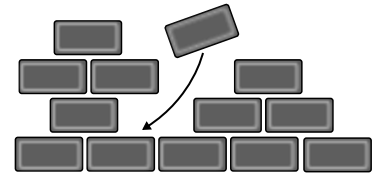


EM3 The Planewave

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This handout is provided to fill in the gaps that inevitably arise when presenting the highlights of a long derivation through the medium of PowerPoint, just in case it helps anyone who likes to see each brick go into the wall. The derivations themselves are most definitely NOT examinable - but you'll need be able to use the results to calculate the fields of a wave in a dielectric and in a conductive medium, as well as knowing the phase relationship between the electric and magnetic fields in both cases. So this document contains *some* examinable material (mainly toward the end).



Introduction

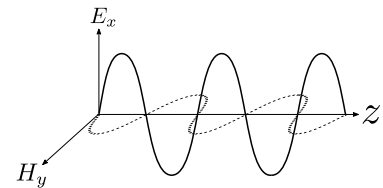
Here we rehearse the combined solution of Gauss's Laws, Ampere's Law, and Faraday's Law to show the existence of electromagnetic waves (let there be light!) We'll work out the fields of a plane wave travelling in the $+z$ direction in an infinite region to be

$$E_x(z, t) = E_0 e^{j\omega t - \gamma z}$$
$$H_y(z, t) = \frac{E_0}{\gamma} e^{j\omega t - \gamma z} (\sigma + j\omega\epsilon)$$

where the propagation constant γ is

$$\gamma = [j\omega\mu (\sigma + j\omega\epsilon)]^{\frac{1}{2}},$$

ω is the radian frequency such that $\omega = 2\pi f$ where f is frequency in Hertz, σ is conductivity in Siemens per metre, $\epsilon = \epsilon_r \epsilon_0$ is the permittivity and $\mu = \mu_r \mu_0$ is the permeability.



Getting Started

We are going to mess about¹ with Ampere's and Faraday's Laws until we get something sensible that tells us how a plane wave travels about the place. Let's start with the derivative-form of Maxwell's equations from the EM3 formula sheet:

$$\nabla \cdot \vec{D} = \rho \quad (1)$$

$$\nabla \cdot \vec{B} = 0 \quad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (3)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (4)$$

¹ Some of the maths will involve tricks and identities that will be at the very least unfamiliar, and may even look downright shady at times, but rest assured that even the scary/exotic div, grad and curl identities are well founded, and not just rabbits we pulled out of hats. Someone else pulled them out of the hat, and it took them a lot of work, so we'll say a breezy "thank-you" and get on with putting their hard work to use in getting us to where we need to go.

We are going to focus on figuring out plane waves that are traveling through wide open empty space where there is nothing to bump into - that avoids complicating things any more than they already are². To continue keeping the maths as simple as possible, we're going to state we are not expecting any sources, or charges in this wide open region. So $\rho = 0$, and we can substitute that and $D = \epsilon E$ into Eq. 1 to give something that will come in handy soon:

$$\nabla \cdot \vec{E} = 0. \quad (5)$$

Let's start by tweaking Eqs. 3 & 4 to contain only E and H , so that we can substitute one into another without getting our fluxes and fields mixed up. We recall that $B = \mu H$ and that $D = \epsilon E$, and we know that with no sources in vacuum there are no currents, so $\vec{J} = 0$. We can tidy them up thus

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}, \quad (6)$$

$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t}. \quad (7)$$

We now move onto bit where we do a bunch of non-obvious things as if they were obvious all along - don't worry, the steps *are* exactly as unobvious as they seem to you. This is just the refined procedure that is left over after a bunch of great minds have spent a lot of graphite and laminar pinus radiata. if you suspend disbelief for a moment you'll see it is an elegant way to get where we want to go.

