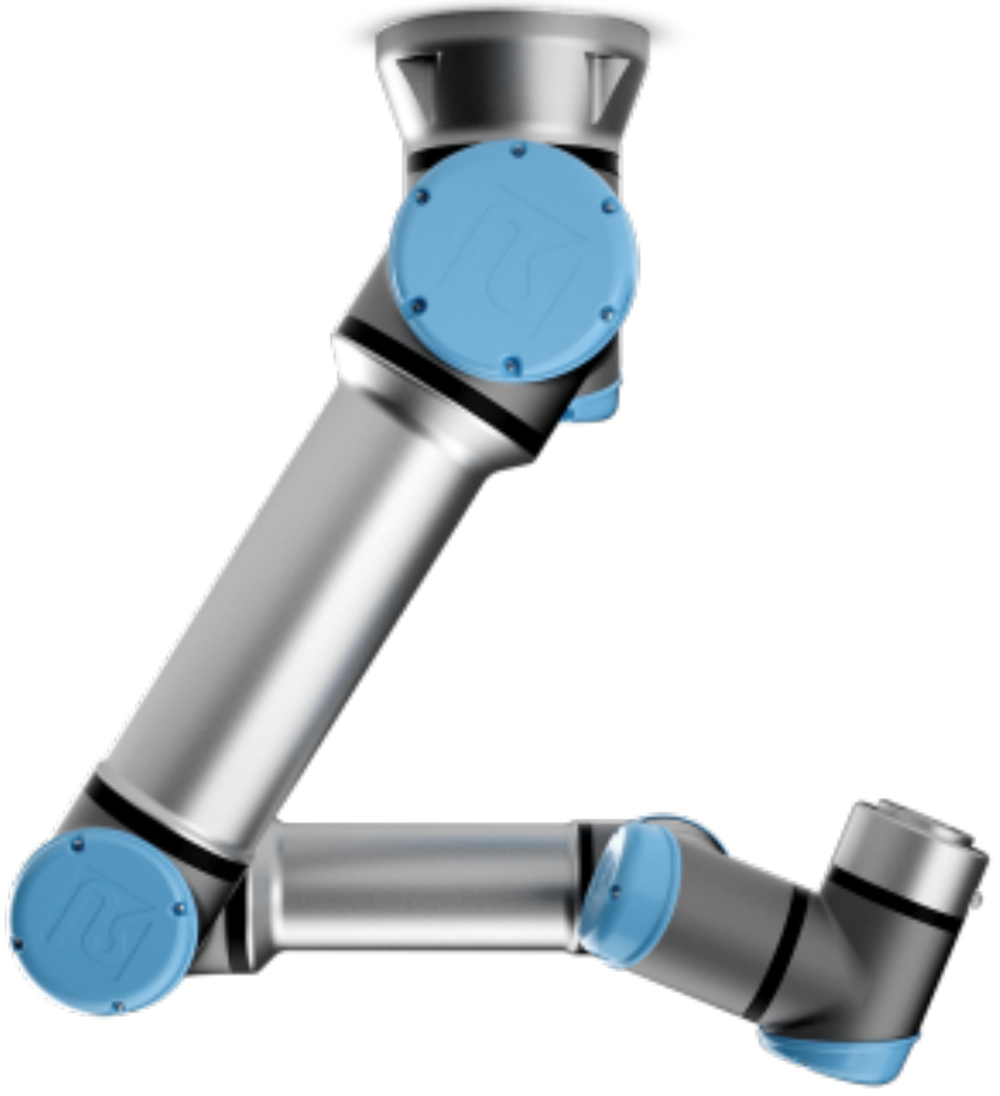


Formalism for Dynamical Systems

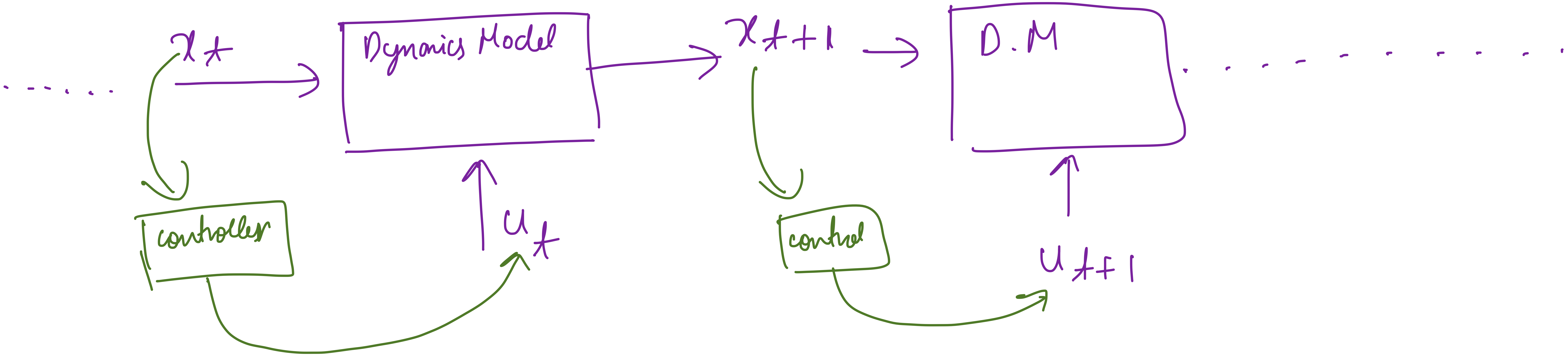
