On specific log integrals, polylog integrals and alternating Euler sums

Ming Hao Zhao

2020.2.22

Abstract

The main purpose of this article is the evaluation of 85 specific logarithmic integrals, 89 alternating Euler sums and 263 polylogarithmic generalizations with their weights ≤ 5. By establishing linear relations between 3 kinds of values, we discover the common pattern on their closed-forms and present a systematic proof. Among these results, 7 weight 5 sums and over 200 polylog integrals are new. Based on previous results, we solved series of problems on related integrals and series which are also unknown in literatures.

Keywords. Logarithmic Integrals, Polylogarithmic Integrals, Euler Sums, Multiple Zeta Values.

0. Introduction

Recall that polylogarithm function is defined as $\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$ in the unit circle and its unique analytic continuation (via inversion formula) outside. For $s = 2, 3$ it is called dilogarithm and trilogarithm respectively (abbr. dilog and trilog). We also have the definition of Riemann Zeta function $\zeta(s) = \mathrm{Li}_s(1)$ for $\Re(s) > 1$ and its analytic continuation (via reflection formula) otherwise, and similarly Dirichlet Eta function $\eta(s) = -\mathrm{Li}_s(-1) = (1 - 2^{1-s})\zeta(s)$ for $\Re(s) > 0$. Based on this, we introduce some basic notations about natural logarithms, polylogarithms (abbr. log and polylog), generalized harmonic numbers and its alternating analogue:

$$f(0;x) = 1-x, f(1;x) = x, f(2;x) = x+1 \tag{0.1}$$

$$l(0;x) = \log(1-x), l(1;x) = \log(x), l(2;x) = \log(x+1) \tag{0.2}$$

$$L(n,1;x) = \mathrm{Li}_n(x), L(n,2;x) = \mathrm{Li}_n(-x), L(n,3;x) = \mathrm{Li}_n(\frac{1-x}{2}), n \geq 2 \tag{0.3a}$$

$$L(n,4;x)=\mathrm{Li}_n(\frac{1+x}{2}), L(n,5;x)=\mathrm{Li}_n(\frac{1-x}{1+x}), L(n,6;x)=\mathrm{Li}_n(\frac{x-1}{1+x}), n\geq 2 \tag{0.3b}$$

$$H_n^{(k)}=\sum_{j=1}^{n}\frac{1}{j^k}, \widetilde{H_n^{(k)}}=\sum_{j=1}^{n}\frac{(-1)^{j-1}}{j^k}, H_n=H_n^{(1)}, \widetilde{H_n}=\widetilde{H_n^{(1)}} \tag{0.4}$$

Now we define log, polylog integrals (abbr. LIs and PLIs) and Euler sums (abbr. ESs) in this article as follow:

$$LI(a(0),a(1),a(2);p)=\int_0^1 \frac{\prod_{m=0}^{2} l(m;x)^{a(m)}}{f(p;x)}\,dx \tag{0.5}$$

$$\begin{aligned} &PLI(a(0),a(1),a(2);b(2,1),\cdots,b(2,6);\cdots;b(N,1),\cdots,b(N,6);p)\\ &=\int_0^1 \frac{\prod_{m=0}^{2} l(m;x)^{a(m)}\prod_{k=1}^{6}\prod_{n=2}^{N} L(n,k;x)^{b(n,k)}}{f(p;x)}\,dx \end{aligned} \tag{0.6}$$

$$ES(a(1),\cdots,a(M),-b(1),\cdots,-b(N);\pm p)=\sum_{n=1}^{\infty}\frac{(\pm 1)^{n-1}\prod_{k=1}^{M} H_n^{(a(k))}\prod_{j=1}^{N}\widetilde{H_n^{(b(j))}}}{n^p} \tag{0.7}$$

Parameters in above expressions are natural numbers or 0. The notations of ESs come from Flajolet & Salvy [6] but differ a little, and we may assume the sequence $a(1),\cdots,a(M)$ and $b(1),\cdots,b(N)$ are non-decreasing since a permutation doesn't change the ES value. If their exists no $b(k)$, we call an ES non-alternating, otherwise alternating. We will also use same notations on multiple zeta values (abbr. MZVs) in their articles below and trivially extend it to multiple zeta star values (abbr. MZSVs) and alternating analogues. For instance, $\zeta(1,-2,3)=\sum_{l=1}^{\infty}\sum_{k=1}^{l-1}\sum_{j=1}^{k-1}\frac{(-1)^{k-1}}{jk^2l^3}$ and $\zeta^*(1,-2,3)=\sum_{l=1}^{\infty}\sum_{k=1}^{l}\sum_{j=1}^{k}\frac{(-1)^{k-1}}{jk^2l^3}$.

In later discussion, the number N and parameters $b(n,k)$ will be small, and many of them equal to 0. In this case, we omit those zero parameters and simply repeat the word 'nk' $b(n,k)$ times for abbreviation. For instance, $LI(1,2,3;0)=\int_0^1\frac{\log(1-x)\log^2(x)\log^3(x+1)}{1-x}\,dx$, $PLI(0,0,1;24,24;32;2)=\int_0^1\frac{\mathrm{Li}_2\left(\frac{x+1}{2}\right)^2\mathrm{Li}_3(-x)\log(x+1)}{x+1}\,dx$, and $ES(2,-3,-3;-4)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}H_n^{(2)}\left(\widetilde{H_n^{(3)}}\right)^2}{n^4}$.

We define the weight (denoted by W) of an LI/PLI/ES above as an integer $\sum_{m=0}^{2}a(m)+1$, $\sum_{k=1}^{6}\sum_{n=2}^{N}nb(n,k)+\sum_{m=0}^{2}a(m)+1$ and $\sum_{k=1}^{M}a(k)+\sum_{j=1}^{N}b(j)+p$ respectively. MZVs' weight is simply the sum of the absolute value of all indexes. Constants are also given weights; precisely, rational numbers have weight 0, $\pi,\log(2)$ have weight 1, $\zeta(n),\mathrm{Li}_n(\frac{1}{2})$ have weight n, and the weight given to product of these constants equals to sum of each components' weight. For example, the LI, PLI and ES mentioned above have weight 7, 9 and 12, $\zeta^*(2,-3,-1)$ has weight 6, while the constant

$\pi^2\mathrm{Li}_4\left(\frac{1}{2}\right)\zeta(3)\log(2)$ has weight 10. Moreover, the depth of an MZV is defined as the quantity of indexes; similarly we define the depth of an ES as $M+N+1$. By this definition $ES(2,-3,-3;-4)$ is weight 12 depth 4 and $\zeta^*(2,-3,-1)$ weight 6 depth 3.

A linear relation is an equality in form 'rational combination of LIs/PLIs/ESs= log/polylog constants'. If all items in LHS have the same weight, we call it a homogeneous relation; otherwise we call it a non-homogeneous relation. According to Au [1], by combining various methods that offer linear relations between LIs, we are able to express all LIs with weight W no more than 5, in rational combination of at most $t(W)$ weight W polylog constants, which are product of $\log(2), \pi, \zeta(m)$ and $\mathrm{Li}_n(\frac{1}{2})$ that conjectured to be linearly independent over $\mathbb{Q}$. Specifically, $t(1)=1, t(2)=2, t(3)=3, t(4)=5, t(5)=8$, and the set of these homogeneous constants generating LIs are known as Fibonacci basis, due to similarity between $t(n)$ and Fibonacci numbers F_n. For LIs with weight higher than 5, we need MZVs (or ESs) for closed-form expressions. In other words, some of the LIs are irreducible. The counting of $t(W)$ is much more complicated here, and $t(W)$ is not proved to have reached its possible minimum as in the case $W \leq 5$. See Au for details on higher weight LIs.

For need in this article, we denote A_W the weight W Fibonacci basis containing $t(W)$ full-simplified elements. By full-simplified we mean $\zeta(2n), n>0$ and $\mathrm{Li}_n\left(\frac{1}{2}\right), n<4$ won't appear, due to simplification formulas $\mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{\log^2(2)}{2}, \mathrm{Li}_3\left(\frac{1}{2}\right)=\frac{7\zeta(3)}{8}+\frac{\log^3(2)}{6}-\frac{1}{12}\pi^2\log(2)$ and $\zeta(2n)=\frac{2^{2n-1}\pi^{2n}|B_{2n}|}{(2n)!}$ [8, 9], where B_n are Bernoulli numbers (apparently these simplifications keep the weight unchanged). Thus, we may write down A_W explicitly for $W \leq 5$:

$$A_1=\{\log(2)\}, A_2=\left\{\pi^2, \log^2(2)\right\}, A_3=\left\{\pi^2\log(2), \log^3(2), \zeta(3)\right\}$$

$$A_4=\left\{\pi^4, \pi^2\log^2(2), \log^4(2), \zeta(3)\log(2), \mathrm{Li}_4\left(\frac{1}{2}\right)\right\}$$

$$A_5=\left\{\pi^4\log(2), \pi^2\log^3(2), \log^5(2), \pi^2\zeta(3), \zeta(3)\log^2(2), \zeta(5), \mathrm{Li}_4\left(\frac{1}{2}\right)\log(2), \mathrm{Li}_5\left(\frac{1}{2}\right)\right\}$$

For $W=2,3,4,5$, we abbreviate Au's remarkable result as: all weight W LIs can be generated by A_W over $\mathbb{Q}$. A possible weight 6 representation is:

$$\begin{gathered}A_6=\left\{\pi^6, \pi^4\log^2(2), \pi^2\log^4(2), \log^6(2), \pi^2\zeta(3)\log(2), \zeta(3)\log^3(2), \zeta(5)\log(2),\right.\\ \left.\zeta(3)^2, \mathrm{Li}_4\left(\frac{1}{2}\right)\log^2(2), \pi^2\mathrm{Li}_4\left(\frac{1}{2}\right), \mathrm{Li}_5\left(\frac{1}{2}\right)\log(2), \mathrm{Li}_6\left(\frac{1}{2}\right), S\right\}\end{gathered}$$

Where the constant $S=ES(1;-5)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}H_k}{k^5}$ is very likely to be irreducible. Due to purposes of this article, we omit higher weights' representations.

Now we state the main result of this article:

Theorem 1. All ESs and PLIs with weight $W \leq 5$ can be generated by A_W over $\mathbb{Q}$.

1. LI evaluation

First of all, we summarize known methods generating linear relations between LIs based on our discoveries and other references, a plenty of which can be modified to apply in PLIs later. By solving the linear system given by these methods, explicit closed-forms of LIs are deduced and tabulated in appendix 1. It's a little strange that few ES/PLI-related literature pay attention to the evaluation of LIs which is apparently powerful, thus we hold the view that a thorough summary is urgently needed.

Before the summary, we'd like to point out that all LIs with weight no more than 3 can be evaluated by brute force, that is, calculating their polylog primitives directly, which can be verified using CAS like Mathematica. This is due to the fact that $\int \log(a+x)\log(b+x)\log(c+x)\,dx$ and $\int \frac{\log(a+x)\log(b+x)}{c+x}\,dx$ are solvable via polylog identities. Some weight 4 integrals can be solved similarly if we apply formulas for $\int \frac{\log^2(a+x)\log(b+x)}{a+x}\,dx$, $\int \frac{\log(a+x)\log^2(b+x)}{a+x}\,dx$, etc, see Lewin [9] for more. Nevertheless, this method cannot be generalized to high weight cases, therefore we won't discuss about it until section 4.

1-1. General formulas

Proposition 1. The following formulas hold:

$$\int_0^1 x^m \log^n(x)\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}} \tag{1.1.1}$$

$$LI(0,0,n;2) = \frac{\log^{n+1}(2)}{n+1} \tag{1.1.2}$$

$$LI(0,n,0;0) = LI(n,0,0;1) = (-1)^n n!\zeta(n+1) \tag{1.1.3}$$

$$LI(0,n,0;2) = \left(1-2^{-n}\right)(-1)^n n!\zeta(n+1) \tag{1.1.4}$$

$$LI(n,0,0;2) = (-1)^n n!\mathrm{Li}_{n+1}\left(\frac{1}{2}\right) \tag{1.1.5}$$

$$LI(1,n,0;1) = LI(n,1,0,0) = (-1)^{n-1} n!\zeta(n+2) \tag{1.1.6}$$

$$LI(0,n,1;1)=\left(1-2^{-(n+1)}\right)(-1)^n n!\zeta(n+2) \tag{1.1.7}$$

$$LI(0,0,n;1)=-n!\sum_{k=0}^{n-1}\frac{\log^k(2)\mathrm{Li}_{-k+n+1}\left(\frac{1}{2}\right)}{k!}+n!\zeta(n+1)-\frac{n\log^{n+1}(2)}{n+1} \tag{1.1.8}$$

$$LI(0,1,n;2)=n!\sum_{k=0}^{n}\frac{\log^k(2)\mathrm{Li}_{-k+n+2}\left(\frac{1}{2}\right)}{k!}-n!\zeta(n+2)+\frac{\log^{n+2}(2)}{n+2} \tag{1.1.9}$$

$$LI(1,0,n;2)=n!\sum_{k=2}^{n+2}\frac{(-1)^{k-1}\zeta(k)\log^{-k+n+2}(2)}{(-k+n+2)!}+(-1)^n n!\mathrm{Li}_{n+2}\left(\frac{1}{2}\right)+\frac{\log^{n+2}(2)}{n+1} \tag{1.1.10}$$

$$LI(1,n,0;0)=LI(n,1,0;1)=\frac{1}{2}(-1)^{n-1}n!\left((n+1)\zeta(n+2)-\sum_{k=2}^{n}\zeta(k)\zeta(-k+n+2)\right) \tag{1.1.11}$$

$$\begin{aligned}LI(0,n,1;0)&=\frac{1}{2}(-1)^n n!\sum_{k=2}^{n}\left(1-2^{1-k}\right)\left(1-2^{k-n-1}\right)\zeta(k)\zeta(-k+n+2)\\&-(-1)^n n!\left(\frac{1}{2}(n+1)\zeta(n+2)-2\left(1-2^{-(n+1)}\right)\log(2)\zeta(n+1)\right)\end{aligned} \tag{1.1.12}$$

$$\begin{aligned}LI(0,2n-1,1;2)&=-(2n-1)!\sum_{k=1}^{n-1}\left(1-2^{1-2k}\right)\zeta(2k)\zeta(-2k+2n+1)\\&-(2n-1)!\left(2^{-(2n+1)}\zeta(2n+1)-(n-1)\left(1-2^{-2n}\right)\zeta(2n+1)\right)\end{aligned} \tag{1.1.13}$$

$$\begin{aligned}LI(1,2n-1,0;2)&=-(2n-1)!\sum_{k=1}^{n-1}\left(1-2^{2k-2n}\right)\zeta(2k)\zeta(-2k+2n+1)\\&+(2n-1)!\left(\left((n+1)\left(1-2^{-2n}\right)+2^{-(2n+1)}\right)\zeta(2n+1)-2\left(1-2^{-2n}\right)\log(2)\zeta(2n)\right)\end{aligned} \tag{1.1.14}$$

$$LI(2,n,0;1)=(-1)^n n!\left((n+2)\zeta(n+3)-\sum_{k=2}^{n+1}\zeta(k)\zeta(-k+n+3)\right) \tag{1.1.15}$$

$$\begin{aligned}LI(0,2n,2;1)&=2(2n)!\sum_{k=1}^{n}\left(1-2^{1-2k}\right)\zeta(2k)\zeta(-2k+2n+3)\\&-2(2n)!\left(\left(1-2^{-(2n+2)}\right)n\zeta(2n+3)-2^{-(2n+3)}\zeta(2n+3)\right)\end{aligned} \tag{1.1.16}$$

$$\begin{aligned}LI(1,2n,1;1)&=-(2n)!2^{-(2n+1)}\sum_{k=1}^{n}\left(1-2^{-2k+2n+2}\right)\zeta(2k)\zeta(-2k+2n+3)\\&+(2n)!\left((2n+3)2^{-(2n+3)}-1\right)\zeta(2n+3)\end{aligned} \tag{1.1.17}$$

Proof. Using $\Gamma(s)=\int_0^\infty e^{-x}x^{s-1}\,dx$ and $\Gamma(n+1)=n!$ one immediately comes to (1.1.1). (1.1.2) is elementary via Newton-Leibniz formula. Noticing the geometric series identity $\sum_{n=0}^\infty x^n=\frac{1}{1-x}$ holds on (0,1), exchange the order, invoke (1.1.1) and a reflection $t=1-x$ gives (1.1.3). Similarly, $\sum_{n=0}^\infty (-x)^n=\frac{1}{1+x}$ yields (1.1.4), where we've used $\eta(s)=\left(1-2^{1-s}\right)\zeta(s)$ for Eta. A reflection $t=1-x$ and scaling $\sum_{n=0}^\infty \left(\frac{x}{2}\right)^n=\frac{1}{1-\frac{x}{2}}$ yields (1.1.5). For (1.1.6) and (1.1.7), apply $\sum_{n=1}^\infty \frac{(\pm x)^n}{n}=-\log(1\mp x)$ and do the same as above. (1.1.8) is the celebrated Nielsen-Ramanujan integral, and can be shown via substitution $t=\frac{1}{1+x}$, and the formula $\int_{\frac{1}{2}}^1 x^m\log^n(x)\,dx=\frac{\partial^n}{\partial m^n}\frac{1-2^{-(m+1)}}{m+1}$, or directly calculating polylog primitives by induction. (1.1.9) is deduced easily from (1.1.8) an integration by parts, or calculating primitives as above. Note that this also works for (1.1.10). (1.1.11) together with (1.1.12) can be proved by using $\sum_{n=1}^\infty H_n(\pm x)^n=-\frac{\log(1\mp x)}{1\mp x}$, (1.1.1), Fubini theorem and Flajolet & Salvy's identities [6] for ESs. For (1.1.13)-(1.1.17) their methods in (1.1.12) (that is, expanding some log terms, exchange the order, and use general formulas for ESs) still work, although they didn't generalize their ES result to LIs which we proposed here. We refer readers to their article for more (the statement of their ES-theorems are in section 2-1 and corresponding series expansion results are tabulated in section 2-5). See also Valean [10, 11]. □

1-2. Integration by parts

By using Newton-Leibniz formula $\int_a^b f(x)\,dx=F(b)-F(a)$ (where $F'(x)=f(x)$) on differentiable function $F(x)=(l(2;x)-\log(2))^k\prod_{m=0}^2 l(m;x)^{a(m)}$, such that $|F(0^+)|,|F(1^-)|<\infty$ to obtain linear a linear relation between LIs. If $k=0$, the relation is homogeneous. Otherwise, we may have to apply integration by parts to reduce integrals containing $(l(2;x)-\log(2))$ term to ordinary LIs. As an example, let $k=0$ we have:

$$-aLI(a-1,b,c;0)+bLI(a,b-1,c;1)+cLI(a,b,c-1;2)=0 \tag{1.2.1}$$

Whenever $a,b,c>0$, and when some parameters are 0 the formula is simpler. Fix the weight W, it's natural to consider all multiple index $(k,a(0),a(1),a(2))$ such that $k+a(0)+a(1)+a(2)=W$ to obtain relations.

Now we explain why we introduce an extra index k. From another point of view, we pay attention to every single LI. Integrate by parts to lift up every LI's denominator to a log term and differentiate the numerator; this offer relations equivalent to the process above (in this article, by lifting up u we mean transforming the LHS to RHS in equality $\int u'v=uv-\int uv'$). If an LI contains $l(0;x)$ and no $l(1;x)$ in numerator, moreover the denominator is $1+x$, due to convergence issue we cannot lift up the denominator to $l(2;x)$ but $l(2;x)-\log(2)$ instead, which shows the necessity of k. What's more, a little analysis shows that we don't need to modify $l(0;x),l(1;x)$ similarly, so that F is in simplest form.

1-3. Fractional linear transformation

Consider $F(x)=\frac{\prod_{m=0}^{2} l(m;x)^{a(m)}}{f(p;x)}$ such that the corresponding LI is convergent, by a substitution $t=\frac{1-x}{x+1}$ it's direct that $\int_0^1 F(x)\,dx=\int_0^1 \frac{2F\left(\frac{1-x}{x+1}\right)}{(x+1)^2}\,dx$ holds. Fix the weight W, apply this substitution to all weight W LIs. For every LI, expanding $F\left(\frac{1-x}{x+1}\right)$ into log monomials using:

$$l\left(0;\frac{1-x}{x+1}\right)=l(1;x)-l(2;x)+\log(2),\, l\left(1;\frac{1-x}{x+1}\right)=l(0;x)-l(2;x),\, l\left(2;\frac{1-x}{x+1}\right)=\log(2)-l(2;x)$$

We may obtain a homo/non-homo linear relation. Note that only a part of them are linear independent and they're usually non-homogeneous, therefore it is essential to evaluate LIs with lower weights first for later use.

1-4. Beta derivatives

Proposition 2. The following formulas hold:

$$LI(n,m,0;1)=LI(m,n,0;0)=\lim_{\{a,b\}\to\{0,1\}}\frac{\partial^{m+n}B(a,b)}{\partial a^m\,\partial b^n} \tag{1.4.1}$$

$$\sum_{k=0}^{n}\binom{n}{k}LI(k,m,n-k;1)=2^{-(m+1)}\lim_{\{a,b\}\to\{0,1\}}\frac{\partial^{m+n}B(a,b)}{\partial a^m\,\partial b^n} \tag{1.4.2}$$

$$\sum_{k=0}^{n}\binom{n}{k}LI(k,m,n-k;0)=2^{-(m+1)}\left(\lim_{\{a,b\}\to\{1,0\}}\frac{\partial^{m+n}B(a,b)}{\partial a^m\,\partial b^n}+\lim_{\{a,b\}\to\left\{\frac{1}{2},0\right\}}\frac{\partial^{m+n}B(a,b)}{\partial a^m\,\partial b^n}\right) \tag{1.4.3}$$

$$\sum_{k=0}^{n}\binom{n}{k}LI(k,m,n-k;2)=2^{-(m+1)}\left(\lim_{\{a,b\}\to\left\{\frac{1}{2},0\right\}}\frac{\partial^{m+n}B(a,b)}{\partial a^m\,\partial b^n}-\lim_{\{a,b\}\to\{1,0\}}\frac{\partial^{m+n}B(a,b)}{\partial a^m\,\partial b^n}\right) \tag{1.4.4}$$

$$LI(0,m,1;0)=\left(2^{-(m+1)}-1\right)\lim_{\{a,b\}\to\{1,0\}}\frac{\partial^{m+1}B(a,b)}{\partial a^m\,\partial b^1}+2^{-(m+1)}\lim_{\{a,b\}\to\left\{\frac{1}{2},0\right\}}\frac{\partial^{m+1}B(a,b)}{\partial a^m\,\partial b^1} \tag{1.4.5}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}LI(k,n-k,0;2)=\lim_{s\to1}\frac{\partial^n}{\partial s^n}\frac{\left(2^{1-s}-1\right)\pi}{\sin(\pi s)} \tag{1.4.6}$$

$$\sum_{j=0}^{n}\sum_{k=0}^{n+1}\binom{n}{j}\binom{n+1}{k}(-\log(2))^{j+k}LI(-k+n+1,0,n-j;2)=\frac{1}{2}\lim_{\{x,y\}\to\{0,1\}}\frac{\partial^{2n+1}B(a,b)}{\partial a^n\,\partial b^{n+1}}-\frac{\log^{2n+2}(2)}{2n+2} \tag{1.4.7}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}LI(0,n-k,k;2)=(-1)^n\sum_{l=0}^{n}l!\binom{n}{l}\mathrm{Li}_{l+1}\left(\frac{1}{2}\right)\log^{n-l}(2) \tag{1.4.8}$$

Proof. By differentiating Beta function we have: $\int_0^1 x^{a-1}(1-x)^{b-1}\log^m(x)\log^n(1-x)\,dx=\frac{\partial^{m+n}B(a,b)}{\partial a^m\,\partial b^n}$. Now plug in $a=0,b=1$ and apply reflection $t=1-x$, we arrive at (1.4.1). Moreover, consider $\int_0^1\frac{\log^m(x)\log^n\left(1-x^2\right)}{x}\,dx$, $\int_0^1\frac{\log^m(x)\log^n\left(1-x^2\right)}{1-x}\,dx$, $\int_0^1\frac{\log^m(x)\log^n\left(1-x^2\right)}{1+x}\,dx$. Noticing $\frac{1}{1-x}=\frac{x+1}{1-x^2},\frac{1}{x+1}=\frac{1-x}{1-x^2}$ and the substitution $t=x^2$, we may transform these integrals into RHSs of next three equalities with the help of (1.4.1). On the other hand, by using $\log(1-x^2)=\log(1-x)+\log(1+x)$ and applying the binomial theorem on $\log^n(1-x^2)$, they are also equal to corresponding LHSs, and this yields (1.4.2)-(1.4.4). Furthermore, start with $\frac{\log(x+1)}{1-x}=\frac{x\log\left(1-x^2\right)}{1-x^2}+\frac{\log\left(1-x^2\right)}{1-x^2}-\frac{\log(1-x)}{1-x}$, we multiply both sides of this equality with $\log^m(x)$ and integrate it on interval $(0,1)$. This together with substitution $t=x^2$ gives (1.4.5). For the next one consider $F(x)=\frac{\log^n\left(\frac{x}{1-x}\right)}{x+1}$. By substitution $t=\frac{x}{1-x}$ we have $\int_0^1 F(x)\,dx=\int_0^\infty\frac{\log^n(x)}{(x+1)(2x+1)}\,dx$. Then, using the another well-known formula $\int_0^\infty\frac{x^{s-1}}{x+1}\,dx=\frac{\pi}{\sin(\pi s)}$ (which comes from another Beta representation $\int_0^\infty\frac{x^{a-1}}{(x+1)^{a+b}}\,dx=B(a,b)$), the fact that $\frac{1}{(x+1)(2x+1)}=\frac{2}{2x+1}-\frac{1}{x+1}$ and scaling $t=2x$, we have $\int_0^\infty\frac{x^{s-1}}{(x+1)(2x+1)}\,dx=\frac{\pi\left(2^{1-s}-1\right)}{\sin(\pi s)}$. Differentiating w.r.t s and setting $s=1$ gives the RHS of (1.4.6), and binomial theorem on $(\log(x)-\log(1-x))^n$ leads us to its LHS. For (1.4.7) we consider integral $\int_0^{\frac{1}{2}}\frac{\log^n(1-x)\log^{n+1}(x)}{1-x}\,dx$. Integration by parts (lift up a $\frac{\log^{n+1}(1-x)}{n+1}$), a reflection $t=1-x$ and Beta's definition yields the RHS. On the other hand, substitute $x=\frac{1-u}{2}$ and use binomial theorem twice yields the corresponding LHS. Finally, (1.4.8) has no direct relation with Beta derivatives, but its solution is similar to (1.4.6). If we substitute $t=\frac{x}{x+1}$ in $F_1(x)=\frac{\log^n\left(\frac{x}{x+1}\right)}{x+1}$, the integral we are about to calculate is nothing but $\int_0^{\frac{1}{2}}\frac{\log^n(x)}{1-x}\,dx$, which is immediately solved using the same technique in N-R formula (1.1.8). These two fractional substitutions are viewed as complements of the classic one in the section above, and the main idea here is that a LI with numerator $\log^n(t)$ always enjoy polylog primitive (which will be used in section 4). □

Now we explain how Beta derivatives are calculated analytically. Recall the definition of polygamma $\psi^{(n)}(x)=\frac{\partial^{n+1}\log(\Gamma(x))}{\partial x^{n+1}}$ and that $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, it's not difficult to see that all order N mixed partial derivative of Beta function are expressible as:

$$B(a,b)R(\psi^{(0)}(a),\psi^{(0)}(b),\psi^{(0)}(a+b),...,\psi^{(N-1)}(a),\psi^{(N-1)}(b),\psi^{(N-1)}(a+b))$$

Where $R(...)$ is a polynomial with $3N$ variables. Now, suppose a will tend to 0 and b won't, since all formulas above are of this form. Plug in b's value directly to get a function $g(a)$, then expand all polygamma terms of g in Laurent series w.r.t a at the origin. By calculating the asymptotic expansion, we will see that origin is a movable singularity of g, hence the double limit, i.e. $g(0)$ is uniquely determined. Furthermore, based on $\psi^{(n)}(z)=(-1)^{n-1}n!\zeta(n+1,z)$ and basic properties of

Hurwitz Zeta $\zeta(s,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s}$, we have $\psi^{(n)}(k+1) = (-1)^{n-1} n! \left(\zeta(n+1) - H_k^{(n+1)}\right)$. Due to this fact, all coefficients of Taylor expansion of $\psi^{(n)}(1+a)$ at origin is expressible in $\pi, \log(2)$ and zeta values modulo $\mathbb{Q}$, for arbitrary integer n. Moreover, the case $\psi^{(n)}(\frac{1}{2}+a)$ is similar if we make use of the relation between Zeta and Eta function cleverly (left to the readers), whose definitions are recalled at the beginning of the article. Now, all values at integers or half-integers of Beta derivatives can be generated by the Fibonacci basis over $\mathbb{Q}$, hence also all limits of mixed derivatives in the proposition above. Because of this, we are able to conclude that formulas in the above proposition offers linear relations between LIs. This procedure is easily carried out with the help of CAS. Fix the weight W, for every formula, consider all possibilities of multi-indexes that the maximum weight of LIs appear do not exceed W to obtain relations.

1-5. Contour integration

Consider the function defined on complex plane, in which all logarithms are in their principal branches (i.e. the absolute value of the argument do not exceed π):

$$F(z) = \frac{l(0;z)^k l(1;z)^{m_1} l(1;-z)^{m_2} l(2;z)^n}{z(1 \pm z)}$$

We have ten types of contour to get linear relations between LIs. The most important one is the large semicircle contour on the upper half plane (with 3 small semicircle indents at singularities $0, 1, -1$). Integral on large semicircle trivially vanishes as the radius tends to infinity due to the quadratic denominator of F. Moreover, if $k+n>0$ and $m_1>0$ (resp. $m_2>0$) for the case the denominator is $z(1-z)$ (resp. $z(1+z)$), all 3 log singularities are mild enough, that integrals along small semicircle also vanishes as the radius tends to 0 with rate $O(r^p \log(r))$. Therefore, by Cauchy's theorem we can say that $\int_{-\infty}^{\infty} F(x)\, dx = 0$ where the contour is actually the upper side of the real axis and the argument jumps are clear. Now, separate the axis into 4 parts by 3 singularities, by using substitution $t=-x$ and $t=\frac{1}{x}$, we can map the other 3 parts bijectively into $(0,1)$. Therefore after simplification we arrive at a formula $\int_0^1 (A(x)+B(x)-C(x)-D(x))\, dx = 0$, where:

$$A(x) = \frac{l(0;x)^k l(1;x)^{m_1} (l(1;x)-i\pi)^{m_2} l(2;x)^n}{x(1 \pm x)}$$

$$B(x) = (-1)^{m_1+m_2} \frac{(l(0;x)-l(1;x)-i\pi)^k l(1;x)^{m_1} (l(1;x)+i\pi)^{m_2} (l(2;x)-l(1;x))^n}{(x \pm 1)}$$

$$C(x) = \frac{l(2;x)^k l(1;x)^{m_2} (l(1;x)+i\pi)^{m_1} l(0;x)^n}{x(1 \mp x)}$$

$$D(x) = (-1)^{m_1+m_2} \frac{(l(0;x)-l(1;x)-i\pi)^n l(1;x)^{m_2} (l(1;x)-i\pi)^{m_1} (l(2;x)-l(1;x))^k}{(x \mp 1)}$$

Expand this formula just like in method 3, decompose the denominator into partial fractions and take real/imaginary part, we obtain 2 linear relations. They are non-homogeneous and not necessarily of same weight.

Now consider the chain $-\infty < -1 < 0 < 1 < \infty$ and all ordered pair (a, b) where $a < b$ are 2 of the 5 elements. Excluding the case $(-\infty, \infty)$ (which is in fact the case above), we have 9 pairs in total. Each pair defines a unique keyhole contour that's composed of two large semicircles on upper/lower plane and two keyhole encircling the interval $(-\infty, a)$ and (b, ∞) with possible indents at singularities, and if one of a, b is not finite the corresponding contour vanishes. What's more, by choosing the interval of arguments of $l(0; z), l(1; \pm z), l(2; z)$ as $(0, 2\pi)$ or $(-\pi, \pi)$ respectively, we can choose the direction of their branch cuts so that all cuts lies in $X = (-\infty, a) \cup (b, \infty)$, therefore F as product of above logarithms is meromorphic inside the contour (if both 2 directions are invalid, we simply restrict the degree of this term to be 0, for instance, if $(a, b) = (-\infty, 0)$ no $l(2; z) = \log(1 + z)$ term should exist). Now, we restrict some of the parameters k, m_1, m_2, n to be non-zero (dependent on the pair chosen), in order to ensure singularities of F that belong to X is movable. Now apply residue theorem to get an integral equality on the real line, and the remaining steps are exactly the same as the upper semicircle case and we'll omit the details. Using all kinds of these contours with all possible multiple index (k, m_1, m_2, n) in F, we are able to obtain much relations as the weight become higher.

1-6. Double integration

According to Au [1], by considering two ways of evaluating $\int_0^1 \int_0^1 \frac{\log^{2n}\left(\frac{1-x}{1-y}\right)}{(x+1)(y+1)} dxdy$ (one using binomial theorem and (1.1.5), another using substitution $u = x, v = \frac{1-x}{1-y}$), it can be shown that (the original version is implicit and we modified it here):

$$LI(2n, 0, 1; 1) = (2n)! \left(\sum_{k=0}^{n-1} (-1)^k \mathrm{Li}_{k+1}\left(\frac{1}{2}\right) \mathrm{Li}_{-k+2n+1}\left(\frac{1}{2}\right) + \frac{1}{2}(-1)^n \mathrm{Li}_{n+1}\left(\frac{1}{2}\right)^2 + \mathrm{Li}_{2n+2}\left(\frac{1}{2}\right) \right) \tag{1.6.1}$$

Another formula from Au that offers one relation between LIs arises from evaluating the integral $\int_0^1 \int_0^1 \frac{\log(1-x)\log^{2n}\left(\frac{1-y}{1+x}\right)}{(x+1)(y+1)} dxdy$. It can't be rewritten into a explicit general formula and will not be used later, thus we omitted it here. However, some similar operations will be considered in section 4.

1-7. Hypergeometric identities

It's also worthy to mention that Au [1] derived an ingenious method that resembles method 4. Using an classical hypergeometric identity, the Beta integral representation of hypergeometric function and differentiation of parameters, we may obtain further relations between high weight LIs. Same as the reason above, we omit it and refer the readers to Au's original paper.

1-8. Series expansion/Iterated integral

Expanding some part of the LI integrand (for instance $\frac{\log(1-x)}{x+1}, \log(1-x)\log(x+1)$, see section 2-5 for more) into several power series, and using identity (1.1.1) or more complicated results on $\int_0^1 x^m \log^n(1-x)\,dx, \int_0^1 x^m \log^k(x)\log^n(1-x)\,dx$, we may convert an LI into single/multiple alternating sums. Some of them can be evaluated successfully via ES or MZV identities. Another thought is to write the LI into iterated integral which resembles the canonical form of MZV iterated integral representation (see Zagier [16]), then use various shuffle relations and MZV identities to obtain a closed-form. However, we won't display much about evaluating LIs via ESs or MZVs according to the route of this article (other than those in section 1-1), that we use LI results as a weapon that assist us to carry out the mutual transformation between ESs and PLIs, instead of deducing LI values based on more complicated results. Recording this method is merely for the sake of completeness of LI reduction; on the contrary, the method of series expansion is an important step on establishing PLI relations in section 3.

1-9. Obtaining closed-forms for LIs

For each weight W, a simple counting shows that there are exactly $\frac{1}{2}\left(3W^2+W-2\right)$ convergent LIs. Thus, the total number of LIs with weight $W \leq 5$ (i.e. low weight LIs) is $85(= 1+6+14+25+39)$. Now we restate Au's result [1] here:

Lemma 1. All LIs with weight $W \leq 5$ can be generated by A_W over $\mathbb{Q}$.

Proof. All linear relations deduced from methods in section 1-1 to 1-7 (in fact 1-1 to 1-6 is enough), satisfying that the maximum weight in the relation does not exceed 5, yields a system containing 85 linear independent equations of 85 variables (that is, the exact value of 85 low weight LIs). Solving this system completes the proof. □

We don't claim much originality on this result but regard it as a verification of Au instead. In fact, section 1-1 to 1-5, the former part of 1-6 as well as whole 1-8 are independently developed, but the latter part of section 1-6 and whole 1-7 are new to us. All 85 low weight LIs are tabulated in appendix 1.

2. ES evaluation

Comparing to LIs, ESs receive much more attention in literatures. Therefore, we won't summarize thoroughly like in previous section, but simply review some typical methods and refer readers to corresponding references for others.

2-1. General formulas

All classical general formulas for ESs are recorded in Flajolet & Salvy [6]. We summarize their statements below:

Lemma 2. $ES(\pm1;p), ES(\pm1;-p)$ (p even), $ES(p;p), ES(p;q)$ ($p+q$ odd or smaller than 7), $ES(1,1;p)$ (p odd or smaller than 5) are reducible to Zeta values (directly or via Dirichlet Eta function).