The first seemingly shady trick is that we take the curl of both sides of Eq. 6, but the motivation is that it sets us up to turn the curl into a vector Laplacian (so long as we are in a source-free region)

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\mu \frac{\partial \vec{H}}{\partial t} \right) \quad (8)$$

$$= -\mu \frac{\partial}{\partial t} \nabla \times \vec{H}. \quad (9)$$

Working on the right hand side, we can substitute in Eq. 7

$$\nabla \times (\nabla \times \vec{E}) = -\mu \frac{\partial}{\partial t} \nabla \times \vec{H} \quad (10)$$

$$= -\mu \frac{\partial}{\partial t} \left(\epsilon \frac{\partial \vec{E}}{\partial t} \right) \quad (11)$$

$$= -\mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}. \quad (12)$$

Now working on the left hand side, we can use this curl-of-curl identity

$$\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \quad (13)$$

² If it helps, imagine we are in a distant region of the Universe that is devoid of even space dust for light weeks around us - that's long enough to delay any boundary condition problems until the second part of the course, literally!

In a vacuum, $\mu = \mu_0$ and $\epsilon = \epsilon_0$ everywhere, but for convenience we'll just continue to use μ and ϵ without the subscripts.

$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t}$$

to obtain

$$\nabla \left(\nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} = -\mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}. \quad (14)$$

Now we substitute in Eq. 5 to trim away some of the complicated stuff and leave us with a (relatively!) straightforward differential equation

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}. \quad (15)$$

Equation 15 relates space to time³, so if we solve it, it will tell us how \vec{E} behaves as a function of position and time - and it's going to be a wave. Let's explore the space (left hand side) a bit more by expanding it in Cartesian coordinates⁴

$$\nabla^2 \vec{E} = \nabla^2 E_x \hat{x} + \nabla^2 E_y \hat{y} + \nabla^2 E_z \hat{z}, \quad (16)$$

Since $E_x \neq 0$ and $E_y = E_z = 0$

$$\nabla^2 \vec{E} = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \hat{x}, \quad (17)$$

but we know that E_x can only vary⁵ in the direction of propagation z , so two of the double derivatives are zero, leaving

$$\nabla^2 \vec{E} = \frac{\partial^2 E_x}{\partial z^2} \hat{x}. \quad (18)$$

Hence our wave equation becomes

$$\frac{\partial^2 E_x}{\partial z^2} \hat{x} = \mu\epsilon \frac{d^2 \vec{E}_x}{dt^2}. \quad (19)$$

Solutions to the wave equation can be done in a number of ways⁶ but it can be proven by direct substitution⁷ that a valid solution is

$$\vec{E}_x(z, t) = A e^{j(\omega t - \beta z)} \quad (20)$$

where A is an arbitrary and constant magnitude, $\omega = 2\pi f$ is the radian frequency, f is the frequency in Hertz, $\beta = \frac{2\pi}{\lambda}$ is the phase constant, $\lambda = \frac{c}{f}$ is the wavelength, and the speed of the wave is $c = \frac{1}{\sqrt{\mu\epsilon}} \approx 3 \times 10^8 \text{ m s}^{-1}$.

Where does the speed of light fit into the equation? In a wave equation of standard form, the coefficient of the time derivative is equal to the inverse square of the speed of the wave, so

$$\frac{1}{c^2} = \mu\epsilon \quad (21)$$

$$c = \frac{1}{\sqrt{\mu\epsilon}} \quad (22)$$

$$c = \frac{1}{\sqrt{4\pi \times 10^{-7} 8.854 \times 10^{-12}}} \quad \text{in a vacuum} \quad (23)$$

$$c \approx 3 \times 10^8 \text{ m s}^{-1}. \quad (24)$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla^2 = 0$$

³ This type of linear partial differential equation is known in classical physics as a wave equation. Wave equations are used for mechanical waves like sound, water and seismic waves, as well as electromagnetic waves, where the speed of the wave is determined by the coefficients in the equation.

⁴ The expansion is based on

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

⁵ If E_x varied in the in-plane directions of x and y the wave would no longer be a plane wave, by definition.