- Examples of non linear systems

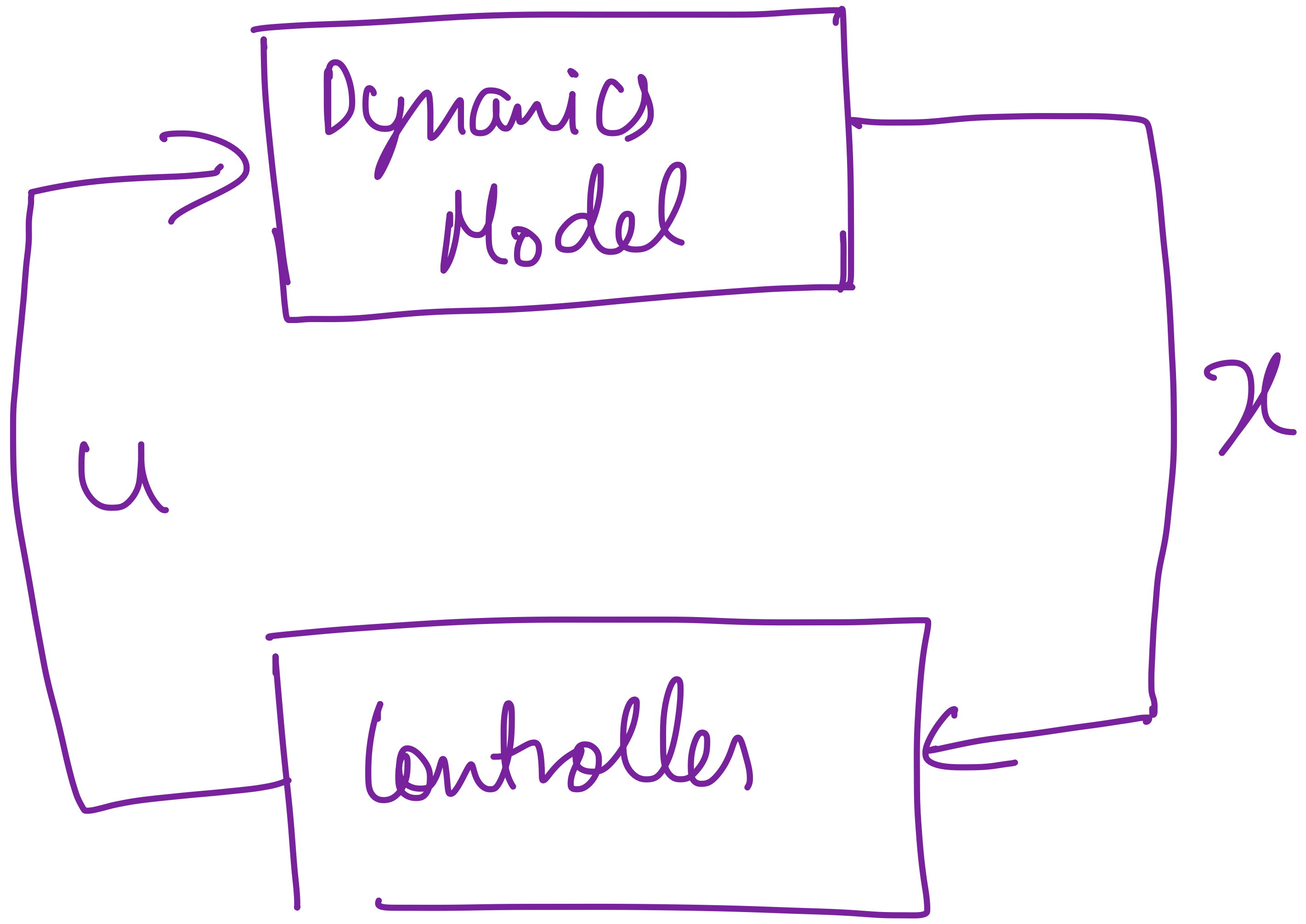






What is a controller?





