INTER-UNIVERSAL TEICHMÜLLER THEORY I: CONSTRUCTION OF HODGE THEATERS

Shinichi Mochizuki

March 2020

Abstract. The present paper is the first in a series of four papers, the goal of which is to establish an *arithmetic* version of *Teichmüller theory* for **number fields** equipped with an **elliptic curve** — which we refer to as **"inter-universal Teichmüller theory"** — by applying the theory of *semi-graphs of anabelioids*, *Frobenioids*, the *étale theta function*, and *log-shells* developed in earlier papers by the author. We begin by fixing what we call *"initial* Θ*-data"*, which consists of an *elliptic curve* E_F over a *number field* F, and a *prime number* $l \geq 5$, as well as some other technical data satisfying certain technical properties. This data determines various *hyperbolic orbicurves* that are related via finite étale coverings to the once-punctured elliptic curve X_F determined by E_F. These finite étale coverings admit various *symmetry properties* arising from the **additive** and **multiplicative** structures on the ring $\mathbb{F}_l = \mathbb{Z}/l\mathbb{Z}$ acting on the l*-torsion points* of the elliptic curve. We then construct *"*$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters"* associated to the given Θ-data. These $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters may be thought of as *miniature models of* **conventional scheme theory** in which the **two underlying combinatorial dimensions** of a number field — which may be thought of as corresponding to the **additive** and **multiplicative** structures of a ring or, alternatively, to the **group of units** and **value group** of a local field associated to the number field — are, in some sense, **"dismantled"** or **"disentangled"** from one another. All $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters are isomorphic to one another, but may also be related to one another by means of a **"Θ-link"**, which relates certain *Frobenioid-theoretic* portions of one $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater to another in a fashion that is **not compatible** *with the respective* **conventional ring/scheme theory structures**. In particular, it is a *highly nontrivial problem to relate the ring structures* on either side of the Θ-link to one another. This will be achieved, up to certain *"relatively mild indeterminacies"*, in future papers in the series by applying the **absolute anabelian geometry** developed in earlier papers by the author. The resulting *description of an* **"alien ring structure"** [associated, say, to the *domain* of the Θ-link] in terms of a given ring structure [associated, say, to the *codomain* of the Θ-link] will be applied in the final paper of the series to obtain results in *diophantine geometry*. Finally, we discuss certain technical results concerning **profinite conjugates** *of decomposition and inertia groups in the* **tempered fundamental group** of a p-adic hyperbolic curve that will be of use in the development of the theory of the present series of papers, but are also of independent interest.

Contents:

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Introduction

§I1. Summary of Main Results

The present paper is the first in a series of four papers, the goal of which is to establish an **arithmetic** version of **Teichmüller theory** for **number fields** equipped with an **elliptic curve**, by applying the theory of *semi-graphs of anabelioids*, *Frobenioids*, the *étale theta function*, and *log-shells* developed in [SemiAnbd], [FrdI], [FrdII], [EtTh], and [AbsTopIII] [cf., especially, [EtTh] and [AbsTopIII]]. Unlike many mathematical papers, which are devoted to verifying properties of mathematical objects that are either well-known or easily constructed from well-known mathematical objects, in the present series of papers, most of our efforts will be devoted to **constructing** *new mathematical objects*. It is only in the final portion of the third paper in the series, i.e., [IUTchIII], that we turn to the task of *proving properties of interest* concerning the mathematical objects constructed. In the fourth paper of the series, i.e., [IUTchIV], we show that these properties may be combined with certain elementary computations to obtain *diophantine results* concerning elliptic curves over number fields.

We refer to §0 below for more on the *notations* and *conventions* applied in the present series of papers. The starting point of our constructions is a collection of **initial Θ-data** [cf. Definition 3.1]. Roughly speaking, this data consists, essentially, of

- an **elliptic curve** E_F over a **number field** F,
- an **algebraic closure** $\overline{F}$ of F,
- a **prime number** $l \geq 5$,
- a *collection of* **valuations** $\underline{\mathbb{V}}$ of a certain subfield $K \subseteq \overline{F}$, and
- a *collection of* **valuations** $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$ of a certain subfield $F_{\mathrm{mod}} \subseteq F$

that satisfy certain technical conditions — we refer to Definition 3.1 for more details. Here, we write $F_{\mathrm{mod}} \subseteq F$ for the *field of moduli* of E_F, $K \subseteq \overline{F}$ for the extension field of F determined by the *l-torsion points* of E_F, $X_F \subseteq E_F$ for the *once-punctured elliptic curve* obtained by removing the origin from E_F, and $X_F \to C_F$ for the *hyperbolic orbicurve* obtained by forming the stack-theoretic quotient of X_F by the natural action of $\{\pm 1\}$. Then F is assumed to be *Galois* over F_{mod}, $\mathrm{Gal}(K/F)$ is assumed to be isomorphic to a subgroup of $GL_2(\mathbb{F}_l)$ that *contains* $SL_2(\mathbb{F}_l)$, E_F is assumed to have *stable reduction* at all of the nonarchimedean valuations of F, $C_K \stackrel{\mathrm{def}}{=} C_F \times_F K$ is assumed to be a K*-core* [cf. [CanLift], Remark 2.1.1], $\underline{\mathbb{V}}$ is assumed to be a collection of valuations of K such that the natural inclusion $F_{\mathrm{mod}} \subseteq F \subseteq K$ induces a *bijection* $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$ between $\underline{\mathbb{V}}$ and the set $\mathbb{V}_{\mathrm{mod}}$ of all valuations of the number field F_{mod}, and

$$\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \subseteq \mathbb{V}_{\mathrm{mod}}$$

is assumed to be some *nonempty set* of nonarchimedean valuations of *odd* residue characteristic over which E_F has *bad [i.e., multiplicative] reduction* — i.e., roughly speaking, the *subset* of the set of valuations where E_F has bad multiplicative reduction that will be *"of interest"* to us in the context of the theory of the present series of papers. Then we shall write $\underline{\mathbb{V}}^{\mathrm{bad}} \stackrel{\mathrm{def}}{=} \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \times_{\mathbb{V}_{\mathrm{mod}}} \underline{\mathbb{V}} \subseteq \underline{\mathbb{V}}$, $\mathbb{V}^{\mathrm{good}}_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \mathbb{V}_{\mathrm{mod}} \setminus \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$, $\underline{\mathbb{V}}^{\mathrm{good}} \stackrel{\mathrm{def}}{=} \underline{\mathbb{V}} \setminus \underline{\mathbb{V}}^{\mathrm{bad}}$. Also, we shall apply the superscripts "non" and "arc" to $\underline{\mathbb{V}}$, $\mathbb{V}_{\mathrm{mod}}$ to denote the subsets of *nonarchimedean* and *archimedean* valuations, respectively.

This data determines, up to K-isomorphism [cf. Remark 3.1.3], a **finite étale covering** $\underline{C}_K \to C_K$ of degree l such that the base-changed covering

$$\underline{X}_K \stackrel{\mathrm{def}}{=} \underline{C}_K \times_{C_F} X_F \rightarrow X_K \stackrel{\mathrm{def}}{=} X_F \times_F K$$

arises from a *rank one quotient* $E_K[l] \twoheadrightarrow Q$ $(\cong \mathbb{Z}/l\mathbb{Z})$ of the module $E_K[l]$ of l-torsion points of $E_K(K)$ [where we write $E_K \stackrel{\mathrm{def}}{=} E_F \times_F K$] which, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, restricts to the quotient arising from *coverings of the dual graph of the special fiber*. Moreover, the above data also determines a **cusp**

$$\underline{\epsilon}$$

of $\underline{C}_K$ which, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, corresponds to the *canonical generator*, up to ± 1, of Q [i.e., the generator determined by the unique *loop* of the dual graph of the special fiber]. Furthermore, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, one obtains a natural finite étale covering of degree l

$$\underline{\underline{X}}_{\underline{v}} \rightarrow \underline{X}_{\underline{v}} \stackrel{\mathrm{def}}{=} \underline{X}_K \times_K K_{\underline{v}} \quad (\rightarrow \underline{C}_{\underline{v}} \stackrel{\mathrm{def}}{=} \underline{C}_K \times_K K_{\underline{v}})$$

by extracting l-th roots of the theta function; at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, one obtains a natural finite étale covering of degree l

$$\underline{\overrightarrow{X}}_{\underline{v}} \rightarrow \underline{X}_{\underline{v}} \stackrel{\mathrm{def}}{=} \underline{X}_K \times_K K_{\underline{v}} \quad (\rightarrow \underline{C}_{\underline{v}} \stackrel{\mathrm{def}}{=} \underline{C}_K \times_K K_{\underline{v}})$$

determined by $\underline{\epsilon}$. More details on the structure of the coverings $\underline{C}_K$, $\underline{X}_K$, $\underline{\underline{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], $\underline{\overrightarrow{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$] may be found in [EtTh], §2, as well as in §1 of the present paper.

In this situation, the objects

$$l^* \stackrel{\mathrm{def}}{=} (l-1)/2; \quad l^{\pm} \stackrel{\mathrm{def}}{=} (l+1)/2; \quad \mathbb{F}_l^* \stackrel{\mathrm{def}}{=} \mathbb{F}_l^{\times}/\{\pm 1\}; \quad \mathbb{F}_l^{\rtimes\pm} \stackrel{\mathrm{def}}{=} \mathbb{F}_l \rtimes \{\pm 1\}$$

[cf. the discussion at the beginning of §4; Definitions 6.1, 6.4] will play an important role in the discussion to follow. The natural action of the stabilizer in $\mathrm{Gal}(K/F)$ of the quotient $E_K[l] \twoheadrightarrow Q$ on Q determines a *natural poly-action* of $\mathbb{F}_l^{\ast}$ on $\underline{C}_K$, i.e., a natural isomorphism of $\mathbb{F}_l^{\ast}$ with some *subquotient* of $\mathrm{Aut}(\underline{C}_K)$ [cf. Example 4.3, (iv)]. The $\mathbb{F}_l^{\ast}$**-symmetry** constituted by this poly-action of $\mathbb{F}_l^{\ast}$ may be thought of as being essentially **arithmetic** in nature, in the sense that the subquotient of $\mathrm{Aut}(\underline{C}_K)$ that gives rise to this poly-action of $\mathbb{F}_l^{\ast}$ is induced, via the natural map $\mathrm{Aut}(\underline{C}_K) \to \mathrm{Aut}(K)$, by a subquotient of $\mathrm{Gal}(K/F) \subseteq \mathrm{Aut}(K)$. In a similar vein, the natural action of the automorphisms of the scheme $\underline{X}_K$ on the *cusps* of $\underline{X}_K$ determines a *natural poly-action* of $\mathbb{F}_l^{\rtimes\pm}$ on $\underline{X}_K$, i.e., a natural isomorphism of $\mathbb{F}_l^{\rtimes\pm}$ with some *subquotient* of $\mathrm{Aut}(\underline{X}_K)$ [cf. Definition 6.1, (v)]. The $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** constituted by this poly-action of $\mathbb{F}_l^{\rtimes\pm}$ may be thought of as being essentially **geometric** in nature, in the sense that the subgroup $\mathrm{Aut}_K(\underline{X}_K) \subseteq \mathrm{Aut}(\underline{X}_K)$ [i.e., of K-linear automorphisms] maps *isomorphically* onto the subquotient of $\mathrm{Aut}(\underline{X}_K)$ that gives rise to this poly-action of $\mathbb{F}_l^{\rtimes\pm}$. On the other hand, the **global** $\mathbb{F}_l^{\ast}$-*symmetry* of $\underline{C}_K$ only extends to a "$\{1\}$-symmetry" [i.e., in essence, fails to extend!] of the *local* coverings $\underline{\underline{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] and $\underrightarrow{X}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$], while the **global** $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* of $\underline{X}_K$ only extends to a "$\{\pm 1\}$-symmetry" [i.e., in essence, fails to extend!] of the *local* coverings $\underline{\underline{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] and $\underrightarrow{X}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$] — cf. Fig. I1.1 below.

$$
\begin{array}{ccc}
 & \overset{\{\pm 1\}}{\curvearrowright} \quad \{\underline{\underline{X}}_{\underline{v}} \text{ or } \underrightarrow{X}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}} & \\
\diagup & & \diagdown \\
\overset{\mathbb{F}_l^{\rtimes\pm}}{\curvearrowright} \quad \underline{X}_K & & \underline{C}_K \quad \overset{\mathbb{F}_l^{\ast}}{\curvearrowleft}
\end{array}
$$

Fig. I1.1: Symmetries of coverings of X_F

We shall write $\Pi_{\underline{v}}$ for the *tempered fundamental group* of $\underline{\underline{X}}_{\underline{v}}$, when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. Definition 3.1, (e)]; we shall write $\Pi_{\underline{v}}$ for the *étale fundamental group* of $\underrightarrow{X}_{\underline{v}}$, when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ [cf. Definition 3.1, (f)]. Also, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, we shall write $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$ for the quotient determined by the *absolute Galois group* of the base field $K_{\underline{v}}$. Often, in the present series of papers, we shall consider various types of collections of data — which we shall refer to as **"prime-strips"** — indexed by $\underline{v} \in \underline{\mathbb{V}}$ ($\xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$) that are isomorphic to certain data that arise naturally from $\underline{\underline{X}}_{\underline{v}}$ [when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] or $\underrightarrow{X}_{\underline{v}}$ [when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$]. The main types of prime-strips that will be considered in the present series of papers are summarized in Fig. I1.2 below.

Perhaps the most basic kind of prime-strip is a $\mathcal{D}$**-prime-strip**. When $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the portion of a $\mathcal{D}$-prime-strip labeled by $\underline{v}$ is given by a category equivalent to [the full subcategory determined by the connected objects of] the category of *tempered coverings* of $\underline{\underline{X}}_{\underline{v}}$ [when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] or *finite étale coverings* of $\underrightarrow{X}_{\underline{v}}$ [when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$]. When $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, an analogous definition may be obtained by applying the theory of Aut-*holomorphic orbispaces* developed in [AbsTopIII], §2. One variant of the notion of a $\mathcal{D}$-prime-strip is the notion of a $\mathcal{D}^{\vdash}$**-prime-strip**. When $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the portion of a $\mathcal{D}^{\vdash}$-prime-strip labeled by $\underline{v}$ is given by a category equivalent to [the full subcategory determined by the connected objects of] the Galois category associated to $G_{\underline{v}}$; when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, an analogous definition may be given. In some sense, $\mathcal{D}$-prime-strips may be thought of as abstractions of the **"local arithmetic holomorphic structure"** of [copies of] F_{mod} [which we regard as equipped with the once-punctured elliptic curve X_F] — cf. the discussion of [AbsTopIII], §I3. On the other hand, $\mathcal{D}^{\vdash}$-prime-strips may be thought of as **"mono-analyticizations"** [i.e., roughly speaking, the arithmetic version of the underlying real analytic structure associated to a holomorphic structure] of $\mathcal{D}$-prime-strips — cf. the discussion of [AbsTopIII], §I3. Throughout the present series of papers, we shall use the notation

$$\vdash$$

to denote *mono-analytic* structures.

Next, we recall the notion of a *Frobenioid* over a *base category* [cf. [FrdI] for more details]. Roughly speaking, a **Frobenioid** [typically denoted "$\mathcal{F}$"] may be thought of as a category-theoretic abstraction of the notion of a category of line bundles or monoids of divisors over a **base category** [typically denoted "$\mathcal{D}$"] of topological localizations [i.e., in the spirit of a *"topos"*] such as a *Galois category*. In addition to $\mathcal{D}$- and $\mathcal{D}^{\vdash}$-prime-strips, we shall also consider various types of prime-strips that arise from considering various natural Frobenioids — i.e., more concretely, various natural *monoids equipped with a Galois action* — at $\underline{v} \in \underline{\mathbb{V}}$. Perhaps the most basic type of prime-strip arising from such a natural monoid is an **$\mathcal{F}$-prime-strip**. Suppose, for simplicity, that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Then $\underline{v}$ and $\overline{F}$ determine, up to conjugacy, an *algebraic closure* $\overline{F}_{\underline{v}}$ of $K_{\underline{v}}$. Write

- $\mathcal{O}_{\overline{F}_{\underline{v}}}$ for the ring of integers of $\overline{F}_{\underline{v}}$;
- $\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}} \subseteq \mathcal{O}_{\overline{F}_{\underline{v}}}$ for the multiplicative monoid of nonzero integers;
- $\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \subseteq \mathcal{O}_{\overline{F}_{\underline{v}}}$ for the multiplicative monoid of units;
- $\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \subseteq \mathcal{O}_{\overline{F}_{\underline{v}}}$ for the multiplicative monoid of roots of unity;
- $\mathcal{O}^{\boldsymbol{\mu}_{2l}}_{\overline{F}_{\underline{v}}} \subseteq \mathcal{O}_{\overline{F}_{\underline{v}}}$ for the multiplicative monoid of $2l$-th roots of unity;
- $\underline{\underline{q}}_{\underline{v}} \in \mathcal{O}_{\overline{F}_{\underline{v}}}$ for a $2l$-th root of the q-parameter of E_F at $\underline{v}$.

Thus, $\mathcal{O}_{\overline{F}_{\underline{v}}}$, $\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$, $\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}$, $\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$, and $\mathcal{O}^{\boldsymbol{\mu}_{2l}}_{\overline{F}_{\underline{v}}}$ are equipped with *natural $G_{\underline{v}}$-actions*. The portion of an $\mathcal{F}$-prime-strip labeled by $\underline{v}$ is given by data isomorphic to the monoid $\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$, equipped with its natural $\Pi_{\underline{v}}$ ($\twoheadrightarrow G_{\underline{v}}$)-action [cf. Fig. I1.2]. There are various *mono-analytic* versions of the notion of an $\mathcal{F}$-prime-strip; perhaps the most basic is the notion of an **$\mathcal{F}^{\vdash}$-prime-strip**. The portion of an $\mathcal{F}^{\vdash}$-prime-strip labeled by $\underline{v}$ is given by data isomorphic to the monoid $\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \times \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}}$, equipped with its natural $G_{\underline{v}}$-action [cf. Fig. I1.2]. Often we shall regard these various mono-analytic versions of an $\mathcal{F}$-prime-strip as being equipped with an additional **global realified Frobenioid**, which, at a concrete level, corresponds, essentially, to considering various *arithmetic degrees* $\in \mathbb{R}$ at $\underline{v} \in \underline{\mathbb{V}}$ ($\xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$) that are related to one another by means of the *product formula*. Throughout the present series of papers, we shall use the notation

$$\Vdash$$

to denote such prime-strips.

Type of prime-strip	*Model at $v \in \mathbb{V}^{\mathrm{bad}}$*	*Reference*
$\mathcal{D}$	$\Pi_{\underline{v}}$	I, 4.1, (i)
$\mathcal{D}^{\vdash}$	$G_{\underline{v}}$	I, 4.1, (iii)
$\mathcal{F}$	$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$	I, 5.2, (i)
$\mathcal{F}^{\vdash}$	$G_{\underline{v}} \curvearrowright \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \times \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}}$	I, 5.2, (ii)
$\mathcal{F}^{\vdash\times}$	$G_{\underline{v}} \curvearrowright \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}$	II, 4.9, (vii)
$\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$	$G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}/\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$	II, 4.9, (vii)
$\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}$	$G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \times \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}}$	II, 4.9, (vii)
$\mathcal{F}^{\vdash\blacktriangleright}$	$G_{\underline{v}} \curvearrowright \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}}$	III, 2.4, (ii)
$\mathcal{F}^{\vdash\perp}$	$G_{\underline{v}} \curvearrowright \mathcal{O}^{\boldsymbol{\mu}_{2l}}_{\overline{F}_{\underline{v}}} \times \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}}$	III, 2.4, (ii)

$$\mathcal{F}^{\Vdash\cdots} \quad = \quad \mathcal{F}^{\vdash\cdots} \quad + \quad \left\{\text{global realified Frobenioid associated to } F_{\mathrm{mod}}\right\}$$

Fig. I1.2: Types of prime-strips

In some sense, the main goal of the present paper may be thought of as the *construction of* $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** [cf. Definition 6.13, (i)]

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

— which may be thought of as *"miniature models of* **conventional scheme theory**" — given, roughly speaking, by **systems of Frobenioids**. To any such

$\Theta^{\pm \text{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \text{ell}}\text{NF}}$, one may associate a **$\mathcal{D}$-$\Theta^{\pm \text{ell}}$NF-Hodge theater** [cf. Definition 6.13, (ii)]

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \text{ell}}\text{NF}}$$

— i.e., the associated **system of base categories**.

One may think of a $\Theta^{\pm \text{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \text{ell}}\text{NF}}$ as the result of **gluing** *together a* $\Theta^{\pm \text{ell}}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \text{ell}}}$ *to a* ΘNF*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta\text{NF}}$ [cf. Remark 6.12.2, (ii)]. In a similar vein, one may think of a $\mathcal{D}$-$\Theta^{\pm \text{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \text{ell}}\text{NF}}$ as the result of gluing together a $\mathcal{D}$-$\Theta^{\pm \text{ell}}$-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \text{ell}}}$ to a $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\text{NF}}$. A $\mathcal{D}$-$\Theta^{\pm \text{ell}}$-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \text{ell}}}$ may be thought of as a **bookkeeping device** that allows one to keep track of the action of the **$\mathbb{F}_l^{\rtimes \pm}$-symmetry** on the **labels**

$$(-l^{*} \ < \ \dots \ < \ -1 \ < 0 \ < \ 1 \ < \ \dots \ < \ l^{*})$$

— which we think of as elements $\in \mathbb{F}_l$ — in the context of the [orbi]curves $\underline{X}_K$, $\underline{\underline{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$], and $\underline{X}_{\rightarrow \underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$]. The $\mathbb{F}_l^{\rtimes \pm}$-symmetry is represented in a $\mathcal{D}$-$\Theta^{\pm \text{ell}}$-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \text{ell}}}$ by a category equivalent to [the full subcategory determined by the connected objects of] the Galois category of finite étale coverings of $\underline{X}_K$. On the other hand, each of the *labels* referred to above is represented in a $\mathcal{D}$-$\Theta^{\pm \text{ell}}$-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \text{ell}}}$ by a **$\mathcal{D}$-prime-strip**. In a similar vein, a $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\text{NF}}$ may be thought of as a bookkeeping device that allows one to keep track of the action of the **$\mathbb{F}_l^{*}$-symmetry** on the **labels**

$$(1 \ < \ \dots \ < \ l^{*})$$

— which we think of as elements $\in \mathbb{F}_l^{*}$ — in the context of the orbicurves $\underline{C}_K$, $\underline{\underline{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$], and $\underline{X}_{\rightarrow \underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$]. The $\mathbb{F}_l^{*}$-symmetry is represented in a $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\text{NF}}$ by a category equivalent to [the full subcategory determined by the connected objects of] the Galois category of finite étale coverings of $\underline{C}_K$. On the other hand, each of the *labels* referred to above is represented in a $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\text{NF}}$ by a **$\mathcal{D}$-prime-strip**. The *combinatorial structure* of $\mathcal{D}$-ΘNF- and $\mathcal{D}$-$\Theta^{\pm \text{ell}}$-Hodge theaters summarized above [cf. also Fig. I1.3 below] is one of the *main topics* of the present paper and is discussed in detail in §4 and §6. The left-hand portion of Fig. I1.3 corresponds to the $\mathcal{D}$-$\Theta^{\pm \text{ell}}$-Hodge theater; the right-hand portion of Fig. I1.3 corresponds to the $\mathcal{D}$-ΘNF-Hodge theater; these left-hand and right-hand portions are glued together by *identifying $\mathcal{D}$-prime-strips* in such a way that the labels $0 \neq \pm t \in \mathbb{F}_l$ on the left are identified with the corresponding label $j \in \mathbb{F}_l^{*}$ on the right [cf. Proposition 6.7; Remark 6.12.2; Fig. 6.5].

In this context, we remark that many of the constructions of [AbsTopIII] were intended as **prototypes** for constructions of the present series of papers. For instance, the *global theory* of [AbsTopIII], §5, was intended as a sort of simplified prototype for the $\Theta^{\pm \text{ell}}$*NF-Hodge theaters* of the present paper, i.e., *except with the various label bookkeeping devices deleted.* The various *panalocal* objects of [AbsTopIII], §5, were intended as prototypes for the various types of *prime-strips* that appear in the present series of papers. Perhaps most importantly, the theory of the *log-Frobenius functor* and *log-shells* developed in [AbsTopIII], §3, §4, §5, was intended as a prototype for the theory of the $\mathfrak{log}$-link that is developed in [IUTchIII]. In particular, although most of the main **ideas** and **techniques** of [AbsTopIII], §3, §4, §5, will play an important role in the present series of papers, many of the constructions performed in [AbsTopIII], §3, §4, §5, will **not** be applied in a **direct**, **literal** sense in the present series of papers.

The $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* has the advantange that, being *geometric* in nature, it allows one to *permute* various copies of "$G_{\underline{v}}$" [where $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$] associated to distinct *labels* $\in \mathbb{F}_l$ without inducing *conjugacy indeterminacies*. This phenomenon, which we shall refer to as **conjugate synchronization**, will play a key role in the **Kummer theory** surrounding the *Hodge-Arakelov-theoretic evaluation of the* **theta function** *at* ***l*-torsion points** that is developed in [IUTchII]— cf. the discussion of Remark 6.12.6; [IUTchII], Remark 3.5.2, (ii), (iii); [IUTchII], Remark 4.5.3, (i). By contrast, the $\mathbb{F}_l^{\divideontimes}$-*symmetry* is more suited to situations in which one must *descend* from K to F_{mod}. In the present series of papers, the most important such situation involves the **Kummer theory** surrounding the **reconstruction** of the **number field** F_{mod} from the étale fundamental group of $\underline{C}_K$ — cf. the discussion of Remark 6.12.6; [IUTchII], Remark 4.7.6. This reconstruction will be discussed in Example 5.1 of the present paper. Here, we note that such situations necessarily induce *global Galois permutations* of the various copies of "$G_{\underline{v}}$" [where $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$] associated to distinct *labels* $\in \mathbb{F}_l^{\divideontimes}$ that are *only well-defined up to conjugacy indeterminacies*. In particular, the $\mathbb{F}_l^{\divideontimes}$-symmetry is ill-suited to situations, such as those that appear in the theory of *Hodge-Arakelov-theoretic evaluation* that is developed in [IUTchII], that require one to establish *conjugate synchronization*.

$$
\begin{array}{ccccc}
\overset{\{\pm 1\}}{\curvearrowright} \begin{pmatrix} -l^{\divideontimes} & < & \dots & < & -1 & < 0 \\ & & < & 1 & < \dots & < l^{\divideontimes} \end{pmatrix} & \Rightarrow & \begin{bmatrix} 1 & < & \dots \\ & < & l^{\divideontimes} \end{bmatrix} & \Leftarrow & \begin{pmatrix} 1 & < & \dots \\ & < & l^{\divideontimes} \end{pmatrix} \\
\Downarrow & & & & \Downarrow \\
\begin{array}{ccc} \pm & \rightarrow & \pm \\ \uparrow & \overset{\mathbb{F}_l^{\rtimes\pm}}{\curvearrowright} & \downarrow \\ \pm & \leftarrow & \pm \end{array} & & & & \begin{array}{ccc} \divideontimes & \rightarrow & \divideontimes \\ \uparrow & \overset{\mathbb{F}_l^{\divideontimes}}{\curvearrowright} & \downarrow \\ \divideontimes & \leftarrow & \divideontimes \end{array}
\end{array}
$$

Fig. I1.3: The combinatorial structure of a $\mathcal{D}$-$\Theta^{\pm\text{ell}}$NF-Hodge theater
[cf. Figs. 4.4, 4.7, 6.1, 6.3, 6.5 for more details]

Ultimately, when, in [IUTchIV], we consider *diophantine applications* of the theory developed in the present series of papers, we will take the prime number l to be *"large"*, i.e., roughly of the order of the *square root of the height* of the elliptic curve E_F [cf. [IUTchIV], Corollary 2.2, (ii), (C1)]. When l is regarded as large, the arithmetic of the finite field $\mathbb{F}_l$ "tends to approximate" the arithmetic of the ring of rational integers $\mathbb{Z}$. That is to say, the decomposition that occurs in a $\Theta^{\pm\text{ell}}$NF-Hodge theater into the *"additive"* [i.e., $\mathbb{F}_l^{\rtimes\pm}$-] and *"multiplicative"* [i.e., $\mathbb{F}_l^{\divideontimes}$-] symmetries of the *ring* $\mathbb{F}_l$ may be regarded as a sort of rough, approximate approach to the issue of **"disentangling"** the **multiplicative** and **additive** structures, i.e., "dismantling" the **"two underlying combinatorial dimensions"** [cf.

the discussion of [AbsTopIII], §I3], of the **ring** $\mathbb{Z}$ — cf. the discussion of Remarks 6.12.3, 6.12.6.

Alternatively, this decomposition into additive and multiplicative symmetries in the theory of $\Theta^{\pm\text{ell}}$NF-Hodge theaters may be compared to groups of **additive** and **multiplicative symmetries** of the **upper half-plane** [cf. Fig. I1.4 below]. Here, the **"cuspidal"** geometry expressed by the additive symmetries of the upper half-plane admits a natural "associated coordinate", namely, the classical ***q*-parameter**, which is reminiscent of the way in which the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is well-adapted to the **Kummer theory** surrounding the *Hodge-Arakelov-theoretic evaluation of the* **theta function** *at* ***l*-torsion points** [cf. the above discussion]. By contrast, the **"toral"**, or **"nodal"** [cf. the classical theory of the structure of *Hecke correspondences* modulo p], geometry expressed by the multiplicative symmetries of the upper half-plane admits a natural "associated coordinate", namely, the classical biholomorphic isomorphism of the upper half-plane with the **unit disc**, which is reminiscent of the way in which the $\mathbb{F}_l^{\circledast}$-symmetry is well-adapted to the **Kummer theory** surrounding the **number field** F_{mod} [cf. the above discussion]. For more details, we refer to the discussion of Remark 6.12.3, (iii).

From the point of view of the *scheme-theoretic* Hodge-Arakelov theory developed in [HASurI], [HASurII], the theory of the *combinatorial structure* of a $\Theta^{\pm\text{ell}}$NF-Hodge theater — and, indeed, the theory of the present series of papers! — may be regarded as a sort of

> **solution** to the problem of constructing **"global multiplicative subspaces"** and **"global canonical generators"** [cf. the *quotient* "Q" and the *cusp* "$\underline{\epsilon}$" that appear in the above discussion!]

— the *nonexistence* of which in a "naive, scheme-theoretic sense" constitutes the *main obstruction* to applying the theory of [HASurI], [HASurII] to diophantine geometry [cf. the discussion of Remark 4.3.1]. Indeed, **prime-strips** may be thought of as **"local analytic sections"** of the natural morphism $\text{Spec}(K) \to \text{Spec}(F_{\text{mod}})$. Thus, it is precisely by working with such "local analytic sections" — i.e., more concretely, by working with the collection of valuations $\underline{\mathbb{V}}$, as opposed to the set of *all* valuations of K — that one can, in some sense, *"simulate"* the notions of a *"global multiplicative subspace"* or a *"global canonical generator"*. On the other hand, such "simulated global objects" may only be achieved at the cost of

> **"dismantling"**, or performing **"surgery"** on, the **global prime structure** *of the number fields* involved [cf. the discussion of Remark 4.3.1]

— a quite *drastic* operation, which has the effect of precipitating *numerous technical difficulties*, whose resolution, via the theory of *semi-graphs of anabelioids*, *Frobenioids*, the *étale theta function*, and *log-shells* developed in [SemiAnbd], [FrdI], [FrdII], [EtTh], and [AbsTopIII], constitutes the bulk of the theory of the present series of papers! From the point of view of "performing surgery on the global prime structure of a number field", the **labels** $\in \mathbb{F}_l^{\circledast}$ that appear in the **"arithmetic"** $\mathbb{F}_l^{\circledast}$**-symmetry** may be thought of as a sort of **"miniature finite approximation"** *of this global prime structure*, in the spirit of the idea of *"Hodge theory at finite resolution"* discussed in [HASurI], §1.3.4. On the other hand, the **labels** $\in \mathbb{F}_l$ that appear in the **"geometric"** $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** may be thought of as a sort of **"miniature finite approximation"** *of the natural tempered* $\mathbb{Z}$*-coverings* [i.e., tempered coverings with Galois group $\mathbb{Z}$] of the Tate curves determined by E_F at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, again in the spirit of the idea of "Hodge theory at finite resolution" discussed in [HASurI], §1.3.4.

	Classical upper half-plane	$\Theta^{\pm\text{ell}}$*NF-Hodge theaters in inter-universal Teichmüller theory*
Additive symmetry	$z \mapsto z + a$, $z \mapsto -\overline{z} + a$ $(a \in \mathbb{R})$	$\mathbb{F}_l^{\rtimes\pm}$-**symmetry**
"Functions" assoc'd to *add. symm.*	$q \stackrel{\text{def}}{=} e^{2\pi i z}$	**theta fn.** evaluated at $\boldsymbol{l}$**-tors.** [cf. I, 6.12.6, (ii)]
Basepoint assoc'd to *add. symm.*	*single* cusp at infinity	$\underline{\mathbb{V}}^{\pm}$ [cf. I, 6.1, (v)]
Combinatorial prototype assoc'd to *add. symm.*	**cusp**	**cusp**
Multiplicative symmetry	$z \mapsto \frac{z\cdot\cos(t)-\sin(t)}{z\cdot\sin(t)+\cos(t)}$, $z \mapsto \frac{\overline{z}\cdot\cos(t)+\sin(t)}{\overline{z}\cdot\sin(t)-\cos(t)}$ $(t \in \mathbb{R})$	$\mathbb{F}_l^{\circledast}$-**symmetry**
"Functions" assoc'd to *mult. symm.*	$w \stackrel{\text{def}}{=} \frac{z-i}{z+i}$	elements of the **number field** F_{mod} [cf. I, 6.12.6, (iii)]
Basepoints assoc'd to *mult. symm.*	$\begin{pmatrix}\cos(t) & -\sin(t)\\ \sin(t) & \cos(t)\end{pmatrix}$, $\begin{pmatrix}\cos(t) & \sin(t)\\ \sin(t) & -\cos(t)\end{pmatrix}$ $\curvearrowright$ {*entire* boundary of $\mathfrak{H}$ }	$\mathbb{F}_l^{\circledast} \curvearrowright \underline{\mathbb{V}}^{\text{Bor}} = \mathbb{F}_l^{\circledast} \cdot \underline{\mathbb{V}}^{\pm\text{un}}$ [cf. I, 4.3, (i)]
Combinatorial prototype assoc'd to *mult. symm.*	**nodes** of mod p Hecke correspondence [cf. II, 4.11.4, (iii), (c)]	**nodes** of mod p Hecke correspondence [cf. II, 4.11.4, (iii), (c)]

Fig. I1.4: Comparison of $\mathbb{F}_l^{\rtimes\pm}$-, $\mathbb{F}_l^{\circledast}$-symmetries with the geometry of the upper half-plane

As discussed above in our explanation of the models at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ for $\mathcal{F}^{\vdash}$-prime-strips, by considering the $2l$-th roots of the **q-parameters** of the elliptic curve E_F at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, and, roughly speaking, extending to $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ in such a way as to satisfy the *product formula*, one may construct a natural **$\mathcal{F}^{\Vdash}$-prime-strip** "$\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$" [cf. Example 3.5, (ii); Definition 5.2, (iv)]. This construction admits an *abstract, algorithmic formulation* that allows one to apply it to the underlying *"Θ-Hodge theater"* of an *arbitrary* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ so as to obtain an $\mathcal{F}^{\Vdash}$-prime-strip

$$ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} $$

[cf. Definitions 3.6, (c); 5.2, (iv)]. On the other hand, by *formally replacing* the $2l$-th roots of the q-parameters that appear in this construction by the *reciprocal of the l-th root of the Frobenioid-theoretic* **theta function**, which we shall denote "$\underline{\underline{\Theta}}_{\underline{v}}$" [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], studied in [EtTh] [cf. also Example 3.2, (ii), of the present paper], one obtains an *abstract, algorithmic formulation* for the construction of an $\mathcal{F}^{\Vdash}$-prime-strip

$$ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} $$

[cf. Definitions 3.6, (c); 5.2, (iv)] from [the underlying Θ-Hodge theater of] the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.

Now let ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ be *another* $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater [relative to the given initial Θ-data]. Then we shall refer to the "full poly-isomorphism" of [i.e., the collection of all isomorphisms between] $\mathcal{F}^{\Vdash}$*-prime-strips*

$$ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \quad \overset{\sim}{\to} \quad {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} $$

as the **Θ-link** from [the underlying Θ-Hodge theater of] ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ to [the underlying Θ-Hodge theater of] ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ [cf. Corollary 3.7, (i); Definition 5.2, (iv)]. One fundamental property of the Θ-link is the property that it induces a collection of isomorphisms [in fact, the full poly-isomorphism] between the $\mathcal{F}^{\vdash\times}$*-prime-strips*

$$ {}^{\dagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}} \quad \overset{\sim}{\to} \quad {}^{\ddagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}} $$

associated to ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ and ${}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ [cf. Corollary 3.7, (ii), (iii); [IUTchII], Definition 4.9, (vii)].

Now let $\{{}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n\in\mathbb{Z}}$ be a *collection of distinct* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters* [relative to the given initial Θ-data] indexed by the integers. Thus, by applying the constructions just discussed, we obtain an **infinite chain**

$$ \ldots \quad \xrightarrow{\Theta} \quad {}^{(n-1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta} \quad {}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta} \quad {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta} \quad \ldots $$

of **Θ-linked $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** [cf. Corollary 3.8], which will be referred to as the **Frobenius-picture** [associated to the Θ-link]. One fundamental property of this Frobenius-picture is the property that it *fails to admit* **permutation automorphisms** *that switch adjacent indices n, $n+1$*, but leave the remaining indices $\in \mathbb{Z}$ fixed [cf. Corollary 3.8]. Roughly speaking, the Θ-link ${}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta} {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ may be thought of as a *formal correspondence*

$$ {}^{n}\underline{\underline{\Theta}}_{\underline{v}} \quad \mapsto \quad {}^{(n+1)}\underline{\underline{q}}_{\underline{v}} $$

[cf. Remark 3.8.1, (i)], which is depicted in Fig. I1.5 below.

In fact, the Θ-link discussed in the present paper is only a **simplified version** of the "Θ-link" that will ultimately play a central role in the present series of papers. The construction of the version of the Θ-link that we shall ultimately be interested in is quite *technically involved* and, indeed, occupies the greater part of the theory to be developed in [IUTchII], [IUTchIII]. On the other hand, the simplified version discussed in the present paper is of interest in that it allows one to give a relatively straightforward introduction to many of the important **qualitative properties** of the Θ-link — such as the *Frobenius-picture* discussed above and the *étale-picture* to be discussed below — that will continue to be of *central importance* in the case of the versions of the Θ-link that will be developed in [IUTchII], [IUTchIII].

$$\dots \quad ---- \quad \boxed{\begin{array}{c} {}^{n}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \\ \\ {}^{n}\underline{\underline{q}}_{\underline{v}} \quad \rightsquigarrow \quad {}^{n}\underline{\underline{\Theta}}_{\underline{v}} \end{array}} \quad ---- \quad \boxed{\begin{array}{c} {}^{n+1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \\ \\ {}^{(n+1)}\underline{\underline{q}}_{\underline{v}} \quad \rightsquigarrow \quad {}^{(n+1)}\underline{\underline{\Theta}}_{\underline{v}} \end{array}} \quad ---- \quad \dots$$

$${}^{n}\underline{\underline{\Theta}}_{\underline{v}} \qquad \mapsto \qquad {}^{(n+1)}\underline{\underline{q}}_{\underline{v}}$$

Fig. I1.5: Frobenius-picture associated to the Θ-link

Now let us return to our discussion of the *Frobenius-picture* associated to the Θ-link. The $\mathcal{D}^{\vdash}$-prime-strip associated to the $\mathcal{F}^{\vdash\times}$-prime-strip ${}^{\dagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}}$ may, in fact, be naturally identified with the $\mathcal{D}^{\vdash}$-prime-strip ${}^{\dagger}\mathfrak{D}^{\vdash}_{>}$ associated to a certain $\mathcal{F}$-prime-strip ${}^{\dagger}\mathfrak{F}_{>}$ [cf. the discussion preceding Example 5.4] that arises from the Θ-Hodge theater underlying the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$. The $\mathcal{D}$-prime-strip ${}^{\dagger}\mathfrak{D}_{>}$ associated to the $\mathcal{F}$-prime-strip ${}^{\dagger}\mathfrak{F}_{>}$ is precisely the $\mathcal{D}$-prime-strip depicted as "$[1 < \dots < l^{*}]$" in Fig. I1.3. Thus, the Frobenius-picture discussed above induces an infinite chain of full poly-isomorphisms

$$\dots \quad \stackrel{\sim}{\to} \quad {}^{(n-1)}\mathfrak{D}^{\vdash}_{>} \quad \stackrel{\sim}{\to} \quad {}^{n}\mathfrak{D}^{\vdash}_{>} \quad \stackrel{\sim}{\to} \quad {}^{(n+1)}\mathfrak{D}^{\vdash}_{>} \quad \stackrel{\sim}{\to} \quad \dots$$

of $\mathcal{D}^{\vdash}$-prime-strips. That is to say, when regarded up to isomorphism, the $\mathcal{D}^{\vdash}$-prime-strip "${}^{(-)}\mathfrak{D}^{\vdash}_{>}$" may be regarded as an **invariant** — i.e., a **"mono-analytic core"** — of the various $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters that occur in the Frobenius-picture [cf. Corollaries 4.12, (ii); 6.10, (ii)]. Unlike the case with the Frobenius-picture, the *relationships* of the various $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters ${}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ to this mono-analytic core — relationships that are depicted by *spokes* in Fig. I1.6 below — are compatible with **arbitrary permutation symmetries** among the spokes [i.e., among the labels $n \in \mathbb{Z}$ of the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters] — cf. Corollaries 4.12, (iii); 6.10, (iii), (iv). The diagram depicted in Fig. I1.6 below will be referred to as the **étale-picture**.

Thus, the étale-picture may, in some sense, be regarded as a collection of **canonical splittings** of the Frobenius-picture. The existence of such splittings suggests that

> by applying various results from **absolute anabelian geometry** to the various tempered and étale fundamental groups that constitute each $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theater in the étale-picture, one may obtain **algorithmic descriptions** of — i.e., roughly speaking, one may take a **"glimpse"** inside — the **conventional scheme theory** of one $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater ${}^{m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ in terms of the conventional scheme theory associated to another $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater ${}^{n}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ [i.e., where $n \neq m$].

Indeed, this point of view constitutes one of the *main themes* of the theory developed in the present series of papers and will be of particular importance in our treatment in [IUTchIII] of the main results of the theory.

$$\begin{array}{ccccc}
 & & \boxed{{}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}} & & \\
\dots & & | & & \dots \\
\boxed{{}^{n-1}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}} & — & \boxed{{}^{(-)}\mathfrak{D}^{\vdash}_{>}} & — & \boxed{{}^{n+1}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}} \\
\dots & & | & & \dots \\
 & & \boxed{{}^{n+2}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}} & &
\end{array}$$

Fig. I1.6: Étale-picture of $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters

Before proceeding, we recall the "heuristic" notions of **Frobenius-like** — i.e., *"order-conscious"* — and **étale-like** — i.e., *"indifferent to order"* — *mathematical structures* discussed in [FrdI], §I4. These notions will play a *key role* in the theory developed in the present series of papers. In particular, the terms "Frobenius-picture" and "étale-picture" introduced above are motivated by these notions.

The *main result* of the present paper may be summarized as follows.

Theorem A. **($\mathbb{F}_l^{\rtimes\pm}$-/$\mathbb{F}_l^{\circledast}$-Symmetries, Θ-Links, and Frobenius-/Étale-Pictures Associated to $\Theta^{\pm \mathrm{ell}}$NF-Hodge Theaters)** *Fix a collection of* **initial Θ-data** *[cf. Definition 3.1], which determines, in particular, data* (E_F, $\overline{F}$, l, $\underline{\mathbb{V}}$) *as in the above discussion. Then one may construct a* $\Theta^{\pm \mathrm{ell}}$**NF-Hodge theater** *[cf. Definition 6.13, (i)]*

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

— in essence, a system of Frobenioids — associated to this initial Θ*-data, as well as an associated* $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$**NF-Hodge theater** ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *[cf. Definition 6.13, (ii)]*

— *in essence, the system of base categories associated to the system of Frobenioids* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.

(i) **($\mathbb{F}_l^{\rtimes\pm}$- and $\mathbb{F}_l^{*}$-Symmetries)** *The* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *may be obtained as the result of* **gluing** *together a* $\Theta^{\pm\mathrm{ell}}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$ *to a* $\Theta\mathrm{NF}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}}$ *[cf. Remark 6.12.2, (ii)]; a similar statement holds for the* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$. *The* **global portion** *of a* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ *consists of a category equivalent to [the full subcategory determined by the connected objects of] the Galois category of finite étale coverings of the [orbi]curve* $\underline{X}_K$. *This global portion is equipped with an* $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, *i.e., a poly-action by* $\mathbb{F}_l^{\rtimes\pm}$ *on the* **labels**

$$(\,-l^{*}\ <\ \dots\ <\ -1\ < 0\ <\ 1\ <\ \dots\ <\ l^{*}\,)$$

— *which we think of as elements* $\in \mathbb{F}_l$ — *each of which is represented in the* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ *by a* $\mathcal{D}$**-prime-strip** *[cf. Fig. I1.3]. The* **global portion** *of a* $\mathcal{D}$-$\Theta\mathrm{NF}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ *consists of a category equivalent to [the full subcategory determined by the connected objects of] the Galois category of finite étale coverings of the orbicurve* $\underline{C}_K$. *This global portion is equipped with an* $\mathbb{F}_l^{*}$**-symmetry**, *i.e., a poly-action by* $\mathbb{F}_l^{*}$ *on the* **labels**

$$(\,1\ <\ \dots\ <\ l^{*}\,)$$

— *which we think of as elements* $\in \mathbb{F}_l^{*}$ — *each of which is represented in the* $\mathcal{D}$-$\Theta\mathrm{NF}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ *by a* $\mathcal{D}$**-prime-strip** *[cf. Fig. I1.3]. The* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ *is glued to the* $\mathcal{D}$-$\Theta\mathrm{NF}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ *by* **identifying** $\mathcal{D}$**-prime-strips** *in such a way that the labels* $0 \neq \pm t \in \mathbb{F}_l$ *that arise in the* $\mathbb{F}_l^{\rtimes\pm}$*-symmetry are identified with the corresponding label* $j \in \mathbb{F}_l^{*}$ *that arises in the* $\mathbb{F}_l^{*}$*-symmetry [cf. Proposition 6.7; Remark 6.12.2; Fig. 6.5].*

(ii) **(Θ-links)** *By considering the* $2l$*-th roots of the* $\boldsymbol{q}$**-parameters** *"*$\underline{\underline{q}}_{\underline{v}}$*" of the elliptic curve* E_F *at* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *and extending to other* $\underline{v} \in \underline{\mathbb{V}}$ *in such a way as to satisfy the* **product formula**, *one may construct a natural* $\mathcal{F}^{\Vdash}$**-prime-strip** ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ *associated to the* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *[cf. Definitions 3.6, (c); 5.2, (iv)]. In a similar vein, by considering the reciprocal of the* l*-th root of the Frobenioid-theoretic* **theta function** *"*$\underline{\underline{\Theta}}_{\underline{v}}$*" associated to the elliptic curve* E_F *at* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *and extending to other* $\underline{v} \in \underline{\mathbb{V}}$ *in such a way as to satisfy the* **product formula**, *one may construct a natural* $\mathcal{F}^{\Vdash}$**-prime-strip** ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$ *associated to the* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *[cf. Definitions 3.6, (c); 5.2, (iv)]. Now let* ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *be* **another** $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater [relative to the given initial* Θ*-data]. Then we shall refer to the "full poly-isomorphism" of [i.e., the collection of all isomorphisms between]* $\mathcal{F}^{\Vdash}$**-prime-strips**

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \quad \stackrel{\sim}{\rightarrow} \quad {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$$

as the Θ**-link** *from [the underlying* Θ*-Hodge theater of]* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *to [the underlying* Θ*-Hodge theater of]* ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *[cf. Corollary 3.7, (i); Definition 5.2, (iv)]. The* Θ*-link induces the full poly-isomorphism between the* $\mathcal{F}^{\vdash\times}$**-prime-strips**

$${}^{\dagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}} \quad \stackrel{\sim}{\rightarrow} \quad {}^{\ddagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}}$$

associated to ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ *and* ${}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ *[cf. Corollary 3.7, (ii), (iii); [IUTchII], Definition 4.9, (vii)].*

(iii) **(Frobenius-/Étale-Pictures)** *Let* $\{{}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n\in\mathbb{Z}}$ *be a* **collection of distinct** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *[relative to the given initial* Θ*-data] indexed by the integers. Then the* **infinite chain**

$$\dots \xrightarrow{\Theta} {}^{(n-1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta} {}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta} {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta} \dots$$

of Θ**-linked** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *will be referred to as the* **Frobenius-picture** *[associated to the* Θ*-link] — cf. Fig. I1.5; Corollary 3.8. The Frobenius-picture* **fails** *to admit* **permutation automorphisms** *that switch adjacent indices* n, $n+1$, *but leave the remaining indices* $\in \mathbb{Z}$ *fixed. The Frobenius-picture induces an infinite chain of* **full poly-isomorphisms**

$$\dots \ \stackrel{\sim}{\rightarrow}\ {}^{(n-1)}\mathfrak{D}^{\vdash}_{>} \ \stackrel{\sim}{\rightarrow}\ {}^{n}\mathfrak{D}^{\vdash}_{>} \ \stackrel{\sim}{\rightarrow}\ {}^{(n+1)}\mathfrak{D}^{\vdash}_{>} \ \stackrel{\sim}{\rightarrow}\ \dots$$

between the various $\mathcal{D}^{\vdash}$*-prime-strips* ${}^{n}\mathfrak{D}^{\vdash}_{>}$, *i.e., in essence, the* $\mathcal{D}^{\vdash}$*-prime-strips associated to the* $\mathcal{F}^{\vdash\times}$*-prime-strips* ${}^{n}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}}$. *The relationships of the various* $\mathcal{D}$*-*$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters* ${}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *to the* **"mono-analytic core"** *constituted by the* $\mathcal{D}^{\vdash}$*-prime-strip "*${}^{(-)}\mathfrak{D}^{\vdash}_{>}$*" regarded up to isomorphism — relationships that are depicted by* **spokes** *in Fig. I1.6 — are compatible with* **arbitrary permutation symmetries** *among the spokes, i.e., among the labels* $n \in \mathbb{Z}$ *of the* $\mathcal{D}$*-*$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters [cf. Corollaries 4.12, (ii), 6.10, (i)]. The diagram depicted in Fig. I1.6 will be referred to as the* **étale-picture***.*

In addition to the main result discussed above, we also prove a certain *technical result* concerning **tempered fundamental groups** — cf. Theorem B below — that will be of use in our development of the theory of *Hodge-Arakelov-theoretic evaluation* in [IUTchII]. This result is essentially a routine application of the theory of *maximal compact subgroups* of tempered fundamental groups developed in [SemiAnbd] [cf., especially, [SemiAnbd], Theorems 3.7, 5.4, as well as Remark 2.5.3, (ii), of the present paper]. Here, we recall that this theory of [SemiAnbd] may be thought of as a sort of **"Combinatorial Section Conjecture"** [cf. Remark 2.5.1 of the present paper; [IUTchII], Remark 1.12.4] — a point of view that is of particular interest in light of the *historical remarks* made in §I5 below. Moreover, Theorem B is of interest *independently of the theory of the present series of papers* in that it yields, for instance, a *new proof* of the *normal terminality* of the tempered fundamental group in its profinite completion, a result originally obtained in [André], Lemma 3.2.1, by means of other techniques [cf. Remark 2.4.1]. This new proof is of interest in that, unlike the techniques of [André], which are only available in the *profinite* case, this new proof [cf. Proposition 2.4, (iii)] holds in the case of **pro-**$\widehat{\Sigma}$**-completions**, for more general $\widehat{\Sigma}$ [i.e., not just the case of $\widehat{\Sigma} = \mathfrak{Primes}$].

Theorem B. (Profinite Conjugates of Tempered Decomposition and Inertia Groups) *Let* k *be a* **mixed-characteristic [nonarchimedean] local field***,* X *a* **hyperbolic curve** *over* k*. Write*

$$\Pi^{\mathrm{tp}}_{X}$$

for the **tempered fundamental group** $\pi_1^{\mathrm{tp}}(X)$ *[relative to a suitable basepoint] of* X *[cf. [André], §4; [SemiAnbd], Example 3.10];* $\widehat{\Pi}_X$ *for the* **étale fundamental group** *[relative to a suitable basepoint] of* X*. Thus, we have a* **natural inclusion**

$$\Pi_X^{\mathrm{tp}} \quad \hookrightarrow \quad \widehat{\Pi}_X$$

which allows one to identify $\widehat{\Pi}_X$ *with the profinite completion of* Π_X^{tp}*. Then every* **decomposition group** *in* $\widehat{\Pi}_X$ *(respectively,* **inertia group** *in* $\widehat{\Pi}_X$*) associated to a closed point or cusp of* X *(respectively, to a cusp of* X*) is contained in* Π_X^{tp} *if and only if it is a decomposition group in* Π_X^{tp} *(respectively, inertia group in* Π_X^{tp}*) associated to a closed point or cusp of* X *(respectively, to a cusp of* X*). Moreover, a* $\widehat{\Pi}_X$*-conjugate of* Π_X^{tp} *contains a decomposition group in* Π_X^{tp} *(respectively, inertia group in* Π_X^{tp}*) associated to a closed point or cusp of* X *(respectively, to a cusp of* X*) if and only if it is equal to* Π_X^{tp}*.*

Theorem B is [essentially] given as Corollary 2.5 [cf. also Remark 2.5.2] in §2. Here, we note that although, in the statement of Corollary 2.5, the hyperbolic curve X is assumed to admit *stable reduction* over the ring of integers $\mathcal{O}_k$ of k, one verifies immediately [by applying Proposition 2.4, (iii)] that this assumption is, in fact, *unnecessary.*

Finally, we remark that one *important reason* for the need to apply Theorem B in the context of the theory of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters summarized in Theorem A is the following. The $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, which will play a crucial role in the theory of the present series of papers [cf., especially, [IUTchII], [IUTchIII]], depends, in an essential way, on the *synchronization of the* $\pm$*-indeterminacies* that occur locally at each $\underline{v} \in \underline{\mathbb{V}}$ [cf. Fig. I1.1]. Such a synchronization may only be obtained by making use of the *global portion* of the $\Theta^{\pm\mathrm{ell}}$-Hodge theater under consideration. On the other hand, in order to avail oneself of such **global** $\pm$**-synchronizations** [cf. Remark 6.12.4, (iii)], it is necessary to regard the various **labels** of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry

$$(-l^* \ < \ \dots \ < \ -1 \ < 0 \ < \ 1 \ < \ \dots \ < \ l^*)$$

as conjugacy classes of inertia groups of the [necessarily] *profinite* geometric étale fundamental group of $\underline{X}_K$. That is to say, in order to relate such *global profinite conjugacy classes* to the corresponding *tempered conjugacy classes* [i.e., conjugacy classes with respect to the geometric *tempered* fundamental group] of inertia groups at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [i.e., where the crucial *Hodge-Arakelov-theoretic evaluation* is to be performed!], it is necessary to apply Theorem B — cf. the discussion of Remark 4.5.1; [IUTchII], Remark 2.5.2, for more details.