⁶ The wave equation can be solved by an algebraic method, by separation of the space and time variables, or perhaps by other methods. The available numerical solvers can be categorised as either time or frequency domain techniques.

⁷ Direct substitution of the solution shows it is valid

$$\begin{aligned} \frac{d^2}{dz^2} A e^{j(\omega t - \beta z)} &= \mu\epsilon \frac{\partial^2}{\partial t^2} A e^{j(\omega t - \beta z)} \\ A j^2 (-\beta)^2 e^{j(\omega t - \beta z)} &= -A \mu\epsilon j^2 \omega^2 e^{j(\omega t - \beta z)} \\ \beta^2 &= \mu\epsilon \omega^2 \\ \left(\frac{2\pi}{\lambda} \right)^2 &= \frac{1}{c^2} (2\pi f)^2 \\ \left(\frac{2\pi}{\lambda} \right)^2 &= \frac{1}{c^2} \left(\frac{2\pi c}{\lambda} \right)^2 \\ \left(\frac{2\pi}{\lambda} \right)^2 &= \left(\frac{2\pi}{\lambda} \right)^2 \quad \blacksquare \end{aligned}$$

Conductive materials

The chances of us being out in space for all our electromagnetics is vanishingly small. So we should repeat the derivation again but this time assume we are in a 3D volume of conductive material of conductivity σ with infinite extent in all directions, i.e. $\vec{J} = \sigma \vec{E} \neq 0$, such that Ampere's law can be expressed

$$\nabla \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad (25)$$

This way, we get the full benefit of Ampere's Law, but by assuming the region is impossibly large, we still avoid the issues of bumping into the edges of the material (for a while yet, at least).

We return to Eq. 9 and substitute in Eq. 25

$$\nabla \times (\nabla \times \vec{E}) = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}. \quad (26)$$

As before, we use Eqs. 5 & 13 to simplify the left hand side, leaving

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad (27)$$

which can be reduced to our plane wave with $E_x \neq 0$ propagating in the $+z$ direction as before

$$\frac{\partial^2 \vec{E}_x}{\partial z^2} = \mu\sigma \frac{\partial \vec{E}_x}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}_x}{\partial t^2}. \quad (28)$$

We know that current flowing in a material causes heating, which means a conversion of energy from the wave into heat. This can be represented by a small tweak to our solution to the wave equation,⁸ to include a decay term

$$\vec{E}_x(z, t) = Ae^{j(\omega t - \beta z)} e^{-\alpha z} = Ae^{j\omega t - \gamma z} \quad (29)$$

where the newly-introduced propagation constant is complex $\gamma = \alpha + j\beta$, such that β continues to describe the phase constant as before (which relates to the wavelength), but we add in a new term, the attenuation constant α that represents the spatial decay of the wave (a decaying wave does not change wavelength, just amplitude, hence α and β are separate). The more conductive the material, the bigger is α , and the quicker the decay of the wave.

Attenuation constant

The attenuation constant α can be worked out by substituting Eq. 29 into Eq. 28 to obtain the dispersion relation, then solving for γ (of which α is the real part). We first perform the substitution

$$\gamma^2 Ae^{j\omega t - \gamma z} = j\mu\sigma\omega Ae^{j\omega t - \gamma z} - \mu\epsilon\omega^2 Ae^{j\omega t - \gamma z}, \quad (30)$$

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\mu \frac{\partial \vec{H}}{\partial t} \right)$$