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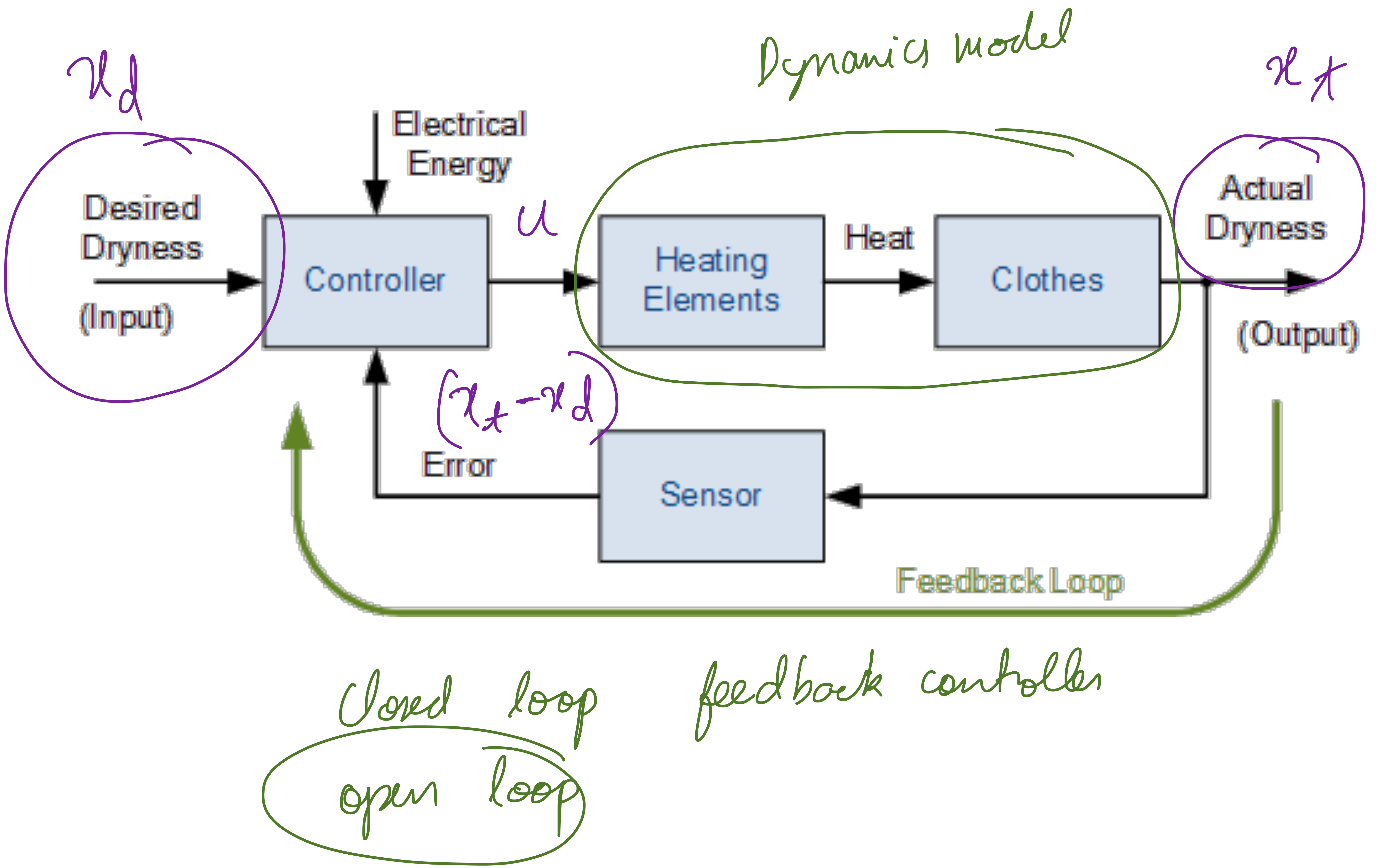