§I2. Gluing Together Models of Conventional Scheme Theory

As discussed in §I1, the system of Frobenioids constituted by a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater is intended to be a sort of miniature model of **conventional scheme theory**. One then **glues** multiple $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters $\{{}^n\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n\in\mathbb{Z}}$ together by means of the full poly-isomorphisms between the *"subsystems of Frobenioids"* constituted by certain $\mathcal{F}^{\Vdash}$*-prime-strips*

$$^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \quad \stackrel{\sim}{\to} \quad ^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$$

to form the **Frobenius-picture**. One fundamental observation in this context is the following:

> these gluing isomorphisms — i.e., in essence, the correspondences
>
> $$^{n}\underline{\underline{\Theta}}_{v} \quad \mapsto \quad ^{(n+1)}\underline{\underline{q}}_{v}$$
>
> — and hence the geometry of the resulting Frobenius-picture *lie* **outside** *the framework of* **conventional scheme theory** *in the sense that they do* **not** *arise from* **ring homomorphisms***!*

In particular, although each particular model ${}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ of conventional scheme theory is constructed within the framework of conventional scheme theory, the relationship between the *distinct* [albeit abstractly isomorphic, as $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters!] conventional scheme theories represented by, for instance, neighboring $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters ${}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, ${}^{n+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *cannot be expressed scheme-theoretically.* In this context, it is also important to note that such gluing operations are possible precisely because of the **relatively simple structure** — for instance, by comparison to the structure of a *ring*! — of the **Frobenius-like structures** constituted by the Frobenioids that appear in the various $\mathcal{F}^{\Vdash}$-prime-strips involved, i.e., in essence, collections of **monoids** isomorphic to $\mathbb{N}$ or $\mathbb{R}_{\geq 0}$ [cf. Fig. I1.2].

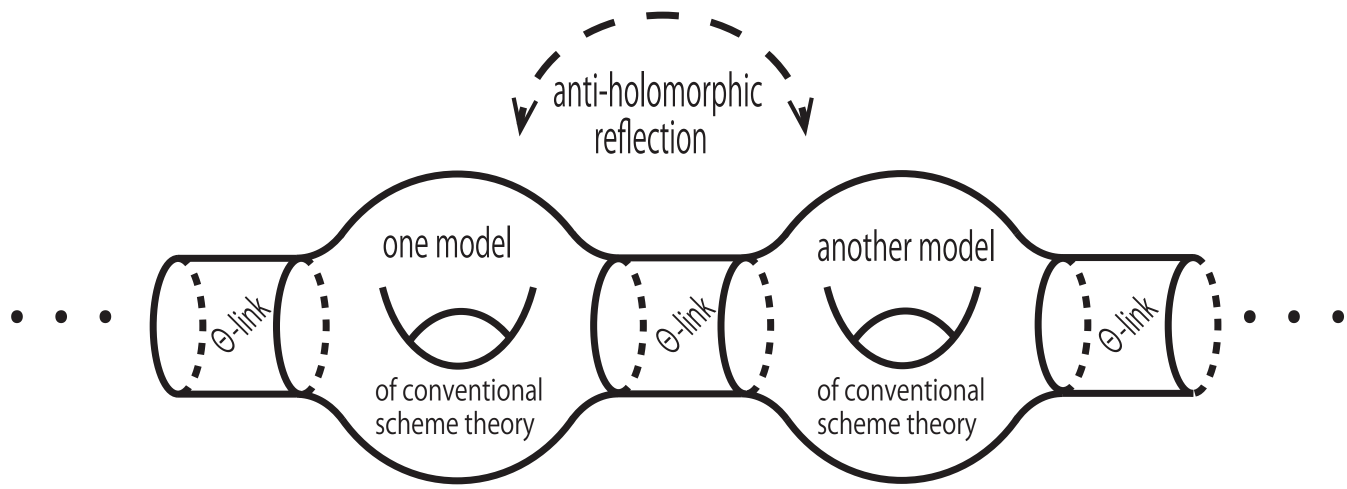

Fig. I2.1: Depiction of Frobenius- and étale-pictures of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters via glued topological surfaces

If one thinks of the geometry of *"conventional scheme theory"* as being analogous to the geometry of *"Euclidean space"*, then the geometry represented by the *Frobenius-picture* corresponds to a *"topological manifold"*, i.e., which is obtained by gluing together various portions of Euclidean space, but which is *not homeomorphic* to Euclidean space. This point of view is illustrated in Fig. I2.1 above, where the various $\Theta^{\pm \mathrm{ell}}$**NF-Hodge theaters** in the *Frobenius-picture* are depicted as [*two-dimensional!* — cf. the discussion of §I1] **twice-punctured topological surfaces of genus one**, glued together along **tubular neighborhoods of cycles**, which correspond to the [*one-dimensional*! — cf. the discussion of §I1] **mono-analytic** data that appears in the isomorphism that constitutes the **Θ-link**. The permutation symmetries in the étale-picture [cf. the discussion of §I1] are depicted in Fig. I2.1 as the **anti-holomorphic reflection** [cf. the discussion of **multiradiality** in [IUTchII], Introduction!] around a gluing cycle between topological surfaces.

Another elementary example that illustrates the *spirit* of the gluing operations discussed in the present series of papers is the following. For $i = 0, 1$, let $\mathbb{R}_i$ be a copy of the *real line*; $I_i \subseteq \mathbb{R}_i$ the *closed unit interval* [i.e., corresponding to $[0, 1] \subseteq \mathbb{R}$]. Write $C_0 \subseteq I_0$ for the *Cantor set* and

$$\phi : C_0 \xrightarrow{\sim} I_1$$

for the *bijection* arising from the **Cantor function**. Then if one thinks of $\mathbb{R}_0$ and $\mathbb{R}_1$ as being **glued** to one another by means of ϕ, then it is a *highly nontrivial* problem

> to describe *structures naturally associated to the* **"alien" ring structure** of $\mathbb{R}_0$ — such as, for instance, the subset of **algebraic numbers** $\in \mathbb{R}_0$ — in terms that only require the use of the ring structure of $\mathbb{R}_1$.

A slightly less elementary example that illustrates the *spirit* of the gluing operations discussed in the present series of papers is the following. This example is *technically* much closer to the theory of the present series of papers than the examples involving topological surfaces and Cantor sets given above. For simplicity, let us write

$$G \curvearrowright \mathcal{O}^{\times}, \quad G \curvearrowright \mathcal{O}^{\triangleright}$$

for the pairs "$G_{\underline{v}} \curvearrowright \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}$", "$G_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$" [cf. the notation of the discussion surrounding Fig. I1.2]. Recall from [AbsTopIII], Proposition 3.2, (iv), that the operation

$$(G \curvearrowright \mathcal{O}^{\triangleright}) \quad \mapsto \quad G$$

of "forgetting $\mathcal{O}^{\triangleright}$" determines a **bijection** from the *group of automorphisms* of the pair $G \curvearrowright \mathcal{O}^{\triangleright}$ — i.e., thought of as an abstract ind-topological monoid equipped with a continuous action by an abstract topological group — to the group of automorphisms of the topological group G. By contrast, we recall from [AbsTopIII], Proposition 3.3, (ii), that the operation

$$(G \curvearrowright \mathcal{O}^{\times}) \quad \mapsto \quad G$$

of "forgetting $\mathcal{O}^{\times}$" only determines a **surjection** from the *group of automorphisms* of the pair $G \curvearrowright \mathcal{O}^{\times}$ — i.e., thought of as an abstract ind-topological monoid equipped with a continuous action by an abstract topological group — to the group of automorphisms of the topological group G; that is to say, the *kernel* of this surjection is given by the **natural action** *of* $\widehat{\mathbb{Z}}^\times$ *on* $\mathcal{O}^\times$. In particular, if one works with *two copies* $G_i \curvearrowright \mathcal{O}_i^\triangleright$, where $i = 0, 1$, of $G \curvearrowright \mathcal{O}^\triangleright$, which one thinks of as being **glued** to one another by means of an **indeterminate isomorphism**

$$(G_0 \curvearrowright \mathcal{O}_0^\times) \quad \stackrel{\sim}{\to} \quad (G_1 \curvearrowright \mathcal{O}_1^\times)$$

[i.e., where one thinks of each $(G_i \curvearrowright \mathcal{O}_i^\times)$, for $i = 0, 1$, as an abstract ind-topological monoid equipped with a continuous action by an abstract topological group], then, in general, it is a *highly nontrivial* problem

> to describe *structures naturally associated to* $(G_0 \curvearrowright \mathcal{O}_0^\triangleright)$ in terms that only require the use of $(G_1 \curvearrowright \mathcal{O}_1^\triangleright)$.

One such structure which is of interest in the context of the present series of papers [cf., especially, the theory of [IUTchII], §1] is the natural **cyclotomic rigidity isomorphism** between the group of torsion elements of $\mathcal{O}_0^\triangleright$ and an analogous group of torsion elements naturally associated to G_0 — i.e., a structure that is manifestly **not preserved** by the natural action of $\widehat{\mathbb{Z}}^\times$ on $\mathcal{O}_0^\times$!

In the context of the above discussion of Fig. I2.1, it is of interest to note the important role played by **Kummer theory** in the present series of papers [cf. the Introductions to [IUTchII], [IUTchIII]]. From the point of view of Fig. I2.1, this role corresponds to the *precise specification* of the gluing cycle within each twice-punctured genus one surface in the illustration. Of course, such a precise specification *depends* on the twice-punctured genus one surface under consideration, i.e., the *same* gluing cycle is subject to *quite different "precise specifications"*, relative to the twice-punctured genus one surface on the *left* and the twice-punctured genus one surface on the *right*. This state of affairs corresponds to the *quite different Kummer theories* to which the monoids/Frobenioids that appear in the Θ-link are subject, relative to the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater in the *domain* of the Θ-link and the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater in the *codomain* of the Θ-link. At first glance, it might appear that the use of *Kummer theory*, i.e., of the correspondence determined by constructing *Kummer classes*, to achieve this precise specification of the relevant monoids/Frobenioids within each $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater is somewhat *arbitrary*, i.e., that one could perhaps use other correspondences [i.e., correspondences not determined by Kummer classes] to achieve such a precise specification. In fact, however, the **rigidity** of the relevant local and global monoids equipped with Galois actions [cf. Corollary 5.3, (i), (ii), (iv)] implies that, if one imposes the natural condition of **Galois-compatibility**, then

> the correspondence furnished by **Kummer theory** is the only acceptable choice for constructing the required "**precise specification** *of the relevant monoids/Frobenioids within each* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater"*

— cf. also the discussion of [IUTchII], Remark 3.6.2, (ii).

The construction of the Frobenius-picture described in §I1 is given in the present paper. More elaborate versions of this Frobenius-picture will be discussed in [IUTchII], [IUTchIII]. Once one constructs the Frobenius-picture, one *natural* *and fundamental problem*, which will, in fact, be one of the *main themes* of the present series of papers, is the problem of

> *describing an* **alien "arithmetic holomorphic structure"** [i.e., an alien "conventional scheme theory"] corresponding to some ${}^{m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *in terms of a "known arithmetic holomorphic structure"* corresponding to ${}^{n}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ [where $n \neq m$]

— a problem, which, as discussed in §I1, will be approached, in the final portion of [IUTchIII], by applying various results from **absolute anabelian geometry** [i.e., more explicitly, the theory of [SemiAnbd], [EtTh], and [AbsTopIII]] to the various tempered and étale fundamental groups that appear in the **étale-picture**.

The relevance to this problem of the extensive theory of *"reconstruction of ring/scheme structures"* provided by absolute anabelian geometry is evident from the statement of the problem. On the other hand, in this context, it is of interest to note that, unlike conventional anabelian geometry, which typically centers on the goal of *reconstructing a "known scheme-theoretic object"*, in the present series of papers, we wish to apply techniques and results from anabelian geometry in order to analyze the structure of an **unknown**, *essentially* **non-scheme-theoretic** *object*, namely, the **Frobenius-picture**, as described above. Put another way, relative to the point of view that *"Galois groups are arithmetic tangent bundles"* [cf. the theory of the *arithmetic Kodaira-Spencer morphism* in [HASurI]], one may think of conventional anabelian geometry as corresponding to the *computation of the automorphisms of a scheme* as

$$H^0(\text{arithmetic tangent bundle})$$

and of the application of absolute anabelian geometry to the analysis of the Frobenius-picture, i.e., to the solution of the problem discussed above, as corresponding to the computation of

$$H^1(\text{arithmetic tangent bundle})$$

— i.e., the *computation of* **"deformations of the arithmetic holomorphic structure"** of a number field equipped with an elliptic curve.

In the context of the above discussion, we remark that the word *"Hodge"* in the term *"Hodge theater"* was intended as a reference to the use of the word "Hodge" in such classical terminology as *"variation of* **Hodge structure***"* [cf. also the discussion of *Hodge filtrations* in [AbsTopIII], §I5], for instance, in discussions of *Torelli maps* [the most fundamental special case of which arises from the tautological family of one-dimensional complex tori parametrized by the upper half-plane!], where a "Hodge structure" corresponds precisely to the *specification of a particular holomorphic structure* in a situation in which one considers variations of the holomorphic structure on a fixed underlying real analytic structure. That is to say, later, in [IUTchIII], we shall see that the position occupied by a *"Hodge theater"* within a much larger framework that will be referred to as the *"log-theta-lattice"* [cf. the discussion of §I4 below] corresponds precisely to the **specification of a particular arithmetic holomorphic structure** in a situation in which such arithmetic holomorphic structures are **subject to deformation**.

§I3. Basepoints and Inter-universality

As discussed in §I2, the present series of papers is concerned with considering *"deformations of the arithmetic holomorphic structure"* of a number field — i.e., so to speak, with performing **"surgery on the number field"**. At a more concrete level, this means that one must consider situations in which *two distinct "theaters" for conventional ring/scheme theory* — i.e., two distinct $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters — are related to one another by means of a *"correspondence"*, or *"filter"*, that **fails** to be compatible with the respective **ring structures**. In the discussion so far of the portion of the theory developed in the present paper, the main example of such a "filter" is given by the **Θ-link**. As mentioned earlier, more elaborate versions of the Θ-link will be discussed in [IUTchII], [IUTchIII]. The other main example of such a non-ring/scheme-theoretic "filter" in the present series of papers is the **$\mathfrak{log}$-link**, which we shall discuss in [IUTchIII] [cf. also the theory of [AbsTopIII]].

One important aspect of such non-ring/scheme-theoretic filters is the property that they are **incompatible** with various constructions that depend on the **ring structure** of the theaters that constitute the domain and codomain of such a filter. From the point of view of the present series of papers, perhaps the most important example of such a construction is given by the various **étale fundamental groups** — e.g., **Galois groups** — that appear in these theaters. Indeed, these groups are defined, essentially, as **automorphism groups** *of some separably closed* **field**, i.e., the field that arises in the definition of the *fiber functor* associated to the **basepoint** determined by a *geometric point* that is used to define the étale fundamental group — cf. the discussion of [IUTchII], Remark 3.6.3, (i); [IUTchIII], Remark 1.2.4, (i); [AbsTopIII], Remark 3.7.7, (i). In particular, unlike the case with ring homomorphisms or morphisms of schemes with respect to which the étale fundamental group satisfies well-known *functoriality* properties, in the case of non-ring/scheme-theoretic filters, the only *"type of mathematical object"* that makes sense *simultaneously* in both the domain and codomain theaters of the filter is the notion of a *topological group*. In particular, the only data that can be considered in relating étale fundamental groups on either side of a filter is the **étale-like structure** constituted by the underlying **abstract topological group** associated to such an étale fundamental group, i.e., devoid of any *auxiliary data* arising from the construction of the group *"as an étale fundamental group associated to a* **basepoint** *determined by a geometric point of a scheme"*. It is this fundamental aspect of the theory of the present series of papers — i.e.,

> of relating the distinct *set-theoretic universes* associated to the distinct fiber functors/basepoints on either side of such a non-ring/scheme-theoretic filter

— that we refer to as **inter-universal**. This inter-universal aspect of the theory manifestly leads to the issue of considering

> the extent to which one can understand *various ring/scheme structures* by considering only the underlying **abstract topological group** of some étale fundamental group arising from such a ring/scheme structure

— i.e., in other words, of considering the **absolute anabelian geometry** [cf. the Introductions to [AbsTopI], [AbsTopII], [AbsTopIII]] of the rings/schemes under consideration.

At this point, the careful reader will note that the above discussion of the inter-universal aspects of the theory of the present series of papers depends, in an essential way, on the issue of *distinguishing different* **"types of mathematical objects"** and hence, in particular, on the *notion of a "type of mathematical object"*. This notion may be formalized via the language of **"species"**, which we develop in the final portion of [IUTchIV].

Another important *"inter-universal"* phenomenon in the present series of papers — i.e., phenomenon which, like the absolute anabelian aspects discussed above, arises from a *"deep sensitivity to particular choices of* **basepoints***"* — is the phenomenon of **conjugate synchronization**, i.e., of synchronization between conjugacy indeterminacies of distinct copies of various local Galois groups, which, as was mentioned in §I1, will play an important role in the theory of [IUTchII], [IUTchIII]. The various **rigidity properties** of the *étale theta function* established in [EtTh] constitute yet another inter-universal phenomenon that will play an important role in theory of [IUTchII], [IUTchIII].

§I4. Relation to Complex and p-adic Teichmüller Theory

In order to understand the sense in which the theory of the present series of papers may be thought of as a sort of *"Teichmüller theory"* of number fields equipped with an elliptic curve, it is useful to recall certain basic, well-known facts concerning the **classical complex Teichmüller theory** of Riemann surfaces of finite type [cf., e.g., [Lehto], Chapter V, §8]. Although such a Riemann surface is **one-dimensional** from a *complex, holomorphic* point of view, this single complex dimension may be thought of consisting of **two** *underlying real analytic dimensions.* Relative to a suitable canonical holomorphic coordinate $z = x + iy$ on the Riemann surface, the **Teichmüller deformation** may be written in the form

$$z \ \mapsto \ \zeta \ = \ \xi + i\eta = \ Kx + iy$$

— where $1 < K < \infty$ is the *dilation* factor associated to the deformation. That is to say, the Teichmüller deformation consists of **dilating one** of the two underlying real analytic dimensions, while keeping the **other dimension fixed**. Moreover, the theory of such Teichmüller deformations may be *summarized* as consisting of

> *the explicit description of a* **varying holomorphic structure** *within a* **fixed real analytic "container"**

— i.e., the underlying real analytic surface associated to the given Riemann surface.

On the other hand, as discussed in [AbsTopIII], §I3, one may think of the **ring structure** of a *number field* F as a **single "arithmetic holomorphic dimension"**, which, in fact, consists of **two** *underlying* **"combinatorial dimensions"**, corresponding to

· its **additive** *structure* "$\boxplus$" and its **multiplicative** *structure* "$\boxtimes$".

When, for simplicity, the number field F is *totally imaginary*, one may think of these two combinatorial dimensions as corresponding to the

· **two cohomological dimensions** of the *absolute Galois group* G_F of F.

A similar statement holds in the case of the absolute Galois group G_k of a **nonarchimedean local field** k. In the case of **complex archimedean fields** k [i.e., topological fields isomorphic to the field of complex numbers equipped with its usual topology], the two combinatorial dimensions of k may also be thought of as corresponding to the

· **two** *underlying* **topological/real dimensions** of k.

Alternatively, in both the nonarchimedean and archimedean cases, one may think of the two underlying combinatorial dimensions of k as corresponding to the

· **group of units** $\mathcal{O}_k^\times$ *and* **value group** $k^\times/\mathcal{O}_k^\times$ of k.

Indeed, in the nonarchimedean case, local class field theory implies that this last point of view is consistent with the interpretation of the two underlying combinatorial dimensions via cohomological dimension; in the archimedean case, the consistency of this last point of view with the interpretation of the two underlying combinatorial dimensions via topological/real dimension is immediate from the definitions.

This last interpretation in terms of groups of units and value groups is of particular relevance in the context of the theory of the present series of papers. That is to say, one may think of the **Θ-link**

$$\begin{array}{ccc} {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} & \xrightarrow{\sim} & {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \\ \{\ {}^{\dagger}\underline{\underline{\Theta}}_{\underline{v}} & \mapsto & {}^{\ddagger}\underline{\underline{q}}_{\underline{v}}\ \}_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}} \end{array}$$

— which, as discussed in §I1, induces a *full poly-isomorphism*

$$\begin{array}{ccc} {}^{\dagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}} & \xrightarrow{\sim} & {}^{\ddagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}} \\ \{\ \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} & \xrightarrow{\sim} & \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}\ \}_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}} \end{array}$$

— as a sort of *"***Teichmüller deformation** *relative to a* **Θ-dilation***"*, i.e., a deformation of the **ring structure** of the number field equipped with an elliptic curve constituted by the given *initial* Θ*-data* in which one **dilates** the underlying combinatorial dimension corresponding to the local **value groups** relative to a *"Θ-factor"*, while one leaves **fixed**, up to isomorphism, the underlying combinatorial dimension corresponding to the local **groups of units** [cf. Remark 3.9.3]. This point of view is reminiscent of the discussion in §I1 of **"disentangling/dismantling"** *of various structures associated to a number field.*

In [IUTchIII], we shall consider *two-dimensional diagrams* of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters which we shall refer to as **log-theta-lattices**. The two dimensions of such diagrams correspond precisely to the *two underlying combinatorial dimensions of a ring.* Of these two dimensions, the "theta dimension" consists of the *Frobenius-picture* associated to [more elaborate versions of] the **Θ-link**. Many of the important properties that involve this "theta dimension" are consequences of the theory of [FrdI], [FrdII], [EtTh]. On the other hand, the "log dimension" consists of iterated copies of the **$\mathfrak{log}$-link**, i.e., diagrams of the sort that are studied in [AbsTopIII].

That is to say, whereas the "theta dimension" corresponds to *"deformations of the arithmetic holomorphic structure"* of the given number field equipped with an elliptic curve, this "log dimension" corresponds to *"rotations of the two underlying combinatorial dimensions"* of a ring that leave the arithmetic holomorphic structure *fixed* — cf. the discussion of the **"juggling of $\boxplus$, $\boxtimes$ induced by $\mathfrak{log}$"** in [AbsTopIII], §I3. The *ultimate conclusion* of the theory of [IUTchIII] is that

> the *"a priori unbounded deformations"* of the arithmetic holomorphic structure given by the Θ-link in fact admit **canonical bounds**, which may be thought of as a sort of reflection of the **"hyperbolicity"** of the given number field equipped with an elliptic curve

— cf. [IUTchIII], Corollary 3.12. Such canonical bounds may be thought of as analogues for a number field of canonical bounds that arise from **differentiating Frobenius liftings** in the context of p-adic hyperbolic curves — cf. the discussion in the final portion of [AbsTopIII], §I5. Moreover, such canonical bounds are obtained in [IUTchIII] as a consequence of

> *the explicit description of a* **varying arithmetic holomorphic structure** *within a* **fixed mono-analytic "container"**

— cf. the discussion of §I2! — furnished by [IUTchIII], Theorem 3.11 [cf. also the discussion of [IUTchIII], Remarks 3.12.2, 3.12.3, 3.12.4], i.e., a situation that is *entirely formally analogous* to the summary of complex Teichmüller theory given above.

The significance of the log-theta-lattice is best understood in the context of the analogy between the **inter-universal Teichmüller theory** developed in the present series of papers and the ***p*-adic Teichmüller theory** of [pOrd], [pTeich]. Here, we recall for the convenience of the reader that the p-adic Teichmüller theory of [pOrd], [pTeich] may be summarized, [very!] roughly speaking, as a sort of **generalization**, to the case of **"quite general" *p*-adic hyperbolic curves**, of the classical p-adic theory surrounding the **canonical representation**

$$\pi_1(\ (\mathbb{P}^1 \setminus \{0,1,\infty\})_{\mathbb{Q}_p}\ ) \quad \to \quad \pi_1(\ (\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}_p}\ ) \quad \to \quad PGL_2(\mathbb{Z}_p)$$

— where the "$\pi_1(-)$'s" denote the *étale fundamental group*, relative to a suitable basepoint; $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}_p}$ denotes the *moduli stack of elliptic curves* over $\mathbb{Q}_p$; the first arrow denotes the morphism induced by the elliptic curve over the projective line minus three points determined by the classical *Legendre form* of the Weierstrass equation; the second arrow is the representation determined by the *p-power torsion points* of the *tautological elliptic curve* over $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}_p}$. In particular, the reader who is familiar with the theory of the classical representation of the above display, but not with the theory of [pOrd], [pTeich], may nevertheless appreciate, to a substantial degree, the analogy between the *inter-universal Teichmüller theory* developed in the present series of papers and the *p-adic Teichmüller theory* of [pOrd], [pTeich] by

> thinking in terms of the
> *well-known classical properties of this classical representation.*

In some sense, the *gap* between the "quite general" p-adic hyperbolic curves that appear in p-adic Teichmüller theory and the classical case of $(\mathbb{P}^1 \setminus \{0,1,\infty\})_{\mathbb{Q}_p}$ may be thought of, roughly speaking, as corresponding, relative to the analogy with the theory of the present series of papers, to the gap between **arbitrary number fields** and the **rational number field** $\mathbb{Q}$. This point of view is especially interesting in the context of the discussion of §I5 below.

The analogy between the **inter-universal Teichmüller theory** developed in the present series of papers and the ***p*-adic Teichmüller theory** of [pOrd], [pTeich] is described to a substantial degree in the discussion of [AbsTopIII], §I5, i.e., where the *"future Teichmüller-like extension of the mono-anabelian theory"* may be understood as referring precisely to the inter-universal Teichmüller theory developed in the present series of papers. The starting point of this analogy is the correspondence between a *number field equipped with a [once-punctured] elliptic curve* [in the present series of papers] and a *hyperbolic curve over a positive characteristic perfect field equipped with a nilpotent ordinary indigenous bundle* [in p-adic Teichmüller theory] — cf. Fig. I4.1 below. That is to say, in this analogy, the *number field* — which may be regarded as being equipped with a *finite collection of "exceptional" valuations*, namely, in the notation of §I1, the valuations lying over $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$ — corresponds to the *hyperbolic curve over a positive characteristic perfect field* — which may be thought of as a *one-dimensional function field* over a positive characteristic perfect field, equipped with a *finite collection of "exceptional" valuations*, namely, the valuations corresponding to the cusps of the curve.

On the other hand, the *[once-punctured] elliptic curve* in the present series of papers corresponds to the *nilpotent ordinary indigenous bundle* in p-adic Teichmüller theory. Here, we recall that an indigenous bundle may be thought of as a sort of *"virtual analogue"* of the first cohomology group of the tautological elliptic curve over the moduli stack of elliptic curves. Indeed, the canonical indigenous bundle over the moduli stack of elliptic curves arises precisely as the first de Rham cohomology module of this tautological elliptic curve. Put another way, from the point of view of *fundamental groups*, an indigenous bundle may be thought of as a sort of "virtual analogue" of the **abelianized fundamental group** of the tautological elliptic curve over the moduli stack of elliptic curves. By contrast, in the present series of papers, it is of crucial importance to use the **entire nonabelian profinite étale fundamental group** — i.e., not just its abelizanization! — of the given once-punctured elliptic curve over a number field. Indeed, only by working with the entire profinite étale fundamental group can one avail oneself of the crucial **absolute anabelian theory** developed in [EtTh], [AbsTopIII] [cf. the discussion of §I3]. This state of affairs prompts the following question:

> To what extent can one extend the indigenous bundles that appear in *classical complex* and *p-adic Teichmüller theory* to objects that serve as *"virtual analogues" of the* **entire nonabelian fundamental group** *of the tautological once-punctured elliptic curve* over the moduli stack of [once-punctured] elliptic curves?

Although this question lies beyond the scope of the present series of papers, it is the hope of the author that this question may be addressed in a future paper.

Inter-universal Teichmüller theory	*p-adic Teichmüller theory*
number field F	**hyperbolic curve** C over a *positive characteristic perfect field*
[once-punctured] **elliptic curve** X over F	*nilpotent ordinary* **indigenous bundle** P over C
Θ-link arrows of the *log-theta-lattice*	**mixed characteristic extension** structure of a ring of *Witt vectors*
$\mathfrak{log}$-link arrows of the *log-theta-lattice*	the **Frobenius morphism** in *positive characteristic*
the entire **log-theta-lattice**	the resulting **canonical lifting** + **canonical Frobenius action**; **canonical Frobenius lifting** over the ordinary locus
relatively straightforward *original construction of* **log-theta-lattice**	relatively straightforward *original construction of* **canonical liftings**
highly nontrivial *description of* **alien arithmetic** **holomorphic structure** via *absolute anabelian geometry*	highly nontrivial *absolute anabelian* *reconstruction of* **canonical liftings**

Fig. I4.1: Correspondence between inter-universal Teichmüller theory and p-adic Teichmüller theory

Now let us return to our discussion of the log-theta-lattice, which, as discussed above, consists of two types of arrows, namely, **Θ-link** arrows and **$\mathfrak{log}$-link** arrows. As discussed in [IUTchIII], Remark 1.4.1, (iii) — cf. also Fig. I4.1 above, as well as Remark 3.9.3, (i), of the present paper — the Θ-link arrows correspond to the *"transition from $p^n\mathbb{Z}/p^{n+1}\mathbb{Z}$ to $p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$"*, i.e., the **mixed characteristic extension** *structure of a ring of Witt vectors*, while the $\mathfrak{log}$-link arrows, i.e., the portion of theory that is developed in detail in [AbsTopIII], and which will be incorporated into the theory of the present series of papers in [IUTchIII], correspond to the **Frobenius morphism** *in positive characteristic*. As we shall see in [IUTchIII], these two types of arrows *fail to commute* [cf. [IUTchIII], Remark 1.4.1, (i)]. This noncommutativity, or **"intertwining"**, of the Θ-link and $\mathfrak{log}$-link arrows of the log-theta-lattice may be thought of as the analogue, in the context of the theory of the present series of papers, of the well-known "intertwining between the mixed characteristic extension structure of a ring of Witt vectors and the Frobenius morphism in positive characteristic" that appears in the classical p-adic theory. In particular, taken as a whole, the log-theta-lattice in the theory of the present series of papers may be thought of as an analogue, for number fields equipped with a [once-punctured] elliptic curve, of the **canonical lifting**, *equipped with a* **canonical Frobenius action** — hence also the **canonical Frobenius lifting** over the ordinary locus of the curve — associated to a positive characteristic hyperbolic curve equipped with a nilpotent ordinary indigenous bundle in p-adic Teichmüller theory [cf. Fig. I4.1 above; the discussion of [IUTchIII], Remarks 3.12.3, 3.12.4].

Finally, we observe that it is of particular interest in the context of the present discussion that a theory is developed in [CanLift], §3, that yields an **absolute anabelian reconstruction** *for the canonical liftings of p-adic Teichmüller theory*. That is to say, whereas the *original construction* of such canonical liftings given in [pOrd], §3, is *relatively straightforward*, the *anabelian reconstruction* given in [CanLift], §3, of, for instance, the canonical lifting modulo p^2 of the logarithmic special fiber consists of a *highly nontrivial anabelian argument*. This state of affairs is strongly reminiscent of the stark contrast between the *relatively straightforward* construction of the log-theta-lattice given in the present series of papers and the description of an "alien arithmetic holomorphic structure" given in [IUTchIII], Theorem 3.11 [cf. the discussion in the earlier portion of the present §I4], which is achieved by applying *highly nontrivial results in absolute anabelian geometry* — cf. Fig. I4.1 above. In this context, we observe that the *absolute anabelian theory* of [AbsTopIII], §1, which plays a central role in the theory surrounding [IUTchIII], Theorem 3.11, corresponds, in the theory of [CanLift], §3, to the *absolute anabelian reconstruction of the logarithmic special fiber* given in [AbsAnab], §2 [i.e., in essence, the theory of absolute anabelian geometry over finite fields developed in [Tama1]; cf. also [Cusp], §2]. Moreover, just as the absolute anabelian theory of [AbsTopIII], §1, follows essentially by combining a version of *"Uchida's Lemma"* [cf. [AbsTopIII], Proposition 1.3] with the theory of *Belyi cuspidalization* — i.e.,

[AbsTopIII], §1 = Uchida Lem. + Belyi cuspidalization

— the absolute anabelian geometry over finite fields of [Tama1], [Cusp], follows essentially by combining a version of *"Uchida's Lemma"* with an application [to the counting of rational points] of the *Lefschetz trace formula for [powers of] the Frobenius morphism on a curve over a finite field* — i.e.,

[Tama1], [Cusp] = Uchida Lem. + Lefschetz trace formula for Frob.

— cf. the discussion of [AbsTopIII], §I5. That is to say, it is perhaps worthy of note that in the analogy between the inter-universal Teichmüller theory developed in the present series of papers and the p-adic Teichmüller theory of [pOrd], [pTeich], [CanLift], the application of the theory of Belyi cuspidalization over number fields and mixed characteristic local fields may be thought of as corresponding to the Lefschetz trace formula for [powers of] the Frobenius morphism on a curve over a finite field, i.e.,

$$\textbf{Belyi cuspidalization} \quad \longleftrightarrow \quad \textbf{Lefschetz trace formula for Frobenius}$$

[Here, we note in passing that this correspondence may be related to the correspondence discussed in [AbsTopIII], §I5, between Belyi cuspidalization and the Verschiebung on positive characteristic indigenous bundles by considering the geometry of Hecke correspondences modulo p, i.e., in essence, graphs of the Frobenius morphism in characteristic p!] It is the hope of the author that these analogies and correspondences might serve to stimulate further developments in the theory.

§I5. Other Galois-theoretic Approaches to Diophantine Geometry

The notion of **anabelian geometry** dates back to a famous *"letter to Faltings"* [cf. [Groth]], written by Grothendieck in response to Faltings' work on the *Mordell Conjecture* [cf. [Falt]]. Anabelian geometry was apparently originally conceived by Grothendieck as a new approach to obtaining results in **diophantine geometry** such as the Mordell Conjecture. At the time of writing, the author is not aware of any expositions by Grothendieck that expose this approach in detail. Nevertheless, it appears that the thrust of this approach revolves around applying the *Section Conjecture* for hyperbolic curves over number fields to obtain a contradiction by applying this Section Conjecture to the *"limit section"* of the Galois sections associated to any *infinite sequence of rational points* of a proper hyperbolic curve over a number field [cf. [MNT], §4.1(B), for more details]. On the other hand, to the knowledge of the author, at least at the time of writing, it does not appear that any rigorous argument has been obtained either by Grothendieck or by other mathematicians for deriving a new proof of the Mordell Conjecture from the [as yet unproven] Section Conjecture for hyperbolic curves over number fields. Nevertheless, one result that has been obtained is a new proof by M. Kim [cf. [Kim]] of *Siegel's theorem* concerning $\mathbb{Q}$-rational points of the projective line minus three points — a proof which proceeds by obtaining certain bounds on the cardinality of the set of Galois sections, *without* applying the Section Conjecture or any other results from anabelian geometry.

In light of the historical background just discussed, the theory exposed in the present series of papers — which yields, in particular, a method for applying results in **absolute anabelian geometry** to obtain **diophantine results** such as those given in [IUTchIV] — occupies a *somewhat curious position*, relative to the historical development of the mathematical ideas involved. That is to say, at a purely formal level, the implication

$$\textit{anabelian geometry} \quad \Longrightarrow \quad \textit{diophantine results}$$

at first glance looks something like a *"confirmation"* of Grothendieck's original intuition. On the other hand, closer inspection reveals that the approach of the theory of the present series of papers — that is to say, the **precise content** *of the relationship between anabelian geometry and diophantine geometry* established in the present series of papers — *differs quite fundamentally* from the sort of approach that was apparently envisioned by Grothendieck.

Perhaps the most characteristic aspect of this difference lies in the central role played by **anabelian geometry over p-adic fields** in the present series of papers. That is to say, unlike the case with number fields, one central feature of anabelian geometry over p-adic fields is the *fundamental gap* between **relative** and **absolute** results [cf., e.g., [AbsTopI], Introduction]. This fundamental gap is closely related to the notion of an **"arithmetic Teichmüller theory for number fields"** [cf. the discussion of §I4 of the present paper; [AbsTopIII], §I3, §I5] — i.e., a theory of deformations *not* for the "arithmetic holomorphic structure" of a hyperbolic *curve* over a number field, but rather for the "arithmetic holomorphic structure" of the *number field itself*! To the knowledge of the author, there does not exist any mention of such ideas [i.e., relative vs. absolute p-adic anabelian geometry; the notion of an arithmetic Teichmüller theory for number fields] in the works of Grothendieck.

As discussed in §I4, one fundamental theme of the theory of the present series of papers is the issue of the

> *explicit description of the relationship between the* **additive** *structure and the* **multiplicative** *structure of a ring/number field/local field.*

Relative to the above discussion of the relationship between anabelian geometry and diophantine geometry, it is of interest to note that this issue of understanding/describing the relationship between *addition* and *multiplication* is, on the one hand, a central theme in the *proofs* of various results in *anabelian geometry* [cf., e.g., [Tama1], [pGC], [AbsTopIII]] and, on the other hand, a central aspect of the *diophantine results* obtained in [IUTchIV].

From a historical point of view, it is also of interest to note that results from absolute anabelian geometry are applied in the present series of papers in the context of the **canonical splittings** *of the Frobenius-picture* that arise by considering the *étale-picture* [cf. the discussion in §I1 preceding Theorem A]. This state of affairs is reminiscent — relative to the point of view that the Grothendieck Conjecture constitutes a sort of *"anabelian version"* of the *Tate Conjecture* for abelian varieties [cf. the discussion of [MNT], §1.2] — of the role played by the Tate Conjecture for abelian varieties in obtaining the *diophantine results* of [Falt], namely, by means of the various **semi-simplicity** properties of the Tate module that arise as formal consequences of the Tate Conjecture. That is to say, such semi-simplicity properties may also be thought of as *"canonical splittings"* that arise from Galois-theoretic considerations [cf. the discussion of "canonical splittings" in the final portion of [CombCusp], Introduction].

Certain aspects of the relationship between the inter-universal Teichmüller theory of the present series of papers and other Galois-theoretic approaches to diophantine geometry are best understood in the context of the **analogy**, discussed in §I4, between inter-universal Teichmüller theory and **p-adic Teichmüller theory**. One way to think of the starting point of p-adic Teichmüller is as an attempt to construct a *p-adic analogue* of the theory of the action of $SL_2(\mathbb{Z})$ on the *upper half-plane*, i.e., of the natural embedding

$$\rho_{\mathbb{R}}: \quad SL_2(\mathbb{Z}) \quad \hookrightarrow \quad SL_2(\mathbb{R})$$

of $SL_2(\mathbb{Z})$ as a discrete subgroup. This leads naturally to consideration of the *representation*

$$\rho_{\widehat{\mathbb{Z}}} = \prod_{p} \rho_{\mathbb{Z}_p} : \quad SL_2(\mathbb{Z})^\wedge \quad \to \quad SL_2(\widehat{\mathbb{Z}}) = \prod_{p \in \mathfrak{Primes}} SL_2(\mathbb{Z}_p)$$

— where we write $SL_2(\mathbb{Z})^\wedge$ for the *profinite completion* of $SL_2(\mathbb{Z})$. If one thinks of $SL_2(\mathbb{Z})^\wedge$ as the *geometric étale fundamental group* of the *moduli stack of elliptic curves* over a field of characteristic zero, then the p-adic Teichmüller theory of [pOrd], [pTeich] does indeed constitute a generalization of $\rho_{\mathbb{Z}_p}$ to more general p-adic hyperbolic curves.

From a **representation-theoretic** point of view, the next natural direction in which to further develop the theory of [pOrd], [pTeich] consists of attempting to generalize the theory of representations into $SL_2(\mathbb{Z}_p)$ obtained in [pOrd], [pTeich] to a theory concerning representations into $SL_n(\mathbb{Z}_p)$ for *arbitrary* $n \geq 2$. This is precisely the motivation that lies, for instance, behind the work of Joshi and Pauly [cf. [JP]].

On the other hand, unlike the *original motivating representation* $\rho_{\mathbb{R}}$, the representation $\rho_{\widehat{\mathbb{Z}}}$ is *far from injective*, i.e., put another way, the so-called *Congruence Subgroup Property* fails to hold in the case of SL_2. This failure of injectivity means that working with

$\rho_{\widehat{\mathbb{Z}}}$ *only allows one to access a* **relatively limited portion** *of* $SL_2(\mathbb{Z})^\wedge$.

From this point of view, a more natural direction in which to further develop the theory of [pOrd], [pTeich] is to consider the **"anabelian version"**

$$\rho_\Delta : \quad SL_2(\mathbb{Z})^\wedge \quad \to \quad \mathrm{Out}(\Delta_{1,1})$$

of $\rho_{\widehat{\mathbb{Z}}}$ — i.e., the natural outer representation on the *geometric étale fundamental group* $\Delta_{1,1}$ of the tautological family of *once-punctured elliptic curves* over the moduli stack of elliptic curves over a field of characteristic zero. Indeed, unlike the case with $\rho_{\widehat{\mathbb{Z}}}$, one knows [cf. [Asada]] that ρ_Δ is **injective**. Thus, the **"arithmetic Teichmüller theory** *for* **number fields** *equipped with a* **[once-punctured] elliptic curve"** constituted by the inter-universal Teichmüller theory developed in the present series of papers may [cf. the discussion of §I4!] be regarded as a realization of this sort of "anabelian" approach to further developing the p-adic Teichmüller theory of [pOrd], [pTeich].

In the context of these two distinct possible directions for the further development of the p-adic Teichmüller theory of [pOrd], [pTeich], it is of interest to recall the following elementary fact:

If G is a *free pro-p group* of rank ≥ 2, then a [continuous] *representation*

$$\rho_G : G \to GL_n(\mathbb{Q}_p)$$

can never be injective!

Indeed, assume that ρ_G is injective and write $\ldots \subseteq H_j \subseteq \ldots \subseteq \mathrm{Im}(\rho_G) \subseteq GL_n(\mathbb{Q}_p)$ for an exhaustive sequence of open normal subgroups of the image of ρ_G. Then since the H_j are closed subgroups of $GL_n(\mathbb{Q}_p)$, hence p-adic Lie groups, it follows that the $\mathbb{Q}_p$-*dimension* $\dim(H_j^{\mathrm{ab}} \otimes \mathbb{Q}_p)$ of the tensor product with $\mathbb{Q}_p$ of the abelianization of H_j may be computed at the level of *Lie algebras*, hence is bounded by the $\mathbb{Q}_p$-dimension of the p-adic Lie group $GL_n(\mathbb{Q}_p)$, i.e., we have $\dim(H_j^{\mathrm{ab}} \otimes \mathbb{Q}_p) \leq n^2$, in contradiction to the well-known fact since $G \cong \mathrm{Im}(\rho_G)$ is *free pro-p of rank* ≥ 2, it holds that $\dim(H_j^{\mathrm{ab}} \otimes \mathbb{Q}_p) \to \infty$ as $j \to \infty$. Note, moreover, that

> this sort of argument, i.e., concerning the **asymptotic behavior** of the **abelianizations** — or, more generally, of the **Lie algebras** associated to the pro-algebraic groups determined by associated Tannakian categories of representations — of **open subgroups**, is **characteristic** of the sort of proofs that typically occur in **anabelian geometry** [cf., e.g., the proofs of [Tama1], [pGC], [CombGC], as well as [Cusp], §3!].

That is to say, the above argument to the effect that ρ_G can never be injective is a typical instance of the more general phenomenon that

> so long as one restricts oneself to **representation theory** into $GL_n(\mathbb{Q}_p)$ [or even more general groups that arise as groups of $\mathbb{Q}_p$-valued points of pro-algebraic groups], one can never access the sort of **asymptotic phenomena** that form the **"technical core"** [cf., e.g., the proofs of [Tama1], [pGC], [CombGC], as well as [Cusp], §3!] of various important results in anabelian geometry.

Put another way, the two "directions" discussed above — i.e., **representation-theoretic** and **anabelian** — appear to be **essentially mutually alien** to one another.

In this context, it is of interest to observe that the *diophantine results* derived in [IUTchIV] from the inter-universal Teichmüller theory developed in the present series of papers concern essentially **asymptotic behavior**, i.e., they do not concern properties of "a specific rational point over a specific number field", but rather properties of the asymptotic behavior of *"varying rational points over varying number fields"*. One important aspect of this asymptotic nature of the diophantine results derived in [IUTchIV] is that there are **no distinguished number fields** that occur in the theory, i.e., the theory — being *essentially asymptotic* in nature! — is "invariant" with respect to the operation of passing to finite extensions of the number field involved [which, from the point of view of the *absolute Galois group* $G_{\mathbb{Q}}$ of $\mathbb{Q}$, corresponds precisely to the operation of passing to *smaller and smaller open subgroups*, as in the above discussion!]. This contrasts sharply with the *"representation-theoretic approach to diophantine geometry"* constituted by such works as [Wiles], where *specific rational points over the specific number field* $\mathbb{Q}$ — or, for instance, in generalizations of [Wiles] involving Shimura varieties, over *specific number fields* characteristically associated to the Shimura varieties involved — play a central role.

Acknowledgements:

The research discussed in the present paper profited enormously from the generous support that the author received from the *Research Institute for Mathematical*

Sciences, a Joint Usage/Research Center located in Kyoto University. At a personal level, I would like to thank *Fumiharu Kato*, *Akio Tamagawa*, *Go Yamashita*, *Mohamed Saïdi*, *Yuichiro Hoshi*, *Ivan Fesenko*, *Fucheng Tan*, *Emmanuel Lepage*, *Arata Minamide*, and *Wojciech Porowski* for many stimulating discussions concerning the material presented in this paper. In particular, I would like to thank *Emmanuel Lepage* for his stimulating comments [summarized in Remark 2.5.3] on [SemiAnbd]. Also, I feel deeply indebted to *Go Yamashita*, *Mohamed Saïdi*, and *Yuichiro Hoshi* for their meticulous reading of and numerous comments concerning the present paper. In particular, the introduction of the theory of *κ-coric functions* was motivated by various stimulating discussions with *Yuichiro Hoshi*. Finally, I would like to express my deep gratitude to *Ivan Fesenko* for his quite substantial efforts to disseminate — for instance, in the form of a survey that he wrote — the theory discussed in the present series of papers.

Section 0: Notations and Conventions

Monoids and Categories:

We shall use the notation and terminology concerning *monoids* and *categories* of [FrdI], §0.

We shall refer to a topological space P equipped with a continuous map

$$P \times P \supseteq S \ \rightarrow \ P$$

as a *topological pseudo-monoid* if there exists a topological abelian group M [whose group operation will be written *multiplicatively*] and an embedding of topological spaces $\iota : P \hookrightarrow M$ such that $S = \{(a, b) \in P \times P \mid \iota(a) \cdot \iota(b) \in \iota(P) \subseteq M\}$, and the map $S \to P$ is obtained by restricting the group operation $M \times M \to M$ on M to P by means of ι. Here, if M is equipped with the *discrete topology*, then we shall refer to the resulting P simply as a *pseudo-monoid*. In particular, every topological pseudo-monoid determines, in an evident fashion, an underlying pseudo-monoid. Let P be a *pseudo-monoid*. Then we shall say that the pseudo-monoid P is *divisible* if M and ι may be taken such that for each positive integer n, every element of M admits an n-th root in M, and, moreover, an element $a \in M$ lies in $\iota(P)$ if and only if a^n lies in $\iota(P)$. We shall say that the pseudo-monoid P is *cyclotomic* if M and ι may be taken such that the subgroup $\boldsymbol{\mu}_M \subseteq M$ of torsion elements of M is isomorphic to the group $\mathbb{Q}/\mathbb{Z}$, $\boldsymbol{\mu}_M \subseteq \iota(P)$, and $\boldsymbol{\mu}_M \cdot \iota(P) \subseteq \iota(P)$.

We shall refer to an isomorphic copy of some object as an *isomorph* of the object.

If $\mathcal{C}$ and $\mathcal{D}$ are *categories*, then we shall refer to as an *isomorphism* $\mathcal{C} \to \mathcal{D}$ any isomorphism class of equivalences of categories $\mathcal{C} \to \mathcal{D}$. [Note that this termniology *differs* from the standard terminology of category theory, but will be *natural in the context of the theory of the present series of papers*.] Thus, from the point of view of *"coarsifications of* 2*-categories of* 1*-categories"* [cf. [FrdI], Appendix, Definition A.1, (ii)], an "isomorphism $\mathcal{C} \to \mathcal{D}$" is precisely an "isomorphism in the usual sense" of the [1-]category constituted by the coarsification of the 2-category of all small 1-categories relative to a suitable universe with respect to which $\mathcal{C}$ and $\mathcal{D}$ are small.

Let $\mathcal{C}$ be a *category*; $A, B \in \mathrm{Ob}(\mathcal{C})$. Then we define a *poly-morphism* $A \to B$ to be a collection of morphisms $A \to B$ [i.e., a subset of the set of morphisms $A \to B$]; if all of the morphisms in the collection are isomorphisms, then we shall refer to the poly-morphism as a *poly-isomorphism*; if $A = B$, then we shall refer to a poly-isomorphism $A \stackrel{\sim}{\to} B$ as a *poly-automorphism*. We define the *full poly-isomorphism* $A \stackrel{\sim}{\to} B$ to be the poly-morphism given by the collection of all isomorphisms $A \stackrel{\sim}{\to} B$. The composite of a poly-morphism $\{f_i : A \to B\}_{i \in I}$ with a poly-morphism $\{g_j : B \to C\}_{j \in J}$ is defined to be the poly-morphism given by the *set* [i.e., where "multiplicities" are ignored] $\{g_j \circ f_i : A \to C\}_{(i,j) \in I \times J}$.

Let $\mathcal{C}$ be a *category*. We define a *capsule of objects of* $\mathcal{C}$ to be a finite collection $\{A_j\}_{j \in J}$ [i.e., where J is a finite index set] of objects A_j of $\mathcal{C}$; if $|J|$ denotes the cardinality of J, then we shall refer to a capsule with index set J as a $|J|$-*capsule*; also, we shall write $\pi_0(\{A_j\}_{j\in J}) \stackrel{\mathrm{def}}{=} J$. A *morphism of capsules of objects of* $\mathcal{C}$

$$\{A_j\}_{j\in J} \to \{A'_{j'}\}_{j'\in J'}$$

is defined to consist of an injection $\iota : J \hookrightarrow J'$, together with, for each $j \in J$, a morphism $A_j \to A'_{\iota(j)}$ of objects of $\mathcal{C}$. Thus, the capsules of objects of $\mathcal{C}$ form a *category* Capsule($\mathcal{C}$). A *capsule-full poly-morphism*

$$\{A_j\}_{j\in J} \to \{A'_{j'}\}_{j'\in J'}$$

between two objects of Capsule($\mathcal{C}$) is defined to be the poly-morphism associated to some [fixed] *injection* $\iota : J \hookrightarrow J'$ which consists of the set of morphisms of Capsule($\mathcal{C}$) given by collections of *[arbitrary] isomorphisms* $A_j \stackrel{\sim}{\to} A'_{\iota(j)}$, for $j \in J$. A *capsule-full poly-isomorphism* is a capsule-full poly-morphism for which the associated injection between index sets is a *bijection*.

If X is a *connected noetherian algebraic stack* which is *generically scheme-like* [cf. [Cusp], §0], then we shall write

$$\mathcal{B}(X)$$

for the *category of finite étale coverings* of X [and morphisms over X]; if A is a *noetherian [commutative] ring [with unity]*, then we shall write $\mathcal{B}(A) \stackrel{\mathrm{def}}{=} \mathcal{B}(\mathrm{Spec}(A))$. Thus, [cf. [FrdI], §0] the subcategory of connected objects $\mathcal{B}(X)^0 \subseteq \mathcal{B}(X)$ may be thought of as the subcategory of *connected finite étale coverings* of X [and morphisms over X].

Let Π be a *topological group*. Then let us write

$$\mathcal{B}^{\mathrm{temp}}(\Pi)$$

for the *category* whose *objects* are *countable* [i.e., of cardinality $\leq$ the cardinality of the set of natural numbers], *discrete* sets equipped with a continuous Π-action, and whose *morphisms* are morphisms of Π-sets [cf. [SemiAnbd], §3]. If Π may be written as an inverse limit of an inverse system of surjections of countable discrete topological groups, then we shall say that Π is *tempered* [cf. [SemiAnbd], Definition 3.1, (i)]. A category $\mathcal{C}$ equivalent to a category of the form $\mathcal{B}^{\mathrm{temp}}(\Pi)$, where Π is a tempered topological group, is called a *connected temperoid* [cf. [SemiAnbd], Definition 3.1, (ii)]. Thus, if $\mathcal{C}$ is a connected temperoid, then $\mathcal{C}$ is naturally equivalent to $(\mathcal{C}^0)^\top$ [cf. [FrdI], §0]. Moreover, if Π is *Galois-countable* [cf. Remark 2.5.3, (i), (T1)], then one can *reconstruct* [cf. Remark 2.5.3, (i), (T5)] *the topological group* Π*, up to inner automorphism, category-theoretically from* $\mathcal{B}^{\mathrm{temp}}(\Pi)$ *or* $\mathcal{B}^{\mathrm{temp}}(\Pi)^0$ [i.e., the subcategory of connected objects of $\mathcal{B}^{\mathrm{temp}}(\Pi)$]; in particular, for any *Galois-countable* [cf. Remark 2.5.3, (i), (T1)] connected temperoid $\mathcal{C}$, it makes sense to write

$$\pi_1(\mathcal{C}), \quad \pi_1(\mathcal{C}^0)$$

for the *topological groups*, up to inner automorphism, obtained by applying this reconstruction algorithm [cf. Remark 2.5.3, (i), (T5)].

