EE1 Impedances for ac circuits

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We start from first principles by dividing voltage by current to find the impedance of a resistor (Z_R), capacitor (Z_C) and inductor (Z_L) as

$$\begin{aligned} Z_R &= R, \\ Z_C &= \frac{1}{j\omega C}, \\ Z_L &= j\omega L. \end{aligned}$$

Resistor

We begin with a resistor so as to show the pattern of our analysis with a familiar and straightforward component. Given an ac voltage:

$$V_{ac} = A_0 e^{j\omega t} \tag{1}$$

where A_0 is the magnitude,[1] ω is the radian frequency, and t is time in seconds, then we can find the ac current I_{ac} using a rearranged form of Ohm's law

$$\begin{aligned} I_{ac} &= \frac{V_{ac}}{R} \qquad (2) \\ &= \frac{A_0 e^{j\omega t}}{R}. \qquad (3) \end{aligned}$$

And, fairly trivially, if we compare[2] the voltage and current waveforms we find they differ only by a factor of R as we would expect

$$\begin{aligned} \frac{V_{ac}}{I_{ac}} &= \frac{A_0 e^{j\omega t}}{\frac{A_0 e^{j\omega t}}{R}} \qquad (4) \\ &= R. \qquad (5) \end{aligned}$$

Since the impedance is defined as the ratio of voltage to current

$$Z = \frac{V_{ac}}{I_{ac}} \tag{6}$$

we can say that

$$\boxed{Z_R = R.} \tag{7}$$

Capacitor

We repeat the process for a capacitor, starting from our knowledge that the voltage V and charge Q in a capacitor are related by

$$Q = CV \tag{8}$$

[1] The magnitude of $\mathrm{Re}(A_0 e^{j\omega t}) = A_0 \cos \omega t$ is A_0, but the peak-to-peak difference is $A_0 - (-A_0) = 2A_0$

[2] we put voltage on the top line because the definition of the impedance Z of a component is

$$Z = \frac{V}{I}$$

where Q is the total charge in Coulombs, and C is the capacitance in Farads. The only time a current can flow in the terminal wires of a capacitor is when the total charge is changing[3], and since we know that current involves the transport of charge it is not surprising that the current at any instant is exactly qual to the rate of change of charge

$$\frac{dQ}{dt} = I. \tag{9}$$

We'd like to get I into Eq. **??**, so we take the time derivative of both sides of Eq. **??**

$$\frac{d}{dt}[Q] = \frac{d}{dt}[CV] \tag{10}$$

$$\frac{dQ}{dt} = C\frac{dV}{dt} \tag{11}$$

where C can go outside the derivative because C is constant with time for a standard capacitor - it's dimensions are usually mechanically defined and unchanging.[4] We substitute Eq. **??** into Eq. **??** to obtain a relationship between current and voltage

$$I = C\frac{dV}{dt}. \tag{12}$$

We substitute in our same voltage as before from Eq. **??** to get the current as

$$I_{ac} = C\frac{d}{dt}[V_{ac}] \tag{13}$$

$$= C\frac{d}{dt}\left[A_0 e^{j\omega t}\right] \tag{14}$$

$$= CA_0\frac{d}{dt}\left[e^{j\omega t}\right] \tag{15}$$

$$= j\omega C A_0 e^{j\omega t}. \tag{16}$$

The impedance Z_c of the capacitor is

$$Z_c = \frac{V_{ac}}{I_{ac}} \tag{17}$$

$$= \frac{A_0 e^{j\omega t}}{j\omega C A_0 e^{j\omega t}}, \tag{18}$$

thus giving us the result that

$$\boxed{Z_c = \frac{1}{j\omega C}.} \tag{19}$$

Inductor

We repeat the analysis again, starting from our knowledge that an inductor generates a voltage whenever the current changes[5]

[3] Q tells us the total charge of the capacitor, although we are not actually storing charges; instead we are pushing charges onto, or pulling them off, the plates such that one plate becomes positively charged to $+Q$ and the other plate becomes negatively charged to $-Q$. The energy is stored in the attractive force between the opposite charges on each plate, which stay separated because charges cannot hop the gap between the plates - unless the dielectric breaks down, which only happens at high voltages, e.g. $3\,\mathrm{V\,\mu m^{-1}}$ for air. As a general rule, we avoid analysing lightning in this course, so you can ignore that possibility.

[4] Technically, there *are* some capacitors which vary with time, such as pressure sensors that rely on using a flexible plate that deforms under load, but even for those, the rate of change of C with time is usually much slower than the rate of change of V with time, so you would still pull this same trick.

[5] the inductor does this in an ultimately doomed, but sometimes exciting, attempt to resist any change in its magnetic flux - remember to apply a flyback diode when using semiconductor switches on inductive loads in a real circuit because they can damage the switch.

