Optimal Online Discrepancy Minimization

STOC2024

Reading Group 2024/8/20

Setting: Online Vector Balancing

- vectors $v_1, \ldots, v_T \in R_n$
- decide the sign $x_i \in \{-1,1\}$
- keep the signed sum $\sum_{i=1}^{T} x_i v_i$ small in some norm

Related Work

- Online vector balancing
  - $\|v\|_\infty \leq 1$:
    - l_∞: $O(\sqrt{T \log n})$
    - l_2: $\Omega(\sqrt{(n-1)T})$

Related Work

- v_i samples from distribution p:
  - $p \in \{-1,1\}^n$: l_∞
    - $O(\sqrt{n})$
    - $O(\sqrt{n}\log T)$ prefix
  - $p \in [-1,1]^n$: l_∞
    - $O(n^2 \log nT) \rightarrow O(\sqrt{n}\log^4 nT) \rightarrow O_n(\sqrt{\log T})$

Related Work

- Oblivious adversary:
  - Edge orientation:
    - $O(\log T)$ w.h.p
  - Subgaussian norm:
    - $O(\sqrt{\log nT})$-subgaussian

Results

Theorem 1. *There is an online algorithm that against any oblivious adversary and for any sequence of vectors* $v_1, \ldots, v_T \in \mathbb{R}^n$ *with* $\|v_i\|_2 \le 1$, *arriving one at a time, decides random signs* $x_1, \ldots, x_T \in \{-1, 1\}$ *so that for every* $t \in [T]$, *the prefix sum* $\sum_{i=1}^{t} x_i v_i$ *is* 10*-subgaussian.*

Results

Theorem 2. *Given a symmetric convex body $K \subseteq \mathbb{R}^n$, there is an online algorithm that against any oblivious adversary and for any sequence of vectors $v_1, \ldots, v_T \in \mathbb{R}^n$ with $\|v_i\|_2 \leq 1$, arriving one at a time, decides random signs $x_1, \ldots, x_T \in \{-1, 1\}$ so that each of the following hold with probability at least $1/2$:*

(a) $\sum_{i=1}^{T} x_i v_i \in O(1) \cdot K$ *under the assumption* $\gamma_n(K) \geq \frac{1}{2}$.
(b) $\sum_{i=1}^{t} x_i v_i \in O(1) \cdot K$ *for all* $t \in [T]$ *under the assumption* $\gamma_n(K) \geq 1 - \frac{1}{2T}$.

Theorem 7 (Banaszczyk [Ban12]). *There is a constant $\alpha < 5$, so that for any $v_1, \ldots, v_T \in \mathbb{R}^n$ with $\|v_i\|_2 \leq 1$ for $i = 1, \ldots, T$ and any convex body $K \subseteq \mathbb{R}^n$ with $\gamma_n(K) \geq 1 - \frac{1}{2T}$, there are signs $x_1, \ldots, x_T \in \{-1, 1\}$ so that*

$$\sum_{i=1}^{t} x_i v_i \in \alpha K \quad \forall t = 1, \ldots, T.$$

Results

Corollary 3. *There is an online algorithm that against any oblivious adversary and for any sequence of vectors $v_1, \ldots, v_T \in \mathbb{R}^n$ with $\|v_i\|_2 \le 1$, arriving one at a time, decides random signs $x_1, \ldots, x_T \in \{-1, 1\}$ so that each of the following hold with probability at least $1 - \delta$ for any $\delta \in (0, \frac{1}{2}]$ and any $p \ge 2$:*

(a) $\|\sum_{i=1}^{T} x_i v_i\|_p \lesssim \sqrt{p}\min(n,T)^{1/p} + \sqrt{\log(1/\delta)}$;

(b) $\max_{t\in[T]} \|\sum_{i=1}^{t} x_i v_i\|_p \lesssim \sqrt{p}\min(n,T)^{1/p} + \sqrt{\log T} + \sqrt{\log(1/\delta)}$.

Furthermore,

(c) $\|\sum_{i=1}^{T} x_i v_i\|_\infty \lesssim \sqrt{\log\min(n,T)} + \sqrt{\log(1/\delta)}$;

(d) $\max_{t\in[T]} \|\sum_{i=1}^{t} x_i v_i\|_\infty \lesssim \sqrt{\log T} + \sqrt{\log(1/\delta)}$.