See their article for proofs using contour integration and Borwein's alternative solutions [2,3]. Note that these formulas together with those in section 2-5 are applied in proofs of (1.1.13)-(1.1.17).

2-2. Symmetry

To some extent, this is an ES-analogue of shuffle relations of MZVs. Rewrite an ES into restricted multiple sums, where the restriction is that one index is always the biggest, for example $\mathrm{ES}(2,3;4) = \sum_{0<j,k\leq l} \frac{1}{j^2k^3l^4} = \sum_{l-max} \frac{1}{j^2k^3l^4}$. By inclusion-exclusion (abbr. I-E) principle, we have:

$$\sum_{0<j,k,l} = \sum_{0<j,k\leq l} + \sum_{0<j,l\leq k} + \sum_{0<k,l\leq j} - \sum_{0<j\leq k=l} - \sum_{0<k\leq j=l} - \sum_{0<l\leq j=k} + \sum_{0<j=k=l}$$

Apply this identity as an operator, on all triple summands $\frac{(\pm1)^{j-1}(\pm1)^{k-1}(\pm1)^{l-1}}{j^ak^bl^c}$ to obtain linear relations between ESs. Notice that when 2 indexes are equal the triple summand reduce to double and similarly single when 3 indexes meets together. If we set all 3 '$\pm$' to be '+' and assume $a,b,c>1$, we obtain:

$$\begin{aligned}\zeta(a)\zeta(b)\zeta(c) = ES(a,b;c) + ES(a,c;b) + ES(b,c;a) - ES(a;b+c)\\ -ES(b;a+c) - ES(c;a+b) + ES(a+b+c)\end{aligned} \tag{2.1.1}$$

Formulas of many other choices of 3 $'\pm'$ and a,b,c are similar and we omit the details (for alternating summands some of a,b,c can be 1). This method can be extended to n-ple sums (i.e. weight n ESs) easily, namely $n=2$ yields symmetry formulas of ESs immediately, and applications of higher depth version can be found in Xu [14,15]. Also see Borwein [5] for similar contents on weight 3 MZVs.

2-3. Partial fraction decomposition

Consider Cauchy product of the double sum $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j^ak^b} = \zeta(a)\zeta(b)$. By partial fraction decomposition we know that all sums of the form $\sum_{k=1}^{n-1}\frac{1}{k^a(n-k)^b}$ are expressible in a combination of homogeneous $\frac{H_n^{(k)}}{n^{a+b-k}}, k = 1, ..., a+b-1$, we found that $\zeta(a)\zeta(b)$ is expressed in a corresponding combination of ESs, which is a linear relation. The consideration on $\frac{(\pm 1)^{j-1}(\pm 1)^{k-1}}{j^a k^b}$ gives similar results, but extension to weight 3 is more subtle. In fact in this case we need MZV again, see Borwein [5] for details.

2-4. Contour integration

Flajolet & Salvy [6] used residue theorem on large circular contour and specific functions to obtain more independent relations for ESs. These functions are of the form FG, where $F = \frac{1}{z^p}$ and G is a product of cotangent(resp. cosecant, depending on whether ESs in the relation alternating) and polygamma (resp. poly-Nielsen Beta). Please see their article for further reference.

2-5 Obtaining closed-forms for ESs

Similar to LI case, we simply state that there are 2,9,24,54 different ESs for weight 2,3,4,5(the simple depth 1 case $\sum_{n=1}^{\infty} \frac{(\pm 1)^{n-1}}{n^p}$ excluded), thus there are 89 (non-alter/alter) ESs with weight $W \leq 5$ in total. While the total number of weight W non-alternating ESs is apparently $\sum_{k=1}^{W-2} p(k)$ ($p(k)$ partition numbers), there doesn't seem to be a general formula for alternating case.

Xu [14,15] had applied various methods (some of them listed above) that generate linear relations to evaluate these ESs. As a result, he remarkably gave explicit closed-forms to all but 1 weight 3 ES, 2 weight 4 ESs and 7 weight 5 ESs. Based on tools above together with what he applied, we verified much of his results successfully. As to solve the remaining problems, we succeeded at weight 3, 4 but partially failed at weight 5. After a considerable time of trying, by the method of ES-PLI mutual transformation, we successfully obtained values of seven remaining high-depth weight 5 ESs, namely:

$$ES(-1,-1,-1;-2), ES(-1,-1,-1,-1;-1), ES(1,1,-2;-1), ES(1,1,-1,-1;-1)$$

$$ES(1,-1,-1;-2), ES(1,-1,-2;-1), ES(1,-1,-1,-1;-1)$$

These 7 evaluations are not found in known literatures on ESs (especially, independent of MZV theory), therefore we consider them to be new. Now we will sketch the route of proving them. All LIs deduced from section 1, low weight ESs together with 49 weight 5 ESs recorded in [6, 14, 15], are supposed to be known and will be used in the proof. Also, several 2-Euler Sum (that is, adding a $\frac{1}{2^n}$ in the summand of an non-alternating ES, e.g. $\sum_{n=1}^{\infty} \frac{H_n}{n^3 2^n}$) evaluations derived by Xu [13] will be applied in later calculations either. We abbreviate 2-Euler Sums as 2-ESs.

Proposition 3. The following formulas hold where $|x| < 1$:

$$\frac{1}{1\mp x}=\sum_{n=0}^{\infty}(\pm x)^n\,,-\log(1\mp x)=\sum_{n=1}^{\infty}\frac{(\pm x)^n}{n} \tag{2.5.1}$$

$$-\frac{\log(1\mp x)}{1\mp x}=\sum_{n=1}^{\infty}H_n\,(\pm x)^n\,,\frac{\log(1\pm x)}{1\mp x}=\sum_{n=1}^{\infty}\widetilde{H}_n\,(\pm x)^n \tag{2.5.2}$$

$$\log^2(1\pm x)=\sum_{n=1}^{\infty}2(-1)^{n-1}\left(\frac{1}{n^2}-\frac{H_n}{n}\right)(\pm x)^n \tag{2.5.3}$$

$$\log(1-x)\log(x+1)=-\sum_{n=1}^{\infty}\left(\frac{\widetilde{H_{2n}}}{n}+\frac{1}{2n^2}\right)x^{2n} \tag{2.5.4}$$

$$\sum_{n=1}^{\infty}\frac{(\pm x)^n}{n^k}=\mathrm{Li}_k(\pm x),\sum_{n=1}^{\infty}H_n^{(k)}\,(\pm x)^n=\frac{\mathrm{Li}_k(\pm x)}{1\mp x},\sum_{n=1}^{\infty}\widetilde{H_n^{(k)}}\,(\pm x)^n=-\frac{\mathrm{Li}_k(\mp x)}{1\mp x} \tag{2.5.5}$$

$$\sum_{n=1}^{\infty}\frac{H_nx^n}{n}=\mathrm{Li}_2(x)+\frac{1}{2}\log^2(1-x) \tag{2.5.6}$$

$$\sum_{n=1}^{\infty}\frac{\widetilde{H}_nx^n}{n}=\mathrm{Li}_2\left(\frac{1-x}{2}\right)-\mathrm{Li}_2(-x)-\mathrm{Li}_2\left(\frac{1}{2}\right)-\log(2)\log(1-x) \tag{2.5.7}$$

$$\sum_{n=1}^{\infty}(H_n)^2x^n=\frac{\mathrm{Li}_2(x)}{1-x}+\frac{\log^2(1-x)}{1-x} \tag{2.5.8}$$

$$\sum_{n=1}^{\infty}\left(\widetilde{H}_n\right)^2x^n=\frac{2\left(\mathrm{Li}_2\left(\frac{1-x}{2}\right)-\mathrm{Li}_2\left(\frac{1}{2}\right)+\log\left(\frac{x+1}{2}\right)\log(1-x)\right)+\mathrm{Li}_2(x)}{1-x} \tag{2.5.9}$$

$$\sum_{n=1}^{\infty}H_n\widetilde{H}_nx^n=\frac{\mathrm{Li}_2\left(\frac{1-x}{2}\right)-\mathrm{Li}_2(-x)-\mathrm{Li}_2\left(\frac{1}{2}\right)-\frac{1}{2}\log^2(x+1)-\log(2)\log(1-x)}{1-x} \tag{2.5.10}$$

$$\sum_{n=1}^{\infty}H_nH_n^{(2)}x^n=\frac{\mathrm{Li}_3(1-x)+\mathrm{Li}_3(x)+\frac{1}{2}\log(x)\log^2(1-x)-\frac{1}{6}\pi^2\log(1-x)-\zeta(3)}{1-x} \tag{2.5.11}$$

$$\begin{aligned}\sum_{n=1}^{\infty}(H_n)^3x^n=\frac{1}{1-x}&\Big(3\mathrm{Li}_3(1-x)+\mathrm{Li}_3(x)-\log^3(1-x)\\&+\frac{3}{2}\log(x)\log^2(1-x)-\frac{1}{2}\pi^2\log(1-x)-3\zeta(3)\Big)\end{aligned} \tag{2.5.12}$$

$$\sum_{n=1}^{\infty}\frac{(H_n)^2x^n}{n}=\mathrm{Li}_3(x)-\mathrm{Li}_2(x)\log(1-x)-\frac{1}{3}\log^3(1-x) \tag{2.5.13}$$

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}x^n}{n}=2\mathrm{Li}_3(1-x)+\mathrm{Li}_3(x)-2\mathrm{Li}_2(1-x)\log(1-x)$$
$$-\mathrm{Li}_2(x)\log(1-x)-\log(x)\log^2(1-x)-2\zeta(3) \tag{2.5.14}$$

$$\sum_{n=1}^{\infty}\frac{H_n x^n}{n^2}=-\mathrm{Li}_3(1-x)+\mathrm{Li}_3(x)+\mathrm{Li}_2(1-x)\log(1-x)+\frac{1}{2}\log(x)\log^2(1-x)+\zeta(3) \tag{2.5.15}$$

Proof. (2.5.1) is elementary. By considering Cauchy products of formulas in (2.5.1) and use a little partial fraction decomposition, we easily come to (2.5.2)-(2.5.4). Note that (2.5.5) follows similarly using the definition of polylogarithms. By applying the operator $L(f)=\frac{\int_0^x f(t)\,dt}{x}$ on specific equalities in (2.5.5), we readily deduce (2.5.6) and (2.5.7). For the next one, we need the technique of calculating differences. Denote LHS of (2.5.8) as $F(x)$ and suppose it's of form $F=\frac{G}{1-x}$, then by Cauchy product we have $G=G(x)=\sum_{n=1}^{\infty}\left((H_n)^2-(H_{n-1})^2\right)x^n$ where $H_{n-1}=H_n-\frac{1}{n}$. Expanding RHS of G and making use of (2.5.5), (2.5.6) gives (2.5.8). (2.5.9)-(2.5.12) can be proved using the same technique as in (2.5.8) if we notice that $\widetilde{H_{n-1}}=\widetilde{H_n}-\frac{(-1)^{n-1}}{n}$. Finally (2.5.13), (2.5.14) and (2.5.15) are consequences of (2.5.8), (2.5.5), (2.5.6) respectively as we make use of $L(f)$ again. All (indefinite) integration operations can be carry out by CAS, or manually using $\int\frac{\mathrm{Li}_{n-1}(x)}{x}\,dx=\mathrm{Li}_n(x)$ and basic integration methods flexibly, therefore we omit the details. □

Proposition 4. The following formulas hold:

$$\int_0^1 x^{n-1}\log^k(x)\,dx=\frac{(-1)^k k!}{n^{k+1}} \tag{2.5.16}$$

$$\int_0^1 x^{n-1}\log(1-x)\,dx=-\sum_{j=1}^{\infty}\frac{1}{j(j+n)}=-\frac{H_n}{n} \tag{2.5.17}$$

$$\int_0^1 x^{n-1}\log^2(1-x)\,dx=\sum_{j=1}^{\infty}\frac{1}{j(j+n)^2}=\frac{(H_n)^2+H_n^{(2)}}{n} \tag{2.5.18}$$

$$\int_0^1 x^{n-1}\log(x)\log(1-x)\,dx=\frac{H_n}{n^2}+\frac{H_n^{(2)}}{n}-\frac{\pi^2}{6n} \tag{2.5.19}$$

$$\int_0^1 x^{n-1}\log^3(1-x)\,dx=-\frac{3H_nH_n^{(2)}+(H_n)^3+2H_n^{(3)}}{n} \tag{2.5.20}$$

$$\sum_{j=1}^{\infty}\frac{(-1)^j}{j(j+n)}=\frac{(-1)^{n-1}\widetilde{H_n}+(-1)^n\log(2)-\log(2)}{n} \tag{2.5.21}$$

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j(j+n)^2} = \frac{(-1)^{n-1}\widetilde{H_n^{(2)}}}{n} + \frac{(-1)^{n-1}\widetilde{H}_n}{n^2} - \frac{(-1)^{n-1}\log(2)}{n^2} - \frac{\log(2)}{n^2} - \frac{\pi^2(-1)^{n-1}}{12n} \tag{2.5.22}$$

Proof. (2.5.16) is just (1.1.1) with indexes rewritten. (2.5.17)-(2.5.20) follows differentiating Beta functions and use relationships between polygamma and generalized harmonic numbers (see section 1-4), possibly with a expansion of log terms recorded in proposition 3. (2.5.21) and (2.5.22) are direct consequences of partial fraction decomposition and $\eta(1) = \log(2), \eta(2) = \frac{\pi^2}{12}$. □

Now we are able to obtain new closed-forms for 7 weight 5 ESs mentioned above. 1 weight 3 ES and 2 weight 4 ESs, which are not contained in literatures either, are relatively trivial and we simply omit their proofs.

Proposition 5. The following formulas hold:

$$\begin{aligned} ES(-1,-1,-1;-2) &= -18\mathrm{Li}_5\left(\frac{1}{2}\right) - \frac{\pi^2\zeta(3)}{8} + \frac{1229\zeta(5)}{64} \\ &+\frac{21}{8}\zeta(3)\log^2(2) + \frac{3\log^5(2)}{20} + \frac{3}{4}\pi^2\log^3(2) - \frac{29}{160}\pi^4\log(2) \end{aligned} \tag{2.5.23}$$

$$\begin{aligned} ES(-1,-1,-1,-1;-1) &= -48\mathrm{Li}_5\left(\frac{1}{2}\right) - 4\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) \\ &+\frac{5\pi^2\zeta(3)}{48} + \frac{733\zeta(5)}{16} + \frac{13\log^5(2)}{30} + \frac{3}{2}\pi^2\log^3(2) - \frac{71}{180}\pi^4\log(2) \end{aligned} \tag{2.5.24}$$

$$\begin{aligned} ES(1,1,-2;-1) &= 6\mathrm{Li}_5\left(\frac{1}{2}\right) - \frac{\pi^2\zeta(3)}{48} - \frac{93\zeta(5)}{64} \\ &-\frac{1}{20}\log^5(2) + \frac{1}{36}\pi^2\log^3(2) - \frac{13\pi^4\log(2)}{1440} \end{aligned} \tag{2.5.25}$$

$$\begin{aligned} ES(1,1,-1,-1;-1) &= 12\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{\pi^2\zeta(3)}{16} - \frac{201\zeta(5)}{16} \\ &-\frac{3}{8}\zeta(3)\log^2(2) + \frac{\log^5(2)}{10} + \frac{1}{12}\pi^2\log^3(2) + \frac{97}{720}\pi^4\log(2) \end{aligned} \tag{2.5.26}$$

$$\begin{aligned} ES(1,-1,-1;-2) &= 10\mathrm{Li}_5\left(\frac{1}{2}\right) - 2\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) + \frac{\pi^2\zeta(3)}{96} \\ &-10\zeta(5) - \frac{7}{8}\zeta(3)\log^2(2) - \frac{1}{6}\log^5(2) + \frac{2}{9}\pi^2\log^3(2) + \frac{1}{10}\pi^4\log(2) \end{aligned} \tag{2.5.27}$$

$$\begin{aligned} ES(1,-1,-2;-1) &= 4\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) + \frac{7}{16}\zeta(3)\log^2(2) \\ &+\frac{\log^5(2)}{6} - \frac{5}{72}\pi^2\log^3(2) - \frac{1}{480}\pi^4\log(2) \end{aligned} \tag{2.5.28}$$

$$ES(1,-1,-1,-1;-1) = 2\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{13\pi^2\zeta(3)}{96} - \frac{83\zeta(5)}{32} \\ + \frac{9}{16}\zeta(3)\log^2(2) - \frac{13}{60}\log^5(2) + \frac{41}{72}\pi^2\log^3(2) - \frac{1}{180}\pi^4\log(2) \tag{2.5.29}$$

Proof. (2.5.23) and (2.5.24) follows immediately by symmetry, we take the latter one as an example. Following section 2-2, we have:

$$ES(-1,-1,-1,-1;-1) = \sum_{0<j,k,l,m\leq n} \frac{(-1)^{j-1}(-1)^{k-1}(-1)^{l-1}(-1)^{m-1}(-1)^{n-1}}{jklmn}$$

Consider the analogous sum without restrictions, by I-E principle we have:

$$\log^5(2) = 5ES(-1,-1,-1,-1;-1) - 10ES(-1,-1,-1;2) + 10ES(-1,-1;-3) - 5ES(-1;4) + \eta(5)$$

In this equality, everything but $ES(-1,-1,-1,-1;-1)$ are known, therefore we've readily deduced (2.5.24). (2.5.23) is similar if we consider the I-E equality for $\sum \frac{(-1)^{j-1}(-1)^{k-1}(-1)^{l-1}(-1)^{m-1}}{jklm^2}$. This method won't work for the other 5 ESs, for if we write one of them as multiple sums like (2.5.24), the corresponding I-E equality will contain a harmonic factor $\sum \frac{1}{k}$ one side and $\sum \frac{H(k)}{k}$ ($H(k)$ product of alter/non-alter harmonic numbers) another, which are both divergent apparently.

Now we deal with (2.5.25). By multiplying both sides of (2.5.22) with $(H_n)^2$ and sum it for all positive integer n, we have:

$$-\frac{\pi^2}{12}ES(1,1;-1) - \log(2)ES(1,1;2) - \log(2)ES(1,1;-2) \\ +ES(1,1,-1;-2) + ES(1,1,-2;-1) = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^j (H_n)^2}{j(j+n)^2}$$

But by (2.5.16), Fubini theorem (conditions satisfied), (2.5.1) and (2.5.8) we have:

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^j (H_n)^2}{j(j+n)^2} = \int_0^1 \frac{\log(x)\left(\sum_{j=1}^{\infty}\frac{(-1)^{j-1}x^j}{j}\right)\left(\sum_{n=1}^{\infty}(H_n)^2 x^n\right)}{x}\,dx \\ = \int_0^1 \left(\frac{1}{x} + \frac{1}{1-x}\right)\log(x)\log(x+1)\left(\mathrm{Li}_2(x) + \log^2(1-x)\right)\,dx$$

After expanding the RHS, we obtain a relation between 5 ESs, 2 LIs and 2 PLIs, where 4 ESs other than the desired one and 2 LIs are already known. Therefore, we only need to get the value for two PLIs, namely $PLI(0,1,1;21;0), PLI(0,1,1;21;1)$ to finish (2.5.25) (i.e. this ES depends on two PLIs modulo known LIs and ESs). This leads us to one theme of this article: mutual transformations between PLIs and ESs. Although we present our solution on ESs via solving particular problems and LI/PLIs via offering general patterns, the idea behind is the same: find enough relations, solve them all. Since we would like to solve all ESs before investigating PLIs, it's clear that we should apply double series expansion to go back to ESs, using (2.5.1), (2.5.2) and (2.5.5). That is:

$$PLI(0,1,1;21;1) = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^{j-1}\int_0^1 \log(x)x^{j+n-1}\,dx}{jn^2} = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^j}{jn^2(j+n)^2}$$

$$PLI(0,1,1;21;0) = ... = -\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \frac{\widetilde{H_j}}{n^2(j+n+1)^2}$$

For the first one, sum either n or j first elementarily and the PLI reduces to ESs already known. For the second one, sum n first and after a index change we get:

$$-\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \frac{\widetilde{H_j}}{n^2(j+n+1)^2} = -\sum_{j=1}^{\infty}\left(\widetilde{H_j} - \frac{(-1)^{j-1}}{j}\right)\left(-\frac{2H_j}{j^3} - \frac{H_j^{(2)}}{j^2} + \frac{\pi^2}{3j^2}\right)$$

Expand the RHS and use known ES values again, we find the second PLI and the desired (2.5.25) is established, and we add (2.5.23)-(2.5.25) to the set of known ESs. Furthermore, solution of (2.5.25) can be modified to fit for (2.5.26)-(2.5.28). We won't present a step-by-step solution like in (2.5.25) but sketch the route and point out the important steps:

For (2.5.26), start from (2.5.18), multiply both sides with $(-1)^{n-1}\widetilde{H_n}^2$ and sum n, using (2.5.9) we know that the desired ES follows from 4 PLIs – $PLI(2,0,0;22;1/2), PLI(2,0,0;24;1/2)$ modulo known ESs and LIs. For $PLI(2,0,0;22;1/2)$, use (2.5.5) for single expansion to reduce them to known ESs. For $PLI(2,0,0;24;1/2)$, recall reflection formula of dilogarithm to obtain: $\mathrm{Li}_2\left(\frac{1-x}{2}\right) + \mathrm{Li}_2\left(\frac{x+1}{2}\right) = \log(2)\log(1-x) + \log(2)\log(x+1) - \log(1-x)\log(x+1) + \frac{\pi^2}{6} - \log^2(2)$. This allow us to reduce them to $PLI(2,0,0;23;1/2)$ modulo LIs again. Now, by substitution $t = 1-x$, single/double expansion for $\frac{\mathrm{Li}_2\left(\frac{x}{2}\right)}{1-\frac{x}{2}} / \frac{\mathrm{Li}_2\left(\frac{x}{2}\right)}{1-x}$ (see (2.5.1) and (2.5.5)) and Xu's result on 2-Euler Sums [13] reduce them to known ESs.

Techniques used in (2.5.27) are exactly the same (e.g. multiplying $(-1)^{n-1}\widetilde{H_n}^2$, reflection formula and double expansion), except that we start from (2.5.19) instead. Readers may verify that all ESs appeared during the reduction are known and all PLIs can be reduced to known ESs. Thus, we add (2.5.26) and (2.5.27) to the set of known ESs. (2.5.28) is similar to (2.5.25): also start from (2.5.22), choose $H_n\widetilde{H_n}$ as the multiplier, the remaining steps are alike to (2.5.25)–(2.5.27). If readers carry out the whole process, they will find that all unknown PLIs reduces to known ESs satisfactorily because among them the result of (2.5.27) is used, which had been shown just now.

Finally we manage to derive (2.5.29). Denote $f(x) = \sum_{n=1}^{\infty}\left(\widetilde{H_n}\right)^3 x^n$, by calculating differences of the generating function (see proposition 3) and a change of variable, one can show that $f(-x) =$

$\frac{-3P(x)-3Q(x)-\mathrm{Li}_3(x)}{x+1}$, where $P(x)=\sum_{n=1}^{\infty}\frac{(\widetilde{H_n})^2x^n}{n}$, $Q(x)=\sum_{n=1}^{\infty}\frac{(-1)^n\widetilde{H_n}x^n}{n^2}$. By (2.5.17) we have:

$$ES(1,-1,-1,-1;-1)=\int_0^1\frac{f(-x)\log(1-x)}{x}\,dx=\int_0^1\frac{\log(1-x)(-3P(x)-\mathrm{Li}_3(x)-3Q(x))}{x(x+1)}\,dx$$

Now integrate by parts with the help of $\int\frac{\log(1-x)}{x(x+1)}\,dx=u(x)=C-\mathrm{Li}_2\left(\frac{1-x}{2}\right)-\mathrm{Li}_2(x)-\log(1-x)\log\left(\frac{x+1}{2}\right)$ (where we take $C=\frac{\pi^2}{6}$ for convergence issues), we know that the desired ES value equals to the following integral:

$$\int_0^1\frac{u(x)\left(3\sum_{n=1}^{\infty}\frac{(-1)^n\widetilde{H_n}x^n}{n}+3\sum_{n=1}^{\infty}\left(\widetilde{H_n}\right)^2x^n+\mathrm{Li}_2(x)\right)}{x}\,dx$$

Using (2.5.7) and (2.5.9), the integral can be furtherly decomposed into 3 parts:

$$3\int_0^1\frac{u(x)\left(2\mathrm{Li}_2\left(\frac{1-x}{2}\right)+\mathrm{Li}_2(x)-2\mathrm{Li}_2\left(\frac{1}{2}\right)+2\log\left(\frac{x+1}{2}\right)\log(1-x)\right)}{x(1-x)}\,dx$$
$$+3\int_0^1\frac{u(x)\left(\mathrm{Li}_2\left(\frac{x+1}{2}\right)-\mathrm{Li}_2(x)-\mathrm{Li}_2\left(\frac{1}{2}\right)-\log(2)\log(x+1)\right)}{x}\,dx+\int_0^1\frac{\mathrm{Li}_2(x)u(x)}{x}\,dx$$

Now we expand all parentheses (including the one containing $u(x)$) in the above expression. Note that some parts of the expanded integrand are divergent and we have to put them together again, such as $\int_0^1\frac{\mathrm{Li}_2(x)\left(\mathrm{Li}_2\left(\frac{x+1}{2}\right)-\mathrm{Li}_2\left(\frac{1}{2}\right)\right)}{x}\,dx$. Apply integration by parts on all these composite integrals (in the example lift up a $\mathrm{Li}_3(x)$) to convert it into convergent PLIs. After this we reduce the intricate problem to the evaluation of plenty of LIs and PLIs. Now integrate by parts and use techniques in (2.5.25)-(2.5.28) over and over again, we've verified manually that all PLIs can be reduced to lower weight ESs and 53 weight 5 ESs already deduced, hence we get (2.5.29) and the proof of proposition is finished. $\square$

In proof of (2.5.27)-(2.5.29) (especially the last one) much details are omitted and left to interested readers. One may verify these proofs more easily after reading section 3, that is, the systematic approach of establishing relations between PLIs and reducing them to ESs. In fact, the 54th ES in (2.5.29) is one of the most complicated results in this article and will not appear in all reductions of PLIs, which confirms that we are not arguing in a circle. All 89 ESs with weight ≤ 5 are tabulated in appendix 2.

3.PLI Evaluation

It's straightforward to see that a PLI's weight is at least 3 due to the existence of a polylog term. After a counting we know that there are 11, 55, 197 distinct PLIs for weight 3, 4, 5. As we are going to show, they all lies in the $\mathbb{Q}$-linear space spanned by the Fibonacci basis with corresponding weights. Parallel to LI case, we point out that some of the PLIs are evaluated directly by calculating polylog primitives, for instance, $PLI(1,0,0;24;2) = \int_0^1 \frac{\mathrm{Li}_2\left(\frac{x+1}{2}\right)\log(1-x)}{x+1}\,dx$ has primitive $\log(2)\mathrm{Li}_3\left(\frac{x+1}{2}\right) - \frac{1}{2}\mathrm{Li}_2\left(\frac{x+1}{2}\right)^2$. However, this only works for a few PLIs with low weight, so we exclude it from the systematic approach below.

Most results obtained in this section are new.

3.1 General formulas

Similar to ES case, we refer the reader to Freitas [7] and Valean [10] on for 3 well-known PLI formulas involving $L(n,1;x)$. We summarize their statements as:

Lemma 3. $\int_0^1 \frac{\mathrm{Li}_m(x)\mathrm{Li}_n(x)}{x}\,dx$, $\int_0^1 \frac{\log^m(x)\mathrm{Li}_n(x)}{1-x}\,dx$ ($m+n$ even) and $\int_0^1 \frac{\mathrm{Li}_{n+1}(x)\log^n(x)}{x+1}\,dx$ are reducible to zeta values.

Note that using proposition 3 we can modify these appropriately, to obtain general formulas for other PLIs containing $L(n,2;x)$, such as $\int_0^1 \frac{\mathrm{Li}_m(x)\mathrm{Li}_n(-x)}{x}\,dx$ and $\int_0^1 \frac{\log^m(x)\mathrm{Li}_n(\pm x)}{x+1}\,dx$. The crucial idea here is (multiple) series expansion, which will be explained more clearly in section 3-5, so we left the simple generalization to readers.

3-2. Integration by parts

Completely analogous to section 1-2, apply N-L formula where the differentiable F is of form $(l(2;x)-\log(2))^r \prod_{m=0}^2 l(m;x)^{a(m)} \prod_{k=1}^6 \prod_{n=2}^N L(n,k;x)^{b(n,k)}$, such that $|F(0^+)|, |F(1^-)| < \infty$ as before. For instance, taking F to be $\mathrm{Li}_2(x)\log(1-x)\log(x+1)\log\left(\frac{x+1}{2}\right)$ gives a homogeneous relation of weight 5, and generally we consider all F with $r + \sum_{k=1}^6 \sum_{n=2}^N nb(n,k) + \sum_{m=0}^2 a(m) = W$ to obtain weight W relations, possibly involving all 6 single polylog terms $L(n,k;x)$ and 15 polylog products $L(n,j;x)L(n,k;x)$. Note that polylog terms can appear more than once, so there're plenty of choices on the kernel F. From the view of solving single PLIs, the above process can be explained as doing integration by parts flexibly, that is, choosing different parts of the PLI integrand to lift up. As an example, we may lift up either $\frac{1}{2}\log^2(x+1) = \int \frac{\log(x+1)}{x+1}\,dx$ or $\mathrm{Li}_3\left(\frac{x+1}{2}\right) = \int \frac{\mathrm{Li}_2\left(\frac{x+1}{2}\right)}{x+1}\,dx$ in $PLI(0,1,1;24;2) = \int_0^1 \frac{\mathrm{Li}_2\left(\frac{x+1}{2}\right)\log(x)\log(x+1)}{x+1}\,dx$, which leads to different relations.

3-3. Fractional transformations

Similar to section 1-3, take $F = \frac{\prod_{m=0}^2 l(m;x)^{a(m)} \prod_{k=1}^6 \prod_{n=2}^N L(n,k;x)^{b(n,k)}}{f(p;x)}$ with $\int_0^1 F(x)\,dx$ convergent and $b(n,3) = b(n,4) = 0$ for all n (equivalently, $\mathrm{Li}_n\left(\frac{1\pm x}{2}\right)$ terms won't appear). Fix the weight W, use the same substitution $t = \frac{1-x}{x+1}$ on every suitable F, then expand all log terms (via formulas in section 1-3) and plug in LI values to obtain weight W relations, involving $L(n,1;x), L(n,5;x)$ or $L(n,2;x), L(n,6;x)$ or both of them.

Here is another way to obtain relations. Denote $P(x)=\prod_{n=2}^{N}\mathrm{Li}_n(x)^{a(n)}$, by splitting $(0,1)$ and substituting $t=\frac{1\pm x}{2}$ for 2 parts respectively we have:

$$\begin{aligned}\int_0^1\left(\frac{P\left(\frac{1-x}{2}\right)\log^m\left(\frac{1-x}{2}\right)\log^n\left(\frac{x+1}{2}\right)}{1\mp x}+\frac{P\left(\frac{x+1}{2}\right)\log^m\left(\frac{x+1}{2}\right)\log^n\left(\frac{1-x}{2}\right)}{1\pm x}\right)dx\\=\int_0^1\frac{P(x)\log^m(x)\log^n(1-x)}{f_\pm(x)}\,dx\end{aligned}\tag{3.3.1}$$

Where $f_+(x)=x, f_-(x)=1-x$. Vary $a(n)$, this formula offer relations that connect PLIs involving $L(n,1;x), L(n,3;x)$ and $L(n,4;x)$.

3-4. Functional equations of polylogarithm

Proposition 6. The following formulas hold as $0<x<1$, where all logs/polylogs are on their principle branches:

$$\mathrm{Li}_n(-x)+\mathrm{Li}_n(x)=\frac{\mathrm{Li}_n\left(x^2\right)}{2^{n-1}}\tag{3.4.1}$$

$$\mathrm{Li}_2(1-x)+\mathrm{Li}_2(x)=\frac{\pi^2}{6}-\log(x)\log(1-x)\tag{3.4.2}$$

$$\mathrm{Li}_2\left(\frac{x}{x-1}\right)+\mathrm{Li}_2(x)=-\frac{1}{2}\log^2(1-x)\tag{3.4.3}$$

$$\mathrm{Li}_2\left(-\frac{1}{x}\right)+\mathrm{Li}_2(-x)=-\frac{\pi^2}{6}-\frac{1}{2}\log^2(x)\tag{3.4.4}$$

$$\begin{aligned}\mathrm{Li}_2\left(\frac{1-x}{2}\right)+\mathrm{Li}_2\left(\frac{x+1}{2}\right)=\log(2)\log(x+1)\\+\log(2)\log(1-x)-\log(1-x)\log(x+1)+\frac{\pi^2}{6}-\log^2(2)\end{aligned}\tag{3.4.5}$$

$$\mathrm{Li}_2\left(\frac{1-x}{2}\right)+\mathrm{Li}_2\left(\frac{x-1}{x+1}\right)=-\frac{1}{2}\log^2(x+1)+\log(2)\log(x+1)-\frac{1}{2}\log^2(2)\tag{3.4.6}$$

$$\mathrm{Li}_3(-x)-\mathrm{Li}_3\left(-\frac{1}{x}\right)=-\frac{1}{6}\log^3(x)-\frac{1}{6}\pi^2\log(x)\tag{3.4.7}$$

$$\mathrm{Li}_3\left(\frac{x}{x-1}\right)+\mathrm{Li}_3(1-x)+\mathrm{Li}_3(x)=\frac{1}{6}\log^3(1-x)-\frac{1}{2}\log(x)\log^2(1-x)+\frac{1}{6}\pi^2\log(1-x)+\zeta(3)\tag{3.4.8}$$

$$
\begin{aligned}
\mathrm{Li}_3\left(\frac{1-x}{2}\right)+\mathrm{Li}_3\left(\frac{x+1}{2}\right)+\mathrm{Li}_3\left(\frac{x-1}{x+1}\right) &= \frac{1}{6}\log^3(x+1)-\frac{1}{2}\log^2(2)\log(1-x) \\
&-\frac{1}{2}\log^2(x+1)\log(1-x)+\left(\frac{\pi^2}{6}-\frac{\log^2(2)}{2}\right)\log(x+1) \\
&+\log(2)\log(1-x)\log(x+1)+\zeta(3)+\frac{\log^3(2)}{3}-\frac{1}{6}\pi^2\log(2)
\end{aligned} \tag{3.4.9}
$$

Proof. (3.4.1)-(3.4.4), (3.4.7)-(3.4.8) are classical formulas for polylogarithms and can be found in Lewin [9]. (3.4.5) is easily deduced from (3.4.2) if we apply the substitution $t=\frac{1-x}{2}$. The same substitution in (3.4.3) gives (3.4.6), and (3.4.8) gives (3.4.9). For the last one a little simplification is needed. □

Now we can obtain relations via proposition 6. Fix m,k,n, multiply both sides of (3.4.1) with $\frac{\log^m\left(1-x^2\right)\log^k(x)}{x}$ and integrate it on $(0,1)$. Substitute $t=x^2$ on RHS and expand $(\log(1-x)+\log(x+1))^m$ on LHS using binomial theorem, we obtain a relation. Another choice of the multiplicator is $\frac{x\log^m\left(1-x^2\right)\log^k(x)}{1-x^2}$, where we still substitute $t=x^2$ for RHS, but decompose the LHS with binomial theorem and partial fraction decomposition $\frac{x}{1-x^2}=\frac{1}{2}\left(\frac{1}{1-x}-\frac{1}{x+1}\right)$. Fix the weight W, consider 2 choices of multiplicators and all suitable F of this type, such that $\int_0^1 LHS\cdot F(x)\,dx, \int_0^1 RHS\cdot F(x)\,dx$ are combinations of weight W PLIs, to obtain relations connecting PLIs involving $L(n,1;x), L(n,2;x)$.

Moreover, (3.4.5), (3.4.6) and (3.4.9) can offer many nontrivial relations either. For instance, multiply both sides of (3.4.9) with a log kernel $F(x)=\frac{\prod_{m=0}^{2} l(m;x)^{a(m)}}{f(p;x)}$ such that $\int_0^1 F(x)L(3,k;x)\,dx$ $(k=3,4,6)$ on LHS and all LIs on RHS are convergent. It gives a relation between 3 PLIs involving trilogarithms already, since the value of LIs on RHS are already known. For (3.4.5) and (3.4.6) the process is similar, except that we can broaden the range of F, that is, allowing F to contain polylog terms. For our case (weight ≤ 5), they are single dilogarithmic terms $\frac{L(2,k;x)}{f(p;x)}$. Fix the weight W, for each of the 3 formulas, consider all F such that $\int_0^1 LHS\cdot F(x)\,dx, \int_0^1 RHS\cdot F(x)\,dx$ are combinations of weight W LI/PLIs, to obtain relations connecting PLIs involving $L(n,3;x), L(n,4;x)$ and $L(n,6;x)$.