⁸ We can show the new solution is valid by substituting it back into Eq. 28. First we calculate the individual parts

$$\begin{aligned} \frac{\partial^2}{\partial z^2} Ae^{j\omega t - \gamma z} &= (-\gamma)^2 Ae^{j\omega t - \gamma z} \\ \frac{\partial}{\partial t} Ae^{j\omega t - \gamma z} &= j\omega Ae^{j\omega t - \gamma z} \\ \frac{\partial^2}{\partial t^2} Ae^{j\omega t - \gamma z} &= (j\omega)^2 Ae^{j\omega t - \gamma z} \end{aligned}$$

then substitute into Eq. 28 and eliminate common factors

$$\begin{aligned} (-\gamma)^2 Ae^{j\omega t - \gamma z} &= Ae^{j\omega t - \gamma z} (\sigma\mu(j\omega) + \mu\epsilon(j\omega)^2) \\ \gamma^2 &= (j\omega\mu\sigma - \omega^2\mu\epsilon) \blacksquare \end{aligned}$$

then remove the common factor of $Ae^{j\omega t - \gamma z}$ to give

$$\gamma^2 = j\omega\mu\sigma - \omega^2\mu\epsilon \quad (31)$$

which rearranges to⁹

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} \quad (32)$$

Things get a bit tricky now because there is no simple algebraic manipulation that will result in a general case we can use across all types of media that gives us nice expressions for α and β . So we need to consider the material we are in, simplify γ and then solve¹⁰ for α and β . For the conductive material we consider in this section, the case is at least relatively straightforward, because in a lossy material $\frac{\sigma}{\omega} \gg \epsilon$, although for the arrangement of this equation on the EM3 formula sheet, it is easier to think of it as $\epsilon \rightarrow 0$, giving¹¹

$$\gamma \approx \frac{1}{\sqrt{2}} \sqrt{\omega\mu\sigma}(j+1) \quad (33)$$

If you are struggling to figure out how to re-arrange the equation, your bail-out option is to substitute everything into Eq. 32 and work out the complex number numerically. There will still be a j under the square root sign, so you will still need this identity to be able to find α and β (this *is* examinable, and not on the formula sheet)

$$\sqrt{j} = \frac{(1+j)}{\sqrt{2}}. \quad (34)$$

Skin depth

The attenuation constant is large in a conductive medium - this is why we shield radio frequency components in metal boxes when we want to reduce interference from external waves. Consider the example of a wave of frequency ¹² $f = 100$ MHz travelling in a gold, a good conductor with $\sigma = 6 \times 10^7 \text{ S m}^{-1}$. Substituting these values into Eq. 33 we get

$$\gamma \approx \frac{1}{\sqrt{2}} \sqrt{2\pi \times 10^8 \times 4\pi \times 10^{-7} \times 6 \times 10^7}(j+1) \quad (35)$$

where

$$\alpha = \text{Re}(\gamma) \approx 1.54 \times 10^5 \text{ Np m}^{-1} \quad (36)$$

You'll see a new unit used here, the Neper (Np), which is dimensionless and used to express ratios in terms of natural logarithms, much as decibels do so for logarithms to the base 10. To get a better feel for how the α value of a given material affects a wave at a given frequency, we often translate it into a measure of how far the wave will

⁹ spot that we play around with j terms to neaten up the equation

In the EM3 formula sheet, γ is given using a different notation for the square root, but is the same equation

$$\gamma = [j\omega\mu(\sigma + j\omega\epsilon)]^{\frac{1}{2}}$$

¹⁰ The attenuation and phase constants can be found by

$$\alpha = \text{Re}(\gamma)$$

$$\beta = \text{Im}(\gamma)$$

¹¹ We'll need to use this identity

$$\sqrt{j} = \frac{(1+j)}{\sqrt{2}}$$

¹² Remember to convert to radian frequency $\omega = 2\pi f$

penetrate before falling to a certain fraction of its original voltage. That fraction is, by convention, $\frac{1}{e}$ for the skin depth δ ,

$$\frac{1}{e} = \frac{E_x(z = \delta)}{E_x(z = 0)} \quad (37)$$

$$= \frac{e^{-\alpha\delta}}{e^0} \quad (38)$$

hence we can solve for δ by taking the natural logarithm of both sides

$$\ln\left(\frac{1}{e}\right) = -\alpha\delta \quad (39)$$

$$\ln 1 - \ln e = -\alpha\delta \quad (40)$$

$$0 - 1 = -\alpha\delta \quad (41)$$

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f\mu\sigma}} \quad (42)$$

The skin depth for our example of gold at 100 MHz is $\delta = 6.5 \mu\text{m}$. A complete wavelength¹³ is $2\pi\delta$, yet the wave is considered fully attenuated when it reaches 1% of its original amplitude, at 4.6δ . Since $4.6 < 2\pi$, not even a full wavelength of the 100 MHz wave penetrates into the gold.