Results

Theorem 4. *For any $n \geq 2$, there is a strategy for an oblivious adversary that yields a sequence of unit vectors $v_1, \ldots, v_T \in \mathbb{R}^n$ so that for any online algorithm, with probability at least $1 - 2^{-T^{\Omega(1)}}$, one has $\max_{t \in [T]} \|\sum_{i=1}^t x_i v_i\|_\infty \gtrsim \sqrt{\log T}$.*

This improves upon the $\Omega\left(\sqrt{\frac{\log T}{\log\log T}}\right)$ lower bound of [BJSS19].

Results

Corollary 5 (Online edge orientation). *There exists an online algorithm that for any set of n vertices and any sequence of edges, arriving one at a time, decides orientations so that at every vertex, the absolute difference between indegree and outdegree always remains bounded by $O(\sqrt{\log T})$ with high probability.*

- Edge orientation:
  - $O(\log T)$ w.h.p

Main Techniques

A *convex body* is a set $K \subseteq \mathbb{R}^n$ that is convex, compact (closed and bounded) and full-dimensional. Let $B_2^n := \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$ be the *Euclidean ball* and let $S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$ be the *sphere*. A set $W \subseteq S^{n-1}$ is called an ε*-net* if for all $x \in S^{n-1}$, there is a $y \in W$ with $\|x - y\|_2 \leq \varepsilon$.

Lemma 8. *For any* $0 < \varepsilon \leq 1$, *there is an* ε*-net* $W \subseteq S^{n-1}$ *of size* $|W| \leq (\frac{3}{\varepsilon})^n$.

Proof. Pick any maximal set of points $W \subseteq S^{n-1}$ that have $\|\cdot\|_2$-distance at least ε to each other. Then W is an ε-net. Moreover the balls $x + \frac{\varepsilon}{2} B_2^n$ are disjoint for $x \in W$ and contained in $(1 + \frac{\varepsilon}{2}) B_2^n$. Hence

$$|W| \leq \frac{\mathrm{Vol}_n((1 + \frac{\varepsilon}{2}) \cdot B_2^n)}{\mathrm{Vol}_n(\frac{\varepsilon}{2} \cdot B_2^n)} = \left(\frac{1 + \frac{\varepsilon}{2}}{\frac{\varepsilon}{2}}\right)^n \leq \left(\frac{3}{\varepsilon}\right)^n. \qquad \square$$

Main Techniques

Theorem 15. *There exists a constant $\alpha < 5$ such that the following holds. Let $\mathcal{T} = (V, E)$ be a tree with a distinguished root $r \in V$ and $|E| \geq 1$, where each edge $e \in E$ is assigned a vector $v_e \in \mathbb{R}^n$ with $\|v_e\|_2 \leq 1$. Let $K \subseteq \mathbb{R}^n$ be a convex body with $\gamma_n(K) \geq 1 - \frac{1}{2|E|}$. Then there are signs $x \in \{-1,1\}^E$ so that*

$$\sum_{e \in P_i} x_e v_e \in \alpha K \quad \forall i \in V$$

where $P_i \subseteq E$ are the edges on the path from the root to i.

$$K_i := \Big(\bigcap_{j \in C_i} (K_j * \beta v_{\{i,j\}}) \Big) \cap K.$$

for any leaf $i \in V$ one simply has $K_i = K$.

Claim I. *For all $i \in V$ one has $\gamma_n(K_i) \geq 1 - \frac{|D_i|}{2|E|}$.*

Claim II. *There are signs $x \in \{-1,1\}^E$ so that $\sum_{e \in P_i} x_e v_e \in \frac{1}{\beta} K_i$ for all $i \in V \setminus \{r\}$.*

$$a \in K_i \overset{\text{Def } K_i}{\subseteq} K_j * \beta v_{\{i,j\}} \overset{\text{Thm } 6}{\subseteq} (K_j + \beta v_{\{i,j\}}) \cup (K_j - \beta v_{\{i,j\}})$$

Then we may pick a sign $x_{\{i,j\}} \in \{-1,1\}$ so that $a + \beta x_{\{i,j\}} v_{\{i,j\}} \in K_j$.