In this context, if $\mathcal{C}_1$, $\mathcal{C}_2$ are *connected temperoids*, then it is natural to define a *morphism*

$$\mathcal{C}_1 \to \mathcal{C}_2$$

to be an isomorphism class of functors $\mathcal{C}_2 \to \mathcal{C}_1$ that preserves finite limits and countable colimits. [Note that this differs — but only slightly! — from the definition given in [SemiAnbd], Definition 3.1, (iii). This difference does not, however, have any effect on the applicability of results of [SemiAnbd] in the context of the present series of papers.] In a similar vein, we define a *morphism*

$$\mathcal{C}_1^0 \to \mathcal{C}_2^0$$

to be a morphism $(\mathcal{C}_1^0)^\top \to (\mathcal{C}_2^0)^\top$ [where we recall that we have natural equivalences of categories $\mathcal{C}_i \xrightarrow{\sim} (\mathcal{C}_i^0)^\top$ for $i = 1, 2$]. One verifies immediately that an *"isomorphism"* relative to this terminology is equivalent to an "isomorphism of categories" in the sense defined at the beginning of the present discussion of **"Monoids and Categories"**. Finally, if Π_1, Π_2 are *Galois-countable* [cf. Remark 2.5.3, (i), (T1)] tempered topological groups, then we recall that there is a *natural bijective correspondence* between

(a) the set of continuous outer homomorphisms $\Pi_1 \to \Pi_2$,
(b) the set of morphisms $\mathcal{B}^{\mathrm{temp}}(\Pi_1) \to \mathcal{B}^{\mathrm{temp}}(\Pi_2)$, and
(c) the set of morphisms $\mathcal{B}^{\mathrm{temp}}(\Pi_1)^0 \to \mathcal{B}^{\mathrm{temp}}(\Pi_2)^0$

— cf. Remark 2.5.3, (ii), (E7); [SemiAnbd], Proposition 3.2.

Suppose that for $i = 1, 2$, $\mathcal{C}_i$ and $\mathcal{C}'_i$ are *categories*. Then we shall say that two isomorphism classes of functors $\phi : \mathcal{C}_1 \to \mathcal{C}_2$, $\phi' : \mathcal{C}'_1 \to \mathcal{C}'_2$ are *abstractly equivalent* if, for $i = 1, 2$, there exist isomorphisms $\alpha_i : \mathcal{C}_i \xrightarrow{\sim} \mathcal{C}'_i$ such that $\phi' \circ \alpha_1 = \alpha_2 \circ \phi$. We shall also apply this terminology to morphisms between [connected] temperoids, as well as to morphisms between subcategories of connected objects of [connected] temperoids.

Numbers:

We shall use the abbreviations NF ("number field"), MLF ("mixed-characteristic [nonarchimedean] local field"), CAF ("complex archimedean field"), as defined in [AbsTopI], §0; [AbsTopIII], §0. We shall denote the *set of prime numbers* by $\mathfrak{Primes}$.

Let F be a *number field* [i.e., a finite extension of the field of rational numbers]. Then we shall write

$$\mathbb{V}(F) \;=\; \mathbb{V}(F)^{\mathrm{arc}} \bigcup \mathbb{V}(F)^{\mathrm{non}}$$

for the set of *valuations* of F, that is to say, the union of the sets of *archimedean* [i.e., $\mathbb{V}(F)^{\mathrm{arc}}$] and *nonarchimedean* [i.e., $\mathbb{V}(F)^{\mathrm{non}}$] valuations of F. Here, we note that this terminology "valuation", as it is applied in the present series of papers, corresponds to such terminology as *"place"* or *"absolute value"* in the work of other authors. Let $v \in \mathbb{V}(F)$. Then we shall write F_v for the *completion* of F at v and say that an element of F or F_v is *integral [at v]* if it is *of norm* ≤ 1 with respect to the valuation v; if, moreover, L is any [possibly infinite] *Galois extension* of F, then, by a slight abuse of notation, we shall write L_v for the *completion* of L at some valuation $\in \mathbb{V}(L)$ that lies over v. If $v \in \mathbb{V}(F)^{\mathrm{non}}$, then we shall write p_v for the *residue characteristic* of v. If $v \in \mathbb{V}(F)^{\mathrm{arc}}$, then we shall write $p_v \in F_v$ for the unique positive real element of F_v whose *natural logarithm is equal to* 1 [i.e., "$e = 2.71828\ldots$"]. By passing to appropriate *projective* or *inductive* limits, we shall also apply the notation "$\mathbb{V}(F)$", "F_v", "p_v" in situations where "F" is an *infinite extension* of $\mathbb{Q}$.

Curves:

We shall use the terms *hyperbolic curve*, *cusp*, *stable log curve*, and *smooth log curve* as they are defined in [SemiAnbd], §0. We shall use the term *hyperbolic orbicurve* as it is defined in [Cusp], §0.

Section 1: Complements on Coverings of Punctured Elliptic Curves

In the present §1, we discuss certain routine complements — which will be of use in the present series of papers — to the theory of *coverings of once-punctured elliptic curves*, as developed in [EtTh], §2.

Let $l \geq 5$ be an *integer prime to* 6; X a *hyperbolic curve of type* $(1,1)$ over a *field* k *of characteric zero*; $\underline{C}$ a *hyperbolic orbicurve of type* $(1, l\text{-tors})_{\pm}$ [cf. [EtTh], Definition 2.1] over k, whose k*-core* C [cf. [CanLift], Remark 2.1.1; [EtTh], the discussion at the beginning of §2] also forms a k-core of X. Thus, $\underline{C}$ determines, up to k-isomorphism, a *hyperbolic orbicurve* $\underline{X} \stackrel{\text{def}}{=} \underline{C} \times_C X$ *of type* $(1, l\text{-tors})$ [cf. [EtTh], Definition 2.1] over k. Moreover, if we write G_k for the *absolute Galois group of* k [relative to an appropriate choice of basepoint], $\Pi_{(-)}$ for the *arithmetic fundamental group* of a geometrically connected, geometrically normal, generically scheme-like k-algebraic stack of finite type "$(-)$" [i.e., the étale fundamental group $\pi_1((-))$], and $\Delta_{(-)}$ for the *geometric fundamental group* of "$(-)$" [i.e., the kernel of the natural surjection $\Pi_{(-)} \twoheadrightarrow G_k$], then we obtain *natural cartesian diagrams*

$$\begin{array}{ccc} \underline{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \underline{C} & \longrightarrow & C \end{array} \qquad \begin{array}{ccc} \Pi_{\underline{X}} & \longrightarrow & \Pi_X \\ \downarrow & & \downarrow \\ \Pi_{\underline{C}} & \longrightarrow & \Pi_C \end{array} \qquad \begin{array}{ccc} \Delta_{\underline{X}} & \longrightarrow & \Delta_X \\ \downarrow & & \downarrow \\ \Delta_{\underline{C}} & \longrightarrow & \Delta_C \end{array}$$

of finite étale coverings of hyperbolic orbicurves and open immersions of profinite groups. Finally, let us make the following assumption:

($*$) The natural action of G_k on $\Delta_{\underline{X}}^{\text{ab}} \otimes (\mathbb{Z}/l\mathbb{Z})$ [where the superscript "ab" denotes the abelianization] is *trivial.*

Next, let $\underline{\epsilon}$ be a *nonzero cusp* of $\underline{C}$ — i.e., a cusp that arises from a *nonzero element* of the quotient "Q" that appears in the definition of a "hyperbolic orbicurve of type $(1, l\text{-tors})_{\pm}$" given in [EtTh], Definition 2.1. Write $\underline{\epsilon}^0$ for the unique *"zero cusp"* [i.e., "non-nonzero cusp"] of $\underline{X}$; $\underline{\epsilon}'$, $\underline{\epsilon}''$ for the two cusps of $\underline{X}$ that lie over $\underline{\epsilon}$; and

$$\Delta_{\underline{X}} \twoheadrightarrow \Delta_{\underline{X}}^{\text{ab}} \otimes (\mathbb{Z}/l\mathbb{Z}) \twoheadrightarrow \Delta_{\underline{\epsilon}}$$

for the quotient of $\Delta_{\underline{X}}^{\text{ab}} \otimes (\mathbb{Z}/l\mathbb{Z})$ by the images of the *inertia groups of all nonzero cusps* $\neq \underline{\epsilon}', \underline{\epsilon}''$ *of* $\underline{X}$. Thus, we obtain a *natural exact sequence*

$$0 \longrightarrow I_{\underline{\epsilon}'} \times I_{\underline{\epsilon}''} \longrightarrow \Delta_{\underline{\epsilon}} \longrightarrow \Delta_{\underline{E}} \otimes (\mathbb{Z}/l\mathbb{Z}) \longrightarrow 0$$

— where we write $\underline{E}$ for the genus one compactification of $\underline{X}$, and $I_{\underline{\epsilon}'}$, $I_{\underline{\epsilon}''}$ for the respective images in $\Delta_{\underline{\epsilon}}$ of the inertia groups of the cusps $\underline{\epsilon}'$, $\underline{\epsilon}''$ [so we have *noncanonical* isomorphisms $I_{\underline{\epsilon}'} \cong \mathbb{Z}/l\mathbb{Z} \cong I_{\underline{\epsilon}''}$].

Next, let us observe that G_k, $\text{Gal}(\underline{X}/\underline{C})$ ($\cong \mathbb{Z}/2\mathbb{Z}$) *act naturally on the above exact sequence*. Write $\iota \in \text{Gal}(\underline{X}/\underline{C})$ for the unique nontrivial element. Then ι induces an isomorphism $I_{\underline{\epsilon}'} \cong I_{\underline{\epsilon}''}$; if we use this isomorphism to identify $I_{\underline{\epsilon}'}$, $I_{\underline{\epsilon}''}$, then one verifies immediately that ι acts on the term "$I_{\underline{\epsilon}'} \times I_{\underline{\epsilon}''}$" of the above exact sequence by *switching the two factors*. Moreover, one verifies immediately that ι

acts on $\Delta_{\underline{E}} \otimes (\mathbb{Z}/l\mathbb{Z})$ via multiplication by -1. In particular, since l is *odd*, it follows that the action by ι on $\Delta_{\underline{\epsilon}}$ determines a *decomposition into eigenspaces*

$$\Delta_{\underline{\epsilon}} \overset{\sim}{\to} \Delta^+_{\underline{\epsilon}} \times \Delta^-_{\underline{\epsilon}}$$

— i.e., where ι acts on $\Delta^+_{\underline{\epsilon}}$ (respectively, $\Delta^-_{\underline{\epsilon}}$) by multiplication by $+1$ (respectively, -1). Moreover, the natural composite maps

$$I_{\underline{\epsilon}'} \hookrightarrow \Delta_{\underline{\epsilon}} \twoheadrightarrow \Delta^+_{\underline{\epsilon}}; \quad I_{\underline{\epsilon}''} \hookrightarrow \Delta_{\underline{\epsilon}} \twoheadrightarrow \Delta^+_{\underline{\epsilon}}$$

determine *isomorphisms* $I_{\underline{\epsilon}'} \overset{\sim}{\to} \Delta^+_{\underline{\epsilon}}$, $I_{\underline{\epsilon}''} \overset{\sim}{\to} \Delta^+_{\underline{\epsilon}}$. Since the natural action of G_k on $\Delta_{\underline{\epsilon}}$ clearly *commutes* with the action of ι, we thus conclude that the quotient $\Delta_{\underline{X}} \twoheadrightarrow \Delta_{\underline{\epsilon}} \twoheadrightarrow \Delta^+_{\underline{\epsilon}}$ determines quotients

$$\Pi_{\underline{X}} \twoheadrightarrow J_{\underline{X}}; \quad \Pi_{\underline{C}} \twoheadrightarrow J_{\underline{C}}$$

— where the surjections $\Pi_{\underline{X}} \twoheadrightarrow G_k$, $\Pi_{\underline{C}} \twoheadrightarrow G_k$ induce natural exact sequences $1 \to \Delta^+_{\underline{\epsilon}} \to J_{\underline{X}} \to G_k \to 1$, $1 \to \Delta^+_{\underline{\epsilon}} \times \mathrm{Gal}(\underline{X}/\underline{C}) \to J_{\underline{C}} \to G_k \to 1$; we have a natural inclusion $J_{\underline{X}} \hookrightarrow J_{\underline{C}}$.

Next, let us consider the cusp "$2\underline{\epsilon}$" of $\underline{C}$ — i.e., the cusp whose inverse images in $\underline{X}$ correspond to the points of $\underline{E}$ obtained by multiplying $\underline{\epsilon}'$, $\underline{\epsilon}''$ by 2, relative to the group law of the elliptic curve determined by the pair $(\underline{X}, \epsilon^0)$. Since $2 \neq \pm 1 \pmod{l}$ [a consequence of our assumption that $l \geq 5$], it follows that the *decomposition group* associated to this cusp "$2\underline{\epsilon}$" determines a *section*

$$\sigma : G_k \to J_{\underline{C}}$$

of the natural surjection $J_{\underline{C}} \twoheadrightarrow G_k$. Here, we note that although, a priori, σ is only determined by $2\underline{\epsilon}$ up to composition with an inner automorphism of $J_{\underline{C}}$ determined by an element of $\Delta^+_{\underline{\epsilon}} \times \mathrm{Gal}(\underline{X}/\underline{C})$, in fact, since [in light of the assumption $(*)$!] the natural [outer] action of G_k on $\Delta^+_{\underline{\epsilon}} \times \mathrm{Gal}(\underline{X}/\underline{C})$ is *trivial*, we conclude that σ is *completely determined* by $2\underline{\epsilon}$, and that the subgroup $\mathrm{Im}(\sigma) \subseteq J_{\underline{C}}$ determined by the image of σ is *normal* in $J_{\underline{C}}$. Moreover, by considering the decomposition groups associated to the cusps of $\underline{X}$ lying over $2\underline{\epsilon}$, we conclude that $\mathrm{Im}(\sigma)$ lies inside the subgroup $J_{\underline{X}} \subseteq J_{\underline{C}}$. Thus, the subgroups $\mathrm{Im}(\sigma) \subseteq J_{\underline{X}}$, $\mathrm{Im}(\sigma) \times \mathrm{Gal}(\underline{X}/\underline{C}) \subseteq J_{\underline{C}}$ determine [the horizontal arrows in] *cartesian diagrams*

$$\begin{array}{ccc} \underrightarrow{X} & \longrightarrow & \underline{X} \\ \downarrow & & \downarrow \\ \underrightarrow{C} & \longrightarrow & \underline{C} \end{array} \qquad \begin{array}{ccc} \Pi_{\underrightarrow{X}} & \longrightarrow & \Pi_{\underline{X}} \\ \downarrow & & \downarrow \\ \Pi_{\underrightarrow{C}} & \longrightarrow & \Pi_{\underline{C}} \end{array} \qquad \begin{array}{ccc} \Delta_{\underrightarrow{X}} & \longrightarrow & \Delta_{\underline{X}} \\ \downarrow & & \downarrow \\ \Delta_{\underrightarrow{C}} & \longrightarrow & \Delta_{\underline{C}} \end{array}$$

of *finite étale cyclic coverings* of hyperbolic orbicurves and open immersions [with normal image] of profinite groups; we have $\mathrm{Gal}(\underrightarrow{C}/\underline{C}) \cong \mathbb{Z}/l\mathbb{Z}$, $\mathrm{Gal}(\underline{X}/\underline{C}) \cong \mathbb{Z}/2\mathbb{Z}$, and $\mathrm{Gal}(\underrightarrow{X}/\underline{C}) \overset{\sim}{\to} \mathrm{Gal}(\underline{X}/\underline{C}) \times \mathrm{Gal}(\underrightarrow{C}/\underline{C}) \cong \mathbb{Z}/2l\mathbb{Z}$.

Definition 1.1. We shall refer to a hyperbolic orbicurve over k that arises, up to isomorphism, as the hyperbolic orbicurve $\underrightarrow{X}$ (respectively, $\underrightarrow{C}$) constructed above for some choice of l, $\underline{\epsilon}$ as being *of type* $(1, l\text{-}\underrightarrow{\mathrm{tors}})$ (respectively, $(1, l\text{-}\underrightarrow{\mathrm{tors}})_{\pm}$).

Remark 1.1.1. The arrow "$\rightarrow$" in the notation "$\underrightarrow{X}$", "$\underrightarrow{C}$", "$(1, l\text{-}\underrightarrow{\mathrm{tors}})$", "$(1, l\text{-}\underrightarrow{\mathrm{tors}})_{\pm}$" may be thought of as denoting the *"archimedean, ordered labels* $1, 2, \ldots$*"* — i.e., determined by the *choice of* $\underline{\epsilon}$*!* — on the $\{\pm 1\}$-orbits of elements of the quotient "Q" that appears in the definition of a "hyperbolic orbicurve of type $(1, l\text{-tors})_{\pm}$" given in [EtTh], Definition 2.1.

Remark 1.1.2. We observe that $\underrightarrow{X}$, $\underrightarrow{C}$ are *completely determined, up to k-isomorphism, by the data* $(X/k,\ \underline{C},\ \underline{\epsilon})$.

Corollary 1.2. **(Characteristic Nature of Coverings)** *Suppose that k is an NF or an MLF. Then there exists a* **functorial group-theoretic algorithm** *[cf. [AbsTopIII], Remark 1.9.8, for more on the meaning of this terminology] to reconstruct*

$$\Pi_{\underline{X}},\ \Pi_{\underrightarrow{C}},\ \Pi_{\underline{C}}\ (\textit{respectively, } \Pi_{\underline{C}})$$

together with the conjugacy classes of the decomposition group(s) determined by the set(s) of cusps $\{\underline{\epsilon}', \underline{\epsilon}''\}$*;* $\{\underline{\epsilon}\}$ *(respectively,* $\{\underline{\epsilon}\}$*) from* $\Pi_{\underrightarrow{X}}$ *(respectively,* $\Pi_{\underrightarrow{C}}$*). Here, the asserted functoriality is with respect to isomorphisms of topological groups; we reconstruct* $\Pi_{\underline{X}}$, $\Pi_{\underrightarrow{C}}$, $\Pi_{\underline{C}}$ *(respectively,* $\Pi_{\underline{C}}$*) as a subgroup of* $\mathrm{Aut}(\Pi_{\underrightarrow{X}})$ *(respectively,* $\mathrm{Aut}(\Pi_{\underrightarrow{C}})$*).*

Proof. For simplicity, we consider the non-resp'd case; the resp'd case is entirely similar [but slightly easier]. The argument is similar to the arguments applied in [EtTh], Proposition 1.8; [EtTh], Proposition 2.4. First, we recall that $\Pi_{\underrightarrow{X}}$, $\Pi_{\underline{X}}$, and $\Pi_{\underline{C}}$ are *slim* [cf., e.g., [AbsTopI], Proposition 2.3, (ii)], hence *embed naturally* into $\mathrm{Aut}(\Pi_{\underrightarrow{X}})$, and that one may recover the subgroup $\Delta_{\underrightarrow{X}} \subseteq \Pi_{\underrightarrow{X}}$ via the algorithms of [AbsTopI], Theorem 2.6, (v), (vi). Next, we recall that the algorithms of [AbsTopII], Corollary 3.3, (i), (ii) — which are applicable in light of [AbsTopI], Example 4.8 — allow one to reconstruct Π_C [together with the natural inclusion $\Pi_{\underrightarrow{X}} \hookrightarrow \Pi_C$], as well as the subgroups $\Delta_X \subseteq \Delta_C \subseteq \Pi_C$. In particular, l may be recovered via the formula $l^2 = [\Delta_X : \Delta_{\underline{X}}] \cdot [\Delta_{\underline{X}} : \Delta_{\underrightarrow{X}}] = [\Delta_X : \Delta_{\underrightarrow{X}}] = [\Delta_C : \Delta_{\underrightarrow{X}}]/2$. Next, let us set $H \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Delta_X \twoheadrightarrow \Delta_X^{\mathrm{ab}} \otimes (\mathbb{Z}/l\mathbb{Z}))$. Then $\Pi_{\underline{X}} \subseteq \Pi_C$ may be recovered via the [easily verified] equality of subgroups $\Pi_{\underline{X}} = \Pi_{\underrightarrow{X}} \cdot H$. The conjugacy classes of the decomposition groups of $\underline{\epsilon}^0$, $\underline{\epsilon}'$, $\underline{\epsilon}''$ in $\Pi_{\underline{X}}$ may be recovered as the *decomposition groups of cusps* [cf. [AbsTopI], Lemma 4.5, as well as Remark 1.2.2, (ii), below] whose image in $\mathrm{Gal}(\underrightarrow{X}/\underline{X}) = \Pi_{\underline{X}}/\Pi_{\underrightarrow{X}}$ is *nontrivial*. Next, to reconstruct $\Pi_{\underline{C}} \subseteq \Pi_C$, it suffices to reconstruct the *splitting* of the surjection $\mathrm{Gal}(\underline{X}/C) = \Pi_C/\Pi_{\underline{X}} \twoheadrightarrow \Pi_C/\Pi_X = \mathrm{Gal}(X/C)$ determined by $\mathrm{Gal}(\underline{X}/\underline{C}) = \Pi_{\underline{C}}/\Pi_{\underline{X}}$; but [since l is *prime to* 3!] this splitting may be characterized [group-theoretically!] as the unique splitting that stabilizes the collection of conjugacy classes of subgroups of $\Pi_{\underline{X}}$ determined by the decomposition groups of $\underline{\epsilon}^0$, $\underline{\epsilon}'$, $\underline{\epsilon}''$. Now $\Pi_{\underrightarrow{C}} \subseteq \Pi_{\underline{C}}$ may be reconstructed by applying the observation that $(\mathbb{Z}/2\mathbb{Z} \cong)\ \mathrm{Gal}(\underrightarrow{X}/\underrightarrow{C}) \subseteq \mathrm{Gal}(\underrightarrow{X}/\underline{C})\ (\cong \mathbb{Z}/2l\mathbb{Z})$ is the *unique maximal subgroup of odd index*. Finally, the conjugacy classes of the decomposition groups of $\underline{\epsilon}'$, $\underline{\epsilon}''$ in $\Pi_{\underline{X}}$ may be recovered as the *decomposition groups of cusps* [cf. [AbsTopI], Lemma 4.5, as well as Remark 1.2.2, (ii), below] whose image in $\mathrm{Gal}(\underrightarrow{X}/\underline{X}) = \Pi_{\underline{X}}/\Pi_{\underrightarrow{X}}$ is *nontrivial*, but which are *not fixed* [up to conjugacy] by the outer action of $\mathrm{Gal}(\underline{X}/\underline{C}) = \Pi_{\underline{C}}/\Pi_{\underline{X}}$ on $\Pi_{\underline{X}}$. This completes the proof of Corollary 1.2. ○

Remark 1.2.1. It follows immediately from Corollary 1.2 that

$$\mathrm{Aut}_k(\underset{\to}{X}) = \mathrm{Gal}(\underset{\to}{X}/\underline{C}) \ (\cong \mathbb{Z}/2l\mathbb{Z}); \quad \mathrm{Aut}_k(\underset{\to}{C}) = \mathrm{Gal}(\underset{\to}{C}/\underline{C}) \ (\cong \mathbb{Z}/l\mathbb{Z})$$

[cf. [EtTh], Remark 2.6.1].

Remark 1.2.2. The group-theoretic algorithm for reconstructing the *decomposition groups of cusps* given [AbsTopI], Lemma 4.5 — which is based on the argument given in the proof of [AbsAnab], Lemma 1.3.9 — contains *some minor, inessential inaccuracies*. In light of the *importance* of this group-theoretic algorithm for the theory of the present series of papers, we thus pause to discuss how these inaccuracies may be amended.

(i) The final portion [beginning with the *third sentence*] of the *second paragraph* of the proof of [AbsAnab], Lemma 1.3.9, should be replaced by the following text:

> Since r_i may be recovered group-theoretically, given any finite étale coverings
> $$Z_i \to V_i \to X_i$$
> such that Z_i is *cyclic* [hence *Galois*], of degree a *power of* l, over V_i, one may determine group-theoretically whether or not $Z_i \to V_i$ is *totally ramified* [i.e., at some point of Z_i], since this condition is easily verified to be equivalent to the condition that the covering $Z_i \to V_i$ admit a *factorization* $Z_i \to W_i \to V_i$, where $W_i \to V_i$ is finite étale of degree l, and $r_{W_i} < l \cdot r_{V_i}$. Moreover, this group-theoreticity of the condition that a cyclic covering be *totally ramified* extends immediately to the case of *pro-*l cyclic coverings $Z_i \to V_i$. Thus, by Lemma 1.3.7, we conclude that *the inertia groups of cusps in* $(\Delta_{X_i})^{(l)}$ [i.e., the maximal pro-l quotient of Δ_{X_i}] *may be characterized [group-theoretically!]* as the maximal subgroups of $(\Delta_{X_i})^{(l)}$ that correspond to [profinite] coverings satisfying this condition.

(ii) The final portion [beginning with the *third sentence*] of the statement of [AbsTopI], Lemma 4.5, (iv), should be replaced by the following text:

> *Then the* **decomposition groups of cusps** $\subseteq H^*$ *may be characterized [***"group-theoretically"***] as the* **maximal** *closed subgroups* $I \subseteq H^*$ *isomorphic to* $\mathbb{Z}_l$ *which satisfy the following condition: We have*
> $$d_{\chi_G^{\mathrm{cyclo}}}((I^l \cdot J)^{\mathrm{ab}} \otimes \mathbb{Q}_l) + 1 < l \cdot \{d_{\chi_G^{\mathrm{cyclo}}}((I \cdot J)^{\mathrm{ab}} \otimes \mathbb{Q}_l) + 1\}$$
> *[i.e., "the covering of curves corresponding to* $J \subseteq I \cdot J$ *is* **totally ramified** *at some cusp"] for every characteristic open subgroup* $J \subseteq H^*$ *such that* $J \neq I \cdot J$.

Remark 1.2.3. The minor, inessential inaccuracies in the group-theoretic algorithms of [AbsAnab], Lemma 1.3.9; [AbsTopI], Lemma 4.5, that were discussed in Remark 1.2.2 are closely related to certain *minor, inessential inaccuracies* in the theory of [CombGC]. Thus, it is of interest, in the context of the discussion of Remark 1.2.2, to pause to discuss how these inaccuracies may be amended. These inaccuracies arise in the arguments applied in [CombGC], Definition 1.4, (v), (vi), and [CombGC], Remarks 1.4.2, 1.4.3, and 1.4.4, to prove [CombGC], Theorem 1.6. These arguments are formulated in a somewhat confusing way and should be modified as follows:

(i) First of all, we remark that in [CombGC], as well as in the following discussion, a *"Galois"* finite étale covering is to be understood as being *connected*.

(ii) In the *second sentence* of [CombGC], Definition 1.4, (v), the *cuspidal* and *nodal* cases of the notion of a *purely totally ramified* covering are in fact *unnecessary* and *may be deleted*. Also, the terminology introduced in [CombGC], Definition 1.4, (vi), concerning finite étale coverings that *descend* is *unnecessary* and *may be deleted*.

(iii) The text of [CombGC], Remark 1.4.2, should be replaced by the following text:

> Let $\mathcal{G}' \to \mathcal{G}$ be a Galois finite étale covering of degree a *positive power of* l, where $\mathcal{G}$ is of pro-Σ PSC-type, $\Sigma = \{l\}$. Then one verifies immediately that, if we assume further that the covering $\mathcal{G}' \to \mathcal{G}$ is *cyclic*, then $\mathcal{G}' \to \mathcal{G}$ is *cuspidally totally ramified* if and only if the inequality
>
> $$\underline{r}(\mathcal{G}'') < l \cdot \underline{r}(\mathcal{G})$$
>
> — where we write $\mathcal{G}' \to \mathcal{G}'' \to \mathcal{G}$ for the *unique* [up to isomorphism] *factorization* of the finite étale covering $\mathcal{G}' \to \mathcal{G}$ as a composite of finite étale coverings such that $\mathcal{G}'' \to \mathcal{G}$ is of degree l — is satisfied. Suppose further that $\mathcal{G}' \to \mathcal{G}$ is a [*not necessarily cyclic*!] $\Pi_{\mathcal{G}}^{\mathrm{unr}}$*-covering* [so $\underline{n}(\mathcal{G}') = \deg(\mathcal{G}'/\mathcal{G}) \cdot \underline{n}(\mathcal{G})$]. Then one verifies immediately that $\mathcal{G}' \to \mathcal{G}$ is *verticially purely totally ramified* if and only if the equality
>
> $$\underline{i}(\mathcal{G}') = \deg(\mathcal{G}'/\mathcal{G}) \cdot (\underline{i}(\mathcal{G}) - 1) + 1$$
>
> is satisfied. Also, we observe that this last inequality is equivalent to the following equality involving the expression "$\underline{i}(\dots) - \underline{n}(\dots)$" [cf. Remark 1.1.3]:
>
> $$\underline{i}(\mathcal{G}') - \underline{n}(\mathcal{G}') = \deg(\mathcal{G}'/\mathcal{G}) \cdot (\underline{i}(\mathcal{G}) - \underline{n}(\mathcal{G}) - 1) + 1$$

(iv) The text of [CombGC], Remark 1.4.3, should be replaced by the following text:

> Suppose that $\mathcal{G}$ is of pro-Σ PSC-type, $\Sigma = \{l\}$. Then one verifies immediately that the *cuspidal edge-like subgroups* of $\Pi_{\mathcal{G}}$ may be *characterized* as the *maximal* [cf. Proposition 1.2, (i)] closed subgroups $A \subseteq \Pi_{\mathcal{G}}$ isomorphic to $\mathbb{Z}_l$ which satisfy the following *condition*:
>
> > for every characteristic open subgroup $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, if we write $\mathcal{G}' \to \mathcal{G}'' \to \mathcal{G}$ for the finite étale coverings corresponding to

> $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}''} \stackrel{\text{def}}{=} A \cdot \Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, then the *cyclic* finite étale covering $\mathcal{G}' \to \mathcal{G}''$ is *cuspidally totally ramified*.

[Indeed, the *necessity* of this *characterization* is immediate from the definitions; the *sufficiency* of this *characterization* follows by observing that since the set of cusps of a finite étale covering of $\mathcal{G}$ is always *finite*, the above *condition* implies that there exists a *compatible system of cusps* of the various $\mathcal{G}'$ that arise, each of which is *stabilized* by the action of A.] On the other hand, in order to characterize the *unramified verticial subgroups* of $\Pi_{\mathcal{G}}^{\text{unr}}$, it suffices — by considering *stabilizers* of *vertices* of underlying semi-graphs of finite étale $\Pi_{\mathcal{G}}^{\text{unr}}$-coverings of $\mathcal{G}$ — to give a *functorial characterization* of the *set of vertices* of $\mathcal{G}$ [i.e., which may also be applied to finite étale $\Pi_{\mathcal{G}}^{\text{unr}}$-coverings of $\mathcal{G}$]. This may be done, for *sturdy* $\mathcal{G}$, as follows. Write $M_{\mathcal{G}}^{\text{unr}}$ for the *abelianization* of $\Pi_{\mathcal{G}}^{\text{unr}}$. For each vertex v of the underlying semi-graph $\mathbb{G}$ of $\mathcal{G}$, write $M_{\mathcal{G}}^{\text{unr}}[v] \subseteq M_{\mathcal{G}}^{\text{unr}}$ for the image of the $\Pi_{\mathcal{G}}^{\text{unr}}$-conjugacy class of unramified verticial subgroups of $\Pi_{\mathcal{G}}^{\text{unr}}$ associated to v. Then one verifies immediately, by *constructing suitable abelian* $\Pi_{\mathcal{G}}^{\text{unr}}$*-coverings* of $\mathcal{G}$ via suitable *gluing* operations [i.e., as in the proof of Proposition 1.2], that the inclusions $M_{\mathcal{G}}^{\text{unr}}[v] \subseteq M_{\mathcal{G}}^{\text{unr}}$ determine a *split injection*

$$\bigoplus_{v} M_{\mathcal{G}}^{\text{unr}}[v] \hookrightarrow M_{\mathcal{G}}^{\text{unr}}$$

[where v ranges over the vertices of $\mathbb{G}$], whose image we denote by $M_{\mathcal{G}}^{\text{unr-vert}} \subseteq M_{\mathcal{G}}^{\text{unr}}$. Now we consider *elementary abelian quotients*

$$\phi : M_{\mathcal{G}}^{\text{unr}} \twoheadrightarrow Q$$

— i.e., where Q is an *elementary abelian group*. We *identify* such quotients whenever their *kernels coincide* and *order* such quotients by means of the relation of *"domination"* [i.e., inclusion of kernels]. Then one verifies immediately that such a quotient $\phi : M_{\mathcal{G}}^{\text{unr}} \twoheadrightarrow Q$ corresponds to a *verticially purely totally ramified* covering of $\mathcal{G}$ if and only if there exists a vertex v of $\mathbb{G}$ such that $\phi(M_{\mathcal{G}}^{\text{unr}}[v]) = Q$, $\phi(M_{\mathcal{G}}^{\text{unr}}[v']) = 0$ for all vertices $v' \neq v$ of $\mathbb{G}$. In particular, one concludes immediately that

> the elementary abelian quotients $\phi : M_{\mathcal{G}}^{\text{unr}} \twoheadrightarrow Q$ whose restriction to $M_{\mathcal{G}}^{\text{unr-vert}}$ surjects onto Q and has the same kernel as the quotient
>
> $$M_{\mathcal{G}}^{\text{unr-vert}} \twoheadrightarrow M_{\mathcal{G}}^{\text{unr}}[v] \twoheadrightarrow M_{\mathcal{G}}^{\text{unr}}[v] \otimes \mathbb{F}_l$$
>
> — where the first "$\twoheadrightarrow$" is the natural projection; the second "$\twoheadrightarrow$" is given by reduction modulo l — may be characterized as the *maximal quotients* [i.e., relative to the relation of domination] among those elementary abelian quotients of $M_{\mathcal{G}}^{\text{unr}}$ that correspond to *verticially purely totally ramified* coverings of $\mathcal{G}$.

Thus, since $\mathcal{G}$ is *sturdy*, the *set of vertices* of $\mathcal{G}$ may be characterized as the *set of [nontrivial!] quotients* $M_{\mathcal{G}}^{\text{unr-vert}} \twoheadrightarrow M_{\mathcal{G}}^{\text{unr}}[v] \otimes \mathbb{F}_l$.

(v) The text of [CombGC], Remark 1.4.4, should be replaced by the following text:

> Suppose that $\mathcal{G}$ is of pro-Σ PSC-type, where $\Sigma = \{l\}$, and that $\mathcal{G}$ is *noncuspidal*. Then, in the spirit of the *cuspidal* portion of Remark 1.4.3, we observe the following: One verifies immediately that the *nodal edge-like subgroups* of $\Pi_{\mathcal{G}}$ may be *characterized* as the *maximal* [cf. Proposition 1.2, (i)] closed subgroups $A \subseteq \Pi_{\mathcal{G}}$ isomorphic to $\mathbb{Z}_l$ which satisfy the following *condition*:
>
> > for every characteristic open subgroup $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, if we write $\mathcal{G}' \to \mathcal{G}'' \to \mathcal{G}$ for the finite étale coverings corresponding to $\Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}''} \stackrel{\text{def}}{=} A \cdot \Pi_{\mathcal{G}'} \subseteq \Pi_{\mathcal{G}}$, then the *cyclic* finite étale covering $\mathcal{G}' \to \mathcal{G}''$ is *nodally totally ramified*.
>
> Here, we note further that [one verifies immediately that] the finite étale covering $\mathcal{G}' \to \mathcal{G}''$ is *nodally totally ramified* if and only if it is *module-wise nodal*.

(vi) The text of the *second paragraph* of the proof of [CombGC], Theorem 1.6, should be replaced by the following text [which may be thought as being appended to the end of the *first paragraph* of the proof of [CombGC], Theorem 1.6]:

> Then the fact that α is *group-theoretically cuspidal* follows formally from the characterization of *cuspidal edge-like subgroups* given in Remark 1.4.3 and the characterization of *cuspidally totally ramified* cyclic finite étale coverings given in Remark 1.4.2.

(vii) The text of the *final paragraph* of the proof of [CombGC], Theorem 1.6, should be replaced by the following text [which may be thought of as a sort of *"easy version"* of the argument given in the proof of the implication "(iii) $\Longrightarrow$ (i)" of [CbTpII], Proposition 1.5]:

> Finally, we consider assertion (iii). *Sufficiency* is immediate. On the other hand, *necessity* follows formally from the characterization of *unramified verticial subgroups* given in Remark 1.4.3 and the characterization of *verticially purely totally ramified* finite étale coverings given in Remark 1.4.2.

Section 2: Complements on Tempered Coverings

In the present §2, we discuss certain routine complements — which will be of use in the present series of papers — to the theory of *tempered coverings of graphs of anabelioids*, as developed in [SemiAnbd], §3 [cf. also the closely related theory of [CombGC]].

Let Σ, $\widehat{\Sigma}$ be *nonempty sets of prime numbers* such that $\Sigma \subseteq \widehat{\Sigma}$;

$$\mathcal{G}$$

a *semi-graph of anabelioids of pro-*Σ *PSC-type* [cf. [CombGC], Definition 1.1, (i)], whose underlying semi-graph we denote by $\mathbb{G}$. Write $\Pi^{\mathrm{tp}}_{\mathcal{G}}$ for the *tempered fundamental group* of $\mathcal{G}$ [cf. the discussion preceding [SemiAnbd], Proposition 3.6, as well as Remark 2.5.3, (i), (T6), of the present paper] and $\widehat{\Pi}_{\mathcal{G}}$ for the *pro-*$\widehat{\Sigma}$ [i.e., maximal pro-$\widehat{\Sigma}$ quotient of the profinite] *fundamental group* of $\mathcal{G}$ [cf. the discussion preceding [SemiAnbd], Definition 2.2] — both taken with respect to appropriate choices of basepoints. Thus, since discrete free groups of finite rank inject into their pro-l completions for any prime number l [cf., e.g., [RZ], Proposition 3.3.15], it follows that we have a *natural injection* [cf. [SemiAnbd], Proposition 3.6, (iii), as well as Remark 2.5.3, (ii), (E7), of the present paper, when $\widehat{\Sigma} = \mathfrak{Primes}$; the proof in the case of arbitrary $\widehat{\Sigma}$ is entirely similar]

$$\Pi^{\mathrm{tp}}_{\mathcal{G}} \hookrightarrow \widehat{\Pi}_{\mathcal{G}}$$

that we shall use to regard $\Pi^{\mathrm{tp}}_{\mathcal{G}}$ as a *subgroup* of $\widehat{\Pi}_{\mathcal{G}}$ and $\widehat{\Pi}_{\mathcal{G}}$ as the *pro-*$\widehat{\Sigma}$ *completion* of $\Pi^{\mathrm{tp}}_{\mathcal{G}}$.

Next, let

$$\mathcal{H}$$

be the semi-graph of anabelioids associated to a **connected** *sub-semi-graph* $\mathbb{H} \subseteq \mathbb{G}$. One verifies immediately that the *restriction* of $\mathcal{H}$ to the *maximal subgraph* [cf. the discussion at the beginning of [SemiAnbd], §1] of $\mathbb{H}$ *coincides* with the restriction to the maximal subgraph of the underlying semi-graph of some semi-graph of anabelioids of pro-Σ PSC-type. That is to say, roughly speaking, up to the possible omission of some of the cuspidal edges, $\mathcal{H}$ "is" a semi-graph of anabelioids of pro-Σ PSC-type. In particular, since the omission of cuspidal edges clearly does not affect either the tempered or pro-$\widehat{\Sigma}$ fundamental groups, we shall apply the notation introduced above for "$\mathcal{G}$" to $\mathcal{H}$. We thus obtain a *natural commutative diagram*

$$\begin{array}{ccc} \Pi^{\mathrm{tp}}_{\mathcal{H}} & \longrightarrow & \widehat{\Pi}_{\mathcal{H}} \\ \downarrow & & \downarrow \\ \Pi^{\mathrm{tp}}_{\mathcal{G}} & \longrightarrow & \widehat{\Pi}_{\mathcal{G}} \end{array}$$

of [outer] inclusions [cf. [SemiAnbd], Proposition 2.5, (i), when $\widehat{\Sigma} = \mathfrak{Primes}$; in light of the well-known structure of fundamental groups of hyperbolic Riemann surfaces of finite type, a similar proof may be given in the case of arbitrary $\widehat{\Sigma}$, i.e., by considering successive composites of finite étale Galois coverings that restrict to *trivial coverings* over the closed edges and finite étale abelian [Galois] coverings obtained by *gluing* together suitable *abelian coverings*] *of topological groups*, which we shall use to regard all of the groups in the diagram as subgroups of $\widehat{\Pi}_{\mathcal{G}}$. In particular, one may think of $\Pi^{\text{tp}}_{\mathcal{H}}$ (respectively, $\widehat{\Pi}_{\mathcal{H}}$) as the *decomposition subgroup* in $\Pi^{\text{tp}}_{\mathcal{G}}$ (respectively, $\widehat{\Pi}_{\mathcal{G}}$) [which is well-defined up to $\Pi^{\text{tp}}_{\mathcal{G}}$- (respectively, $\widehat{\Pi}_{\mathcal{G}}$-)conjugacy] associated to the sub-semi-graph $\mathcal{H}$.

The following result is the *central technical result* underlying the theory of the present §2.

Proposition 2.1. (Profinite Conjugates of Nontrivial Compact Subgroups) *In the notation of the above discussion, let* $\Lambda \subseteq \Pi^{\text{tp}}_{\mathcal{G}}$ *be a* **nontrivial compact subgroup**, $\gamma \in \widehat{\Pi}_{\mathcal{G}}$ *an element such that* $\gamma \cdot \Lambda \cdot \gamma^{-1} \subseteq \Pi^{\text{tp}}_{\mathcal{G}}$ *[or, equivalently,* $\Lambda \subseteq \gamma^{-1} \cdot \Pi^{\text{tp}}_{\mathcal{G}} \cdot \gamma$*]. Then* $\gamma \in \Pi^{\text{tp}}_{\mathcal{G}}$.

Proof. Write $\widehat{\Gamma}$ for the "pro-$\widehat{\Sigma}$ semi-graph" associated to the *universal pro-*$\widehat{\Sigma}$ *étale covering* of $\mathcal{G}$ [i.e., the covering corresponding to the subgroup $\{1\} \subseteq \widehat{\Pi}_{\mathcal{G}}$]; Γ^{tp} for the "pro-semi-graph" associated to the *universal tempered covering* of $\mathcal{G}$ [i.e., the covering corresponding to the subgroup $\{1\} \subseteq \Pi^{\text{tp}}_{\mathcal{G}}$]. Thus, we have a natural dense map $\Gamma^{\text{tp}} \to \widehat{\Gamma}$. Let us refer to a ["pro-"]vertex of $\widehat{\Gamma}$ that occurs as the image of a ["pro-"]vertex of Γ^{tp} as *tempered*. Since Λ, $\gamma \cdot \Lambda \cdot \gamma^{-1}$ are *compact* subgroups of $\Pi^{\text{tp}}_{\mathcal{G}}$, it follows from [SemiAnbd], Theorem 3.7, (iii) [cf. also [SemiAnbd], Example 3.10, as well as Remark 2.5.3, (ii), (E7), of the present paper], that there exist *verticial subgroups* $\Lambda', \Lambda'' \subseteq \Pi^{\text{tp}}_{\mathcal{G}}$ such that $\Lambda \subseteq \Lambda'$, $\gamma \cdot \Lambda \cdot \gamma^{-1} \subseteq \Lambda''$. Thus, Λ', Λ'' correspond to *tempered vertices* v', v'' of $\widehat{\Gamma}$; $\{1\} \neq \gamma \cdot \Lambda \cdot \gamma^{-1} \subseteq \gamma \cdot \Lambda' \cdot \gamma^{-1}$, so $(\gamma \cdot \Lambda' \cdot \gamma^{-1}) \bigcap \Lambda'' \neq \{1\}$. Since Λ'', $\gamma \cdot \Lambda' \cdot \gamma^{-1}$ are both verticial subgroups of $\widehat{\Pi}_{\mathcal{G}}$, it thus follows either from [AbsTopII], Proposition 1.3, (iv), or from [NodNon], Proposition 3.9, (i), that the corresponding vertices v'', $(v')^{\gamma}$ of $\widehat{\Gamma}$ are either *equal* or *adjacent*. In particular, since v'' is *tempered*, we thus conclude that $(v')^{\gamma}$ is *tempered*. Thus, v', $(v')^{\gamma}$ are tempered, so $\gamma \in \Pi^{\text{tp}}_{\mathcal{G}}$, as desired. ◯

Next, relative to the notation "C", "N" and related terminology concerning *commensurators* and *normalizers* discussed, for instance, in [SemiAnbd], §0; [CombGC], §0, we have the following result.

Proposition 2.2. (Commensurators of Decomposition Subgroups Associated to Sub-semi-graphs) *In the notation of the above discussion,* $\widehat{\Pi}_{\mathcal{H}}$ *(respectively,* $\Pi^{\text{tp}}_{\mathcal{H}}$*) is* **commensurably terminal** *in* $\widehat{\Pi}_{\mathcal{G}}$ *(respectively,* $\widehat{\Pi}_{\mathcal{G}}$ *[hence, also in* $\Pi^{\text{tp}}_{\mathcal{G}}$*]). In particular,* $\Pi^{\text{tp}}_{\mathcal{G}}$ *is commensurably terminal in* $\widehat{\Pi}_{\mathcal{G}}$.

Proof. First, let us observe that by allowing, in Proposition 2.1, Λ to range over the open subgroups of any verticial [hence, in particular, *nontrivial compact*!] subgroup of $\Pi^{\text{tp}}_{\mathcal{G}}$, we conclude from Proposition 2.1 that

$$\Pi^{\text{tp}}_{\mathcal{G}} \text{ is } \textit{commensurably terminal} \text{ in } \widehat{\Pi}_{\mathcal{G}}$$

— cf. Remark 2.2.2 below. In particular, by applying this fact to $\mathcal{H}$ [cf. the discussion preceding Proposition 2.1], we conclude that $\Pi^{\mathrm{tp}}_{\mathcal{H}}$ is *commensurably terminal* in $\widehat{\Pi}_{\mathcal{H}}$. Next, let us observe that it is immediate from the definitions that

$$\Pi^{\mathrm{tp}}_{\mathcal{H}} \subseteq C_{\Pi^{\mathrm{tp}}_{\mathcal{G}}}(\Pi^{\mathrm{tp}}_{\mathcal{H}}) \subseteq C_{\widehat{\Pi}_{\mathcal{G}}}(\Pi^{\mathrm{tp}}_{\mathcal{H}}) \subseteq C_{\widehat{\Pi}_{\mathcal{G}}}(\widehat{\Pi}_{\mathcal{H}})$$

[where we think of $\widehat{\Pi}_{\mathcal{H}}$, $\widehat{\Pi}_{\mathcal{G}}$, respectively, as the pro-$\widehat{\Sigma}$ completions of $\Pi^{\mathrm{tp}}_{\mathcal{H}}$, $\Pi^{\mathrm{tp}}_{\mathcal{G}}$]. On the other hand, by the evident pro-$\widehat{\Sigma}$ analogue of [SemiAnbd], Corollary 2.7, (i) [cf. also the argument involving *gluing* of *abelian coverings* in the discussion preceding Proposition 2.1], we have $C_{\widehat{\Pi}_{\mathcal{G}}}(\widehat{\Pi}_{\mathcal{H}}) = \widehat{\Pi}_{\mathcal{H}}$. Thus, by the *commensurable terminality* of $\Pi^{\mathrm{tp}}_{\mathcal{H}}$ in $\widehat{\Pi}_{\mathcal{H}}$, we conclude that

$$\Pi^{\mathrm{tp}}_{\mathcal{H}} \subseteq C_{\widehat{\Pi}_{\mathcal{G}}}(\Pi^{\mathrm{tp}}_{\mathcal{H}}) \subseteq C_{\widehat{\Pi}_{\mathcal{H}}}(\Pi^{\mathrm{tp}}_{\mathcal{H}}) = \Pi^{\mathrm{tp}}_{\mathcal{H}}$$

— as desired. ◯

Remark 2.2.1. It follows immediately from the theory of [SemiAnbd] [cf., e.g., [SemiAnbd], Corollary 2.7, (i)] that, in fact, Propositions 2.1 and 2.2 can be proven for much more general semi-graphs of anabelioids $\mathcal{G}$ than the sort of $\mathcal{G}$ that appears in the above discussion. We leave the routine details of such generalizations to the interested reader.

Remark 2.2.2. Recall that when $\widehat{\Sigma} = \mathfrak{Primes}$, the fact that

$$\Pi^{\mathrm{tp}}_{\mathcal{G}} \text{ is } \textit{normally terminal} \text{ in } \widehat{\Pi}_{\mathcal{G}}$$

may also be derived from the fact that any nonabelian finitely generated free group is *normally terminal* [cf. [André], Lemma 3.2.1; [SemiAnbd], Lemma 6.1, (i)] in its profinite completion. In particular, the proof of the commensurable terminality of $\Pi^{\mathrm{tp}}_{\mathcal{G}}$ in $\widehat{\Pi}_{\mathcal{G}}$ that is given in the proof of Proposition 2.2 may be thought of as a *new proof* of this normal terminality that *does not require one to invoke* [André], Lemma 3.2.1, which is essentially an immediate consequence of the rather difficult *conjugacy separability* result given in [Stb1], Theorem 1. This relation of Proposition 2.1 to the theory of [Stb1] is interesting in light of the *discrete analogue* given in Theorem 2.6 below of [the "tempered version of Theorem 2.6" constituted by] Proposition 2.4 [which is essentially a formal consequence of Proposition 2.1].

Now let k be an *MLF*, $\overline{k}$ an algebraic closure of k, $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$, X a *hyperbolic curve* over k that admits *stable reduction* over the ring of integers $\mathcal{O}_k$ of k. Write

$$\Pi^{\mathrm{tp}}_X, \quad \Delta^{\mathrm{tp}}_X$$

for the respective *"$\widehat{\Sigma}$-tempered"* quotients of the *tempered fundamental groups* $\pi_1^{\mathrm{tp}}(X)$, $\pi_1^{\mathrm{tp}}(X_{\overline{k}})$ [relative to suitable basepoints] of X, $X_{\overline{k}} \stackrel{\mathrm{def}}{=} X \times_k \overline{k}$ [cf. [André], §4; [SemiAnbd], Example 3.10]. That is to say, $\pi_1^{\mathrm{tp}}(X_{\overline{k}}) \twoheadrightarrow \Delta^{\mathrm{tp}}_X$ is the quotient determined by the intersection of the kernels of all continuous surjections of $\pi_1^{\mathrm{tp}}(X_{\overline{k}})$ onto extensions of a finite group of order a product [possibly with multiplicities] of primes

$\in \widehat{\Sigma}$ by a discrete free group of finite rank; $\pi_1^{\text{tp}}(X) \twoheadrightarrow \Pi_X^{\text{tp}}$ is the quotient of $\pi_1^{\text{tp}}(X)$ determined by the kernel of the quotient of $\pi_1^{\text{tp}}(X_{\overline{k}}) \twoheadrightarrow \Delta_X^{\text{tp}}$. Write $\widehat{\Delta}_X$ for the *pro-*$\widehat{\Sigma}$ [i.e., maximal pro-$\widehat{\Sigma}$ quotient of the profinite] *fundamental group* of $X_{\overline{k}}$; $\widehat{\Pi}_X$ for the quotient of the profinite fundamental group of X by the subgroup of the profinite fundamental group of $X_{\overline{k}}$ that determines the quotient $\widehat{\Delta}_X$. Thus, since discrete free groups of finite rank inject into their pro-l completions for any prime number l [cf., e.g., [RZ], Proposition 3.3.15], we have *natural inclusions*

$$\Pi_X^{\text{tp}} \quad \hookrightarrow \quad \widehat{\Pi}_X, \quad \Delta_X^{\text{tp}} \quad \hookrightarrow \quad \widehat{\Delta}_X$$

[cf., e.g., [SemiAnbd], Proposition 3.6, (iii), as well as Remark 2.5.3, (ii), (E7), of the present paper, when $\widehat{\Sigma} = \mathfrak{Primes}$]; $\widehat{\Delta}_X$ may be identified with the *pro-*$\widehat{\Sigma}$ *completion* of Δ_X^{tp}; $\widehat{\Pi}_X$ is generated by the images of Π_X^{tp} and $\widehat{\Delta}_X$.

Now suppose that the **residue characteristic** p of k is **not contained** in Σ; that the semi-graph of anabelioids $\mathcal{G}$ of the above discussion is the *pro-*Σ *semi-graph of anabelioids associated to the geometric special fiber of the stable model* $\mathcal{X}$ *of* X *over* $\mathcal{O}_k$ [cf., e.g., [SemiAnbd], Example 3.10]; and that the *sub-semi-graph* $\mathbb{H} \subseteq \mathbb{G}$ is *stabilized* by the natural action of G_k on $\mathbb{G}$. Thus, we have *natural surjections*

$$\Delta_X^{\text{tp}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\text{tp}}; \quad \widehat{\Delta}_X \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}}$$

of topological groups.

Corollary 2.3. (Subgroups of Tempered Fundamental Groups Associated to Sub-semi-graphs) *In the notation of the above discussion:*

(i) The closed subgroups

$$\Delta_{X,\mathbb{H}}^{\text{tp}} \stackrel{\text{def}}{=} \Delta_X^{\text{tp}} \times_{\Pi_{\mathcal{G}}^{\text{tp}}} \Pi_{\mathcal{H}}^{\text{tp}} \subseteq \Delta_X^{\text{tp}}; \quad \widehat{\Delta}_{X,\mathbb{H}} \stackrel{\text{def}}{=} \widehat{\Delta}_X \times_{\widehat{\Pi}_{\mathcal{G}}} \widehat{\Pi}_{\mathcal{H}} \subseteq \widehat{\Delta}_X$$

are **commensurably terminal**. *In particular, the natural outer actions of* G_k *on* Δ_X^{tp}, $\widehat{\Delta}_X$ *determine* **natural outer actions** *of* G_k *on* $\Delta_{X,\mathbb{H}}^{\text{tp}}$, $\widehat{\Delta}_{X,\mathbb{H}}$.