So far so good - we've worked out the electric field for the wave in a dielectric and conductor. Let's look at the counterpart magnetic field now.

Magnetic Field

We can use Faraday's Law (Eq. 4) to work out the magnetic field for our plane wave, by substituting in Eq. 29, and recalling that $\vec{D} = \epsilon\vec{E}$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (43)$$

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) \hat{x} = \left(\sigma E_x + \epsilon \frac{\partial E_x}{\partial t}\right) \hat{x}. \quad (44)$$

Since a plane wave must be everywhere the same in the in-plane directions (for this wave, x and y) $\frac{\partial H_z}{\partial y} = 0$, so we can simplify and drop the unit vectors

¹³ In all materials

$$\lambda = \frac{2\pi}{\beta}$$

In a conductor

$$\beta = \frac{1}{\delta}$$

so in a conductor

$$\lambda = 2\pi\delta$$

$$e^{-4.6} = 0.01$$

$$-\frac{\partial H_y}{\partial z} = \sigma E_x + \epsilon \frac{\partial E_x}{\partial t} \quad (45)$$

$$-H_y = \int \left(\sigma E_x + \epsilon \frac{\partial E_x}{\partial t} \right) \partial z \quad (46)$$

$$= \int \sigma A e^{j\omega t - \gamma z} \partial z + \int j\omega \epsilon A e^{j\omega t - \gamma z} \partial z \quad (47)$$

$$H_y = \frac{\sigma}{\gamma} A e^{j\omega t - \gamma z} + \frac{j\omega \epsilon}{\gamma} A e^{j\omega t - \gamma z} \quad (48)$$

$$H_y = A e^{j\omega t - \gamma z} \left(\frac{\sigma}{\gamma} + \frac{j\omega \epsilon}{\gamma} \right) \quad (49)$$

Our troubles are not quite yet over because γ is complex, so we can't just eyeball the phase relationship between the electric and magnetic field (yet). But we can return to our assumptions that the material is either a dielectric ($\sigma = 0$) or a conductor ($\sigma \gg \omega \epsilon$) and treat each case separately. For the dielectric case

For a dielectric,

$$H_y = A e^{j\omega t - \gamma z} \frac{j\omega \epsilon}{\gamma} \quad (50)$$

$$= A e^{j\omega t - \gamma z} \frac{j\omega \epsilon}{j\omega \sqrt{\mu \epsilon}} \quad (51)$$

$$= A e^{j\omega t - \gamma z} \sqrt{\frac{\epsilon}{\mu}} \quad (52)$$

$$\gamma^2 = -\omega^2 \mu \epsilon$$

$$= j^2 \omega^2 \mu \epsilon.$$

where we recalled that $-1 = j^2$. Hence

$$\gamma = j\omega \sqrt{\mu \epsilon}$$

The ratio between the electric and magnetic fields gives us the impedance Z , which is

$$Z_{diel} = \frac{E}{H} \quad (53)$$

$$= \frac{A e^{j\omega t - \gamma z}}{A e^{j\omega t - \gamma z} \sqrt{\frac{\epsilon}{\mu}}} \quad (54)$$

$$= \sqrt{\frac{\mu}{\epsilon}} \quad (55)$$

$$= \sqrt{\frac{4\pi \times 10^{-7}}{8.854 \times 10^{-12}}} \text{ in a vacuum} \quad (56)$$

$$\approx 377 \Omega \quad (57)$$

Both the electric and magnetic fields have only real coefficients, so they are both in phase in a dielectric, i.e.