Main Techniques

$$K := \left\{ (y^{(1)}, \ldots, y^{(N)}) \in \mathbb{R}^{Nn} \mid \|Y\|_{\psi_{2,\infty}} \leq 2 + \delta \text{ where } Y \sim \{y^{(1)}, \ldots, y^{(N)}\} \right\}. \quad (1)$$

Intuitively, the vectors in K consist of N many blocks of dimension n with the property that a uniform random block generates a subgaussian random vector. Since $\|\cdot\|_{\psi_{2,\infty}}$ is a norm, K is a symmetric convex body. The main result for this section will be that K has a large Gaussian measure if N is large enough.

Proposition 16. *For any $\delta > 0$, there is a constant $C_\delta > 0$ so that for all $n, N \in \mathbb{N}$ one has $\gamma_{Nn}(K) \geq 1 - \frac{C_\delta^n}{N^{1+\delta}}$.*

Main Techniques

Theorem 19. *There exists a constant $\gamma < 10$ such that the following holds. Let $\mathcal{T} = (V, E)$ be a tree with a distinguished root, where each edge $e \in E$ is assigned a vector $v_e \in \mathbb{R}^n$ with $\|v_e\|_2 \leq 1$. Then there is a distribution $\mathcal{D}$ over $\{-1, 1\}^E$ so that for $x \sim \mathcal{D}$, $\sum_{e \in P_i} x_e v_e$ is γ-subgaussian for every $i \in V$ where $P_i \subseteq V$ are the edges on the path from the root to i.*

Main Techniques

Theorem 20. *For every $T \in \mathbb{N}$, there exists a randomized online algorithm which, upon receiving a vector $v_i \in \mathbb{R}^n$ with $\|v_i\|_2 \leq 1$ for each $i \in [T]$, outputs a random sign $x_i \in \{-1, 1\}$ so that the prefix sum $\sum_{j=1}^{i} x_j v_j$ is 10-subgaussian. The algorithm runs in time $\exp(T^{CnT})$ for some universal constant $C > 0$.*

Open Questions

Conjecture 1. *Does there exist an online algorithm that for any sequence of vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ with $\|v_i\|_\infty \le 1$, arriving one at a time, decides random signs $x_1, \ldots, x_n \in \{-1, 1\}$ so that $\|\sum_{i=1}^n x_i v_i\|_\infty \le O(\sqrt{n})$ with high probability?*

Conjecture 2. *Does there exist an online algorithm that for any sequence of vectors $v_1, \ldots, v_T \in \mathbb{R}^n$, each with two nonzero coordinates (one equal to 1 and the other -1) and arriving one at a time, decides random signs $x_1, \ldots, x_T \in \{-1, 1\}$ so that $\|\sum_{i=1}^t x_i v_i\|_\infty \le O(\sqrt[3]{\log T})$ for all $t \in [T]$ with high probability?*

Conjecture 3. *Does there exist a polynomial time online algorithm that against any oblivious adversary, for any sequence of vectors $v_1, \ldots, v_T \in \mathbb{R}^n$ with $\|v_i\|_2 \le 1$, decides random signs $x_1, \ldots, x_T \in \{-1, 1\}$ so that for every $t \in [T]$, the prefix sum $\sum_{i=1}^t x_i v_i$ is $O(1)$-subgaussian?*

Online Edge Coloring is (Nearly) as Easy as Offline

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Setting: Online Edge Coloring

- A graph is revealed piece by piece (either edge-by-edge or vertex-by-vertex)
- Algorithm: assign colors to edges upon their arrival irrevocably so that no two adjacent edges are assigned the same color.
- Objective: use few colors in any graph of maximum degree Δ

Related Work

- Offline: Δ or $\Delta + 1$
- Online:
  - $\Delta = O(\log n)$: $2\Delta - 1$ (greedy)
  - $\Delta = \omega(\log n)$: $(\frac{e}{e+1} + o(1))\Delta$

Conjecture 1.1 ([BNMN92]). *There exists an online edge-coloring algorithm for n-vertex graphs that colors the edges of the graph online using $(1 + o(1))\Delta$ colors, assuming known maximum degree $\Delta = \omega(\log n)$.*[1]

Knowledge of Δ is necessary: no algorithm can use fewer than $\frac{e}{e-1}\Delta \approx 1.582\Delta$ colors otherwise [CPW19].