(ii) The **closure** *of* $\Delta_{X,\mathbb{H}}^{\text{tp}} \subseteq \Delta_X^{\text{tp}} \subseteq \widehat{\Delta}_X$ *in* $\widehat{\Delta}_X$ *is equal to* $\widehat{\Delta}_{X,\mathbb{H}}$.

(iii) Suppose that [at least] one of the following conditions holds: (a) $\widehat{\Sigma}$ *contains a prime number* $l \notin \Sigma \bigcup \{p\}$*; (b)* $\widehat{\Sigma} = \mathfrak{Primes}$. *Then* $\widehat{\Delta}_{X,\mathbb{H}}$ *is* **slim**. *In particular, the natural outer actions of* G_k *on* $\Delta_{X,\mathbb{H}}^{\text{tp}}$, $\widehat{\Delta}_{X,\mathbb{H}}$ *[cf. (i)] determine* **natural exact sequences of center-free topological groups** *[cf. (ii); the slimness of* $\widehat{\Delta}_{X,\mathbb{H}}$*; [AbsAnab], Theorem 1.1.1, (ii)]*

$$1 \to \Delta_{X,\mathbb{H}}^{\text{tp}} \to \Pi_{X,\mathbb{H}}^{\text{tp}} \to G_k \to 1$$

$$1 \to \widehat{\Delta}_{X,\mathbb{H}} \to \widehat{\Pi}_{X,\mathbb{H}} \to G_k \to 1$$

— where $\Pi_{X,\mathbb{H}}^{\text{tp}}$, $\widehat{\Pi}_{X,\mathbb{H}}$ *are* **defined** *so as to render the sequences exact.*

(iv) Suppose that the hypothesis of (iii) holds. Then the images of the natural inclusions $\Pi_{X,\mathbb{H}}^{\text{tp}} \hookrightarrow \Pi_X^{\text{tp}}$, $\widehat{\Pi}_{X,\mathbb{H}} \hookrightarrow \widehat{\Pi}_X$ *are* **commensurably terminal**.

(v) We have: $\widehat{\Delta}_{X,\mathbb{H}} \bigcap \Delta_X^{\mathrm{tp}} = \Delta_{X,\mathbb{H}}^{\mathrm{tp}} \subseteq \widehat{\Delta}_X$.

(vi) Let

$$I_x \subseteq \Delta_X^{\mathrm{tp}} \text{ (respectively, } I_x \subseteq \widehat{\Delta}_X\text{)}$$

be an **inertia group** *associated to a cusp x of X. Write ξ for the cusp of the stable model $\mathcal{X}$ corresponding to x. Then the following conditions are equivalent:*

(a) I_x lies in a Δ_X^{tp}- (respectively, $\widehat{\Delta}_X$-) conjugate of $\Delta_{X,\mathbb{H}}^{\mathrm{tp}}$ (respectively, $\widehat{\Delta}_{X,\mathbb{H}}$);

(b) ξ meets an irreducible component of the special fiber of $\mathcal{X}$ that is **contained** *in $\mathbb{H}$.*

Proof. Assertion (i) follows immediately from Proposition 2.2. Assertion (ii) follows immediately from the definitions of the various tempered fundamental groups involved, together with the following elementary *observation*: If $G \twoheadrightarrow F$ is a surjection of finitely generated free discrete groups, which induces a surjection $\widehat{G} \twoheadrightarrow \widehat{F}$ between the respective pro-$\widehat{\Sigma}$ completions [so, since discrete free groups of finite rank inject into their pro-l completions for any prime number l [cf., e.g., [RZ], Proposition 3.3.15], we think of G and F as subgroups of $\widehat{G}$ and $\widehat{F}$, respectively], then $H \stackrel{\mathrm{def}}{=} \mathrm{Ker}(G \twoheadrightarrow F)$ is *dense* in $\widehat{H} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\widehat{G} \twoheadrightarrow \widehat{F})$, relative to the pro-$\widehat{\Sigma}$ topology of $\widehat{G}$. Indeed, let $\iota : F \hookrightarrow G$ be a *section* of the given surjection $G \twoheadrightarrow F$ [which exists since F is *free*]. Then if $\{g_i\}_{i\in\mathbb{N}}$ is a sequence of elements of G that converges, in the pro-$\widehat{\Sigma}$ topology of $\widehat{G}$, to a given element $h \in \widehat{H}$, and maps to a sequence of elements $\{f_i\}_{i\in\mathbb{N}}$ of F [which necessarily converges, in the pro-$\widehat{\Sigma}$ topology of $\widehat{F}$, to the *identity element* $1 \in \widehat{F}$], then one verifies immediately that $\{g_i \cdot \iota(f_i)^{-1}\}_{i\in\mathbb{N}}$ is a sequence of elements of H that converges, in the pro-$\widehat{\Sigma}$ topology of $\widehat{G}$, to h. This completes the proof of the *observation* and hence of assertion (ii).

Next, we consider assertion (iii). In the following, we give, in effect, *two distinct proofs* of the slimness of $\widehat{\Delta}_{X,\mathbb{H}}$: one is *elementary*, but requires one to assume that *condition (a)* holds; the other *depends on the highly nontrivial theory of [Tama2]* and requires one to assume that *condition (b)* holds. If condition (a) holds, then let us set $\Sigma^* \stackrel{\mathrm{def}}{=} \Sigma \bigcup \{l\}$. If condition (b) holds, but condition (a) does not hold [so $\widehat{\Sigma} = \mathfrak{Primes} = \Sigma \bigcup \{p\}$], then let us set $\Sigma^* \stackrel{\mathrm{def}}{=} \Sigma$. Thus, in either case, $p \notin \Sigma^*$, and $\Sigma \subseteq \Sigma^* \subseteq \widehat{\Sigma}$.

Let $J \subseteq \widehat{\Delta}_X$ be a *normal open subgroup*. Write $J_{\mathbb{H}} \stackrel{\mathrm{def}}{=} J \bigcap \widehat{\Delta}_{X,\mathbb{H}}$; $J \twoheadrightarrow J^*$ for the *maximal pro-Σ^* quotient*; $J^*_{\mathbb{H}} \subseteq J^*$ for the image of $J_{\mathbb{H}}$ in J^*. Now suppose that $\alpha \in \widehat{\Delta}_{X,\mathbb{H}}$ *commutes* with $J_{\mathbb{H}}$. Let v be a vertex of the dual graph of the geometric special fiber of a stable model $\mathcal{X}_J$ of the covering X_J of $X_{\overline{k}}$ determined by J. Write $J_v \subseteq J$ for the decomposition group [well-defined up to conjugation in J] associated to v; $J^*_v \subseteq J^*$ for the image of J_v in J^*. Then let us observe that

(†) there exists an open subgroup $J_0 \subseteq \widehat{\Delta}_X$ which is *independent* of J, v, and α such that if $J \subseteq J_0$, then for arbitrary v [and α] as above, it holds that $J^*_v \bigcap J^*_{\mathbb{H}}$ ($\subseteq J^*$) is *infinite* and *nonabelian*.

Indeed, suppose that condition (a) holds. Now it follows immediately from the definitions that the image of the homomorphism $J_v \subseteq J \subseteq \widehat{\Delta}_X \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}}$ is *pro-Σ*; in particular, since $l \notin \Sigma$, and $\mathrm{Ker}(J_v \subseteq J \subseteq \widehat{\Delta}_X \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}}) \subseteq J_v \bigcap J_{\mathbb{H}}$, it follows that $J_v \bigcap J_{\mathbb{H}}$, hence also $J_v^* \bigcap J_{\mathbb{H}}^*$, *surjects* onto the *maximal pro-l quotient* of J_v, which is isomorphic to the pro-l completion of the fundamental group of a hyperbolic Riemann surface, hence [as is well-known] is *infinite* and *nonabelian* [so we may take $J_0 \stackrel{\mathrm{def}}{=} \widehat{\Delta}_X$]. Now suppose that condition (b) holds, but condition (a) does *not* hold. Then it follows immediately from [Tama2], Theorem 0.2, (v), that, for an appropriate choice of J_0, if $J \subseteq J_0$, then every v corresponds to an irreducible component that either maps to a point in $\mathcal{X}$ or contains a node that maps to a smooth point of $\mathcal{X}$. In particular, it follows that for every choice of v, there exists at least one *pro-Σ, torsion-free, pro-cyclic subgroup* $F \subseteq J_v$ that *lies in* $\mathrm{Ker}(J_v \subseteq J \subseteq \widehat{\Delta}_X \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}}) \subseteq J_v \bigcap J_{\mathbb{H}}$ and, moreover, *maps injectively into* J^*. Thus, we obtain an injection $F \hookrightarrow J_v^* \bigcap J_{\mathbb{H}}^*$; a similar statement holds when F is replaced by any J_v-conjugate of F. Moreover, it follows from the well-known structure of the pro-Σ completion of the fundamental group of a hyperbolic Riemann surface [such as J_v^*] that the image of the J_v-conjugates of such a group F *topologically generate* a closed subgroup of $J_v^* \bigcap J_{\mathbb{H}}^*$ which is *infinite* and *nonabelian*. This completes the proof of (†).

Next, let us observe that it follows by applying either [AbsTopII], Proposition 1.3, (iv), or [NodNon], Proposition 3.9, (i), to the various $\widehat{\Delta}_X$-conjugates in J^* of $J_v^* \bigcap J_{\mathbb{H}}^*$ as in (†) that the fact that α *commutes* with $J_v^* \bigcap J_{\mathbb{H}}^*$ implies that α *fixes* v. If condition (a) holds, then the fact that conjugation by α on the *maximal pro-l quotient* of J_v [which, as we saw above, is a quotient of $J_v^* \bigcap J_{\mathbb{H}}^*$] is *trivial* implies [cf. the argument concerning the inertia group "$I_v \subseteq D_v$" in the latter portion of the proof of [SemiAnbd], Corollary 3.11] that α not only fixes v, but also *acts trivially on the irreducible component of the special fiber of $\mathcal{X}_J$ determined by v*; since v as in (†) is *arbitrary*, we thus conclude that α acts on the *abelianization* $(J^*)^{\mathrm{ab}}$ of J^* as a *unipotent* automorphism of *finite order*, hence that α acts *trivially* on $(J^*)^{\mathrm{ab}}$; since J as in (†) is *arbitrary*, we thus conclude [cf., e.g., the proof of [Config], Proposition 1.4] that α is the *identity element*, as desired. Now suppose that condition (b) holds, but condition (a) does *not* hold. Then since J and v as in (†) are *arbitrary*, we thus conclude again from [Tama2], Theorem 0.2, (v), that α fixes not only v, but also *every closed point* on the irreducible component of the special fiber of $\mathcal{X}_J$ determined by v, hence that α *acts trivially on this irreducible component*. Again since J and v as in (†) are *arbitrary*, we thus conclude that α is the *identity element*, as desired. This completes the proof of assertion (iii). In light of the exact sequences of assertion (iii), assertion (iv) follows immediately from assertion (i). Assertion (vi) follows immediately from a similar argument to the argument applied in the proof of [CombGC], Proposition 1.5, (i), by passing to pro-Σ completions.

Finally, it follows immediately from the definitions of the various tempered fundamental groups involved that to verify assertion (v), it suffices to verify the following analogue of assertion (v) for a *nonabelian finitely generated free discrete group G*: for any finitely generated subgroup $F \subseteq G$, if we use the notation "$\wedge$" to denote the pro-$\widehat{\Sigma}$ completion, then $\widehat{F} \bigcap G = F$. But to verify this assertion concerning G, it follows immediately from [SemiAnbd], Corollary 1.6, (ii), that we may assume without loss of generality that the inclusion $F \subseteq G$ admits a *splitting* $G \twoheadrightarrow F$ [i.e., such that the composite $F \hookrightarrow G \twoheadrightarrow F$ is the identity on F], in which case the desired equality "$\widehat{F} \bigcap G = F$" follows immediately. This completes the proof of assertion (v), and hence of Corollary 2.3. ◯

Next, we observe the following *arithmetic analogue* of Proposition 2.1.

Proposition 2.4. (Profinite Conjugates of Nontrivial Arithmetic Compact Subgroups) *In the notation of the above discussion:*

(i) Let $\Lambda \subseteq \Delta_X^{\mathrm{tp}}$ *be a* **nontrivial pro-**Σ **compact subgroup**, $\gamma \in \widehat{\Pi}_X$ *an element such that* $\gamma \cdot \Lambda \cdot \gamma^{-1} \subseteq \Delta_X^{\mathrm{tp}}$ *[or, equivalently,* $\Lambda \subseteq \gamma^{-1} \cdot \Delta_X^{\mathrm{tp}} \cdot \gamma$*]. Then* $\gamma \in \Pi_X^{\mathrm{tp}}$.

(ii) Suppose that $\widehat{\Sigma} = \mathfrak{Primes}$*. Let* $\Lambda \subseteq \Pi_X^{\mathrm{tp}}$ *be a* **[nontrivial] compact subgroup** *whose image in* G_k *is* **open**, $\gamma \in \widehat{\Pi}_X$ *an element such that* $\gamma \cdot \Lambda \cdot \gamma^{-1} \subseteq \Pi_X^{\mathrm{tp}}$ *[or, equivalently,* $\Lambda \subseteq \gamma^{-1} \cdot \Pi_X^{\mathrm{tp}} \cdot \gamma$*]. Then* $\gamma \in \Pi_X^{\mathrm{tp}}$.

(iii) Δ_X^{tp} *(respectively,* Π_X^{tp}*) is* **commensurably terminal** *in* $\widehat{\Delta}_X$ *(respectively,* $\widehat{\Pi}_X$*).*

Proof. First, we consider assertion (i). We begin by *observing* that since [as is well-known — cf., e.g., [Config], Remark 1.2.2] $\widehat{\Delta}_X$ is *strongly torsion-free*, it follows that there exists a finite index characteristic open subgroup $J \subseteq \Delta_X^{\mathrm{tp}}$ such that, if we write $\mathcal{G}_J$ for the *pro-*Σ *semi-graph of anabelioids* associated to the special fiber of the stable model [i.e., over the ring of integers $\mathcal{O}_{\overline{k}}$ of $\overline{k}$] of the finite étale covering of $X \times_k \overline{k}$ determined by J, then $J \bigcap \Lambda$ has *nontrivial image* in the *pro-*Σ *completion of the abelianization* of J, hence in $\Pi_{\mathcal{G}_J}^{\mathrm{tp}}$ [since, as is well-known, our assumption that $p \notin \Sigma$ implies that the surjection $J \twoheadrightarrow \Pi_{\mathcal{G}_J}^{\mathrm{tp}}$ induces an isomorphism between the pro-Σ completions of the respective abelianizations]. Since the quotient Π_X^{tp} *surjects* onto G_k, and J is open of finite index in Δ_X^{tp}, we may assume without loss of generality that γ lies in the closure $\widehat{J}$ of J in $\widehat{\Pi}_X$. Since $J \bigcap \Lambda$ has *nontrivial image* in $\Pi_{\mathcal{G}_J}^{\mathrm{tp}}$, it thus follows from Proposition 2.1 [applied to $\mathcal{G}_J$] that the image of γ via the natural surjection on pro-$\widehat{\Sigma}$ completions $\widehat{J} \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}_J}$ lies in $\Pi_{\mathcal{G}_J}^{\mathrm{tp}}$. Since, by allowing J to *vary*, Π_X^{tp} (respectively, $\widehat{\Pi}_X$) may be written as an inverse limit of the topological groups $\Pi_X^{\mathrm{tp}}/\mathrm{Ker}(J \twoheadrightarrow \Pi_{\mathcal{G}_J}^{\mathrm{tp}})$ (respectively, $\widehat{\Pi}_X/\mathrm{Ker}(\widehat{J} \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}_J})$), we thus conclude that [the original] γ lies in Π_X^{tp}, as desired.

Next, we consider assertion (ii). First, let us *observe* that it follows from a similar argument to the argument applied to prove Proposition 2.1 — where, instead of applying [SemiAnbd], Theorem 3.7, (iii), we apply its *arithmetic analogue*, namely, [SemiAnbd], Theorem 5.4, (ii); [SemiAnbd], Example 5.6 [cf. also Remark 2.5.3, (ii), (E5), (E7), of the present paper] — that the image of γ in $\widehat{\Pi}_X/\mathrm{Ker}(\widehat{\Delta}_X \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}^*})$ lies in $\Pi_X^{\mathrm{tp}}/\mathrm{Ker}(\Delta_X^{\mathrm{tp}} \twoheadrightarrow \Pi_{\mathcal{G}^*}^{\mathrm{tp}})$, where [by invoking the hypothesis that $\widehat{\Sigma} = \mathfrak{Primes}$] we take $\mathcal{G}^*$ to be a *semi-graph of anabelioids* as in [SemiAnbd], Example 5.6, i.e., the semi-graph of anabelioids whose finite étale coverings correspond to *arbitrary admissible coverings* of the geometric special fiber of the stable model $\mathcal{X}$. Here, we note that when one applies either [AbsTopII], Proposition 1.3, (iv), or [NodNon], Proposition 3.9, (i) — after, say, restricting the outer action of G_k on $\Pi^{\mathrm{tp}}_{\mathcal{G}^*}$ to a closed pro-Σ subgroup of the inertia group I_k of G_k that maps isomorphically onto the maximal pro-Σ quotient of I_k — to the vertices "v''", "$(v')^\gamma$", one may only conclude that these two vertices either *coincide*, are *adjacent*, or *admit a common adjacent vertex*; but this is still sufficient to conclude the *temperedness* of "$(v')^\gamma$" from that of "v''". Now [just as in the proof of assertion (i)] by applying [the evident analogue of] this *observation* to the quotients $\Pi^{\mathrm{tp}}_X \twoheadrightarrow \Pi^{\mathrm{tp}}_X/\mathrm{Ker}(J \twoheadrightarrow \Pi^{\mathrm{tp}}_{\mathcal{G}^*_J})$ — where $J \subseteq \Delta^{\mathrm{tp}}_X$ is a finite index characteristic open subgroup, and $\mathcal{G}^*_J$ is the *semi-graph of anabelioids* whose finite étale coverings correspond to *arbitrary admissible coverings* of the special fiber of the stable model over $\mathcal{O}_{\overline{k}}$ of the finite étale covering of $X \times_k \overline{k}$ determined by J — we conclude that $\gamma \in \Pi^{\mathrm{tp}}_X$, as desired.

Finally, we consider assertion (iii). Just as in the proof of Proposition 2.2, the commensurable terminality of Δ^{tp}_X in $\widehat{\Delta}_X$ follows immediately from assertion (i), by allowing, in assertion (i), Λ to range over the open subgroups of a pro-Σ Sylow subgroup of a decomposition group $\subseteq \Delta^{\mathrm{tp}}_X$ associated to an irreducible component of the special fiber of $\mathcal{X}$. The commensurable terminality of Π^{tp}_X in $\widehat{\Pi}_X$ then follows immediately from the commensurable terminality of Δ^{tp}_X in $\widehat{\Delta}_X$. $\bigcirc$

Remark 2.4.1. Thus, when $\widehat{\Sigma} = \mathfrak{Primes}$, the proof given above of Proposition 2.4, (iii), yields a *new proof* of [André], Corollary 6.2.2 [cf. also [SemiAnbd], Lemma 6.1, (ii), (iii)] which is *independent* of [André], Lemma 3.2.1, hence also of [Stb1], Theorem 1 [cf. the discussion of Remark 2.2.2].

Corollary 2.5. (Profinite Conjugates of Tempered Decomposition and Inertia Groups) *In the notation of the above discussion, suppose further that $\widehat{\Sigma} = \mathfrak{Primes}$. Then every* **decomposition group** *in $\widehat{\Pi}_X$ (respectively,* **inertia group** *in $\widehat{\Pi}_X$) associated to a closed point or cusp of X (respectively, to a cusp of X) is contained in Π^{tp}_X if and only if it is a decomposition group in Π^{tp}_X (respectively, inertia group in Π^{tp}_X) associated to a closed point or cusp of X (respectively, to a cusp of X). Moreover, a $\widehat{\Pi}_X$-conjugate of Π^{tp}_X contains a decomposition group in Π^{tp}_X (respectively, inertia group in Π^{tp}_X) associated to a closed point or cusp of X (respectively, to a cusp of X) if and only if it is equal to Π^{tp}_X.*

Proof. Let $D_x \subseteq \Pi^{\mathrm{tp}}_X$ be the decomposition group in Π^{tp}_X associated to a closed point or cusp x of X; $I_x \stackrel{\mathrm{def}}{=} D_x \bigcap \Delta^{\mathrm{tp}}_X$. Then the decomposition groups of $\widehat{\Pi}_X$ associated to x are precisely the $\widehat{\Pi}_X$-conjugates of D_x; the decomposition groups of Π^{tp}_X associated to x are precisely the Π^{tp}_X-conjugates of D_x. Since D_x is compact and surjects onto an open subgroup of G_k, it thus follows from Proposition 2.4, (ii), that a $\widehat{\Pi}_X$-conjugate of D_x is contained in Π^{tp}_X if and only if it is, in fact, a Π^{tp}_X-conjugate of D_x, and that a $\widehat{\Pi}_X$-conjugate of Π^{tp}_X contains D_x if and only if it is, in fact, equal to Π^{tp}_X. In a similar vein, when x is a cusp of X [so $I_x \cong \widehat{\mathbb{Z}}$], it follows — i.e., by applying Proposition 2.4, (i), to the unique maximal pro-Σ subgroup of I_x — that a $\widehat{\Pi}_X$-conjugate of I_x is contained in Π^{tp}_X if and only if it is, in fact, a Π^{tp}_X-conjugate of I_x, and that a $\widehat{\Pi}_X$-conjugate of Π^{tp}_X contains I_x if and only if it is, in fact, equal to Π^{tp}_X. This completes the proof of Corollary 2.5. $\bigcirc$

Remark 2.5.1. The content of Corollary 2.5 may be regarded as a sort of [very weak!] version of the *"Section Conjecture"* of anabelian geometry — i.e., as the assertion that certain sections of the tempered fundamental group [namely, those that arise from geometric sections of the profinite fundamental group] are geometric as sections of the tempered fundamental group. This point of view is reminiscent of the point of view of [SemiAnbd], Remark 6.9.1. Perhaps one way of summarizing this circle of ideas is to state that one may think of

(i) the *classification of maximal compact subgroups of tempered fundamental groups* given in [SemiAnbd], Theorem 3.7, (iv); [SemiAnbd], Theorem 5.4, (ii) [cf. also Remark 2.5.3, (ii), (E5), (E7), of the present paper], or, for that matter,

(ii) the more elementary fact that *"any finite group acting on a tree [without inversion] fixes at least one vertex"* [cf. [SemiAnbd], Lemma 1.8, (ii)] from which these results of [SemiAnbd] are derived

as a sort of **combinatorial version of the Section Conjecture**.

Remark 2.5.2. Ultimately, when we apply Corollary 2.5 in [IUTchII], it will only be necessary to apply the portion of Corollary 2.5 that concerns *inertia groups of cusps*, i.e., the portion whose proof only requires the use of Proposition 2.4, (i), which is essentially an immediate consequence of Proposition 2.1. That is to say, the theory developed in [IUTchII] [and indeed throughout the present series of papers] will never require the application of Proposition 2.4, (ii), i.e., whose proof depends on a slightly more complicated version of the proof of Proposition 2.1.

Remark 2.5.3. In light of the importance of the theory of [SemiAnbd] in the present §2, we pause to discuss certain minor oversights on the part of the author in the exposition of [SemiAnbd].

(i) Certain pathologies occur in the theory of tempered fundamental groups if one does not impose suitable *countability* hypotheses. In order to discuss these countability hypotheses, it will be convenient to introduce some *terminology* as follows:

(T1) We shall say that a tempered group is *Galois-countable* if its topology admits a countable basis. We shall say that a connected temperoid is *Galois-countable* if it arises from a Galois-countable tempered group. We shall say that a temperoid is *Galois-countable* if it arises from a collection of Galois-countable connected temperoids. We shall say that a connected quasi-temperoid is *Galois-countable* if it arises from a Galois-countable connected temperoid. We shall say that a quasi-temperoid is *Galois-countable* if it arises from a collection of Galois-countable connected quasi-temperoids.

(T2) We shall say that a semi-graph of anabelioids $\mathcal{G}$ is *Galois-countable* if it is countable, and, moreover, admits a countable collection of finite étale coverings $\{\mathcal{G}_i \to \mathcal{G}\}_{i \in I}$ such that for any finite étale covering $\mathcal{H} \to \mathcal{G}$, there exists an $i \in I$ such that the base-changed covering $\mathcal{H} \times_{\mathcal{G}} \mathcal{G}_i \to \mathcal{G}_i$ *splits* over the constituent anabelioid associated to each component of [the underlying semi-graph of] $\mathcal{G}_i$.

(T3) We shall say that a semi-graph of anabelioids $\mathcal{G}$ is *strictly coherent* if it is coherent [cf. [SemiAnbd], Definition 2.3, (iii)], and, moreover, each of the profinite groups associated to components c of [the underlying semi-graph of] $\mathcal{G}$ [cf. the final sentence of [SemiAnbd], Definition 2.3, (iii)] is *topologically generated by* N *generators*, for some positive integer N that is *independent* of c. In particular, it follows that if $\mathcal{G}$ is *finite* and *coherent*, then it is *strictly coherent*.

(T4) One verifies immediately that every *strictly coherent*, countable semi-graph of anabelioids is *Galois-countable*.

(T5) One verifies immediately that if, in [SemiAnbd], Remark 3.2.1, one assumes in addition that the temperoid $\mathcal{X}$ is *Galois-countable*, then it follows that its associated *tempered fundamental group* $\pi_1^{\mathrm{temp}}(\mathcal{X})$ is *well-defined* and *Galois-countable*.

(T6) One verifies immediately that if, in the discussion of the paragraph preceding [SemiAnbd], Proposition 3.6, one assumes in addition that the semi-graph of anabelioids $\mathcal{G}$ is *Galois-countable*, then it follows that its associated *tempered fundamental group* $\pi_1^{\mathrm{temp}}(\mathcal{G})$ and *temperoid* $\mathcal{B}^{\mathrm{temp}}(\mathcal{G})$ are *well-defined* and *Galois-countable*.

Here, we note that, in (T5) and (T6), the *Galois-countability* assumption is necessary in order to ensure that the index sets of *"universal covering pro-objects"* implicit in the definition of the *tempered fundamental group* may to be taken to be *countable*. This countability of the index sets involved implies that the various objects that constitute such a universal covering pro-object admit a *compatible system of basepoints*, i.e., that the *obstruction* to the existence of such a compatible system — which may be thought of as an element of a sort of "nonabelian $\mathbb{R}^1 \varprojlim$" — *vanishes*. In order to define the tempered fundamental group in an intrinsically meaningful fashion, it is necessary to know the existence of such a compatible system of basepoints.

(ii) The *effects* of the *omission* of *Galois-countability hypotheses* in [SemiAnbd], §3 [cf. the discussion of (i)], on the remainder of [SemiAnbd], as well as on subsequent papers of the author, may be summarized as follows:

(E1) First of all, we observe that all topological subquotients of *absolute Galois groups of fields of countable cardinality* are *Galois-countable*.

(E2) Also, we observe that if k is a field whose absolute Galois group is *Galois-countable*, and U is a nonempty open subscheme of a connected proper k-scheme X that arises as the underlying scheme of a log scheme that is log smooth over k [where we regard Spec(k) as equipped with the trivial log structure], and whose interior is equal to U, then the *tamely ramified arithmetic fundamental group* of U [i.e., that arises by considering finite étale coverings of U with *tame ramification* over the divisors that lie in the complement of U in X] is itself *Galois-countable* [cf., e.g., [AbsTopI], Proposition 2.2].

(E3) Next, we observe, with regard to [SemiAnbd], Examples 2.10, 3.10, and 5.6, that the tempered groups and temperoids that appear in these Examples are *Galois-countable* [cf. (E1), (E2)], while the semi-graphs of anabelioids that appear in these Examples are *strictly coherent* [cf. item (T3) of (i)], hence [cf. item (T4) of (i)] *Galois-countable*. In particular, there is *no effect* on the theory of objects discussed in these Examples.

(E4) It follows immediately from (E3) that there is *no effect* on [SemiAnbd], §6.

(E5) It follows immediately from items (T3), (T4) of (i), together with the assumptions of *finiteness* and *coherence* in the discussion of the paragraph immediately preceding [SemiAnbd], Definition 4.2, the assumption of *coherence* in [SemiAnbd], Definition 5.1, (i), and the assumption of [SemiAnbd], Definition 5.1, (i), (d), that there is *no effect* on [SemiAnbd], §4, §5. [Here, we note that since the notion of a *tempered covering* of a semi-graph of anabelioids is only defined in the case where the semi-graph of anabelioids is *countable*, it is implicit in [SemiAnbd], Proposition 5.2, and [SemiAnbd], Definition 5.3, that the semi-graphs of anabelioids under consideration are *countable*.]

(E6) There is *no effect* on [SemiAnbd], §1, §2, or the Appendix of [SemiAnbd], since *tempered fundamental groups* are *never discussed* in these portions of [SemiAnbd].

(E7) In the Definitions/Propositions/Theorems/Corollaries of [SemiAnbd] that are numbered 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, one must assume that all tempered groups, temperoids, and semi-graphs of anabelioids that appear are *Galois-countable*. On the other hand, it follows immediately from (E1), (E2), and (E3) that there is *no effect* on the remaining portions of [SemiAnbd], §3.

(E8) In [QuCnf] and [FrdII], one must assume that all tempered groups and [quasi-]temperoids that appear are *Galois-countable*.

(E9) There is *no effect* on any papers of the author other than [SemiAnbd] and the papers discussed in (E8).

(iii) The *assertion* stated in the second display of [SemiAnbd], Remark 2.4.2, is *false* as stated. [The automorphisms of the semi-graphs of anabelioids in [SemiAnbd], Example 2.10, that arise from *"Dehn twists"* constitute a well-known counterexample to this *assertion*.] This *assertion* should be replaced by the following slightly modified version of this *assertion*:

> The isomorphism classes of the ϕ_v completely determine the isomorphism class of each of the ϕ_e, as well as each isomorphism ϕ_b, up to composition with an automorphism of the composite 1-morphism of anabelioids $\mathcal{G}_e \to \mathcal{H}_f \to \mathcal{H}_w$ that arises from an automorphism of the 1-morphism of anabelioids $\mathcal{G}_e \to \mathcal{H}_f$.

Also, in the *discussion* following this *assertion* [as well as the various places where this *discussion* is applied, i.e., [SemiAnbd], Remark 3.5.2; the second paragraph of [SemiAnbd], §4; [SemiAnbd], Definition 5.1, (iv)], it is necessary to assume further that the semi-graphs of anabelioids that appear satisfy the condition that *every edge abuts to at least one vertex*.

(iv) The phrase "is *Galois*" at the end of the first sentence of the proof of [SemiAnbd], Proposition 3.2, should read "is a countable coproduct of *Galois* objects".

(v) In the first sentence of [SemiAnbd], Definition 3.5, (ii), the phrase "Suppose that" should read "Suppose that each connected component of"; the phrase "*splits* the restriction of" should read "*splits* the restriction of this connected component of".

(vi) In order to carry out the argument stated in the proof of [SemiAnbd], Proposition 5.2, (i), it is necessary to *strengthen* the conditions (c) and (d) of [SemiAnbd], Definition 5.1, (i), as follows. This strengthening of the conditions (c) and (d) of [SemiAnbd], Definition 5.1, (i), has *no effect* either on the remainder of [SemiAnbd] or on subsequent papers of the author. Suppose that $\mathcal{G}$ is as in [SemiAnbd], Definition 5.1, (i). Then we begin by making the following *observation*:

(O1) Suppose that $\mathcal{G}$ is *finite*. Then $\mathcal{G}$ admits a *cofinal, countable* collection of *connected finite étale Galois coverings* $\{\mathcal{G}^i \to \mathcal{G}\}_{i \in I}$, each of which is *characteristic* [i.e., any pull-back of the covering via an element of $\mathrm{Aut}(\mathcal{G})$ is isomorphic to the original covering]. [For instance, one verifies immediately, by applying the *finiteness* and *coherence* of $\mathcal{G}$, that such a collection of coverings may be obtained by considering, for n a positive integer, the *composite* of all connected finite étale Galois coverings of degree $\leq n$.] We may assume, without loss of generality, that this collection of coverings arises from a *projective system*, which we denote by $\widetilde{\mathcal{G}}$. Thus, we obtain a *natural exact sequence*

$$1 \longrightarrow \mathrm{Gal}(\widetilde{\mathcal{G}}/\mathcal{G}) \longrightarrow \mathrm{Aut}(\widetilde{\mathcal{G}}/\mathcal{G}) \longrightarrow \mathrm{Aut}(\mathcal{G}) \longrightarrow 1$$

— where we write "$\mathrm{Aut}(\widetilde{\mathcal{G}}/\mathcal{G})$" for the group of pairs of *compatible* automorphisms of $\widetilde{\mathcal{G}}$ and $\mathcal{G}$.

This observation (O1) has the following immediate consequence:

(O2) Suppose that we are in the situation of (O1). Consider, for $i \in I$, the *finite index normal subgroup*

$$\mathrm{Aut}^i(\widetilde{\mathcal{G}}/\mathcal{G}) \quad \subseteq \quad \mathrm{Aut}(\widetilde{\mathcal{G}}/\mathcal{G})$$

of elements of $\mathrm{Aut}(\widetilde{\mathcal{G}}/\mathcal{G})$ that induce the *identity* automorphism on the underlying semi-graph $\mathbb{G}^i$ of $\mathcal{G}^i$, as well as on $\mathrm{Gal}(\mathcal{G}^i/\mathcal{G})$. Then one verifies immediately [from the definition of a *semi-graph of anabelioids*; cf. also [SemiAnbd], Proposition 2.5, (i)] that the intersection of the $\mathrm{Aut}^i(\widetilde{\mathcal{G}}/\mathcal{G})$, for $i \in I$, is $= \{1\}$. Thus, the $\mathrm{Aut}^i(\widetilde{\mathcal{G}}/\mathcal{G})$, for $i \in I$, determine a *natural profinite topology* on $\mathrm{Aut}(\widetilde{\mathcal{G}}/\mathcal{G})$ and hence also on the quotient $\mathrm{Aut}(\mathcal{G})$, which is easily seen to be compatible with the profinite topology on $\mathrm{Gal}(\widetilde{\mathcal{G}}/\mathcal{G})$ and, moreover, *independent* of the choice of $\widetilde{\mathcal{G}}$.

The *new version* of the condition (c) of [SemiAnbd], Definition 5.1, (i), that we wish to consider is the following:

($\mathrm{c}^{\mathrm{new}}$) The action of H on $\mathbb{G}$ is trivial; the resulting homomorphism $H \to \mathrm{Aut}(\mathcal{G}[c])$, where c ranges over the *components* [i.e., vertices and edges] of $\mathbb{G}$, is *continuous* [i.e., relative to the natural profinite group topology defined in (O2) on $\mathrm{Aut}(\mathcal{G}[c])$].

It is immediate that ($\mathrm{c}^{\mathrm{new}}$) implies (c). Moreover, we observe in passing that:

(O3) In fact, since H is *topologically finitely generated* [cf. [SemiAnbd], Definition 5.1, (i), (a)], it holds [cf. [NS], Theorem 1.1] that *every finite index subgroup of* H *is open in* H. Thus, the conditions (c) and ($\mathrm{c}^{\mathrm{new}}$) in fact *hold automatically.*

The *new version* of the condition (d) of [SemiAnbd], Definition 5.1, (i), that we wish to consider is the following:

($\mathrm{d}^{\mathrm{new}}$) There is a *finite* set C^* of *components* [i.e., vertices and edges] of $\mathbb{G}$ such that for every component c of $\mathbb{G}$, there exists a $c^* \in C^*$ and an *isomorphism* of semi-graphs of anabelioids $\mathcal{G}[c] \xrightarrow{\sim} \mathcal{G}[c^*]$ that is *compatible* with the action of H on both sides.

It is immediate that ($\mathrm{d}^{\mathrm{new}}$) implies (d). The reason that, in the context of the proof of [SemiAnbd], Proposition 5.2, (i), it is necessary to consider the *stronger conditions* ($\mathrm{c}^{\mathrm{new}}$) and ($\mathrm{d}^{\mathrm{new}}$) is as follows. It suffices to show that, given a *connected finite étale covering* $\mathcal{G}' \to \mathcal{G}$, after possibly replacing H by an open subgroup of H, the action of H on $\mathcal{G}$ *lifts* to an action on $\mathcal{G}'$ that satisfies the conditions of [SemiAnbd], Definition 5.1, (i). Such a lifting of the action of H on $\mathcal{G}$ to an action on the portion of $\mathcal{G}'$ that lies over the *vertices* of $\mathbb{G}$ follows in a straightforward manner from the *original* conditions (a), (b), (c), and (d). On the other hand, in order to conclude that such a lifting is [after possibly replacing H by an open subgroup of H] *compatible with the gluing conditions* arising from the structure of $\mathcal{G}'$ over the *edges* of $\mathbb{G}$, it is necessary to assume further that the *"component-wise versions (*$\mathrm{c}^{\mathrm{new}}$*), (*$\mathrm{d}^{\mathrm{new}}$*)"* of the original *"vertex-wise conditions (c), (d)"* hold. This issue is closely related to the issue discussed in (iii) above.

Finally, we observe that Proposition 2.4, Corollary 2.5 admit the following *discrete analogues*, which may be regarded as generalizations of [André], Lemma 3.2.1 [cf. Theorem 2.6 below in the case where $H = F = G$ is free]; [EtTh], Lemma 2.17, (i).

Theorem 2.6. **(Profinite Conjugates of Discrete Subgroups)** *Let F be a group that contains a subgroup of finite index $G \subseteq F$ such that G is either a* **free discrete group of finite rank** *or an* **orientable surface group** *[i.e., a fundamental group of a compact orientable topological surface of genus ≥ 2]; $H \subseteq F$ an infinite subgroup. Since F is residually finite [cf., e.g., [Config], Proposition 7.1, (ii)], we shall write $H, G \subseteq F \subseteq \widehat{F}$, where $\widehat{F}$ denotes the* **profinite completion** *of F. Let $\gamma \in \widehat{F}$ be an element such that*

$$\gamma \cdot H \cdot \gamma^{-1} \subseteq F \quad \textit{[or, equivalently, } H \subseteq \gamma^{-1} \cdot F \cdot \gamma\textit{].}$$

Write $H_G \stackrel{\mathrm{def}}{=} H \bigcap G$. Then $\gamma \in F \cdot N_{\widehat{F}}(H_G)$, i.e., $\gamma \cdot H_G \cdot \gamma^{-1} = \delta \cdot H_G \cdot \delta^{-1}$, for some $\delta \in F$. If, moreover, H_G is **nonabelian***, then $\gamma \in F$.*

Proof. Let us first consider the case where H_G is *abelian.* In this case, it follows from Lemma 2.7, (iv), below, that H_G is *cyclic.* Thus, by applying Lemma 2.7, (ii), it follows that by replacing G by an appropriate finite index subgroup of G, we may assume that the natural composite homomorphism $H_G \hookrightarrow G \twoheadrightarrow G^{\mathrm{ab}}$ is a *split injection*. In particular, by Lemma 2.7, (v), we conclude that $N_{\widehat{G}}(H_G) = \widehat{H}_G$, where we write $\widehat{H}_G$ for the closure of H_G in the profinite completion $\widehat{G}$ of G. Next, let us observe that by multiplying γ on the left by an appropriate element of F, we may assume that $\gamma \in \widehat{G}$. Thus, we have $\gamma \cdot H_G \cdot \gamma^{-1} \subseteq F \bigcap \widehat{G} = G$. Next, let us recall that G is *conjugacy separable*. Indeed, this is precisely the content of [Stb1], Theorem 1, when G is *free*; [Stb2], Theorem 3.3, when G is an *orientable surface group*. Since G is conjugacy separable, it follows that $\gamma \cdot H_G \cdot \gamma^{-1} = \epsilon \cdot H_G \cdot \epsilon^{-1}$ for some $\epsilon \in G$, so $\gamma \in G \cdot N_{\widehat{G}}(H_G) = G \cdot \widehat{H}_G \subseteq F \cdot N_{\widehat{F}}(H_G)$, as desired. This completes the proof of Theorem 2.6 when H_G is *abelian*.

Thus, let us assume for the remainder of the proof of Theorem 2.6 that H_G is *nonabelian*. Then, by applying Lemma 2.7, (iii), it follows that, after replacing G by an appropriate finite index subgroup of G, we may assume that there exist elements $x, y \in H_G$ that generate a *free abelian subgroup of rank two* $M \subseteq G^{\mathrm{ab}}$ such that the injection $M \hookrightarrow G^{\mathrm{ab}}$ *splits*. Write $H_x, H_y \subseteq H_G$ for the subgroups generated, respectively, by x and y; $\widehat{H}_x, \widehat{H}_y \subseteq \widehat{G}$ for the respective closures of H_x, H_y. Then by Lemma 2.7, (v), we conclude that $N_{\widehat{G}}(H_x) = \widehat{H}_x$, $N_{\widehat{G}}(H_y) = \widehat{H}_y$. Next, let us observe that by multiplying γ on the left by an appropriate element of F, we may assume that $\gamma \in \widehat{G}$. Thus, we have $\gamma \cdot H_G \cdot \gamma^{-1} \subseteq F \bigcap \widehat{G} = G$. In particular, by applying the portion of Theorem 2.6 that has already been proven to the subgroups $H_x, H_y \subseteq G$, we conclude that $\gamma \in G \cdot N_{\widehat{G}}(H_x) = G \cdot \widehat{H}_x$, $\gamma \in G \cdot N_{\widehat{G}}(H_y) = G \cdot \widehat{H}_y$. Thus, by projecting to $\widehat{G}^{\mathrm{ab}}$, and applying the fact that M is of *rank two*, we conclude that $\gamma \in G$, as desired. This completes the proof of Theorem 2.6. ◯

Remark 2.6.1. Note that in the situation of Theorem 2.6, if H_G is *abelian*, then — unlike the tempered case discussed in Proposition 2.4! — it is *not necessarily* the case that $F = \gamma^{-1} \cdot F \cdot \gamma$.

Lemma 2.7. (Well-known Properties of Free Groups and Orientable Surface Groups) *Let G be a group as in Theorem 2.6. Write $\widehat{G}$ for the profinite completion of G. Then:*

(i) Any subgroup of G generated by two elements of G is **free***.*

(ii) Let $x \in G$ be an element $\neq 1$. Then there exists a finite index subgroup $G_1 \subseteq G$ such that $x \in G_1$, and x has **nontrivial image** *in the abelianization G_1^{ab} of G_1.*

(iii) Let $x, y \in G$ be **noncommuting** *elements of G. Then there exists a finite index subgroup $G_1 \subseteq G$ and a positive integer n such that $x^n, y^n \in G_1$, and the images of x^n and y^n in the abelianization G_1^{ab} of G_1 generate a* **free abelian subgroup of rank two***.*

(iv) Any abelian subgroup of G is **cyclic***.*

(v) Let $\widehat{T} \subseteq \widehat{G}$ be a closed subgroup such that there exists a continuous surjection of topological groups $\widehat{G} \twoheadrightarrow \widehat{\mathbb{Z}}$ that induces an isomorphism $\widehat{T} \xrightarrow{\sim} \widehat{\mathbb{Z}}$. Then $\widehat{T}$ is **normally terminal** *in $\widehat{G}$.*

(vi) Suppose that G *is* **nonabelian**. *Write* $\widehat{N} \subseteq \widehat{G}$ *for the kernel of the natural surjection* $\widehat{G} \twoheadrightarrow \widehat{G}^{\mathrm{ab}}$ *to the abelianization* $\widehat{G}^{\mathrm{ab}}$ *of* $\widehat{G}$. *Then the* **centralizer** $Z_{\widehat{G}}(\widehat{N})$ *of* $\widehat{N}$ *in* $\widehat{G}$ *is* **trivial**.

(vii) In the notation of (vi), let α *be an automorphism of the profinite group* $\widehat{G}$ *that preserves and restricts to the identity on the subgroup* $\widehat{N}$. *Then* α *is the* **identity** *automorphism of* $\widehat{G}$.

Proof. First, we consider assertion (i). If G is *free*, then assertion (i) follows from the well-known fact that any subgroup of a free group is free. If G is an *orientable surface group*, then assertion (i) follows immediately — i.e., by considering the *noncompact* covering of a compact surface that corresponds to an *infinite index* subgroup of G of the sort discussed in assertion (i) — from a classical result concerning the *fundamental group of a noncompact surface* due to *Johansson* [cf. [Stl], p. 142; the discussion preceding [FRS], Theorem A1]. This completes the proof of assertion (i). Next, we consider assertion (ii). Since G is *residually finite* [cf., e.g., [Config], Proposition 7.1, (ii)], it follows that there exists a finite index normal subgroup $G_0 \subseteq G$ such that $x \notin G_0$. Thus, it suffices to take G_1 to be the subgroup of G generated by G_0 and x. This completes the proof of assertion (ii).

Next, we consider assertion (iii). By applying assertion (i) to the subgroup J of G generated by x and y, it follows from the fact that x and y are *noncommuting* elements of G that J is a *free group of rank* 2, hence that $x^a \cdot y^b \neq 1$, for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $(a, b) \neq (0, 0)$. Next, let us recall the well-known fact that the abelianization of any finite index subgroup of G is *torsion-free*. Thus, by applying assertion (ii) to x and y, we conclude that there exists a finite index subgroup $G_0 \subseteq G$ and a positive integer m such that $x^m, y^m \in G_0$, and x^m and y^m have *nontrivial image* in the abelianization G_0^{ab} of G_0. Now suppose that $x^{ma} \cdot y^{mb}$ lies in the *kernel* of the natural surjection $G_0 \twoheadrightarrow G_0^{\mathrm{ab}}$ for some $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $(a, b) \neq (0, 0)$. Since G is *residually finite*, and [as we observed above] $x^{ma} \cdot y^{mb} \neq 1$, it follows, by applying assertion (ii) to G_0, that there exists a finite index subgroup $G_1 \subseteq G_0$ and a positive integer n that is divisible by m such that $x^n, y^n, x^{ma} \cdot y^{mb} \in G_1$, and the image of $x^{ma} \cdot y^{mb}$ in G_1^{ab} is *nontrivial*. Since G_1^{ab} is *torsion-free*, it thus follows that the image of $x^{na} \cdot y^{nb}$ in G_1^{ab} is *nontrivial*. On the other hand, by considering the natural homomorphism $G_1^{\mathrm{ab}} \to G_0^{\mathrm{ab}}$, we thus conclude that the images of x^n and y^n in G_1^{ab} generate a *free abelian subgroup of rank two*, as desired. This completes the proof of assertion (iii).

Next, we consider assertion (iv). By assertion (i), it follows that any abelian subgroup of G generated by two elements is *free*, hence *cyclic*. In particular, we conclude that any abelian subgroup J of G is equal to the union of the groups that appear in some chain $G_1 \subseteq G_2 \subseteq \ldots \subseteq G$ of cyclic subgroups of G. On the other hand, by applying assertion (ii) to some generator of G_1, it follows that there exists a finite index subgroup G_0 and a positive integer n such that $G_j^n \subseteq G_0$ for all $j = 1, 2, \ldots$, and, moreover, G_1^n has *nontrivial image* in G_0^{ab}. Thus, by considering the image in [the finitely generated abelian group] G_0^{ab} of the chain of cyclic subgroups $G_1^n \subseteq G_2^n \subseteq \ldots$, we conclude that this chain, hence also the original chain $G_1 \subseteq G_2 \subseteq \ldots$, must *terminate*. Thus, J is *cyclic*, as desired. This completes the proof of assertion (iv).

Next, we consider assertion (v). By considering the surjection $\widehat{G} \twoheadrightarrow \widehat{\mathbb{Z}}$, we conclude immediately that the normalizer $N_{\widehat{G}}(\widehat{T})$ of $\widehat{T}$ in $\widehat{G}$ is equal to the centralizer $Z_{\widehat{G}}(\widehat{T})$ of $\widehat{T}$ in $\widehat{G}$. If $Z_{\widehat{G}}(\widehat{T}) \neq \widehat{T}$, then it follows immediately that, for some prime number l, there exists a closed [abelian] subgroup $\widehat{T}_1 \subseteq Z_{\widehat{G}}(\widehat{T})$ containing the pro-l portion of $\widehat{T}$ such that there exists a continuous surjection $\mathbb{Z}_l \times \mathbb{Z}_l \twoheadrightarrow \widehat{T}_1$ whose kernel lies in $l \cdot (\mathbb{Z}_l \times \mathbb{Z}_l)$. In particular, one computes easily that the *l-cohomological dimension* of $\widehat{T}_1$ is ≥ 2. On the other hand, since $\widehat{T}_1$ is of *infinite index* in $\widehat{G}$, it follows immediately that there exists an open subgroup $\widehat{G}_1 \subseteq \widehat{G}$ of $\widehat{G}$ such that $\widehat{T}_1 \subseteq \widehat{G}_1$, and, moreover, there exists a continuous surjection $\phi : \widehat{G}_1 \twoheadrightarrow \mathbb{Z}_l$ whose kernel $\mathrm{Ker}(\phi)$ contains $\widehat{T}_1$. In particular, since the cohomology of $\widehat{T}_1$ may be computed as the direct limit of the cohomologies of open subgroups of $\widehat{G}$ containing $\widehat{T}_1$, it follows immediately from the existence of ϕ, together with the well-known structure of the cohomology of open subgroups of $\widehat{G}$, that the *l-cohomological dimension* of $\widehat{T}_1$ is 1, a contradiction. This completes the proof of assertion (v).

Next, we consider assertion (vi). Write $N \subseteq G$ for the kernel of the natural surjection $G \twoheadrightarrow G^{\mathrm{ab}}$ to the abelianization G^{ab} of G. It follows immediately from the *"tautological universal property"* of a free group or an orientable surface group [i.e., regarded as the quotient of a free group by a single relation] that N is *not cyclic*, hence by assertion (iv), that N is *nonabelian*. Thus, by assertion (iii), there exist a finite index subgroup $G_1 \subseteq G$ equipped with a surjection $\beta : G_1 \twoheadrightarrow \mathbb{Z} \times \mathbb{Z}$ and elements $x, y \in N \bigcap G_1$ such that $\beta(x) = (1,0)$ and $\beta(y) = (0,1)$. In particular, it follows from assertion (v) that the closed subgroups $\widehat{T}_x, \widehat{T}_y \subseteq \widehat{G}$ topologically generated by x and y, respectively, are *normally terminal* in the profinite completion $\widehat{G}_1 \subseteq \widehat{G}$ of G_1. But this implies formally that $Z_{\widehat{G}}(\widehat{N}) \bigcap \widehat{G}_1 \subseteq Z_{\widehat{G}_1}(\widehat{T}_x) \bigcap Z_{\widehat{G}_1}(\widehat{T}_y) \subseteq \widehat{T}_x \bigcap \widehat{T}_y = \{1\}$ [where the last equality follows from the existence of the surjection $\widehat{G}_1 \twoheadrightarrow \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}$ induced by β]. Since [as is well-known] the abelianizations of all open subgroups of $\widehat{G}$ are *torsion-free*, we thus conclude that $Z_{\widehat{G}}(\widehat{N}) = \{1\}$, as desired. This completes the proof of assertion (vi). Finally, we consider assertion (vii). If $x \in \widehat{G}$, $y \in \widehat{N}$ [so $x \cdot y \cdot x^{-1} \in \widehat{N}$], then $x \cdot y \cdot x^{-1} = \alpha(x \cdot y \cdot x^{-1}) = \alpha(x) \cdot \alpha(y) \cdot \alpha(x)^{-1} = \alpha(x) \cdot y \cdot \alpha(x)^{-1}$. We thus conclude from assertion (vi) that $\alpha(x) \cdot x^{-1} \in Z_{\widehat{G}}(\widehat{N}) = \{1\}$, i.e., that $\alpha(x) = x$. This completes the proof of assertion (vii). ◯

Corollary 2.8. (Subgroups of Topological Fundamental Groups of Complex Hyperbolic Curves) *Let Z be a* **hyperbolic curve** *over $\mathbb{C}$. Write Π_Z for the usual topological fundamental group of Z; $\widehat{\Pi}_Z$ for the profinite completion of Π_Z. Let $H \subseteq \Pi_Z$ be an infinite subgroup [such as a* **cuspidal inertia group***!]; $\gamma \in \widehat{\Pi}_Z$ an element such that*

$$\gamma \cdot H \cdot \gamma^{-1} \subseteq \Pi_Z \quad \textit{[or, equivalently, } H \subseteq \gamma^{-1} \cdot \Pi_Z \cdot \gamma\textit{].}$$

Then $\gamma \in \Pi_Z \cdot N_{\widehat{\Pi}_Z}(H)$, i.e., $\gamma \cdot H \cdot \gamma^{-1} = \delta \cdot H \cdot \delta^{-1}$, for some $\delta \in \Pi_Z$. If, moreover, H is **nonabelian***, then $\gamma \in \Pi_Z$.*

Remark 2.8.1. Corollary 2.8 is an immediate consequence of Theorem 2.6. In fact, in the present series of papers, we shall only apply Corollary 2.8 in the case where Z is *non-proper*, and H is a *cuspidal inertia group*. In this case, the proof of Theorem 2.6 may be simplified somewhat, but we chose to include the general version given here, for the sake of completeness.

Section 3: Chains of Θ-Hodge Theaters

In the present §3, we construct *chains of "Θ-Hodge theaters"*. Each "Θ-Hodge theater" is to be thought of as a sort of **miniature model of the conventional scheme-theoretic arithmetic geometry** that surrounds the **theta function**. This miniature model is formulated via the theory of *Frobenioids* [cf. [FrdI]; [FrdII]; [EtTh], §3, §4, §5]. On the other hand, the *link* [cf. Corollary 3.7, (i)] between adjacent members of such chains is *purely Frobenioid-theoretic*, i.e., it lies outside the framework of ring theory/scheme theory. It is these chains of Θ-Hodge theaters that form the *starting point* of the theory of the present series of papers.

Definition 3.1. We shall refer to as *initial* Θ*-data* any collection of data

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$$

that satisfies the following conditions:

(a) F is a *number field* such that $\sqrt{-1} \in F$; $\overline{F}$ is an *algebraic closure* of F. Write $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}/F)$.