$$\phi = \arctan\left(\frac{\text{Im}(Z)}{\text{Re}(Z)}\right) = 0. \quad (58)$$

In the case of the conductor, $\sigma \gg \omega \epsilon$, and $\gamma = \sqrt{j\omega \mu \sigma}$ so

$$H_y = A e^{j\omega t - \gamma z} \frac{\sigma}{\gamma} \quad (59)$$

$$= A e^{j\omega t - \gamma z} \sqrt{\frac{\sigma}{2\omega \mu}} (1 - j). \quad (60)$$

The impedance becomes

$$Z_{cond} = \frac{E}{H} \quad (61)$$

$$= \frac{Ae^{j\omega t - \gamma z}}{Ae^{j\omega t - \gamma z} \sqrt{\frac{\sigma}{2\omega\mu}} (1-j)} \quad (62)$$

$$= \frac{1}{\sqrt{\frac{\sigma}{2\omega\mu}} (1-j)} \quad (63)$$

$$= \sqrt{\frac{2\omega\mu}{\sigma}} \frac{(1+j)}{2} \quad (64)$$

$$= \sqrt{\frac{\omega\mu}{2\sigma}} (1+j) \quad (65)$$

We can see from the presence of $1+j$ in the impedance that the time-phase relationship is now that the electric field is $\pi/4$ ahead of the magnetic field, due to the conductive medium.

$$\phi = \arctan\left(\frac{\text{Im}(Z)}{\text{Re}(Z)}\right) \quad (66)$$

$$= \arctan\left(\frac{1}{1}\right) \quad (67)$$

$$= \pi/4. \quad (68)$$

To eliminate complex numbers in the denominator, multiply by the conjugate/conjugate

$$\begin{aligned} \frac{1}{1+j} &= \frac{1}{1+j} \frac{1-j}{1-j} \\ &= \frac{1-j}{2} \end{aligned}$$

Hence

$$\begin{aligned} \frac{\sigma}{\gamma} &= \frac{\sigma\sqrt{2}}{(1+j)\sqrt{\omega\mu\sigma}} \\ &= \frac{\sqrt{2\sigma}}{(1+j)\sqrt{\omega\mu}} \\ &= \frac{\sqrt{2\sigma}}{\sqrt{\omega\mu}} \frac{1-j}{2} \\ &= \sqrt{\frac{\sigma}{2\omega\mu}} (1-j) \end{aligned}$$

You can arrive at same impedance result almost directly from the EM3 formula sheet by setting $j\omega\epsilon = 0$ and then taking the j out of the square root using the same identity we used earlier.

The time-phase relationships for both cases are illustrated in Figure 1.

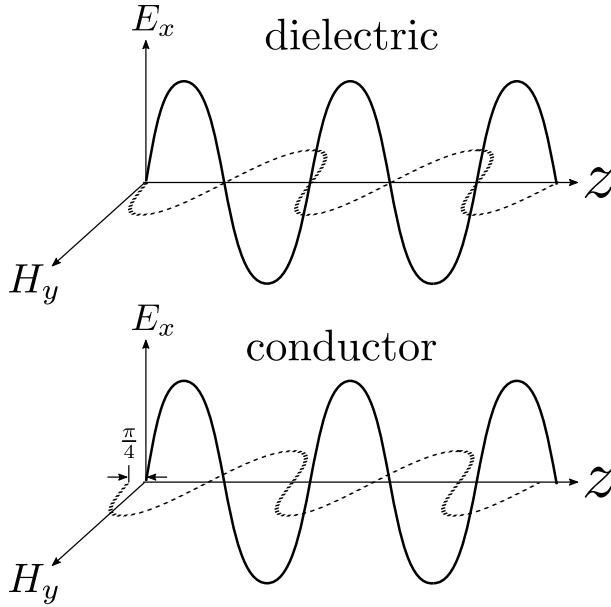


Figure 1: Illustration of the time-phase relationship in dielectric (same), and conductor (electric leads by $\frac{\pi}{4}$)