Related Work

- Online Edge Coloring to Online Matching:
  - $\alpha\Delta \to \frac{1}{\alpha\Delta}$
- Online Matching to Online Edge Coloring:
  - $\frac{1}{\alpha\Delta} \to (\alpha + O((\frac{\log n}{\Delta})^{\frac{1}{4}}))\Delta$
  - $\Delta = \omega(\log n) \to (\alpha + o(1))\Delta$

Results

Theorem 1.2 (See exact bounds in Theorem 4.11). *There exists an online algorithm that, on n-vertex graphs with known maximum degree $\Delta = \omega(\log n)$, outputs a $(1 + o(1))\Delta$-edge-coloring with high probability.*

Via the aforementioned reduction, we obtain the above from our following key technical contribution.

Theorem 1.3. *There exists an online matching algorithm that on graphs with known maximum degree Δ, outputs a random matching M satisfying*

$$\Pr[e \in M] \geqslant \frac{1}{\Delta + \Theta(\Delta^{3/4} \log^{1/2} \Delta)} = \frac{1}{(1 + o(1)) \cdot \Delta} \qquad \forall e \in E.$$

Main Techniques

Algorithm 1 (NaturalMatchingAlgorithm).

When an edge $e_t = (u, v)$ *arrives, match it with probability*

$$P(e_t) \leftarrow \begin{cases} \frac{1}{\Delta+q} \cdot \frac{1}{\prod_{j=1}^{k}(1-P(e_{t_j}))} & \text{if } u \text{ and } v \text{ are still unmatched,} \\ 0 & \text{otherwise,} \end{cases}$$

where $e_{t_1}, \ldots, e_{t_k}$ *are those previously-arrived edges incident to the endpoints of* e_t.

Main Techniques

$$F_t(u) := \prod_{e_{t_j} \in \delta_t(u)} (1 - P(e_{t_j})) \quad \text{and} \quad F_t(v) := \prod_{e_{t_j} \in \delta_t(v)} (1 - P(e_{t_j})).$$

Algorithm 2 (MatchingAlgorithm).

Initialization: *Set* $F_1(v) \leftarrow 1$ *for every vertex* v *and* $M_1 \leftarrow \emptyset$.

At the arrival of edge $e_t = (u, v)$ at time t:

- *Sample* $X_t \sim Uni[0,1]$.
- *Define*

$$P(e_t) = \begin{cases} \frac{1}{\Delta+q} \cdot \frac{1}{F_t(u) \cdot F_t(v)} & \text{if } u \text{ and } v \text{ are unmatched in } M_t, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\hat{P}(e_t) = \begin{cases} P(e_t) & \text{if } \min\{F_t(u), F_t(v)\} \cdot (1 - P(e_t)) \geqslant q/(4\Delta) \\ 0 & \text{otherwise.} \end{cases}$$

- *Set*
  - $F_{t+1}(u) \leftarrow F_t(u) \cdot (1 - \hat{P}(e_t))$;
  - $F_{t+1}(v) \leftarrow F_t(v) \cdot (1 - \hat{P}(e_t))$;
  - $M_{t+1} \leftarrow \begin{cases} M_t \cup \{e_t\} & \text{if } X_t < \hat{P}(e_t), \\ M_t & \text{otherwise.} \end{cases}$

Main Techniques

$$q = \sqrt{200} \cdot \Delta^{3/4} \ln^{1/2} \Delta.$$

Lemma 4.2. *For any edge* $e_t = (u, v)$ *it holds that*

$$\Pr[X_t < P(e_t)] = \frac{1}{\Delta + q}.$$

Observation 4.4. *If* $e_t = (u, v)$ *and* $\min\{F_t(u),\ F_t(v)\} \geqslant q/(3\Delta)$, *then* $\hat{P}(e_t) = P(e_t)$.