(b) X_F is a *once-punctured elliptic curve* [i.e., a hyperbolic curve of type $(1,1)$] over F that admits *stable reduction* over all $v \in \mathbb{V}(F)^{\mathrm{non}}$. Write E_F for the *elliptic curve* over F determined by X_F [so $X_F \subseteq E_F$];

$$X_F \to C_F$$

for the *hyperbolic orbicurve* [cf. §0] over F obtained by forming the stack-theoretic quotient of X_F by the unique F-involution [i.e., automorphism of order two] "-1" of X_F; $F_{\mathrm{mod}} \subseteq F$ for the *field of moduli* [cf., e.g., [AbsTopIII], Definition 5.1, (ii)] of X_F; $F_{\mathrm{sol}} \subseteq \overline{F}$ for the *maximal solvable extension* of F_{mod} in $\overline{F}$; $\mathbb{V}_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \mathbb{V}(F_{\mathrm{mod}})$. Then

$$\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \subseteq \mathbb{V}_{\mathrm{mod}}$$

is a *nonempty set* of nonarchimedean valuations of F_{mod} of *odd* residue characteristic such that X_F has *bad [i.e., multiplicative] reduction* at the elements of $\mathbb{V}(F)$ that lie over $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \subseteq \mathbb{V}_{\mathrm{mod}}$. Write $\mathbb{V}^{\mathrm{good}}_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \mathbb{V}_{\mathrm{mod}} \setminus \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$ [where we note that X_F may in fact have *bad* reduction at some of the elements of $\mathbb{V}(F)$ that lie over $\mathbb{V}^{\mathrm{good}}_{\mathrm{mod}} \subseteq \mathbb{V}_{\mathrm{mod}}$!]; $\mathbb{V}(F)^{\Box} \stackrel{\mathrm{def}}{=} \mathbb{V}^{\Box}_{\mathrm{mod}} \times_{\mathbb{V}_{\mathrm{mod}}} \mathbb{V}(F)$ for $\Box \in \{\mathrm{bad}, \mathrm{good}\}$;

$$\Pi_{X_F} \stackrel{\mathrm{def}}{=} \pi_1(X_F) \subseteq \Pi_{C_F} \stackrel{\mathrm{def}}{=} \pi_1(C_F)$$
$$\Delta_X \stackrel{\mathrm{def}}{=} \pi_1(X_F \times_F \overline{F}) \subseteq \Delta_C \stackrel{\mathrm{def}}{=} \pi_1(C_F \times_F \overline{F})$$

for the *étale fundamental groups* [relative to appropriate choices of basepoints] of X_F, C_F, $X_F \times_F \overline{F}$, $C_F \times_F \overline{F}$. [Thus, we have *natural exact sequences* $1 \to \Delta_{(-)} \to \Pi_{(-)_F} \to G_F \to 1$ for "$(-)$" taken to be either "X" or "C".] Here, we suppose further that the field extension F/F_{mod}

is *Galois* of degree *prime to* l, and that the $2 \cdot 3$-*torsion points* of E_F are *rational* over F.

(c) l is a *prime number* ≥ 5 such that the image of the outer homomorphism

$$G_F \to GL_2(\mathbb{F}_l)$$

determined by the l-*torsion points* of E_F *contains the subgroup* $SL_2(\mathbb{F}_l) \subseteq GL_2(\mathbb{F}_l)$; write $K \subseteq \overline{F}$ for the finite Galois extension of F determined by the kernel of this homomorphism. Also, we suppose that l is *prime* to the [residue characteristics of the] elements of $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$, as well as to the *orders of the* q-*parameters* of E_F [i.e., in the terminology of [GenEll], Definition 3.3, the *"local heights"* of E_F] at the primes of $\mathbb{V}(F)^{\mathrm{bad}}$.

(d) $\underline{C}_K$ is a *hyperbolic orbicurve of type* $(1, l\text{-tors})_{\pm}$ [cf. [EtTh], Definition 2.1] over K, with K-*core* [cf. [CanLift], Remark 2.1.1; [EtTh], the discussion at the beginning of §2] given by $C_K \stackrel{\mathrm{def}}{=} C_F \times_F K$. [Thus, by (c), it follows that $\underline{C}_K$ is *completely determined*, up to isomorphism over F, by C_F.] In particular, $\underline{C}_K$ determines, up to K-isomorphism, a *hyperbolic orbicurve* $\underline{X}_K$ *of type* $(1, l\text{-tors})$ [cf. [EtTh], Definition 2.1] over K, together with *natural cartesian diagrams*

$$\begin{array}{ccc} \underline{X}_K & \longrightarrow & X_F \\ \downarrow & & \downarrow \\ \underline{C}_K & \longrightarrow & C_F \end{array} \qquad \begin{array}{ccc} \Pi_{\underline{X}_K} & \longrightarrow & \Pi_{X_F} \\ \downarrow & & \downarrow \\ \Pi_{\underline{C}_K} & \longrightarrow & \Pi_{C_F} \end{array} \qquad \begin{array}{ccc} \Delta_{\underline{X}} & \longrightarrow & \Delta_X \\ \downarrow & & \downarrow \\ \Delta_{\underline{C}} & \longrightarrow & \Delta_C \end{array}$$

of finite étale coverings of hyperbolic orbicurves and corresponding open immersions of profinite groups. Finally, we recall from [EtTh], Proposition 2.2, that $\Delta_{\underline{C}}$ admits *uniquely determined* open subgroups $\Delta_{\underline{\underline{X}}} \subseteq \Delta_{\underline{\underline{C}}} \subseteq \Delta_{\underline{C}}$, which may be thought of as corresponding to finite étale coverings of $\underline{C}_{\overline{F}} \stackrel{\mathrm{def}}{=} \underline{C} \times_F \overline{F}$ by *hyperbolic orbicurves* $\underline{\underline{X}}_{\overline{F}}$, $\underline{\underline{C}}_{\overline{F}}$ *of type* $(1, l\text{-tors}^{\Theta})$, $(1, l\text{-tors}^{\Theta})_{\pm}$, respectively [cf. [EtTh], Definition 2.3].

(e) $\underline{\mathbb{V}} \subseteq \mathbb{V}(K)$ is a subset that induces a *natural bijection*

$$\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$$

— i.e., a *section* of the natural surjection $\mathbb{V}(K) \twoheadrightarrow \mathbb{V}_{\mathrm{mod}}$. Write $\underline{\mathbb{V}}^{\mathrm{non}} \stackrel{\mathrm{def}}{=} \underline{\mathbb{V}} \bigcap \mathbb{V}(K)^{\mathrm{non}}$, $\underline{\mathbb{V}}^{\mathrm{arc}} \stackrel{\mathrm{def}}{=} \underline{\mathbb{V}} \bigcap \mathbb{V}(K)^{\mathrm{arc}}$, $\underline{\mathbb{V}}^{\mathrm{good}} \stackrel{\mathrm{def}}{=} \underline{\mathbb{V}} \bigcap \mathbb{V}(K)^{\mathrm{good}}$, $\underline{\mathbb{V}}^{\mathrm{bad}} \stackrel{\mathrm{def}}{=} \underline{\mathbb{V}} \bigcap \mathbb{V}(K)^{\mathrm{bad}}$. For each $\underline{v} \in \mathbb{V}(K)$, we shall use the *subscript* $\underline{v}$ to denote the result of base-changing hyperbolic orbicurves over F or K to $K_{\underline{v}}$. Thus, for each $\underline{v} \in \mathbb{V}(K)$ lying under a $\overline{v} \in \mathbb{V}(\overline{F})$, we have *natural cartesian diagrams*

$$\begin{array}{ccccc} \underline{\underline{X}}_{\overline{v}} & \longrightarrow & \underline{X}_{\underline{v}} & \longrightarrow & X_{\underline{v}} \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\underline{C}}_{\overline{v}} & \longrightarrow & \underline{C}_{\underline{v}} & \longrightarrow & C_{\underline{v}} \end{array} \qquad \begin{array}{ccccc} \Delta_{\underline{\underline{X}}} & \longrightarrow & \Pi_{\underline{X}_{\underline{v}}} & \longrightarrow & \Pi_{X_{\underline{v}}} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{\underline{\underline{C}}} & \longrightarrow & \Pi_{\underline{C}_{\underline{v}}} & \longrightarrow & \Pi_{C_{\underline{v}}} \end{array}$$

of profinite étale coverings of hyperbolic orbicurves and corresponding injections of profinite groups [i.e., *étale fundamental groups*]. Here, the *subscript* $\overline{v}$ denotes base-change with respect to $\overline{F} \hookrightarrow \overline{F}_{\overline{v}}$; the various profinite groups "$\Pi_{(-)}$" admit *natural outer surjections* onto the decomposition group $G_{\underline{v}} \subseteq G_K \stackrel{\text{def}}{=} \text{Gal}(\overline{F}/K)$ determined, up to G_K-conjugacy, by $\underline{v}$. If $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, then we assume further that the hyperbolic orbicurve $\underline{C}_{\underline{v}}$ is *of type* $(1, \mathbb{Z}/l\mathbb{Z})_{\pm}$ [cf. [EtTh], Definition 2.5, (i)]. [Here, we note that it follows from the portion of (b) concerning 2-*torsion points* that the base field $K_{\underline{v}}$ satisfies the assumption "$K = \ddot{K}$" of [EtTh], Definition 2.5, (i).] Finally, we observe that when $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, it follows from the theory of [EtTh], §2 — i.e., roughly speaking, *"by extracting an l-th root of the theta function"* — that $\underline{\underline{X}}_{\overline{v}}$, $\underline{\underline{C}}_{\overline{v}}$ admit *natural models*

$$\underline{\underline{X}}_{\underline{v}}, \quad \underline{\underline{C}}_{\underline{v}}$$

over $K_{\underline{v}}$, which are hyperbolic orbicurves *of type* $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$, $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})_{\pm}$, respectively [cf. [EtTh], Definition 2.5, (i)]; these models determine open subgroups $\Pi_{\underline{\underline{X}}_{\underline{v}}} \subseteq \Pi_{\underline{\underline{C}}_{\underline{v}}} \subseteq \Pi_{\underline{C}_{\underline{v}}}$. If $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, then, relative to the notation of Remark 3.1.1 below, we shall write $\Pi_{\underline{v}} \stackrel{\text{def}}{=} \Pi^{\text{tp}}_{\underline{\underline{X}}_{\underline{v}}}$.

(f) $\underline{\epsilon}$ is a *cusp* of the hyperbolic orbicurve $\underline{C}_K$ [cf. (d)] that arises from a *nonzero element* of the quotient "Q" that appears in the definition of a "hyperbolic orbicurve of type $(1, l\text{-tors})_{\pm}$" given in [EtTh], Definition 2.1. If $\underline{v} \in \underline{\mathbb{V}}$, then let us write $\underline{\epsilon}_{\underline{v}}$ for the cusp of $\underline{C}_{\underline{v}}$ determined by $\underline{\epsilon}$. If $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, then we assume that $\underline{\epsilon}_{\underline{v}}$ is the cusp that arises from the *canonical generator* [up to sign] "± 1" of the quotient "$\widehat{\mathbb{Z}}$" that appears in the definition of a "hyperbolic orbicurve of type $(1, \mathbb{Z}/l\mathbb{Z})_{\pm}$" given in [EtTh], Definition 2.5, (i). Thus, the data $(X_K \stackrel{\text{def}}{=} X_F \times_F K, \underline{C}_K, \underline{\epsilon})$ determines *hyperbolic orbicurves*

$$\underset{\rightarrow}{X}_K, \quad \underset{\rightarrow}{C}_K$$

of type $(1, l\text{-}\underrightarrow{\text{tors}})$, $(1, l\text{-}\underrightarrow{\text{tors}})_{\pm}$, respectively [cf. Definition 1.1, Remark 1.1.2], as well as open subgroups $\Pi_{\underset{\rightarrow}{X}_K} \subseteq \Pi_{\underset{\rightarrow}{C}_K} \subseteq \Pi_{C_F}$, $\Delta_{\underset{\rightarrow}{X}} \subseteq \Delta_{\underset{\rightarrow}{C}} \subseteq \Delta_C$, and, for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$, $\Pi_{\underset{\rightarrow}{X}_{\underline{v}}} \subseteq \Pi_{\underset{\rightarrow}{C}_{\underline{v}}} \subseteq \Pi_{C_{\underline{v}}}$. If $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$, then we shall write $\Pi_{\underline{v}} \stackrel{\text{def}}{=} \Pi_{\underset{\rightarrow}{X}_{\underline{v}}}$.

Remark 3.1.1. Relative to the notation of Definition 3.1, (e), suppose that $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$. Then in addition to the various profinite groups $\Pi_{(-)_{\underline{v}}}$, $\Delta_{(-)}$, one also has corresponding *tempered fundamental groups*

$$\Pi^{\text{tp}}_{(-)_{\underline{v}}}; \quad \Delta^{\text{tp}}_{(-)_{\underline{v}}}$$

[cf. [André], §4; [SemiAnbd], Example 3.10], whose profinite completions may be identified with $\Pi_{(-)_{\underline{v}}}$, $\Delta_{(-)}$. Here, we note that unlike "$\Delta_{(-)}$", the topological group $\Delta^{\text{tp}}_{(-)_{\underline{v}}}$ *depends, a priori, on* $\underline{v}$.

Remark 3.1.2.

(i) Observe that the open subgroup $\Pi_{\underline{X}_K} \subseteq \Pi_{\underline{C}_K}$ may be *constructed group-theoretically from the topological group* $\Pi_{\underline{C}_K}$. Indeed, it follows immediately from the construction of the coverings "$\underline{X}$", "$\underline{C}$" in the discussion at the beginning of [EtTh], §2 [cf. also [AbsAnab], Lemma 1.1.4, (i)], that the closed subgroup $\Delta_{\underline{X}} \subseteq \Pi_{\underline{C}_K}$ may be characterized by a rather simple explicit algorithm. Since the decomposition groups of $\Pi_{\underline{C}_K}$ at the *nonzero cusps* — i.e., the cusps whose inertia groups are contained in $\Delta_{\underline{X}}$ [cf. the discussion at the beginning of §1] — are also *group-theoretic* [cf., e.g., [AbsTopI], Lemma 4.5, as well as Remark 1.2.2, (ii), of the present paper], the above observation follows immediately from the easily verified fact that the image of any of these decomposition groups associated to nonzero cusps coincides with the image of $\Pi_{\underline{X}_K}$ in $\Pi_{\underline{C}_K}/\Delta_{\underline{X}}$.

(ii) In light of the observation of (i), it makes sense to adopt the following convention:

> Instead of applying the *group-theoretic reconstruction algorithm* of [AbsTopIII], Theorem 1.9 [cf. also the discussion of [AbsTopIII], Remark 2.8.3], directly to $\Pi_{\underline{C}_K}$ [or topological groups isomorphic to $\Pi_{\underline{C}_K}$], we shall apply this reconstruction algorithm to the *open subgroup* $\Pi_{\underline{X}_K} \subseteq \Pi_{\underline{C}_K}$ to reconstruct the *function field of* $\underline{X}_K$, equipped with its *natural* $\mathrm{Gal}(\underline{X}_K/\underline{C}_K) \cong \Pi_{\underline{C}_K}/\Pi_{\underline{X}_K}$*-action.*

In this context, we shall refer to this approach of applying [AbsTopIII], Theorem 1.9, as the **Θ-approach** to [AbsTopIII], Theorem 1.9. Note that, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ (respectively, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$), one may also adopt a "Θ-approach" to applying [AbsTopIII], Theorem 1.9, to $\Pi_{\underline{C}_{\underline{v}}}$ or [by applying Corollary 1.2] $\Pi_{\underrightarrow{X}_{\underline{v}}}$, $\Pi_{\underrightarrow{C}_{\underline{v}}}$ (respectively, to $\Pi^{\mathrm{tp}}_{\underline{C}_{\underline{v}}}$ or [by applying [EtTh], Proposition 2.4] $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}}$). In the present series of papers, we shall always think of [AbsTopIII], Theorem 1.9 [as well as the other results of [AbsTopIII] that arise as consequences of [AbsTopIII], Theorem 1.9] as being applied to [isomorphs of] $\Pi_{\underline{C}_K}$ or, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ (respectively, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$), $\Pi_{\underline{C}_{\underline{v}}}$, $\Pi_{\underrightarrow{X}_{\underline{v}}}$, $\Pi_{\underrightarrow{C}_{\underline{v}}}$ (respectively, $\Pi^{\mathrm{tp}}_{\underline{C}_{\underline{v}}}$, $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}}$) via the "$\Theta$-approach" [cf. also Remark 3.4.3, (i), below].

(iii) Recall from the discussion at the beginning of [EtTh], §2, the *tautological extension*

$$1 \to \Delta_\Theta \to \Delta_X^\Theta \to \Delta_X^{\mathrm{ell}} \to 1$$

— where $\Delta_\Theta \stackrel{\mathrm{def}}{=} [\Delta_X, \Delta_X]/[\Delta_X, [\Delta_X, \Delta_X]]$; $\Delta_X^\Theta \stackrel{\mathrm{def}}{=} \Delta_X/[\Delta_X, [\Delta_X, \Delta_X]]$; $\Delta_X^{\mathrm{ell}} \stackrel{\mathrm{def}}{=} \Delta_X^{\mathrm{ab}}$. The extension class $\in H^2(\Delta_X^{\mathrm{ell}}, \Delta_\Theta)$ of this extension determines a *tautological isomorphism*

$$M_X \xrightarrow{\sim} \Delta_\Theta$$

— where we recall from [AbsTopIII], Theorem 1.9, (b), that the module "M_X" of [AbsTopIII], Theorem 1.9, (b) [cf. also [AbsTopIII], Proposition 1.4, (ii); [AbsTopIII], Remark 1.10.1, (ii)], may be *naturally identified* with $\mathrm{Hom}(H^2(\Delta_X^{\mathrm{ell}}, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}})$. In particular, we obtain a *tautological isomorphism*

$$M_{\underline{X}} \xrightarrow{\sim} (l \cdot \Delta_\Theta)$$

[i.e., since $[\Delta_X : \Delta_{\underline{X}}] = l$]. In particular, we observe that if we write $\Pi_{C_{F_{\mathrm{mod}}}}$ for the étale fundamental group of the orbicurve $C_{F_{\mathrm{mod}}}$ discussed in Remark 3.1.7, (i), below, then $M_{\underline{X}} \stackrel{\sim}{\to} (l \cdot \Delta_\Theta)$ may be regarded as a *characteristic subquotient* of $\Pi_{C_{F_{\mathrm{mod}}}}$, hence admits a *natural conjugation action* by $\Pi_{C_{F_{\mathrm{mod}}}}$. From the point of view of the theory of the present series of papers, the **significance** *of the "Θ-approach"* lies precisely in the existence of this tautological isomorphism $M_{\underline{X}} \stackrel{\sim}{\to} (l \cdot \Delta_\Theta)$, which will be applied in [IUTchII] at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. That is to say, the Θ-approach involves applying the reconstruction algorithm of [AbsTopIII], Theorem 1.9, via the cyclotome $M_{\underline{X}}$, which may be identified, via the above tautological isomorphism, with the cyclotome $(l \cdot \Delta_\Theta)$, which plays a central role in the theory of [EtTh] — cf., especially, the discussion of *"cyclotomic rigidity"* in [EtTh], Corollary 2.19, (i).

(iv) If one thinks of the prime number l as being *"large"*, then the role played by the covering $\underline{X}$ in the above discussion of the *"Θ-approach"* is reminiscent of the role played by the *universal covering of a complex elliptic curve by the complex plane* in the holomorphic reconstruction theory of [AbsTopIII], §2 [cf., e.g., [AbsTopIII], Propositions 2.5, 2.6].

Remark 3.1.3. Since $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \neq \emptyset$ [cf. Definition 3.1, (b)], it follows immediately from Definition 3.1, (d), (e), (f), that the data $(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$ is, in fact, *completely determined* by the data $(\overline{F}/F,\ X_F,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}})$, and that $\underline{C}_K$ is *completely determined up to K-isomorphism* by the data $(\overline{F}/F,\ X_F,\ l,\ \underline{\mathbb{V}})$. Finally, we remark that for given data $(X_F,\ l,\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}})$, distinct choices of "$\underline{\mathbb{V}}$" will not affect the theory in any significant way.

Remark 3.1.4. It follows immediately from the definitions that at each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [which is necessarily *prime to l* — cf. Definition 3.1, (c)] (respectively, each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ which is *prime to l*; each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$), $\underline{\underline{X}}_{\underline{v}}$ (respectively, $\underline{X}_{\rightarrow \underline{v}}$; $\underline{X}_{\underline{v}}$) admits a *stable model* over the ring of integers of $K_{\underline{v}}$.

Remark 3.1.5. Note that since the 3-*torsion points* of E_F are *rational* over F, and F is *Galois* over F_{mod} [cf. Definition 3.1, (b)], it follows [cf., e.g., [IUTchIV], Proposition 1.8, (iv)] that K is *Galois* over F_{mod}. In addition to working with the *field* F_{mod} and various extensions of F_{mod} contained in $\overline{F}$, we shall also have occasion to work with the *algebraic stack*

$$S_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathcal{O}_K) \ // \ \mathrm{Gal}(K/F_{\mathrm{mod}})$$

obtained by forming the *stack-theoretic quotient* [i.e., "//"] of the spectrum of the ring of integers $\mathcal{O}_K$ of K by the Galois group $\mathrm{Gal}(K/F_{\mathrm{mod}})$. Thus, any finite extension $L \subseteq \overline{F}$ of F_{mod} in $\overline{F}$ determines, by forming the integral closure of S_{mod} in L, an algebraic stack $S_{\mathrm{mod},L}$ over S_{mod}. In particular, by considering *arithmetic line bundles* over such $S_{\mathrm{mod},L}$, one may associate to any finite quotient $\mathrm{Gal}(\overline{F}/F_{\mathrm{mod}}) \twoheadrightarrow Q$ a *Frobenioid* via [the easily verified *"stack-theoretic version"* of] the construction of [FrdI], Example 6.3. One verifies immediately that an appropriate analogue of [FrdI], Theorem 6.4, holds for such stack-theoretic versions of the Frobenioids constructed in [FrdI], Example 6.3. Also, we observe that upon passing to either the *perfection* or the *realification*, such stack-theoretic versions become *naturally isomorphic* to the non-stack-theoretic versions [i.e., of [FrdI], Example 6.3, as stated].

Remark 3.1.6. In light of the important role played by the various orbicurves constructed in [EtTh], §2, in the present series of papers, we take the opportunity to correct an unfortunate — albeit in fact *irrelevant*! — error in [EtTh]. In the discussion preceding [EtTh], Definition 2.1, one must in fact assume that the integer l is *odd* in order for the quotient $\overline{\Delta}_X$ to be *well-defined*. Since, ultimately, in [EtTh] [cf. the discussion following [EtTh], Remark 5.7.1], as well as in the present series of papers, this is the only case that is of interest, this oversight does not affect either the present series of papers or the bulk of the remainder of [EtTh]. Indeed, the only places in [EtTh] where the case of *even* l is used are [EtTh], Remark 2.2.1, and the application of [EtTh], Remark 2.2.1, in the proof of [EtTh], Proposition 2.12, for the orbicurves "$\underline{\underline{\dot{C}}}$". Thus, [EtTh], Remark 2.2.1, must be *deleted*; in [EtTh], Proposition 2.12, one must in fact *exclude* the case where the orbicurve under consideration is "$\underline{\underline{\dot{C}}}$". On the other hand, this theory involving [EtTh], Proposition 2.12 [cf., especially, [EtTh], Corollaries 2.18, 2.19] is only applied *after* the discussion following [EtTh], Remark 5.7.1, i.e., which only treats the curves "$\underline{\underline{X}}$". That is to say, ultimately, in [EtTh], as well as in the present series of papers, one is only interested in the curves "$\underline{\underline{X}}$", whose treatment only requires the case of *odd* l.

Remark 3.1.7.

(i) Observe that it follows immediately from the definition of F_{mod} and the K-*coricity* of C_K [cf. Definition 3.1, (b), (d)] that C_F admits a *unique* [up to unique isomorphism] *model*

$$C_{F_{\mathrm{mod}}}$$

over F_{mod}. If $v \in \mathbb{V}_{\mathrm{mod}}$, then we shall write C_v for the result of base-changing this model to $(F_{\mathrm{mod}})_v$. When applying the *group-theoretic reconstruction algorithm* of [AbsTopIII], Theorem 1.9 [cf. Remark 3.1.2, (ii)], it will frequently be useful to consider certain **special types of rational functions** on $C_{F_{\mathrm{mod}}}$ and C_v, as follows. Let L be a *field* which is equal either to F_{mod} or to $(F_{\mathrm{mod}})_v$ for some $v \in \mathbb{V}_{\mathrm{mod}}$. Write C_L for the model just discussed of C_F over L. Thus, one verifies immediately that the *coarse space* $|C_L|$ associated to the algebraic stack C_L is isomorphic to the **affine line** over L. Now suppose that we are given an *algebraic closure* $\overline{L}_C$ of the *function field* L_C of C_L. Write $\overline{L}$ for the algebraic closure of L determined by $\overline{L}_C$. We shall refer to a closed point of the proper smooth curve determined by some finite subextension $\subseteq \overline{L}_C$ of L_C as a *critical point* if it maps to a closed point of the [proper smooth] compactification $|C_L|^{\mathrm{cpt}}$ of $|C_L|$ that arises from one of the 2-*torsion points* of E_F; we shall refer to a critical point which does *not* map to the closed point of $|C_L|^{\mathrm{cpt}}$ that arises from the unique *cusp* of C_L as *strictly critical*. Thus, as one might imagine from the *central importance of* 2-*torsion points in the elementary theory of elliptic curves*, the strictly critical points of $|C_L|^{\mathrm{cpt}}$ may be thought of as the **"most fundamental/canonical non-cuspidal points"** of $|C_L|^{\mathrm{cpt}}$. We shall refer to a rational function $f \in L_C$ on C_L as κ-*coric* — where we think of the κ as standing for **"Kummer"** — if

- whenever $f \notin L$, it holds that, over $\overline{L}$, f has *precisely one pole* [of unrestricted order], but *at least two distinct zeroes*;
- the divisor of *zeroes and poles* of f is *defined over a number field* and *avoids* the *critical points*;
- f restricts to a *root of unity* at *every strictly critical* point of $|C_L|^{\text{cpt}}$.

Thus, the first displayed condition, taken together with the latter portion of the second displayed condition, may be understood as the condition that *there exist a unique non-critical L-rational point* of $|C_L|^{\text{cpt}}$ with respect to which [i.e., if one takes this L-rational point to be the "point at infinity"] f may be thought of as a **polynomial** on the **affine line** over L with *non-critical zeroes*. In particular, it follows from the first displayed condition that, whenever $f \notin L$, it is *never* the case that *both* f and f^{-1} are *κ-coric*. By contrast, the third displayed condition may be understood as the condition that *restriction to the strictly critical* points determines a sort of **canonical splitting up to roots of unity** [which will play an *important role* in the present series of papers — cf., e.g., the discussion of Example 5.1, (v); Definition 5.2, (vi), (viii); Remark 5.2.3, below] of the set of nonzero constant [i.e., L-] multiples of κ-coric functions into a direct product, up to roots of unity, of the set of κ-coric functions and the set of nonzero elements of L. In particular, it follows from the third displayed condition that if $c \in L$ and $f \in L_C$ are such that *both* f and $c \cdot f$ are *κ-coric*, then c is a *root of unity*.

(ii) We maintain the notation of (i). Let $L^{\square}$ be an *intermediate field* between L and $\overline{L}$ that is *solvably closed* [cf. [GlSol], Definition 1, (i)], i.e., has no nontrivial abelian extensions. Observe that, since $|C_L|^{\text{cpt}}$ has *precisely* 4 *critical points*, it follows immediately from the **elementary theory of polynomial functions on the affine line over** L [i.e., the complement in $|C_L|^{\text{cpt}}$ of some L-rational point $|C_L|^{\text{cpt}}$] that there exists a *κ-coric* $f_{\text{sol}} \in L_C$ [i.e., a rational function on the affine line over L] of degree 4. In particular, it follows immediately *from the elementary theory of polynomial functions on the affine line [i.e., $|C_L|$] over L* [together with *"Hensel's lemma"* — cf., e.g., the method of proof of [AbsTopII], Lemma 2.1] (respectively, from the existence of f_{sol} [together with the well-known fact that the *symmetric group on* 4 *letters* is *solvable*]) that

> every element of L (respectively, $L^{\square}$) appears as a value of some *κ-coric* rational function on C_L at some L- (respectively, $L^{\square}$-) valued point of C_L that is *not critical*.

If $L = F_{\text{mod}}$, then write $\mathcal{U}_{\overline{L}}$ for the group $\overline{L}^{\times}$ of nonzero elements of $\overline{L}$; if $L = (F_{\text{mod}})_v$ for some $v \in \mathbb{V}_{\text{mod}}$, then write $\mathcal{U}_{\overline{L}}$ for the group of units [i.e., relative to the unique valuation on $\overline{L}$ that extends v] of $\overline{L}$. We shall say that an element $f \in \overline{L}_C$ is *${}_\infty\kappa$-coric* if there exists a positive integer n such that f^n is a *κ-coric* element of L_C; we shall say that an element $f \in \overline{L}_C$ is *${}_\infty\kappa\times$-coric* if there exists an element $c \in \mathcal{U}_{\overline{L}}$ such that $c \cdot f \in \overline{L}_C$ is *${}_\infty\kappa$-coric*. Thus, an element $f \in L_C$ is *κ-coric* if and only if it is *${}_\infty\kappa$-coric*. Also, one verifies immediately that

> an *${}_\infty\kappa\times$-coric* element $f \in \overline{L}_C$ is *${}_\infty\kappa$-coric* if and only if it restricts to a *root of unity* at *some* [or, equivalently, *every*] *strictly critical* point of the proper smooth curve determined by some finite subextension $\subseteq \overline{L}_C$ of the function field L_C that contains f.

Finally, one verifies immediately that the operation of multiplication determines a structure of *pseudo-monoid* [cf. §0] on the sets of κ-, ${}_\infty\kappa$-, and ${}_\infty\kappa\times$-coric rational functions; moreover, in the case of ${}_\infty\kappa$- and ${}_\infty\kappa\times$-coric rational functions, the resulting pseudo-monoid is *divisible* and *cyclotomic*. These pseudo-monoids will be of use in discussions concerning the **Kummer theory** of rational functions on C_L [cf. Example 5.1, (i), (v); Definition 5.2, (v), (vi), (vii), (viii), below].

(iii) We maintain the notation of (i) and (ii) and assume further that $L = F_{\mathrm{mod}}$, $\overline{L} = \overline{F}$. We shall say that an element $f \in \overline{L}_C$ is *κ-solvable* if it is an $F^{\times}_{\mathrm{sol}}$-multiple [cf. Definition 3.1, (b)] of a ${}_\infty\kappa$-coric element of $\overline{L}_C$. Thus, one verifies immediately that an element $f \in \overline{L}_C$ is *κ-solvable* if and only if there exists a positive integer n such that f^n is a ${}_\infty\kappa\times$-coric element of $F_{\mathrm{sol}} \cdot L_C$. Write $F(\boldsymbol{\mu}_l) \subseteq K$ for the subextension of K generated by the l-th roots of unity; $L_C(\kappa\text{-sol}) \subseteq \overline{L}_C$ for the subfield of $\overline{L}_C$ generated by the *κ-solvable* elements of $\overline{L}_C$; $L_C(\underline{C}_K) \subseteq \overline{L}_C$ for the subfield of $\overline{L}_C$ generated over L_C by the images of the $F(\boldsymbol{\mu}_l)\cdot L_C$-linear embeddings into $\overline{L}_C$ of the *function field* of $\underline{C}_K$. Thus, the fact that the extension F/F_{mod} is *Galois* of degree *prime to l* [cf. Definition 3.1, (b)] implies that

> the subgroup $\mathrm{Gal}(K/F(\boldsymbol{\mu}_l)) \subseteq \mathrm{Gal}(K/F_{\mathrm{mod}})$ is *normal* and may be characterized as the **unique subgroup** of $\mathrm{Gal}(K/F_{\mathrm{mod}})$ that is [abstractly] **isomorphic to** $SL_2(\mathbb{F}_l)$

[cf. Remark 3.1.5; [GenEll], Lemma 3.1, (i)]. Moreover, we observe that it follows immediately from the well-known fact that the finite group $SL_2(\mathbb{F}_l)$ is *perfect* [cf. Definition 3.1, (c); [GenEll], Lemma 3.1, (ii)], together with the definition of the term "*${}_\infty\kappa\times$-coric*" [cf., especially, the fact that the *zeroes and poles avoid the critical points*!], that

> the subfields $L_C(\underline{C}_K) \subseteq \overline{L}_C \supseteq F(\boldsymbol{\mu}_l) \cdot L_C(\kappa\text{-sol})$ are **linearly disjoint** over $F(\boldsymbol{\mu}_l) \cdot L_C$.

In particular, it follows that there is a *natural isomorphism*

$$\mathrm{Gal}(L_C(\underline{C}_K)/F(\boldsymbol{\mu}_l) \cdot L_C) \xrightarrow{\sim} \mathrm{Gal}(L_C(\underline{C}_K) \cdot L_C(\kappa\text{-sol})/F(\boldsymbol{\mu}_l) \cdot L_C(\kappa\text{-sol}))$$

— i.e., one may regard $\mathrm{Gal}(L_C(\underline{C}_K)/F(\boldsymbol{\mu}_l) \cdot L_C)$ as being equipped with an *action on $L_C(\underline{C}_K) \cdot L_C(\kappa\text{-sol})$* that *restricts to the trivial action* on $F(\boldsymbol{\mu}_l) \cdot L_C(\kappa\text{-sol})$.

(iv) We maintain the notation of (iii). In the following, we shall write "$\mathrm{Out}(-)$" for the group of outer automorphisms of the topological group in parentheses. Consider the *tautological exact sequence of Galois groups*

$$1 \to \mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol})) \to \mathrm{Gal}(\overline{L}_C/L_C) \to \mathrm{Gal}(L_C(\kappa\text{-sol})/L_C) \to 1$$

[cf. the discussion of (iii)]. Let us refer to a subgroup of $\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))$ as a κ-sol-*open subgroup* if it is the intersection with $\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))$ of a normal open subgroup of $\mathrm{Gal}(\overline{L}_C/L_C)$. Thus, the subgroups

$$\mathrm{Aut}^{\kappa\text{-sol}}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))) \subseteq \mathrm{Aut}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol})))$$
$$\mathrm{Out}^{\kappa\text{-sol}}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))) \subseteq \mathrm{Out}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol})))$$

of automorphisms/outer automorphisms of the topological group $\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))$ that preserve each κ-sol-open subgroup — i.e., of *"κ-sol-automorphisms/κ-sol-outer automorphisms"* — admit *natural compatible homomorphisms*

$$\mathrm{Aut}^{\kappa\text{-sol}}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))) \to \mathrm{Aut}(Q)$$
$$\mathrm{Out}^{\kappa\text{-sol}}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))) \to \mathrm{Out}(Q)$$

for each quotient $\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol})) \twoheadrightarrow Q$ by a κ-sol-open subgroup. The kernels of these natural homomorphisms [for varying "Q"] determine *natural profinite topologies* on $\mathrm{Aut}^{\kappa\text{-sol}}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol})))$, $\mathrm{Out}^{\kappa\text{-sol}}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol})))$, with respect to which each arrow of the *commutative diagram of homomorphisms*

$$\begin{array}{ccc}
\mathrm{Gal}(\overline{L}_C/L_C) & \longrightarrow & \mathrm{Aut}^{\kappa\text{-sol}}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))) \\
\downarrow & & \downarrow \\
\mathrm{Gal}(L_C(\kappa\text{-sol})/L_C) & \longrightarrow & \mathrm{Out}^{\kappa\text{-sol}}(\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol})))
\end{array}$$

that arises, via conjugation, from the *exact sequence* considered above is *continuous*. Finally, we observe that

> $\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))$ is **center-free**; in particular, the above commutative diagram of homomorphisms of topological groups is **cartesian**.

Indeed, let us first observe that it follows immediately from the definitions that $\mathrm{Gal}(\overline{F}\cdot L_C(\kappa\text{-sol})/\overline{F}\cdot L_C)$ is *abelian*. Thus, it follows formally, by applying Lemma 2.7, (vi), (vii), to the *geometric fundamental groups* of the various *genus zero affine hyperbolic curves* whose *function field* is equal to L_C, that the *conjugacy action* by any element α in the *center* of $\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))$ on such a [center-free] geometric fundamental group is *trivial* and hence, by [the special case that was already known to *Belyi* of] the *Galois injectivity* result discussed in [NodNon], Theorem C, that α is the identity element of $\mathrm{Gal}(\overline{L}_C/L_C(\kappa\text{-sol}))$, as desired.

Given *initial Θ-data* as in Definition 3.1, the theory of Frobenioids given in [FrdI], [FrdII], [EtTh] allows one to construct various *associated Frobenioids*, as follows.

Example 3.2. Frobenioids at Bad Nonarchimedean Primes. Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}} = \underline{\mathbb{V}} \bigcap \mathbb{V}(K)^{\mathrm{bad}}$. The theory of the *"Frobenioid-theoretic theta function"* discussed in [EtTh], §5, may be thought of as a sort of *formal, category-theoretic* way to formulate various *elementary classical facts* [which are reviewed in [EtTh], §1] concerning the theory of the *line bundles* and *divisors* related to the *classical theta function* on a *Tate curve* over an MLF. We give a brief review of this theory of [EtTh], §5, as follows:

(i) By the theory of [EtTh], the hyperbolic curve $\underline{\underline{X}}_{\underline{v}}$ determines a *tempered Frobenioid*

$$\underline{\underline{\mathcal{F}}}_{\underline{v}}$$

[i.e., the Frobenioid denoted "$\mathcal{C}$" in the discussion at the beginning of [EtTh], §5; cf. also the discussion of Remark 3.2.4 below] over a *base category*

$$\mathcal{D}_{\underline{v}}$$

[i.e., the category denoted "$\mathcal{D}$" in the discussion at the beginning of [EtTh], §5]. We recall from the theory of [EtTh] that $\mathcal{D}_{\underline{v}}$ may be thought of as the *category of connected tempered coverings* — i.e., "$\mathcal{B}^{\text{temp}}(\underline{\underline{X}}_{\underline{v}})^0$" in the notation of [EtTh], Example 3.9 — of the hyperbolic curve $\underline{\underline{X}}_{\underline{v}}$. In the following, we shall write

$$\mathcal{D}^{\vdash}_{\underline{v}} \stackrel{\text{def}}{=} \mathcal{B}(K_{\underline{v}})^0$$

[cf. the notational conventions concerning categories discussed in §0]. Also, we observe that $\mathcal{D}^{\vdash}_{\underline{v}}$ may be naturally regarded [by pulling back finite étale coverings via the structure morphism $\underline{\underline{X}}_{\underline{v}} \to \text{Spec}(K_{\underline{v}})$] as a *full subcategory*

$$\mathcal{D}^{\vdash}_{\underline{v}} \subseteq \mathcal{D}_{\underline{v}}$$

of $\mathcal{D}_{\underline{v}}$, and that we have a *natural functor* $\mathcal{D}_{\underline{v}} \to \mathcal{D}^{\vdash}_{\underline{v}}$, which is *left-adjoint* to the natural inclusion functor $\mathcal{D}^{\vdash}_{\underline{v}} \hookrightarrow \mathcal{D}_{\underline{v}}$ [cf. [FrdII], Example 1.3, (ii)]. If $(-)$ is an object of $\mathcal{D}_{\underline{v}}$, then we shall denote by "$\mathbb{T}_{(-)}$" the *Frobenius-trivial object* [a notion which is *category-theoretic* — cf. [FrdI], Definition 1.2, (iv); [FrdI], Corollary 4.11, (iv); [EtTh], Proposition 5.1] of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [which is completely determined up to isomorphism] that lies over "$(-)$".

(ii) Next, let us recall [cf. [EtTh], Proposition 5.1; [FrdI], Corollary 4.10] that the *birationalization*

$$\underline{\underline{\mathcal{F}}}^{\div}_{\underline{v}} \stackrel{\text{def}}{=} \underline{\underline{\mathcal{F}}}^{\text{birat}}_{\underline{v}}$$

may be *reconstructed category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [cf. Remark 3.2.1 below]. Write

$$\underline{\underline{\ddot{Y}}}_{\underline{v}} \to \underline{\underline{X}}_{\underline{v}}$$

for the tempered covering determined by the object "$\underline{\underline{\ddot{Y}}}^{\log}$" in the discussion at the beginning of [EtTh], §5. Thus, we may think of $\underline{\underline{\ddot{Y}}}_{\underline{v}}$ as an object of $\mathcal{D}_{\underline{v}}$ [cf. the object "$A^{\text{bs}}_{\circledcirc}$" of [EtTh], §5, in the "double underline case"]. Then let us recall the *"Frobenioid-theoretic l-th root of the theta function"*, which is *normalized* so as to attain the value 1 at the point "$\sqrt{-1}$" [cf. [EtTh], Theorem 5.7]; we shall denote the *reciprocal* of [i.e., "1 over"] this theta function by

$$\underline{\underline{\Theta}}_{\underline{v}} \in \mathcal{O}^{\times}(\mathbb{T}^{\div}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$$

— where we use the superscript "$\div$" to denote the image in $\underline{\underline{\mathcal{F}}}^{\div}_{\underline{v}}$ of an object of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. Here, we recall that $\underline{\underline{\Theta}}_{\underline{v}}$ is *completely determined* up to *multiplication by a $2l$-th root of unity* [i.e., an element of $\boldsymbol{\mu}_{2l}(\mathbb{T}^{\div}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$] and the *action of the group of automorphisms*

$l \cdot \underline{\mathbb{Z}} \subseteq \mathrm{Aut}(\mathbb{T}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$ [i.e., we write $\underline{\mathbb{Z}}$ for the group denoted "$\underline{\mathbb{Z}}$" in [EtTh], Theorem 5.7; cf. also the discussion preceding [EtTh], Definition 1.9]. Moreover, we recall from the theory of [EtTh], §5 [cf. the discussion at the beginning of [EtTh], §5; [EtTh], Theorem 5.7] that

$$\mathbb{T}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}} \quad \text{[regarded up to isomorphism] and}$$

$$\underline{\underline{\Theta}}_{\underline{v}} \quad \text{[regarded up to the } \boldsymbol{\mu}_{2l}(\mathbb{T}_{\underline{\underline{\dot{\ddot{Y}}}}_{\underline{v}}}),\ l \cdot \underline{\mathbb{Z}} \text{ indeterminacies discussed above]}$$

may be *reconstructed category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [cf. Remark 3.2.1 below].

(iii) Next, we recall from [EtTh], Corollary 3.8, (ii) [cf. also [EtTh], Proposition 5.1], that the $p_{\underline{v}}$*-adic Frobenioid* constituted by the *"base-field-theoretic hull"* [cf. [EtTh], Remark 3.6.2]

$$\mathcal{C}_{\underline{v}} \subseteq \underline{\underline{\mathcal{F}}}_{\underline{v}}$$

[i.e., we write $\mathcal{C}_{\underline{v}}$ for the subcategory "$\mathcal{C}^{\text{bs-fld}}$" of [EtTh], Definition 3.6, (iv)] may be *reconstructed category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [cf. Remark 3.2.1 below].

(iv) Write $q_{\underline{v}}$ for the *q-parameter* of the elliptic curve $E_{\underline{v}}$ over $K_{\underline{v}}$. Thus, we may think of $q_{\underline{v}}$ as an element $q_{\underline{v}} \in \mathcal{O}^{\rhd}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$ $(\cong \mathcal{O}^{\rhd}_{K_{\underline{v}}})$. Note that it follows from our assumption concerning 2-*torsion* [cf. Definition 3.1, (b)], together with the definition of "K" [cf. Definition 3.1, (c)], that $q_{\underline{v}}$ admits a $2l$*-th root* in $\mathcal{O}^{\rhd}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$ $(\cong \mathcal{O}^{\rhd}_{K_{\underline{v}}})$. Then one computes immediately from the final formula of [EtTh], Proposition 1.4, (ii), that the *value of* $\underline{\underline{\Theta}}_{\underline{v}}$ *at* $\sqrt{-q_{\underline{v}}}$ is equal to

$$\underline{\underline{q}}_{\underline{v}} \stackrel{\text{def}}{=} q_{\underline{v}}^{1/2l} \in \mathcal{O}^{\rhd}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$$

— where the notation "$q_{\underline{v}}^{1/2l}$" [hence also $\underline{\underline{q}}_{\underline{v}}$] is *completely determined up to a* $\boldsymbol{\mu}_{2l}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$*-multiple.* Write $\Phi_{\mathcal{C}_{\underline{v}}}$ for the *divisor monoid* [cf. [FrdI], Definition 1.1, (iv)] of the $p_{\underline{v}}$-adic Frobenioid $\mathcal{C}_{\underline{v}}$. Then the image of $\underline{\underline{q}}_{\underline{v}}$ determines a *constant section* [i.e., a sub-monoid on $\mathcal{D}_{\underline{v}}$ isomorphic to $\mathbb{N}$] "$\log_{\Phi}(\underline{\underline{q}}_{\underline{v}})$" of $\Phi_{\mathcal{C}_{\underline{v}}}$. Moreover, the resulting submonoid [cf. Remark 3.2.2 below]

$$\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}} \stackrel{\text{def}}{=} \mathbb{N} \cdot \log_{\Phi}(\underline{\underline{q}}_{\underline{v}})|_{\mathcal{D}^{\vdash}_{\underline{v}}} \subseteq \Phi_{\mathcal{C}_{\underline{v}}}|_{\mathcal{D}^{\vdash}_{\underline{v}}}$$

determines a $p_{\underline{v}}$*-adic Frobenioid* with *base category* given by $\mathcal{D}^{\vdash}_{\underline{v}}$ [cf. [FrdII], Example 1.1, (ii)]

$$\mathcal{C}^{\vdash}_{\underline{v}} \quad (\subseteq \mathcal{C}_{\underline{v}} \subseteq \underline{\underline{\mathcal{F}}}_{\underline{v}} \to \underline{\underline{\mathcal{F}}}^{\div}_{\underline{v}})$$

— which may be thought of as a subcategory of $\mathcal{C}_{\underline{v}}$. Also, we observe that [since the q-parameter $\underline{\underline{q}}_{\underline{v}} \in K_{\underline{v}}$, it follows that] $\underline{\underline{q}}_{\underline{v}}$ determines a $\boldsymbol{\mu}_{2l}(-)$*-orbit* of *characteristic splittings* [cf. [FrdI], Definition 2.3]

$$\tau^{\vdash}_{\underline{v}}$$

on $\mathcal{C}^{\vdash}_{\underline{v}}$.

(v) Next, let us recall that the *base field* of $\ddot{\underline{\underline{Y}}}_{\underline{v}}$ is equal to $K_{\underline{v}}$ [cf. the discussion of Definition 3.1, (e)]. Write

$$\mathcal{D}^{\Theta}_{\underline{v}} \subseteq (\mathcal{D}_{\underline{v}})_{\ddot{\underline{\underline{Y}}}_{\underline{v}}}$$

for the *full subcategory* of the category $(\mathcal{D}_{\underline{v}})_{\ddot{\underline{\underline{Y}}}_{\underline{v}}}$ [cf. the notational conventions concerning categories discussed in §0] determined by the *products* in $\mathcal{D}_{\underline{v}}$ of $\ddot{\underline{\underline{Y}}}_{\underline{v}}$ with objects of $\mathcal{D}^{\vdash}_{\underline{v}}$. Thus, one verifies immediately that "forming the product with $\ddot{\underline{\underline{Y}}}_{\underline{v}}$" determines a *natural equivalence of categories* $\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{D}^{\Theta}_{\underline{v}}$. Moreover, for $A^{\Theta} \in \mathrm{Ob}(\mathcal{D}^{\Theta}_{\underline{v}})$, the assignment

$$A^{\Theta} \mapsto \mathcal{O}^{\times}(\mathbb{T}_{A^{\Theta}}) \cdot (\underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}|_{\mathbb{T}_{A^{\Theta}}}) \subseteq \mathcal{O}^{\times}(\mathbb{T}^{\div}_{A^{\Theta}})$$

determines a *monoid* $\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ *on* $\mathcal{D}^{\Theta}_{\underline{v}}$ [in the sense of [FrdI], Definition 1.1, (ii)]; write $\mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-) \subseteq \mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ for the submonoid determined by the invertible elements. Next, let us observe that, relative to the natural equivalence of categories $\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{D}^{\Theta}_{\underline{v}}$ — which we think of as mapping $\mathrm{Ob}(\mathcal{D}^{\vdash}_{\underline{v}}) \ni A \mapsto A^{\Theta} \stackrel{\mathrm{def}}{=} \ddot{\underline{\underline{Y}}}_{\underline{v}} \times A \in \mathrm{Ob}(\mathcal{D}^{\Theta}_{\underline{v}})$ — we have *natural isomorphisms*

$$\mathcal{O}^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) \xrightarrow{\sim} \mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-); \quad \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) \xrightarrow{\sim} \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$$

[where $\mathcal{O}^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-)$, $\mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-)$ are the monoids associated to the Frobenioid $\mathcal{C}^{\vdash}_{\underline{v}}$ as in [FrdI], Proposition 2.2] which are *compatible* with the assignment

$$\underline{\underline{q}}_{\underline{v}}|_{\mathbb{T}_A} \mapsto \underline{\underline{\Theta}}_{\underline{v}}|_{\mathbb{T}_{A^{\Theta}}}$$

and the natural isomorphism [i.e., induced by the natural projection $A^{\Theta} = \ddot{\underline{\underline{Y}}}_{\underline{v}} \times A \to A$] $\mathcal{O}^{\times}(\mathbb{T}_A) \xrightarrow{\sim} \mathcal{O}^{\times}(\mathbb{T}_{A^{\Theta}})$. In particular, we conclude that the monoid $\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ determines — in a fashion consistent with the notation of [FrdI], Proposition 2.2! — a $p_{\underline{v}}$*-adic Frobenioid* with *base category* given by $\mathcal{D}^{\Theta}_{\underline{v}}$ [cf. [FrdII], Example 1.1, (ii)]

$$\mathcal{C}^{\Theta}_{\underline{v}} \quad (\subseteq \underline{\underline{\mathcal{F}}}^{\div}_{\underline{v}})$$

— which may be thought of as a subcategory of $\underline{\underline{\mathcal{F}}}^{\div}_{\underline{v}}$, and which is equipped with a $\boldsymbol{\mu}_{2l}(-)$*-orbit* of *characteristic splittings* [cf. [FrdI], Definition 2.3]

$$\tau^{\Theta}_{\underline{v}}$$

determined by $\underline{\underline{\Theta}}_{\underline{v}}$. Moreover, we have a *natural equivalence of categories*

$$\mathcal{C}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{C}^{\Theta}_{\underline{v}}$$

that maps $\tau^{\vdash}_{\underline{v}}$ to $\tau^{\Theta}_{\underline{v}}$. This fact may be stated more succinctly by writing

$$\mathcal{F}^{\vdash}_{\underline{v}} \stackrel{\sim}{\to} \mathcal{F}^{\Theta}_{\underline{v}}$$

— where we write $\mathcal{F}^{\vdash}_{\underline{v}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}})$; $\mathcal{F}^{\Theta}_{\underline{v}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$. In the following, we shall refer to a pair such as $\mathcal{F}^{\vdash}_{\underline{v}}$ or $\mathcal{F}^{\Theta}_{\underline{v}}$ consisting of a Frobenioid equipped with a collection of characteristic splittings as a *split Frobenioid*.

(vi) Here, it is useful to recall [cf. Remark 3.2.1 below] that:

(a) the subcategory $\mathcal{D}^{\vdash}_{\underline{v}} \subseteq \mathcal{D}_{\underline{v}}$ may be *reconstructed category-theoretically from* $\mathcal{D}_{\underline{v}}$ [cf. [AbsAnab], Lemma 1.3.8];

(b) the category $\mathcal{D}^{\Theta}_{\underline{v}}$ may be *reconstructed category-theoretically from* $\mathcal{D}_{\underline{v}}$ [cf. (a); the discussion at the beginning of [EtTh], §5];

(c) the category $\mathcal{D}^{\vdash}_{\underline{v}}$ (respectively, $\mathcal{D}^{\Theta}_{\underline{v}}$) may be *reconstructed category-theoretically from* $\mathcal{C}^{\vdash}_{\underline{v}}$ (respectively, $\mathcal{C}^{\Theta}_{\underline{v}}$) [cf. [FrdI], Theorem 3.4, (v); [FrdII], Theorem 1.2, (i); [FrdII], Example 1.3, (i); [AbsAnab], Theorem 1.1.1, (ii)];

(d) the category $\mathcal{D}_{\underline{v}}$ may be *reconstructed category-theoretically either from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [cf. [EtTh], Theorem 4.4; [EtTh], Proposition 5.1] *or from* $\mathcal{C}_{\underline{v}}$ [cf. [FrdI], Theorem 3.4, (v); [FrdII], Theorem 1.2, (i); [FrdII], Example 1.3, (i); [SemiAnbd], Example 3.10; [SemiAnbd], Remark 3.4.1].

Next, let us observe that by (b), (d), together with the discussion of (ii) concerning the *category-theoreticity* of $\underline{\underline{\Theta}}_{\underline{v}}$, it follows [cf. Remark 3.2.1 below] that

(e) one may *reconstruct the split Frobenioid* $\mathcal{F}^{\Theta}_{\underline{v}}$ [up to the $l \cdot \underline{\mathbb{Z}}$ indeterminacy in $\underline{\underline{\Theta}}_{\underline{v}}$ discussed in (ii); cf. also Remark 3.2.3 below] *category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [cf. [FrdI], Theorem 3.4, (i), (v); [EtTh], Proposition 5.1].

Next, let us recall that the *values of* $\underline{\underline{\Theta}}_{\underline{v}}$ may be computed by restricting the corresponding *Kummer class*, i.e., the *"étale theta function"* [cf. [EtTh], Proposition 1.4, (iii); the proof of [EtTh], Theorem 1.10, (ii); the proof of [EtTh], Theorem 5.7], which may be reconstructed *category-theoretically* from $\mathcal{D}_{\underline{v}}$ [cf. [EtTh], Corollary 2.8, (i)]. Thus, by applying the *isomorphisms of cyclotomes* of [AbsTopIII], Corollary 1.10, (c); [AbsTopIII], Remark 3.2.1 [cf. also [AbsTopIII], Remark 3.1.1], to these Kummer classes, one concludes from (a), (d) that

(f) one may *reconstruct the split Frobenioid* $\mathcal{F}^{\vdash}_{\underline{v}}$ *category-theoretically from* $\mathcal{C}_{\underline{v}}$, hence also [cf. (iii)] from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [cf. Remark 3.2.1 below].