3-5. Series expansion

Now we come to the last part of building the PLI relation system–reduce some of PLIs to combination of ESs to obtain their value directly. Because all PLI we deal with have weight no more than 5, we may classify them clearly. For further explanation, we have 4 kinds of weight 5 PLIs in total, named as (2,1,1) class, (3,1) class, (4) class and (2,2) class respectively. The index indicates the quantity as well as order of polylog terms, for instance, $PLI(1,1,0;22;2)$ belongs to (2,1,1) class for it contains one Li_2 term and two log terms in the integrand's numerator, and the classification for other weight 5 PLIs is similar. Note that class (1,1,1,1,1) is excluded for integrals belong to this class are nothing but LIs determined in section 1.

From now on we only discuss about weight 5 PLIs, for lower weight PLIs are relevantly trivial. Moreover, we restrict the use of this method to (2,1,1), (3,1) and (4) class PLIs only, for relations generated by these PLIs via series expansion, together with all relations given by sections above, have lead to closed-forms of all 197 weight 5 PLIs already. Furthermore, we require the only polylog term $L(n,k;x)$ $(n=2,3)$ contained in the PLI to satisfy $1 \le k \le 3$ as other polylog terms are not convenient to apply series expansion. Denote the set of these PLIs as Q.

We'll only present general procedure of converting the PLI into combination of ESs. Note that in this section 'ES' represents all ordinary ESs together with all Xu's 2-ES results [13].

Step 1. Reflection. If the only polylog term involved is $\mathrm{Li}_n\left(\frac{1-x}{2}\right)$, substitute $t=1-x$ in the integral. If the only polylog term involved is $\mathrm{Li}_n(-x)$, keep it unchanged. If the only polylog term involved is $\mathrm{Li}_n(x)$, keep it unchanged, or reduce the integral to the corresponding one involving $\mathrm{Li}_n(1-x)$ via (3.4.2) modulo known LIs, then substitute $t=1-x$. After this operation, the argument in polylog term is either x or $\frac{x}{2}$. In this step we have 2 choices for case $\mathrm{Li}_n(x)$ and 1 choice for other 2 cases.

Step 2. Determine the invariant part of integrand. Since we will use (2.5.16)-(2.5.19) to convert the integral to ESs, we choose a part of the integrand to be 'invariant', which means we won't expand them into series later. This part should be of form $F_1(x)=\log^m(x)\log^n(1-x)$, with m equal to the degree of $\log(x)$ in the original integrand, n do not exceed the degree of $\log(1-x)$ in the original integrand. In this step we have at most 3 choices on determining n, namely $n=0,1,2$.

Step 3. Multiple expansion. Denote $F=F_1F_2$ where F is the whole integrand (after possible manipulations in step 1), F_1 the invariant part determined in step 2. We have F_2 a combination of products of functional terms belong to the set:

$$\left\{\frac{1}{x},\frac{1}{1-x},\frac{1}{x+1},\frac{1}{1-\frac{x}{2}},\log(1-x),\log(x+1),\log\left(1-\frac{x}{2}\right),\mathrm{Li}_n(x),\mathrm{Li}_n\left(\frac{x}{2}\right)\right\}$$

Where by combination we mean the integral is composed of several integrals with different weights (see examples below). Separate F_2, expand all terms (other than $\frac{1}{x}$) for each into elementary power series using (2.5.1)/(2.5.5) and exchange the order of evaluation, we get a multiple sum (at most quadruple) with the summand an integral consistent with (2.5.16)-(2.5.19). In this step we have only 1 choice for the process of expansion is definite.

Step 4. Converting into multiple number series. This is straightforward if we use (2.5.16)-(2.5.19) to evaluate the integral in rational/harmonic terms. In this step we have only 1 choice for the process of integration is definite.

Step 5. Cauchy's simplification. After step 4 we arrive at a multiple sum with at most 4 indexes. If there's only 1 index, it's already a combination of ESs hence the procedure is finished. If there are more than 2 indexes, sum w.r.t all indexes with rational summands to reduce the weight, and go to the next step if the order reduces to 2. Otherwise (order greater than 2 and summand harmonic to all indexes), consider all index pairs (j, k) and check whether their Cauchy product is expressible in harmonic terms with a new index m. If so, replace (j, k) with m in the summation, so that the order of the sum is reduced by 1, and we should consider index pairs again for Cauchy simplification until the order of summation reduce to 2, or still larger than 2 but cannot be Cauchy-simplified any more (this can be done equivalently in step 3, where we should consider Cauchy type generating functions instead, namely (2.5.2), (2.5.3) and (2.5.5)). For the first case go to step 6, and for the second discard this sum and stop the procedure. In this step we have a great deal of choices to simplify the sum, but much of them will be discarded.

Step 6. Reduction to ESs. If we go through step 5, we will have a double series in hand. Denote the index of summation as (j, k). If the summand is rational functions of j, k, sum w.r.t either of the indexes first to finish the procedure; if the summand is rational to j but harmonic to k, sum w.r.t j also finishes; see proof of proposition 5 for examples. If we have all harmonic terms with index $j + k$ and other parts rational, invoke Cauchy product again to finish the procedure. For other cases (e.g. the sum involves $H_k H_{j+k}$), discard it. In this step we have 1 or 2 choices for using iterated summation or Cauchy simplification.

Fix a weight 5 PLI that belongs to Q, consider the tree of choices that we would make through step 1-6 of the procedure above. If there exist a branch of this tree that is not ended with 'discarded' in step 5-6, we have reduced the PLI to ESs successfully. Since [6, 14, 15] together with section 2 already cover all weight 5 ESs, we readily obtained the closed-form of this PLI. If all branches are discarded, we get nothing new from the procedure. Now carry out the procedure for every PLI belong to Q to obtain independent relations which give PLI values directly. The whole process is extremely lengthy and we only show some typical examples:

A successful example: Consider $PLI(1, 0, 1; 21; 2) = \int_0^1 \frac{\mathrm{Li}_2(x)\log(1-x)\log(x+1)}{x+1}\,dx$. Step 1: Modulo known LIs, we only need to evaluate $\int_0^1 \frac{\mathrm{Li}_2(1-x)\log(1-x)\log(x+1)}{x+1}$ then plug in (3.4.2), so substitute $t = 1-x$. Step 2: After reflection, the problem boils down to evaluating $\int_0^1 \log(2)\frac{\mathrm{Li}_2(x)\log(x)}{1-\frac{x}{2}}\,dx$ and $\int_0^1 \frac{\mathrm{Li}_2(x)\log(x)\log\left(1-\frac{x}{2}\right)}{1-\frac{x}{2}}\,dx$. Set $\log(x)$ to be invariant in both 2 integrals. Making use of double expansion, (2.5.16) and [13]'s results, we come to the closed-form of the former weight 4 integral.

Now we deal with the latter one. Step 3: Expand $\mathrm{Li}_2(x), \log\left(1-\frac{x}{2}\right), \frac{1}{1-\frac{x}{2}}$ with 3 independent index j,k,l. Step 4: Make use of (2.5.16). Step 5: Consider Cauchy product of the summation w.r.t k,l. The convolution term is harmonic hence the triple sum is reduced to double, which can be deduced either if we expand $\frac{\log\left(1-\frac{x}{2}\right)}{1-\frac{x}{2}}$ directly via (2.5.2) in step 3. Step 6: Denote the single Cauchy index replacing j,k as m. The double sum we have in hand is $\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\frac{H_m}{j^2 2^m (j+m+1)^2}$. Sum w.r.t j (this is nothing but the elementary sum occurred in proof of (2.5.25), proposition 5), we reduced this sum to a combination of three 2-ESs, which are also solved in [13]. The result is:

$$\begin{aligned} PLI(1,0,1;21;2) = & -2\mathrm{Li}_5\left(\frac{1}{2}\right) - \mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) + \frac{23\pi^2\zeta(3)}{96} - \frac{15\zeta(5)}{64} \\ & -\frac{21}{16}\zeta(3)\log^2(2) - \frac{1}{40}\log^5(2) + \frac{5}{72}\pi^2\log^3(2) - \frac{11}{720}\pi^4\log(2) \end{aligned} \tag{3.5.1}$$

Unsuccessful examples: Readers may verify that $PLI(1,0,1;22;2), PLI(2,0,0;22;2)$ are not solvable using this procedure. At first glance they are not much more difficult than the PLI above, but it's impossible to make choices in the procedure above to avoid the multiple sum containing summand of form $H_k H_{j+k}$ or its alternating analogue, hence in step 6 they must be discarded. Actually, these two integrals are evaluated nontrivially when we put all relations generated from section 3 together. From the viewpoint of solving single problems, we have to integrate by parts repeatedly, then use fractional transformation and connection formulas in section 3-4, finally evaluate those PLIs involving $\mathrm{Li}_3(-x), \mathrm{Li}_3\left(\frac{1-x}{2}\right)$ by multiple expansions. This is a highly circuitous process which reveals the necessity of a systematic approach providing linear relations, that is, the whole section 3.

3-6. Obtaining closed-forms for PLIs

Finally we are able to prove the main result:

Proof (of Theorem 1). In section 2 we showed that the theorem hold for all 89 ESs with weight ≤ 5. Moreover, all linear relations deduced from methods in section 3-1 to 3-5, satisfying that the maximum weight of PLIs in the relation does not exceed 5, forms a linear equation system from which all 263 PLIs with weight ≤ 5 are explicitly solved. □

All closed-forms of PLIs with weight ≤ 5, most of which are not discussed in literatures, are tabulated in appendix 3.

4. Applications

We can apply results obtained in section 1 to section 3 on many topics. In section 4-1 to 4-5 we present some systematic applications, and in 4-6 10 specific problems are solved. Furthermore, some typical pattern-displaying results, instead of all closed-form integrals and series we deduced, are tabulated in appendix 4.

Most results obtained in this section are new.

4-1. Logsine integrals, poly-logsine integrals and others

The first natural application of LIs is the evaluation of a special kind of logsine integrals (abbr. LSIs).

$$LSI(a,b,c;d) = \int_0^{\frac{\pi}{2}} x^a \log^b(2\sin(x)) \log^c(2\cos(x)) \cot^d(x)\, dx \tag{4.1.1}$$

For 'logsine' we mean the interval is $(0, \frac{\pi}{2})$ and integrand a product of functions belong to set

$$\{x, \log(2\sin(x)), \log(2\cos(x)), \cot(x), \tan(x)\}$$

Where $\cot, \tan$ do not appear at the same time, and the weight of the logsine integral ($d = 0$ or ± 1) is defined as $W = a + b + c + 1$. At first sight, if $a = d = 0$ the LSI is trivially evaluated via substitution $t = \sin^2(x)$ and Beta's derivative; also the reflection $t = \frac{\pi}{2} - x$ may offer several LSI relations. Notwithstanding, we have a far more elegant and systematic approach, in fact:

Proposition 7. All convergent LSIs with weight $W \leq 5$ can be generated by $A_W \cup \pi A_{W-1}$ over $\mathbb{Q}$. Here $\pi S = \{\pi s | s \in S\}, A_0 = \{1\}$.

Proof. Consider LI kernel $F_0(x) = \frac{l(0;x)^k l(1;x)^m l(2;x)^n}{f(p;x)}$, with all logarithms on their principle branches. Integrate F along a semicircle contour whose diameter is interval $(-1, 1)$. More precisely, separate the contour into 3 parts by $0, \pm 1$, substitute $t = -x$ for the $(-1, 0)$ part and parametrize the semicircle part by $t = e^{2ix}$, we obtain a relation between LIs and LSIs. Noticing that we have $\log(-x) = \log(x) + i\pi$ on $(-1, 0)$ and $\log(1 + e^{2ix}) = \log(2\cos(x)) + ix, \log(1 - e^{2ix}) = \log(2\sin(x)) + i\left(x - \frac{\pi}{2}\right)$ on $(0, \frac{\pi}{2})$ due to definition of logs, we have that:

$$\int_0^{\frac{\pi}{2}} F(x) G_l(x)\, dx = LI(k, m, n; l) + \sum_{j=0}^{m} \binom{m}{j} (i\pi)^j LI(n, m-j, k; 2-l) \tag{4.1.2}$$

Where $F(x) = (2ix)^m \left(\log(2\sin(x)) + i\left(x - \frac{\pi}{2}\right)\right)^k (\log(2\cos(x)) + ix)^n, G_0(x) = \cot(x) + i, G_1(x) = -2i$ and $G_2(x) = \tan(x) - i$, whenever all integrals on both sides are convergent. Now consider all LI kernel F_0 with $k + m + n + 1 \leq 5$, apply this formula (and take real or imaginary part of the identity) to obtain relations between LI/LSIs. Solving the whole system yields the result. In fact, according to LSI-LI dependence and Au's results on higher weight LIs [1], we are able to express higher weight (say, weight 6 to weight 12) LSIs via ESs or MZVs using the connection formula above. □

Apart from reducing LSIs to LIs via deforming contours, we may apply Fourier expansion on LSI evaluation. Since $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\cos(2kx)}{k} = \log(2\cos(x)), -\sum_{k=1}^{\infty} \frac{\cos(2kx)}{k} = \log(2\sin(x))$ hold on $(0, \frac{\pi}{2})$, expand the integrand in $\int_0^{\frac{\pi}{2}} x^n \log(2\sin(x))\,dx$ or $\int_0^{\frac{\pi}{2}} x^n \log(2\cos(x))\,dx$, switch the order, evaluate the elementary integral $\int_0^{\frac{\pi}{2}} x^m e^{inx}\,dx$ then sum up. By this alternative way we give Zeta closed-forms to this class of LSI (see section 4-6 (8) for more). Note that the interval $(0, \frac{\pi}{2})$ can be replaced by any sub-interval, say $(0, \frac{\pi}{4})$, which will be useful in next section.

Another method solving LSIs is to apply Feynman's trick on various Beta and polygamma identities. For instance, by contour integration we can show that [8]:

$$\int_0^{\pi} e^{iax}\sin^{v-1}(x)\,dx = \frac{\pi e^{\frac{i\pi a}{2}}}{2^{v-1} v B\left(\frac{a+v+1}{2}, \frac{-a+v+1}{2}\right)} \tag{4.1.3}$$

$$\int_0^{\frac{\pi}{2}} \cos(ax)\cos^{v-1}(x)\,dx = \frac{\pi}{2^{v} v B\left(\frac{a+v+1}{2}, \frac{-a+v+1}{2}\right)} \tag{4.1.4}$$

Now, differentiate and take limits on parameters v, a appropriately, we are able to obtain LSI relations or closed-forms. For instance, all closed-forms of LSIs of form $\int_0^{\frac{\pi}{2}} x^{2n}\log^m(2\cos(x))\,dx$ or $\int_0^{\frac{\pi}{2}} x^{2n+1}\tan(x)\log^m(2\cos(x))\,dx$ are deducible from differentiating (4.1.4).

Similarly, substitute $t = \tan(x)$ and make use of the Mellin transform in section 4-2, paragraph 'General formulas' (independent from current argument), we may show that:

$$\int_0^{\frac{\pi}{2}} x\tan^p(x)\,dx = \frac{1}{4}\pi\csc\left(\frac{\pi p}{2}\right)\left(\psi^{(0)}\left(1-\frac{p}{2}\right) - 2\psi^{(0)}(1-p) - \gamma\right) \tag{4.1.5}$$

Where γ denotes Euler-Mascheroni constant. Differentiate w.r.t p then take limit, we obtain closed-forms of $\int_0^{\frac{\pi}{2}} x\log^n(\tan(x))\,dx$ and $\int_0^{\frac{\pi}{2}} x\cot(x)\log^n(\tan(x))\,dx$. After splitting $\log^n(\tan(x))$, for each integral there is a LSI relation.

We also dealt with poly-logsine integrals (abbr. PLSIs), that is, the integrand F_0G_0 with F_0 defined above, and G_0 a product whose terms belong to the set

$$\left\{\mathrm{Cl}_2(2x), \mathrm{Cl}_2(4x), \mathrm{Li}_2\left(-\tan^2(x)\right), \mathrm{Ti}_2(\tan(x))\right\}$$

Where $\mathrm{Cl}_2(x) = \sum_{n=1}^{\infty}\frac{\sin(nx)}{n^2}$ Clausen function and $\mathrm{Ti}_2(x) = \Im(\mathrm{Li}_2(ix))$ inverse tangent integral. These actually arise in parametrizations of $L(2,k;x), k = 1,2,5,6$ via substitution $x = e^{2it}$ in some PLIs, that is:

$$\mathrm{Li}_2(x) = \mathrm{Li}_2\left(e^{2it}\right) = i\mathrm{Cl}_2(2t) + t^2 - \pi t + \frac{\pi^2}{6} \tag{4.1.6}$$

$$\mathrm{Li}_2(-x) = \mathrm{Li}_2\left(-e^{2it}\right) = i\left(\frac{1}{2}\mathrm{Cl}_2(4t) - \mathrm{Cl}_2(2t)\right) + t^2 - \frac{\pi^2}{12} \tag{4.1.7}$$

$$\mathrm{Li}_2\left(\pm\frac{1-x}{x+1}\right) = \mathrm{Li}_2(\mp i\tan(t)) = \frac{1}{4}\mathrm{Li}_2\left(-\tan^2(t)\right) \mp i\mathrm{Ti}_2(\tan(t)) \tag{4.1.8}$$

All of which are based upon basic identities of Clausen function, inverse tangent and Bernoulli polynomials (see [9]). These PLSIs with weight no more than 5 can be evaluated similarly as LSIs thus we pause here. Nevertheless, as the general structure of high weight PLIs (i.e. reducibility to polylog or multiple-polylog terms) are not quite understood, numerous high weight PLSIs remain unknown.

Furthermore, by substitution $x = \tan(t)$ and repeated integration by parts, it's possible to convert the family (with suitable parameters ensuring convergence) $\int_0^\infty \frac{\tan^{-1}(x)^k \log^m\left(x^2+1\right)}{x^l (x^2+1)^n}\,dx$ to combination of LSIs hence obtain their closed-forms. Also, the consideration of differentiating $\int_0^\infty \frac{x^{a-1}}{(x+1)^{a+b}}\,dx = B(a,b)$ or integrating $\frac{\log^m(1+iz)}{z^n}$ along the large upper semicircle on the complex plane may help (we omit the details). See appendix 4 for some of these results, together with typical LSIs and PLSIs and other integrals using techniques on trigonometric functions.

4-2. Quadratic log integrals and trigonometric analogue

Another way to generalize LI evaluations is considering 'quadratic log integrals' (abbr. QLIs):

$$QLI(a,b,c,d,e;p) = \int_0^1 \frac{\log^a(1-x)\log^b(x)\log^c(x+1)\log^d\left(x^2+1\right)\tan^{-1}(x)^e}{g(p;x)}\,dx \tag{4.2.1}$$

Here the range is $p \in \{1,2,3,4,5\}$, where:

$$g(1;x) = 1-x,\, g(2;x) = x,\, g(3;x) = 1+x,\, g(4;x) = x^2+1,\, g(5;x) = \frac{x^2+1}{x} \tag{4.2.2}$$

The weight of a QLI is naturally defined as $W = a+b+c+d+e+1$. For their name, note that taking real/imaginary part of principle $\log(1+ix)$ gives 'quadratic' terms $\frac{1}{2}\log\left(x^2+1\right), \tan^{-1}(x)$. This is a far more general topic than LI but few investigations are found in literatures. In later discussion, parameters a,b,c,d,e will be small, and many of them equal to 0. In this case, we omit those zero parameters and simply repeat the word '1' a times, the word '2' b times, $\cdots$, for abbreviation. For instance we have $QLI(355;5) = \int_0^1 \frac{x\log(x+1)\tan^{-1}(x)^2}{x^2+1}\,dx$ and $QLI(345;4) = \int_0^1 \frac{\log(x+1)\log\left(x^2+1\right)\tan^{-1}(x)}{x^2+1}\,dx$. If we exclude those QLIs that already evaluated in section 1, after a simple counting we find that the number of convergent QLIs of weight 2, 3, 4 is 14, 50, 129 respectively. We have:

Proposition 8. All 193 QLIs with weight $W \leq 4$ can be generated by B_W over $\mathbb{Q}$, where:

$$B_2 = A_2 \cup \{C, \pi \log(2)\}$$

$$B_3 = A_3 \cup \left\{C \log(2), \pi C, \pi \log^2(2), \pi^3, \Im(\mathrm{Li}_3(1+i))\right\}$$

$$B_4 = A_4 \cup \left\{C^2, \pi C \log(2), C \log^2(2), \pi^2 C, \pi \log^3(2), \pi^3 \log(2), \log(2)\Im(\mathrm{Li}_3(1+i)),\right.$$
$$\left.\pi\zeta(3), \pi\Im(\mathrm{Li}_3(1+i)), \Im(\mathrm{Li}_4(1+i)), \psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right)\right\}$$

Where $C = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ denotes the Catalan constant and $\psi^{(k)}$ polygamma function (section 1-4). If we denote $\beta(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s} = \frac{1}{2}\left(\psi^{(0)}\left(\frac{s+1}{2}\right) - \psi^{(0)}\left(\frac{s}{2}\right)\right)$ the Dirichlet Beta function, then $\beta(2) = C, \beta(4) = \frac{\psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right)}{1536}$. In our cases, each B_W contains 2^W elements that is likely to be linear independent.

Proof. We list all methods generating relations between QLIs. Combining all of them yields the proposition.

Polylog identities. These are simple identities that meant to simplify closed-forms, resulting from formulas in section 3-4, together with Lewin [9].

$$\Re(\mathrm{Li}_3(1+i)) = \frac{35\zeta(3)}{64} + \frac{1}{32}\pi^2 \log(2), \Re\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right) = \frac{35\zeta(3)}{64} + \frac{\log^3(2)}{48} - \frac{5}{192}\pi^2 \log(2)$$

$$\Im\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right) + \Im(\mathrm{Li}_3(1+i)) = \frac{7\pi^3}{128} + \frac{3}{32}\pi \log^2(2)$$

$$\Im(\mathrm{Li}_4(1+i)) - \Im\left(\mathrm{Li}_4\left(\frac{1+i}{2}\right)\right) = \frac{1}{64}\pi \log^3(2) + \frac{7}{256}\pi^3 \log(2)$$

See appendix 4(2) for a pair of nontrivial evaluation. They are also used in QLI's simplification.

Brute force. By writing $\log\left(x^2+1\right), \tan^{-1}(x)$ as real/imaginary part of $\log(1+ix)$, all QLIs with weight 2 or 3 can be decomposed into several integrals of form $\int \frac{\log(a+x)\log(b+x)}{c+x}\,dx$ over interval (0,1), all whose primitives are polylog-expressible according to (the beginning part of) section 1.

Moreover, $\int_0^1 \frac{\log^n(1\pm x)}{x^2+1}\,dx, \int_0^1 \frac{x\log^n(1\pm x)}{x^2+1}\,dx$ can be solved by substituting $t = 1 \pm x$ and calculating primitives either, which actually belongs to a larger class of QLI relations generated by brute force. In fact, due to section 1 again, all integrals of form $\int \frac{\log^2(A(x))\log(B(x))}{A(x)}\,dx, \int \frac{\log^2(A(x))\log(B(x))}{B(x)}\,dx$ and $\int \frac{\log^n(A(x))}{B(x)}\,dx$ are trivially evaluated by obtaining primitives directly, where $A(x), B(x)$ belong to the monomial set $\{x, 1 \pm x, 1 \pm ix, x \pm i\}$ and logs are on their principle branches. Consider all possible choices of A, B then take real/imaginary part and simplify, we get QLI relations; see appendix 4(2) for a weight 5 example. Some other QLIs (and QLSIs below) are also conquered this way but we omit those non-typical cases. All these procedures can be done without much effort with the help of CAS.

General formulas. It is apparent that the following integrals are evaluable in a general sense, that is, expressible in terms of Zeta, Eta, (Dirichlet) Beta and polylog constants:

$$\int_0^1 \frac{\log^n(x)}{x^2+1}\,dx, \int_0^1 \frac{x\log^n(x)}{x^2+1}\,dx, \int_0^1 \frac{x\log^n\left(x^2+1\right)}{x^2+1}\,dx, \int_0^1 \frac{\tan^{-1}(x)^n}{x^2+1}\,dx$$

For former two we simply expand their denominator into Taylor series, then plug in (1.1.1). For latter two their primitives are obvious. Here are some more complicated examples.

$$\int_0^1 \frac{\log^n\left(x^2+1\right)}{x}\,dx, \int_0^1 \frac{x\log(x)\log^n\left(x^2+1\right)}{x^2+1}\,dx, \int_0^1 \frac{\log\left(x^2+1\right)\log^n(x)}{x}\,dx$$

$$\int_0^1 \frac{\log^2\left(x^2+1\right)\log^{2n}(x)}{x}\,dx, \int_0^1 \frac{x\log\left(x^2+1\right)\log^{2n-1}(x)}{x^2+1}\,dx$$

The substitution $t=x^2$ and integration by parts reduce them to former four, or general formulas recorded in section 1-1. Interested readers may write down explicit formulas themselves.

Now we present 3 nontrivial formulas independent of section 1. Denote

$$f(s)=\frac{1}{4}\pi\sec\left(\frac{\pi s}{2}\right)\left(\psi^{(0)}\left(1-\frac{s}{2}\right)-\psi^{(0)}\left(\frac{1}{2}\right)\right)$$

$$g(s)=2\beta\left(\frac{1-s}{2}\right)\csc\left(\frac{\pi s}{2}\right)+2\beta\left(\frac{2-s}{2}\right)\sec\left(\frac{\pi s}{2}\right)-2\pi\csc(\pi s)$$

Then we have

$$\int_0^1 \frac{\tan^{-1}(x)\log^{2n+1}(x)}{x^2+1}\,dx=\frac{1}{2}\lim_{s\to 0}\frac{\partial^{2n+1}f(s)}{\partial s^{2n+1}}-\frac{1}{4}\pi(2n+1)!\beta(2n+2) \tag{4.2.3}$$

$$\int_0^1 \frac{x\tan^{-1}(x)\log^{2n}(x)}{x^2+1}\,dx=\frac{1}{2}(2n)!\beta(2n+2)+\frac{\pi\left(1-2^{-2n}\right)(2n)!\zeta(2n+1)}{2^{2n+3}}-\frac{1}{2}\lim_{s\to 0}\frac{\partial^{2n}f(s)}{\partial s^{2n}} \tag{4.2.4}$$

$$\int_0^1 \frac{\tan^{-1}(x)\log^{2n}(x)}{x+1}\,dx=\frac{1}{2}(2n)!\beta(2n+2)+\frac{1}{4}\pi\left(1-2^{-2n}\right)(2n)!\zeta(2n+1)-\frac{1}{16}\pi\lim_{s\to 0}\frac{\partial^{2n}g(s)}{\partial s^{2n}} \tag{4.2.5}$$

These formulas arise from the Mellin transform of $\frac{\tan^{-1}(x)}{x+1}, \frac{\tan^{-1}(x)}{x^2+1}$, we take $\frac{\tan^{-1}(x)}{x^2+1}$ as the example. Let $M(s)=\int_0^\infty \frac{x^{s-1}\tan^{-1}(x)}{x^2+1}\,dx, J(a,s)=\int_0^\infty \frac{x^{s-1}\tan^{-1}(ax)}{x^2+1}\,dx$, by Feynman's trick we have

$\frac{\partial J}{\partial a}=\int_0^\infty \frac{x^s}{(x^2+1)(a^2x^2+1)}\,dx=\frac{\pi\left(1-a^{1-s}\right)\sec\left(\frac{\pi s}{2}\right)}{2(1-a^2)}$ according to substitution $t=x^2$, partial fraction decomposition and Beta's representation on real axis (see proof of (1.4.6)). Hence $M(s)=f(s)$ is readily deduced if we integrate this expression on $a\in(0,1)$, substitute $t=a^2$ again, then plug in the classical result $\int_0^1 \frac{x^u-x^v}{1-x}\,dx=\psi^{(0)}(v+1)-\psi^{(0)}(u+1)$ [8]. Now differentiate with respect to the parameter s even/odd times to generate log terms, separate the real axis by 1, substitute $t=\frac{1}{x}$ to settle the log-inverse tangent integral in $(0,1)$. This yields former two formulas since all other terms are on the list above and can be expressed via ζ,η,β. The last one is similar and we refer readers to Valean [10] for details. Remaining work (i.e. calculation of derivatives of f,g) is covered in section 1-4, again, CAS is used in order to save time.

Integration by parts. Similar to section 1-2, apply N-L formula to

$$F=\log^a(1-x)\log^b(x)\log^{c_1}(x+1)\log^{c_2}(\frac{x+1}{2})\log^d\left(x^2+1\right)\tan^{-1}(x)^e$$

then vary parameters.

Fractional linear transformation. Similar to section 1-3, for each QLI, apply substitution $t=\frac{1-x}{x+1}$. Note that $\tan^{-1}(x)+\tan^{-1}\left(\frac{1-x}{x+1}\right)=\frac{\pi}{4}$ and $\log\left(\left(\frac{1-x}{x+1}\right)^2+1\right)=\log\left(\frac{2\left(x^2+1\right)}{(x+1)^2}\right)$ holds, we may split the new integrand to QLIs again.

Power substitution. For each LI, substitute $x=t^2$, then expand $\log^n(1-t^2)$ via binomial theorem (and possibly decompose $\frac{t}{1-t^2}$) to obtain QLI relation.

Weierstrass substitution. For each LSI, substitute $x=2\tan^{-1}(t)$. Note that $\sin(x),\cos(x)$ are transformed into $\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}$ respectively and $dx=2\frac{dt}{1+t^2}$, it is possible to expand the logs (and decompose denominator if needed) to obtain QLI relation.

Trigonometric substitution. By substituting $x=\tan(t)$ we have that $\int_0^1 \frac{\log^m(x)\log^n\left(x^2+1\right)}{x^2+1}\,dx=(-2)^n\int_0^{\frac{\pi}{4}}\log^m(\tan(t))\log^n(\cos(t))\,dt$. Here we introduce analogue of QLIs on trigonometric case, that is, logsine integrals on the interval $(0,\frac{\pi}{4})$ with integrands unchanged, i.e. QLSIs. Now, for m,n small, we may evaluate this QLSI by brute force. In other words, write $\log(\tan(t))$ as $\log(\sin(t))-\log(\cos(t))$, expand the integrand, deform contour like in section 4-1 to convert them to complex log integrals (on quarter of unit arc from 1 to i), then calculate polylog primitives with help of CAS to apply N-L formula.

Here is another method. Start from a integral belongs to the Arctan family in the end of section 4-1, map the $(1,\infty)$ part to (0,1) via $t=\frac{1}{x}$, after routine simplification we obtain a QLI relation. This is related to trig-substitution because the Arctan family is solved by making use of the same transformation $x = \tan(t)$.

Some integrals slightly different from the Arctan family can be treated as QLI relation generator, though no trig-substitutions are used. For instance, since by contour integration we have $\int_0^\infty \frac{\log^2(x)}{(x^2+1)(a+x)}\,dx = \frac{\pi^2 a}{8(a^2+1)} - \frac{\log^3(a)}{3(a^2+1)} - \frac{\pi^2\log(a)}{3(a^2+1)}$, a integration w.r.t a on (0,1) and series expansion on denominators give closed-form to $\int_0^\infty \frac{\log(x+1)\log^2(x)}{x^2+1}\,dx$. Now do the same as last paragraph. Readers may give alternate examples.

We may also solve $\int_0^1 \frac{\tan^{-1}(x)^n}{x}\,dx$ and $\int_0^1 \frac{x\tan^{-1}(x)^n}{x^2+1}\,dx$ by trigonometric transformations. The former one becomes $\int_0^{\frac{\pi}{4}} t^n(\tan(t)+\cot(t))\,dt$ after substitution, now an integration by parts reduce them to the evaluation of $\int_0^{\frac{\pi}{4}} t^{n-1}\log(\sin(t))\,dt$ and$\int_0^{\frac{\pi}{4}} t^{n-1}\log(\cos(t))\,dt$, which is completely solved by Fourier expansion in section 4-1. The latter one boils down to $\int_0^{\frac{\pi}{4}} t^n\tan(t)\,dt$ either.

Beta derivatives. Similar to section 1-4, we know that all integrals of form $\frac{\log^m(x)\log^n\left(1-x^4\right)}{g(p;x)}$ $(p=1,2,3,4,5)$ are expressible via Beta derivatives, since all possible denominators of QLI integrand are included in $1-x^4 = (1-x)(x+1)\left(1-x^2\right)$. For instance, when $p=3$ we write $\frac{1}{x+1} = \frac{-x^3+x^2-x+1}{1-x^4}$, substitute $x^4 = t$ then split it into Beta integrals. On the other hand, we may expand $\log^n(1-x^4)$ via trinomial formula to reduce it to QLIs. Now consider all 5 cases and vary m,n to obtain QLI relations.

Another generalization of section 1-4 is to multiply both sides of $\frac{\log\left(x^2+1\right)}{1\mp x} = \frac{\left(x^3\pm x^2\pm x+1\right)\log\left(1-x^4\right)}{1-x^4} - \frac{(\pm x+1)\log\left(1-x^2\right)}{1-x^2}$ with $\log^m(x)$ and integrate. Therefore the pair $\int_0^1 \frac{\log\left(x^2+1\right)\log^m(x)}{1\mp x}\,dx$ can be evaluated by Beta derivatives by 2 power substitutions on RHS.

The third generalization is to substitute $t = \frac{x}{1\pm x}$ for $\int_0^1 \frac{\log^n\left(\frac{x}{1\pm x}\right)}{x^2+1}\,dx$ and $\int_0^1 \frac{x\log^n\left(\frac{x}{1\pm x}\right)}{x^2+1}\,dx$. After substitution these integrals are of form $\int_0^{\frac{1}{2}} R(t)\log^n(t)\,dt$ or $\int_0^\infty R(t)\log^n(t)\,dt$ (R rational), all of which can be evaluated by calculating their (complex) polylog primitives. However, they cannot be expressed in Beta derivatives explicitly. A little analysis shows that this doesn't work for denominator x or $1-x$ due to convergence issues.

The fourth generalization is

$$\int_{-1}^1 \frac{(t+1)^{2x-1}(1-t)^{2y-1}\log^m\left(\frac{(t+1)^2}{t^2+1}\right)\log^n\left(\frac{(1-t)^2}{t^2+1}\right)}{(t^2+1)^{x+y}}\,dt = \frac{\partial^{m+n}\left(2^{x+y-2}B(x,y)\right)}{\partial x^m\,\partial y^n} \qquad (4.2.6)$$

This can be shown by Beta's definition, substitution $x = \frac{1+2t+t^2}{2(1+t^2)}$ and differentiation w.r.t x,y. Now let x,y tend to $\frac{1}{2}$, fold the interval into $(0,1)$ via reflection then vary m,n to obtain relations. This is an interesting technique similar to (1.4.7).

Contour integration. Similar to section 1-5, consider all 10 kinds of keyhole contour, and all choices of suitable range of log's argument (such that all branch cuts lie out of the contour), for 2 types of modified log kernels

$$F_1(z)=\frac{l(0;z)^k l(1;z)^{m_1} l(1;-z)^{m_2} l(2;z)^n}{1+z^2}, F_2(z)=\frac{F_1(z)}{z}$$

Then vary parameters and use the residue theorem. Map all to (0,1) (and do partial fraction decomposition for F_2 type) yields QLI relations. Note that $\pm i$ are simple poles and their residues are unique according to choices of the branch cuts.

A even more general version of complex integrand is $F(z)=\frac{\prod_{k=1}^{N}\log(A_k(x))}{\prod_{l=1}^{M}B_l(x)}$, where all $A_k, B_l \in \{\pm x, \pm ix, 1\pm x, 1\pm ix, x\pm i\}, M>1$, and all branch cuts of logs lie in real/imaginary axis. Consider all types of contour (consist of large full/semi/quarter-circle on the plane, with possible keyholes and indents on two axes), all choices of log's branch cut, vary N, M, A_k, B_l then apply residue theorem, discard those divergent on small arcs. Finally, map all integrals on 2 axes to (0,1) and decompose to obtain non-homogeneous QLI relations.

Another variation is to consider contour $C=i\mathbb{R}^+\cup(0,1)\cup(i\mathbb{R}^++1)$ and $f=\frac{\log^m(z)\log^n(1-z)}{z(1-z)}$. After parametrization we have a Beta integral and 2 complex log integrals on $(0,\infty)$. By mapping the outer integral to (0,1) we get QLI relations since real/imaginary parts of $\log(1+ix)$ are 2 quadratic terms. Now vary m, n.

Series expansion. We naturally extend the procedure of multiple series expansion in section 3-5 here since Taylor formulas for $\log\left(x^2+1\right), \tan^{-1}(x), \frac{1}{x^2+1}, \frac{x}{x^2+1}$ are elementary. For each QLI, carry out this procedure and discard those Cauchy-irreducible multiple sums likewise. To deal with remaining single harmonic sums, apply F&S's systematic method [6] to see if they are directly solvable, or pull them back to QLIs via extended representations of integral/series kernel (generalizations of those in section 2-5 and 4-5) hence obtain QLI relation.

Now, based on all possible relations generated by these methods, all QLIs we aimed at are given closed-form and the solution is finalized. □

It is interesting to see, that all QLIs of weight W are generated by less than a half of elements in B_W. Take $W=4$, then 68 QLIs lie in the subspace generated by 8 constants in B_4 over $\mathbb{Q}$, in which those five in A_4 are included. On the contrary, the remaining 61 QLIs are completely expressible by other 8 constants in B_4. See appendix 4 for all QLIs with weight ≤ 4, some typical QLSIs and exotic polylog identities.