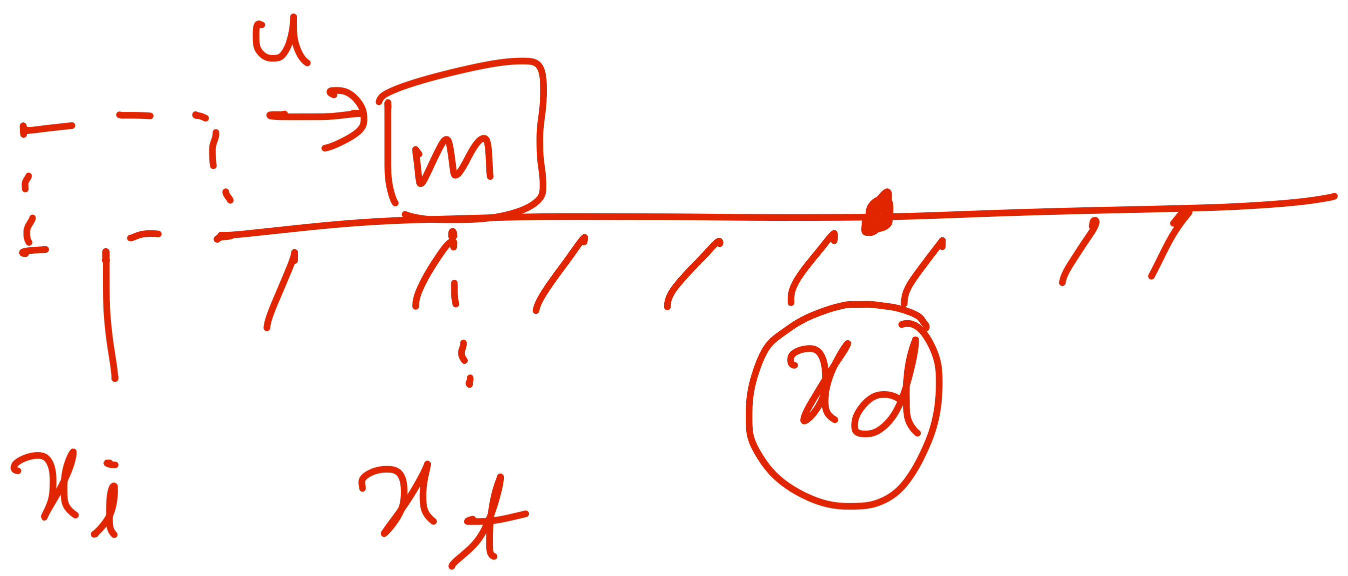
How do we measure optimality?

$-u_{max} < u_t < u_{max}$

$C_t(x_t, u_t)$

$$\min_{u_0, u_1, \dots, u_H} \sum_{t=0}^{H} C_t(x_t, u_t)$$

$x_{t+1} = f(x_t, u_t) \rightarrow$ Dynamics Model



$C_t = (x_d - x_t)^2$

LQR – One class of control problem

$$x_{t+1} = f(x_t, u_t)$$

$$\underbrace{x_{t+1} = A x_t + B u_t}_{\text{Linear Dynamics}}$$

$$\underbrace{\begin{aligned} c_t(x_t, u_t) &= Qx^2 + Ru^2 \\ &= x_t^T Q x_t + u_t^T R u_t \end{aligned}}_{\text{Quadratic}}$$