Remark 3.2.1.

(i) In [FrdI], [FrdII], and [EtTh] [cf. [EtTh], Remark 5.1.1], the phrase *"reconstructed category-theoretically"* is interpreted as meaning *"preserved by equivalences of categories"*. From the point of view of the theory of [AbsTopIII] — i.e., the discussion of *"mono-anabelian"* versus *"bi-anabelian"* geometry [cf. [AbsTopIII], §I2, (Q2)] — this sort of definition is "bi-anabelian" in nature. In fact, it is not difficult to verify that the techniques of [FrdI], [FrdII], and [EtTh] all result in *explicit reconstruction algorithms*, whose *input data* consists solely of the category structure of the given category, of a *"mono-anabelian"* nature that *do not require the use of some* **fixed reference model** *that arises from scheme theory* [cf. the discussion of [AbsTopIII], §I4]. For more on the *foundational aspects* of such "mono-anabelian reconstruction algorithms", we refer to the discussion of [IUTchIV], Example 3.5.

(ii) One reason that we do not develop in detail here a "mono-anabelian approach to the geometry of categories" along the lines of [AbsTopIII] is that, unlike the case with the mono-anabelian theory of [AbsTopIII], which plays a *quite essential role* in the theory of the present series of papers, much of the category-theoretic reconstruction theory of [FrdI], [FrdII], and [EtTh] is *not of essential importance* in the development of the theory of the present series of papers. That is to say, for instance, instead of quoting results to the effect that the base categories or divisor monoids of various Frobenioids may be reconstructed category-theoretically, one could instead simply work with the data consisting of "the category constituted by the Frobenioid equipped with its pre-Frobenioid structure" [cf. [FrdI], Definition 1.1, (iv)]. Nevertheless, we chose to apply the theory of [FrdI], [FrdII], and [EtTh] partly because it *simplifies the exposition* [i.e., reduces the number of auxiliary structures that one must carry around], but more importantly because it renders explicit precisely which structures arising from scheme-theory are *"categorically intrinsic"* and which merely amount to *"arbitrary, non-intrinsic choices"* which, when formulated intrinsically, correspond to various *"indeterminacies"*. This explicitness is of particular importance with respect to phenomena related to the *unit-linear Frobenius functor* [cf. [FrdI], Proposition 2.5] and the *Frobenioid-theoretic indeterminacies* studied in [EtTh], §5.

Remark 3.2.2. Although the submonoid $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ is not "absolutely primitive" in the sense of [FrdII], Example 1.1, (ii), it is "very close to being absolutely primitive", in the sense that [as is easily verified] there exists a positive integer N such that $N \cdot \Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ is *absolutely primitive*. This proximity to absolute primitiveness may also be seen in the existence of the *characteristic splittings* $\tau^{\vdash}_{\underline{v}}$.

Remark 3.2.3.

(i) Let $\alpha \in \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{v})$. Then observe that α determines, in a natural way, an *automorphism* $\alpha_{\mathcal{D}}$ *of the functor* $\mathcal{D}^{\vdash}_{\underline{v}} \to \mathcal{D}_{\underline{v}}$ obtained by composing the equivalence of categories $\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{D}^{\Theta}_{\underline{v}}$ [i.e., which maps $\mathrm{Ob}(\mathcal{D}^{\vdash}_{\underline{v}}) \ni A \mapsto A^{\Theta} \in \mathrm{Ob}(\mathcal{D}^{\Theta}_{\underline{v}})$] discussed in Example 3.2, (v), with the natural functor $\mathcal{D}^{\Theta}_{\underline{v}} \subseteq (\mathcal{D}_{\underline{v}})_{\underline{\underline{\ddot{Y}}}_{v}} \to \mathcal{D}_{\underline{v}}$. Moreover, $\alpha_{\mathcal{D}}$ induces, in a natural way, an *isomorphism* $\alpha_{\mathcal{O}^{\triangleright}}$ of the *monoid* $\mathcal{O}^{\triangleright}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ *on* $\mathcal{D}^{\Theta}_{\underline{v}}$ *associated to* $\underline{\underline{\Theta}}_{v}$ in Example 3.2, (v), onto the *corresponding monoid on* $\mathcal{D}^{\Theta}_{\underline{v}}$ *associated to the* α*-conjugate* $\underline{\underline{\Theta}}^{\alpha}_{v}$ of $\underline{\underline{\Theta}}_{v}$. Thus, it follows immediately from the discussion of Example 3.2, (v), that

$\alpha_{\mathcal{O}^{\triangleright}}$ — hence also α — induces an isomorphism of the *split Frobenioid*

$\mathcal{F}^{\Theta}_{\underline{v}}$ *associated to* $\underline{\underline{\Theta}}_{\underline{v}}$ onto the *split Frobenioid* $\mathcal{F}^{\Theta^{\alpha}}_{\underline{v}}$ *associated to* $\underline{\underline{\Theta}}^{\alpha}_{\underline{v}}$ which lies over the *identity functor on* $\mathcal{D}^{\Theta}_{\underline{v}}$.

In particular, the expression *"*$\mathcal{F}^{\Theta}_{\underline{v}}$*, regarded up to the* $l \cdot \underline{\mathbb{Z}}$ *indeterminacy in* $\underline{\underline{\Theta}}_{\underline{v}}$ *discussed in Example 3.2, (ii)"* may be understood as referring to the various split Frobenioids "$\mathcal{F}^{\Theta^{\alpha}}_{\underline{v}}$", as α ranges over the elements of $\mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{\underline{v}})$, relative to the identifications given by these isomorphisms of split Frobenioids induced by the various elements of $\mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{\underline{v}})$.

(ii) Suppose that $A \in \mathrm{Ob}(\mathcal{D}_{\underline{v}})$ lies in the image of the natural functor $\mathcal{D}^{\Theta}_{\underline{v}} \subseteq (\mathcal{D}_{\underline{v}})_{\underline{\underline{\ddot{Y}}}_{\underline{v}}} \to \mathcal{D}_{\underline{v}}$, and that $\psi : B \to \mathbb{T}_A$ is a linear morphism in the Frobenioid $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. Then ψ induces an injective homomorphism

$$\mathcal{O}^{\times}(\mathbb{T}^{\div}_A) \hookrightarrow \mathcal{O}^{\times}(B^{\div})$$

[cf. [FrdI], Proposition 1.11, (iv)]. In particular, one may pull-back sections of the *monoid* $\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ *on* $\mathcal{D}^{\Theta}_{\underline{v}}$ of Example 3.2, (v), to B. Such pull-backs are useful, for instance, when one considers the *roots of* $\underline{\underline{\Theta}}_{\underline{v}}$, as in the theory of [EtTh], §5.

Remark 3.2.4. Before proceeding, we pause to discuss certain minor oversights on the part of the author in the discussion of the theory of *tempered Frobenioids* in [EtTh], §3, §4. Let $\mathfrak{Z}^{\log}_{\infty}$ be as in the discussion at the beginning of [EtTh], §3. Here, we recall that $\mathfrak{Z}^{\log}_{\infty}$ is obtained as the *"universal combinatorial covering"* of the formal log scheme associated to a stable log curve with split special fiber over the ring of integers of a finite extension of an MLF of residue characteristic p [cf. *loc. cit.* for more details]; we write $Z^{\log}$ for the generic fiber of the stable log curve under consideration.

(i) First, let us consider the following conditions on a nonzero meromorphic function f on $\mathfrak{Z}^{\log}_{\infty}$:

(a) For every $N \in \mathbb{N}_{\geq 1}$, it holds that f admits an N*-th root* over some tempered covering of $Z^{\log}$.

(b) For every $N \in \mathbb{N}_{\geq 1}$ which is *prime to* p, it holds that f admits an N*-th root* over some tempered covering of $Z^{\log}$.

(c) The divisor of zeroes and poles of f is a *log-divisor*.

It is immediate that (a) implies (b). Moreover, one verifies immediately, by considering the ramification divisors of the tempered coverings that arise from extracting roots of f, that (b) implies (c). When N is *prime to* p, if f satisfies (c), then it follows immediately from the theory of *admissible coverings* [cf., e.g., [PrfGC], §2, §8] that there exists a *finite* log étale covering $Y^{\log} \to Z^{\log}$ whose pull-back $Y^{\log}_{\infty} \to Z^{\log}_{\infty}$ to the generic fiber $Z^{\log}_{\infty}$ of $\mathfrak{Z}^{\log}_{\infty}$ is sufficient

(R1) to annihilate all ramification over the cusps or special fiber of $\mathfrak{Z}^{\log}_{\infty}$ that might arise from extracting an N-th root of f, as well as

(R2) to split all extensions of the function fields of irreducible components of the special fiber of $\mathfrak{Z}_{\infty}^{\log}$ that might arise from extracting an N-th root of f.

That is to say, in this situation, it follows that f *admits an N-th root over the tempered covering of $Z^{\log}$ given by the "universal combinatorial covering" of* $Y^{\log}$. In particular, it follows that (c) implies (b). Thus, in summary, we have:

$$\text{(a)} \implies \text{(b)} \iff \text{(c)}.$$

On the other hand, unfortunately, it is not clear to the author at the time of writing *whether or not (c) [or (b)] implies (a).*

(ii) Observe that it follows from the theory of [EtTh], §1 [cf., especially, [EtTh], Proposition 1.3] that the *theta function* that forms the main topic of interest of [EtTh] *satisfies condition (a)* of (i).

(iii) In [EtTh], Definition 3.1, (ii), a meromorphic function f as in (i) is defined to be *"log-meromorphic"* if it satisfies condition (c) of (i). On the other hand, in the proof of [EtTh], Proposition 4.2, (iii), it is necessary to use property (a) of (i) — i.e., despite the fact that, as remarked in (i), it is not clear whether or not property (c) implies property (a). The author apologizes for any confusion caused by this oversight on his part.

(iv) The problem pointed out in (iii) may be remedied — at least *from the point of view of the theory of [EtTh]* — via either of the following two approaches:

(A) One may modify [EtTh], Definition 3.1, (ii), by taking the definition of a *"log-meromorphic"* function to be a function that satisfies condition (a) [i.e., as opposed to condition (c)] of (i). [In light of the content of this modified definition, perhaps a better term for this class of meromorphic functions would be *"tempered-meromorphic"*.] Then the remainder of the text of [EtTh] goes through without change.

(B) One may modify [EtTh], Definition 4.1, (i), by assuming that the meromorphic function "$f \in \mathcal{O}^{\times}(A^{\mathrm{birat}})$" of [EtTh], Definition 4.1, (i), satisfies the following "Frobenioid-theoretic version" of condition (a):

(d) For every $N \in \mathbb{N}_{\geq 1}$, there exists a linear morphism $A' \to A$ in $\mathcal{C}$ such that the pull-back of f to A' admits an N-th root.

[Here, we recall that, as discussed in (ii), the Frobenioid-theoretic theta functions that appear in [EtTh] satisfy (d).] Note that since the rational function monoid of the Frobenioid $\mathcal{C}$, as well as the linear morphisms of $\mathcal{C}$, are *category-theoretic* [cf. [FrdI], Theorem 3.4, (iii), (v); [FrdI], Corollary 4.10], this condition (d) is *category-theoretic*. Thus, if one modifies [EtTh], Definition 4.1, (i), in this way, then the remainder of the text of [EtTh] goes through without change, except that one must replace the reference to the definition of "log-meromorphic" [i.e., [EtTh], Definition 3.1, (ii)] that occurs in the proof of [EtTh], Proposition 4.2, (iii), by a reference to condition (d) [i.e., in the modified version of [EtTh], Definition 4.1, (i)].

(v) In the discussion of (iv), we note that the approach of (A) results in a slightly different definition of the notion of a *"tempered Frobenioid"* from the original definition given in [EtTh]. Put another way, the approach of (B) has the advantage that it does *not* result in any modification of the definition of the notion of a *"tempered Frobenioid"*; that is to say, the approach of (B) only results in a slight reduction in the range of applicability of the theory of [EtTh], §4, which is essentially *irrelevant* from the point of view of the present series of papers, since [cf. (ii)] *theta functions* lie within this reduced range of applicability. On the other hand, the approach of (A) has the advantage that one may consider the *Kummer theory of arbitrary rational functions* of the tempered Frobenioid *without imposing any further hypotheses.* Thus, for the sake of simplicity, in the present series of papers, *we shall interpret the notion of a "tempered Frobenioid" via the approach of (A).*

(vi) Strictly speaking, the definition of the monoid "Φ_W^{ell}" given in [EtTh], Example 3.9, (iii), leads to certain technical difficulties, which are, in fact, *entirely irrelevant* to the theory of [EtTh]. These technical difficulties may be averted by making the following slight modifications to the text of [EtTh], Example 3.9, as follows:

(1) In the discussion following the first display of [EtTh], Example 3.9, (i), the phrase "$Y^{\log}$ is of *genus* 1" should be replaced by the phrase "$Y^{\log}$ is of *genus* 1 and has either *precisely one cusp* or *precisely two cusps* whose difference is a 2-torsion element of the underlying elliptic curve".

(2) In the discussion following the first display of [EtTh], Example 3.9, (i), the phrase

the lower arrow of the diagram to be "$\underline{\dot{X}}^{\log} \to \underline{\dot{C}}^{\log}$"

should be replaced by the phrase

the lower arrow of the diagram to be "$\dot{X}^{\log} \to \dot{C}^{\log}$".

(3) In the discussion following the first display of [EtTh], Example 3.9, (ii), the phrase "*unramified over the cusps* of ..." should be replaced by the phrase "*unramified over the cusps* as well as *over the generic points* of the irreducible components of the special fibers of the stable models of ...". Also, the phrase "tempered coverings of the underlying ..." should be replaced by the phrase "tempered admissible coverings of the underlying ...".

In a word, the thrust of both the original text and the slight modifications just discussed is that the monoid "Φ_W^{ell}" is to be defined to be just large enough to include precisely those divisors which are necessary in order to treat the *theta functions* that appear in [EtTh].

Example 3.3. Frobenioids at Good Nonarchimedean Primes. Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$. Then:

(i) Write

$$\mathcal{D}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{B}(\underline{X}_{\rightarrow\underline{v}})^0; \quad \mathcal{D}^{\vdash}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{B}(K_{\underline{v}})^0$$

[cf. §0]. Thus, $\mathcal{D}^{\vdash}_{\underline{v}}$ may be naturally regarded [by pulling back finite étale coverings via the structure morphism $\underline{X}_{\to_{\underline{v}}} \to \mathrm{Spec}(K_{\underline{v}})$] as a *full subcategory*

$$\mathcal{D}^{\vdash}_{\underline{v}} \subseteq \mathcal{D}_{\underline{v}}$$

of $\mathcal{D}_{\underline{v}}$, and we have a *natural functor* $\mathcal{D}_{\underline{v}} \to \mathcal{D}^{\vdash}_{\underline{v}}$, which is *left-adjoint* to the natural inclusion functor $\mathcal{D}^{\vdash}_{\underline{v}} \hookrightarrow \mathcal{D}_{\underline{v}}$ [cf. [FrdII], Example 1.3, (ii)]. For $\mathrm{Spec}(L) \in \mathrm{Ob}(\mathcal{D}^{\vdash}_{\underline{v}})$ [i.e., L is a finite separable extension of $K_{\underline{v}}$], write $\mathrm{ord}(\mathcal{O}^{\triangleright}_L) \stackrel{\mathrm{def}}{=} \mathcal{O}^{\triangleright}_L/\mathcal{O}^{\times}_L$ as in [FrdII], Example 1.1, (i). Thus, the assignment [cf. §0]

$$\Phi_{\mathcal{C}_{\underline{v}}} : \mathrm{Spec}(L) \mapsto \mathrm{ord}(\mathcal{O}^{\triangleright}_L)^{\mathrm{pf}}$$

determines a *monoid* $\Phi_{\mathcal{C}_{\underline{v}}}$ on $[\mathcal{D}^{\vdash}_{\underline{v}}$, hence, by pull-back via the natural functor $\mathcal{D}_{\underline{v}} \to \mathcal{D}^{\vdash}_{\underline{v}}$, on] $\mathcal{D}_{\underline{v}}$; the assignment

$$\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}} : \mathrm{Spec}(L) \mapsto \mathrm{ord}(\mathbb{Z}^{\triangleright}_{p_{\underline{v}}}) \ (\subseteq \mathrm{ord}(\mathcal{O}^{\triangleright}_L)^{\mathrm{pf}})$$

determines an *absolutely primitive* [cf. [FrdII], Example 1.1, (ii)] *submonoid* $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}} \subseteq \Phi_{\mathcal{C}_{\underline{v}}}|_{\mathcal{D}^{\vdash}_{\underline{v}}}$ on $\mathcal{D}^{\vdash}_{\underline{v}}$; these monoids $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$, $\Phi_{\mathcal{C}_{\underline{v}}}$ determine $p_{\underline{v}}$*-adic Frobenioids*

$$\mathcal{C}^{\vdash}_{\underline{v}} \subseteq \mathcal{C}_{\underline{v}}$$

[cf. [FrdII], Example 1.1, (ii), where we take "Λ" to be $\mathbb{Z}$], whose *base categories* are given by $\mathcal{D}^{\vdash}_{\underline{v}}$, $\mathcal{D}_{\underline{v}}$ [in a fashion compatible with the natural inclusion $\mathcal{D}^{\vdash}_{\underline{v}} \subseteq \mathcal{D}_{\underline{v}}$], respectively. Also, we shall write

$$\underline{\underline{\mathcal{F}}}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{C}_{\underline{v}}$$

[cf. the notation of Example 3.2, (i)]. Finally, let us observe that the element $p_{\underline{v}} \in \mathbb{Z}^{\triangleright}_{p_{\underline{v}}} \subseteq \mathcal{O}^{\triangleright}_{K_{\underline{v}}}$ determines a *characteristic splitting*

$$\tau^{\vdash}_{\underline{v}}$$

on $\mathcal{C}^{\vdash}_{\underline{v}}$ [cf. [FrdII], Theorem 1.2, (v)]. Write $\mathcal{F}^{\vdash}_{\underline{v}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}})$ for the resulting *split Frobenioid*.

(ii) Next, let us write $\log(p_{\underline{v}})$ for the element $p_{\underline{v}}$ of (i) *considered additively* and consider the monoid on $\mathcal{D}^{\vdash}_{\underline{v}}$

$$\mathcal{O}^{\triangleright}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) = \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) \times \ (\mathbb{N} \cdot \log(p_{\underline{v}}))$$

associated to $\mathcal{C}^{\vdash}_{\underline{v}}$ [cf. [FrdI], Proposition 2.2]. By replacing "$\log(p_{\underline{v}})$" by the *formal symbol* "$\log(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}}) = \log(p_{\underline{v}}^{\log(\underline{\underline{\Theta}})})$", we obtain a monoid

$$\mathcal{O}^{\triangleright}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-) \stackrel{\mathrm{def}}{=} \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-) \times \ (\mathbb{N} \cdot \log(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}}))$$

[i.e., where $\mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-) \stackrel{\text{def}}{=} \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-)$], which is *naturally isomorphic* to $\mathcal{O}^{\triangleright}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ and which arises as the monoid "$\mathcal{O}^{\triangleright}(-)$" of [FrdI], Proposition 2.2, associated to some $p_{\underline{v}}$*-adic Frobenioid* $\mathcal{C}^{\Theta}_{\underline{v}}$ with base category $\mathcal{D}^{\Theta}_{\underline{v}} \stackrel{\text{def}}{=} \mathcal{D}^{\vdash}_{\underline{v}}$ equipped with a *characteristic splitting* $\tau^{\Theta}_{\underline{v}}$ determined by $\log(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}})$. In particular, we have a *natural equivalence*

$$\mathcal{F}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{F}^{\Theta}_{\underline{v}}$$

— where $\mathcal{F}^{\Theta}_{\underline{v}} \stackrel{\text{def}}{=} (\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$ — of *split Frobenioids*.

(iii) Here, it is useful to recall that

(a) the subcategory $\mathcal{D}^{\vdash}_{\underline{v}} \subseteq \mathcal{D}_{\underline{v}}$ may be *reconstructed category-theoretically from* $\mathcal{D}_{\underline{v}}$ [cf. [AbsAnab], Lemma 1.3.8];

(b) the category $\mathcal{D}^{\vdash}_{\underline{v}}$ (respectively, $\mathcal{D}^{\Theta}_{\underline{v}}$) may be *reconstructed category-theoretically from* $\mathcal{C}^{\vdash}_{\underline{v}}$ (respectively, $\mathcal{C}^{\Theta}_{\underline{v}}$) [cf. [FrdI], Theorem 3.4, (v); [FrdII], Theorem 1.2, (i); [FrdII], Example 1.3, (i); [AbsAnab], Theorem 1.1.1, (ii)];

(c) the category $\mathcal{D}_{\underline{v}}$ may be *reconstructed category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}} = \mathcal{C}_{\underline{v}}$ [cf. [FrdI], Theorem 3.4, (v); [FrdII], Theorem 1.2, (i); [FrdII], Example 1.3, (i); [AbsAnab], Lemma 1.3.1].

Note that it follows immediately from the *category-theoreticity of the divisor monoid* $\Phi_{\mathcal{C}_{\underline{v}}}$ [cf. [FrdI], Corollary 4.11, (iii); [FrdII], Theorem 1.2, (i)], together with (a), (c), and the definition of $\mathcal{C}^{\vdash}_{\underline{v}}$ [cf. also [AbsAnab], Proposition 1.2.1, (v)], that

(d) $\mathcal{C}^{\vdash}_{\underline{v}}$ may be *reconstructed category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$.

Finally, by applying the *algorithmically constructed field structure* on the image of the *Kummer map* of [AbsTopIII], Proposition 3.2, (iii) [cf. Remark 3.1.2; Remark 3.3.2 below], it follows that one may construct the element "$p_{\underline{v}}$" of $\mathcal{O}^{\triangleright}_{K_{\underline{v}}}$ category-theoretically from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, hence that the characteristic splitting $\tau^{\vdash}_{\underline{v}}$ may be *reconstructed category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. [Here, we recall that the curve X_F is *"of strictly Belyi type"* — cf. [AbsTopIII], Remark 2.8.3.] In particular,

(e) one may *reconstruct the split Frobenioids* $\mathcal{F}^{\vdash}_{\underline{v}}$, $\mathcal{F}^{\Theta}_{\underline{v}}$ *category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$.

Remark 3.3.1. A similar remark to Remark 3.2.1 [i.e., concerning the phrase *"reconstructed category-theoretically"*] applies to the Frobenioids $\mathcal{C}_{\underline{v}}$, $\mathcal{C}^{\vdash}_{\underline{v}}$ constructed in Example 3.3.

Remark 3.3.2. Note that the $p_{\underline{v}}$-adic Frobenioid $\mathcal{C}_{\underline{v}}$ (respectively, $\mathcal{C}^{\vdash}_{\underline{v}}$) of Example 3.3, (i), consists of *essentially the same data* as an *"MLF-Galois* $\mathbb{TM}$*-pair of strictly Belyi type"* (respectively, *"MLF-Galois* $\mathbb{TM}$*-pair of mono-analytic type"*), in the sense of [AbsTopIII], Definition 3.1, (ii) [cf. [AbsTopIII], Remark 3.1.1]. A similar remark applies to the $p_{\underline{v}}$-adic Frobenioid $\mathcal{C}_{\underline{v}}$ (respectively, $\mathcal{C}^{\vdash}_{\underline{v}}$) of Example 3.2, (iii), (iv) [cf. [AbsTopIII], Remark 3.1.3].

Example 3.4. Frobenioids at Archimedean Primes. Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Then:

(i) Write

$$\mathbb{X}_{\underline{v}},\ \mathbb{C}_{\underline{v}},\ \underline{\mathbb{X}}_{\underline{v}},\ \underline{\mathbb{C}}_{\underline{v}},\ \underset{\rightarrow}{\mathbb{X}}_{\underline{v}},\ \underset{\rightarrow}{\mathbb{C}}_{\underline{v}}$$

for the Aut-*holomorphic orbispaces* [cf. [AbsTopIII], Definition 2.1, (i); [AbsTopIII], Remark 2.1.1] determined, respectively, by the hyperbolic orbicurves X_K, C_K, $\underline{X}_K$, $\underline{C}_K$, $\underset{\rightarrow}{X}_K$, $\underset{\rightarrow}{C}_K$ at $\underline{v}$. Thus, for $\Box \in \{\mathbb{X}_{\underline{v}}, \mathbb{C}_{\underline{v}}, \underline{\mathbb{X}}_{\underline{v}}, \underline{\mathbb{C}}_{\underline{v}}, \underset{\rightarrow}{\mathbb{X}}_{\underline{v}}, \underset{\rightarrow}{\mathbb{C}}_{\underline{v}}\}$, we have a *complex archimedean topological field* [i.e., a "CAF" — cf. §0]

$$\overline{\mathcal{A}}_{\Box}$$

[cf. [AbsTopIII], Definition 4.1, (i)] which may be *algorithmically constructed from* $\Box$; write $\mathcal{A}_{\Box} \stackrel{\mathrm{def}}{=} \overline{\mathcal{A}}_{\Box} \setminus \{0\}$ [cf. Remark 3.4.3, (i), below]. Next, let us write

$$\mathcal{D}_{\underline{v}} \stackrel{\mathrm{def}}{=} \underset{\rightarrow}{\mathbb{X}}_{\underline{v}}$$

and

$$\mathcal{C}_{\underline{v}}$$

for the *archimedean Frobenioid* as in [FrdII], Example 3.3, (ii) [i.e., "$\mathcal{C}$" of *loc. cit.*], where we take the *base category* [i.e., "$\mathcal{D}$" of *loc. cit.*] to be the one-morphism category determined by $\mathrm{Spec}(K_{\underline{v}})$. Thus, the *linear morphisms* among the pseudo-terminal objects of $\mathcal{C}$ determine *unique isomorphisms* [cf. [FrdI], Definition 1.3, (iii), (c)] among the respective *topological monoids* "$\mathcal{O}^{\rhd}(-)$" — where we recall [cf. [FrdI], Theorem 3.4, (iii); [FrdII], Theorem 3.6, (i), (vii)] that these topological monoids may be *reconstructed category-theoretically* from $\mathcal{C}$. In particular, it makes sense to write "$\mathcal{O}^{\rhd}(\mathcal{C}_{\underline{v}})$", "$\mathcal{O}^{\times}(\mathcal{C}_{\underline{v}}) \subseteq \mathcal{O}^{\rhd}(\mathcal{C}_{\underline{v}})$". Moreover, we observe that, by construction, there is a *natural isomorphism*

$$\mathcal{O}^{\rhd}(\mathcal{C}_{\underline{v}}) \stackrel{\sim}{\rightarrow} \mathcal{O}^{\rhd}_{K_{\underline{v}}}$$

of *topological monoids*. Thus, one may also think of $\mathcal{C}_{\underline{v}}$ as a *"Frobenioid-theoretic representation"* of the topological monoid $\mathcal{O}^{\rhd}_{K_{\underline{v}}}$ [cf. [AbsTopIII], Remark 4.1.1]. Observe that there is a *natural topological isomorphism* $K_{\underline{v}} \stackrel{\sim}{\rightarrow} \overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$, which may be restricted to $\mathcal{O}^{\rhd}_{K_{\underline{v}}}$ to obtain an *inclusion of topological monoids*

$$\kappa_{\underline{v}} : \mathcal{O}^{\rhd}(\mathcal{C}_{\underline{v}}) \hookrightarrow \mathcal{A}_{\mathcal{D}_{\underline{v}}}$$

— which we shall refer to as the *Kummer structure* on $\mathcal{C}_{\underline{v}}$ [cf. Remark 3.4.2 below]. Write

$$\underline{\underline{\mathcal{F}}}_{\underline{v}} \stackrel{\mathrm{def}}{=} (\mathcal{C}_{\underline{v}}, \mathcal{D}_{\underline{v}}, \kappa_{\underline{v}})$$

[cf. Example 3.2, (i); Example 3.3, (i)].

(ii) Next, recall the category $\mathbb{TM}^{\vdash}$ of *"split topological monoids"* of [AbsTopIII], Definition 5.6, (i) — i.e., the category whose *objects* $(C, \overrightarrow{C})$ consist of a topological monoid C isomorphic to $\mathcal{O}^{\triangleright}_{\mathbb{C}}$ and a topological submonoid $\overrightarrow{C} \subseteq C$ [necessarily isomorphic to $\mathbb{R}_{\geq 0}$] such that the natural inclusions $C^{\times} \hookrightarrow C$ [where $C^{\times}$, which is necessarily isomorphic to $\mathbb{S}^1$, denotes the topological submonoid of *invertible elements*], $\overrightarrow{C} \hookrightarrow C$ determine an isomorphism $C^{\times} \times \overrightarrow{C} \xrightarrow{\sim} C$ of topological monoids, and whose *morphisms* $(C_1, \overrightarrow{C}_1) \to (C_2, \overrightarrow{C}_2)$ are isomorphisms of topological monoids $C_1 \xrightarrow{\sim} C_2$ that induce isomorphisms $\overrightarrow{C}_1 \xrightarrow{\sim} \overrightarrow{C}_2$. Note that the CAF's $K_{\underline{v}}$, $\overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$ determine, in a natural way, objects of $\mathbb{TM}^{\vdash}$. Write

$$\tau^{\vdash}_{\underline{v}}$$

for the resulting *characteristic splitting* of the Frobenioid $\mathcal{C}^{\vdash}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{C}_{\underline{v}}$, i.e., so that we may think of the pair $(\mathcal{O}^{\triangleright}(\mathcal{C}^{\vdash}_{\underline{v}}), \tau^{\vdash}_{\underline{v}})$ as the object of $\mathbb{TM}^{\vdash}$ determined by $K_{\underline{v}}$;

$$\mathcal{D}^{\vdash}_{\underline{v}}$$

for the object of $\mathbb{TM}^{\vdash}$ determined by $\overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$;

$$\mathcal{F}^{\vdash}_{\underline{v}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\vdash}_{\underline{v}}, \mathcal{D}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}})$$

for the [ordered] *triple* consisting of $\mathcal{C}^{\vdash}_{\underline{v}}$, $\mathcal{D}^{\vdash}_{\underline{v}}$, and $\tau^{\vdash}_{\underline{v}}$. Thus, the object $(\mathcal{O}^{\triangleright}(\mathcal{C}^{\vdash}_{\underline{v}}), \tau^{\vdash}_{\underline{v}})$ of $\mathbb{TM}^{\vdash}$ is *isomorphic* to $\mathcal{D}^{\vdash}_{\underline{v}}$. Moreover, $\mathcal{C}^{\vdash}_{\underline{v}}$ (respectively, $\mathcal{D}^{\vdash}_{\underline{v}}$; $\mathcal{F}^{\vdash}_{\underline{v}}$) may be *algorithmically reconstructed* from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ (respectively, $\mathcal{D}_{\underline{v}}$; $\underline{\underline{\mathcal{F}}}_{\underline{v}}$).

(iii) Next, let us observe that $p_{\underline{v}} \in K_{\underline{v}}$ [cf. §0] may be thought of as a(n) [non-identity] element of the *noncompact factor* $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ [i.e., the factor denoted by a "$\rightarrow$" in the definition of $\mathbb{TM}^{\vdash}$] of the object $(\mathcal{O}^{\triangleright}(\mathcal{C}^{\vdash}_{\underline{v}}), \tau^{\vdash}_{\underline{v}})$ of $\mathbb{TM}^{\vdash}$. This noncompact factor $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ is isomorphic, as a topological monoid, to $\mathbb{R}_{\geq 0}$; let us write $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ *additively* and denote by $\log(p_{\underline{v}})$ the element of $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ determined by $p_{\underline{v}}$. Thus, relative to the natural action [by multiplication!] of $\mathbb{R}_{\geq 0}$ on $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$, it follows that $\log(p_{\underline{v}})$ is a *generator* of $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$. In particular, we may form a *new topological monoid*

$$\Phi_{\mathcal{C}^{\Theta}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathbb{R}_{\geq 0} \cdot \log(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}})$$

isomorphic to $\mathbb{R}_{\geq 0}$ that is generated by a *formal symbol* "$\log(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}}) = \log(p_{\underline{v}}^{\log(\underline{\underline{\Theta}})})$". Moreover, if we denote by $\mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ the *compact factor* of the object $(\mathcal{O}^{\triangleright}(\mathcal{C}^{\vdash}_{\underline{v}}), \tau^{\vdash}_{\underline{v}})$ of $\mathbb{TM}^{\vdash}$, and set $\mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$, then we obtain a *new split Frobenioid* $(\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$, isomorphic to $(\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}})$, such that

$$\mathcal{O}^{\triangleright}(\mathcal{C}^{\Theta}_{\underline{v}}) = \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}} \times \Phi_{\mathcal{C}^{\Theta}_{\underline{v}}}$$

— where we note that this equality gives rise to a *natural isomorphism of split Frobenioids* $(\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}}) \xrightarrow{\sim} (\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$, obtained by "forgetting the formal symbol $\log(\underline{\underline{\Theta}})$". In particular, we thus obtain a *natural isomorphism*

$$\mathcal{F}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{F}^{\Theta}_{\underline{v}}$$

— where we write $\mathcal{F}^{\Theta}_{\underline{v}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\Theta}_{\underline{v}}, \mathcal{D}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$ for the [ordered] *triple* consisting of $\mathcal{C}^{\Theta}_{\underline{v}}$, $\mathcal{D}^{\Theta}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{D}^{\vdash}_{\underline{v}}$, $\tau^{\Theta}_{\underline{v}}$. Finally, we observe that $\mathcal{F}^{\Theta}_{\underline{v}}$ may be *algorithmically reconstructed* from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$.

Remark 3.4.1. A similar remark to Remark 3.2.1 [i.e., concerning the phrase *"reconstructed category-theoretically"*] applies to the phrase *"algorithmically reconstructed"* that was applied in the discussion of Example 3.4.

Remark 3.4.2. One way to think of the *Kummer structure*

$$\kappa_{\underline{v}} : \mathcal{O}^{\rhd}(\mathcal{C}_{\underline{v}}) \hookrightarrow \mathcal{A}_{\mathcal{D}_{\underline{v}}}$$

discussed in Example 3.4, (i), is as follows. In the terminology of [AbsTopIII], Definition 2.1, (i), (iv), the structure of CAF on $\overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$ determines, via pull-back by $\kappa_{\underline{v}}$, an Aut-*holomorphic structure* on the *groupification* $\mathcal{O}^{\rhd}(\mathcal{C}_{\underline{v}})^{\mathrm{gp}}$ of $\mathcal{O}^{\rhd}(\mathcal{C}_{\underline{v}})$, together with a [tautological!] *co-holomorphicization* $\mathcal{O}^{\rhd}(\mathcal{C}_{\underline{v}})^{\mathrm{gp}} \to \mathcal{A}_{\mathcal{D}_{\underline{v}}}$. Conversely, if one starts with this Aut-holomorphic structure on [the groupification of] the topological monoid $\mathcal{O}^{\rhd}(\mathcal{C}_{\underline{v}})$, together with the co-holomorphicization $\mathcal{O}^{\rhd}(\mathcal{C}_{\underline{v}})^{\mathrm{gp}} \to \mathcal{A}_{\mathcal{D}_{\underline{v}}}$, then one verifies immediately that one may recover the *inclusion of topological monoids* $\kappa_{\underline{v}}$. [Indeed, this follows immediately from [AbsTopIII], Corollary 2.3, together with the elementary fact that every holomorphic automorphism of the complex Lie group $\mathbb{C}^{\times}$ that preserves the submonoid of elements of norm ≤ 1 is equal to the *identity*.] That is to say, in summary,

> the **Kummer structure** $\kappa_{\underline{v}}$ is completely **equivalent** to the collection of data consisting of the Aut-**holomorphic structure** [induced by $\kappa_{\underline{v}}$] on the *groupification* $\mathcal{O}^{\rhd}(\mathcal{C}_{\underline{v}})^{\mathrm{gp}}$ of $\mathcal{O}^{\rhd}(\mathcal{C}_{\underline{v}})$, together with the **co-holomorphicization** [induced by $\kappa_{\underline{v}}$] $\mathcal{O}^{\rhd}(\mathcal{C}_{\underline{v}})^{\mathrm{gp}} \to \mathcal{A}_{\mathcal{D}_{\underline{v}}}$.

The significance of thinking of Kummer structures in this way lies in the *observation* that [unlike inclusions of topological monoids!]

> the **co-holomorphicization** induced by $\kappa_{\underline{v}}$ is **compatible** with the **logarithm** operation discussed in [AbsTopIII], Corollary 4.5.

Indeed, this observation may be thought of as a rough summary of a substantial portion of the content of [AbsTopIII], Corollary 4.5. Put another way, thinking of Kummer structures in terms of co-holomorphicizations allows one to *separate* out the portion of the structures involved that is *not compatible* with this logarithm operation — i.e., the *monoid* structures! — from the portion of the structures involved that *is compatible* with this logarithm operation — i.e., the tautological *co-holomorphicization*.

Remark 3.4.3.

(i) In the notation of Example 3.4, write $\mathcal{A}^{\times}_{\Box} \subseteq \overline{\mathcal{A}}_{\Box}$ for the topological group of *units* [i.e., of elements of norm 1] of the CAF $\overline{\mathcal{A}}_{\Box}$ [so $\mathcal{A}^{\times}_{\Box}$ is noncanonically isomorphic to the unit circle $\mathbb{S}^1$]; $\mathcal{A}^{\boldsymbol{\mu}}_{\Box} \subseteq \mathcal{A}^{\times}_{\Box}$ for the subgroup of *torsion elements* [so $\mathcal{A}^{\boldsymbol{\mu}}_{\Box}$ is noncanonically isomorphic to $\mathbb{Q}/\mathbb{Z}$]; $\underline{\mathbb{E}}_{\underline{v}}$ for the Aut-*holomorphic space* [cf. [AbsTopIII], Definition 2.1, (i)] determined by the *elliptic curve* obtained by *compactifying* $\underline{X}_K$ at $\underline{v}$. Now recall from the *construction* of "$\overline{\mathcal{A}}_{\Box}$" in [AbsTopIII], Corollary 2.7 [cf. also [AbsTopIII], Definition 4.1, (i)] via the technique of **"holomorphic elliptic cuspidalization"**, that one has a *natural isomorphism of CAF's*

$$\overline{\mathcal{A}}_{\underline{\overrightarrow{\mathbb{X}}}_{\underline{v}}} = \overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}} \xrightarrow{\sim} \overline{\mathcal{A}}_{\underline{\mathbb{X}}_{\underline{v}}}$$

— which may be used to *"identify"* $\overline{\mathcal{A}}_{\underline{\overrightarrow{\mathbb{X}}}_{\underline{v}}} = \overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$ with $\overline{\mathcal{A}}_{\underline{\mathbb{X}}_{\underline{v}}}$. Indeed, thinking of "$\overline{\mathcal{A}}_{\underline{\overrightarrow{\mathbb{X}}}_{\underline{v}}} = \overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$" as "$\overline{\mathcal{A}}_{\underline{\mathbb{X}}_{\underline{v}}}$" is natural from the point of view of the **"Θ-approach"** discussed in Remark 3.1.2, (ii). Moreover, by allowing $\mathcal{A}^{\times}_{\underline{\mathbb{X}}_{\underline{v}}}$ to "act" [cf. the algorithm discussed in [AbsTopIII], Corollary 2.7, (e)] on points in a *sufficiently small neighborhood* of [but *not equal to*!] a given point "x" of $\underline{\mathbb{E}}_{\underline{v}}$, one may regard the "circle" $\mathcal{A}^{\times}_{\underline{\mathbb{X}}_{\underline{v}}}$ as a *deformation retract* of the complement of x in a suitable small neighborhood of x in $\underline{\mathbb{E}}_{\underline{v}}$. In particular,

> from the point of view of the **"Θ-approach"** discussed in Remark 3.1.2, (ii), it is natural to think of "$\overline{\mathcal{A}}_{\underline{\overrightarrow{\mathbb{X}}}_{\underline{v}}} = \overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$" as "$\overline{\mathcal{A}}_{\underline{\mathbb{X}}_{\underline{v}}}$" and to regard
>
> $$\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathcal{A}^{\boldsymbol{\mu}}_{\underline{\mathbb{X}}_{\underline{v}}}) = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathcal{A}^{\times}_{\underline{\mathbb{X}}_{\underline{v}}})$$
>
> [a profinite group which is noncanonically isomorphic to $\widehat{\mathbb{Z}}$] as the result of identifying the **cuspidal inertia groups** of the various points "x" of $\underline{\mathbb{E}}_{\underline{v}}$

— cf. discussion of the cuspidal inertia groups "I_x" in [AbsTopIII], Proposition 1.4, (i), (ii). Indeed, this *interpretation of* $\overline{\mathcal{A}}_{\underline{\overrightarrow{\mathbb{X}}}_{\underline{v}}} = \overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$ *via cuspidal inertia groups* may be thought of as a sort of *archimedean version* of the *"Θ-approach"* discussed in Remark 3.1.2, (ii).

(ii) We observe that just as the theory of *elliptic cuspidalization* [cf. [AbsTopII], Example 3.2; [AbsTopII], Corollaries 3.3, 3.4] admits a straightforward *holomorphic analogue*, i.e., the theory of *"holomorphic elliptic cuspidalization"* [cf. [AbsTopIII], Corollary 2.7] referred to in (i) above, the theory of **Belyi cuspidalization** [cf. [AbsTopII], Example 3.6; [AbsTopII], Corollaries 3.7, 3.8; [AbsTopIII], Remark 2.8.3] admits a straightforward *holomorphic analogue*, i.e., a theory of **"holomorphic Belyi cuspidalization"**. We leave the routine details to the reader. Here, we observe that one immediate consequence of such "holomorphic Belyi cuspidalizations" may be stated as follows:

> the set of **NF-points** [i.e., points defined over a *number field*] of the underlying topological space of the Aut-holomorphic space $\mathcal{D}_{\underline{v}}$ may be **reconstructed** via a **functorial algorithm** from the [abstract] Aut-holomorphic space $\mathcal{D}_{\underline{v}}$.

Example 3.5. Global Realified Frobenioids.

(i) Write

$$\mathcal{C}^{\Vdash}_{\mathrm{mod}}$$

for the *realification* [cf. [FrdI], Theorem 6.4, (ii)] of the *Frobenioid* of [FrdI], Example 6.3 [cf. also Remark 3.1.5 of the present paper], associated to the *number field* F_{mod} and the *trivial Galois extension* [i.e., the Galois extension of degree 1] of F_{mod} [so the base category of $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ is, in the terminology of [FrdI], equivalent to a *one-morphism category*]. Thus, the *divisor monoid* $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}}$ of $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ may be thought of as a single abstract monoid, whose *set of primes*, which we denote $\mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{mod}})$ [cf. [FrdI], §0], is in *natural bijective correspondence* with $\mathbb{V}_{\mathrm{mod}}$ [cf. the discussion of [FrdI], Example 6.3]. Moreover, the submonoid $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v}$ of $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}}$ corresponding to $v \in \mathbb{V}_{\mathrm{mod}}$ is *naturally isomorphic* to $\mathrm{ord}(\mathcal{O}^{\rhd}_{(F_{\mathrm{mod}})_v})^{\mathrm{pf}} \otimes \mathbb{R}_{\geq 0}$ ($\cong \mathbb{R}_{\geq 0}$) [i.e., to $\mathrm{ord}(\mathcal{O}^{\rhd}_{(F_{\mathrm{mod}})_v})$ ($\cong \mathbb{R}_{\geq 0}$) if $v \in \mathbb{V}^{\mathrm{arc}}_{\mathrm{mod}}$]. In particular, p_v determines an element $\log^{\vdash}_{\mathrm{mod}}(p_v) \in \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v}$. Write $\underline{v} \in \underline{\mathbb{V}}$ for the element of $\underline{\mathbb{V}}$ that corresponds to v. Then observe that regardless of whether $\underline{v}$ belongs to $\underline{\mathbb{V}}^{\mathrm{bad}}$, $\underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$, or $\underline{\mathbb{V}}^{\mathrm{arc}}$, the *realification* $\Phi^{\mathrm{rlf}}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ of the divisor monoid $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ of $\mathcal{C}^{\vdash}_{\underline{v}}$ [which, as is easily verified, is a *constant monoid* over the corresponding base category] may be regarded as a *single abstract monoid* isomorphic to $\mathbb{R}_{\geq 0}$. Write $\log_{\Phi}(p_{\underline{v}}) \in \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ for the element defined by $p_{\underline{v}}$ and

$$\mathcal{C}_{\rho_{\underline{v}}} : \mathcal{C}^{\Vdash}_{\mathrm{mod}} \to (\mathcal{C}^{\vdash}_{\underline{v}})^{\mathrm{rlf}}$$

for the *natural restriction functor* [cf. the theory of poly-Frobenioids developed in [FrdII], §5] to the *realification* of the Frobenioid $\mathcal{C}^{\vdash}_{\underline{v}}$ [cf. [FrdI], Proposition 5.3]. Thus, one verifies immediately that $\mathcal{C}_{\rho_{\underline{v}}}$ is determined, up to isomorphism, by the *isomorphism of topological monoids* [which are isomorphic to $\mathbb{R}_{\geq 0}$]

$$\rho_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v} \xrightarrow{\sim} \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\vdash}_{\underline{v}}}$$

induced by $\mathcal{C}_{\rho_{\underline{v}}}$ — which, by considering the natural *"volume interpretations"* of the arithmetic divisors involved, is easily computed to be given by the assignment $\log^{\vdash}_{\mathrm{mod}}(p_v) \mapsto \frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]} \log_{\Phi}(p_{\underline{v}})$.

(ii) In a similar vein, one may construct a *"Θ-version"* [i.e., as in Examples 3.2, (v); 3.3, (ii); 3.4, (iii)] of the various data constructed in (i). That is to say, we set

$$\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht}}} \stackrel{\mathrm{def}}{=} \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}} \cdot \log(\underline{\underline{\Theta}})$$

— i.e., an isomorphic copy of $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}}$ generated by a *formal symbol* $\log(\underline{\underline{\Theta}})$. This monoid $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht}}}$ thus determines a *Frobenioid* $\mathcal{C}^{\Vdash}_{\mathrm{tht}}$, equipped with a *natural equivalence of categories* $\mathcal{C}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{C}^{\Vdash}_{\mathrm{tht}}$ and a *natural bijection* $\mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{tht}}) \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$. For $v \in \mathbb{V}_{\mathrm{mod}}$, the element $\log^{\vdash}_{\mathrm{mod}}(p_v)$ of the submonoid $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v} \subseteq \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}}$ thus determines an element $\log^{\vdash}_{\mathrm{mod}}(p_v) \cdot \log(\underline{\underline{\Theta}})$ of a submonoid $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht}},v} \subseteq \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht}}}$. Write $\underline{v} \in \underline{\mathbb{V}}$ for the element of $\underline{\mathbb{V}}$ that corresponds to v. Then the *realification* $\Phi^{\mathrm{rlf}}_{\mathcal{C}^{\Theta}_{\underline{v}}}$ of the divisor monoid $\Phi_{\mathcal{C}^{\Theta}_{\underline{v}}}$ of $\mathcal{C}^{\Theta}_{\underline{v}}$ [which, as is easily verified, is a *constant monoid* over the corresponding base category] may be regarded as a *single abstract monoid* isomorphic to $\mathbb{R}_{\geq 0}$. Write

$$\mathcal{C}_{\rho^{\Theta}_{\underline{v}}} : \mathcal{C}^{\Vdash}_{\mathrm{tht}} \to (\mathcal{C}^{\Theta}_{\underline{v}})^{\mathrm{rlf}}$$

for the *natural restriction functor* [cf. (i) above; the theory of poly-Frobenioids developed in [FrdII], §5] to the *realification* of the Frobenioid $\mathcal{C}^{\Theta}_{\underline{v}}$ [cf. [FrdI], Proposition 5.3]. Thus, one verifies immediately that $\mathcal{C}_{\rho^{\Theta}_{\underline{v}}}$ is determined, up to isomorphism, by the *isomorphism of topological monoids* [which are isomorphic to $\mathbb{R}_{\geq 0}$]

$$\rho^{\Theta}_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht}}, v} \xrightarrow{\sim} \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\Theta}_{\underline{v}}}$$

induced by $\mathcal{C}_{\rho^{\Theta}_{\underline{v}}}$. If $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, then write $\log_{\Phi}(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}}) \in \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\Theta}_{\underline{v}}}$ for the element determined by $\log_{\Phi}(p_{\underline{v}})$; thus, [cf. (i)] $\rho^{\Theta}_{\underline{v}}$ is given by the assignment $\log^{\vdash}_{\mathrm{mod}}(p_v) \cdot \log(\underline{\underline{\Theta}}) \mapsto \frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}\log_{\Phi}(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}})$. On the other hand, if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, then let us write

$$\log_{\Phi}(\underline{\underline{\Theta}}_{\underline{v}}) \in \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\Theta}_{\underline{v}}}$$

for the element determined by $\underline{\underline{\Theta}}_{\underline{v}}$ [cf. Example 3.2, (v)] and $\log_{\Phi}(p_{\underline{v}})$ for the constant section of $\Phi_{\mathcal{C}_{\underline{v}}}$ determined by $p_{\underline{v}}$ [cf. the notation "$\log_{\Phi}(\underline{\underline{q}}_{\underline{v}})$" of Example 3.2, (iv)]; in particular, it makes sense to write $\log_{\Phi}(p_{\underline{v}})/\log_{\Phi}(\underline{\underline{q}}_{\underline{v}}) \in \mathbb{Q}_{>0}$; thus, [cf. (i)] $\rho^{\Theta}_{\underline{v}}$ is given by the assignment

$$\log^{\vdash}_{\mathrm{mod}}(p_v) \cdot \log(\underline{\underline{\Theta}}) \mapsto \frac{\log_{\Phi}(p_{\underline{v}})}{[K_{\underline{v}} : (F_{\mathrm{mod}})_v]} \cdot \frac{\log_{\Phi}(\underline{\underline{\Theta}}_{\underline{v}})}{\log_{\Phi}(\underline{\underline{q}}_{\underline{v}})}$$

— cf. Remark 3.5.1, (i), below. Note that, for arbitrary $\underline{v} \in \underline{\mathbb{V}}$, the various $\rho_{\underline{v}}$, $\rho^{\Theta}_{\underline{v}}$ are *compatible* with the *natural isomorphisms* $\mathcal{C}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{C}^{\Vdash}_{\mathrm{tht}}$, $\mathcal{C}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{C}^{\Theta}_{\underline{v}}$ [cf. §0]. This fact may be expressed as a *natural isomorphism between collections of data* [consisting of a category, a bijection of sets, a collection of data indexed by $\underline{\mathbb{V}}$, and a collection of isomorphisms indexed by $\underline{\mathbb{V}}$]

$$\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \quad \xrightarrow{\sim} \quad \mathfrak{F}^{\Vdash}_{\mathrm{tht}}$$

— where we write

$$\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\Vdash}_{\mathrm{mod}},\ \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ \{\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$
$$\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\Vdash}_{\mathrm{tht}},\ \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{tht}}) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{\mathcal{F}^{\Theta}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ \{\rho^{\Theta}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

[and we apply the natural bijection $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$]; cf. Remark 3.5.2 below.

(iii) One may also construct a *"$\mathcal{D}$-version"* — which, from the point of view of the theory of [AbsTopIII], one may also think of as a *"log-shell version"* — of the various data constructed in (i), (ii). To this end, we write

$$\mathcal{D}^{\Vdash}_{\mathrm{mod}}$$

for a [i.e., another] copy of $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$. Thus, one may associate to $\mathcal{D}^{\Vdash}_{\mathrm{mod}}$ various objects $\Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod}}}$, $\mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{mod}}) \stackrel{\sim}{\to} \mathbb{V}_{\mathrm{mod}}$, $\log^{\mathcal{D}}_{\mathrm{mod}}(p_v) \in \Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod}},v} \subseteq \Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod}}}$ [for $v \in \mathbb{V}_{\mathrm{mod}}$] that map to the corresponding objects associated to $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ under the *tautological equivalence of categories* $\mathcal{C}^{\Vdash}_{\mathrm{mod}} \stackrel{\sim}{\to} \mathcal{D}^{\Vdash}_{\mathrm{mod}}$. Write $\underline{v} \in \underline{\mathbb{V}}$ for the element of $\underline{\mathbb{V}}$ that corresponds to v. Next, suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$; then let us recall from [AbsTopIII], Proposition 5.8, (iii), that [since the profinite group associated to $\mathcal{D}^{\vdash}_{\underline{v}}$ is the absolute Galois group of an MLF] one may *construct algorithmically* from $\mathcal{D}^{\vdash}_{\underline{v}}$ a *topological monoid* isomorphic to $\mathbb{R}_{\geq 0}$

$$(\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$$

[i.e., the topological monoid determined by the nonnegative elements of the ordered topological group "$\mathbb{R}_{\mathrm{non}}(G)$" of *loc. cit.*] equipped with a *distinguished "Frobenius element"* $\in (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$; if $e_{\underline{v}}$ is the *absolute ramification index* of the MLF $K_{\underline{v}}$, then we shall write $\log^{\mathcal{D}}_{\Phi}(p_{\underline{v}}) \in (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$ for the result of multiplying this Frobenius element by [the positive real number] $e_{\underline{v}}$. Next, suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$; then let us recall from [AbsTopIII], Proposition 5.8, (vi), that [since, by definition, $\mathcal{D}^{\vdash}_{\underline{v}} \in \mathrm{Ob}(\mathbb{TM}^{\vdash})$] one may *construct algorithmically* from $\mathcal{D}^{\vdash}_{\underline{v}}$ a *topological monoid* isomorphic to $\mathbb{R}_{\geq 0}$

$$(\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$$

[i.e., the topological monoid determined by the nonnegative elements of the ordered topological group "$\mathbb{R}_{\mathrm{arc}}(G)$" of *loc. cit.*] equipped with a *distinguished "Frobenius element"* $\in (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$; we shall write $\log^{\mathcal{D}}_{\Phi}(p_{\underline{v}}) \in (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$ for the result of dividing this Frobenius element by [the positive real number] 2π. In particular, for every $\underline{v} \in \underline{\mathbb{V}}$, we obtain a uniquely determined *isomorphism of topological monoids* [which are isomorphic to $\mathbb{R}_{\geq 0}$]

$$\rho^{\mathcal{D}}_{\underline{v}} : \Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod}},v} \stackrel{\sim}{\to} (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$$

by assigning $\log^{\mathcal{D}}_{\mathrm{mod}}(p_v) \mapsto \frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]} \log^{\mathcal{D}}_{\Phi}(p_{\underline{v}})$. Thus, we obtain *data* [consisting of a Frobenioid, a bijection of sets, a collection of data indexed by $\underline{\mathbb{V}}$, and a collection of isomorphisms indexed by $\underline{\mathbb{V}}$]

$$\mathfrak{F}^{\Vdash}_{\mathcal{D}} \stackrel{\mathrm{def}}{=} (\mathcal{D}^{\Vdash}_{\mathrm{mod}},\ \mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{mod}}) \stackrel{\sim}{\to} \underline{\mathbb{V}},\ \{\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}},\ \{\rho^{\mathcal{D}}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}})$$

[where we apply the natural bijection $\underline{\mathbb{V}} \stackrel{\sim}{\to} \mathbb{V}_{\mathrm{mod}}$], which, by [AbsTopIII], Proposition 5.8, (iii), (vi), may be *reconstructed algorithmically* from the *data* $\{\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$.