4-3. Non-homogeneous patterns

Now we present some non-homogeneous integrals. We define non-homogeneous log integrals (abbr. NLIs) as

$$NLI(a(0), a(1), a(2); k) = \int_0^1 x^k \prod_{m=0}^{2} l(m; x)^{a(m)}\, dx \tag{4.3.1}$$

Whenever the order $k \neq -1$ is an integer and the integral convergent. We have:

Proposition 9. All convergent NLIs (infinite-many, according to arbitrariness of order k) with weight $W = a(0) + a(1) + a(2) \leq 5$ can be generated by $\cup_{w=0}^{W} A_w$ over $\mathbb{Q}$.

Proof. This is elementary by using integration by parts repeatedly. Suppose k a positive integer, lift up x^k to $\frac{x^{k+1}}{k+1}$ (or $\frac{x^{k+1}-1}{k+1}$ sometimes, for convergence issues) and differentiate the log terms, after splitting and partial fraction decomposition on $\frac{x^{k+1}}{1\pm x}$ we arrive at a combination of several weight W LIs and weight $W-1$ NLIs. Now an induction completes the proof, and for negative k the technique is similar. $\square$

Moreover, this kind of variation is also valid for non-homo polylog integrals (abbr. NPLIs). For instance, we state that all 219 order zero NPLIs (count it) of form

$$\int_0^1 \prod_{m=0}^{2} l(m; x)^{a(m)} \prod_{k=1}^{6} \prod_{n=2}^{N} L(n, k; x)^{b(n,k)}\, dx$$

with weight $W = \sum_{k=1}^{6} \sum_{n=2}^{N} nb(n, k) + \sum_{m=0}^{2} a(m) = 5$ are generated by $\cup_{w=0}^{5} A_w$ over $\mathbb{Q}$. The method applied in proof of Prop, 9 remains available here. Readers may find the process of reducing them to PLIs boring but results attractive, since most of them are generated by over 10 polylog constants in $\cup_{w=0}^{5} A_w$, and some of them 20(=1+1+2+3+5+8).

Furthermore, we get non-homo quadratic log integrals (abbr. NQLIs) from QLIs, if we replace the denominator $g(p; x)$ in QLI definitions with a moment function x^k. By using exactly the same method, they have shown a pattern similar to NLIs/NPLIs, that weight W NQLIs can be generated by $\cup_{w=0}^{W} B_w$ for $W = 2, 3, 4$.

Last but not least, we deal with non-homo logsine integrals (abbr. NLSIs). They are of form $\int_0^{\frac{\pi}{2}} x^a \sin^m(x) \cos^n(x) \log^b(2\sin(x)) \log^c(2\cos(x))\, dx$, m, n integers, which arise from parametrization $x = e^{2it}$ in NLIs. Plenty of NLSIs can be evaluated by using 3 main methods recorded in section 4-1. For instance, by differentiating $\int_0^{\frac{\pi}{2}} x \sin(x) \cos^{p-1}(x)\, dx = \frac{\pi\Gamma(p+1)}{2^{p+1} p \Gamma\left(\frac{p}{2}+1\right)^2}$ [8] w.r.t p, we may evaluate $\int_0^{\frac{\pi}{2}} x \sin(x) \log^n(\cos(x))\, dx$; this is analogous to those LSIs. Partially they show non-homogeneous pattern. See appendix 4 for some typical NLIs, NPLIs, NQLIs, NLSIs and variations.

4-4. PLIs and ESs with other arguments

We may extend the range of PLIs to, for example:

$$\int_0^1 \frac{\prod_{m=0}^{2} l(m;x)^{a(m)} \prod_{k=1}^{10} \prod_{n=2}^{N} L(n,k;x)^{b(n,k)}}{f(p;x)}\, dx$$

In which 4 new polylog terms $L(n,7;x),\cdots,L(n,10;x)$ stand for $\mathrm{Li}_n\left(x^2\right), \mathrm{Li}_n(1-x), \mathrm{Li}_n\left(\frac{x}{x-1}\right)$ and $\mathrm{Li}_n\left(\frac{x}{x+1}\right)$ respectively. With the definition of weight of PLIs trivially extended, Theorem 1 also hold for all these new PLIs with weight ≤ 5. This is apparent since we can apply formula (3.4.1)-(3.4.3) (possibly a substitution $t=-x$ in the last one) to reduce these new PLIs to ordinary ones, which are solved completely in section 3. Moreover, by applying series expansion on these new terms, we may obtain closed-forms of some new ESs. For instance, by expanding $\mathrm{Li}_2\left(\pm x^2\right)$ in some new PLIs, plenty of new ESs containing H_{2n} and its alternating analogue can be evaluated, for instance:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(H_n)^2 H_{2n}}{n^2} &= -16\mathrm{Li}_5\left(\frac{1}{2}\right) - 16\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) + \frac{3\pi^2\zeta(3)}{8} \\ &+ \frac{421\zeta(5)}{16} - 7\zeta(3)\log^2(2) - \frac{1}{15}8\log^5(2) + \frac{4}{9}\pi^2\log^3(2)\end{aligned} \tag{4.4.1}$$

Readers may give further generalizations.

4-5. Restricted sums and multiple log integrals

The evaluation of MZVs is the central topic in numerous literatures, for instance [5, 6, 16]. Shuffle relations, duality, partial fraction decomposition and residue methods are frequently used as obtaining their closed-forms. Nevertheless, we won't discuss about systematic evaluation of MZVs, but only show some examples on connection between restricted MZV-like sums and known ESs.

Consider a general n-ple sum $\sum_R \prod_{i=1}^{n} f_i(k_i)$, where for all i, $f_i(k) = \frac{(\pm 1)^{k-1}}{k^{s_i}}$, s_i natural numbers, and R some restrictions on n positive summation indexes. If R is empty (i.e. no restrictions between indexes given), the sum is simply a Zeta–Eta product. If R: indexes are completely ordered in a '$<$' (resp. '$\leq$') chain, we get MZVs (resp. MZSVs). If R: one of the indexes is always not smaller than all others, we get ESs. For other restrictions R, we get some restricted MZV-like sums (abbr. RSs), for instance:

$$RS_1 = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} H_m \sum_{n=1}^{m} \frac{H_n}{n^2}}{m} = \sum_{1\leq n\leq m, 1\leq q\leq p\leq m} \frac{(-1)^{m-1}}{mnp^2 q}$$

Some of PLIs that are not able to be calculated directly by multiple expansion (i.e. discarded in the algorithm in section 3-5), but nontrivially evaluated by solving the system of all relations obtained in section 3, can be used here to give closed-forms to several RSs. For instance, by expansion via (2.5.2), (2.5.5), (2.5.17) and Cauchy simplification, we have:

$$\int_0^1 \frac{\mathrm{Li}_2(-x)\log^2(1-x)}{x(x+1)}\,dx = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^{j+k-1}\widetilde{H_j}H_{j+k}}{k^2(j+k)} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}H_m \sum_{k=1}^{m-1} \frac{\widetilde{H_{m-k}}}{k^2}}{m}$$

The LHS integral is easily reduced to 2 PLIs via partial fraction decomposition, hence the value of sum on RHS is also known. To simplify $\sum_{k=1}^{m-1} \frac{\widetilde{H_{m-k}}}{k^2}$, rewrite it into a double sum, do finite Cauchy simplification again. Using the fact that $\sum_{k=1}^{m-1} \frac{(-1)^{m-k}}{k^2(m-k)}$ is expressible in alternating harmonic numbers (carry it out), and the symmetric identity: $\sum_{n=1}^{m}(-1)^{n-1}\left(\frac{\widetilde{H_n}}{n^2}+\frac{\widetilde{H_n^{(2)}}}{n}\right) = \widetilde{H_n}\widetilde{H_n^{(2)}}+H_n^{(3)}$, we know that $\sum_{k=1}^{m-1} \frac{\widetilde{H_{m-k}}}{k^2} = \sum_{n=1}^{m} \frac{\widetilde{H_n}}{n^2} - \widetilde{H_m}\widetilde{H_m^{(2)}} - \widetilde{H_m^{(3)}} + H_m^{(3)}$. Now the known RHS is decomposed into 1 RS and 3 weight 5 ESs, namely $ES(1,\pm 3;-1), ES(1,-1,-2;-1)$. Plug in ES values, we obtain the closed-form of this RS:

$$\begin{aligned} RS_2 = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}H_m \sum_{n=1}^{m} \frac{\widetilde{H_n}}{n^2}}{m} &= -10\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{\pi^2\zeta(3)}{96} + \frac{277\zeta(5)}{32} \\ &+ \frac{9}{16}\zeta(3)\log^2(2) + \frac{\log^5(2)}{12} - \frac{1}{72}\pi^2\log^3(2) - \frac{1}{18}\pi^4\log(2) \end{aligned} \tag{4.5.1}$$

The sum RS_1 can be evaluated similarly if we consider expanding $\int_0^1 \frac{\mathrm{Li}_2(-x)\log(1-x)\log(1+x)}{x(x+1)}\,dx$ instead.

Symmetric relations are also helpful in solving restricted sums. Recall that for ESs we have I-E relations (section 2-2) and for MZVs shuffle and duality formulas [6]. Analogously, plenty of relations between RSs are deduced from outer and inner symmetry. Generally we have:

$$\sum_{m=1}^{\infty} g(m)\sum_{n=1}^{m} f(n) + \sum_{m=1}^{\infty} f(m)\sum_{n=1}^{m} g(n) = \sum_{m=1}^{\infty} f(m)\sum_{m=1}^{\infty} g(m) + \sum_{m=1}^{\infty} f(m)g(m) \tag{4.5.2}$$

$$\sum_{m=1}^{\infty} f(m)\sum_{n=1}^{m} g(n)\sum_{k=1}^{n} h(k) + \sum_{m=1}^{\infty} f(m)\sum_{n=1}^{m} h(n)\sum_{k=1}^{n} g(k) = \sum_{m=1}^{\infty} f(m)\left(\sum_{n=1}^{m} g(n)\sum_{n=1}^{m} h(n) + \sum_{n=1}^{m} g(n)h(n)\right) \tag{4.5.3}$$

Both of them are apparent according to section 2-2. Now apply 2 formulas and n-ple sum generalizations with all $f, g, h, \ldots$ arbitrary harmonic terms and repeat alternatively, we obatin relations between RSs, for example, all weight 5 RSs of form $\sum_{m=1}^{\infty} f(m)\sum_{n=1}^{m} \frac{H_n}{n}, \sum_{m=1}^{\infty} f(m)\sum_{n=1}^{m}(-1)^{n-1}\frac{\widetilde{H_n}}{n}$

with f harmonic term can be reduced to ESs directly via inner symmetric identity $\sum_{n=1}^{m}\frac{H_n}{n}=\frac{1}{2}\left((H_m)^2+H_m^{(2)}\right)$ and its alternating analogue, while $\sum_{m=1}^{\infty}\frac{\widetilde{H_m}\sum_{n=1}^{m}\frac{(-1)^{n-1}H_n}{n}}{m^2}$ can be evaluated via outer symmetry identity, with the help of value of weight 5 ESs and RS_2 above. However, based upon our observation the inner structure of RSs is far more complicated than ESs, which means expanding PLIs and using symmetry cannot provide us enough relations to evaluate all weight 5 RSs in closed-forms (by weight we mean $\sum_{i=1}^{n}s_i$ in the restricted sum definition). We only present some result on RSs obtained by methods above in appendix 4.

Armed with ES/RSs we already known, we can evaluate plentiful multiple integrals using simple formulas, namely (2.5.16), (2.5.17). We take RS_2 as an example. Since $\sum_{n=1}^{m}\frac{\widetilde{H_n}}{n^2}=\int_0^1\frac{H_m^{(2)}-\sum_{n=1}^{m}\frac{(-x)^n}{n^2}}{x+1}dx$, with help of (2.5.17), (2.5.5) and repeated use of Fubini theorem we have:

$$
\begin{aligned}
RS_2&=\int_0^1\int_0^1\frac{\log(1-y)\left(\sum_{m=1}^{\infty}H_m^{(2)}(-y)^m-\sum_{m=1}^{\infty}\sum_{n=1}^{m}\frac{(-y)^m(-x)^n}{n^2}\right)}{(x+1)y}dxdy\\
&=\int_0^1\int_0^1\frac{\log(1-y)(\mathrm{Li}_2(-y)-\mathrm{Li}_2(-xy))}{(x+1)y(y+1)}dxdy
\end{aligned}
$$

For ESs or deeper MZVs the procedure is similar, that is, use (2.5.16) and (2.5.17) to replace elementary (i.e. rational $\frac{1}{k^n}$ and harmonic $\frac{H_k}{k}$) terms into log integrals and exchange order repeatedly (and reasonably due to convergence theorems). An advantage of representing RSs as iterated LI/PLIs is that integrals, compared to alternating sums, are easier to be approximated numerically by CAS. We've verified the correctness of closed-forms of 7 ESs evaluated in section 2-5 and extra RSs through their integral representations. In appendix 4 some typical multiple log integrals are also recorded. Readers who know MZV theory well may observe the similarity between these multiple LIs and MZVs' classical integral representation [16].

4-6. Some exotic applications

Based on closed-forms of LI/ES/PLIs we obtained, some exotic problems can be solved. We sketch their proofs briefly, details and possible generalizations are left to interested readers.

(1) LI convolution

Using the same technique in section 1-6, consider $\int_0^1\int_0^1\frac{\log^n((1-x)(1-y))}{(x+1)(y+1)}dxdy$. By binomial theorem and (1.1.5) we may express this integral in terms of polylogs. On the other hand, substitute $u=(1-x)(1-y)$ for y while remain x unchanged, integrate w.r.t x first, we find that the integral also equals to $\int_0^1\frac{(2\log(2-u)-\log(u))\log^n(u)}{4-u}du$. Now a reflection $x=1-u$ for the former part and

$\int_0^1 \frac{\log^{n+1}(u)}{4-u}\,du = (-1)^{n-1}(n+1)!\mathrm{Li}_{n+2}\left(\frac{1}{4}\right)$ for the latter yields the formula:

$$\int_0^1 \frac{\log(x+1)\log^n(1-x)}{x+3}\,dx = \frac{(-1)^n(n+1)!}{2}\left(\frac{1}{n+1}\sum_{k=0}^{n}\mathrm{Li}_{k+1}\left(\frac{1}{2}\right)\mathrm{Li}_{-k+n+1}\left(\frac{1}{2}\right) - \mathrm{Li}_{n+2}\left(\frac{1}{4}\right)\right) \tag{4.6.1a}$$

Formula (1.1.5) plays an important role in the proof above. Also, similar approaches based on (1.1.8) give some complicated but not so beautiful results. By using (1.1.8) we mean starting from $\int_0^1\int_0^1 \frac{\log(x+1)\log(y+1)(\log(x+1)\pm\log(y+1))^n}{xy}dxdy$ or $\int_0^1\int_0^1 \frac{\log(x+1)(\log(x+1)\pm\log(1-y))^n}{x(y+1)}dxdy$. For instance, evaluating $\int_0^1\int_0^1 \frac{\log(x+1)\log(y+1)\log(xy+x+y+1)^n}{xy}dxdy$ in 2 ways yields a log-moment integral:

$$\int_1^2 f(x)\log^m(x)\,dx + \int_2^4 g(x)\log^m(x)\,dx = m!\sum_{k=0}^{m}(k+1)(-k+m+1)J(k)J(m-k) \tag{4.6.1b}$$

Where

$$J(k) = -\sum_{j=0}^{k}\frac{\log^j(2)\mathrm{Li}_{-j+k+2}\left(\frac{1}{2}\right)}{j!} - \frac{\log^{k+2}(2)}{(k+2)k!} + \zeta(k+2)$$

$$f(x) = \frac{1}{6(x-1)}\left(12\mathrm{Li}_3\left(\frac{1}{x}\right) + 12\mathrm{Li}_3(x) - 6\mathrm{Li}_2(1-x)\log(x) + 6\mathrm{Li}_2\left(\frac{1}{x}\right)\log(x)\right.$$
$$\left.-12\mathrm{Li}_2(x)\log(x) - 6\log(x-1)\log^2(x) - 6i\pi\log^2(x) + \pi^2\log(x) - 24\zeta(3)\right)$$

$$g(x) = \frac{1}{12(x-1)}\left(24\mathrm{Li}_3\left(\frac{x}{2}\right) + 24\mathrm{Li}_3\left(\frac{2}{x}\right) - 24\mathrm{Li}_2\left(\frac{x}{2}\right)\log\left(\frac{x}{2}\right) - 12\mathrm{Li}_2\left(1-\frac{x}{2}\right)\log(x)\right.$$
$$+12\mathrm{Li}_2\left(\frac{2}{x}\right)\log\left(\frac{x}{4}\right) - 24\mathrm{Li}_3(2) + 12\mathrm{Li}_2(2)\log(4) + 6\log^2(2)\log\left(\frac{x}{2}\right) + 12i\pi\log^2(2)$$
$$\left.-12\log\left(1-\frac{x}{2}\right)\log^2\left(\frac{x}{2}\right) + 12\log(2)\log\left(\frac{x}{2}\right)\log\left(\frac{x-2}{x}\right) - 21\zeta(3) + 2\log^3(2)\right)$$

Functions F, G, which are generated by brute force with help of CAS, can be simplified via polylog identities recorded in section 3-4 and [9], and detailed simplifications and generalizations are left to the readers. Note that all integrals of this type share the structure of 'convolution'.

(2) A PLI master formula

Formula (3.4.1) is used on $L(n,1;x), L(n,2;x)$ for extending PLIs in section 4-4. Noticing that $L(n,5;x), L(n,6;x)$ have opposite argument either, we have the following general formula:

$$\int_0^1 F(x)\left(\mathrm{Li}_2\left(\frac{x-1}{x+1}\right) + \mathrm{Li}_2\left(\frac{1-x}{x+1}\right)\right)dx = \int_0^1 \frac{\left(1-\sqrt{1-x^2}\right)F\left(\frac{1-\sqrt{1-x^2}}{x}\right)\mathrm{Li}_2\left(\frac{1-x}{x+1}\right)}{2x^2\sqrt{1-x^2}}\,dx \tag{4.6.2}$$

Where F is chosen so that both sides are convergent. The proof is simple: LHS can be rewritten as $\frac{1}{2}\int_0^1 F(x)\mathrm{Li}_2\left(\frac{x^2-2x+1}{x^2+2x+1}\right)dx$ via (3.4.1), then a substitution $\frac{1}{t}=\frac{x^2+1}{2x}$ transforms this to RHS. By choosing appropriate F we can deduce some beautiful formulas using PLI/NPLI values on LHS, for instance, set $F=\frac{\mathrm{Li}_2\left(\frac{1-x}{x+1}\right)}{x+1}$ then plug in closed-forms of $PLI(0,0,0;25,25;2), PLI(0,0,0;25,26;2)$ recorded in appendix 3 gives:

$$\begin{aligned}\int_0^1 \frac{\left(1-\sqrt{1-x^2}\right)\mathrm{Li}_2\left(\frac{1-x}{x+1}\right)\mathrm{Li}_2\left(\frac{x+\sqrt{1-x^2}-1}{x-\sqrt{1-x^2}+1}\right)}{x\sqrt{1-x^2}\left(-\sqrt{1-x^2}+x+1\right)}dx &= 8\mathrm{Li}_5\left(\frac{1}{2}\right)+8\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2)\\ -\frac{13\pi^2\zeta(3)}{16}-\frac{\zeta(5)}{4}+\frac{7}{2}\zeta(3)\log^2(2)+\frac{4\log^5(2)}{15}-\frac{2}{9}\pi^2\log^3(2)+\frac{1}{36}\pi^4\log(2)\end{aligned} \tag{4.6.3}$$

(3) Exotic multiple sum

In section 3-5 we only expand PLIs containing $L(n,k;x), k=1,2,3$ into multiple series. If we deal with expansions on other $L(n,k;x)$, some exotic sums can be evaluated. For example:

$$\begin{aligned}\sum_{k=1}^\infty\sum_{j=1}^\infty\sum_{n=0}^j \frac{(-1)^n(j-1)!}{j2^kk^2(k+n)(j-n)!(2n)!!} &= 3\mathrm{Li}_5\left(\frac{1}{2}\right)+3\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2)-\frac{81\zeta(5)}{64}\\ -\frac{7\pi^2\zeta(3)}{96}+\frac{21}{16}\zeta(3)\log^2(2)+\frac{\log^5(2)}{20}-\frac{1}{18}\pi^2\log^3(2)-\frac{1}{144}\pi^4\log(2)\end{aligned} \tag{4.6.4}$$

By expanding 2 Li_2 terms simultaneously in $PLI(0,0,0;23,24;0)=\int_0^1\frac{\mathrm{Li}_2\left(\frac{1-x}{2}\right)\mathrm{Li}_2\left(\frac{x+1}{2}\right)}{1-x}dx$, we get a double sum with the summand a rational integral. Now substitute $t=1-x$, use binomial theorem on $(2-t)^k$ then simplify to finish the proof. Many more interesting exotic identity can be obtained if we apply direct expansion on PLIs.

(4) High weight PLI

Consider 4 integrals

$$\int_0^1\frac{\mathrm{Li}_3(x)\log(x)\log(1-x)}{x}dx, \int_0^1\frac{\mathrm{Li}_3(x)\log(x)\log(1-x)}{1-x}dx$$

$$\int_0^1\frac{\mathrm{Li}_3(x)\log^2(x)}{1-x}dx, \int_0^1\frac{\mathrm{Li}_3(x)\log^2(1-x)}{x}dx$$

Using methods introduced in section 3-5 (or MZV theory), it's not difficult to check that they are all reducible to zeta values. By manipulating (3.4.7) and (3.4.8) we know that $\mathrm{Li}_3\left(1-\frac{1}{x}\right)$ is expressible in $\mathrm{Li}_3(x), \mathrm{Li}_3(1-x)$ and log terms. Similar to section 3-4 we have 4 corresponding integral $\int_0^1\frac{\mathrm{Li}_3\left(1-\frac{1}{x}\right)\log(x)\log(1-x)}{x}dx$ also zeta expressible. Substitute $-t=1-\frac{1}{x}$, we arrive at 4 integrals on $(0,\infty)$ with integrands composed of $\mathrm{Li}_3(-x)$ and logs. Combining 2 of them we get that $\int_0^\infty \frac{\mathrm{Li}_3(-x)\log(x+1)\log\left(\frac{1}{x}+1\right)}{x}\,dx = -2\zeta(3)^2-\frac{31\pi^6}{5670}$. Now separate $(0,\infty)$ into $(0,1),(1,\infty)$, substitute $t=\frac{1}{x}$ in the latter part, use (3.4.7) again and plug in weight 6 LI values [1], we readily arrive at:

$$\begin{aligned}&\int_0^1 \frac{\mathrm{Li}_3(-x)\log(x+1)\log\left(\frac{1}{x}+1\right)}{x}\,dx = -ES(1;-5)+\frac{1}{3}\pi^2\mathrm{Li}_4\left(\frac{1}{2}\right)\\&+\frac{7}{24}\pi^2\zeta(3)\log(2)-\frac{137\pi^6}{90720}+\frac{1}{72}\pi^2\log^4(2)-\frac{1}{72}\pi^4\log^2(2)-\zeta(3)^2\end{aligned}\tag{4.6.5}$$

(5) Mixed harmonic sum

Consider the integral $\int_0^1 \frac{\log(x)\log\left(1-x^2\right)\sin^{-1}(x)^4}{\sqrt{1-x^2}}\,dx$. Expand the power of $\sin^{-1}$ into Taylor series using Borwein's formula $\sin^{-1}(x)^4=\frac{3}{2}\sum_{n=1}^\infty \frac{H_{n-1}^{(2)}(2x)^{2n}}{n^2\binom{2n}{n}}$ [4], exchange the order of integration and summation, substitute $t=x^2$. To evaluate the inner integral in closed-form, apply identities of Beta derivatives (see section 1-4 for general case), namely $\psi^{(0)}(n+1)=H_n-\gamma$ and $\psi^{(0)}\left(n+\frac{1}{2}\right)=2\left(-H_n+H_{2n}-\log(2)\right)$. Therefore we've completely transformed the integral to a mixed harmonic sum. On the other hand, by a simple substitution $x=\sin(t)$ we have the integral also equals to $2\int_0^{\frac{\pi}{2}} t^4(\log(2\sin(t))-\log(2))(\log(2\cos(t))-\log(2))\,dt$, which can be reduced to several LSIs with weight at most 6. Now general method in section 4-1 and Au's results [1] lead to the following formula:

$$\begin{aligned}&\sum_{n=1}^\infty \frac{H_{n-1}^{(2)}\left(2\left(H_n-H_{2n}+\log(2)\right)\left(H_n+2\log(2)\right)+H_n^{(2)}-\frac{\pi^2}{6}\right)}{n^2}=-4ES(1;-5)-\frac{4}{3}\pi^2\mathrm{Li}_4\left(\frac{1}{2}\right)\\&-\frac{15\zeta(3)^2}{8}-\pi^2\zeta(3)\log(2)-\frac{1}{4}\zeta(5)\log(2)+\frac{281\pi^6}{15120}-\frac{1}{18}\pi^2\log^4(2)+\frac{4}{45}\pi^4\log^2(2)\end{aligned}\tag{4.6.6}$$

(6) Inverse ES and multiple integral

Some 'inverse ESs' admit a special pattern, where by inverse we mean the summand contains something like $\zeta(k)-H_n^{(k)}$ or $\eta(k)-\widetilde{H_n^{(k)}}$, that is, the tail of generalized harmonic sums.

Consider the convergent inverse ES $\sum_{n=1}^\infty (-1)^n\left(\log(2)-\widetilde{H_n}\right)^5$. By Abel's summation by parts (lift up $(-1)^n$ by partial summation and take difference of the other part) we have it equals to:

$$\sum_{n=1}^\infty \frac{1}{2}\left((-1)^{n-1}-1\right)\left(\left(\frac{(-1)^{n-1}}{n}+\log(2)-\widetilde{H_n}\right)^5-\left(\log(2)-\widetilde{H_n}\right)^5\right)$$

Expand the RHS, we find that all terms are weight 5 ESs expect another inverse ES $\sum_{n=1}^\infty \frac{\left(\log(2)-\widetilde{H_n}\right)^4}{n}$. Sum by parts again and split the summand, the new sum can be reduced to ESs either. Therefore we have:

$$\begin{aligned}\sum_{n=1}^\infty(-1)^n\left(\log(2)-\widetilde{H_n}\right)^5 &= 10\mathrm{Li}_5\left(\frac{1}{2}\right)+10\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2)\\ &-\frac{157\zeta(5)}{16}+\frac{35}{8}\zeta(3)\log^2(2)-\frac{2\log^5(2)}{3}-\frac{5}{18}\pi^2\log^3(2)\end{aligned}\tag{4.6.7}$$

Using the same method we are able to deduce that:

$$\sum_{n=0}^\infty\left(\log(2)-\widetilde{H_n}\right)^5=-20\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{45}{4}\zeta(3)\log(2)+\frac{259\pi^4}{1440}+\frac{5\log^4(2)}{3}+\frac{5}{12}\pi^2\log^2(2)\tag{4.6.8}$$

The reason why a 'quintic' inverse ES yields a 'quartic' weight 4 value is that the existence of n after summation by parts once. Note that $\int_0^1\frac{x^n}{x+1}\,dx=(-1)^n\left(\log(2)-\widetilde{H_n}\right)$, combine 2 results above and apply an expansion, we conclude:

$$\begin{aligned}\sum_{n=0}^\infty\left(\log(2)-\widetilde{H_{2n}}\right)^5 &= \int_{(0,1)^5}\frac{dudvdxdydz}{(u+1)(v+1)(x+1)(y+1)(z+1)(1-(uvxyz)^2)}\\ &= -10\mathrm{Li}_4\left(\frac{1}{2}\right)+5\mathrm{Li}_5\left(\frac{1}{2}\right)+5\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2)-\frac{157\zeta(5)}{32}+\frac{35}{16}\zeta(3)\log^2(2)\\ &-\frac{45}{8}\zeta(3)\log(2)+\frac{259\pi^4}{2880}+\frac{\log^5(2)}{6}+\frac{5\log^4(2)}{6}-\frac{5}{36}\pi^2\log^3(2)+\frac{5}{24}\pi^2\log^2(2)\end{aligned}\tag{4.6.9}$$

(7) Parseval LSI

Generating functions recorded in section 2-5 may take part in some surprising identities. For instance we have:

$$\begin{aligned}&\int_0^{2\pi}\left(\left(\frac{1}{12}\pi^2f(x)-\frac{1}{3}f(x)^3+\frac{1}{2}(x-\pi)g(x)+h(x)\right)^2\right.\\ &\left.+\left(\frac{(\pi-x)}{2}f(x)^2-f(x)g(x)+\frac{\pi^2(\pi-x)}{24}\right)^2\right)dx=6\pi\zeta(3)^2+\frac{979\pi^7}{11340}\end{aligned}\tag{4.6.10}$$

Where $f(x)=\log\left(2\sin\left(\frac{x}{2}\right)\right), g(x)=\mathrm{Cl}_2(x), h(x)=\mathrm{Cl}_3(x)$ are logsine and Clausen functions. This is a direct consequence of Parseval identity applied to $f(x)=\sum_{n=1}^\infty\frac{(H_n)^2e^{inx}}{n}$ on $(0,2\pi)$, formula (2.5.13) and the value of $ES(1,1,1,1;2)$ (easily deduced from section 3).

(8) Valean PLI

We generalize Valean's elegant formulas of PLIs with argument $\pm\frac{2x}{x^2+1}$ [12]. Denote

$$X = (-1)^k \left(1 - 2^{k-2n}\right) \zeta(-k+2n+1) + \sum_{j=0}^{-k+2n-1} \frac{(-1)^{j-1}\zeta(j+2)(i\pi)^{-j-k+2n-1}}{(-j-k+2n-1)!}$$

$$Y = \Re\left(\frac{X}{(2i)^{2n-k}}\right) - \frac{\log(2)\left(\frac{\pi}{2}\right)^{2n-k}}{(2n-k)!}$$

Then we have:

$$\int_0^1 \frac{\operatorname{Li}_2\left(\frac{2x}{x^2+1}\right)\log^{2n-2}(x)}{x}\,dx = (-1)^{n-1}(2n-2)!\sum_{k=0}^{2n-1}\frac{B_k(2\pi)^k Y}{k!} \tag{4.6.11}$$

$$\int_0^1 \frac{\operatorname{Li}_2\left(-\frac{2x}{x^2+1}\right)\log^{2n-2}(x)}{x}\,dx = (-1)^{n}(2n-2)!\sum_{k=0}^{2n-1}\frac{B_k(2\pi)^k(1-2^{1-k}) Y}{k!} \tag{4.6.12}$$

Here is the proof. Following Valean [12], by applying double integration, together with classical Fourier expansion of Poisson kernel and logsine functions, we have the formula:

$$\int_0^1 \frac{\operatorname{Li}_2\left(\frac{2x}{x^2+1}\right)\log^{2n-2}(x)}{x}\,dx = 2(-1)^{2n-1}(2n-1)!\int_0^{\frac{\pi}{2}} \log(\cos(t))\left(\sum_{k=1}^{\infty}\frac{\sin(kt)}{k^{2n-1}}\right)dt$$

Gradshteyn & Ryzhik [8] offers the Fourier expansion $\sum_{k=1}^\infty \frac{\sin(kt)}{k^{2n-1}} = \frac{(2\pi)^{2n-1}(-1)^n B_{2n-1}\left(\frac{t}{2\pi}\right)}{2(2n-1)!}$, where $B_n(x) = \sum_{k=0}^n B_k\binom{n}{k}x^{n-k}$ are Bernoulli polynomials and B_n Bernoulli numbers. Plugging in these results we find that:

$$\int_0^1 \frac{\operatorname{Li}_2\left(\frac{2x}{x^2+1}\right)\log^{2n-2}(x)}{x}\,dx = \frac{(2\pi)^{2n-1}(-1)^{n-1}}{2n-1}\sum_{k=0}^{2n-1}\frac{B_k\binom{2n-1}{k}\int_0^{\frac{\pi}{2}} t^{-k+2n-1}\log(\cos(t))\,dt}{(2\pi)^{-k+2n-1}}$$

Finally, by simple induction we have $\int \frac{\log(z+1)\log^n(z)}{z}\,dz = \sum_{k=0}^n \frac{(-1)^{k-1}n!\operatorname{Li}_{k+2}(-z)\log^{n-k}(z)}{(n-k)!}$. Integrate $\frac{\log(z+1)\log^n(z)}{z}$ along the upper half of unit circle (counter clockwise), parametrize $z = e^{2ix}$ just like section 4-1, it is direct that $\int_0^{\frac{\pi}{2}} x^n\log(2\cos(x))\,dx = \Re\left(\frac{Z}{(2i)^{n+1}}\right)$, where

$$Z = (-1)^{n-1}\left(1-2^{-n-1}\right)n!\zeta(n+2) + \sum_{j=0}^{n}\frac{(-1)^{j-1}n!(i\pi)^{n-j}\zeta(j+2)}{(n-j)!}$$

Making use of this general LSI result completes the proof of (4.6.11). (4.6.12)'s proof is exactly the same and we omit it.

(9) Hypergeometric sum

Recall the definition of generalized hypergeometric function

$$ {}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;z)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n z^n}{(b_1)_n\cdots(b_q)_n n!} $$

For $|z|<1$ and its unique analytic continuation outside, where $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol. Now consider $\int_0^1 \frac{\log^3(x)}{\sqrt{2-x^2}}\,dx=\int_0^{\frac{\pi}{4}}\left(\log(2\sin(t))-\frac{\log(2)}{2}\right)^3 dt$. The RHS is clearly combination of QLSIs which are known according to section 4-2 and appendix 4. Meanwhile, expanding $\frac{1}{\sqrt{2-x^2}}$ into power series and using (1.1.1) transforms the LHS into a generalized hypergeometric series. After simplification we arrive at:

$$ \begin{aligned} &{}_5F_4\left(\frac12,\frac12,\frac12,\frac12,\frac12;\frac32,\frac32,\frac32,\frac32;\frac12\right)=\frac12\sqrt{2}\Im(\mathrm{Li}_4(1+i))\\ &+\frac{1}{16}\pi\sqrt{2}\zeta(3)+\frac{1}{768}\pi^3\sqrt{2}\log(2)-\frac{\sqrt{2}\left(\psi^{(3)}\left(\frac14\right)-\psi^{(3)}\left(\frac34\right)\right)}{12288} \end{aligned} \tag{4.6.13} $$

If we consider $\int_0^1 \frac{\log^2(x)\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)}{\sqrt{2-x^2}}\,dx=\int_0^{\frac{\pi}{4}} t\left(\log(2\sin(t))-\frac{\log(2)}{2}\right)^2 dt$ instead and apply the same method above, we will have the following analogous result (note that $\frac{\sin^{-1}(x)}{\sqrt{1-x^2}}$ also has a simple hypergeometric power series by differentiating the well known $\sin^{-1}(x)^2=\frac12\sum_{n=1}^\infty \frac{(2x)^{2n}}{n^2\binom{2n}{n}}$):

$$ {}_5F_4\left(1,1,1,1,1;\frac32,2,2,2;\frac12\right)=2\pi\Im(\mathrm{Li}_3(1+i))+\frac{5\mathrm{Li}_4\left(\frac12\right)}{2}-\frac{11\pi^4}{144}+\frac{5\log^4(2)}{48}-\frac16\pi^2\log^2(2) \tag{4.6.14} $$

Finally, make an imaginary substitution in the $\sin^{-1}(x)^2$ series above we have $\sinh^{-1}\left(\sqrt{\frac{x}{8}}\right)^2=\frac12\sum_{n=1}^\infty\frac{(-1)^{n-1}x^n}{n^2 2^n\binom{2n}{n}}$. Multiply both sides by $\frac{\log^2(x)}{x}$ then integrate it on $(0,1)$. For LHS substitute $t=\sinh^{-1}\left(\sqrt{\frac{x}{8}}\right)$, for RHS exchange the order to use (1.1.1), we deduce $\sum_{k=1}^\infty\frac{(-1)^{k-1}}{k^5 2^k\binom{2k}{k}}=2\int_0^{\frac{\log(2)}{2}} x^2\coth(x)\log^2\left(8\sinh^2(x)\right)dx$. Therefore we only need to evaluate log-hyperbolic integrals $\int_0^{\frac{\log(2)}{2}} x^2\coth(x)\log^k(\sinh(x))\,dx$ for $k=0,1,2$, which is direct if we substitute $x=\frac12\log(y+1)$ to reduce them to known LIs. Rewrite the binomial sum into hypergeometric form and simplify, we have:

$$ \begin{aligned} &{}_6F_5\left(1,1,1,1,1,1;\frac32,2,2,2,2;-\frac18\right)=40\mathrm{Li}_5\left(\frac12\right)+24\mathrm{Li}_4\left(\frac12\right)\log(2)-38\zeta(5)\\ &+4\zeta(3)\log^2(2)+\frac{19\log^5(2)}{30}-\frac49\pi^2\log^3(2)+\frac{7}{45}\pi^4\log(2) \end{aligned} \tag{4.6.15} $$

(10) Symmetric PLI

We conclude by a simple but fascinating integral. Denote D the subset of $\mathbb{R}^2$ where $0 < x < 1, 0 < y < \frac{1-x}{x+1}$. Because the fractional transformation $f(x) = \frac{1-x}{x+1}$ is self-dual (i.e. $f(f(x)) = x$), we have the region D symmetric about the line $y = x$. Let D_1 denote the part of D that's above $y = x$, by symmetry and direct integration w.r.t y, we have $-\int_0^1 \frac{\mathrm{Li}_2\left(\frac{1-x}{x+1}\right)\log(1-x)}{x}\,dx = \int_D \frac{\log(1-x)\log(1-y)}{xy}dxdy = 2\int_{D_1} \frac{\log(1-x)\log(1-y)}{xy}dxdy = 2\int_0^{\sqrt{2}-1} \frac{\left(\mathrm{Li}_2(x)-\mathrm{Li}_2\left(\frac{1-x}{x+1}\right)\right)\log(1-x)}{x}\,dx$. Hence by using $\int \frac{\mathrm{Li}_2(x)\log(1-x)}{x}\,dx = -\frac{\mathrm{Li}_2(x)^2}{2}$ and the value of known $PLI(1,0,0;25;1)$ we come to the result:

$$\begin{aligned}\int_0^{\sqrt{2}-1} \frac{\mathrm{Li}_2\left(\frac{1-x}{x+1}\right)\log(1-x)}{x}\,dx &= -\frac{1}{2}\mathrm{Li}_2\left(\sqrt{2}-1\right)^2 - \frac{\mathrm{Li}_4\left(\frac{1}{2}\right)}{2} \\ &-\frac{7}{16}\zeta(3)\log(2) + \frac{7\pi^4}{2880} - \frac{1}{48}\log^4(2) + \frac{1}{48}\pi^2\log^2(2)\end{aligned} \tag{4.6.16}$$

Note that a similar consideration for integrand $\frac{\mathrm{Li}_n(x)\mathrm{Li}_n(y)}{xy}$ yields the general formula:

$$\left(\int_0^{\sqrt{2}-1} - \int_{\sqrt{2}-1}^1\right) \frac{\mathrm{Li}_n(x)\mathrm{Li}_{n+1}\left(\frac{1-x}{x+1}\right)}{x}dx = \mathrm{Li}_{n+1}\left(\sqrt{2}-1\right)^2 \tag{4.6.17}$$

Appendix 1. LI closed-forms

We tabulate all LI with weight ≤ 5 here. These integrals are independently solved by Au[1] and the author. Our investigation on weight 6 and higher cases are consistent with Au, therefore we only refer the readers to his paper. Some weight 6 results not explicitly shown here are applied in derivation of some results in appendix 4.