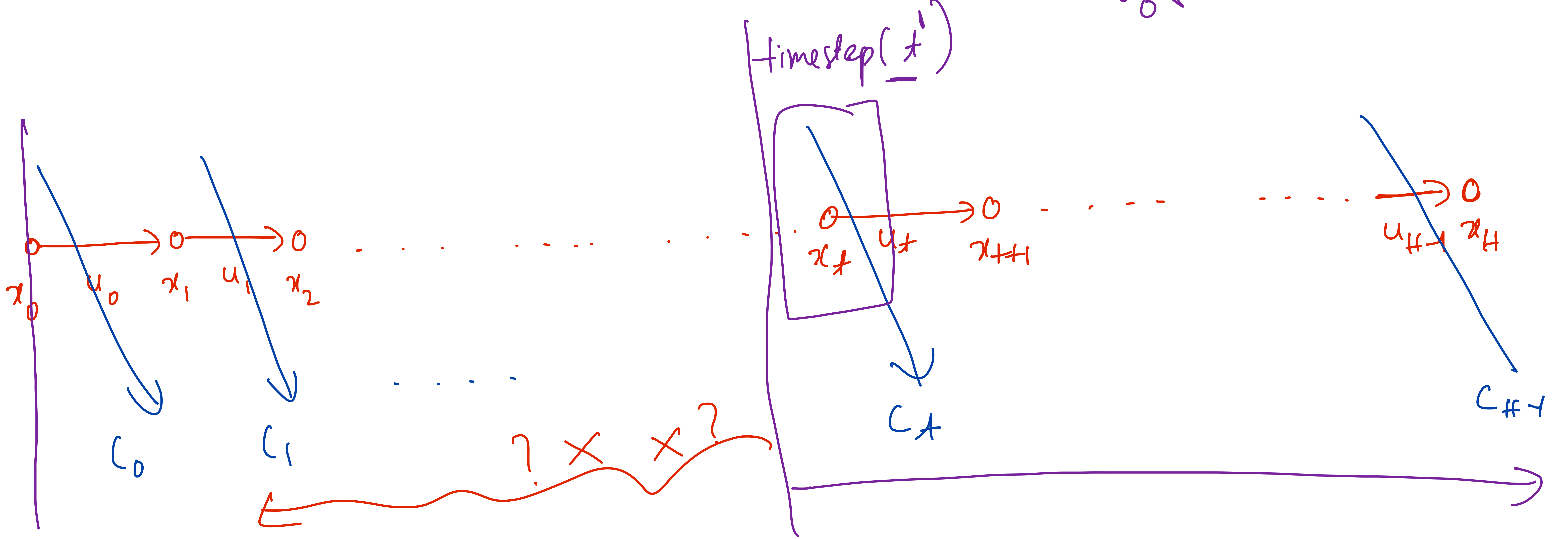
Linear Quadratic Regulator

Value – Cost to go function

$$\sum_{t=0}^{H} c_t(x_t, u_t)$$

$$V_{t'} = \sum_{t=t'}^{H} c_t(x_t, u_t)$$

V_0



LQR solving through dynamic programming

$$\min_{u_{0:H-1}} \sum_{t=0}^{H} c_t(x_t, u_t) \;\Big|\; \text{s.t.} \quad x_{t+1} = Ax_t + Bu_t$$

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \cdots \xrightarrow{u_{t-1}} x_t \xrightarrow{u_t} x_{t+1} \cdots x_{H-1} \xrightarrow{u_{H-1}} x_H$$

V^*_{t+1}

$$V_H = \sum_{t=H}^{H} c_H(x_t, u_t) = x_H^T Q x_H = x_H^T P_H x_H \;\Big|\; P_H = Q \qquad u_H = 0$$

$$V^*_{t'} = \min \sum_{t=t'}^{H} c_t(x_t, u_t)$$

$$= \min c_{t'}(x_{t'}, u_{t'}) + \sum_{t=t'+1}^{H} c_t(x_t, u_t)$$

$$= \min \quad c_{t'}(x_{t'}, u_{t'}) + V^*_{t'+1}(x_{t'+1})$$

LQR solving through dynamic programming