Remark 3.5.1.

(i) The formal symbol "$\log(\underline{\Theta})$" may be thought of as the result of *identifying* the various formal quotients "$\log_{\Phi}(\underline{\underline{\Theta}}_{\underline{v}})/\log_{\Phi}(\underline{\underline{q}}_{\underline{v}})$", as $\underline{v}$ varies over the elements of $\underline{\mathbb{V}}^{\mathrm{bad}}$.

(ii) The global Frobenioids $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$, $\mathcal{C}^{\Vdash}_{\mathrm{tht}}$ of Example 3.5 may be thought of as *"devices for currency exchange"* between the various *"local currencies"* constituted by the divisor monoids at the various $\underline{v} \in \underline{\mathbb{V}}$.

(iii) One may also formulate the data contained in $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$, $\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$ via the language of *poly-Frobenioids* as developed in [FrdII], §5, but we shall not pursue this topic in the present series of papers.

Remark 3.5.2. In Example 3.5, as well as in the following discussion, we shall often speak of *"isomorphisms of collections of data"*, relative to the following conventions.

(i) Such isomorphisms are always assumed to satisfy various *evident compatibility conditions*, relative to the various relationships stipulated between the various constituent data, whose explicit mention we shall omit for the sake of simplicity.

(ii) In situations where the collections of data consist partially of various *categories*, the portion of the "isomorphism of collections of data" involving corresponding categories is to be understood as an *isomorphism class of equivalences of categories* [cf. §0].

Definition 3.6. Fix a collection of *initial* Θ-*data* $(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$ as in Definition 3.1. In the following, we shall use the various notations introduced in Definition 3.1 for various objects associated to this initial Θ-data. Then we define a Θ-*Hodge theater* [relative to the given initial Θ-data] to be a collection of data

$$^{\dagger}\mathcal{HT}^{\Theta} = (\{^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ ^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$$

that satisfies the following conditions:

(a) If $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is a *category* which admits an equivalence of categories $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}} \xrightarrow{\sim} \underline{\underline{\mathcal{F}}}_{\underline{v}}$ [where $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is as in Examples 3.2, (i); 3.3, (i)]. In particular, $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ admits a natural *Frobenioid structure* [cf. [FrdI], Corollary 4.11, (iv)], which may be *constructed solely from the category-theoretic structure of* $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$. Write $^{\dagger}\mathcal{D}_{\underline{v}}$, $^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$, $^{\dagger}\mathcal{D}^{\Theta}_{\underline{v}}$, $^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$, $^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}$ for the objects *constructed category-theoretically from* $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ that correspond to the objects without a "$\dagger$" discussed in Examples 3.2, 3.3 [cf., especially, Examples 3.2, (vi); 3.3, (iii)].

(b) If $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is a *collection of data* $(^{\dagger}\mathcal{C}_{\underline{v}}, {}^{\dagger}\mathcal{D}_{\underline{v}}, {}^{\dagger}\kappa_{\underline{v}})$ — where $^{\dagger}\mathcal{C}_{\underline{v}}$ is a *category* equivalent to the category $\mathcal{C}_{\underline{v}}$ of Example 3.4, (i); $^{\dagger}\mathcal{D}_{\underline{v}}$ is an Aut-*holomorphic orbispace*; and $^{\dagger}\kappa_{\underline{v}} : \mathcal{O}^{\rhd}(^{\dagger}\mathcal{C}_{\underline{v}}) \hookrightarrow \mathcal{A}_{^{\dagger}\mathcal{D}_{\underline{v}}}$ is an inclusion of topological monoids, which we shall refer to as the *Kummer structure* on $^{\dagger}\mathcal{C}_{\underline{v}}$ — such that there exists an isomorphism of collections of data $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}} \xrightarrow{\sim} \underline{\underline{\mathcal{F}}}_{\underline{v}}$ [where $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is as in Example 3.4, (i)]. Write $^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$, $^{\dagger}\mathcal{D}^{\Theta}_{\underline{v}}$, $^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$, $^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}$ for the objects *constructed algorithmically from* $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ that correspond to the objects without a "$\dagger$" discussed in Example 3.4, (ii), (iii).

(c) $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ is a *collection of data*

$$(^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}},\ \mathrm{Prime}(^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ \{^{\dagger}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

— where ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ is a *category* which admits an equivalence of categories ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{C}^{\Vdash}_{\mathrm{mod}}$ [which implies that ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ admits a natural category-theoretically constructible *Frobenioid structure* — cf. [FrdI], Corollary 4.11, (iv); [FrdI], Theorem 6.4, (i)]; $\mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}}$ is a bijection of sets, where we write $\mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}})$ for the set of primes constructed from the category ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ [cf. [FrdI], Theorem 6.4, (iii)]; ${}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$ is as discussed in (a), (b) above; ${}^{\dagger}\rho_{\underline{v}} : \Phi_{{}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}},\underline{v}} \xrightarrow{\sim} \Phi^{\mathrm{rlf}}_{{}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}}$ [where we use notation as in the discussion of Example 3.5, (i)] is an *isomorphism of topological monoids*. Moreover, we require that there exist an *isomorphism of collections of data* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\sim} \mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ [where $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ is as in Example 3.5, (ii)]. Write ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$, ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathcal{D}}$ for the objects *constructed algorithmically from* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ that correspond to the objects without a "†" discussed in Example 3.5, (ii), (iii).

Remark 3.6.1. When we discuss various collections of Θ-Hodge theaters, labeled by some symbol "□" in place of a "†", we shall apply the notation of Definition 3.6 with "†" replaced by "□" to denote the various objects associated to the Θ-Hodge theater labeled by "□".

Remark 3.6.2. If ${}^{\dagger}\mathcal{HT}^{\Theta}$ and ${}^{\ddagger}\mathcal{HT}^{\Theta}$ are Θ-*Hodge theaters*, then there is an evident notion of *isomorphism of* Θ-*Hodge theaters* ${}^{\dagger}\mathcal{HT}^{\Theta} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{\Theta}$ [cf. Remark 3.5.2]. We leave the routine details to the interested reader.

Corollary 3.7. **(Θ-Links Between Θ-Hodge Theaters)** *Fix a collection of* **initial Θ-data** $(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$ *as in Definition 3.1. Let*

$$ {}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}); \quad {}^{\ddagger}\mathcal{HT}^{\Theta} = (\{{}^{\ddagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}) $$

be **Θ-Hodge theaters** *[relative to the given initial Θ-data]. Then:*

(i) **(Θ-Link)** *The* **full poly-isomorphism** *[cf. §0] between collections of data [cf. Remark 3.5.2]*

$$ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} $$

is **nonempty** *[cf. Remark 3.7.1 below]. We shall refer to this full poly-isomorphism as the* **Θ-link**

$$ {}^{\dagger}\mathcal{HT}^{\Theta} \quad \xrightarrow{\Theta} \quad {}^{\ddagger}\mathcal{HT}^{\Theta} $$

from ${}^{\dagger}\mathcal{HT}^{\Theta}$ *to* ${}^{\ddagger}\mathcal{HT}^{\Theta}$.

(ii) **(Preservation of "$\mathcal{D}^{\vdash}$")** *Let* $\underline{v} \in \underline{\mathbb{V}}$. *Recall the* **tautological isomorphisms** ${}^{\square}\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} {}^{\square}\mathcal{D}^{\Theta}_{\underline{v}}$ *for* $\square = \dagger, \ddagger$ *— i.e., which arise from the definitions when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ *[cf. Examples 3.3, (ii); 3.4, (iii)], and which arise from a natural product functor [cf. Example 3.2, (v)] when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. *Then we obtain a composite [full] poly-isomorphism*

$$ {}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} {}^{\dagger}\mathcal{D}^{\Theta}_{\underline{v}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}} $$

by composing the tautological isomorphism just mentioned with the poly-isomorphism induced by the **Θ-link poly-isomorphism** *of (i).*

(iii) **(Preservation of "$\mathcal{O}^\times$")** *Let* $\underline{v} \in \underline{\mathbb{V}}$. *Recall the* **tautological isomorphisms** $\mathcal{O}^\times_{\square\mathcal{C}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^\times_{\square\mathcal{C}^{\Theta}_{\underline{v}}}$ *[where we omit the notation "$(-)$"] for* $\square = \dagger, \ddagger$ *— i.e., which arise from the definitions when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ *[cf. Examples 3.3, (ii); 3.4, (iii)], and which are induced by the natural product functor [cf. Example 3.2, (v)] when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. *Then, relative to the corresponding composite isomorphism of (ii), we obtain a composite [full] poly-isomorphism*

$$\mathcal{O}^\times_{\dagger\mathcal{C}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^\times_{\dagger\mathcal{C}^{\Theta}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^\times_{\ddagger\mathcal{C}^{\vdash}_{\underline{v}}}$$

by composing the tautological isomorphism just mentioned with the poly-isomorphism induced by the **Θ-link poly-isomorphism** *of (i).*

Proof. The various assertions of Corollary 3.7 follow immediately from the definitions and the discussion of Examples 3.2, 3.3, 3.4, and 3.5. ○

Remark 3.7.1. One verifies immediately that there exist *many distinct isomorphisms* ${}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \xrightarrow{\sim} {}^\ddagger\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ as in Corollary 3.7, (i), *none* of which is conferred a *"distinguished"* status, i.e., in the fashion of the *"natural isomorphism* $\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\sim} \mathfrak{F}^{\Vdash}_{\mathrm{tht}}$*"* discussed in Example 3.5, (ii).

The following result follows formally from Corollary 3.7.

Corollary 3.8. (Frobenius-pictures of Θ-Hodge Theaters) *Fix a collection of initial Θ-data as in Corollary 3.7. Let* $\{{}^n\mathcal{HT}^\Theta\}_{n\in\mathbb{Z}}$ *be a* **collection of distinct Θ-Hodge theaters** *indexed by the integers. Then by applying Corollary 3.7, (i), with* ${}^\dagger\mathcal{HT}^\Theta \stackrel{\mathrm{def}}{=} {}^n\mathcal{HT}^\Theta$, ${}^\ddagger\mathcal{HT}^\Theta \stackrel{\mathrm{def}}{=} {}^{(n+1)}\mathcal{HT}^\Theta$, *we obtain an* **infinite chain**

$$\ldots \xrightarrow{\Theta} {}^{(n-1)}\mathcal{HT}^\Theta \xrightarrow{\Theta} {}^n\mathcal{HT}^\Theta \xrightarrow{\Theta} {}^{(n+1)}\mathcal{HT}^\Theta \xrightarrow{\Theta} \ldots$$

of **Θ-linked Θ-Hodge theaters**. *This infinite chain may be represented symbolically as an* **oriented graph** $\vec{\Gamma}$ *[cf. [AbsTopIII], §0]*

$$\ldots \to \bullet \to \bullet \to \bullet \to \ldots$$

— i.e., where the arrows correspond to the "$\xrightarrow{\Theta}$'s", and the "$\bullet$'s" correspond to the "${}^n\mathcal{HT}^\Theta$". This oriented graph $\vec{\Gamma}$ *admits a natural action by* $\mathbb{Z}$ *— i.e., a* **translation symmetry** *— but it does* **not admit arbitrary permutation symmetries**. *For instance,* $\vec{\Gamma}$ *does not admit an automorphism that switches two adjacent vertices, but leaves the remaining vertices fixed. Put another way, from the point of view of the discussion of [FrdI], §I4, the mathematical structure constituted by this infinite chain is* **"Frobenius-like"**, *or* **"order-conscious"**. *It is for this reason that we shall refer to this infinite chain in the following discussion as the* **Frobenius-picture**.

Remark 3.8.1.

(i) Perhaps the central defining aspect of the Frobenius-picture is the fact that the Θ-link maps

$$ {}^{n}\underline{\underline{\Theta}}_{\underline{v}} \quad \mapsto \quad {}^{(n+1)}\underline{\underline{q}}_{\underline{v}} $$

[i.e., where $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ — cf. the discussion of Example 3.2, (v)]. From this point of view, the Frobenius-picture may be depicted as in Fig. 3.1 below — i.e., each box is a Θ-Hodge theater; the "$\rightsquigarrow$" may be thought of as denoting the scheme theory that lies between "$\underline{\underline{q}}_{\underline{v}}$" and "$\underline{\underline{\Theta}}_{\underline{v}}$"; the "- - - -" denotes the Θ-link.

$$ \cdots \quad \text{- - - -} \quad \boxed{{}^{n}\underline{\underline{q}}_{\underline{v}} \ \rightsquigarrow \ {}^{n}\underline{\underline{\Theta}}_{\underline{v}}} \quad \text{- - - -} \quad \boxed{{}^{(n+1)}\underline{\underline{q}}_{\underline{v}} \ \rightsquigarrow \ {}^{(n+1)}\underline{\underline{\Theta}}_{\underline{v}}} \quad \text{- - - -} \quad \cdots $$

$$ {}^{n}\underline{\underline{\Theta}}_{\underline{v}} \qquad \mapsto \qquad {}^{(n+1)}\underline{\underline{q}}_{\underline{v}} $$

Fig. 3.1: Frobenius-picture of Θ-Hodge theaters

(ii) It is perhaps not surprising [cf. the theory of [FrdI]] that the Frobenius-picture involves, in an essential way, the *divisor monoid* portion [i.e., "$\underline{\underline{q}}_{\underline{v}}$" and "$\underline{\underline{\Theta}}_{\underline{v}}$"] of the various Frobenioids that appear in a Θ-Hodge theater. Put another way,

> it is as if the *"Frobenius-like nature"* of the divisor monoid portion of the Frobenioids involved *induces the "Frobenius-like nature" of the Frobenius-picture.*

By contrast, *observe* that for $\underline{v} \in \underline{\mathbb{V}}$, the isomorphisms

$$ \ldots \ \stackrel{\sim}{\rightarrow} \ {}^{n}\mathcal{D}^{\vdash}_{\underline{v}} \ \stackrel{\sim}{\rightarrow} \ {}^{(n+1)}\mathcal{D}^{\vdash}_{\underline{v}} \ \stackrel{\sim}{\rightarrow} \ \ldots $$

of Corollary 3.7, (ii), imply that if one thinks of the various ${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$ as being only known *up to isomorphism*, then

> *one may regard ${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$ as a sort of* **constant invariant** *of the various Θ-Hodge theaters that constitute the Frobenius-picture*

— cf. Remark 3.9.1 below. This *observation* is the starting point of the theory of the *étale-picture* [cf. Corollary 3.9, (i), below]. Note that by Corollary 3.7, (iii), we also obtain isomorphisms

$$ \ldots \ \stackrel{\sim}{\rightarrow} \ \mathcal{O}^{\times}_{{}^{n}\mathcal{C}^{\vdash}_{\underline{v}}} \ \stackrel{\sim}{\rightarrow} \ \mathcal{O}^{\times}_{{}^{(n+1)}\mathcal{C}^{\vdash}_{\underline{v}}} \ \stackrel{\sim}{\rightarrow} \ \ldots $$

lying over the isomorphisms involving the "${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$" discussed above.

(iii) In the situation of (ii), suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$. Then ${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$ is simply the category of connected objects of the Galois category associated to the profinite group $G_{\underline{v}}$. That is to say, one may think of ${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$ as representing *"$G_{\underline{v}}$ up to* *isomorphism"*. Then each ${}^{n}\mathcal{D}_{\underline{v}}$ represents an *"isomorph of the topological group* $\Pi_{\underline{v}}$, **labeled** *by* n*, which is regarded as an extension of some isomorph of* $G_{\underline{v}}$ *that is* **independent** *of* n*"*. In particular, the quotients corresponding to $G_{\underline{v}}$ of the copies of $\Pi_{\underline{v}}$ that arise from ${}^{n}\mathcal{HT}^{\Theta}$ for different n are only related to one another via some *indeterminate* isomorphism. Thus, from the point of view of the theory of [AbsTopIII] [cf. [AbsTopIII], §I3; [AbsTopIII], Remark 5.10.2, (ii)], each $\Pi_{\underline{v}}$ gives rise to a *well-defined ring structure* — i.e., a *"holomorphic structure"* — which is *obliterated* by the indeterminate isomorphism between the quotient isomorphs of $G_{\underline{v}}$ arising from ${}^{n}\mathcal{HT}^{\Theta}$ for distinct n.

(iv) In the situation of (ii), suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Then ${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$ is an object of $\mathbb{TM}^{\vdash}$; each ${}^{n}\mathcal{D}_{\underline{v}}$ represents an *"isomorph of the* Aut*-holomorphic orbispace* $\underrightarrow{\mathbb{X}}_{\underline{v}}$, **labeled** *by* n*, whose associated [complex archimedean] topological field* $\overline{\mathcal{A}}_{\underrightarrow{\mathbb{X}}_{\underline{v}}}$ *gives rise to an isomorph of* $\mathcal{D}^{\vdash}_{\underline{v}}$ *that is* **independent** *of* n*"*. In particular, the various isomorphs of $\mathcal{D}^{\vdash}_{\underline{v}}$ associated to the copies of $\underrightarrow{\mathbb{X}}_{\underline{v}}$ that arise from ${}^{n}\mathcal{HT}^{\Theta}$ for different n are only related to one another via some *indeterminate* isomorphism. Thus, from the point of view of the theory of [AbsTopIII] [cf. [AbsTopIII], §I3; [AbsTopIII], Remark 5.10.2, (ii)], each $\underrightarrow{\mathbb{X}}_{\underline{v}}$ gives rise to a *well-defined ring structure* — i.e., a *"holomorphic structure"* — which is *obliterated* by the indeterminate isomorphism between the isomorphs of $\mathcal{D}^{\vdash}_{\underline{v}}$ arising from ${}^{n}\mathcal{HT}^{\Theta}$ for distinct n.

The discussion of Remark 3.8.1, (iii), (iv), may be summarized as follows.

Corollary 3.9. (Étale-pictures of Θ-Hodge Theaters) *In the situation of Corollary 3.8, let* $\underline{v} \in \underline{\mathbb{V}}$*. Then:*

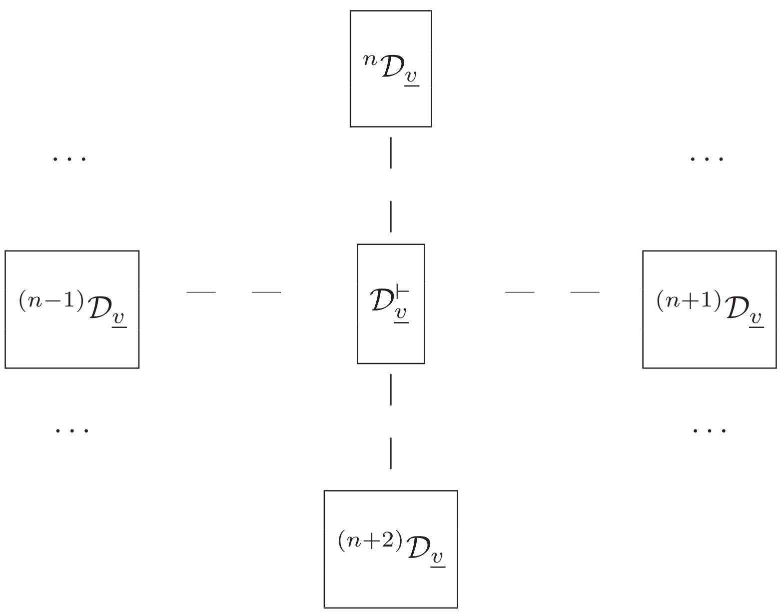

Fig. 3.2: Étale-picture of Θ-Hodge theaters

$$\begin{array}{ccccc} & & \boxed{{}^{n}\mathcal{D}_{\underline{v}}} & & \\ \dots & & | & & \dots \\ & & | & & \\ \boxed{{}^{(n-1)}\mathcal{D}_{\underline{v}}} & — \quad — & \boxed{\mathcal{D}^{\vdash}_{\underline{v}} \ \curvearrowright \ \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}} & — \quad — & \boxed{{}^{(n+1)}\mathcal{D}_{\underline{v}}} \\ & & | & & \\ \dots & & & & \dots \\ & & | & & \\ & & \boxed{{}^{(n+2)}\mathcal{D}_{\underline{v}}} & & \end{array}$$

Fig. 3.3: Étale-picture plus units

(i) We have a **diagram** *as in Fig. 3.2 above, which we refer to as the* **étale-picture**. *Here, each horizontal and vertical "— —" denotes the relationship between* ${}^{(-)}\mathcal{D}_{\underline{v}}$ *and* $\mathcal{D}^{\vdash}_{\underline{v}}$ *— i.e., an extension of topological groups when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*, or the underlying object of* $\mathbb{TM}^{\vdash}$ *arising from the associated topological field when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ *— discussed in Remark 3.8.1, (iii), (iv). The étale-picture [unlike the Frobenius-picture!] admits* **arbitrary permutation symmetries** *among the labels* $n \in \mathbb{Z}$ *corresponding to the various* Θ*-Hodge theaters. Put another way, the étale-picture may be thought of as a sort of* **canonical splitting** *of the Frobenius-picture.*

(ii) In a similar vein, we have a **diagram** *as in Fig. 3.3 above, obtained by replacing the "*$\mathcal{D}^{\vdash}_{\underline{v}}$*" in the middle of Fig. 3.2 by "*$\mathcal{D}^{\vdash}_{\underline{v}} \curvearrowright \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$*". Here, each horizontal and vertical "— —" denotes the relationship between* ${}^{(-)}\mathcal{D}_{\underline{v}}$ *and* $\mathcal{D}^{\vdash}_{\underline{v}}$ *discussed in (i); when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*, the notation "*$\mathcal{D}^{\vdash}_{\underline{v}} \curvearrowright \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$*" denotes an isomorph of the pair consisting of the category* $\mathcal{D}^{\vdash}_{\underline{v}}$ *together with the group-like monoid* $\mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ *on* $\mathcal{D}^{\vdash}_{\underline{v}}$*; when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*, the notation "*$\mathcal{D}^{\vdash}_{\underline{v}} \curvearrowright \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$*" denotes an isomorph of the pair consisting of the object* $\mathcal{D}^{\vdash}_{\underline{v}} \in \mathrm{Ob}(\mathbb{TM}^{\vdash})$ *and the topological group* $\mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ *[which is isomorphic — but not canonically! — to the compact factor of* $\mathcal{D}^{\vdash}_{\underline{v}}$*]. Just as in the case of (i), this diagram admits* **arbitrary permutation symmetries** *among the labels* $n \in \mathbb{Z}$ *corresponding to the various* Θ*-Hodge theaters.*

Remark 3.9.1. If one formulates things relative to the language of [AbsTopIII], Definition 3.5, then ${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$ constitutes a **core**. Relative to the theory of [AbsTopIII], §5, this core is essentially the **mono-analytic core** discussed in [AbsTopIII], §I3; [AbsTopIII], Remark 5.10.2, (ii). Indeed, the symbol "$\vdash$" is intended — both in [AbsTopIII] and in the present series of papers! — as an abbreviation for the term *"mono-analytic"*.

Remark 3.9.2. Whereas the *étale-picture* of Corollary 3.9, (i), will remain valid throughout the development of the remainder of the theory of the present series of papers, the local units "$\mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$" that appear in Corollary 3.9, (ii), will ultimately *cease to be a constant invariant* of various enhanced versions of the Frobenius-picture that will arise in the theory of [IUTchIII]. In a word, these enhancements revolve around the incorporation into each Hodge theater of the *"rotation of addition [i.e., '⊞'] and multiplication [i.e., '⊠']"* in the style of the theory of [AbsTopIII].

Remark 3.9.3.

(i) As discussed in [AbsTopIII], §I3; [AbsTopIII], Remark 5.10.2, (ii), the *"mono-analytic core"* $\{\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ may be thought of as a sort of **fixed underlying real-analytic surface** associated to a number field on which various *holomorphic structures* are imposed. Then the Frobenius-picture in its entirety may be thought of as a sort of **global arithmetic analogue** of the notion of a **Teichmüller geodesic** *in classical complex Teichmüller theory* or, alternatively, as a global arithmetic analogue of the **canonical liftings** *of p-adic Teichmüller theory* [cf. the discussion of [AbsTopIII], §I5].

(ii) Recall that in classical complex Teichmüller theory, **one** of the **two** real dimensions of the surface is **dilated** as one moves along a Teichmüller geodesic, while the **other** of the two real dimensions is **held fixed**. In the case of the Frobenius-picture of Corollary 3.8, the **local units "$\mathcal{O}^{\times}$"** correspond to the dimension that is **held fixed**, while the **local value groups** are subject to **"Θ-dilations"** as one moves along the diagram constituted by the Frobenius-picture. Note that in order to construct such a mathematical structure in which the local units and local value groups are treated **independently**, it is of crucial importance to avail oneself of the various **characteristic splittings** that appear in the split Frobenioids of Examples 3.2, 3.3, 3.4. Here, we note in passing that, in the case of Example 3.2, this splitting corresponds to the **"constant multiple rigidity"** *of the étale theta function*, which forms a central theme of the theory of [EtTh].

(iii) In classical complex Teichmüller theory, the two real dimensions of the surface that are treated independently of one another correspond to the **real** and **imaginary** parts of the coordinate obtained by locally integrating the square root of a given square differential. In particular, it is of crucial importance in classical complex Teichmüller theory that these real and imaginary parts *not be "subject to confusion with one another"*. In the case of the square root of a square differential, the only indeterminacy that arises is indeterminacy with respect to *multiplication by* -1, an operation that satisfies the *crucial property of preserving the real and imaginary parts of a complex number*. By contrast, it is interesting to note that

> if, for $n \geq 3$, one attempts to construct Teichmüller deformations in the fashion of classical complex Teichmüller theory by means of coordinates obtained by *locally integrating the n-th root of a given section of the n-th tensor power of the sheaf of differentials*, then one must contend with an indeterminacy with respect to *multiplication by an n-th root of unity*, an operation that results in an *essential confusion between the real and imaginary parts of a complex number*.

(iv) Whereas linear movement along the oriented graph $\vec{\Gamma}$ of Corollary 3.8 corresponds to the *linear flow* along a Teichmüller geodesic, the *"rotation of addition [i.e., '$\boxplus$'] and multiplication [i.e., '$\boxtimes$']"* in the style of the theory of [AbsTopIII] — which will be incorporated into the theory of the present series of papers in [IUTchIII] [cf. Remark 3.9.2] — corresponds to *rotations around a fixed point* in the complex geometry arising from Teichmüller theory [cf., e.g., the discussion of [AbsTopIII], §I3; the hyperbolic geometry of the upper half-plane, regarded as the "Teichmüller space" of compact Riemann surfaces of genus 1]. Alternatively, in the analogy with p-adic Teichmüller theory, this "rotation of $\boxplus$ and $\boxtimes$" corresponds to the *Frobenius morphism in positive characteristic* — cf. the discussion of [AbsTopIII], §I5.

Remark 3.9.4. At first glance, the assignment "${}^{n}\underline{\underline{\Theta}}_{\underline{v}} \mapsto {}^{(n+1)}\underline{\underline{q}}_{\underline{v}}$" [cf. Remark 3.8.1, (i)] may strike the reader as being nothing more than a *"conventional evaluation map"* [i.e., of the theta function at a torsion point — cf. the discussion of Example 3.2, (iv)]. Although we shall ultimately be interested, in the theory of the present series of papers, in such *"Hodge-Arakelov-style evaluation maps"* [within a fixed Hodge theater!] *of the theta function at torsion points"* [cf. the theory of [IUTchII]], the Θ-link considered here *differs quite fundamentally* from such conventional evaluation maps in the following respect:

> *the* **value** ${}^{(n+1)}\underline{\underline{q}}_{\underline{v}}$ *belongs to a* **distinct scheme theory** *— i.e., the scheme theory represented by the distinct* Θ*-Hodge theater* ${}^{(n+1)}\mathcal{HT}^{\Theta}$ *— from the* **base** ${}^{n}\underline{\underline{q}}_{\underline{v}}$ *[which belongs to the scheme theory represented by the* Θ*-Hodge theater* ${}^{n}\mathcal{HT}^{\Theta}$*] over which the theta function* ${}^{n}\underline{\underline{\Theta}}_{\underline{v}}$ *is constructed.*

The distinctness of the ring/scheme theories of distinct Θ-Hodge theaters may be seen, for instance, in the *indeterminacy* of the isomorphism between the associated isomorphs of $\mathcal{D}^{\vdash}_{\underline{v}}$, an indeterminacy which has the effect of *obliterating* the ring structure — i.e., the *"arithmetic holomorphic structure"* — associated to ${}^{n}\mathcal{D}_{\underline{v}}$ for distinct n [cf. the discussion of Remark 3.8.1, (iii), (iv)].

Section 4: Multiplicative Combinatorial Teichmüller Theory

In the present §4, we begin to prepare for the construction of the various "enhancements" to the Θ-Hodge theaters of §3 that will be made in §5. More precisely, in the present §4, we discuss the *combinatorial aspects* of the *"$\mathcal{D}$"* — i.e., in the terminology of the theory of Frobenioids, the *"base category"* — *portion* of the notions to be introduced in §5 below. In a word, these combinatorial aspects revolve around the **"functorial dynamics"** *imposed upon the various number fields and local fields involved by the* **"labels"**

$$\in \quad \mathbb{F}_l^{\circledast} \quad \stackrel{\mathrm{def}}{=} \quad \mathbb{F}_l^{\times}/\{\pm 1\}$$

— where we note that the set $\mathbb{F}_l^{\circledast}$ is of *cardinality* $l^{\circledast} \stackrel{\mathrm{def}}{=} (l-1)/2$ — of the l-torsion points at which we intend to conduct, in [IUTchII], the *"Hodge-Arakelov-theoretic evaluation"* of the étale theta function studied in [EtTh] [cf. Remarks 4.3.1; 4.3.2; 4.5.1, (v); 4.9.1, (i)].

In the following, we fix a collection of *initial* Θ*-data*

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}},\ \underline{\epsilon})$$

as in Definition 3.1; also, we shall use the various notations introduced in Definition 3.1 for various objects associated to this initial Θ-data.

Definition 4.1.

(i) We define a *holomorphic base-prime-strip*, or $\mathcal{D}$*-prime-strip*, [relative to the given initial Θ-data] to be a collection of data

$$^{\dagger}\mathfrak{D} = \{^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

that satisfies the following conditions: (a) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then $^{\dagger}\mathcal{D}_{\underline{v}}$ is a *category* which admits an equivalence of categories $^{\dagger}\mathcal{D}_{\underline{v}} \xrightarrow{\sim} \mathcal{D}_{\underline{v}}$ [where $\mathcal{D}_{\underline{v}}$ is as in Examples 3.2, (i); 3.3, (i)]; (b) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then $^{\dagger}\mathcal{D}_{\underline{v}}$ is an Aut-*holomorphic orbispace* such that there exists an isomorphism of Aut-holomorphic orbispaces $^{\dagger}\mathcal{D}_{\underline{v}} \xrightarrow{\sim} \mathcal{D}_{\underline{v}}$ [where $\mathcal{D}_{\underline{v}}$ is as in Example 3.4, (i)]. Observe that if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then $\pi_1(^{\dagger}\mathcal{D}_{\underline{v}})$ determines, in a functorial fashion, a *topological* [in fact, *profinite* if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$] *group corresponding to "*$\underline{C}_{\underline{v}}$*"* [cf. Corollary 1.2 if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$; [EtTh], Proposition 2.4, if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], which contains $\pi_1(^{\dagger}\mathcal{D}_{\underline{v}})$ as an open subgroup; thus, if we write $^{\dagger}\underline{\mathcal{D}}_{\underline{v}}$ for $\mathcal{B}(-)^0$ of this topological group, then we obtain a *natural morphism* $^{\dagger}\mathcal{D}_{\underline{v}} \to {^{\dagger}\underline{\mathcal{D}}_{\underline{v}}}$ [cf. §0]. In a similar vein, if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then since $\underline{X}_{\to\underline{v}}$ admits a $K_{\underline{v}}$-core, a routine translation into the "language of Aut-holomorphic orbispaces" of the argument given in the proof of Corollary 1.2 [cf. also [AbsTopIII], Corollary 2.4] reveals that $^{\dagger}\mathcal{D}_{\underline{v}}$ determines, in a functorial fashion, an Aut-holomorphic orbispace $^{\dagger}\underline{\mathcal{D}}_{\underline{v}}$ corresponding to "$\underline{C}_{\underline{v}}$", together with a *natural morphism* $^{\dagger}\mathcal{D}_{\underline{v}} \to {^{\dagger}\underline{\mathcal{D}}_{\underline{v}}}$ *of* Aut-*holomorphic orbispaces.* Thus, in summary, one obtains a collection of data

$$^{\dagger}\underline{\mathfrak{D}} = \{^{\dagger}\underline{\mathcal{D}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

completely determined by ${}^{\dagger}\mathfrak{D}$.

(ii) Suppose that we are in the situation of (i). Then observe that by applying the *group-theoretic algorithm* of [AbsTopI], Lemma 4.5 [cf., especially, [AbsTopI], Lemma 4.5, (v), as well as Remark 1.2.2, (ii), of the present paper], to construct the set of conjugacy classes of cuspidal decomposition groups of the topological group $\pi_1({}^{\dagger}\mathcal{D}_{\underline{v}})$ when $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, or by considering $\pi_0(-)$ of a cofinal collection of "neighborhoods of infinity" [i.e., complements of compact subsets] of the underlying topological space of ${}^{\dagger}\mathcal{D}_{\underline{v}}$ when $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, it makes sense to speak of the *set of cusps* of ${}^{\dagger}\mathcal{D}_{\underline{v}}$; a similar observation applies to ${}^{\dagger}\underline{\mathcal{D}}_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}$. If $\underline{v} \in \underline{\mathbb{V}}$, then we define a *label class of cusps* of ${}^{\dagger}\mathcal{D}_{\underline{v}}$ to be the set of cusps of ${}^{\dagger}\mathcal{D}_{\underline{v}}$ that lie over a single *"nonzero cusp"* [i.e., a cusp that arises from a *nonzero element* of the quotient "Q" that appears in the definition of a "hyperbolic orbicurve of type $(1, l\text{-tors})_{\pm}$" given in [EtTh], Definition 2.1] of ${}^{\dagger}\underline{\mathcal{D}}_{\underline{v}}$; write

$$\text{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{v}})$$

for the *set of label classes of cusps* of ${}^{\dagger}\mathcal{D}_{\underline{v}}$. Thus, for each $\underline{v} \in \underline{\mathbb{V}}$, $\text{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{v}})$ admits a natural $\mathbb{F}_l^{\circledast}$*-torsor structure* [i.e., which arises from the natural action of $\mathbb{F}_l^{\times}$ on the quotient "Q" of [EtTh], Definition 2.1]. Moreover, [for any $\underline{v} \in \underline{\mathbb{V}}$!] one may construct, solely from ${}^{\dagger}\mathcal{D}_{\underline{v}}$, a *canonical element*

$${}^{\dagger}\underline{\eta}_{\underline{v}} \in \text{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{v}})$$

determined by "$\underline{\epsilon}_{\underline{v}}$" [cf. the notation of Definition 3.1, (f)]. [Indeed, this follows from [EtTh], Corollary 2.9, for $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, from Corollary 1.2 for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \bigcap \underline{\mathbb{V}}^{\text{non}}$, and from the evident translation into the "language of Aut-holomorphic orbispaces" of Corollary 1.2 for $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$.]

(iii) We define a *mono-analytic base-prime-strip*, or $\mathcal{D}^{\vdash}$*-prime-strip*, [relative to the given initial Θ-data] to be a collection of data

$${}^{\dagger}\mathfrak{D}^{\vdash} = \{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$$

that satisfies the following conditions: (a) if $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, then ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$ is a *category* which admits an equivalence of categories ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{D}^{\vdash}_{\underline{v}}$ [where $\mathcal{D}^{\vdash}_{\underline{v}}$ is as in Examples 3.2, (i); 3.3, (i)]; (b) if $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, then ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$ is an object of the category $\mathbb{TM}^{\vdash}$ [so, if $\mathcal{D}^{\vdash}_{\underline{v}}$ is as in Example 3.4, (ii), then there exists an isomorphism ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{D}^{\vdash}_{\underline{v}}$ in $\mathbb{TM}^{\vdash}$].

(iv) A *morphism of* $\mathcal{D}$*- (respectively,* $\mathcal{D}^{\vdash}$*-) prime-strips* is defined to be a collection of morphisms, indexed by $\underline{\mathbb{V}}$, between the various constituent objects of the prime-strips. Following the conventions of §0, one thus has a notion of *capsules of* $\mathcal{D}$*- (respectively,* $\mathcal{D}^{\vdash}$*-)* and *morphisms of capsules of* $\mathcal{D}$*- (respectively,* $\mathcal{D}^{\vdash}$*-) prime-strips*. Note that to any $\mathcal{D}$*-prime-strip* ${}^{\dagger}\mathfrak{D}$, one may associate, in a natural way, a $\mathcal{D}^{\vdash}$*-prime-strip* ${}^{\dagger}\mathfrak{D}^{\vdash}$ — which we shall refer to as the *mono-analyticization* of ${}^{\dagger}\mathfrak{D}$ — by considering appropriate subcategories at the *nonarchimedean primes* [cf. Examples 3.2, (i), (vi); 3.3, (i), (iii)], or by applying the construction of Example 3.4, (ii), at the *archimedean primes*.

(v) Write

$$\mathcal{D}^{\circledcirc} \stackrel{\mathrm{def}}{=} \mathcal{B}(\underline{C}_K)^0$$

[cf. §0]. Then recall from [AbsTopIII], Theorem 1.9 [cf. Remark 3.1.2], that there exists a *group-theoretic algorithm* for reconstructing, from $\pi_1(\mathcal{D}^{\circledcirc})$ [cf. §0], the algebraic closure "$\overline{F}$" of the base field "K", hence also the *set of valuations* "$\mathbb{V}(\overline{F})$" [e.g., as a collection of topologies on $\overline{F}$ — cf., e.g., [AbsTopIII], Corollary 2.8]. Moreover, for $\underline{w} \in \mathbb{V}(K)^{\mathrm{arc}}$, let us recall [cf. Remark 3.1.2; [AbsTopIII], Corollaries 2.8, 2.9] that one may *reconstruct group-theoretically*, from $\pi_1(\mathcal{D}^{\circledcirc})$, the Aut-*holomorphic orbispace* $\underline{\mathbb{C}}_{\underline{w}}$ associated to $\underline{C}_{\underline{w}}$. Let ${}^{\dagger}\mathcal{D}^{\circledcirc}$ be a category equivalent to $\mathcal{D}^{\circledcirc}$. Then let us write

$$\overline{\mathbb{V}}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

for the set of valuations [i.e., "$\mathbb{V}(\overline{F})$"], equipped with its natural $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$-action,

$$\mathbb{V}({}^{\dagger}\mathcal{D}^{\circledcirc}) \quad \stackrel{\mathrm{def}}{=} \quad \overline{\mathbb{V}}({}^{\dagger}\mathcal{D}^{\circledcirc})/\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$$

for the quotient of $\overline{\mathbb{V}}({}^{\dagger}\mathcal{D}^{\circledcirc})$ by $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$ [i.e., "$\mathbb{V}(K)$"], and, for $\underline{w} \in \mathbb{V}({}^{\dagger}\mathcal{D}^{\circledcirc})^{\mathrm{arc}}$,

$$\underline{\mathbb{C}}({}^{\dagger}\mathcal{D}^{\circledcirc}, \underline{w})$$

[i.e., "$\underline{\mathbb{C}}_{\underline{w}}$" — cf. the discussion of [AbsTopIII], Definition 5.1, (ii)] for the Aut-holomorphic orbispace obtained by applying these group-theoretic reconstruction algorithms to $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$. Now if $\mathbb{U}$ is an arbitrary Aut-*holomorphic orbispace*, then let us define a *morphism*

$$\mathbb{U} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$$

to be a morphism of Aut-holomorphic orbispaces [cf. [AbsTopIII], Definition 2.1, (ii)] $\mathbb{U} \to \underline{\mathbb{C}}({}^{\dagger}\mathcal{D}^{\circledcirc}, \underline{w})$ for some $\underline{w} \in \mathbb{V}({}^{\dagger}\mathcal{D}^{\circledcirc})^{\mathrm{arc}}$. Thus, it makes sense to speak of the pre-composite (respectively, post-composite) of such a morphism $\mathbb{U} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$ with a morphism of Aut-holomorphic orbispaces (respectively, with an isomorphism [cf. §0] ${}^{\dagger}\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^{\ddagger}\mathcal{D}^{\circledcirc}$ [i.e., where ${}^{\ddagger}\mathcal{D}^{\circledcirc}$ is a category equivalent to $\mathcal{D}^{\circledcirc}$]). Finally, just as in the discussion of (ii) in the case of "$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$", we may apply [AbsTopI], Lemma 4.5 [cf. also Remark 1.2.2, (ii), of the present paper], to conclude that it makes sense to speak of the *set of cusps* of ${}^{\dagger}\mathcal{D}^{\circledcirc}$, as well as the *set of label classes of cusps*

$$\mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

of ${}^{\dagger}\mathcal{D}^{\circledcirc}$, which admits a natural $\mathbb{F}_l^{\circledast}$-*torsor structure*.

(vi) Let ${}^{\dagger}\mathcal{D}^{\circledcirc}$ be a category equivalent to $\mathcal{D}^{\circledcirc}$, ${}^{\dagger}\mathfrak{D} = \{{}^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ a $\mathcal{D}$-*prime-strip*. If $\underline{v} \in \underline{\mathbb{V}}$, then we define a *poly-morphism* ${}^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$ to be a collection of morphisms ${}^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$ [cf. §0 when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$; (v) when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$]. We define a *poly-morphism*

$${}^{\dagger}\mathfrak{D} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$$

to be a collection of poly-morphisms $\{{}^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\mathcal{D}^{\circledcirc}\}_{\underline{v} \in \underline{\mathbb{V}}}$. Finally, if $\{{}^{e}\mathfrak{D}\}_{e \in E}$ is a *capsule of* $\mathcal{D}$-*prime-strips*, then we define a *poly-morphism*

$$\{{}^{e}\mathfrak{D}\}_{e \in E} \to {}^{\dagger}\mathcal{D}^{\circledcirc} \text{ (respectively, } \{{}^{e}\mathfrak{D}\}_{e \in E} \to {}^{\dagger}\mathfrak{D})$$

to be a collection of poly-morphisms $\{{}^e\mathfrak{D} \to {}^\dagger\mathcal{D}^\circledcirc\}_{e \in E}$ (respectively, $\{{}^e\mathfrak{D} \to {}^\dagger\mathfrak{D}\}_{e \in E}$).

The following result follows immediately from the discussion of Definition 4.1, (ii).

Proposition 4.2. **(The Set of Label Classes of Cusps of a Base-Prime-Strip)** *Let* ${}^\dagger\mathfrak{D} = \{{}^\dagger\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ *be a* $\mathcal{D}$**-prime-strip**. *Then for any* $\underline{v}, \underline{w} \in \underline{\mathbb{V}}$*, there exist* **bijections**

$$\mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{w}})$$

that are **uniquely determined** *by the condition that they be* **compatible** *with the assignments* ${}^\dagger\underline{\eta}_{\underline{v}} \mapsto {}^\dagger\underline{\eta}_{\underline{w}}$ *[cf. Definition 4.1, (ii)], as well as with the* $\mathbb{F}_l^{\circledast}$**-torsor structures** *on either side. In particular, these bijections are* **preserved** *by arbitrary* **isomorphisms** *of* $\mathcal{D}$*-prime-strips. Thus, by identifying the various "*$\mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}})$*" via these bijections, it* **makes sense to write** $\mathrm{LabCusp}({}^\dagger\mathfrak{D})$. *Finally,* $\mathrm{LabCusp}({}^\dagger\mathfrak{D})$ *is equipped with a* **canonical element**, *arising from the* ${}^\dagger\underline{\eta}_{\underline{v}}$ *[for* $\underline{v} \in \underline{\mathbb{V}}$*], as well as a natural* $\mathbb{F}_l^{\circledast}$**-torsor structure**; *in particular, this canonical element and* $\mathbb{F}_l^{\circledast}$*-torsor structure determine a* **natural bijection**

$$\mathrm{LabCusp}({}^\dagger\mathfrak{D}) \quad \xrightarrow{\sim} \quad \mathbb{F}_l^{\circledast}$$

that is preserved by isomorphisms of $\mathcal{D}$*-prime-strips.*

Remark 4.2.1. Note that if, in Examples 3.3, 3.4 — i.e., at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ — *one defines "*$\mathcal{D}_{\underline{v}}$*" by means of "*$\underline{C}_{\underline{v}}$*" instead of "*$\underline{X}_{\to\underline{v}}$", then there does *not* exist a system of bijections as in Proposition 4.2. Indeed, by the *Tchebotarev density theorem* [cf., e.g., [Lang], Chapter VIII, §4, Theorem 10], it follows immediately that there exist $\underline{v} \in \underline{\mathbb{V}}$ such that, for a suitable embedding $\mathrm{Gal}(K/F) \hookrightarrow GL_2(\mathbb{F}_l)$, the decomposition subgroup in $\mathrm{Gal}(K/F) \hookrightarrow GL_2(\mathbb{F}_l)$ determined [up to conjugation] by $\underline{v}$ is equal to the *subgroup of diagonal matrices* with determinant 1. Thus, if ${}^\dagger\mathfrak{D} = \{{}^\dagger\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$, ${}^\dagger\underline{\mathfrak{D}} = \{{}^\dagger\underline{\mathcal{D}}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ are as in Definition 4.1, (i), then for such a $\underline{v}$, the automorphism group of ${}^\dagger\underline{\mathcal{D}}_{\underline{v}}$ acts *transitively* on the set of label classes of cusps of ${}^\dagger\underline{\mathcal{D}}_{\underline{v}}$, while the automorphism group of ${}^\dagger\underline{\mathcal{D}}_{\underline{w}}$ acts *trivially* [by [EtTh], Corollary 2.9] on the set of label classes of cusps of ${}^\dagger\underline{\mathcal{D}}_{\underline{w}}$ for any $\underline{w} \in \underline{\mathbb{V}}^{\mathrm{bad}}$.

Example 4.3. **Model Base-NF-Bridges.** In the following, we construct the *"models"* for the notion of a "base-NF-bridge" [cf. Definition 4.6, (i), below].

(i) Write

$$\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \quad \subseteq \quad \mathrm{Aut}(\underline{C}_K) \quad \cong \quad \mathrm{Out}(\Pi_{\underline{C}_K}) \quad \cong \quad \mathrm{Aut}(\mathcal{D}^\circledcirc)$$

— where the first "$\cong$" follows, for instance, from [AbsTopIII], Theorem 1.9 — for the subgroup of elements which *fix the cusp* $\underline{\epsilon}$. Now let us recall that the profinite group Δ_X may be *reconstructed group-theoretically* from $\Pi_{\underline{C}_K}$ [cf. [AbsTopII], Corollary 3.3, (i), (ii); [AbsTopII], Remark 3.3.2; [AbsTopI], Example 4.8]. Since inner automorphisms of $\Pi_{\underline{C}_K}$ clearly act *by multiplication by* ± 1 on the l-torsion points of $E_{\overline{F}}$ [i.e., on $\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l$], we obtain a natural homomorphism $\mathrm{Out}(\Pi_{\underline{C}_K}) \to \mathrm{Aut}(\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l)/\{\pm 1\}$. Thus, it follows immediately from the discussion of the notation "K", "$\underline{C}_K$", and "$\underline{\epsilon}$" in Definition 3.1, (c), (d), (f) [cf. also Remark 3.1.5; the discussion preceding [EtTh], Definition 2.1; the discussion of [EtTh], Remark 2.6.1], that, relative to an isomorphism $\mathrm{Aut}(\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l)/\{\pm 1\} \xrightarrow{\sim} GL_2(\mathbb{F}_l)/\{\pm 1\}$ arising from a suitable choice of *basis* for $\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l$, if we write $\mathrm{Im}(G_{F_{\mathrm{mod}}}) \subseteq GL_2(\mathbb{F}_l)/\{\pm 1\}$ for the *image* of the *natural action* [i.e., modulo $\{\pm 1\}$] of $G_{F_{\mathrm{mod}}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}/F_{\mathrm{mod}})$ on the *l-torsion points* of E_F [cf. the homomorphism of the display of Definition 3.1, (c); the model "$C_{F_{\mathrm{mod}}}$" discussed in Remark 3.1.7], then the images of the groups $\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K)$, $\mathrm{Aut}(\underline{C}_K)$ may be identified with the subgroups consisting of elements of the form

$$\left\{ \begin{pmatrix} * & * \\ 0 & \pm 1 \end{pmatrix} \right\} \subseteq \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subseteq \mathrm{Im}(G_{F_{\mathrm{mod}}}) \; \Big(\supseteq SL_2(\mathbb{F}_l)/\{\pm 1\} \Big)$$

— i.e., *"semi-unipotent, up to* ± 1*"* and *"Borel"* subgroups — of $\mathrm{Im}(G_{F_{\mathrm{mod}}}) \subseteq GL_2(\mathbb{F}_l)/\{\pm 1\}$. Write

$$\mathrm{Aut}^{SL}_{\underline{\epsilon}}(\underline{C}_K) \subseteq \mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K), \quad \mathrm{Aut}^{SL}(\underline{C}_K) \subseteq \mathrm{Aut}(\underline{C}_K)$$

for the respective subgroups of elements that *act trivially* on the subfield $F(\boldsymbol{\mu}_l) \subseteq K$ [cf. Remark 3.1.7, (iii)] and

$$\underline{\mathbb{V}}^{\pm \mathrm{un}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \cdot \underline{\mathbb{V}} \quad \subseteq \quad \underline{\mathbb{V}}^{\mathrm{Bor}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\underline{C}_K) \cdot \underline{\mathbb{V}} \quad \subseteq \quad \mathbb{V}(K)$$

for the resulting subsets of $\mathbb{V}(K)$. Thus, one verifies immediately that the subgroup $\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \subseteq \mathrm{Aut}(\underline{C}_K)$ is *normal*, and that we have *natural isomorphisms*

$$\mathrm{Aut}^{SL}(\underline{C}_K)/\mathrm{Aut}^{SL}_{\underline{\epsilon}}(\underline{C}_K) \xrightarrow{\sim} \mathrm{Aut}(\underline{C}_K)/\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \xrightarrow{\sim} \mathbb{F}_l^{\circledast}$$

— so we may think of $\underline{\mathbb{V}}^{\mathrm{Bor}}$ as the $\mathbb{F}_l^{\circledast}$*-orbit of* $\underline{\mathbb{V}}^{\pm \mathrm{un}}$. Also, we observe that [in light of the above discussion] it follows immediately that there exists a *group-theoretic algorithm* for reconstructing, from $\pi_1(\mathcal{D}^{\circledcirc})$ [i.e., an isomorph of $\Pi_{\underline{C}_K}$] the subgroup

$$\mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \subseteq \mathrm{Aut}(\mathcal{D}^{\circledcirc})$$

determined by $\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K)$.

(ii) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$. Then the natural restriction functor on finite étale coverings arising from the natural composite morphism $\underline{X}_{\to \underline{v}} \to \underline{C}_{\underline{v}} \to \underline{C}_K$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ (respectively, $\underline{\underline{X}}_{\underline{v}} \to \underline{C}_{\underline{v}} \to \underline{C}_K$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$) determines [cf. Examples 3.2, (i); 3.3, (i)] a *natural morphism* $\phi^{\mathrm{NF}}_{\bullet,\underline{v}} : \mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc}$ [cf. §0 for the definition of the term "morphism"]. Write

$$\phi^{\mathrm{NF}}_{\underline{v}} : \mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc}$$

for the *poly-morphism* given by the collection of morphisms $\mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc}$ of the form

$$\beta \circ \phi^{\mathrm{NF}}_{\bullet,\underline{v}} \circ \alpha$$

— where $\alpha \in \mathrm{Aut}(\mathcal{D}_{\underline{v}}) \cong \mathrm{Aut}(\underline{X}_{\to_{\underline{v}}})$ (respectively, $\alpha \in \mathrm{Aut}(\mathcal{D}_{\underline{v}}) \cong \mathrm{Aut}(\underline{\underline{X}}_{\underline{v}})$); $\beta \in \mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \cong \mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K)$ [cf., e.g., [AbsTopIII], Theorem 1.9].

(iii) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Thus, [cf. Example 3.4, (i)] we have a *tautological morphism* $\mathcal{D}_{\underline{v}} = \underline{\mathbb{X}}_{\to_{\underline{v}}} \to \underline{\mathbb{C}}_{\underline{v}} \xrightarrow{\sim} \underline{\mathbb{C}}(\mathcal{D}^{\circledcirc}, \underline{v})$, hence a morphism $\phi^{\mathrm{NF}}_{\bullet,\underline{v}} : \mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc}$ [cf. Definition 4.1, (v)]. Write

$$\phi^{\mathrm{NF}}_{\underline{v}} : \mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc}$$

for the *poly-morphism* given by the collection of morphisms $\mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc}$ of the form

$$\beta \circ \phi^{\mathrm{NF}}_{\bullet,\underline{v}} \circ \alpha$$

— where $\alpha \in \mathrm{Aut}(\mathcal{D}_{\underline{v}}) \cong \mathrm{Aut}(\underline{\mathbb{X}}_{\to_{\underline{v}}})$ [cf. [AbsTopIII], Corollary 2.3]; $\beta \in \mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \cong \mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K)$.