How to read the table: for each row's LI, do an inner product with the top row (Fibonacci basis) to get the closed-form of this LI. As an example, we find this row in table of weight 3 LIs:

Weight 3 LI	$\pi^2\log(2)$	$\log^3(2)$	$\zeta(3)$
(1,0,1;2)	$-\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{8}$

Which implies that $LI(1,0,1;2) = \int_0^1 \frac{\log(1-x)\log(x+1)}{x+1}\,dx = \frac{\zeta(3)}{8} + \frac{\log^3(2)}{3} - \frac{1}{12}\pi^2\log(2)$. We can read tables in appendix 2, 3, 4 in a similar manner to find the closed-form for a certain ES/PLI/QLI.

Weight 1 LI: $LI(0,0,0;1) = \log(2)$

Weight 2 LI	π^2	$\log^2(2)$
(0,1,0;0)	$-\frac{1}{6}$	0
(1,0,0;1)	$-\frac{1}{6}$	0
(0,0,1;1)	$\frac{1}{12}$	0
(1,0,0;2)	$-\frac{1}{12}$	$\frac{1}{2}$
(0,1,0;2)	$-\frac{1}{12}$	0
(0,0,1;2)	0	$\frac{1}{2}$

Weight 3 LI	$\pi^2\log(2)$	$\log^3(2)$	$\zeta(3)$
(0,2,0;0)	0	0	2
(1,1,0;0)	0	0	1
(0,1,1;0)	$-\frac{1}{4}$	0	1
(2,0,0;1)	0	0	2
(0,0,2;1)	0	0	$\frac{1}{4}$
(1,1,0;1)	0	0	1
(0,1,1;1)	0	0	$-\frac{3}{4}$
(1,0,1;1)	0	0	$-\frac{5}{8}$
(2,0,0;2)	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{7}{4}$
(0,2,0;2)	0	0	$\frac{3}{2}$
(0,0,2;2)	0	$\frac{1}{3}$	0
(1,1,0;2)	$-\frac{1}{4}$	0	$\frac{13}{8}$
(0,1,1;2)	0	0	$-\frac{1}{8}$
(1,0,1;2)	$-\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{8}$

Weight 4 LI	π^4	$\pi^2\log^2(2)$	$\log^4(2)$	$\zeta(3)\log(2)$	$\mathrm{Li}_4(\frac{1}{2})$
(0,3,0;0)	$-\frac{1}{15}$	0	0	0	0
(2,1,0;0)	$-\frac{1}{45}$	0	0	0	0
(1,2,0;0)	$-\frac{1}{180}$	0	0	0	0
(0,2,1;0)	$-\frac{19}{720}$	0	0	$\frac{7}{2}$	0
(0,1,2;0)	$-\frac{7}{144}$	$-\frac{5}{12}$	$\frac{1}{6}$	$\frac{21}{4}$	4
(1,1,1;0)	$\frac{17}{1440}$	$-\frac{1}{24}$	$-\frac{1}{12}$	$\frac{7}{8}$	-2
(3,0,0;1)	$-\frac{1}{15}$	0	0	0	0
(0,0,3;1)	$\frac{1}{15}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{21}{4}$	-6
(2,1,0;1)	$-\frac{1}{180}$	0	0	0	0
(2,0,1;1)	$-\frac{1}{144}$	$-\frac{1}{12}$	$\frac{1}{12}$	$\frac{7}{4}$	2
(1,2,0;1)	$-\frac{1}{45}$	0	0	0	0
(0,2,1;1)	$\frac{7}{360}$	0	0	0	0
(1,0,2;1)	$-\frac{1}{240}$	0	0	0	0
(0,1,2;1)	$\frac{1}{24}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{7}{2}$	-4
(1,1,1;1)	$-\frac{3}{160}$	$-\frac{1}{12}$	$\frac{1}{12}$	$\frac{7}{4}$	2
(3,0,0;2)	0	0	0	0	-6
(0,3,0;2)	$-\frac{7}{120}$	0	0	0	0
(0,0,3;2)	0	0	$\frac{1}{4}$	0	0
(2,1,0;2)	$\frac{11}{360}$	0	$-\frac{1}{4}$	0	-6
(2,0,1;2)	$-\frac{1}{360}$	$-\frac{1}{6}$	$\frac{1}{4}$	2	0
(1,2,0;2)	$\frac{1}{90}$	$\frac{1}{6}$	$-\frac{1}{6}$	0	-4
(0,2,1;2)	$-\frac{1}{24}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{7}{2}$	4
(1,0,2;2)	$-\frac{1}{45}$	$-\frac{1}{6}$	$\frac{1}{3}$	2	2
(0,1,2;2)	$-\frac{1}{45}$	$-\frac{1}{12}$	$\frac{1}{12}$	$\frac{7}{4}$	2
(1,1,1;2)	$-\frac{1}{45}$	$-\frac{5}{24}$	$\frac{1}{12}$	$\frac{21}{8}$	2

Weight 5 LI	$\pi^4\log(2)$	$\pi^2\log^3(2)$	$\log^5(2)$	$\pi^2\zeta(3)$	$\zeta(3)\log^2(2)$	$\zeta(5)$	$\mathrm{Li}_4(\frac{1}{2})\log(2)$	$\mathrm{Li}_5(\frac{1}{2})$
(0,4,0;0)	0	0	0	0	0	24	0	0
(3,1,0;0)	0	0	0	0	0	6	0	0
(1,3,0;0)	0	0	0	-1	0	12	0	0
(0,3,1;0)	$-\frac{1}{8}$	0	0	$-\frac{3}{8}$	0	12	0	0
(0,1,3;0)	$-\frac{17}{80}$	$-\frac{1}{3}$	$-\frac{1}{20}$	$-\frac{7}{16}$	$\frac{63}{8}$	$\frac{117}{4}$	-6	-24
(2,2,0;0)	0	0	0	$-\frac{2}{3}$	0	8	0	0
(0,2,2;0)	$-\frac{49}{360}$	$\frac{1}{9}$	$-\frac{1}{5}$	$-\frac{13}{24}$	$\frac{7}{2}$	$\frac{47}{2}$	-8	-16
(2,1,1;0)	$\frac{1}{120}$	$-\frac{1}{18}$	$\frac{1}{60}$	$-\frac{7}{16}$	$\frac{7}{8}$	$-\frac{7}{4}$	2	8
(1,2,1;0)	$\frac{1}{180}$	$-\frac{1}{18}$	$\frac{1}{10}$	$-\frac{29}{48}$	$\frac{7}{4}$	$\frac{1}{4}$	4	8
(1,1,2;0)	$-\frac{1}{48}$	$-\frac{1}{6}$	0	$-\frac{1}{2}$	$\frac{7}{2}$	6	0	0

Weight 5 LI	$\pi^4\log(2)$	$\pi^2\log^3(2)$	$\log^5(2)$	$\pi^2\zeta(3)$	$\zeta(3)\log^2(2)$	$\zeta(5)$	$\mathrm{Li}_4(\frac{1}{2})\log(2)$	$\mathrm{Li}_5(\frac{1}{2})$
(4,0,0;1)	0	0	0	0	0	24	0	0
(0,0,4;1)	0	$\frac{2}{3}$	$-\frac{4}{5}$	0	$-\frac{21}{2}$	24	-24	-24
(3,1,0;1)	0	0	0	-1	0	12	0	0
(3,0,1;1)	0	$-\frac{1}{6}$	$\frac{1}{5}$	$-\frac{7}{16}$	$\frac{21}{8}$	$-\frac{81}{16}$	6	6
(1,3,0;1)	0	0	0	0	0	6	0	0
(0,3,1;1)	0	0	0	0	0	$-\frac{45}{8}$	0	0
(1,0,3;1)	0	$\frac{1}{6}$	$-\frac{1}{5}$	$\frac{7}{16}$	$-\frac{21}{8}$	$\frac{3}{4}$	-6	-6
(0,1,3;1)	0	$\frac{1}{3}$	$-\frac{2}{5}$	$\frac{1}{2}$	$-\frac{21}{4}$	$\frac{99}{16}$	-12	-12
(2,2,0;1)	0	0	0	$-\frac{2}{3}$	0	8	0	0
(2,0,2;1)	0	$-\frac{1}{9}$	$\frac{2}{15}$	0	$\frac{7}{4}$	$-\frac{25}{8}$	4	4
(0,2,2;1)	0	0	0	$\frac{1}{3}$	0	$-\frac{29}{8}$	0	0
(2,1,1;1)	0	$-\frac{1}{9}$	$\frac{2}{15}$	$-\frac{1}{16}$	$\frac{7}{4}$	$-\frac{7}{2}$	4	4
(1,2,1;1)	0	0	0	$\frac{1}{8}$	0	$-\frac{27}{16}$	0	0
(1,1,2;1)	0	0	0	$\frac{7}{48}$	0	$-\frac{25}{16}$	0	0
(4,0,0;2)	0	0	0	0	0	0	0	24
(0,4,0;2)	0	0	0	0	0	$\frac{45}{2}$	0	0
(0,0,4;2)	0	0	$\frac{1}{5}$	0	0	0	0	0
(3,1,0;2)	$\frac{1}{40}$	0	$-\frac{3}{20}$	$-\frac{7}{8}$	0	$-\frac{3}{16}$	0	18
(3,0,1;2)	$-\frac{1}{120}$	$-\frac{1}{6}$	$\frac{3}{20}$	$-\frac{1}{2}$	3	$\frac{3}{16}$	0	6
(1,3,0;2)	$-\frac{1}{8}$	0	0	$-\frac{3}{4}$	0	$\frac{273}{16}$	0	0
(0,3,1;2)	0	0	0	$-\frac{1}{2}$	0	$\frac{87}{16}$	0	0
(1,0,3;2)	$-\frac{1}{15}$	$-\frac{1}{6}$	$\frac{1}{4}$	0	3	6	0	-6
(0,1,3;2)	0	$-\frac{1}{6}$	$\frac{1}{5}$	0	$\frac{21}{8}$	-6	6	6
(2,2,0;2)	$\frac{1}{90}$	$\frac{1}{9}$	$-\frac{1}{15}$	$-\frac{13}{12}$	0	$\frac{15}{2}$	0	8
(2,0,2;2)	$-\frac{1}{20}$	$-\frac{2}{9}$	$\frac{7}{30}$	$-\frac{1}{3}$	4	$\frac{63}{8}$	0	-4
(0,2,2;2)	0	$-\frac{2}{9}$	$\frac{4}{15}$	$-\frac{1}{3}$	$\frac{7}{2}$	$-\frac{33}{8}$	8	8
(2,1,1;2)	$-\frac{1}{48}$	$-\frac{1}{9}$	$-\frac{1}{15}$	$-\frac{1}{2}$	$\frac{21}{8}$	$\frac{121}{16}$	-2	-2
(1,2,1;2)	$-\frac{49}{720}$	$\frac{1}{18}$	$-\frac{1}{10}$	$-\frac{5}{12}$	$\frac{7}{4}$	$\frac{213}{16}$	-4	-8
(1,1,2;2)	$-\frac{17}{240}$	$-\frac{1}{6}$	$\frac{1}{20}$	$-\frac{7}{24}$	$\frac{7}{2}$	$\frac{19}{2}$	0	-6

Appendix 2. ES closed-forms

We tabulate all ES with weight ≤ 5 here. For weight ≤ 4 we solved all ESs by using ES-PLI transformation independently. For weight 5, based on 17 ESs recorded in (or easily deduced from) Flajolet&Salvy, 30 ESs recorded in Xu (partially verified by the author), we obtain closed-forms of the remaining 7 ESs (section 2-5) using various methods.

$$\text{Weight 2 ESs: } ES(1;-1) = \frac{\pi^2}{12} - \frac{\log^2(2)}{2}, ES(-1;-1) = \frac{\pi^2}{12} + \frac{\log^2(2)}{2}$$

Weight 3 ES	$\pi^2 \log(2)$	$\log^3(2)$	$\zeta(3)$
$(1;2)$	0	0	2
$(-1;2)$	$\frac{1}{4}$	0	$-\frac{1}{4}$
$(1;-2)$	0	0	$\frac{5}{8}$
$(-1;-2)$	$\frac{1}{4}$	0	$-\frac{5}{8}$
$(2;-1)$	$-\frac{1}{12}$	0	1
$(-2;-1)$	$-\frac{1}{6}$	0	$\frac{13}{8}$
$(1,1;-1)$	$-\frac{1}{12}$	$\frac{1}{3}$	$\frac{3}{4}$
$(1,-1;-1)$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{4}$
$(-1,-1;-1)$	$\frac{1}{4}$	$\frac{1}{3}$	$-\frac{1}{2}$

Weight 4 ES	π^4	$\pi^2\log^2(2)$	$\log^4(2)$	$\zeta(3)\log(2)$	$\mathrm{Li}_4(\frac{1}{2})$
$(1;3)$	$\frac{1}{72}$	0	0	0	0
$(-1;3)$	$-\frac{1}{288}$	0	0	$\frac{7}{4}$	0
$(1;-3)$	$\frac{11}{360}$	$\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{7}{4}$	-2
$(-1;-3)$	$\frac{1}{60}$	$\frac{1}{12}$	$-\frac{1}{12}$	0	-2
$(2;2)$	$\frac{7}{360}$	0	0	0	0
$(-2;2)$	$\frac{17}{288}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{7}{2}$	-4
$(2;-2)$	$-\frac{17}{480}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{7}{2}$	4
$(-2;-2)$	$\frac{13}{1440}$	0	0	0	0
$(1,1;2)$	$\frac{17}{360}$	0	0	0	0
$(1,-1;2)$	$\frac{43}{1440}$	$\frac{1}{8}$	$-\frac{1}{8}$	0	-3
$(-1,-1;2)$	$-\frac{13}{720}$	$\frac{5}{12}$	$\frac{1}{12}$	0	2
$(1,1;-2)$	$\frac{41}{1440}$	$\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{7}{4}$	-2
$(1,-1;-2)$	$\frac{29}{1440}$	$\frac{1}{8}$	$-\frac{1}{8}$	0	-3
$(-1,-1;-2)$	$-\frac{61}{1440}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{7}{4}$	4
$(3;-1)$	$\frac{19}{1440}$	0	0	$-\frac{3}{4}$	0
$(-3;-1)$	$-\frac{1}{180}$	$-\frac{1}{12}$	$\frac{1}{12}$	$\frac{3}{4}$	2
$(1,2;-1)$	$-\frac{1}{90}$	$-\frac{1}{24}$	$\frac{1}{12}$	$\frac{7}{8}$	2
$(-1,2;-1)$	$\frac{43}{1440}$	$-\frac{1}{24}$	$-\frac{1}{8}$	0	-3
$(1,-2;-1)$	$-\frac{1}{160}$	$-\frac{1}{24}$	$\frac{1}{8}$	0	3
$(-1,-2;-1)$	$\frac{61}{1440}$	$-\frac{1}{24}$	$-\frac{1}{6}$	$-\frac{7}{8}$	-4
$(1,1,1;-1)$	$\frac{1}{144}$	$\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{9}{8}$	0
$(1,1,-1;-1)$	$\frac{53}{1440}$	$\frac{1}{24}$	$\frac{1}{8}$	0	-3
$(1,-1,-1;-1)$	0	$\frac{7}{24}$	$-\frac{1}{4}$	$\frac{3}{8}$	0
$(-1,-1,-1;-1)$	$-\frac{59}{1440}$	$\frac{13}{24}$	$\frac{11}{24}$	0	5

Weight 5 ES	$\pi^4\log(2)$	$\pi^2\log^3(2)$	$\log^5(2)$	$\pi^2\zeta(3)$	$\zeta(3)\log^2(2)$	$\zeta(5)$	$\mathrm{Li}_4(\frac{1}{2})\log(2)$	$\mathrm{Li}_5(\frac{1}{2})$
$(1;4)$	0	0	0	$-\frac{1}{6}$	0	3	0	0
$(-1;4)$	$\frac{1}{48}$	0	0	$\frac{1}{16}$	0	$-\frac{17}{16}$	0	0
$(1;-4)$	0	0	0	$-\frac{1}{12}$	0	$\frac{59}{32}$	0	0
$(-1;-4)$	$\frac{1}{48}$	0	0	$\frac{1}{8}$	0	$-\frac{59}{32}$	0	0
$(2;3)$	0	0	0	$\frac{1}{2}$	0	$-\frac{9}{2}$	0	0
$(-2;3)$	0	0	0	$-\frac{1}{24}$	0	$\frac{51}{32}$	0	0
$(2;-3)$	0	0	0	$\frac{5}{48}$	0	$-\frac{11}{32}$	0	0
$(-2;-3)$	0	0	0	$-\frac{3}{8}$	0	$\frac{83}{16}$	0	0
$(1,1;3)$	0	0	0	$-\frac{1}{6}$	0	$\frac{7}{2}$	0	0
$(1,-1;3)$	$\frac{2}{45}$	$\frac{1}{36}$	$-\frac{1}{60}$	$\frac{1}{16}$	$-\frac{7}{8}$	$-\frac{193}{64}$	0	2
$(-1,-1;3)$	$\frac{19}{720}$	$\frac{1}{18}$	$-\frac{1}{30}$	$\frac{1}{8}$	$\frac{7}{4}$	$-\frac{167}{32}$	0	4
$(1,1;-3)$	0	$-\frac{1}{9}$	$\frac{2}{15}$	$-\frac{11}{48}$	$\frac{7}{4}$	$-\frac{19}{32}$	4	4
$(1,-1;-3)$	$\frac{2}{45}$	$\frac{1}{36}$	$-\frac{1}{60}$	$-\frac{1}{48}$	$-\frac{7}{8}$	$-\frac{37}{16}$	0	2
$(-1,-1;-3)$	$\frac{19}{720}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{16}$	0	$-\frac{19}{32}$	-4	0
$(3;2)$	0	0	0	$-\frac{1}{3}$	0	$\frac{11}{2}$	0	0
$(-3;2)$	0	0	0	$\frac{1}{48}$	0	$\frac{41}{32}$	0	0
$(3;-2)$	0	0	0	$\frac{1}{8}$	0	$-\frac{21}{32}$	0	0
$(-3;-2)$	0	0	0	$\frac{7}{16}$	0	$-\frac{67}{16}$	0	0
$(1,2;2)$	0	0	0	$\frac{1}{6}$	0	1	0	0
$(-1,2;2)$	$-\frac{23}{1440}$	$\frac{1}{18}$	$-\frac{1}{10}$	$\frac{1}{12}$	0	$\frac{27}{4}$	-4	-8
$(1,-2;2)$	0	$-\frac{1}{9}$	$\frac{2}{15}$	$-\frac{5}{24}$	$\frac{7}{4}$	$\frac{29}{64}$	4	4
$(-1,-2;2)$	$\frac{49}{720}$	$\frac{1}{18}$	$-\frac{1}{30}$	$-\frac{1}{8}$	$-\frac{7}{4}$	$-\frac{23}{8}$	0	4
$(1,2;-2)$	0	$\frac{1}{9}$	$-\frac{2}{15}$	$\frac{5}{32}$	$-\frac{7}{4}$	$\frac{23}{8}$	-4	-4
$(-1,2;-2)$	$-\frac{23}{1440}$	$-\frac{1}{18}$	$\frac{1}{30}$	$\frac{5}{16}$	$\frac{7}{4}$	$-\frac{29}{64}$	0	-4
$(1,-2;-2)$	0	0	0	$-\frac{13}{48}$	0	$\frac{125}{32}$	0	0
$(-1,-2;-2)$	$\frac{49}{720}$	$-\frac{1}{18}$	$\frac{1}{10}$	$\frac{1}{96}$	0	$-\frac{35}{4}$	4	8
$(1,1,1;2)$	0	0	0	$\frac{1}{6}$	0	10	0	0
$(1,1,-1;2)$	$\frac{109}{1440}$	$-\frac{1}{36}$	$\frac{1}{20}$	$\frac{1}{12}$	0	$-\frac{165}{32}$	2	4
$(1,-1,-1;2)$	$\frac{1}{10}$	$\frac{1}{6}$	$-\frac{1}{10}$	$\frac{1}{6}$	0	$-\frac{53}{4}$	0	12
$(-1,-1,-1;2)$	$-\frac{29}{160}$	$\frac{11}{12}$	$-\frac{1}{20}$	$\frac{1}{12}$	0	$\frac{367}{16}$	-6	-24
$(1,1,1;-2)$	0	$-\frac{1}{6}$	$\frac{1}{5}$	$-\frac{9}{32}$	$\frac{21}{8}$	$-\frac{9}{4}$	6	6
$(1,1,-1;-2)$	$\frac{109}{1440}$	$\frac{1}{36}$	$-\frac{1}{60}$	$-\frac{1}{8}$	$-\frac{7}{8}$	$-\frac{197}{64}$	0	2
$(1,-1,-1;-2)$	$\frac{1}{10}$	$\frac{2}{9}$	$-\frac{1}{6}$	$\frac{1}{96}$	$-\frac{7}{8}$	-10	-2	10
$(-1,-1,-1;-2)$	$-\frac{29}{160}$	$\frac{3}{4}$	$\frac{3}{20}$	$-\frac{1}{8}$	$\frac{21}{8}$	$\frac{1229}{64}$	0	-18
$(4;-1)$	$-\frac{7}{720}$	0	0	$-\frac{1}{16}$	0	2	0	0
$(-4;-1)$	$-\frac{1}{90}$	0	0	$-\frac{1}{8}$	0	$\frac{91}{32}$	0	0
$(1,3;-1)$	$-\frac{49}{1440}$	$\frac{1}{36}$	$-\frac{1}{20}$	$-\frac{1}{96}$	$\frac{3}{8}$	$\frac{167}{32}$	-2	-4
$(-1,3;-1)$	$\frac{11}{1440}$	$-\frac{1}{36}$	$\frac{1}{60}$	$-\frac{5}{48}$	$-\frac{3}{8}$	$-\frac{193}{64}$	0	-2
$(1,-3;-1)$	$\frac{1}{360}$	$\frac{1}{36}$	$-\frac{1}{60}$	$\frac{1}{6}$	$-\frac{3}{8}$	$-\frac{37}{16}$	0	2

Weight 5 ES	$\pi^4\log(2)$	$\pi^2\log^3(2)$	$\log^5(2)$	$\pi^2\zeta(3)$	$\zeta(3)\log^2(2)$	$\zeta(5)$	$\mathrm{Li}_4(\frac{1}{2})\log(2)$	$\mathrm{Li}_5(\frac{1}{2})$
$(-1,-3;-1)$	$\frac{11}{1440}$	$-\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{3}{8}$	$-\frac{19}{32}$	2	0
$(2,2;-1)$	$\frac{29}{360}$	$-\frac{1}{9}$	$\frac{1}{5}$	$\frac{5}{48}$	0	$-\frac{259}{16}$	8	16
$(2,-2;-1)$	$-\frac{5}{288}$	0	0	$\frac{1}{16}$	0	$\frac{75}{64}$	0	0
$(-2,-2;-1)$	$-\frac{13}{120}$	$\frac{1}{9}$	$-\frac{1}{5}$	$-\frac{1}{24}$	0	$\frac{299}{16}$	-8	-16
$(1,1,2;-1)$	$\frac{7}{720}$	$\frac{1}{18}$	$-\frac{1}{12}$	$\frac{1}{16}$	$-\frac{7}{8}$	0	-2	0
$(1,-1,2;-1)$	$-\frac{1}{45}$	$-\frac{1}{72}$	$\frac{1}{20}$	$\frac{7}{96}$	$\frac{7}{16}$	$\frac{155}{32}$	0	-6
$(-1,-1,2;-1)$	$\frac{1}{20}$	$-\frac{1}{36}$	$-\frac{1}{4}$	$\frac{1}{48}$	0	0	-6	0
$(1,1,-2;-1)$	$-\frac{13}{1440}$	$\frac{1}{36}$	$-\frac{1}{20}$	$-\frac{1}{48}$	0	$-\frac{93}{64}$	0	6
$(1,-1,-2;-1)$	$-\frac{1}{480}$	$-\frac{5}{72}$	$\frac{1}{6}$	0	$\frac{7}{16}$	0	4	0
$(-1,-1,-2;-1)$	$\frac{193}{1440}$	$-\frac{1}{9}$	$-\frac{7}{60}$	$\frac{1}{24}$	$-\frac{7}{8}$	$-\frac{899}{64}$	0	14
$(1,1,1,1;-1)$	$\frac{11}{360}$	$-\frac{2}{9}$	$\frac{3}{10}$	$-\frac{11}{48}$	$\frac{9}{4}$	$-\frac{83}{16}$	4	8
$(1,1,1,-1;-1)$	$\frac{37}{720}$	$\frac{7}{72}$	$-\frac{11}{60}$	$-\frac{1}{96}$	$-\frac{9}{16}$	$\frac{55}{16}$	0	-2
$(1,1,-1,-1;-1)$	$\frac{97}{720}$	$\frac{1}{12}$	$\frac{1}{10}$	$\frac{1}{16}$	$-\frac{3}{8}$	$-\frac{201}{16}$	0	12
$(1,-1,-1,-1;-1)$	$-\frac{1}{180}$	$\frac{41}{72}$	$-\frac{13}{60}$	$\frac{13}{96}$	$\frac{9}{16}$	$-\frac{83}{32}$	0	2
$(-1,-1,-1,-1;-1)$	$-\frac{71}{180}$	$\frac{3}{2}$	$\frac{13}{30}$	$\frac{5}{48}$	0	$\frac{733}{16}$	-4	-48

Appendix 3. PLI closed-forms

We tabulate the main result of this article here: 263(=11+55+197) convergent PLIs with weight ≤ 5. These integrals are independently solved by the author. Note that there are few literatures dealing with PLIs systematically, especially without usage of MZV theory.

For the sake of simplicity, we leave out 2 commas between $a(0)$, $a(1)$ and $a(2)$, in the definition of PLI. For instance, we have $PLI(100;24;1) = \int_0^1 \frac{\mathrm{Li}_2\left(\frac{x+1}{2}\right)\log(1-x)}{x}\,dx, PLI(000;25,26;0) = \int_0^1 \frac{\mathrm{Li}_2\left(\frac{1-x}{x+1}\right)\mathrm{Li}_2\left(\frac{x-1}{x+1}\right)}{1-x}\,dx$. Moreover, by noticing this row in weight 4 PLI table:

Weight 4 PLI	π^4	$\pi^2\log^2(2)$	$\log^4(2)$	$\zeta(3)\log(2)$	$\mathrm{Li}_4(\frac{1}{2})$
$(100;24;1)$	$-\frac{11}{1440}$	$\frac{5}{24}$	$-\frac{1}{8}$	$-\frac{5}{8}$	-3

We have $\int_0^1 \frac{\mathrm{Li}_2\left(\frac{x+1}{2}\right)\log(1-x)}{x}\,dx = -3\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{5}{8}\zeta(3)\log(2) - \frac{11\pi^4}{1440} - \frac{1}{8}\log^4(2) + \frac{5}{24}\pi^2\log^2(2)$. The rest are similar.

Weight 3 PLI	$\pi^2 \log(2)$	$\log^3(2)$	$\zeta(3)$
$(000;21;1)$	0	0	1
$(000;21;2)$	$\frac{1}{6}$	0	$-\frac{5}{8}$
$(000;22;1)$	0	0	$-\frac{3}{4}$
$(000;22;2)$	$-\frac{1}{12}$	0	$\frac{1}{4}$
$(000;23;0)$	$-\frac{1}{12}$	$\frac{1}{6}$	$\frac{7}{8}$
$(000;23;2)$	$\frac{1}{12}$	$-\frac{1}{6}$	$-\frac{1}{4}$
$(000;24;2)$	$\frac{1}{12}$	$-\frac{1}{6}$	$\frac{1}{8}$
$(000;25;0)$	$-\frac{1}{6}$	0	$\frac{13}{8}$
$(000;25;2)$	$\frac{1}{6}$	0	$-\frac{5}{8}$
$(000;26;0)$	$\frac{1}{12}$	0	-1
$(000;26;2)$	$-\frac{1}{12}$	0	$\frac{1}{4}$

Weight 4 PLI	π^4	$\pi^2 \log^2(2)$	$\log^4(2)$	$\zeta(3)\log(2)$	$\mathrm{Li}_4(\frac{1}{2})$
$(010;21;0)$	$-\frac{1}{120}$	0	0	0	0
$(010;21;1)$	$-\frac{1}{90}$	0	0	0	0
$(010;21;2)$	$-\frac{1}{480}$	0	0	0	0
$(001;21;1)$	$-\frac{1}{60}$	$-\frac{1}{12}$	$\frac{1}{12}$	$\frac{7}{4}$	2
$(001;21;2)$	$-\frac{1}{480}$	$\frac{1}{12}$	0	0	0
$(100;21;1)$	$-\frac{1}{72}$	0	0	0	0
$(100;21;2)$	$-\frac{29}{1440}$	$-\frac{1}{24}$	$\frac{1}{8}$	0	3
$(010;22;0)$	$\frac{71}{1440}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{7}{2}$	-4
$(010;22;1)$	$\frac{7}{720}$	0	0	0	0
$(010;22;2)$	$\frac{13}{288}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{7}{2}$	-4
$(001;22;1)$	$-\frac{1}{288}$	0	0	0	0
$(001;22;2)$	$\frac{1}{30}$	$\frac{1}{12}$	$-\frac{1}{8}$	$-\frac{21}{8}$	-3
$(100;22;1)$	$\frac{11}{360}$	$\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{7}{4}$	-2
$(100;22;2)$	$\frac{1}{24}$	$\frac{1}{8}$	$-\frac{1}{6}$	$-\frac{21}{8}$	-4
$(010;23;0)$	$-\frac{1}{720}$	0	$-\frac{1}{24}$	$\frac{1}{8}$	-1
$(010;23;2)$	$\frac{1}{360}$	$\frac{1}{24}$	0	-1	0
$(001;23;0)$	$-\frac{1}{288}$	$-\frac{1}{24}$	$\frac{1}{24}$	$\frac{7}{8}$	0
$(001;23;1)$	$\frac{11}{240}$	$\frac{1}{8}$	$-\frac{1}{6}$	$-\frac{13}{4}$	-4
$(001;23;2)$	$\frac{1}{30}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{23}{8}$	-3

Weight 4 PLI	π^4	$\pi^2\log^2(2)$	$\log^4(2)$	$\zeta(3)\log(2)$	$\mathrm{Li}_4(\frac{1}{2})$
$(100;23;0)$	0	0	0	0	-1
$(100;23;1)$	$-\frac{19}{1440}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{4}$	1
$(100;23;2)$	$-\frac{1}{1440}$	$\frac{1}{24}$	$-\frac{1}{24}$	$-\frac{1}{4}$	0
$(010;24;0)$	$-\frac{11}{288}$	$-\frac{1}{24}$	$\frac{1}{8}$	1	3
$(010;24;2)$	$\frac{1}{180}$	0	$-\frac{1}{12}$	$-\frac{1}{8}$	-2
$(001;24;1)$	$-\frac{1}{36}$	$-\frac{5}{24}$	$\frac{1}{6}$	$\frac{23}{8}$	4
$(001;24;2)$	$-\frac{1}{90}$	0	0	1	1
$(100;24;1)$	$-\frac{11}{1440}$	$\frac{5}{24}$	$-\frac{1}{8}$	$-\frac{5}{8}$	-3
$(100;24;2)$	$-\frac{1}{96}$	$\frac{1}{24}$	$-\frac{1}{24}$	$\frac{1}{8}$	0
$(010;25;0)$	$\frac{1}{48}$	$\frac{5}{24}$	$-\frac{5}{24}$	$-\frac{7}{4}$	-5
$(010;25;2)$	$-\frac{13}{720}$	$-\frac{1}{8}$	$\frac{1}{8}$	0	3
$(001;25;0)$	$\frac{7}{480}$	0	$-\frac{1}{12}$	$-\frac{1}{8}$	-2
$(001;25;1)$	$\frac{11}{720}$	$\frac{1}{24}$	$-\frac{1}{24}$	$-\frac{7}{8}$	-1
$(001;25;2)$	$\frac{1}{480}$	$\frac{1}{12}$	0	$-\frac{5}{8}$	0
$(100;25;0)$	$\frac{1}{180}$	0	$-\frac{1}{12}$	$-\frac{1}{8}$	-2
$(100;25;1)$	$\frac{7}{1440}$	$\frac{1}{24}$	$-\frac{1}{24}$	$-\frac{7}{8}$	-1
$(100;25;2)$	0	$\frac{1}{12}$	0	$-\frac{5}{8}$	0
$(010;26;0)$	$\frac{37}{1440}$	$\frac{1}{24}$	$-\frac{1}{24}$	$-\frac{7}{4}$	-1
$(010;26;2)$	$\frac{1}{120}$	$\frac{1}{24}$	$-\frac{1}{24}$	0	-1
$(001;26;0)$	$\frac{53}{1440}$	$\frac{1}{6}$	$-\frac{1}{8}$	$-\frac{29}{8}$	-3
$(001;26;1)$	$-\frac{19}{240}$	$-\frac{7}{24}$	$\frac{7}{24}$	$\frac{49}{8}$	7
$(001;26;2)$	$-\frac{1}{30}$	$-\frac{1}{6}$	$\frac{1}{8}$	$\frac{23}{8}$	3
$(100;26;0)$	$\frac{1}{720}$	0	$\frac{1}{24}$	$-\frac{1}{8}$	1
$(100;26;1)$	$\frac{11}{720}$	$\frac{1}{24}$	$-\frac{1}{24}$	$-\frac{7}{8}$	-1
$(100;26;2)$	$\frac{17}{1440}$	0	$-\frac{1}{24}$	$-\frac{5}{8}$	-1
$(000;31;1)$	$\frac{1}{90}$	0	0	0	0
$(000;31;2)$	$\frac{1}{60}$	$\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{3}{4}$	-2
$(000;32;1)$	$-\frac{7}{720}$	0	0	0	0
$(000;32;2)$	$\frac{1}{288}$	0	0	$-\frac{3}{4}$	0
$(000;33;0)$	0	0	0	0	1
$(000;33;2)$	$-\frac{1}{288}$	$-\frac{1}{24}$	$\frac{1}{24}$	$\frac{7}{8}$	0
$(000;34;2)$	$\frac{1}{90}$	0	0	0	-1
$(000;35;0)$	$-\frac{1}{180}$	$-\frac{1}{12}$	$\frac{1}{12}$	$\frac{3}{4}$	2
$(000;35;2)$	$\frac{1}{60}$	$\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{3}{4}$	-2
$(000;36;0)$	$-\frac{19}{1440}$	0	0	$\frac{3}{4}$	0
$(000;36;2)$	$\frac{1}{288}$	0	0	$-\frac{3}{4}$	0