Main Techniques

Lemma 4.5. *Let* $e_{t_1} = (u_1, v), \ldots, e_{t_\ell} = (u_\ell, v)$ *be the edges incident to* v*, arriving at times* $t_1 < \cdots < t_\ell$*. Let* $S := \{u_i \in N(v) \mid u_i \notin M_{t_i}\}$ *be those neighbors* u_i *that are not matched before time* t_i *when the edge* $e_{t_i} = (u_i, v)$ *arrives. Then,*

$$F(v) \geqslant 1 - \sum_{u_i \in S} \frac{1}{\Delta + q} \frac{1}{F_{t_i}(u_i)}.$$

As a consequence, $F(v) \geqslant q/(3\Delta)$ *holds if*

$$\sum_{u_i \in S} \frac{1}{\Delta + q} \frac{1}{F_{t_i}(u_i)} \leqslant \frac{\Delta}{\Delta + q/2}. \tag{3}$$

Main Techniques

$$S_t := \{u_i \in N(v) \mid u_i \notin M_{\min\{t,t_i\}}\} \quad \text{and} \quad Y_{t-1} := \sum_{u_i \in S_t} \frac{1}{\Delta + q} \frac{1}{F_{\min\{t,t_i\}}(u_i)}.$$

Lemma 4.6. *$Y_0, \ldots, Y_m$ form a martingale w.r.t. the random variables $X_1, \ldots, X_m$. Furthermore, the difference $Y_t - Y_{t-1}$ is given by the following two cases:*

- *If e_t is added to M_{t+1}, which happens with probability $\hat{P}(e_t)$, then:*

$$Y_t - Y_{t-1} = -\frac{1}{\Delta + q} \sum_{u_i \in S_t \cap e_t} \frac{1}{F_t(u_i)}. \tag{4}$$

- *If instead e_t is not added to M_{t+1}, which happens with probability $1 - \hat{P}(e_t)$, then:*

$$Y_t - Y_{t-1} = \frac{1}{\Delta + q} \cdot \frac{\hat{P}(e_t)}{1 - \hat{P}(e_t)} \sum_{u_i \in S_t \cap e_t} \frac{1}{F_t(u_i)}. \tag{5}$$

Lemma 4.9 (Observed Variance). *For the martingale Y_t described above, we have:*

$$W_m := \sum_{t=1}^{m} \mathbb{E}[(Y_t - Y_{t-1})^2 \mid X_1, \ldots, X_{t-1}] \leqslant \frac{128\Delta \ln \Delta}{q^2}.$$

Main Techniques

Lemma 4.9 (Observed Variance). *For the martingale Y_t described above, we have:*

$$W_m := \sum_{t=1}^{m} \mathbb{E}[(Y_t - Y_{t-1})^2 \mid X_1, \ldots, X_{t-1}] \leqslant \frac{128\Delta \ln \Delta}{q^2}.$$

Lemma 2.3 (Freedman's Inequality [Fre75]; see also [BDG16, Theorem 12], [HMRAR98, Theorem 3.15]). *Let $Y_0, \ldots, Y_m$ be a martingale with respect to the random variables $X_1, \ldots, X_m$. If $|Y_k - Y_{k-1}| \leqslant A$ for any $k \geqslant 1$ and $W_m \leqslant \sigma^2$ always, then for any real $\lambda \geqslant 0$:*

$$\Pr[|Y_n - Y_0| \geqslant \lambda] \leqslant 2\exp\left(-\frac{\lambda^2}{2(\sigma^2 + A\lambda/3)}\right).$$

Lemma 4.10. $\Pr[F(v) < q/(3\Delta)] \leqslant 2\Delta^{-3}$.

Proof. Let $\lambda := q/(3\Delta)$. By Fact 4.7 and Lemma 4.5, we have that

$$\Pr[F(v) < q/(3\Delta)] \leqslant \Pr\left[|Y - Y_0| \geqslant \frac{\Delta}{\Delta + q/2} - \frac{\Delta}{\Delta + q}\right] \leqslant \Pr[|Y - Y_0| \geqslant \lambda].$$