$$V = L\frac{dI}{dt}. \tag{20}$$

Since we want to start from voltage and calculate current, to follow the same pattern as before, we must integrate

$$\int V dt = \int L \frac{dI}{dt} dt \tag{21}$$
$$= L \int \frac{dI}{dt} dt \tag{22}$$
$$= LI + K \tag{23}$$

where K is a constant of integration. We can let $K = 0$ because there is no dc offset[6] in our signal in this ac analysis at a single frequency ω, hence

$$I = \frac{1}{L} \int V dt. \tag{24}$$

. We substitute Eq. refeq:v into Eq. refeq:li and do the integration as follows

$$I_{ac} = \frac{1}{L} \int V_{ac} dt \tag{25}$$
$$= \frac{1}{L} \int A_0 e^{j\omega t} dt \tag{26}$$
$$= \frac{1}{j\omega L} A_0 e^{j\omega t} + K \tag{27}$$
$$= \frac{1}{j\omega L} A_0 e^{j\omega t} \tag{28}$$

where once again the constant of integration $K = 0$ because there is no dc component in our signal. Hence the impedance is

$$Z = \frac{V_{ac}}{I_{ac}} \tag{29}$$
$$= \frac{A_0 e^{j\omega t}}{\frac{1}{j\omega L} A_0 e^{j\omega t}} \tag{30}$$

which simplifies, by removing common factors, to

$$\boxed{Z_L = j\omega L.} \tag{31}$$

We have now derived the (complex) impedances of resistors, capacitors and inductors.

[6] If there was a DC offset in a circuit, then we would separate the problem into two parts, one part dealing with the dc offset (where $\omega = 0$), and the other part dealing with the ac signal at some single frequency ($\omega > 0$). If it were even more complicated, and there were multiple frequencies, we'd just deal with them one by one. If it started to get crazy, like a square wave, with lots of frequencies, then we'd probably using a frequency domain technique (such as Fourier analysis) to speed things up - but you needn't worry about that in EE1. Back to the task in hand - analysis at a single frequency - the frequency ω we defined in Eq. **??** *includes* zero frequency, so our analysis will be valid for dc, i.e. in the limit that $\omega \to 0$, and for *ac*, so we are totally legit to set $K = 0$ to find the general case for the impedance at a single frequency ω.

Impedance

As we have seen in the preceeding sections, the impedance Z can be purely real, for a resistor, or purely imaginary, for a capacitor or inductor. There are other cases where we might encounter mixed components that comprise both a resistive element, and some element of capacitance or inductance. Thus we tend to think of impedance being a complex number. In the general case, both the real and imaginary parts may be non-zero, i.e.

$$Z = R \pm jX. \tag{32}$$

where X is the reactance of the capacitance (X_C) or inductance (X_L).[7]

You'll note the $\pm$ symbol in Eq. **??**. Why do we have that? That's there because when we are talking about reactances on their own, we usually don't care about the phase information that is contained in the $\pm j$ so we just leave that bit out altogether, like

$$X_C = \frac{1}{\omega C} \tag{33}$$

$$X_L = \omega L. \tag{34}$$

If phase matters, use impedance (Z). If only magnitude matters, use reactance (X).[8] Note that the magnitude of the impedance $Z = R \pm jX$ is

$$|Z| = \sqrt{R^2 + X^2} \tag{35}$$

and that it has a phase angle of

$$\angle Z = \arctan \frac{\mathrm{Im}\{Z\}}{\mathrm{Re}\{Z\}} \tag{36}$$

such that

$$\angle V_{ac} = \angle Z + \angle I_{ac} \tag{37}$$

[7] since we are dealing with the general case, it's possible we have a combination of components. When that happens, it is more precise to talk about the inductance rather than the inductor - it might not be an inductor, but whatever it is, might have some inductance. Same goes for the capacitance. This is not an essential distinction to make, but you will find it transmits a better impression to experienced engineers if you master this subtlety in your communications.

[8] Nothing is ever really that simple in life, so let me add a qualifier - if you have more than one component in a circuit, then even just to calculate the overall magnitude of something in the circuit, you have to keep track of all the phases when you add things up. Beware! Many examples will follow in lectures and tutorial questions.

Phase relationship between voltage and current in components

Resistor

We can use Eq. **??** to help us interpret what the impedance is telling us about the phase relationship between V_{ac} and I_{ac}. For the resistor

$$\angle Z = \arctan \frac{0}{R} \tag{38}$$

$$= \arctan 0 \tag{39}$$

$$= 0 \tag{40}$$

therefore

$$\angle V_{ac} = \angle I_{ac} \tag{41}$$

or in other words, the voltage and current in a resistor are in phase.

Capacitor

For a capacitor the situation is a little more complicated. We have a j on the bottom line, so we need to get it onto the top line. We know[9] that $\frac{1}{j} = -j$ so

[9] $\frac{1}{j} \times \frac{j}{j} = \frac{j}{j^2} = \frac{j}{-1} = -j$

$$\begin{aligned} \angle Z &= \arctan \frac{\frac{-1}{wC}}{0} && (42) \\ &= \arctan \frac{-1}{0} && (43) \\ &= \arctan -\infty && (44) \\ &= -\frac{\pi}{2} && (45) \end{aligned}$$

therefore

$$\angle V_{ac} = -\frac{\pi}{2} + \angle I_{ac} \qquad (46)$$

or in other words, the current leads the voltage by $\frac{\pi}{2}$.

In a **C**apacitor, the current **I** leads the **V**oltage ...

Inductor

For an inductor

$$\begin{aligned} \angle Z &= \arctan \frac{wL}{0} && (47) \\ &= \arctan \frac{1}{0} && (48) \\ &= \arctan \infty && (49) \\ &= \frac{\pi}{2} && (50) \end{aligned}$$

therefore

$$\angle V_{ac} = +\frac{\pi}{2} + \angle I_{ac} \qquad (51)$$

or in other words, the voltage leads the current by $\frac{\pi}{2}$.

... the **V**oltage leads the current **I** in an inductor **L**

Mnemonic

Combining our two bits of knowledge about phase relationships in inductors and capacitors, we get the mnemonic

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