Weight 5 PLI	$\pi^4\log(2)$	$\pi^2\log^3(2)$	$\log^5(2)$	$\pi^2\zeta(3)$	$\zeta(3)\log^2(2)$	$\zeta(5)$	$\mathrm{Li}_4(\frac{1}{2})\log(2)$	$\mathrm{Li}_5(\frac{1}{2})$
$(020;21;0)$	0	0	0	1	0	-11	0	0
$(020;21;1)$	0	0	0	0	0	2	0	0
$(020;21;2)$	0	0	0	$\frac{3}{4}$	0	$-\frac{67}{8}$	0	0
$(002;21;1)$	0	$-\frac{2}{9}$	$\frac{4}{15}$	$-\frac{1}{4}$	$\frac{7}{2}$	$-\frac{39}{8}$	8	8
$(002;21;2)$	0	$\frac{1}{9}$	$-\frac{1}{15}$	$\frac{7}{48}$	$-\frac{7}{8}$	$\frac{1}{4}$	-2	-2
$(200;21;1)$	0	0	0	$\frac{1}{3}$	0	-1	0	0
$(200;21;2)$	$\frac{19}{720}$	$-\frac{1}{36}$	$\frac{1}{20}$	$\frac{7}{24}$	0	$-\frac{23}{32}$	0	-6
$(110;21;0)$	0	0	0	$\frac{2}{3}$	0	$-\frac{13}{2}$	0	0
$(110;21;1)$	0	0	0	$\frac{1}{6}$	0	$-\frac{3}{2}$	0	0
$(100;21;2)$	$-\frac{1}{96}$	0	0	$\frac{31}{48}$	0	$-\frac{421}{64}$	0	0
$(011;21;0)$	$-\frac{1}{96}$	$\frac{1}{9}$	$-\frac{2}{15}$	$\frac{29}{48}$	$-\frac{7}{4}$	$-\frac{21}{8}$	-4	-4
$(011;21;1)$	0	0	0	$-\frac{5}{16}$	0	$\frac{107}{32}$	0	0
$(011;21;2)$	0	$\frac{1}{9}$	$-\frac{2}{15}$	$\frac{19}{96}$	$-\frac{7}{4}$	$\frac{53}{32}$	-4	-4
$(101;21;1)$	0	0	0	$-\frac{5}{48}$	0	$\frac{29}{64}$	0	0
$(101;21;2)$	$-\frac{11}{720}$	$\frac{5}{72}$	$-\frac{1}{40}$	$\frac{23}{96}$	$-\frac{21}{16}$	$-\frac{15}{64}$	-1	-2
$(020;22;0)$	0	0	0	$\frac{1}{12}$	0	$-\frac{21}{16}$	0	0
$(020;22;1)$	0	0	0	0	0	$-\frac{15}{8}$	0	0
$(020;22;2)$	0	0	0	$\frac{5}{24}$	0	$-\frac{41}{16}$	0	0
$(002;22;1)$	0	$-\frac{1}{9}$	$\frac{2}{15}$	$-\frac{1}{48}$	$\frac{7}{4}$	$-\frac{125}{32}$	4	4
$(002;22;2)$	0	$\frac{7}{36}$	$-\frac{4}{15}$	0	$-\frac{7}{2}$	8	-8	-8
$(200;22;1)$	0	$\frac{1}{9}$	$-\frac{2}{15}$	$\frac{1}{8}$	$-\frac{7}{4}$	$\frac{15}{16}$	-4	-4
$(200;22;2)$	$\frac{13}{144}$	$\frac{1}{18}$	$-\frac{1}{60}$	$\frac{7}{24}$	$-\frac{21}{8}$	$-\frac{61}{4}$	2	12
$(110;22;0)$	0	$\frac{1}{9}$	$-\frac{2}{15}$	$\frac{3}{32}$	$-\frac{7}{4}$	$\frac{71}{32}$	-4	-4
$(110;22;1)$	0	0	0	$\frac{5}{48}$	0	$-\frac{3}{2}$	0	0
$(110;22;2)$	$\frac{17}{180}$	0	$\frac{1}{15}$	$\frac{19}{96}$	$-\frac{7}{4}$	$-\frac{237}{16}$	4	12
$(011;22;0)$	$\frac{17}{180}$	$\frac{1}{9}$	$-\frac{1}{15}$	$\frac{5}{48}$	$-\frac{7}{2}$	$-\frac{301}{32}$	0	8
$(011;22;1)$	0	0	0	$\frac{1}{16}$	0	$-\frac{17}{32}$	0	0
$(011;22;2)$	0	$\frac{2}{9}$	$-\frac{4}{15}$	$\frac{25}{96}$	$-\frac{7}{2}$	$\frac{323}{64}$	-8	-8
$(101;22;1)$	0	$\frac{1}{9}$	$-\frac{2}{15}$	$\frac{5}{96}$	$-\frac{7}{4}$	$\frac{123}{32}$	-4	-4
$(101;22;2)$	$\frac{19}{240}$	$\frac{11}{72}$	$-\frac{19}{120}$	$\frac{5}{24}$	$-\frac{63}{16}$	$-\frac{209}{32}$	-3	4
$(020;23;0)$	$-\frac{19}{720}$	$\frac{1}{9}$	$-\frac{2}{15}$	$\frac{7}{48}$	-1	$\frac{151}{32}$	-4	-4
$(020;23;2)$	$\frac{19}{720}$	0	0	$\frac{7}{24}$	$-\frac{3}{4}$	$-\frac{15}{4}$	0	0
$(002;23;0)$	$-\frac{1}{144}$	$-\frac{1}{36}$	$\frac{1}{60}$	$\frac{1}{48}$	$\frac{7}{8}$	$-\frac{3}{32}$	0	0
$(002;23;1)$	$\frac{1}{15}$	$\frac{1}{12}$	$-\frac{1}{20}$	$\frac{1}{48}$	$-\frac{11}{4}$	$-\frac{99}{16}$	0	6
$(002;23;2)$	$\frac{1}{15}$	$\frac{1}{18}$	0	0	-2	-8	2	8
$(200;23;0)$	0	0	0	0	0	0	0	2

Weight 5 PLI	$\pi^4\log(2)$	$\pi^2\log^3(2)$	$\log^5(2)$	$\pi^2\zeta(3)$	$\zeta(3)\log^2(2)$	$\zeta(5)$	$\mathrm{Li}_4(\frac{1}{2})\log(2)$	$\mathrm{Li}_5(\frac{1}{2})$
$(200;23;1)$	$-\frac{1}{144}$	$\frac{5}{36}$	$-\frac{11}{60}$	$\frac{11}{24}$	$-\frac{11}{4}$	$\frac{81}{32}$	-6	-8
$(200;23;2)$	$-\frac{1}{720}$	$\frac{7}{36}$	$-\frac{13}{60}$	$\frac{23}{48}$	$-\frac{23}{8}$	$\frac{23}{32}$	-6	-6
$(110;23;0)$	$\frac{17}{1440}$	$\frac{1}{24}$	$-\frac{1}{12}$	$\frac{1}{96}$	$-\frac{15}{16}$	$\frac{25}{64}$	-2	0
$(110;23;1)$	$-\frac{3}{160}$	$\frac{1}{12}$	$-\frac{7}{60}$	$\frac{3}{8}$	$-\frac{11}{8}$	$\frac{143}{64}$	-4	-6
$(110;23;2)$	$-\frac{1}{96}$	$\frac{5}{36}$	$-\frac{1}{8}$	$\frac{5}{12}$	$-\frac{17}{8}$	$\frac{37}{64}$	-4	-5
$(011;23;0)$	$-\frac{17}{360}$	$\frac{7}{72}$	$-\frac{17}{120}$	$\frac{1}{12}$	$-\frac{1}{2}$	$\frac{245}{32}$	-5	-8
$(011;23;1)$	$\frac{1}{24}$	$-\frac{1}{18}$	$\frac{1}{10}$	$\frac{1}{8}$	$\frac{3}{8}$	$-\frac{79}{8}$	4	8
$(011;23;2)$	$\frac{7}{144}$	$-\frac{1}{12}$	$\frac{2}{15}$	$\frac{5}{24}$	$\frac{1}{16}$	$-\frac{369}{32}$	5	9
$(101;23;0)$	$-\frac{1}{288}$	$\frac{1}{24}$	$-\frac{7}{120}$	$\frac{7}{96}$	$-\frac{7}{16}$	$\frac{27}{64}$	-2	-1
$(101;23;1)$	$-\frac{1}{240}$	$\frac{1}{12}$	$-\frac{1}{10}$	$\frac{1}{6}$	-1	$\frac{69}{64}$	-3	-3
$(101;23;2)$	$\frac{47}{1440}$	$\frac{5}{72}$	$-\frac{1}{24}$	$\frac{25}{96}$	$-\frac{25}{16}$	$-\frac{381}{64}$	0	3
$(020;24;0)$	$-\frac{1}{90}$	$-\frac{1}{18}$	$\frac{1}{30}$	$\frac{19}{24}$	$\frac{3}{4}$	$-\frac{159}{32}$	0	-4
$(020;24;2)$	$\frac{1}{90}$	$-\frac{1}{18}$	$\frac{1}{10}$	$\frac{3}{8}$	1	$-\frac{153}{16}$	4	8
$(022;24;1)$	$-\frac{1}{240}$	0	0	$-\frac{5}{12}$	$-\frac{1}{8}$	$\frac{87}{16}$	0	0
$(022;24;2)$	$-\frac{1}{45}$	0	0	0	1	2	0	-2
$(200;24;1)$	$-\frac{1}{15}$	$-\frac{1}{18}$	$\frac{1}{15}$	$\frac{5}{16}$	$-\frac{1}{8}$	$\frac{81}{32}$	2	2
$(200;24;2)$	$-\frac{1}{48}$	$\frac{1}{36}$	$-\frac{1}{60}$	$\frac{5}{16}$	$\frac{1}{8}$	$-\frac{29}{32}$	0	0
$(110;24;0)$	$-\frac{11}{360}$	$-\frac{1}{36}$	$-\frac{1}{60}$	$\frac{19}{32}$	$-\frac{1}{16}$	$\frac{87}{64}$	-2	-8
$(110;24;1)$	$-\frac{1}{180}$	$-\frac{1}{18}$	$\frac{1}{15}$	$-\frac{7}{48}$	$\frac{3}{8}$	$\frac{81}{64}$	2	2
$(110;24;2)$	$-\frac{1}{480}$	$\frac{1}{72}$	$\frac{1}{40}$	$\frac{17}{48}$	$\frac{1}{2}$	$-\frac{521}{64}$	2	7
$(011;24;0)$	$-\frac{1}{96}$	$-\frac{5}{36}$	$\frac{9}{40}$	$\frac{7}{12}$	$\frac{17}{8}$	$-\frac{437}{32}$	7	8
$(011;24;1)$	$-\frac{3}{160}$	$\frac{5}{36}$	$-\frac{11}{60}$	$-\frac{19}{48}$	$-\frac{11}{8}$	$\frac{183}{16}$	-6	-8
$(011;24;2)$	$-\frac{1}{45}$	$-\frac{1}{24}$	$-\frac{1}{60}$	$\frac{1}{16}$	$\frac{15}{16}$	$\frac{65}{32}$	-1	-3
$(101;24;1)$	$-\frac{1}{144}$	$-\frac{1}{18}$	$\frac{1}{20}$	$-\frac{13}{48}$	$\frac{13}{8}$	$\frac{131}{64}$	1	-1
$(101;24;2)$	$-\frac{31}{1440}$	$-\frac{1}{24}$	$\frac{7}{120}$	$\frac{3}{32}$	$\frac{23}{16}$	$-\frac{123}{64}$	2	1
$(020;25;0)$	$-\frac{41}{720}$	$-\frac{1}{6}$	$\frac{7}{30}$	$\frac{1}{3}$	$\frac{7}{4}$	$-\frac{217}{32}$	8	12
$(020;25;2)$	$\frac{41}{720}$	$-\frac{1}{18}$	$\frac{1}{30}$	$-\frac{1}{24}$	$\frac{7}{4}$	0	0	-4
$(002;25;0)$	$\frac{7}{240}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{19}{48}$	$\frac{5}{2}$	$-\frac{41}{8}$	6	10
$(002;25;1)$	0	$\frac{5}{18}$	$-\frac{1}{3}$	$\frac{5}{8}$	$-\frac{35}{8}$	$\frac{93}{32}$	-10	-10
$(002;25;2)$	$\frac{1}{240}$	$\frac{1}{9}$	$-\frac{1}{15}$	$\frac{7}{48}$	$-\frac{3}{2}$	$\frac{1}{4}$	-2	-2
$(200;25;0)$	$\frac{1}{90}$	$\frac{1}{18}$	$-\frac{1}{10}$	$-\frac{1}{8}$	-1	$\frac{15}{8}$	-2	2
$(200;25;1)$	0	$-\frac{1}{6}$	$\frac{1}{5}$	$\frac{37}{48}$	$\frac{21}{8}$	$-\frac{465}{32}$	6	6
$(200;25;2)$	0	$-\frac{1}{9}$	$\frac{1}{5}$	$\frac{1}{2}$	2	$-\frac{183}{16}$	6	6
$(110;25;0)$	$\frac{23}{1440}$	$\frac{1}{18}$	$-\frac{1}{30}$	$-\frac{1}{48}$	$-\frac{7}{16}$	$-\frac{155}{64}$	0	4
$(110;25;1)$	0	$-\frac{1}{6}$	$\frac{1}{5}$	$\frac{17}{32}$	$\frac{21}{8}$	$-\frac{713}{64}$	6	6
$(110;25;2)$	$-\frac{19}{1440}$	$-\frac{7}{36}$	$\frac{13}{60}$	$\frac{17}{48}$	$\frac{35}{16}$	$-\frac{31}{4}$	6	4

Weight 5 PLI	$\pi^4\log(2)$	$\pi^2\log^3(2)$	$\log^5(2)$	$\pi^2\zeta(3)$	$\zeta(3)\log^2(2)$	$\zeta(5)$	$\mathrm{Li}_4(\frac{1}{2})\log(2)$	$\mathrm{Li}_5(\frac{1}{2})$
$(011;25;0)$	$\frac{1}{180}$	$-\frac{1}{18}$	$\frac{1}{10}$	$-\frac{5}{96}$	$\frac{21}{16}$	$-\frac{93}{16}$	4	8
$(011;25;1)$	0	$\frac{1}{18}$	$-\frac{1}{15}$	$\frac{1}{48}$	$-\frac{7}{8}$	$\frac{31}{32}$	-2	-2
$(011;25;2)$	$-\frac{1}{360}$	$-\frac{1}{12}$	$\frac{1}{12}$	$-\frac{3}{32}$	$\frac{7}{16}$	$\frac{31}{64}$	2	0
$(101;25;0)$	$\frac{29}{1440}$	$-\frac{1}{18}$	$\frac{1}{30}$	$\frac{11}{96}$	$\frac{3}{4}$	$-\frac{109}{16}$	2	6
$(101;25;1)$	0	$\frac{1}{18}$	$-\frac{1}{15}$	$-\frac{17}{96}$	$-\frac{7}{8}$	$\frac{31}{8}$	-2	-2
$(101;25;2)$	$\frac{1}{480}$	0	$\frac{1}{15}$	$-\frac{5}{96}$	$\frac{1}{4}$	$-\frac{45}{32}$	2	2
$(020;26;0)$	$\frac{49}{720}$	$-\frac{1}{6}$	$\frac{7}{30}$	$\frac{1}{8}$	$\frac{7}{4}$	$-\frac{527}{32}$	8	12
$(020;26;2)$	$-\frac{49}{720}$	$-\frac{1}{18}$	$\frac{1}{30}$	$-\frac{1}{8}$	$\frac{7}{4}$	$\frac{93}{16}$	0	-4
$(002;26;0)$	$\frac{53}{720}$	$-\frac{1}{18}$	$\frac{3}{20}$	$-\frac{1}{48}$	-1	$-\frac{381}{32}$	6	12
$(002;26;1)$	0	$-\frac{1}{6}$	$\frac{1}{5}$	$-\frac{1}{48}$	$\frac{21}{8}$	$-\frac{93}{16}$	6	6
$(002;26;2)$	$-\frac{1}{15}$	$-\frac{1}{18}$	$-\frac{1}{60}$	0	2	8	-2	-8
$(200;26;0)$	$\frac{1}{360}$	$\frac{1}{18}$	$-\frac{1}{20}$	$\frac{1}{6}$	-1	$-\frac{1}{16}$	-2	-4
$(200;26;1)$	0	$-\frac{1}{6}$	$\frac{1}{5}$	$-\frac{11}{24}$	$\frac{21}{8}$	$-\frac{31}{32}$	6	6
$(200;26;2)$	$\frac{17}{720}$	$-\frac{1}{6}$	$\frac{11}{60}$	$-\frac{5}{16}$	2	$-\frac{149}{32}$	6	8
$(110;26;0)$	$\frac{1}{96}$	0	0	$\frac{23}{96}$	$-\frac{7}{16}$	$-\frac{217}{64}$	0	0
$(110;26;1)$	0	$-\frac{1}{6}$	$\frac{1}{5}$	$-\frac{43}{96}$	$\frac{21}{8}$	$-\frac{93}{64}$	6	6
$(110;26;2)$	$\frac{17}{720}$	$-\frac{5}{36}$	$\frac{11}{60}$	$-\frac{13}{48}$	$\frac{35}{16}$	$-\frac{341}{64}$	6	8
$(011;26;0)$	$\frac{151}{1440}$	$-\frac{2}{9}$	$\frac{1}{3}$	$\frac{13}{96}$	$\frac{21}{16}$	$-\frac{713}{32}$	12	20
$(011;26;1)$	0	$\frac{1}{18}$	$-\frac{1}{15}$	$-\frac{3}{8}$	$-\frac{7}{8}$	$\frac{217}{32}$	-2	-2
$(011;26;2)$	$-\frac{17}{240}$	$\frac{1}{12}$	$-\frac{3}{20}$	$-\frac{5}{24}$	$\frac{7}{16}$	$\frac{465}{32}$	-6	-12
$(101;26;0)$	$\frac{11}{288}$	0	$\frac{1}{20}$	$\frac{17}{96}$	-1	$-\frac{405}{64}$	2	4
$(101;26;1)$	0	$-\frac{1}{6}$	$\frac{1}{5}$	$-\frac{37}{96}$	$\frac{21}{8}$	$-\frac{93}{64}$	6	6
$(101;26;2)$	$-\frac{31}{1440}$	$-\frac{1}{9}$	$\frac{1}{12}$	$-\frac{25}{96}$	2	$\frac{189}{64}$	2	0
$(010;31;0)$	0	0	0	$\frac{1}{3}$	0	$-\frac{9}{2}$	0	0
$(010;31;1)$	0	0	0	0	0	-1	0	0
$(010;31;2)$	0	0	0	$\frac{7}{16}$	0	$-\frac{83}{16}$	0	0
$(100;31;1)$	0	0	0	$\frac{1}{6}$	0	-3	0	0
$(100;31;2)$	$\frac{1}{360}$	$\frac{1}{36}$	$-\frac{1}{60}$	$\frac{1}{3}$	$-\frac{3}{8}$	$-\frac{85}{16}$	0	2
$(001;31;1)$	0	0	0	$-\frac{1}{8}$	0	$\frac{59}{32}$	0	0
$(001;31;2)$	0	$\frac{1}{9}$	$-\frac{2}{15}$	$\frac{1}{8}$	$-\frac{5}{4}$	$\frac{39}{16}$	-4	-4
$(010;32;0)$	0	0	0	$\frac{1}{48}$	0	$\frac{11}{32}$	0	0
$(010;32;1)$	0	0	0	0	0	$\frac{15}{16}$	0	0
$(010;32;2)$	0	0	0	$-\frac{1}{8}$	0	$\frac{51}{32}$	0	0
$(100;32;1)$	0	0	0	$-\frac{1}{12}$	0	$\frac{59}{32}$	0	0
$(100;32;2)$	$\frac{49}{1440}$	$-\frac{1}{36}$	$\frac{1}{20}$	$-\frac{7}{96}$	$-\frac{3}{8}$	$-\frac{27}{8}$	2	4
$(001;32;1)$	0	0	0	$\frac{1}{16}$	0	$-\frac{17}{16}$	0	0

Weight 5 PLI	$\pi^4\log(2)$	$\pi^2\log^3(2)$	$\log^5(2)$	$\pi^2\zeta(3)$	$\zeta(3)\log^2(2)$	$\zeta(5)$	$\mathrm{Li}_4(\frac{1}{2})\log(2)$	$\mathrm{Li}_5(\frac{1}{2})$
$(001;32;2)$	0	$\frac{1}{18}$	$-\frac{1}{15}$	$\frac{1}{96}$	$-\frac{5}{4}$	$\frac{125}{64}$	-2	-2
$(010;33;0)$	$\frac{1}{720}$	$\frac{1}{36}$	$-\frac{1}{40}$	$\frac{1}{12}$	$-\frac{1}{2}$	$-\frac{1}{32}$	-1	-2
$(010;33;2)$	$-\frac{1}{720}$	$\frac{1}{18}$	$-\frac{3}{40}$	$\frac{1}{16}$	$-\frac{13}{16}$	$\frac{33}{64}$	-2	-1
$(100;33;0)$	0	0	0	0	0	0	0	-1
$(100;33;1)$	$\frac{19}{1440}$	$\frac{5}{72}$	$-\frac{13}{120}$	$\frac{3}{32}$	$-\frac{23}{16}$	$\frac{23}{64}$	-3	-2
$(100;33;2)$	$-\frac{1}{288}$	$\frac{5}{72}$	$-\frac{11}{120}$	$\frac{7}{48}$	$-\frac{7}{8}$	$\frac{81}{64}$	-3	-3
$(001;33;0)$	0	$\frac{1}{36}$	$-\frac{1}{30}$	$\frac{7}{96}$	$-\frac{7}{16}$	$\frac{27}{32}$	-1	-2
$(001;33;1)$	$-\frac{11}{240}$	$\frac{1}{24}$	$-\frac{1}{15}$	$\frac{1}{48}$	$\frac{5}{16}$	$\frac{457}{64}$	-3	-7
$(001;33;2)$	$-\frac{1}{288}$	$-\frac{1}{72}$	$\frac{1}{120}$	$\frac{1}{96}$	$\frac{7}{16}$	$-\frac{3}{64}$	0	0
$(010;34;0)$	$\frac{1}{180}$	$-\frac{1}{18}$	$\frac{1}{20}$	$\frac{5}{32}$	$\frac{13}{16}$	$-\frac{405}{64}$	2	4
$(010;34;2)$	$-\frac{1}{180}$	$-\frac{1}{36}$	$\frac{1}{20}$	$\frac{1}{16}$	$\frac{1}{2}$	$-\frac{15}{16}$	1	-1
$(100;34;1)$	$\frac{11}{1440}$	$-\frac{5}{72}$	$\frac{1}{40}$	$-\frac{19}{96}$	$\frac{5}{16}$	$\frac{29}{16}$	0	-3
$(100;34;2)$	$\frac{1}{90}$	$-\frac{1}{36}$	$\frac{1}{30}$	$\frac{3}{32}$	$\frac{7}{16}$	$-\frac{123}{32}$	1	2
$(001;34;1)$	$\frac{1}{36}$	$\frac{5}{72}$	$-\frac{1}{30}$	$-\frac{1}{8}$	$-\frac{23}{16}$	$-\frac{35}{32}$	0	4
$(001;34;2)$	$\frac{1}{90}$	0	0	0	0	-1	0	1
$(010;35;0)$	$-\frac{1}{360}$	$\frac{1}{12}$	$-\frac{7}{60}$	$\frac{1}{12}$	$-\frac{7}{8}$	$\frac{93}{32}$	-4	-6
$(010;35;2)$	$\frac{1}{360}$	$-\frac{1}{12}$	$\frac{7}{60}$	$\frac{5}{24}$	$\frac{7}{8}$	$-\frac{31}{4}$	4	6
$(100;35;0)$	$-\frac{1}{180}$	$\frac{1}{36}$	$-\frac{1}{20}$	$-\frac{3}{16}$	$-\frac{1}{2}$	$\frac{153}{32}$	-2	-4
$(100;35;1)$	0	$-\frac{1}{18}$	$\frac{1}{15}$	$\frac{5}{12}$	$\frac{7}{8}$	$-\frac{465}{64}$	2	2
$(100;35;2)$	$\frac{1}{60}$	$-\frac{1}{36}$	$\frac{1}{20}$	$\frac{5}{16}$	$\frac{1}{2}$	$-\frac{61}{8}$	2	4
$(001;35;0)$	$-\frac{1}{180}$	$\frac{1}{36}$	$-\frac{1}{20}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{19}{32}$	-2	-4
$(001;35;1)$	0	$-\frac{1}{18}$	$\frac{1}{15}$	$-\frac{17}{48}$	$\frac{7}{8}$	$\frac{155}{64}$	2	2
$(001;35;2)$	$\frac{1}{60}$	$-\frac{1}{36}$	$\frac{1}{20}$	$-\frac{1}{8}$	$\frac{1}{2}$	$-\frac{39}{16}$	2	4
$(010;36;0)$	$-\frac{49}{1440}$	$\frac{1}{12}$	$-\frac{7}{60}$	$-\frac{1}{16}$	$-\frac{7}{8}$	$\frac{527}{64}$	-4	-6
$(010;36;2)$	$\frac{49}{1440}$	$-\frac{1}{12}$	$\frac{7}{60}$	$-\frac{1}{12}$	$\frac{7}{8}$	$-\frac{341}{64}$	4	6
$(100;36;0)$	$-\frac{19}{1440}$	$\frac{1}{18}$	$-\frac{1}{15}$	$\frac{7}{96}$	$-\frac{1}{2}$	$\frac{151}{64}$	-2	-2
$(100;36;1)$	0	$-\frac{1}{18}$	$\frac{1}{15}$	$-\frac{3}{32}$	$\frac{7}{8}$	$-\frac{31}{64}$	2	2
$(100;36;2)$	$\frac{1}{288}$	$-\frac{1}{18}$	$\frac{1}{15}$	$-\frac{13}{96}$	$\frac{1}{2}$	$-\frac{23}{64}$	2	2
$(001;36;0)$	$-\frac{19}{1440}$	$\frac{1}{18}$	$-\frac{1}{15}$	$-\frac{5}{96}$	$-\frac{1}{2}$	$\frac{193}{64}$	-2	-2
$(001;36;1)$	0	$-\frac{1}{18}$	$\frac{1}{15}$	$\frac{1}{96}$	$\frac{7}{8}$	$-\frac{155}{64}$	2	2
$(001;36;2)$	$\frac{1}{288}$	$-\frac{1}{18}$	$\frac{1}{15}$	$-\frac{1}{96}$	$\frac{1}{2}$	$-\frac{125}{64}$	2	2
$(000;41;1)$	0	0	0	0	0	1	0	0
$(000;41;2)$	$\frac{1}{90}$	0	0	$\frac{1}{8}$	0	$-\frac{59}{32}$	0	0
$(000;42;1)$	0	0	0	0	0	$-\frac{15}{16}$	0	0
$(000;42;2)$	$-\frac{7}{720}$	0	0	$-\frac{1}{16}$	0	$\frac{17}{16}$	0	0
$(000;43;0)$	0	0	0	0	0	0	0	1

Weight 5 PLI	$\pi^4\log(2)$	$\pi^2\log^3(2)$	$\log^5(2)$	$\pi^2\zeta(3)$	$\zeta(3)\log^2(2)$	$\zeta(5)$	$\mathrm{Li}_4(\frac{1}{2})\log(2)$	$\mathrm{Li}_5(\frac{1}{2})$
$(000;43;2)$	0	$\frac{1}{36}$	$-\frac{1}{30}$	$\frac{7}{96}$	$-\frac{7}{16}$	$\frac{27}{32}$	-1	-2
$(000;44;2)$	0	0	0	0	0	1	0	-1
$(000;45;0)$	$-\frac{1}{90}$	0	0	$-\frac{1}{8}$	0	$\frac{91}{32}$	0	0
$(000;45;2)$	$\frac{1}{90}$	0	0	$\frac{1}{8}$	0	$-\frac{59}{32}$	0	0
$(000;46;0)$	$\frac{7}{720}$	0	0	$\frac{1}{16}$	0	-2	0	0
$(000;46;2)$	$-\frac{7}{720}$	0	0	$-\frac{1}{16}$	0	$\frac{17}{16}$	0	0
$(000;21,21;1)$	0	0	0	$\frac{1}{3}$	0	-3	0	0
$(000;21,21;2)$	$\frac{1}{36}$	0	0	$-\frac{5}{24}$	0	$\frac{29}{32}$	0	0
$(000;22,22;1)$	0	0	0	$\frac{1}{8}$	0	$-\frac{17}{16}$	0	0
$(000;22,22;2)$	$\frac{1}{144}$	$-\frac{2}{9}$	$\frac{4}{15}$	$-\frac{1}{24}$	$\frac{7}{2}$	$-\frac{125}{16}$	8	8
$(000;23,23;0)$	$-\frac{1}{144}$	$\frac{1}{12}$	$-\frac{7}{60}$	$\frac{7}{48}$	$-\frac{7}{8}$	$\frac{27}{32}$	-2	-2
$(000;23,23;2)$	$\frac{1}{144}$	$-\frac{1}{36}$	$\frac{1}{20}$	$-\frac{1}{24}$	0	$\frac{3}{16}$	0	0
$(000;24,24;2)$	$\frac{1}{144}$	$-\frac{1}{12}$	$\frac{7}{60}$	$\frac{3}{16}$	$\frac{7}{8}$	$-\frac{123}{32}$	2	2
$(000;25,25;2)$	$\frac{1}{36}$	0	0	$-\frac{5}{24}$	0	$\frac{29}{32}$	0	0
$(000;25,25;0)$	$-\frac{1}{36}$	0	0	$\frac{13}{24}$	0	$-\frac{125}{32}$	0	0
$(000;26,26;2)$	$\frac{1}{144}$	$-\frac{2}{9}$	$\frac{4}{15}$	$-\frac{1}{24}$	$\frac{7}{2}$	$-\frac{125}{16}$	8	8
$(000;26,26;0)$	$-\frac{1}{144}$	$\frac{2}{9}$	$-\frac{4}{15}$	$\frac{1}{6}$	$-\frac{7}{2}$	$\frac{27}{4}$	-8	-8
$(000;21,22;1)$	0	0	0	$-\frac{5}{24}$	0	$\frac{59}{32}$	0	0
$(000;21,22;2)$	$-\frac{1}{72}$	$-\frac{1}{9}$	$\frac{2}{15}$	$-\frac{19}{96}$	$\frac{7}{4}$	$-\frac{33}{32}$	4	4
$(000;21,23;0)$	$-\frac{19}{1440}$	$-\frac{5}{72}$	$\frac{13}{120}$	$-\frac{3}{32}$	$\frac{23}{16}$	$-\frac{23}{64}$	3	2
$(000;21,23;1)$	$-\frac{1}{60}$	$-\frac{1}{36}$	$\frac{1}{60}$	$-\frac{11}{48}$	$\frac{3}{8}$	$\frac{311}{64}$	0	-2
$(000;21,23;2)$	$\frac{19}{1440}$	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{5}{16}$	$\frac{13}{8}$	$-\frac{101}{64}$	4	5
$(000;21,24;1)$	$-\frac{1}{72}$	$-\frac{1}{18}$	$\frac{1}{15}$	$\frac{1}{2}$	$\frac{3}{8}$	$-\frac{85}{16}$	2	2
$(000;21,24;2)$	$\frac{11}{1440}$	$-\frac{5}{72}$	$\frac{1}{40}$	$-\frac{1}{32}$	$\frac{5}{16}$	$\frac{29}{16}$	0	-3
$(000;21,25;1)$	0	$\frac{1}{18}$	$-\frac{1}{15}$	$-\frac{5}{12}$	$-\frac{7}{8}$	$\frac{465}{64}$	-2	-2
$(000;21,25;2)$	$-\frac{7}{1440}$	$\frac{5}{72}$	$-\frac{11}{120}$	$-\frac{17}{48}$	$-\frac{7}{8}$	$\frac{31}{4}$	-3	-4
$(000;21,25;0)$	$\frac{7}{1440}$	$-\frac{1}{72}$	$\frac{1}{40}$	$-\frac{1}{16}$	0	$-\frac{31}{64}$	1	2
$(000;21,26;1)$	0	$\frac{1}{18}$	$-\frac{1}{15}$	$\frac{17}{48}$	$-\frac{7}{8}$	$-\frac{155}{64}$	-2	-2
$(000;21,26;2)$	$-\frac{11}{720}$	$\frac{5}{72}$	$-\frac{11}{120}$	$\frac{23}{96}$	$-\frac{7}{8}$	$\frac{93}{64}$	-3	-4
$(000;21,26;0)$	$\frac{11}{720}$	$-\frac{1}{72}$	$\frac{1}{40}$	$-\frac{7}{48}$	0	$-\frac{31}{32}$	1	2
$(000;22,23;0)$	$\frac{11}{240}$	$-\frac{1}{24}$	$\frac{1}{15}$	$-\frac{1}{48}$	$-\frac{5}{16}$	$-\frac{457}{64}$	3	7
$(000;22,23;1)$	$-\frac{1}{288}$	0	0	$\frac{1}{48}$	$\frac{3}{8}$	$-\frac{15}{32}$	0	0
$(000;22,23;2)$	$-\frac{11}{240}$	$\frac{5}{72}$	$-\frac{1}{10}$	$\frac{1}{24}$	$-\frac{1}{8}$	$\frac{61}{8}$	-4	-8
$(000;22,24;1)$	$\frac{11}{360}$	$-\frac{1}{36}$	$\frac{1}{20}$	$-\frac{19}{96}$	$\frac{3}{8}$	$-\frac{27}{8}$	2	4
$(000;22,24;2)$	$\frac{1}{36}$	$\frac{5}{72}$	$-\frac{1}{30}$	$-\frac{5}{24}$	$-\frac{23}{16}$	$-\frac{35}{32}$	0	4
$(000;22,25;0)$	$\frac{11}{720}$	$-\frac{1}{72}$	$\frac{1}{40}$	$\frac{11}{96}$	0	$-\frac{31}{8}$	1	2

Weight 5 PLI	$\pi^4\log(2)$	$\pi^2\log^3(2)$	$\log^5(2)$	$\pi^2\zeta(3)$	$\zeta(3)\log^2(2)$	$\zeta(5)$	$\mathrm{Li}_4(\frac{1}{2})\log(2)$	$\mathrm{Li}_5(\frac{1}{2})$
$(000;22,25;1)$	0	$\frac{1}{18}$	$-\frac{1}{15}$	$\frac{3}{32}$	$-\frac{7}{8}$	$\frac{31}{64}$	-2	-2
$(000;22,25;2)$	$-\frac{11}{720}$	$\frac{5}{72}$	$-\frac{11}{120}$	$\frac{23}{96}$	$-\frac{7}{8}$	$\frac{93}{64}$	-3	-4
$(000;22,26;0)$	$-\frac{19}{240}$	$\frac{7}{72}$	$-\frac{7}{40}$	$\frac{1}{32}$	0	$\frac{899}{64}$	-7	-14
$(000;22,26;1)$	0	$\frac{1}{18}$	$-\frac{1}{15}$	$-\frac{1}{96}$	$-\frac{7}{8}$	$\frac{155}{64}$	-2	-2
$(000;22,26;2)$	$\frac{19}{240}$	$-\frac{1}{24}$	$\frac{13}{120}$	$-\frac{1}{24}$	$-\frac{7}{8}$	$-\frac{93}{8}$	5	12
$(000;23,24;0)$	$-\frac{1}{144}$	$-\frac{1}{18}$	$\frac{1}{20}$	$-\frac{7}{96}$	$\frac{21}{16}$	$-\frac{81}{64}$	3	3
$(000;23,24;2)$	$\frac{1}{144}$	$\frac{1}{18}$	$-\frac{1}{20}$	$-\frac{25}{96}$	$-\frac{21}{16}$	$\frac{369}{64}$	-3	-3
$(000;23,25;0)$	$-\frac{1}{72}$	$\frac{1}{18}$	$-\frac{1}{30}$	$\frac{5}{24}$	$-\frac{7}{16}$	$-\frac{5}{16}$	-1	-1
$(000;23,25;2)$	$\frac{1}{72}$	$\frac{1}{18}$	$-\frac{1}{10}$	$\frac{1}{8}$	$-\frac{21}{16}$	$\frac{29}{32}$	-3	-3
$(000;23,26;0)$	$\frac{1}{144}$	$-\frac{5}{72}$	$\frac{1}{15}$	$-\frac{5}{32}$	$\frac{7}{8}$	$-\frac{51}{64}$	2	2
$(000;23,26;2)$	$-\frac{1}{144}$	$\frac{1}{8}$	$-\frac{2}{15}$	$\frac{1}{24}$	$-\frac{7}{4}$	$\frac{61}{16}$	-4	-4
$(000;24,25;0)$	$-\frac{1}{72}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{5}{96}$	$-\frac{35}{16}$	$\frac{57}{8}$	-5	-5
$(000;24,25;2)$	$\frac{1}{72}$	$-\frac{1}{18}$	$\frac{1}{30}$	$-\frac{17}{96}$	$\frac{7}{16}$	$\frac{1}{2}$	1	1
$(000;24,26;0)$	$\frac{1}{144}$	$\frac{11}{72}$	$-\frac{1}{5}$	$-\frac{3}{16}$	$-\frac{21}{8}$	$\frac{57}{8}$	-6	-6
$(000;24,26;2)$	$-\frac{1}{144}$	$-\frac{7}{72}$	$\frac{2}{15}$	$\frac{25}{96}$	$\frac{7}{4}$	$-\frac{433}{64}$	4	4
$(000;25,26;0)$	$\frac{1}{72}$	$\frac{1}{9}$	$-\frac{2}{15}$	$-\frac{1}{96}$	$-\frac{7}{4}$	$\frac{23}{8}$	-4	-4
$(000;25,26;2)$	$-\frac{1}{72}$	$-\frac{1}{9}$	$\frac{2}{15}$	$-\frac{19}{96}$	$\frac{7}{4}$	$-\frac{33}{32}$	4	4

Appendix 4. Other closed-forms

We present some new and typical examples. Interested readers may derive similar results by themselves using methods recorded in section 4.

(1) LSIs, PLSIs and Arctan integrals

Several weight 4, 5, 6 LSIs involving only log-trigonometric terms:

$$\int_0^{\frac{\pi}{2}} x\log^2(2\sin(x))\,dx$$
$$= \mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{7}{8}\zeta(3)\log(2)-\frac{19\pi^4}{2880}+\frac{\log^4(2)}{24}-\frac{1}{24}\pi^2\log^2(2)$$

$$\int_0^{\frac{\pi}{2}} x\log^3(2\sin(x))\,dx$$

$$= 3\mathrm{Li}_5\left(\frac{1}{2}\right) + 3\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) - \frac{3\pi^2\zeta(3)}{16} - \frac{93\zeta(5)}{128} + \frac{21}{16}\zeta(3)\log^2(2) + \frac{\log^5(2)}{10} - \frac{1}{12}\pi^2\log^3(2)$$

$$\int_0^{\frac{\pi}{2}} x\log^2(2\sin(x))\log(2\cos(x))\,dx$$

$$= \mathrm{Li}_5\left(\frac{1}{2}\right) + \mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) - \frac{155\zeta(5)}{128} - \frac{\pi^2\zeta(3)}{192} + \frac{7}{16}\zeta(3)\log^2(2) + \frac{\log^5(2)}{30} - \frac{1}{36}\pi^2\log^3(2)$$

$$\int_0^{\frac{\pi}{2}} x^2\log^2(2\sin(x))\,dx$$

$$= \pi\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7}{8}\pi\zeta(3)\log(2) - \frac{3\pi^5}{320} + \frac{1}{24}\pi\log^4(2) - \frac{1}{24}\pi^3\log^2(2)$$

$$\int_0^{\frac{\pi}{2}} x^3\log^2(2\sin(x))\,dx$$

$$= \frac{3}{4}ES(1;-5) + \frac{3}{4}\pi^2\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{3\zeta(3)^2}{8} + \frac{21}{32}\pi^2\zeta(3)\log(2) - \frac{89\pi^6}{11520} + \frac{1}{32}\pi^2\log^4(2) - \frac{1}{32}\pi^4\log^2(2)$$

$$\int_0^{\frac{\pi}{2}} x^2\log(2\sin(x))\log^2(2\cos(x))\,dx$$

$$= -\pi\mathrm{Li}_5\left(\frac{1}{2}\right) - \pi\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) + \frac{3\pi^3\zeta(3)}{64} + \frac{121\pi\zeta(5)}{128} - \frac{7}{16}\pi\zeta(3)\log^2(2) - \frac{1}{30}\pi\log^5(2) + \frac{1}{36}\pi^3\log^3(2)$$

$$\int_0^{\frac{\pi}{2}} x\log^4(2\sin(x))\,dx$$

$$= \frac{3}{2}ES(1;-5) + 12\mathrm{Li}_6\left(\frac{1}{2}\right) + 6\mathrm{Li}_4\left(\frac{1}{2}\right)\log^2(2)$$

$$+12\mathrm{Li}_5\left(\frac{1}{2}\right)\log(2) - \frac{3\zeta(3)^2}{4} + \frac{7}{4}\zeta(3)\log^3(2) - \frac{57\pi^6}{4480} + \frac{\log^6(2)}{6} - \frac{1}{8}\pi^2\log^4(2)$$

$$\int_0^{\frac{\pi}{2}} x\log^3(2\sin(x))\log(2\cos(x))\,dx$$

$$= \frac{9}{4}ES(1;-5) - \frac{1}{8}\pi^2\mathrm{Li}_4\left(\frac{1}{2}\right) + 6\mathrm{Li}_6\left(\frac{1}{2}\right) + 3\mathrm{Li}_4\left(\frac{1}{2}\right)\log^2(2) + 6\mathrm{Li}_5\left(\frac{1}{2}\right)\log(2) + \frac{3\zeta(3)^2}{2}$$

$$-\frac{1579\pi^6}{161280} + \frac{7}{8}\zeta(3)\log^3(2) + \frac{\log^6(2)}{12} - \frac{7}{64}\pi^2\zeta(3)\log(2) - \frac{13}{192}\pi^2\log^4(2) + \frac{1}{192}\pi^4\log^2(2)$$

$$\int_0^{\frac{\pi}{2}} x^5\log(2\sin(x))\,dx = \frac{15\pi^4\zeta(3)}{256} - \frac{225\pi^2\zeta(5)}{256} + \frac{1905\zeta(7)}{512}$$

$$\int_0^{\frac{\pi}{2}} x^4\log^2(2\sin(x))\,dx$$

$$= \frac{3}{2}\pi ES(1;-5) + \frac{1}{2}\pi^3 \mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7}{16}\pi^3\zeta(3)\log(2) - \frac{269\pi^7}{40320} + \frac{1}{48}\pi^3\log^4(2) - \frac{1}{48}\pi^5\log^2(2)$$

2 weight 5 LSIs involving $\cot(x)$:

$$\int_0^{\frac{\pi}{2}} x^3\cot(x)\log(2\sin(x))\,dx$$

$$= -\frac{3}{2}\pi \mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{21}{16}\pi\zeta(3)\log(2) + \frac{9\pi^5}{640} - \frac{1}{16}\pi\log^4(2) + \frac{1}{8}\pi^3\log^2(2)$$

$$\int_0^{\frac{\pi}{2}} x^2\cot(x)\log(2\sin(x))\log(2\cos(x))\,dx$$

$$= \frac{\pi^2\zeta(3)}{12} - \frac{31\zeta(5)}{64} - \frac{7}{8}\zeta(3)\log^2(2) + \frac{1}{12}\pi^2\log^3(2) - \frac{1}{192}\pi^4\log(2)$$

Several weight 5 PLSIs (use duplication formula of sine for the last two):

$$\int_0^{\frac{\pi}{2}} \mathrm{Cl}_2(2x)\log^2(2\sin(x))\,dx$$

$$= -2\mathrm{Li}_5\left(\frac{1}{2}\right) - 2\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) + \frac{155\zeta(5)}{64} - \frac{7}{8}\zeta(3)\log^2(2) - \frac{1}{15}\log^5(2) + \frac{1}{18}\pi^2\log^3(2)$$

$$\int_0^{\frac{\pi}{2}} \mathrm{Cl}_2(2x)\log(2\sin(x))\log(2\cos(x))\,dx$$

$$= 2\mathrm{Li}_5\left(\frac{1}{2}\right) + 2\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) - \frac{279\zeta(5)}{128} + \frac{7}{8}\zeta(3)\log^2(2) + \frac{\log^5(2)}{15} - \frac{1}{18}\pi^2\log^3(2)$$

$$\int_0^{\frac{\pi}{2}} \mathrm{Cl}_2(4x)\log^2(2\cos(x))\,dx$$

$$= 16\mathrm{Li}_5\left(\frac{1}{2}\right) + 16\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) - \frac{465\zeta(5)}{32} - \frac{7\pi^2\zeta(3)}{48} + 7\zeta(3)\log^2(2) + \frac{8\log^5(2)}{15} - \frac{4}{9}\pi^2\log^3(2)$$

$$\int_0^{\frac{\pi}{2}} x\mathrm{Ti}_2(\tan(x))\log(2\sin(x))\,dx$$

$$= \frac{7}{4}\pi \mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{49}{32}\pi\zeta(3)\log(2) - \frac{163\pi^5}{11520} + \frac{7}{96}\pi\log^4(2) - \frac{7}{96}\pi^3\log^2(2)$$

$$\int_0^{\frac{\pi}{2}} x\mathrm{Cl}_2(2x)\cot(x)\log(2\sin(2x))\,dx$$

$$= 4\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) - \frac{65\pi^2\zeta(3)}{192} + \frac{341\zeta(5)}{128} + \frac{7}{2}\zeta(3)\log^2(2) + \frac{\log^5(2)}{6} - \frac{1}{6}\pi^2\log^3(2) - \frac{61\pi^4\log(2)}{2880}$$

$$\int_0^{\frac{\pi}{2}} x\mathrm{Cl}_2(4x)\cot(x)\log(2\sin(2x))\,dx$$

$$= 16\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) - \frac{23\pi^2\zeta(3)}{48} + \frac{31\zeta(5)}{4} + 14\zeta(3)\log^2(2) + \frac{2\log^5(2)}{3} - \frac{2}{3}\pi^2\log^3(2) - \frac{151}{720}\pi^4\log(2)$$

Some examples of Arctan family. Note the resemblance between them and QLIs, as well as homogeneous and non-homo patterns:

$$\int_0^\infty \frac{\log^3\left(x^2+1\right)\tan^{-1}(x)}{x^2}\,dx$$

$$= -24\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{83\pi^4}{120} - \log^4(2) + 4\pi^2\log^2(2)$$

$$\int_0^\infty \frac{\log^2\left(x^2+1\right)\tan^{-1}(x)^2}{x^4}\,dx$$

$$= -\frac{\pi\zeta(3)}{6} + \frac{\pi^3}{18} - \frac{4}{9}\pi\log^3(2) + \frac{4}{3}\pi\log^2(2) - \frac{2}{9}\pi^3\log(2) + \frac{4}{3}\pi\log(2)$$

$$\int_0^\infty \frac{\log^2\left(x^2+1\right)\tan^{-1}(x)^3}{x^2}\,dx$$

$$= 24\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{3\pi^2\zeta(3)}{8} - \frac{93\zeta(5)}{2} - \frac{1}{5}\log^5(2) + \frac{4}{3}\pi^2\log^3(2) + \frac{47}{60}\pi^4\log(2)$$

$$\int_0^\infty \frac{\log^2\left(x^2+1\right)\tan^{-1}(x)^2}{x^2+1}\,dx$$

$$= 2\pi\zeta(3)\log(2) + \frac{11\pi^5}{360} + \frac{1}{6}\pi^3\log^2(2)$$

$$\int_0^\infty \frac{\log^2\left(x^2+1\right)\tan^{-1}(x)^2}{x^3+x}\,dx$$

$$= 8\mathrm{Li}_5\left(\frac{1}{2}\right) - \frac{3\pi^2\zeta(3)}{8} - \frac{217\zeta(5)}{16} - \frac{1}{15}\log^5(2) + \frac{4}{9}\pi^2\log^3(2) + \frac{79}{360}\pi^4\log(2)$$

(2) QLIs and QLSIs

We tabulate all QLIs with weight ≤ 4 here. Note that in this part we abbreviate $\Im(\mathrm{Li}_3(1+i)), \Im(\mathrm{Li}_4(1+i)), \psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right)$ as P_3, P_4, Ψ_3 respectively.

Weight 2 QLI	π^2	$\log^2(2)$
$(4;2)$	$\frac{1}{24}$	0
$(4;3)$	$-\frac{1}{48}$	$\frac{3}{4}$
$(5;4)$	$\frac{1}{32}$	0
$(1;5)$	$-\frac{5}{96}$	$\frac{1}{8}$
$(2;5)$	$-\frac{1}{48}$	0
$(3;5)$	$\frac{1}{96}$	$\frac{1}{8}$
$(4;5)$	0	$\frac{1}{4}$

Weight 2 QLI	$\pi\log(2)$	C
$(5;2)$	0	1
$(5;3)$	$\frac{1}{8}$	0
$(1;4)$	$\frac{1}{8}$	-1
$(2;4)$	0	-1
$(3;4)$	$\frac{1}{8}$	0
$(4;4)$	$\frac{1}{2}$	-1
$(5;5)$	$-\frac{1}{8}$	$\frac{1}{2}$

Weight 3 QLI	$\pi^2 \log(2)$	$\log^3(2)$	$\zeta(3)$	πC
$(24;1)$	$-\frac{3}{16}$	0	2	$-\frac{1}{2}$
$(14;2)$	0	0	$\frac{23}{32}$	$-\frac{1}{2}$
$(24;2)$	0	0	$-\frac{3}{16}$	0
$(34;2)$	0	0	$-\frac{33}{32}$	$\frac{1}{2}$
$(44;2)$	0	0	$\frac{1}{8}$	0
$(55;2)$	0	0	$-\frac{7}{8}$	$\frac{1}{2}$
$(14;3)$	$-\frac{5}{32}$	$\frac{11}{24}$	$\frac{7}{4}$	$-\frac{1}{2}$
$(24;3)$	$-\frac{1}{16}$	0	$\frac{3}{2}$	$-\frac{1}{2}$
$(34;3)$	$-\frac{1}{96}$	$\frac{11}{24}$	0	0
$(44;3)$	$-\frac{1}{24}$	$\frac{2}{3}$	$\frac{5}{2}$	-1
$(55;3)$	$\frac{1}{32}$	0	$-\frac{21}{32}$	$\frac{1}{4}$
$(15;4)$	$\frac{1}{64}$	0	$-\frac{7}{64}$	$-\frac{1}{8}$
$(25;4)$	0	0	$\frac{7}{16}$	$-\frac{1}{4}$
$(35;4)$	$\frac{1}{64}$	0	$\frac{21}{64}$	$-\frac{1}{8}$
$(45;4)$	$\frac{1}{16}$	0	$\frac{21}{64}$	$-\frac{1}{4}$
$(11;5)$	$-\frac{5}{96}$	$\frac{1}{24}$	$\frac{35}{32}$	0
$(12;5)$	$-\frac{3}{32}$	0	$\frac{41}{64}$	0
$(13;5)$	$-\frac{1}{48}$	$\frac{1}{24}$	$-\frac{7}{64}$	0
$(14;5)$	$-\frac{5}{96}$	$\frac{1}{12}$	$\frac{11}{16}$	$-\frac{1}{4}$
$(22;5)$	0	0	$\frac{3}{16}$	0
$(23;5)$	$\frac{1}{32}$	0	$-\frac{15}{64}$	0
$(24;5)$	0	0	$-\frac{1}{32}$	0
$(33;5)$	$\frac{1}{96}$	$\frac{1}{24}$	0	0
$(34;5)$	$\frac{1}{96}$	$\frac{1}{12}$	$-\frac{5}{8}$	$\frac{1}{4}$
$(44;5)$	0	$\frac{1}{6}$	0	0
$(55;5)$	$-\frac{1}{32}$	0	$-\frac{21}{64}$	$\frac{1}{4}$

Weight 3 QLI	π^3	$\pi\log^2(2)$	$C\log(2)$	P_3
$(25;1)$	$\frac{3}{64}$	$\frac{1}{8}$	$\frac{1}{2}$	-2
$(15;2)$	0	$\frac{1}{16}$	$\frac{1}{2}$	-1
$(25;2)$	$-\frac{1}{32}$	0	0	0
$(35;2)$	$\frac{3}{32}$	$\frac{3}{16}$	$\frac{3}{2}$	-3
$(45;2)$	$\frac{1}{16}$	$\frac{1}{8}$	1	-2
$(15;3)$	$\frac{19}{384}$	$\frac{7}{32}$	$\frac{1}{2}$	-2
$(25;3)$	$-\frac{1}{64}$	0	$\frac{1}{2}$	0
$(35;3)$	$\frac{7}{128}$	$\frac{7}{32}$	1	-2
$(45;3)$	$\frac{5}{96}$	$\frac{1}{4}$	1	-2
$(11;4)$	$\frac{7}{64}$	$\frac{3}{16}$	0	-2
$(12;4)$	$\frac{3}{64}$	$\frac{1}{16}$	0	-1
$(13;4)$	$-\frac{1}{32}$	0	-1	1
$(14;4)$	$\frac{9}{64}$	$\frac{1}{2}$	$-\frac{1}{2}$	-4
$(22;4)$	$\frac{1}{16}$	0	0	0
$(23;4)$	$-\frac{5}{64}$	$-\frac{3}{16}$	-2	3
$(24;4)$	$\frac{3}{32}$	$\frac{1}{8}$	-1	-2
$(33;4)$	$-\frac{7}{64}$	$-\frac{3}{16}$	-2	4
$(34;4)$	$-\frac{7}{64}$	0	$-\frac{5}{2}$	4
$(44;4)$	$\frac{7}{48}$	$\frac{5}{4}$	-2	-4
$(55;4)$	$\frac{1}{192}$	0	0	0
$(15;5)$	$-\frac{19}{384}$	$-\frac{1}{8}$	$\frac{1}{4}$	1
$(25;5)$	$-\frac{5}{64}$	$-\frac{1}{8}$	0	2
$(35;5)$	$\frac{11}{384}$	0	$\frac{3}{4}$	-1
$(45;5)$	$-\frac{7}{192}$	$-\frac{1}{4}$	$\frac{1}{2}$	1

Weight 4 QLI	π^4	$\pi^2\log^2(2)$	$\log^4(2)$	$\zeta(3)\log(2)$	$\mathrm{Li}_4(\frac{1}{2})$	πP_3	$\pi C\log(2)$	C^2
$(124;1)$	$-\frac{59}{3840}$	$-\frac{5}{192}$	$-\frac{5}{96}$	$\frac{35}{64}$	$-\frac{5}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	1
$(224;1)$	$-\frac{199}{5760}$	0	0	$\frac{35}{16}$	0	0	0	2
$(234;1)$	$\frac{211}{11520}$	$-\frac{11}{64}$	$\frac{1}{32}$	$\frac{315}{64}$	$\frac{3}{4}$	$-\frac{3}{2}$	$-\frac{1}{4}$	1
$(244;1)$	$-\frac{61}{576}$	$-\frac{17}{48}$	$\frac{1}{24}$	$\frac{77}{16}$	1	2	-2	4
$(255;1)$	$\frac{53}{5760}$	$\frac{1}{24}$	$-\frac{1}{24}$	$-\frac{7}{16}$	-1	0	$-\frac{1}{4}$	0
$(114;2)$	$-\frac{521}{11520}$	$-\frac{11}{96}$	$\frac{5}{96}$	$\frac{35}{32}$	$\frac{5}{4}$	1	$-\frac{1}{2}$	2
$(124;2)$	$-\frac{119}{5760}$	$-\frac{5}{96}$	$\frac{5}{96}$	$\frac{35}{32}$	$\frac{5}{4}$	0	0	1
$(134;2)$	$-\frac{37}{1920}$	$-\frac{3}{32}$	$\frac{3}{32}$	$\frac{63}{32}$	$\frac{9}{4}$	0	0	-1
$(144;2)$	$-\frac{877}{5760}$	$-\frac{1}{3}$	$\frac{1}{12}$	$\frac{7}{4}$	2	4	-2	2
$(155;2)$	$-\frac{151}{23040}$	$-\frac{1}{48}$	$\frac{1}{48}$	$\frac{7}{16}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$
$(224;2)$	$\frac{7}{2880}$	0	0	0	0	0	0	0
$(234;2)$	$\frac{23}{1440}$	$\frac{1}{32}$	$-\frac{1}{32}$	$-\frac{21}{32}$	$-\frac{3}{4}$	0	0	-1
$(244;2)$	$\frac{1}{96}$	$\frac{1}{24}$	$-\frac{1}{24}$	$-\frac{7}{8}$	-1	0	0	0
$(255;2)$	$-\frac{151}{11520}$	$-\frac{1}{24}$	$\frac{1}{24}$	$\frac{7}{8}$	1	0	0	0
$(334;2)$	$\frac{185}{2304}$	$\frac{25}{96}$	$-\frac{19}{96}$	$-\frac{133}{32}$	$-\frac{19}{4}$	-1	$\frac{1}{2}$	0
$(344;2)$	$\frac{173}{1152}$	$\frac{1}{3}$	$-\frac{1}{12}$	$-\frac{7}{4}$	-2	-4	2	-2
$(355;2)$	$\frac{1289}{23040}$	$\frac{5}{48}$	$\frac{1}{48}$	$\frac{7}{16}$	$\frac{1}{2}$	-2	1	$-\frac{1}{2}$
$(444;2)$	$\frac{1}{30}$	$\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{21}{8}$	-3	0	0	0
$(455;2)$	$\frac{209}{11520}$	$\frac{1}{48}$	$\frac{1}{24}$	$\frac{7}{8}$	1	-1	$\frac{1}{2}$	0
$(114;3)$	$\frac{277}{4608}$	$\frac{3}{64}$	$\frac{7}{96}$	$\frac{35}{32}$	-6	-1	$-\frac{1}{2}$	2
$(124;3)$	$\frac{331}{5760}$	$\frac{3}{64}$	$-\frac{5}{32}$	$\frac{77}{64}$	$-\frac{15}{4}$	$-\frac{3}{2}$	$-\frac{1}{4}$	2
$(134;3)$	$-\frac{259}{23040}$	$-\frac{19}{96}$	$\frac{5}{12}$	$\frac{239}{64}$	$\frac{9}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0
$(144;3)$	$\frac{193}{11520}$	$-\frac{1}{48}$	$\frac{11}{48}$	3	-5	0	$-\frac{5}{2}$	4
$(155;3)$	$-\frac{557}{23040}$	$-\frac{19}{384}$	$\frac{1}{24}$	$\frac{7}{64}$	1	$\frac{1}{2}$	$-\frac{1}{8}$	0
$(224;3)$	$-\frac{161}{5760}$	0	0	$\frac{21}{16}$	0	0	0	2
$(234;3)$	$-\frac{29}{480}$	$-\frac{43}{192}$	$\frac{17}{96}$	$\frac{245}{64}$	$\frac{17}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0
$(244;3)$	$-\frac{47}{576}$	$-\frac{7}{48}$	$-\frac{1}{24}$	$\frac{35}{16}$	-1	2	-2	4
$(255;3)$	$-\frac{233}{5760}$	$-\frac{5}{48}$	$\frac{1}{24}$	$\frac{7}{16}$	1	1	$-\frac{1}{4}$	0
$(334;3)$	$-\frac{209}{7680}$	$-\frac{7}{64}$	$\frac{41}{96}$	$\frac{35}{16}$	$\frac{5}{2}$	0	0	0
$(344;3)$	$-\frac{209}{3840}$	$-\frac{11}{48}$	$\frac{31}{48}$	$\frac{45}{8}$	5	0	$-\frac{1}{2}$	0
$(355;3)$	$\frac{7}{512}$	$\frac{7}{128}$	0	$-\frac{21}{64}$	0	$-\frac{1}{2}$	$\frac{3}{8}$	0
$(444;3)$	$-\frac{89}{480}$	$-\frac{5}{16}$	$\frac{1}{2}$	$\frac{39}{8}$	-3	6	-6	6
$(455;3)$	$-\frac{677}{23040}$	$-\frac{1}{24}$	$\frac{1}{24}$	$\frac{7}{32}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$

Weight 4 QLI	π^4	$\pi^2\log^2(2)$	$\log^4(2)$	$\zeta(3)\log(2)$	$\mathrm{Li}_4(\frac{1}{2})$	πP_3	$\pi C\log(2)$	C^2
$(115;4)$	$\frac{143}{11520}$	$\frac{11}{384}$	$-\frac{1}{48}$	$-\frac{35}{64}$	$-\frac{1}{2}$	0	$-\frac{1}{8}$	$\frac{1}{2}$
$(125;4)$	$\frac{121}{15360}$	$\frac{1}{32}$	$-\frac{1}{32}$	$-\frac{7}{16}$	$-\frac{3}{4}$	0	$-\frac{1}{8}$	$\frac{1}{4}$
$(135;4)$	$-\frac{29}{15360}$	$\frac{3}{128}$	$-\frac{1}{32}$	$-\frac{35}{64}$	$-\frac{3}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$
$(145;4)$	$\frac{97}{5760}$	$\frac{1}{12}$	$-\frac{7}{192}$	$-\frac{105}{128}$	$-\frac{7}{8}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$
$(225;4)$	$\frac{151}{11520}$	$\frac{1}{24}$	$-\frac{1}{24}$	$-\frac{7}{8}$	-1	0	0	0
$(235;4)$	$-\frac{119}{15360}$	0	$-\frac{1}{32}$	$-\frac{7}{16}$	$-\frac{3}{4}$	$\frac{1}{2}$	$-\frac{3}{8}$	$\frac{1}{4}$
$(245;4)$	$\frac{437}{11520}$	$\frac{11}{96}$	$-\frac{1}{12}$	$-\frac{7}{8}$	-2	$-\frac{1}{2}$	$-\frac{1}{4}$	0
$(335;4)$	$-\frac{7}{512}$	$-\frac{3}{128}$	0	$\frac{21}{64}$	0	$\frac{1}{2}$	$-\frac{3}{8}$	0
$(345;4)$	$\frac{179}{23040}$	$\frac{7}{96}$	$-\frac{11}{192}$	$-\frac{49}{128}$	$-\frac{11}{8}$	$\frac{1}{4}$	$-\frac{1}{2}$	0
$(445;4)$	$\frac{11}{180}$	$\frac{29}{96}$	$-\frac{11}{96}$	$-\frac{35}{32}$	$-\frac{11}{4}$	-1	$-\frac{1}{2}$	0
$(555;4)$	$\frac{1}{1024}$	0	0	0	0	0	0	0
$(111;5)$	$-\frac{343}{15360}$	$\frac{5}{128}$	$-\frac{1}{16}$	0	$-\frac{15}{8}$	0	0	0
$(112;5)$	$\frac{167}{23040}$	$\frac{1}{32}$	$-\frac{5}{64}$	0	$-\frac{15}{8}$	0	0	0
$(113;5)$	$-\frac{23}{5120}$	$-\frac{7}{128}$	$\frac{3}{64}$	$\frac{35}{32}$	$\frac{3}{4}$	0	0	0
$(114;5)$	$\frac{1}{2560}$	$-\frac{1}{96}$	$-\frac{1}{96}$	$\frac{29}{32}$	-1	0	$-\frac{1}{4}$	1
$(122;5)$	$\frac{13}{3840}$	$\frac{5}{96}$	$-\frac{5}{96}$	0	$-\frac{5}{4}$	0	0	0
$(123;5)$	$-\frac{229}{23040}$	$-\frac{1}{16}$	$\frac{3}{64}$	$\frac{7}{8}$	$\frac{9}{8}$	0	0	0
$(124;5)$	$\frac{89}{7680}$	$-\frac{1}{192}$	$-\frac{1}{96}$	$\frac{49}{64}$	$-\frac{1}{4}$	$-\frac{1}{2}$	0	$\frac{1}{2}$
$(133;5)$	$-\frac{77}{46080}$	$-\frac{5}{384}$	$\frac{1}{48}$	0	$\frac{1}{8}$	0	0	0
$(134;5)$	$-\frac{167}{11520}$	$-\frac{3}{32}$	$\frac{5}{48}$	$\frac{93}{64}$	$\frac{7}{4}$	0	0	$-\frac{1}{2}$
$(144;5)$	$-\frac{121}{2880}$	$-\frac{13}{96}$	$\frac{1}{12}$	$\frac{29}{16}$	$\frac{1}{2}$	1	-1	1
$(155;5)$	$-\frac{111}{5120}$	$-\frac{1}{16}$	$\frac{1}{64}$	$\frac{35}{128}$	$\frac{3}{8}$	$\frac{1}{2}$	0	$-\frac{1}{4}$
$(222;5)$	$-\frac{7}{1920}$	0	0	0	0	0	0	0
$(223;5)$	$-\frac{23}{11520}$	$-\frac{1}{32}$	$\frac{1}{32}$	0	$\frac{3}{4}$	0	0	0
$(224;5)$	$-\frac{1}{192}$	$-\frac{1}{48}$	$\frac{1}{48}$	$\frac{7}{16}$	$\frac{1}{2}$	0	0	0
$(233;5)$	$\frac{467}{23040}$	$\frac{3}{32}$	$-\frac{5}{64}$	$-\frac{7}{4}$	$-\frac{15}{8}$	0	0	0
$(234;5)$	$-\frac{79}{4608}$	$-\frac{3}{64}$	$\frac{1}{32}$	$-\frac{7}{64}$	$\frac{3}{4}$	$\frac{1}{2}$	0	$-\frac{1}{2}$
$(244;5)$	$-\frac{1}{180}$	$-\frac{1}{48}$	$\frac{1}{48}$	$\frac{7}{16}$	$\frac{1}{2}$	0	0	0
$(255;5)$	$-\frac{361}{7680}$	$-\frac{1}{8}$	$\frac{1}{16}$	$\frac{7}{16}$	$\frac{3}{2}$	1	0	0
$(333;5)$	$\frac{209}{5120}$	$\frac{21}{128}$	$-\frac{9}{64}$	$-\frac{105}{32}$	$-\frac{15}{4}$	0	0	0
$(334;5)$	$\frac{209}{7680}$	$\frac{11}{96}$	$-\frac{7}{96}$	$-\frac{45}{16}$	$-\frac{5}{2}$	0	$\frac{1}{4}$	0
$(344;5)$	$\frac{89}{2880}$	$\frac{5}{96}$	$\frac{1}{12}$	$-\frac{13}{16}$	$\frac{1}{2}$	-1	1	-1
$(355;5)$	$\frac{319}{46080}$	$-\frac{1}{48}$	$\frac{7}{192}$	$\frac{35}{128}$	$\frac{7}{8}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{4}$
$(444;5)$	0	0	$\frac{1}{8}$	0	0	0	0	0
$(455;5)$	$-\frac{11}{360}$	$-\frac{13}{96}$	$\frac{11}{192}$	$\frac{35}{64}$	$\frac{11}{8}$	$\frac{1}{2}$	$\frac{1}{4}$	0

Weight 4 QLI	$\pi^3\log(2)$	$\pi\log^3(2)$	$\pi^2 C$	$C\log^2(2)$	$\pi\zeta(3)$	$\log(2)P_3$	P_4	Ψ_3
$(125;1)$	$\frac{3}{128}$	$\frac{1}{32}$	$\frac{11}{96}$	$\frac{1}{8}$	$\frac{35}{256}$	$-\frac{1}{2}$	0	$-\frac{1}{1536}$
$(225;1)$	$\frac{1}{32}$	0	$-\frac{1}{24}$	0	$\frac{35}{64}$	0	0	$-\frac{1}{768}$
$(235;1)$	$-\frac{1}{128}$	$\frac{1}{32}$	$-\frac{1}{96}$	$\frac{1}{8}$	$\frac{91}{256}$	$-\frac{1}{2}$	0	$-\frac{1}{1536}$
$(245;1)$	$\frac{1}{64}$	$\frac{1}{16}$	$-\frac{5}{48}$	$\frac{1}{4}$	$\frac{105}{128}$	-1	0	$-\frac{1}{768}$
$(115;2)$	0	$\frac{1}{24}$	$-\frac{5}{48}$	$\frac{1}{4}$	$\frac{35}{128}$	-1	2	0
$(125;2)$	0	0	$-\frac{3}{16}$	0	$\frac{35}{128}$	0	0	$\frac{1}{1536}$
$(135;2)$	0	$\frac{1}{6}$	$-\frac{5}{48}$	1	$\frac{7}{128}$	-4	8	$-\frac{1}{256}$
$(145;2)$	0	$\frac{1}{12}$	$-\frac{1}{6}$	$\frac{1}{2}$	0	-2	4	$-\frac{1}{768}$
$(225;2)$	0	0	0	0	0	0	0	$\frac{1}{768}$
$(235;2)$	0	0	$-\frac{1}{16}$	0	$-\frac{21}{128}$	0	0	$\frac{1}{1536}$
$(245;2)$	0	0	0	0	$-\frac{35}{64}$	0	0	$\frac{1}{768}$
$(335;2)$	0	$\frac{7}{24}$	$-\frac{1}{48}$	$\frac{7}{4}$	$-\frac{21}{128}$	-7	14	$-\frac{5}{768}$
$(345;2)$	0	$\frac{1}{4}$	$\frac{1}{24}$	$\frac{3}{2}$	$-\frac{7}{16}$	-6	12	$-\frac{1}{192}$
$(445;2)$	0	$\frac{1}{6}$	0	1	$-\frac{35}{32}$	-4	8	$-\frac{1}{768}$
$(555;2)$	0	0	$\frac{3}{16}$	0	0	0	0	$-\frac{1}{1024}$
$(115;3)$	$\frac{59}{384}$	$\frac{13}{96}$	$-\frac{5}{48}$	$\frac{1}{4}$	$\frac{35}{128}$	1	-12	$\frac{5}{768}$
$(125;3)$	$\frac{1}{128}$	$-\frac{1}{32}$	$-\frac{7}{96}$	$\frac{1}{8}$	$\frac{77}{256}$	$\frac{3}{2}$	-6	$\frac{5}{1536}$
$(135;3)$	$\frac{17}{192}$	$\frac{1}{6}$	$-\frac{7}{96}$	$\frac{5}{8}$	$\frac{29}{256}$	$-\frac{1}{2}$	-6	$\frac{5}{1536}$
$(145;3)$	$\frac{47}{384}$	$\frac{5}{32}$	$-\frac{1}{6}$	$\frac{1}{2}$	$\frac{41}{128}$	1	-12	$\frac{5}{768}$
$(225;3)$	$-\frac{1}{32}$	0	$-\frac{1}{24}$	0	$\frac{21}{64}$	0	0	0
$(235;3)$	$\frac{5}{128}$	$-\frac{1}{32}$	$-\frac{7}{96}$	$\frac{1}{8}$	$\frac{21}{256}$	$\frac{3}{2}$	-6	$\frac{5}{1536}$
$(245;3)$	$\frac{1}{64}$	$-\frac{1}{48}$	$-\frac{5}{48}$	$\frac{1}{4}$	$\frac{7}{128}$	1	-4	$\frac{1}{384}$
$(335;3)$	$\frac{7}{128}$	$\frac{19}{96}$	0	1	0	-2	0	0
$(345;3)$	$\frac{31}{384}$	$\frac{19}{96}$	0	1	$-\frac{29}{128}$	-1	-4	$\frac{1}{384}$
$(445;3)$	$\frac{5}{48}$	$\frac{5}{24}$	$-\frac{1}{8}$	1	$-\frac{9}{64}$	0	-8	$\frac{1}{192}$
$(555;3)$	$\frac{1}{128}$	0	$\frac{3}{32}$	0	$-\frac{63}{256}$	0	0	0

Weight 4 QLI	$\pi^3\log(2)$	$\pi\log^3(2)$	$\pi^2 C$	$C\log^2(2)$	$\pi\zeta(3)$	$\log(2)P_3$	P_4	Ψ_3
$(111;4)$	$\frac{21}{128}$	$\frac{3}{32}$	0	0	0	0	-6	0
$(112;4)$	$\frac{3}{64}$	$\frac{1}{48}$	$\frac{1}{3}$	0	0	0	-2	$-\frac{1}{768}$
$(113;4)$	$\frac{3}{128}$	$-\frac{5}{96}$	$\frac{1}{6}$	-1	0	2	-4	$\frac{1}{768}$
$(114;4)$	$\frac{23}{64}$	$\frac{7}{24}$	$\frac{5}{48}$	$-\frac{1}{4}$	$\frac{35}{64}$	0	-16	$\frac{1}{192}$
$(122;4)$	$\frac{1}{32}$	0	$\frac{1}{3}$	0	0	0	0	$-\frac{1}{384}$
$(123;4)$	$-\frac{1}{64}$	$-\frac{5}{48}$	$\frac{1}{6}$	-1	0	2	-2	0
$(124;4)$	$\frac{9}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{7}{16}$	0	-6	0
$(133;4)$	$-\frac{11}{128}$	$-\frac{19}{96}$	$\frac{1}{6}$	-2	0	4	-2	0
$(134;4)$	$-\frac{3}{64}$	$\frac{1}{24}$	$\frac{11}{48}$	-2	$\frac{7}{64}$	1	2	$-\frac{1}{384}$
$(144;4)$	$\frac{61}{96}$	$\frac{23}{24}$	$\frac{5}{24}$	-1	$\frac{35}{32}$	-2	-32	$\frac{3}{256}$
$(155;4)$	$\frac{1}{384}$	0	$-\frac{1}{32}$	0	$-\frac{35}{256}$	0	0	$\frac{1}{3072}$
$(222;4)$	0	0	0	0	0	0	0	$-\frac{1}{256}$
$(223;4)$	$\frac{1}{32}$	0	$\frac{1}{6}$	0	0	0	0	$-\frac{1}{768}$
$(224;4)$	$\frac{1}{8}$	0	0	0	$\frac{7}{8}$	0	0	$-\frac{1}{256}$
$(233;4)$	$-\frac{5}{64}$	$-\frac{11}{48}$	$\frac{1}{6}$	-2	0	4	-2	0
$(234;4)$	$-\frac{7}{64}$	$-\frac{1}{48}$	$\frac{1}{8}$	$-\frac{7}{4}$	$\frac{7}{16}$	-2	14	$-\frac{7}{768}$
$(244;4)$	$\frac{3}{8}$	$\frac{1}{3}$	0	-1	$\frac{7}{4}$	-4	-8	$-\frac{1}{768}$
$(255;4)$	0	0	$-\frac{1}{16}$	0	0	0	0	$\frac{1}{3072}$
$(333;4)$	$-\frac{21}{128}$	$-\frac{11}{32}$	0	-3	0	6	0	0
$(334;4)$	$-\frac{21}{64}$	$-\frac{5}{24}$	$\frac{1}{48}$	$-\frac{15}{4}$	$\frac{35}{64}$	2	20	$-\frac{3}{256}$
$(344;4)$	$-\frac{35}{96}$	$\frac{11}{24}$	$\frac{1}{24}$	-4	$\frac{35}{32}$	-6	40	$-\frac{3}{128}$
$(355;4)$	$\frac{1}{384}$	0	$-\frac{1}{32}$	0	$\frac{21}{256}$	0	0	0
$(444;4)$	$\frac{7}{8}$	3	0	-3	3	-12	-24	$\frac{1}{256}$
$(455;4)$	$\frac{1}{96}$	0	$-\frac{1}{16}$	0	$-\frac{3}{128}$	0	0	$\frac{1}{3072}$
$(115;5)$	$-\frac{5}{48}$	$-\frac{7}{96}$	$-\frac{5}{96}$	$\frac{1}{8}$	$\frac{29}{128}$	0	4	$-\frac{1}{768}$
$(125;5)$	$-\frac{1}{16}$	$-\frac{1}{24}$	$-\frac{3}{32}$	0	$\frac{49}{256}$	$\frac{1}{2}$	1	0
$(135;5)$	$\frac{1}{192}$	$-\frac{1}{48}$	$-\frac{5}{96}$	$\frac{1}{2}$	$\frac{9}{256}$	0	-1	$\frac{1}{1536}$
$(145;5)$	$-\frac{53}{384}$	$-\frac{3}{16}$	$-\frac{1}{12}$	$\frac{1}{4}$	$\frac{23}{256}$	$\frac{1}{2}$	6	$-\frac{1}{384}$
$(225;5)$	$-\frac{1}{16}$	0	0	0	$\frac{7}{64}$	0	0	$\frac{1}{1536}$
$(235;5)$	$\frac{3}{64}$	$-\frac{5}{48}$	$-\frac{1}{32}$	0	$-\frac{7}{256}$	$\frac{7}{2}$	-11	$\frac{3}{512}$
$(245;5)$	$-\frac{3}{32}$	$-\frac{1}{8}$	0	0	$-\frac{21}{128}$	2	0	$\frac{1}{1536}$
$(335;5)$	$\frac{1}{12}$	$\frac{1}{32}$	$-\frac{1}{96}$	$\frac{7}{8}$	$-\frac{3}{64}$	0	-6	$\frac{5}{1536}$
$(345;5)$	$\frac{25}{384}$	$-\frac{5}{48}$	$\frac{1}{48}$	$\frac{3}{4}$	$-\frac{61}{256}$	$\frac{3}{2}$	-8	$\frac{7}{1536}$
$(445;5)$	$-\frac{7}{48}$	$-\frac{11}{24}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	2	4	$-\frac{1}{1536}$
$(555;5)$	$-\frac{1}{128}$	0	$\frac{3}{32}$	0	$\frac{9}{256}$	0	0	$-\frac{1}{2048}$

Apart from these integrals, a pair of surprising closed-form polylog special values arise from evaluating a weight 4 QLI by 2 different ways. We have:

$$\Re(\mathrm{Li}_4(1+i)) = -\frac{5\mathrm{Li}_4\left(\frac{1}{2}\right)}{16} + \frac{97\pi^4}{9216} - \frac{5\log^4(2)}{384} + \frac{1}{48}\pi^2\log^2(2)$$

$$\Re\left(\mathrm{Li}_4\left(\frac{1}{2}+\frac{i}{2}\right)\right) = \sum_{k=1}^{\infty}\frac{\sum_{l=0}^{\left[\frac{k}{2}\right]}(-1)^l\binom{k}{2l}}{k^4 2^k} = \frac{5\mathrm{Li}_4\left(\frac{1}{2}\right)}{16} + \frac{343\pi^4}{92160} + \frac{\log^4(2)}{96} - \frac{5}{768}\pi^2\log^2(2)$$

Where $[a]$ denotes the integer part of a. Here is the proof outline: since two formulas are equivalent due to inversion formula of tetralog Li_4 [9], we only need to prove one of them. Consider evaluating QLI(225;4) by:

$$QLSIs \xrightarrow{combination} \int_0^{\frac{\pi}{4}} t\log^2(\tan(t))\,dt \xrightarrow{sub} QLI(225;4)$$

Where initial QLSIs are evaluated by brute force in section 4-2. Nonetheless, an alternative choice exists, that is, to bring this QLI into section 3-5's procedure. Expand $\tan^{-1}(x)$, $\frac{1}{x^2+1}$ into double series, reverse orders then simplify, we arrive at a harmonic series:

$$QLI(225;4) = -\frac{1}{8}ES(1;-3) - \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n H_{2n}}{n^3}$$

In which the latter sum is directly related to $\Re\left(\sum_{n=1}^{\infty}\frac{i^n H_n}{n^3}\right)$. Now apply the operator $L(f)$ in proof of Prop. 3 to the function on LHS of (2.5.15), with help of CAS we know that $\sum_{n=1}^{\infty}\frac{H_n z^n}{n^3}$ is expressible by logs and polylogs up to Li_4 in the unit circle (the expression is too long to present here). Letting $z\to i$ gives another closed-form of QLI with different coefficient of $\Re(\mathrm{Li}_4(1+i))$. Equating 2 forms generated by 2 methods completes the proof.

Some higher weight examples:

$$\int_0^1 \frac{\log^2(x)\tan^{-1}(x)^2}{x^2+1}\,dx$$
$$= \frac{1}{2}\pi\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7}{16}\pi\zeta(3)\log(2) - \frac{121\pi^5}{23040} + \frac{1}{48}\pi\log^4(2) - \frac{1}{48}\pi^3\log^2(2)$$

$$\int_0^1 \frac{\log^3(x)\tan^{-1}(x)}{x^2+1}\,dx$$
$$= \frac{7\pi^2\zeta(3)}{64} + \frac{93\zeta(5)}{32} - \frac{\pi\left(\psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right)\right)}{1024}$$

$$\int_0^1 \frac{x\log^4(x)\tan^{-1}(x)}{x^2+1}\,dx$$

$$= \frac{3\pi^3\zeta(3)}{32} + \frac{93\pi\zeta(5)}{256} - \frac{5}{64}\pi^5\log(2) + \frac{\psi^{(5)}\left(\frac{1}{4}\right) - \psi^{(5)}\left(\frac{3}{4}\right)}{40960}$$

$$\int_0^1 \frac{x\log^2\left(x^2+1\right)\tan^{-1}(x)^2}{x^2+1}\,dx$$

$$= \frac{1}{4}\pi C\log^2(2) + 2\pi\Im(\mathrm{Li}_4(1+i)) + \pi\log(2)\Im(\mathrm{Li}_3(1+i)) + 8\Re\left(\mathrm{Li}_5\left(\frac{1}{2}+\frac{i}{2}\right)\right) - 8\mathrm{Li}_5\left(\frac{1}{2}\right) - \frac{5}{4}\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2)$$

$$-\frac{\pi^2\zeta(3)}{8} + \frac{465\zeta(5)}{512} + \frac{35}{64}\zeta(3)\log^2(2) + \frac{\log^5(2)}{80} - \frac{119}{576}\pi^2\log^3(2) - \frac{2473\pi^4\log(2)}{23040} - \frac{\pi\left(\psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right)\right)}{3072}$$

$$\int_0^1 \frac{\log^2\left(x^2+1\right)\tan^{-1}(x)^2}{x^2+1}\,dx$$

$$= -8\sqrt{2}\,{}_6F_5\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2};\frac{1}{2}\right) - \frac{1}{8}\pi^2 C\log(2) - \frac{1}{4}\pi^2\Im(\mathrm{Li}_3(1+i)) - 8\Im\left(\mathrm{Li}_5\left(\frac{1}{2}+\frac{i}{2}\right)\right)$$

$$+\frac{5}{8}\pi\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{61}{64}\pi\zeta(3)\log(2) + \frac{709\pi^5}{15360} + \frac{1}{24}\pi\log^4(2) + \frac{9}{128}\pi^3\log^2(2) + \frac{\log(2)\left(\psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right)\right)}{1536}$$

They are direct consequences of smart use of methods recorded in section 4-2, which we left to the readers to discover. For instance, the last integral arises in the computation of $\Re\left(\int_0^1 \frac{\log^4(1+ix)}{x^2+1}\,dx\right)$. Since the numerator is power of log monomials, the inner integral can be solved 'polylogarithmically' by brute-force. However, if we separate the complex log then expand it via binomial formula, we reduce the desired integral to the calculation of the nontrivial $I = \int_0^1 \frac{\log^4\left(x^2+1\right)}{x^2+1}\,dx$ and trivial $\int_0^1 \frac{\tan^{-1}(x)^4}{x^2+1}\,dx$. To see why the former one is related to hypergeometric terms, consider the chain:

$$I \xrightarrow{sub} \int_0^{\frac{\pi}{4}} \log^4(\cos(x))\,dx \xrightarrow{ref,Beta} \int_0^{\frac{\pi}{4}} \log^4(\sin(x))\,dx \xrightarrow{binom,weight\ 3\ QLSI} \int_0^{\frac{\pi}{4}} \log^4(\sqrt{2}\sin(x))\,dx$$

$$\xrightarrow{sub} \int_0^1 \frac{\log^4(x)}{\sqrt{2-x^2}}\,dx \xrightarrow{series,(1.1.1)} {}_6F_5(\cdots)$$

Following this route with the help of section 1-1, 4-1, 4-2 and 4-6(9) and $(1-z)^{-a} = {}_1F_0(a;-;z)$ (used in the last step), we are able to give I (hence the original one) a closed-form involving hypergeometric terms. The fourth integral can be computed similarly, but we are not sure whether $\Re\left(\mathrm{Li}_5\left(\frac{1}{2}+\frac{i}{2}\right)\right)$ can be simplified further like $\Re\left(\mathrm{Li}_4\left(\frac{1}{2}+\frac{i}{2}\right)\right)$ above.

3 weight 4, 5 QLSIs:

$$\int_0^{\frac{\pi}{4}} \log^3(2\cos(x))\,dx$$

$$= 3\Im(\mathrm{Li}_4(1+i)) - \frac{3}{2}\log(2)\Im(\mathrm{Li}_3(1+i)) + \frac{3}{8}C\log^2(2) - \frac{3\pi\zeta(3)}{8} + \frac{1}{16}\pi\log^3(2) - \frac{\psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right)}{2048}$$

$$\int_0^{\frac{\pi}{4}} \log(2\sin(x))\log^2(2\cos(x))\,dx$$

$$= \frac{1}{8}C\log^2(2) + \Im(\mathrm{Li}_4(1+i)) + \frac{\pi\zeta(3)}{16} - \frac{1}{2}\log(2)\Im(\mathrm{Li}_3(1+i)) + \frac{1}{48}\pi\log^3(2) - \frac{5\left(\psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right)\right)}{6144}$$

$$\int_0^{\frac{\pi}{4}} x^3\log(2\sin(x))\,dx$$

$$= -\frac{1}{128}\pi^3 C + \frac{9\pi^2\zeta(3)}{2048} - \frac{1581\zeta(5)}{4096} + \frac{\pi\left(\psi^{(3)}\left(\frac{1}{4}\right) - \psi^{(3)}\left(\frac{3}{4}\right)\right)}{8192}$$

(3) NLIs, NPLIs, NQLIs and NLSIs

Weight 3, 4, 5 NLIs, one for each:

$$\int_0^1 \log(1-x)\log(x)\log(x+1)\,dx$$

$$= \frac{21\zeta(3)}{8} + \frac{5\pi^2}{12} - 6 - \log^2(2) - \frac{1}{2}\pi^2\log(2) + 4\log(2)$$

$$\int_0^1 \log(x)\log^3(x+1)\,dx$$

$$= 6\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{3\zeta(3)}{4} + \frac{21}{4}\zeta(3)\log(2) - \frac{\pi^2}{2} - \frac{\pi^4}{15}$$

$$+24 + \frac{\log^4(2)}{4} - 2\log^3(2) - \frac{1}{4}\pi^2\log^2(2) + 12\log^2(2) - 36\log(2)$$

$$\int_0^1 \log^3(1-x)\log^2(x+1)\,dx$$

$$= 24\mathrm{Li}_4\left(\frac{1}{2}\right) + 24\mathrm{Li}_5\left(\frac{1}{2}\right) - 2\pi^2\zeta(3) + 45\zeta(3) + \frac{3\zeta(5)}{4}$$

$$+12\zeta(3)\log^2(2) - 24\zeta(3)\log(2) + \frac{\pi^4}{30} + 6\pi^2 - 120 + \frac{3\log^5(2)}{5} - 3\log^4(2)$$

$$-\frac{2}{3}\pi^2\log^3(2) + 16\log^3(2) + 2\pi^2\log^2(2) - 48\log^2(2) - \frac{1}{30}\pi^4\log(2) - 6\pi^2\log(2) + 96\log(2)$$

And 2 symmetric but monstrous weight 6 NLIs:

$$\int_0^1 \frac{\log^2(1-x)\log^2(x)\log^2(x+1)}{x^2}\,dx$$
$$= -8ES(1;-5) - \frac{8}{3}\pi^2\mathrm{Li}_4\left(\frac{1}{2}\right) + 8\mathrm{Li}_4\left(\frac{1}{2}\right) + 24\mathrm{Li}_5\left(\frac{1}{2}\right) + 16\mathrm{Li}_6\left(\frac{1}{2}\right) + 8\mathrm{Li}_4\left(\frac{1}{2}\right)\log^2(2)$$
$$+24\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) + 16\mathrm{Li}_5\left(\frac{1}{2}\right)\log(2) + \frac{19\pi^2\zeta(3)}{6} - \frac{227\zeta(3)^2}{8} - 62\zeta(5) - 7\zeta(3)\log^3(2)$$
$$-\frac{35}{2}\zeta(3)\log^2(2) - 9\zeta(3)\log(2) - \frac{217}{2}\zeta(5)\log(2) + \frac{35}{6}\pi^2\zeta(3)\log(2) + \frac{\pi^4}{90} + \frac{25\pi^6}{378} + \frac{2\log^6(2)}{9}$$
$$+\frac{4\log^5(2)}{5} - \frac{5\log^4(2)}{3} - \frac{5}{18}\pi^2\log^4(2) + \frac{2}{3}\pi^2\log^3(2) + \pi^2\log^2(2) + \frac{13}{36}\pi^4\log^2(2) + \frac{1}{6}\pi^4\log(2)$$

$$\int_0^1 \log^2(1-x)\log^2(x)\log^2(x+1)\,dx$$
$$= -4ES(1;-5) - \frac{16}{3}\pi^2\mathrm{Li}_4\left(\frac{1}{2}\right) - 32\mathrm{Li}_4\left(\frac{1}{2}\right) + 24\mathrm{Li}_5\left(\frac{1}{2}\right) - 16\mathrm{Li}_6\left(\frac{1}{2}\right) - 8\mathrm{Li}_4\left(\frac{1}{2}\right)\log^2(2)$$
$$+24\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) - 16\mathrm{Li}_5\left(\frac{1}{2}\right)\log(2) + \frac{275\zeta(3)^2}{8} + \frac{34\pi^2\zeta(3)}{3} - 237\zeta(3) - \frac{341\zeta(5)}{2} + 7\zeta(3)\log^3(2)$$
$$-\frac{91}{2}\zeta(3)\log^2(2) - \frac{77}{6}\pi^2\zeta(3)\log(2) + 156\zeta(3)\log(2) + \frac{217}{2}\zeta(5)\log(2) + \frac{73\pi^6}{3024} - 40\pi^2 - \frac{29\pi^4}{45} + 720$$
$$-\frac{2}{9}\log^6(2) + \frac{4\log^5(2)}{5} - \frac{1}{18}\pi^2\log^4(2) + \frac{2\log^4(2)}{3} + \frac{2}{3}\pi^2\log^3(2) - 24\log^3(2) - \frac{1}{36}\pi^4\log^2(2)$$
$$-8\pi^2\log^2(2) + 144\log^2(2) + \frac{2}{3}\pi^4\log(2) + 32\pi^2\log(2) - 480\log(2)$$

Weight 3, 4, 5 NPLIs, two for each:

$$\int_0^1 \mathrm{Li}_2\left(\frac{x+1}{2}\right)\log(x)\,dx$$
$$= -\frac{13\zeta(3)}{8} - \frac{5\pi^2}{12} + 3 - \frac{1}{2}\log^2(2) + 2\log(2) + \frac{1}{6}\pi^2\log(2)$$
$$\int_0^1 \mathrm{Li}_2\left(\frac{1-x}{2}\right)\log(x+1)\,dx$$
$$= -\frac{\zeta(3)}{2} - \frac{\pi^2}{12} + 3 - \frac{1}{3}\log^3(2) + \frac{\log^2(2)}{2} - 4\log(2) + \frac{1}{6}\pi^2\log(2)$$

$$\int_0^1 \mathrm{Li}_2(x)\log(1-x)\log(x+1)\,dx$$

$$= 4\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{49\zeta(3)}{8}-\frac{7}{4}\zeta(3)\log(2)+\frac{5\pi^2}{6}-\frac{\pi^4}{30}-12$$

$$+\frac{\log^4(2)}{6}+\frac{2\log^3(2)}{3}-3\log^2(2)+6\log(2)-\frac{2}{3}\pi^2\log(2)$$

$$\int_0^1 \mathrm{Li}_2(-x)\mathrm{Li}_2\left(\frac{x+1}{2}\right)\,dx$$

$$= 4\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{5\zeta(3)}{8}+\frac{23}{8}\zeta(3)\log(2)-\frac{\pi^2}{2}-\frac{\pi^4}{18}+6$$

$$+\frac{\log^4(2)}{6}-\frac{5\log^2(2)}{2}-\frac{5}{24}\pi^2\log^2(2)-3\log(2)+\frac{5}{12}\pi^2\log(2)$$

$$\int_0^1 \mathrm{Li}_2\left(\frac{1-x}{2}\right)\mathrm{Li}_2\left(\frac{x-1}{x+1}\right)\log(x+1)\,dx$$

$$= 6\mathrm{Li}_4\left(\frac{1}{2}\right)-24\mathrm{Li}_5\left(\frac{1}{2}\right)-18\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2)+\frac{\pi^2\zeta(3)}{12}-\frac{3\zeta(3)}{2}$$

$$+\frac{189\zeta(5)}{8}-\frac{21}{4}\zeta(3)\log^2(2)+\frac{23}{4}\zeta(3)\log(2)-\frac{43\pi^4}{720}-\frac{\pi^2}{4}-\frac{3\log^5(2)}{5}$$

$$+\frac{\log^4(2)}{3}+\log^3(2)+\frac{17}{36}\pi^2\log^3(2)+3\log^2(2)-\frac{3}{8}\pi^2\log^2(2)+\frac{1}{3}\pi^2\log(2)-\frac{29}{360}\pi^4\log(2)$$

$$\int_0^1 \mathrm{Li}_3\left(\frac{x+1}{2}\right)\log(x)\log(1-x)\,dx$$

$$= -7\mathrm{Li}_4\left(\frac{1}{2}\right)+11\mathrm{Li}_5\left(\frac{1}{2}\right)+4\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2)+\frac{49\pi^2\zeta(3)}{96}-10\zeta(3)-\frac{463\zeta(5)}{32}$$

$$+\frac{21}{16}\zeta(3)\log^2(2)-\frac{11}{4}\zeta(3)\log(2)+\frac{121\pi^4}{1440}-\frac{13\pi^2}{6}+20+\frac{3\log^5(2)}{40}-\frac{5\log^4(2)}{24}$$

$$-\frac{1}{24}\pi^2\log^3(2)+\frac{\log^3(2)}{3}+\frac{5}{24}\pi^2\log^2(2)-3\log^2(2)+\frac{1}{288}\pi^4\log(2)+\frac{2}{3}\pi^2\log(2)+12\log(2)$$

2 weight 3 NQLIs with one exotic (use reflection and addition formula of Arctan for the latter):

$$\int_0^1 \log(x)\log(x+1)\log\left(x^2+1\right)\,dx$$

$$= -\frac{\pi C}{2}+2C-4C\log(2)-6\Im\left(\mathrm{Li}_3\left(\frac{1}{2}+\frac{i}{2}\right)\right)+\frac{3\zeta(3)}{2}+\frac{11\pi^3}{64}$$

$$+\frac{3\pi^2}{16}+\pi-12-\frac{1}{4}7\log^2(2)+\frac{3}{16}\pi\log^2(2)-\frac{1}{16}\pi^2\log(2)-\frac{1}{4}\pi\log(2)+10\log(2)$$

$$\int_0^1 \log(x)\log(1-x)\tan^{-1}\left(x^2-x+1\right)\,dx$$

$$=-2C-C\log(2)-4\Im\left(\mathrm{Li}_3\left(\frac{1}{2}+\frac{i}{2}\right)\right)+\frac{41\zeta(3)}{32}$$

$$+\frac{\pi^3}{24}+\frac{7\pi^2}{48}-\frac{1}{4}\log^2(2)+\frac{1}{8}\pi\log^2(2)-\frac{3}{16}\pi^2\log(2)+\frac{1}{4}\pi\log(2)+2\log(2)$$

3 weight 4 NQLIs with the first mixed:

$$\int_0^1 x\log\left(x^2+1\right)\log^3\left(\frac{1-x}{x+1}\right)dx$$

$$=-6\pi C+\frac{105\zeta(3)}{8}+3\pi^2-\frac{3\pi^3}{8}-\frac{7\pi^4}{64}-\frac{7}{4}\pi^2\log(2)$$

$$\int_0^1 \log^4\left(x^2+1\right)dx$$

$$=-192C-24C\log^2(2)+96C\log(2)+192\Im(\mathrm{Li}_3(1+i))-192\Im(\mathrm{Li}_4(1+i))$$

$$-96\log(2)\Im(\mathrm{Li}_3(1+i))+24\pi\zeta(3)-7\pi^3-96\pi+384+\log^4(2)+24\pi\log^3(2)-8\log^3(2)$$

$$-60\pi\log^2(2)+48\log^2(2)+7\pi^3\log(2)+96\pi\log(2)-192\log(2)+\frac{1}{32}\left(\psi^{(3)}\left(\frac{1}{4}\right)-\psi^{(3)}\left(\frac{3}{4}\right)\right)$$

$$\int_0^1 \log(1-x)\log(x+1)\log\left(x^2+1\right)\tan^{-1}(x)\,dx$$

$$=C^2-\frac{5\pi^2C}{24}+\frac{\pi C}{2}-4C-\frac{1}{2}\pi C\log(2)-C\log(2)+\frac{1}{2}\pi\Im(\mathrm{Li}_3(1+i))+10\Im(\mathrm{Li}_3(1+i))$$

$$-24\Im(\mathrm{Li}_4(1+i))+3\log(2)\Im(\mathrm{Li}_3(1+i))-\frac{13\mathrm{Li}_4\left(\frac{1}{2}\right)}{4}+\frac{105\pi\zeta(3)}{128}-\frac{19\zeta(3)}{32}-\frac{163}{64}\zeta(3)\log(2)$$

$$+\frac{247\pi^4}{23040}+\frac{7\pi^2}{24}-\frac{11\pi^3}{48}-3\pi-\frac{1}{6}\log^4(2)+\frac{\log^3(2)}{4}+\frac{5}{16}\pi\log^3(2)+\frac{9}{64}\pi^2\log^2(2)-\frac{3\log^2(2)}{2}$$

$$-\frac{13}{8}\pi\log^2(2)+\frac{5}{24}\pi^3\log(2)-\frac{7}{48}\pi^2\log(2)+\frac{5}{2}\pi\log(2)+6\log(2)+\frac{3}{256}\left(\psi^{(3)}\left(\frac{1}{4}\right)-\psi^{(3)}\left(\frac{3}{4}\right)\right)$$

4 NLSIs. Actually some of their closed-forms are still homogeneous but settled in this class due to their non-homo-like integrands:

$$\int_0^{\frac{\pi}{2}} \left(\frac{x}{\sin(x)}\right)^2 \log(\sin(x))\, dx = -\frac{\pi^3}{12} - \frac{1}{2}\pi \log^2(2) + \pi \log(2)$$

$$\int_0^{\frac{\pi}{2}} \left(\frac{x \log(\sin(x))}{\sin(x)}\right)^2 dx$$

$$= \frac{\pi\zeta(3)}{8} - \frac{\pi^3}{6} + \frac{1}{3}\pi \log^3(2) - \pi \log^2(2) + \frac{1}{6}\pi^3 \log(2) + 2\pi \log(2)$$

$$\int_0^{\frac{\pi}{2}} \frac{x \log(2\sin(x))}{\sin(x)}\, dx$$

$$= 2C \log(2) + 4\Im\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right) - \frac{9\pi^3}{48} - \frac{1}{4}\pi \log^2(2)$$

$$\int_0^{\frac{\pi}{2}} \frac{x^2 \log(2\cos(x))}{\sin(x)}\, dx$$

$$= -4C^2 + 4\pi C \log(2) + 4\pi\Im\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right) - \frac{7}{2}\zeta(3)\log(2) - \frac{3\pi^4}{32} - \frac{1}{8}\pi^2 \log^2(2)$$

(4) RSs and multiple integrals

Integral representation of the most sophisticated weight 5 ES. Denote $f(t) = \frac{t}{1+t}$, then (symmetry may help):

$$ES(1,-1,-1,-1;-1) = \int_{(0,1)^4} \frac{(f(-wxyz) - 3f(xyz) + 3f(-xy) - f(x))\log(1-x)}{x(y+1)(z+1)(w+1)} dxdydzdw$$

$$= 2\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{13\pi^2\zeta(3)}{96} - \frac{83\zeta(5)}{32} + \frac{9}{16}\zeta(3)\log^2(2) - \frac{13}{60}\log^5(2) + \frac{41}{72}\pi^2\log^3(2) - \frac{1}{180}\pi^4\log(2)$$

3 RSs and their multiple integral representations:

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1} H_m \sum_{n=1}^{m} \frac{(-1)^{n-1} H_n^{(2)}}{n}}{m} = \int_0^1 \int_0^1 \frac{\log(x)\log(1-y)\log\left(\frac{1-xy}{1-y}\right)}{(1-x)y(y+1)} dxdy$$

$$= 6\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{11\pi^2\zeta(3)}{48} - \frac{49\zeta(5)}{8} - \frac{1}{2}\zeta(3)\log^2(2) - \frac{1}{20}\log^5(2) + \frac{1}{36}\pi^2\log^3(2) + \frac{13}{360}\pi^4\log(2)$$

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}\left(\widetilde{H_m}\right)^2 \sum_{n=1}^{m} \frac{(-1)^{n-1} H_n}{n}}{m} = \int_0^1 \int_0^1 \int_0^1 \frac{\log(1-z)\log\left(\frac{(1-x)(1-y)(1-z)(1-xyz)}{2(xy+1)(xz+1)(yz+1)}\right)}{(x+1)(y+1)(z+1)} dxdydz$$

$$= 22\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{\pi^2\zeta(3)}{8} - \frac{349\zeta(5)}{16} - \frac{5}{8}\zeta(3)\log^2(2) - \frac{17\log^5(2)}{60} - \frac{7}{72}\pi^2\log^3(2) + \frac{43}{240}\pi^4\log(2)$$

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}\widetilde{H_m}\sum_{n=1}^{m}\frac{(-1)^{n-1}H_n\widetilde{H_n}}{n}}{m}$$

$$= \int_0^1\int_0^1\int_0^1 \frac{\log(1-y)\log\left(\frac{(1-y)(1-z)}{2(yz+1)}\right)}{(x+1)(y+1)(z+1)}dxdydz - \int_0^1\int_0^1\int_0^1 \frac{x\log(1-y)\log\left(\frac{(1-z)(xy+1)}{2(1-xyz)}\right)}{(x+1)(z+1)(1-xy)}dxdydz$$

$$= 14\mathrm{Li}_5\left(\frac{1}{2}\right) + \frac{\pi^2\zeta(3)}{12} - \frac{115\zeta(5)}{8} - \frac{3}{8}\zeta(3)\log^2(2) - \frac{11\log^5(2)}{60} - \frac{1}{18}\pi^2\log^3(2) + \frac{23}{180}\pi^4\log(2)$$

A high-depth symmetric RS (also an MZSV). This can be shown by using inner- and outer-symmetry relations repeatedly:

$$\zeta^*(\{-1\}_5) \overset{def}{=} \zeta^*(-1,-1,-1,-1,-1) = \sum_{1\le j\le k\le l\le m\le n}\frac{(-1)^{j-1}(-1)^{k-1}(-1)^{l-1}(-1)^{m-1}(-1)^{n-1}}{jklmn}$$

$$= \int_{(0,1)^4}\frac{p_1q_1 - p_2q_2 + p_3q_3}{x+1}d\mu = \frac{\pi^2\zeta(3)}{48} + \frac{3\zeta(5)}{16} + \frac{1}{8}\zeta(3)\log^2(2) + \frac{\log^5(2)}{120} + \frac{1}{72}\pi^2\log^3(2) + \frac{1}{160}\pi^4\log(2)$$

Where $d\mu = dxdydzdw, p_1 = \frac{x}{(z+1)(1-xy)} + \frac{1}{(y+1)(z+1)}, p_2 = \frac{x^2y}{(1-xy)(xyz+1)}, p_3 = \frac{y}{(y+1)(1-yz)}, q_1 = f(w) + zf(-zw), q_2 = f(w) + xyzf(-xyzw), q_3 = f(w) - yzf(yzw), f(t) = \frac{\log(2)-\log(1-t)}{1+t}$.

(5) Finally we conclude the appendix by a general formula, which appeared as a by-product while completing weight 4 QLIs via Fourier expansion:

$$\int_0^{\tan(a)}\tan^{-1}(x)^n\,dx = na^{n-1}\log(2\cos(a)) + a^n\tan(a) - n(n-1)\Re(X)$$

$$X = (-1)^{n-1}\left(1-2^{1-n}\right)(n-2)!\zeta(n) - (2ia)^{n-2}\mathrm{Li}_2\left(-e^{2ia}\right) + \frac{Y}{(2i)^{n-1}}$$

$$Y = \sum_{k=1}^{n-2}\frac{(-1)^{k-1}(n-2)!(2ia)^{-k+n-2}\mathrm{Li}_{k+2}\left(-e^{2ia}\right)}{(-k+n-2)!}$$

Thoughts on further developments

LI: Develop other methods generating (possible) new relations between high weight LIs (say, $W \ge 6$) to minimize the Fibonacci basis, or prove that current basis (see [1]) are already linear independent. The latter is much more difficult since we know little about irrationality and transcendence of these constants.

ES/RS: Develop new techniques and compute all RSs with weight ≤ 5. For higher weights the problem for ES/RSs is similar to LIs. High weight 2-ES, inverse ES and their alternating analogue may be developed too. What's more, section 4-6(9) suggest that deeper discoveries might be made on connection formulas of ESs and hypergeometric sums.

PLI: Determine whether PLIs with $W \geq 6$ share the same basis with equal-weight LIs or not. Evaluate high weight PLIs. For polylog functional identities (section 3-4) play an important role in low weight PLI evaluation, and no such formulas seem to exist in the case of high weight, we may have to develop other methods either. 2 possible ways are multiple integration and contour integration. For multiple integration a iterated procedure corresponds to section 3-5 can be written. For contour integration extend section 1-5's method to functions whose PLI integrands containing class 1, 2, 5, 6 kernel (i.e. $\mathrm{Li}_n(\pm x), \mathrm{Li}_n\left(\pm\frac{1-x}{x+1}\right)$), then apply identities recorded in section 3-4 and [9] to simplify. Based on the author's computation, these two methods generate only redundant relations in low weight case (section 3), but possibly useful in high weight cases which needs a thorough investigation.

QLI: Determine the structure of QLI basis for $W \geq 5$ and see how they extend Fibo. bases in [1]. It is challenging since $F_n = O\left(\left(\frac{1}{2}\left(\sqrt{5}+1\right)\right)^n\right) = o\left(2^n\right)$ as n tends to infinity. Based on experiences on weight 2, 3, 4, multiplying the basis with π or $\log(2)$ generates higher weight constants, but they're not enough to keep pace with the growth of 2^n. Due to section 4-2, 4-6(9) and appendix 4(2), some new irreducible logsine, harmonic or hypergeometric constants must be considered to fulfill a larger basis as $W \geq 5$. After this, evaluate high weight QLIs; a multiple integration procedure similar to last paragraph might help. Moreover, success on weight 4 suggests while $\Im(\mathrm{Li}_n(1+i)), n \geq 3$ are irreducible, nontrivial relationships between $\Re(\mathrm{Li}_n(1+i))$ and $\mathrm{Li}_n\left(\frac{1}{2}\right)$ for $n \geq 5$ may exist but remain unknown. They are of great importance to minimize QLI bases.

Extension of the subject: Consider 'quadratic polylog integrals' (abbr. QPLIs), whose integrands contain PLI, QLI components, together with generalizations of 2 quadratic terms i.e. $\mathrm{Li}_n(\pm x^2)$, $\mathrm{Ti}_n(x)(=\int_0^x \frac{\mathrm{Ti}_{n-1}(t)}{t}dt)$. Do they share the same basis with equal-weight QLIs, at least for $W \leq 4$? Valean [10] offered several vivid weight 4 examples consistent with the above conjecture, but we are not aware of a complete solution.

Another generalization is the evaluation of integrals of form $\int_0^1 \frac{\prod_{m=1}^n \log(a_m x+1)}{bx+1}dx$ where all parameters are unit roots of arbitrary order. We may put $\log(p(x))$ terms ($p(x)$ cyclotomic) into consideration to avoid complex logs. To see why this is an extension, simply note that LI/QLIs correspond to a_m order 2/4 unit roots respectively. Determining the general structure seems interesting but extremely challenging. What about its analogue in PLI/QPLIs? Corresponding integrals on $(0, \frac{1}{2})$? Or even, series counterparts? $\cdots$

Acknowledgements

We are grateful to all researchers that this article refers to, for their ingenious thoughts and remarkable discoveries on LI/ES/PLIs, from which this article originate. Moreover, we also appreciate Valean for his challenging problem book [10] which give insights on many related topics. Furthermore, we would like to thank Au for brilliant ideas he presented during our personal communication, and anonymous users on Mathematics Stack Exchange and other mathematical websites for intensive discussions on closed-form evaluations. Last but not least, I say thanks to my family and friends for their support and encouragements during the paper's completion.

References

[1] Au, K. C. (2019). Linear relations between logarithmic integrals of high weight and some closed-form evaluations. *arXiv preprint arXiv*:1910.12113. https://arxiv.org/pdf/1910.12113.pdf

[2] Bailey, D. H., Borwein, J. M., Girgensohn, R. (1994). Experimental Evaluation of Euler Sums. *Experimental Mathematics*, 3(1), 17-30.

[3] Borwein, D., Borwein, J. M., Girgensohn, R. (1995). Explicit Evaluation of Euler Sums. *Proceedings of the Edinburgh Mathematical Society*, 38(02), 277.

[4] Borwein, J. M., Chamberland, M. (2007). Integer Powers of Arcsin. *International Journal of Mathematics and Mathematical Sciences*, Volume 2007, Article ID: 19381.

[5] Borwein, J. M., Girgensohn, R. (1996). Evaluation of Triple Euler Sums. *The Electronic Journal of Combinatorics*, 3, #R23.

[6] Flajolet, P., Salvy, B. (1998). Euler Sums and Contour Integral Representations. *Experimental Mathematics*, 7(1), 15-35.

[7] Freitas, P. (2005). Integrals of polylogarithmic functions, recurrence relations, and associated euler sums. *Mathematics of Computation*, 74(251), 1425-1440.

[8] Gradshteyn, I. S., Ryzhik, I. M. (2015). *Table of Integrals, Series, and Products.* Academic Press, New York, 8th edition.

[9] Lewin, L. (1981). *Polylogarithms and Associated Functions.* New York, North-Holland.

[10] Valean, C. I. (2019). *(Almost) Impossible Integrals, Sums, and Series.* Springer, New York.

[11] Valean, C. I. (2019). A Note Presenting the Generalization of a Special Logarithmic Integral. https://www.researchgate.net/publication/335149209

[12] Valean, C. I. (2019). The derivation of eighteen special challenging logarithmic integrals. https://www.researchgate.net/publication/334598773

[13] Xu, C. (2017). Evaluations of Euler-Type Sums of Weight ≤ 5. *Bulletin of the Malaysian Mathematical Sciences Society*, 1-31.

[14] Xu, C., Cai, Y. L. (2016). On Harmonic Numbers and Nonlinear Euler Sums. *arXiv preprint arXiv*:1609.04924. https://arxiv.org/pdf/1609.04924.pdf

[15] Xu, C., Cheng, J. F. (2016). Some Results on Euler Sums. *Functiones et Approximatio*, 54(1), 25-37.

[16] Zagier D. (1994). Values of Zeta Functions and Their Applications. *Progress in Mathematics*, vol 120, Birkhauser, Basel.