(iv) For each $j \in \mathbb{F}_l^{\circledast}$, let

$$\mathfrak{D}_j = \{\mathcal{D}_{\underline{v}_j}\}_{\underline{v} \in \underline{\mathbb{V}}}$$

— where we use the notation $\underline{v}_j$ to denote the pair $(j, \underline{v})$ — be a *copy* of the "tautological $\mathcal{D}$-prime-strip" $\{\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$. Let us denote by

$$\phi^{\mathrm{NF}}_1 : \mathfrak{D}_1 \to \mathcal{D}^{\circledcirc}$$

[where, by abuse of notation, we write "1" for the element of $\mathbb{F}_l^{\circledast}$ determined by 1] the *poly-morphism* determined by the collection $\{\phi^{\mathrm{NF}}_{\underline{v}_1} : \mathcal{D}_{\underline{v}_1} \to \mathcal{D}^{\circledcirc}\}_{\underline{v} \in \underline{\mathbb{V}}}$ of copies of the poly-morphisms $\phi^{\mathrm{NF}}_{\underline{v}}$ constructed in (ii), (iii). Note that ϕ^{NF}_1 is *stabilized by the action of* $\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K)$ *on* $\mathcal{D}^{\circledcirc}$. Thus, it makes sense to consider, for arbitrary $j \in \mathbb{F}_l^{\circledast}$, the *poly-morphism*

$$\phi^{\mathrm{NF}}_j : \mathfrak{D}_j \to \mathcal{D}^{\circledcirc}$$

obtained [via any isomorphism $\mathfrak{D}_1 \cong \mathfrak{D}_j$] by *post-composing with the "poly-action"* [i.e., action via poly-automorphisms — cf. (i)] *of* $j \in \mathbb{F}_l^{\circledast}$ *on* $\mathcal{D}^{\circledcirc}$. Let us write

$$\mathfrak{D}_{\circledast} \stackrel{\mathrm{def}}{=} \{\mathfrak{D}_j\}_{j \in \mathbb{F}_l^{\circledast}}$$

for the *capsule of $\mathcal{D}$-prime-strips* indexed by $j \in \mathbb{F}_l^{\circledast}$ [cf. Definition 4.1, (iv)] and denote by

$$\phi^{\mathrm{NF}}_{\circledast} : \mathfrak{D}_{\circledast} \to \mathcal{D}^{\circledcirc}$$

the *poly-morphism* given by the collection of poly-morphisms $\{\phi^{\mathrm{NF}}_j\}_{j \in \mathbb{F}_l^{\circledast}}$. Thus, $\phi^{\mathrm{NF}}_{\circledast}$ is *equivariant* with respect to the *natural poly-action of* $\mathbb{F}_l^{\circledast}$ *on* $\mathcal{D}^{\circledcirc}$ and the *natural permutation poly-action* of $\mathbb{F}_l^{\circledast}$, via capsule-full [cf. §0] poly-automorphisms, on the constituents of the capsule $\mathfrak{D}_{\circledast}$. In particular, we obtain a *natural poly-action* of $\mathbb{F}_l^{\circledast}$ on the collection of data $(\mathfrak{D}_{\circledast}, \mathcal{D}^{\circledcirc}, \phi^{\mathrm{NF}}_{\circledast})$.

Remark 4.3.1.

(i) Suppose, for simplicity, in the following discussion that $F = F_{\mathrm{mod}}$. Note that the morphism of schemes $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$ [or, equivalently, the homomorphism of rings $F \hookrightarrow K$] *does not admit a section*. This nonexistence of a section is closely related to the *nonexistence of a "global multiplicative subspace"* of the sort discussed in [HASurII], Remark 3.7. In the context of *loc. cit.*, this nonexistence of a "global multiplicative subspace" may be thought of as a concrete way of representing the *principal obstruction* to applying the scheme-theoretic Hodge-Arakelov theory of [HASurI], [HASurII] to diophantine geometry. From this point of view, if one thinks of the ring structure of F, K as a sort of *"arithmetic holomorphic structure"* [cf. [AbsTopIII], Remark 5.10.2, (ii)], then one may think of the *[$\mathcal{D}$-]prime-strips* that appear in the discussion of Example 4.3 as defining, via the arrows ϕ_j^{NF} of Example 4.3, (iv),

"arithmetic collections of local analytic sections" of $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$

— cf. Fig. 4.1 below, where each "$\cdot - \cdot - \ldots - \cdot - \cdot$" represents a [$\mathcal{D}$-]prime-strip. In fact, if, for the sake of brevity, we abbreviate the phrase "collection of local analytic" by the term *"local-analytic"*, then each of these sections may be thought of as yielding not only an **"arithmetic local-analytic global multiplicative subspace"**, but also an **"arithmetic local-analytic global canonical generator"** [i.e., up to multiplication by ± 1, of the quotient of the module of l-torsion points of the elliptic curve in question by the "arithmetic local-analytic global multiplicative subspace"]. We refer to Remark 4.9.1, (i), below, for more on this point of view.

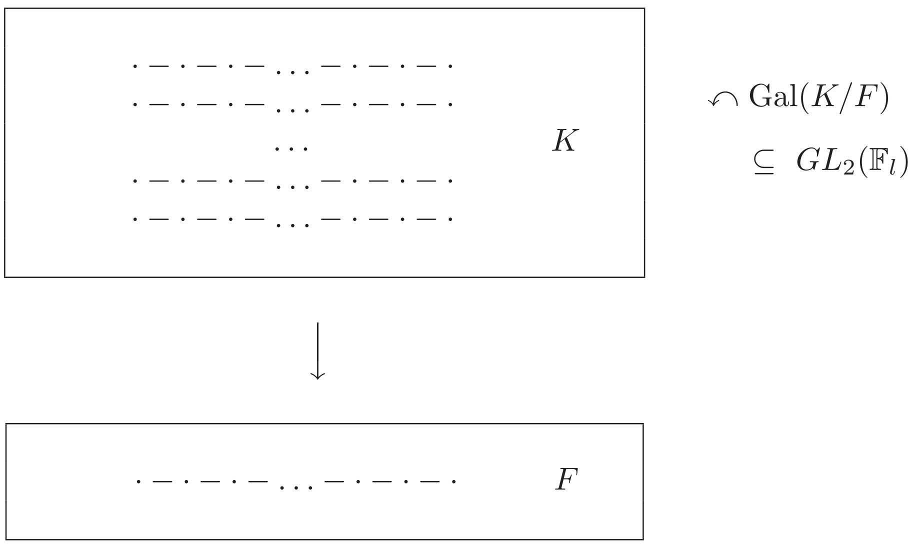

Fig. 4.1: Prime-strips as "sections" of $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$

(ii) The way in which these *"arithmetic local-analytic sections"* constituted by the [$\mathcal{D}$-]prime-strips *fail to be [globally] "arithmetically holomorphic"* may be understood from several closely related points of view. The first point of view was already noted above in (i) — namely:

(a) these sections *fail* to extend to *ring homomorphisms* $K \to F$.

The second point of view involves the classical phenomenon of *decomposition of primes* in extensions of number fields. The decomposition of primes in extensions of number fields may be represented by a *tree*, as in Fig. 4.2, below. If one thinks of the tree in large parentheses of Fig. 4.2 as representing the decomposition of primes over a prime v of F in extensions of F [such as K!], then the "arithmetic local-analytic sections" constituted by the $\mathcal{D}$-prime-strips may be thought of as

(b) an isomorphism, or identification, between v [i.e., a prime of F] and v' [i.e., a prime of K] which [manifestly — cf., e.g., [NSW], Theorem 12.2.5] fails to extend to an *isomorphism between the respective prime decomposition trees* over v and v'.

If one thinks of the relation "$\in$" between sets in axiomatic set theory as determining a *"tree"*, then

> the point of view of (b) is *reminiscent* of the point of view of [IUTchIV], §3, where one is concerned with *constructing some sort of artificial solution to the "membership equation $a \in a$"* [cf. the discussion of [IUTchIV], Remark 3.3.1, (i)].

The third point of view consists of the observation that although the "arithmetic local-analytic sections" constituted by the $\mathcal{D}$-prime-strips involve isomorphisms of the various *local absolute Galois groups*,

(c) these isomorphisms of local absolute Galois groups fail to extend to a *section of global absolute Galois groups* $G_F \twoheadrightarrow G_K$ [i.e., a section of the natural inclusion $G_K \hookrightarrow G_F$].

Here, we note that in fact, by the *Neukirch-Uchida theorem* [cf. [NSW], Chapter XII, §2], one may think of (a) and (c) as *essentially equivalent*. Moreover, (b) is *closely related* to this equivalence, in the sense that the proof [cf., e.g., [NSW], Chapter XII, §2] of the Neukirch-Uchida theorem *depends in an essential fashion* on a careful analysis of the *prime decomposition trees* of the number fields involved.

$$\begin{pmatrix} \dots & & \\ \backslash | / & \dots & \dots \\ v' & v'' & v''' \\ \backslash & | & / \\ & v & \end{pmatrix} \quad \supseteq \quad \begin{pmatrix} \dots \\ \backslash | / \\ v' \end{pmatrix}$$

Fig. 4.2: Prime decomposition trees

(iii) In some sense, understanding more precisely the content of the failure of these "arithmetic local-analytic sections" constituted by the $\mathcal{D}$-prime-strips to be "arithmetically holomorphic" is a *central theme* of the theory of the present series of papers — a theme which is very much in line with the *spirit of classical complex Teichmüller theory*.

Remark 4.3.2. The *incompatibility* of the "arithmetic local-analytic sections" of Remark 4.3.1, (i), with *global prime distributions* and *global absolute Galois groups* [cf. the discussion of Remark 4.3.1, (ii)] is precisely the technical obstacle that will necessitate the application — in [IUTchIII] — of the *absolute p-adic mono-anabelian geometry* developed in [AbsTopIII], in the form of *"panalocalization along the various prime-strips"* [cf. [IUTchIII] for more details]. Indeed,

> the mono-anabelian theory developed in [AbsTopIII] represents the *culmination* of earlier research of the author during the years 2000 to 2007 concerning **absolute *p*-adic anabelian geometry** — research that was motivated precisely by the goal of *developing a geometry* that would allow one to work with the "arithmetic local-analytic sections" constituted by the prime-strips, so as to overcome the principal technical obstruction to applying the Hodge-Arakelov theory of [HASurI], [HASurII] [cf. Remark 4.3.1, (i)].

Note that the "desired geometry" in question will also be subject to other requirements. For instance, in [IUTchIII] [cf. also [IUTchII], §4], we shall make essential use of the *global arithmetic — i.e., the ring structure and absolute Galois groups — of number fields*. As observed above in Remark 4.3.1, (ii), these global arithmetic structures are *not compatible* with the "arithmetic local-analytic sections" constituted by the prime-strips. In particular, this state of affairs imposes the further requirement that the "geometry" in question be *compatible with globalization*, i.e., that it give rise to the global arithmetic of the number fields in question in a fashion that is *independent of the various local geometries* that appear in the "arithmetic local-analytic sections" constituted by the prime-strips, but nevertheless admits *localization operations* to these various local geometries [cf. Fig. 4.3; the discussion of [IUTchII], Remark 4.11.2, (iii); [AbsTopIII], Remark 3.7.6, (iii), (v)].

$$\begin{array}{ccccc} \boxed{\begin{array}{c}\text{local geometry}\\ \text{at } v\end{array}} & \dots & \boxed{\begin{array}{c}\text{local geometry}\\ \text{at } v'\end{array}} & \dots & \boxed{\begin{array}{c}\text{local geometry}\\ \text{at } v''\end{array}} \\ \nwarrow & & \uparrow & & \nearrow \\ & & \boxed{\text{global geometry}} & & \end{array}$$

Fig. 4.3: Globalizability

Finally, in order for the "desired geometry" to be applicable to the theory developed in the present series of papers, it is necessary for it to be based on *"étale-like structures"*, so as to give rise to *canonical splittings*, as in the *étale-picture* discussed in Corollary 3.9, (i). Thus, in summary, the requirements that we wish to impose on the "desired geometry" are the following:

(a) **local independence** *of global structures*,
(b) **globalizability**, in a fashion that is **independent** *of local structures*,
(c) the property of being based on **étale-like structures**.

Note, in particular, that properties (a), (b) at first glance *almost appear to contradict one another*. In particular, the simultaneous realization of (a), (b) is *highly* *nontrivial*. For instance, in the case of a function field of dimension one over a base field, the simultaneous realization of properties (a), (b) appears to require that one restrict oneself essentially to working with structures that *descend to the base field*! It is thus a *highly nontrivial consequence* of the theory of [AbsTopIII] that the *mono-anabelian geometry* of [AbsTopIII] does indeed satisfy all of these requirements (a), (b), (c) [cf. the discussion of [AbsTopIII], §I1].

Remark 4.3.3.

(i) One important theme of [AbsTopIII] is the analogy between the **mono-anabelian theory** of [AbsTopIII] and the theory of *Frobenius-invariant indigenous bundles* of the sort that appear in p-adic Teichmüller theory [cf. [AbsTopIII], §I5]. In fact, [although this point of view is not mentioned in [AbsTopIII]] one may *"compose"* this analogy with the analogy between the p-adic and complex theories discussed in [pOrd], Introduction; [pTeich], Introduction, §0, and consider the analogy between the mono-anabelian theory of [AbsTopIII] and the **classical geometry of the upper half-plane $\mathfrak{H}$**. In addition to being *more elementary* than the p-adic theory, this analogy with the classical geometry of the upper half-plane $\mathfrak{H}$ also has the virtue that

> since it revolves around the **canonical Kähler metric** — i.e., the **Poincaré metric** — on the upper half-plane, it renders more transparent the relationship between the theory of the present series of papers and *classical Arakelov theory* [which also revolves, to a substantial extent, around Kähler metrics at the archimedean primes].

(ii) The essential content of the mono-anabelian theory of [AbsTopIII] may be summarized by the diagram

$$\Pi \curvearrowright \overline{k}^{\times} \xrightarrow{\mathfrak{log}} \overline{k} \curvearrowleft \Pi \qquad (*)$$

— where k is a finite extension of $\mathbb{Q}_p$; $\overline{k}$ is an algebraic closure of k; Π is the arithmetic fundamental group of a hyperbolic orbicurve over k; $\mathfrak{log}$ is the p-adic logarithm [cf. [AbsTopIII], §I1]. On the other hand, if $(\mathcal{E}, \nabla_{\mathcal{E}})$ denotes the *"tautological indigenous bundle"* on $\mathfrak{H}$ [i.e., the first de Rham cohomology of the tautological elliptic curve over $\mathfrak{H}$], then one has a natural *Hodge filtration* $0 \to \omega \to \mathcal{E} \to \tau \to 0$ [where ω, $\tau \stackrel{\mathrm{def}}{=} \omega^{-1}$ are holomorphic line bundles on $\mathfrak{H}$], together with a natural *complex conjugation operation* $\iota_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$. The composite

$$\omega \hookrightarrow \mathcal{E} \xrightarrow{\iota_{\mathcal{E}}} \mathcal{E} \twoheadrightarrow \tau$$

then determines an *Hermitian metric* $|-|_{\omega}$ on ω. For any trivializing section f of ω, the $(1,1)$-form

$$\kappa_{\mathfrak{H}} \stackrel{\mathrm{def}}{=} \frac{1}{2\pi i}\partial\overline{\partial} \log(|f|^2_{\omega})$$

is the **canonical Kähler metric** [i.e., Poincaré metric] on $\mathfrak{H}$. Then one can already identify various *formal similarities* between $\kappa_{\mathfrak{H}}$ and the diagram $(*)$ reviewed above: Indeed, at a purely formal [but by no means coincidental!] level, the "log" that appears in the definition of $\kappa_{\mathfrak{H}}$ is reminiscent of the "log-Frobenius operation" $\mathfrak{log}$. At a less formal level, the "Galois group" Π is reminiscent — cf. the point of view that *"Galois groups are arithmetic tangent bundles"*, a point of view that underlies the theory of the *arithmetic Kodaira-Spencer morphism* discussed in [HASurI]! — of ∂. If one thinks of *complex conjugation* as a sort of *"archimedean Frobenius"* [cf. [pTeich], Introduction, §0], then $\overline{\partial}$ is reminiscent of the "Galois group" Π operating on the *opposite side* [cf. $\iota_{\mathcal{E}}$] of the log-Frobenius operation $\mathfrak{log}$. The Hodge filtration of $\mathcal{E}$ corresponds to the *ring structures* of the copies of $\overline{k}$ on either side of $\mathfrak{log}$ [cf. the discussion of [AbsTopIII], Remark 3.7.2]. Finally, perhaps most importantly from the point of view of the theory of the present series of papers:

> the fact that *log-shells* play the role in the theory of [AbsTopIII] of *"canonical rigid integral structures"* [cf. [AbsTopIII], §I1] — i.e., *"canonical standard units of volume"* — is reminiscent of the fact that the Kähler metric $\kappa_{\mathfrak{H}}$ also plays the role of *determining a canonical notion of volume* on $\mathfrak{H}$.

(iii) From the point of view of the analogy discussed in (ii), property (a) of Remark 4.3.2 may be thought of as corresponding to the **local representability** *via the [positive]* $(1,1)$*-form* $\kappa_{\mathfrak{H}}$ — on, say, a compact quotient S of $\mathfrak{H}$ — of the [positive] **global degree** of [the result of descending to S] the line bundle ω; property (b) of Remark 4.3.2 may be thought of as corresponding to the fact that this $(1,1)$-form $\kappa_{\mathfrak{H}}$ that gives rise to a local representation on S of the notion of a positive global degree not only exists locally on S, but also admits a **canonical global extension** to the entire Riemann surface S which may be related to the **algebraic theory** [i.e., of algebraic rational functions on S].

(iv) The analogy discussed in (ii) may be summarized as follows:

<u>*mono-anabelian theory*</u>	<u>*geometry of the upper-half plane* $\mathfrak{H}$</u>
the Galois group Π	the differential operator ∂
the Galois group Π on the opposite side of $\mathfrak{log}$	the differential operator $\overline{\partial}$
the ring structures of the copies of $\overline{k}$ on either side of $\mathfrak{log}$	the Hodge filtration of $\mathcal{E}$, $\iota_{\mathcal{E}}$, $\lvert - \rvert_{\mathcal{E}}$
log-shells as *canonical units of volume*	the *canonical Kähler volume* $\kappa_{\mathfrak{H}}$

Example 4.4. **Model Base-Θ-Bridges.** In the following, we construct the *"models"* for the notion of a "base-Θ-bridge" [cf. Definition 4.6, (ii), below]. We continue to use the notation of Example 4.3.

(i) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Recall that there is a *natural bijection* between the set

$$|\mathbb{F}_l| \quad \stackrel{\mathrm{def}}{=} \quad \mathbb{F}_l/\{\pm 1\} = 0 \bigcup \mathbb{F}_l^{*}$$

[i.e., the set of $\{\pm 1\}$-orbits of $\mathbb{F}_l$] and the set of *cusps* of the hyperbolic orbicurve $\underline{C}_{\underline{v}}$ [cf. [EtTh], Corollary 2.9]. Thus, [by considering fibers over $\underline{C}_{\underline{v}}$] we obtain *labels* $\in |\mathbb{F}_l|$ of various collections of cusps of $\underline{X}_{\underline{v}}$, $\underline{\underline{X}}_{\underline{v}}$. Write

$$\mu_- \in \underline{X}_{\underline{v}}(K_{\underline{v}})$$

for the *unique torsion point of order* 2 whose closure in any stable model of $\underline{X}_{\underline{v}}$ over $\mathcal{O}_{K_{\underline{v}}}$ intersects the same irreducible component of the special fiber of the stable model as the [unique] cusp labeled $0 \in |\mathbb{F}_l|$. Now observe that it makes sense to speak of the points $\in \underline{X}_{\underline{v}}(K_{\underline{v}})$ obtained as *μ_--translates of the cusps*, relative to the group scheme structure of the elliptic curve determined by $\underline{X}_{\underline{v}}$ [i.e., whose origin is given by the cusp labeled $0 \in |\mathbb{F}_l|$]. We shall refer to these μ_--translates of the cusps with labels $\in |\mathbb{F}_l|$ as the **evaluation points** of $\underline{X}_{\underline{v}}$. Note that the **value** of the **theta function** "$\underline{\underline{\Theta}}_{\underline{v}}$" of Example 3.2, (ii), at a point lying over an evaluation point arising from a cusp with label $j \in |\mathbb{F}_l|$ is contained in the $\boldsymbol{\mu}_{2l}$-orbit of

$$\{ \underline{\underline{q}}_{\underline{v}}^{\underline{\underline{j}}^2} \}_{\underline{\underline{j}} \equiv j}$$

[cf. Example 3.2, (iv); [EtTh], Proposition 1.4, (ii)] — where $\underline{\underline{j}}$ ranges over the elements of $\mathbb{Z}$ that map to $j \in |\mathbb{F}_l|$. In particular, it follows immediately from the *definition* of the covering $\underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}}$ [i.e., by considering l-th roots of the theta function! — cf. [EtTh], Definition 2.5, (i)] that the points of $\underline{\underline{X}}_{\underline{v}}$ that lie over evaluation points of $\underline{X}_{\underline{v}}$ are *all defined over* $K_{\underline{v}}$. We shall refer to the points $\in \underline{\underline{X}}_{\underline{v}}(K_{\underline{v}})$ that lie over the evaluation points of $\underline{X}_{\underline{v}}$ as the *evaluation points* of $\underline{\underline{X}}_{\underline{v}}$ and to the various sections

$$G_{\underline{v}} \to \Pi_{\underline{v}} = \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}}$$

of the natural surjection $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$ that arise from the evaluation points as the **evaluation sections** of $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$. Thus, each evaluation section has an associated **label** $\in |\mathbb{F}_l|$. Note that there is a *group-theoretic algorithm* for constructing the evaluation sections from [isomorphs of] the *topological group* $\Pi_{\underline{v}}$. Indeed, this follows immediately from [the proofs of] [EtTh], Corollary 2.9 [concerning the *group-theoreticity of the labels*]; [EtTh], Proposition 2.4 [concerning the *group-theoreticity of* $\Pi_{\underline{C}_{\underline{v}}}$, $\Pi_{\underline{X}_{\underline{v}}}$]; [SemiAnbd], Corollary 3.11 [concerning the dual semi-graphs of the special fibers of stable models], applied to $\Delta^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}} \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}} = \Pi_{\underline{v}}$; [SemiAnbd], Theorem 6.8, (iii) [concerning the *group-theoreticity of the decomposition groups of μ_--translates of the cusps*].

(ii) We continue to suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Let

$$\mathfrak{D}_{>} = \{\mathcal{D}_{>,\underline{w}}\}_{\underline{w} \in \underline{\mathbb{V}}}$$

be a *copy* of the *"tautological $\mathcal{D}$-prime-strip"* $\{\mathcal{D}_{\underline{w}}\}_{\underline{w} \in \underline{\mathbb{V}}}$. For each $j \in \mathbb{F}_l^{\circledast}$, write

$$\phi^{\Theta}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j} \to \mathcal{D}_{>,\underline{v}}$$

for the *poly-morphism* given by the collection of morphisms [cf. §0] obtained by composing with arbitrary *isomorphisms* $\mathcal{D}_{\underline{v}_j} \xrightarrow{\sim} \mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0$, $\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0 \xrightarrow{\sim} \mathcal{D}_{>,\underline{v}}$ the various morphisms $\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0 \to \mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0$ that arise [i.e., via composition with the natural surjection $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$] from the *evaluation sections labeled* j. Now if $\mathcal{C}$ is any isomorph of $\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0$, then let us write

$$\pi_1^{\mathrm{geo}}(\mathcal{C}) \subseteq \pi_1(\mathcal{C})$$

for the subgroup corresponding to $\Delta^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}} \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}} = \Pi_{\underline{v}}$, a subgroup which we recall may be *reconstructed group-theoretically* [cf., e.g., [AbsTopI], Theorem 2.6, (v); [AbsTopI], Proposition 4.10, (i)]. Then we observe that for each constituent morphism $\mathcal{D}_{\underline{v}_j} \to \mathcal{D}_{>,\underline{v}}$ of the poly-morphism $\phi^{\Theta}_{\underline{v}_j}$, the induced homomorphism $\pi_1(\mathcal{D}_{\underline{v}_j}) \to \pi_1(\mathcal{D}_{>,\underline{v}})$ [well-defined, up to composition with an inner automorphism] is *compatible with the respective outer actions* [of the domain and codomain of this homomorphism] on $\pi_1^{\mathrm{geo}}(\mathcal{D}_{\underline{v}_j})$, $\pi_1^{\mathrm{geo}}(\mathcal{D}_{>,\underline{v}})$ for some [not necessarily unique, but determined up to *finite ambiguity* — cf. [SemiAnbd], Theorem 6.4!] outer isomorphism $\pi_1^{\mathrm{geo}}(\mathcal{D}_{\underline{v}_j}) \xrightarrow{\sim} \pi_1^{\mathrm{geo}}(\mathcal{D}_{>,\underline{v}})$. We shall refer to this fact by saying that "$\phi^{\Theta}_{\underline{v}_j}$ *is compatible with the outer actions on the respective geometric [tempered] fundamental groups*".

(iii) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$. For each $j \in \mathbb{F}_l^{\circledast}$, write

$$\phi^{\Theta}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j} \xrightarrow{\sim} \mathcal{D}_{>,\underline{v}}$$

for the *full poly-isomorphism* [cf. §0].

(iv) For each $j \in \mathbb{F}_l^{\circledast}$, write

$$\phi^{\Theta}_j : \mathfrak{D}_j \to \mathfrak{D}_>$$

for the *poly-morphism* determined by the collection $\{\phi^{\Theta}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j} \to \mathcal{D}_{>,\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ and

$$\phi^{\Theta}_{\circledast} : \mathfrak{D}_{\circledast} \to \mathfrak{D}_>$$

for the *poly-morphism* $\{\phi^{\Theta}_j\}_{j \in \mathbb{F}_l^{\circledast}}$. Thus, whereas the capsule $\mathfrak{D}_{\circledast}$ admits a *natural permutation poly-action by* $\mathbb{F}_l^{\circledast}$, the *"labels"* — i.e., in effect, elements of $\mathrm{LabCusp}(\mathfrak{D}_>)$ [cf. Proposition 4.2] — determined by the various collections of *evaluation sections* corresponding to a given $j \in \mathbb{F}_l^{\circledast}$ are *held fixed by arbitrary automorphisms of* $\mathfrak{D}_>$ [cf. Proposition 4.2].

Example 4.5. Transport of Label Classes of Cusps via Model Base-Bridges. We continue to use the notation of Examples 4.3, 4.4.

(i) Let $j \in \mathbb{F}_l^{\circledast}$, $\underline{v} \in \underline{\mathbb{V}}$. Recall from Example 4.3, (iv), that the data of the arrow $\phi^{\mathrm{NF}}_j : \mathfrak{D}_j \to \mathcal{D}^{\circledcirc}$ at $\underline{v}$ consists of an arrow $\phi^{\mathrm{NF}}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j} \to \mathcal{D}^{\circledcirc}$. If $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then $\phi^{\mathrm{NF}}_{\underline{v}_j}$ induces various outer homomorphisms $\pi_1(\mathcal{D}_{\underline{v}_j}) \to \pi_1(\mathcal{D}^{\circledcirc})$; thus,

> by considering *cuspidal inertia groups of* $\pi_1(\mathcal{D}^{\circledcirc})$ whose unique index l subgroup is *contained in the image* of this homomorphism [cf. Corollary 2.5 when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$; the discussion of Remark 4.5.1 below],

we conclude that these homomorphisms induce a *natural isomorphism of* $\mathbb{F}_l^{\circledast}$*-torsors* $\mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}(\mathcal{D}_{\underline{v}_j})$. In a similar vein, if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then it follows from Definition 4.1, (v), that $\phi^{\mathrm{NF}}_{\underline{v}_j}$ consists of certain morphisms of Aut-holomorphic orbispaces which induce various outer homomorphisms $\pi_1(\mathcal{D}_{\underline{v}_j}) \to \pi_1(\mathcal{D}^{\circledcirc})$ from the [discrete] topological fundamental group $\pi_1(\mathcal{D}_{\underline{v}_j})$ to the profinite group $\pi_1(\mathcal{D}^{\circledcirc})$; thus,

> by considering the *closures in* $\pi_1(\mathcal{D}^{\circledcirc})$ *of the images of cuspidal inertia groups of* $\pi_1(\mathcal{D}_{\underline{v}_j})$ [cf. the discussion of Remark 4.5.1 below],

we conclude that these homomorphisms induce a *natural isomorphism of* $\mathbb{F}_l^{\circledast}$*-torsors* $\mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}(\mathcal{D}_{\underline{v}_j})$. Now let us observe that it follows immediately from the definitions that, as one allows $\underline{v}$ to *vary*, these isomorphisms of $\mathbb{F}_l^{\circledast}$-torsors $\mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}(\mathcal{D}_{\underline{v}_j})$ are compatible with the natural bijections in the first display of Proposition 4.2, hence determine an isomorphism of $\mathbb{F}_l^{\circledast}$-torsors $\mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}(\mathfrak{D}_j)$. Next, let us note that the data of the arrow $\phi^{\Theta}_j : \mathfrak{D}_j \to \mathfrak{D}_>$ at the various $\underline{v} \in \underline{\mathbb{V}}$ determines an isomorphism of $\mathbb{F}_l^{\circledast}$-torsors $\mathrm{LabCusp}(\mathfrak{D}_j) \xrightarrow{\sim} \mathrm{LabCusp}(\mathfrak{D}_>)$ [which may be *composed* with the previous isomorphism of $\mathbb{F}_l^{\circledast}$-torsors $\mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}(\mathfrak{D}_j)$]. Indeed, this is immediate from the definitions when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$; when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, it follows immediately from the discussion of Example 4.4, (ii).

(ii) The discussion of (i) may be summarized as follows:

> for each $j \in \mathbb{F}_l^{\circledast}$, *restriction* at the various $\underline{v} \in \underline{\mathbb{V}}$ via ϕ^{NF}_j, ϕ^{Θ}_j determines an *isomorphism of* $\mathbb{F}_l^{\circledast}$*-torsors*
>
> $$\phi^{\mathrm{LC}}_j : \mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}(\mathfrak{D}_>)$$
>
> such that ϕ^{LC}_j is obtained from ϕ^{LC}_1 by composing with the action by $j \in \mathbb{F}_l^{\circledast}$.

Write $[\underline{\epsilon}] \in \mathrm{LabCusp}(\mathcal{D}^{\circledcirc})$ for the element determined by $\underline{\epsilon}$. Then we observe that

$$\phi^{\mathrm{LC}}_j([\underline{\epsilon}]) \mapsto j; \quad \phi^{\mathrm{LC}}_1(j \cdot [\underline{\epsilon}]) \mapsto j$$

via the *natural bijection* $\mathrm{LabCusp}(\mathfrak{D}_>) \xrightarrow{\sim} \mathbb{F}_l^{\circledast}$ of Proposition 4.2. In particular, the element $[\underline{\epsilon}] \in \mathrm{LabCusp}(\mathcal{D}^{\circledcirc})$ may be *characterized* as the *unique element* $\eta \in \mathrm{LabCusp}(\mathcal{D}^{\circledcirc})$ such that *evaluation* at η yields the assignment $\phi^{\mathrm{LC}}_j \mapsto j$.

Remark 4.5.1.

(i) Let G be a group. If $H \subseteq G$ is a subgroup, $g \in G$, then we shall write $H^g \stackrel{\mathrm{def}}{=} g \cdot H \cdot g^{-1}$. Let $J \subseteq H \subseteq G$ be subgroups. Suppose further that each of the subgroups J, H of G is *only known up to conjugacy in* G. Put another way, we suppose that we are in a situation in which there are **independent G-conjugacy indeterminacies** in the specification of the subgroups J and H. Thus, for instance, there is *no natural way* to distinguish the given inclusion $\iota : J \hookrightarrow H$ from its γ-conjugate $\iota^\gamma : J^\gamma \hookrightarrow H^\gamma$, for $\gamma \in G$. Moreover, it may happen to be the case that for some $g \in G$, not only J, but also $J^g \subseteq H$ [or, equivalently $J \subseteq H^{g^{-1}}$]. Here, the subgroups J, J^g of H are *not necessarily conjugate in* H; indeed, the abstract pairs of a group and a subgroup given by (H, J) and (H, J^g) *need not be isomorphic* [i.e., it is not even necessarily the case that there exists an automorphism of H that maps J onto J^g]. In particular, the existence of the independent G-conjugacy indeterminacies in the specification of J and H means that *one cannot specify the inclusion* $\iota : J \hookrightarrow H$ *independently of the inclusion* $\zeta : J \hookrightarrow H^{g^{-1}}$ [i.e., arising from $J^g \subseteq H$]. One way to express this state of affairs is as follows. Write "$\stackrel{\text{out}}{\hookrightarrow}$" for the outer homomorphism determined by an injective homomorphism between groups. Then the collection of **factorizations** $J \stackrel{\text{out}}{\hookrightarrow} H \stackrel{\text{out}}{\hookrightarrow} G$ of the natural "outer" inclusion $J \stackrel{\text{out}}{\hookrightarrow} G$ through some G-conjugate of H — i.e., put another way,

the collection of **outer homomorphisms**

$$J \stackrel{\text{out}}{\hookrightarrow} H$$

that are **compatible** with the **"structure morphisms"** $J \stackrel{\text{out}}{\hookrightarrow} G$, $H \stackrel{\text{out}}{\hookrightarrow} G$ determined by the natural inclusions

— is **well-defined**, in a fashion that is *compatible with independent G-conjugacy indeterminacies in the specification of J and H*. That is to say, this collection of outer homomorphisms amounts to the collection of inclusions $J^{g_1} \hookrightarrow H^{g_2}$, for $g_1, g_2 \in G$. By contrast, to specify the inclusion $\iota : J \hookrightarrow H$ [together with, say, its G-conjugates $\{\iota^\gamma\}_{\gamma \in G}$] *independently* of the inclusion $\zeta : J \hookrightarrow H^{g^{-1}}$ [and its G-conjugates $\{\zeta^\gamma\}_{\gamma \in G}$] amounts to the imposition of a **partial synchronization** — *i.e., a* **partial deactivation** — *of the [a priori!] independent G-conjugacy indeterminacies in the specification of J and H*. Moreover, such a "partial deactivation" can only be effected at the cost of introducing certain **arbitrary choices** into the construction under consideration.

(ii) Relative to the *factorizations* considered in (i), we make the following observation. Given a G-conjugate H^* of H and a subgroup $I \subseteq H^*$, the condition on I that

$$(*^{\subseteq}) \ \textit{I be a G-conjugate of J}$$

is a condition that is *independent* of the datum H^*, while the condition on I that

$$(*^{\cong}) \ \textit{I be a G-conjugate of J such that } (H^*, I) \cong (H, J)$$

[where the "$\cong$" denotes an isomorphism of pairs consisting of a group and a subgroup — cf. the discussion of (i)] is a condition that *depends*, in an essential fashion, on the datum H^*. Here, $(*^{\subseteq})$ is precisely the condition that one must impose when one considers *arbitrary factorizations* as in (i), while $(*^{\cong})$ is the condition that one must impose when one wishes to restrict one's attention to factorizations whose first arrow gives rise to a pair isomorphic to the pair determined by ι. That is to say, the *dependence* of $(*^{\cong})$ on the datum H^* may be regarded as an explicit formulation of the necessity for the *"imposition of a partial synchronization"* as discussed in (i), while the corresponding *independence*, exhibited by $(*^{\subseteq})$, of the datum H^* may be regarded as an explicit formulation of the *lack* of such a necessity when one considers arbitrary factorizations as in (i). Finally, we note that by reversing the direction of the inclusion "$\subseteq$", one may consider a subgroup $I \subseteq G$ that *contains* a given G-conjugate J^* of J, i.e., $I \supseteq J^*$; then analogous observations may be made concerning the condition $(*^{\supseteq})$ on I that *I be a G-conjugate of H*.

(iii) The abstract situation described in (i) occurs in the discussion of Example 4.5, (i), at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. That is to say, the group "G" (respectively, "H"; "J") of (i) corresponds to the group $\pi_1(\mathcal{D}^{\circledcirc})$ (respectively, the image of $\pi_1(\mathcal{D}_{\underline{v}_j})$ in $\pi_1(\mathcal{D}^{\circledcirc})$; the unique index l open subgroup of a cuspidal inertia group of $\pi_1(\mathcal{D}^{\circledcirc})$) of Example 4.5, (i). Here, we recall that the homomorphism $\pi_1(\mathcal{D}_{\underline{v}_j}) \to \pi_1(\mathcal{D}^{\circledcirc})$ is only known up to composition with an inner automorphism — i.e., up to $\pi_1(\mathcal{D}^{\circledcirc})$*-conjugacy*; a cuspidal inertia group of $\pi_1(\mathcal{D}^{\circledcirc})$ is also only determined by an element $\in \mathrm{LabCusp}(\mathcal{D}^{\circledcirc})$ up to $\pi_1(\mathcal{D}^{\circledcirc})$*-conjugacy*. Moreover, it is immediate from the construction of the *"model $\mathcal{D}$-NF-bridges"* of Example 4.3 [cf. also Definition 4.6, (i), below] that *there is* **no natural** *way to* **synchronize these indeterminacies**. Indeed, from the point of view of the discussion of Remark 4.3.1, (ii), by considering the actions of the absolute Galois groups of the local and global base fields involved on the cuspidal inertia groups that appear, one sees that such a synchronization would amount, roughly speaking, to a *Galois-equivariant splitting* [i.e., relative to the global absolute Galois groups that appear] *of the "prime decomposition trees"* of Remark 4.3.1, (ii) — which is absurd [cf. [IUTchII], Remark 2.5.2, (iii), for a more detailed discussion of this sort of phenomenon]. This phenomenon of the "non-synchronizability" of indeterminacies arising from local and global absolute Galois groups is reminiscent of the discussion of [EtTh], Remark 2.16.2. On the other hand, by Corollary 2.5, one concludes in the present situation the *highly nontrivial* fact that

> a **factorization** "$J \hookrightarrow H \hookrightarrow G$" is **uniquely determined** by the composite $J \hookrightarrow G$, i.e., by the G-conjugate of J that one starts with, **without** resorting to any *a priori* **"synchronization of indeterminacies"**.

(iv) A similar situation to the situation of (iii) occurs in the discussion of Example 4.5, (i), at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. That is to say, in this case, the group "G" (respectively, "H"; "J") of (i) corresponds to the group $\pi_1(\mathcal{D}^{\circledcirc})$ (respectively, the image of $\pi_1(\mathcal{D}_{\underline{v}_j})$ in $\pi_1(\mathcal{D}^{\circledcirc})$; a cuspidal inertia group of $\pi_1(\mathcal{D}_{\underline{v}_j})$) of Example 4.5, (i). In this case, although it does not hold that a *factorization* "$J \hookrightarrow H \hookrightarrow G$" is *uniquely determined* by the composite $J \hookrightarrow G$, i.e., by the G-conjugate of J that one starts with [cf. Remark 2.6.1], it does nevertheless hold, by Corollary 2.8, that the H-conjugacy class of the image of J via the arrow $J \hookrightarrow H$ that occurs in such a factorization is *uniquely determined*.

(v) The property observed at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ in (iv) is somewhat *weaker* than the *rather strong* property observed at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ in (iii). In the present series of papers, however, we shall only be concerned with such subtle factorization properties at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, where we wish to develop, in [IUTchII], the theory of *"Hodge-*

Arakelov-theoretic evaluation" by restricting certain cohomology classes via an arrow "$J \hookrightarrow H$" appearing in a *factorization* "$J \hookrightarrow H \hookrightarrow G$" of the sort discussed in (i). In fact, in the context of the theory of Hodge-Arakelov-theoretic evaluation that will be developed in [IUTchII], a slightly modified version of the phenomenon discussed in (iii) — which involves the *"additive"* version to be developed in §6 of the "multiplicative" theory developed in the present §4 — will be of central importance.

Definition 4.6.

(i) We define a *base-NF-bridge*, or *$\mathcal{D}$-NF-bridge*, [relative to the given initial Θ-data] to be a poly-morphism

$$^{\dagger}\mathfrak{D}_J \quad \xrightarrow{^{\dagger}\phi^{\mathrm{NF}}_{\divideontimes}} \quad {}^{\dagger}\mathcal{D}^{\circledcirc}$$

— where $^{\dagger}\mathcal{D}^{\circledcirc}$ is a *category equivalent to* $\mathcal{D}^{\circledcirc}$; $^{\dagger}\mathfrak{D}_J = \{^{\dagger}\mathfrak{D}_j\}_{j \in J}$ is a *capsule of $\mathcal{D}$-prime-strips*, indexed by a finite index set J — such that there exist isomorphisms $\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^{\dagger}\mathcal{D}^{\circledcirc}$, $\mathfrak{D}_{\divideontimes} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_J$, conjugation by which maps $\phi^{\mathrm{NF}}_{\divideontimes} \mapsto {}^{\dagger}\phi^{\mathrm{NF}}_{\divideontimes}$. We define a(n) *[iso]morphism of $\mathcal{D}$-NF-bridges*

$$(^{\dagger}\mathfrak{D}_J \quad \xrightarrow{^{\dagger}\phi^{\mathrm{NF}}_{\divideontimes}} \quad {}^{\dagger}\mathcal{D}^{\circledcirc}) \quad \to \quad (^{\ddagger}\mathfrak{D}_{J'} \quad \xrightarrow{^{\ddagger}\phi^{\mathrm{NF}}_{\divideontimes}} \quad {}^{\ddagger}\mathcal{D}^{\circledcirc})$$

to be a pair of poly-morphisms

$$^{\dagger}\mathfrak{D}_J \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{J'}; \quad {}^{\dagger}\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^{\ddagger}\mathcal{D}^{\circledcirc}$$

— where $^{\dagger}\mathfrak{D}_J \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{J'}$ is a *capsule-full poly-isomorphism* [cf. §0]; $^{\dagger}\mathcal{D}^{\circledcirc} \to {}^{\ddagger}\mathcal{D}^{\circledcirc}$ is a poly-morphism which is an $\mathrm{Aut}_{\underline{\epsilon}}(^{\dagger}\mathcal{D}^{\circledcirc})$- [or, equivalently, $\mathrm{Aut}_{\underline{\epsilon}}(^{\ddagger}\mathcal{D}^{\circledcirc})$-] *orbit* [cf. the discussion of Example 4.3, (i)] of isomorphisms — which are *compatible* with $^{\dagger}\phi^{\mathrm{NF}}_{\divideontimes}$, $^{\ddagger}\phi^{\mathrm{NF}}_{\divideontimes}$. There is an evident notion of composition of morphisms of $\mathcal{D}$-NF-bridges.

(ii) We define a *base-Θ-bridge*, or *$\mathcal{D}$-Θ-bridge*, [relative to the given initial Θ-data] to be a poly-morphism

$$^{\dagger}\mathfrak{D}_J \quad \xrightarrow{^{\dagger}\phi^{\Theta}_{\divideontimes}} \quad {}^{\dagger}\mathfrak{D}_{>}$$

— where $^{\dagger}\mathfrak{D}_{>}$ is a *$\mathcal{D}$-prime-strip*; $^{\dagger}\mathfrak{D}_J = \{^{\dagger}\mathfrak{D}_j\}_{j \in J}$ is a *capsule of $\mathcal{D}$-prime-strips*, indexed by a finite index set J — such that there exist isomorphisms $\mathfrak{D}_{>} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_{>}$, $\mathfrak{D}_{\divideontimes} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_J$, conjugation by which maps $\phi^{\Theta}_{\divideontimes} \mapsto {}^{\dagger}\phi^{\Theta}_{\divideontimes}$. We define a(n) *[iso]morphism of $\mathcal{D}$-Θ-bridges*

$$(^{\dagger}\mathfrak{D}_J \quad \xrightarrow{^{\dagger}\phi^{\Theta}_{\divideontimes}} \quad {}^{\dagger}\mathfrak{D}_{>}) \quad \to \quad (^{\ddagger}\mathfrak{D}_{J'} \quad \xrightarrow{^{\ddagger}\phi^{\Theta}_{\divideontimes}} \quad {}^{\ddagger}\mathfrak{D}_{>})$$

to be a pair of poly-morphisms

$$^{\dagger}\mathfrak{D}_J \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{J'}; \quad {}^{\dagger}\mathfrak{D}_{>} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{>}$$

— where $^{\dagger}\mathfrak{D}_J \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{J'}$ is a *capsule-full poly-isomorphism*; $^{\dagger}\mathfrak{D}_{>} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{>}$ is the *full poly-isomorphism* — which are *compatible* with $^{\dagger}\phi^{\Theta}_{\divideontimes}$, $^{\ddagger}\phi^{\Theta}_{\divideontimes}$. There is an evident notion of composition of morphisms of $\mathcal{D}$-Θ-bridges.

(iii) We define a *base*-ΘNF-*Hodge theater*, or $\mathcal{D}$-ΘNF-*Hodge theater*, [relative to the given initial Θ-data] to be a collection of data

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} = ({}^{\dagger}\mathcal{D}^{\circledcirc} \quad \xleftarrow{{}^{\dagger}\phi_{*}^{\mathrm{NF}}} \quad {}^{\dagger}\mathfrak{D}_J \quad \xrightarrow{{}^{\dagger}\phi_{*}^{\Theta}} \quad {}^{\dagger}\mathfrak{D}_{>})$$

— where ${}^{\dagger}\phi_{*}^{\mathrm{NF}}$ is a $\mathcal{D}$-NF-bridge; ${}^{\dagger}\phi_{*}^{\Theta}$ is a $\mathcal{D}$-Θ-bridge — such that there exist isomorphisms

$$\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^{\dagger}\mathcal{D}^{\circledcirc}; \quad \mathfrak{D}_{*} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_J; \quad \mathfrak{D}_{>} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_{>}$$

conjugation by which maps $\phi_{*}^{\mathrm{NF}} \mapsto {}^{\dagger}\phi_{*}^{\mathrm{NF}}$, $\phi_{*}^{\Theta} \mapsto {}^{\dagger}\phi_{*}^{\Theta}$. A(n) *[iso]morphism of* $\mathcal{D}$-ΘNF-*Hodge theaters* is defined to be a pair of morphisms between the respective associated $\mathcal{D}$-NF- and $\mathcal{D}$-Θ-bridges that are *compatible* with one another in the sense that they induce the *same bijection* between the index sets of the respective capsules of $\mathcal{D}$-prime-strips. There is an evident notion of composition of morphisms of $\mathcal{D}$-ΘNF-Hodge theaters.

Proposition 4.7. (Transport of Label Classes of Cusps via Base-Bridges) *Let*

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} = ({}^{\dagger}\mathcal{D}^{\circledcirc} \quad \xleftarrow{{}^{\dagger}\phi_{*}^{\mathrm{NF}}} \quad {}^{\dagger}\mathfrak{D}_J \quad \xrightarrow{{}^{\dagger}\phi_{*}^{\Theta}} \quad {}^{\dagger}\mathfrak{D}_{>})$$

be a $\mathcal{D}$-ΘNF-**Hodge theater** *[relative to the given initial* Θ*-data]. Then:*

(i) The structure at the various $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *of the* $\mathcal{D}$-Θ-**bridge** ${}^{\dagger}\phi_{*}^{\Theta}$ *[i.e., involving* **evaluation sections** *— cf. Example 4.4, (i), (ii); Definition 4.6, (ii)] determines a* **bijection**

$$^{\dagger}\chi : \pi_0({}^{\dagger}\mathfrak{D}_J) = J \xrightarrow{\sim} \mathbb{F}_l^{*}$$

— i.e., determines **labels** $\in \mathbb{F}_l^{*}$ *for the constituent* $\mathcal{D}$-**prime-strips** *of the capsule* ${}^{\dagger}\mathfrak{D}_J$*.*

(ii) For each $j \in J$*,* **restriction** *at the various* $\underline{v} \in \underline{\mathbb{V}}$ *[cf. Example 4.5] via the portion of* ${}^{\dagger}\phi_{*}^{\mathrm{NF}}$*,* ${}^{\dagger}\phi_{*}^{\Theta}$ *indexed by* j *determines an* **isomorphism of** $\mathbb{F}_l^{*}$**-torsors**

$$^{\dagger}\phi_j^{\mathrm{LC}} : \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}({}^{\dagger}\mathfrak{D}_{>})$$

such that ${}^{\dagger}\phi_j^{\mathrm{LC}}$ *is obtained from* ${}^{\dagger}\phi_1^{\mathrm{LC}}$ *[where, by abuse of notation, we write "*$1 \in J$*" for the element of* J *that maps via* ${}^{\dagger}\chi$ *to the image of* 1 *in* $\mathbb{F}_l^{*}$*] by composing with the action by* ${}^{\dagger}\chi(j) \in \mathbb{F}_l^{*}$*.*

(iii) There exists a **unique element**

$$[{}^{\dagger}\underline{\epsilon}] \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

such that for each $j \in J$*, the* **natural bijection** $\mathrm{LabCusp}({}^{\dagger}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathbb{F}_l^{*}$ *of the second display of Proposition 4.2 maps* ${}^{\dagger}\phi_j^{\mathrm{LC}}([{}^{\dagger}\underline{\epsilon}]) = {}^{\dagger}\phi_1^{\mathrm{LC}}({}^{\dagger}\chi(j) \cdot [{}^{\dagger}\underline{\epsilon}]) \mapsto {}^{\dagger}\chi(j)$*. In particular, the element* $[{}^{\dagger}\underline{\epsilon}]$ *determines an* **isomorphism of** $\mathbb{F}_l^{*}$**-torsors**

$$^{\dagger}\zeta_{*} : \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} J \quad (\xrightarrow{\sim} \mathbb{F}_l^{*})$$

[where the bijection in parentheses is the bijection $^\dagger\chi$ *of (i)] between* **"global cusps"** *[i.e., "*$^\dagger\chi(j) \cdot [^\dagger\underline{\epsilon}]$*"] and* **capsule indices** *[i.e.,* $j \in J \xrightarrow{\sim} \mathbb{F}_l^{\circledast}$*]. Finally, when considered up to composition with multiplication by an element of* $\mathbb{F}_l^{\circledast}$*, the bijection* $^\dagger\zeta_{\circledast}$ *is* **independent** *of the choice of* $^\dagger\phi_{\circledast}^{\mathrm{NF}}$ *within the* $\mathbb{F}_l^{\circledast}$**-orbit** *of* $^\dagger\phi_{\circledast}^{\mathrm{NF}}$ *relative to the natural poly-action of* $\mathbb{F}_l^{\circledast}$ *on* $^\dagger\mathcal{D}^{\circledcirc}$ *[cf. Example 4.3, (iv); Fig. 4.4 below].*

Proof. Assertion (i) follows immediately from the definitions [cf. Example 4.4, (i), (ii), (iv); Definition 4.6], together with the bijection of the second display of Proposition 4.2. Assertions (ii) and (iii) follow immediately from the *intrinsic nature* of the constructions of Example 4.5. ◯

Remark 4.7.1. The significance of the natural bijection $^\dagger\zeta_{\circledast}$ of Proposition 4.7, (iii), lies in the following observation: Suppose that one wishes to work with the *global data* $^\dagger\mathcal{D}^{\circledcirc}$ in a fashion that is *independent* of the *local data* [i.e., "prime-strip data"] $^\dagger\mathfrak{D}_>$, $^\dagger\mathfrak{D}_J$ [cf. Remark 4.3.2, (b)]. Then

> by replacing the *capsule index set J* by the *set of global label classes of cusps* $\mathrm{LabCusp}(^\dagger\mathcal{D}^{\circledcirc})$ via $^\dagger\zeta_{\circledast}$, one obtains an object — i.e., $\mathrm{LabCusp}(^\dagger\mathcal{D}^{\circledcirc})$ — constructed via [i.e., "native to"] the *global data* that is **immune** to the **"collapsing"** of $J \xrightarrow{\sim} \mathbb{F}_l^{\circledast}$ — i.e., of $\mathbb{F}_l^{\circledast}$-orbits of $\underline{\mathbb{V}}^{\pm\mathrm{un}}$ — *even* at primes $\underline{v} \in \underline{\mathbb{V}}$ of the sort discussed in Remark 4.2.1!

That is to say, this "collapsing" of [i.e., failure of $\mathbb{F}_l^{\circledast}$ to act *freely* on] $\mathbb{F}_l^{\circledast}$-orbits of $\underline{\mathbb{V}}^{\pm\mathrm{un}}$ is a *characteristically global* consequence of the *global prime decomposition trees* discussed in Remark 4.3.1, (ii) [cf. the example discussed in Remark 4.2.1]. We refer to Remark 4.9.3, (ii), below for a discussion of a closely related phenomenon.

Remark 4.7.2.

(i) At the level of *labels* [cf. the content of Proposition 4.7], the structure of a $\mathcal{D}$-ΘNF-*Hodge theater* may be summarized via the diagram of Fig. 4.4 below — i.e., where the expression "$[\,1 \;<\; 2 < \;\ldots\; < \;(l^{\circledast}-1) \;<\; l^{\circledast}\,]$" corresponds to $^\dagger\mathfrak{D}_>$; the expression "$(\,1 \quad 2 \quad \ldots \quad l^{\circledast}-1 \quad l^{\circledast}\,)$" corresponds to $^\dagger\mathfrak{D}_J$; the lower right-hand "$\mathbb{F}_l^{\circledast}$-cycle of $\circledast$'s" corresponds to $^\dagger\mathcal{D}^{\circledcirc}$; the "$\Uparrow$" corresponds to the associated $\mathcal{D}$-Θ-*bridge*; the "$\Rightarrow$" corresponds to the associated $\mathcal{D}$-*NF-bridge*; the "$/{\circledast}$'s" denote $\mathcal{D}$-prime-strips.

(ii) Note that the labels arising from $^\dagger\mathfrak{D}_>$ correspond, ultimately, to various **irreducible components** in the special fiber of a certain tempered covering of a **["geometric"!] Tate curve** [a special fiber which consists of a *chain of copies of the projective line* — cf. [EtTh], Corollary 2.9]. In particular, these labels are obtained by *counting* — in an intuitive, *archimedean*, *additive* fashion — the number of irreducible components between a given irreducible component and the "origin". That is to say, the portion of the diagram of Fig. 4.4 corresponding to $^\dagger\mathfrak{D}_>$ may be described by the following terms:

> **geometric**, **additive**, **archimedean**, hence **Frobenius-like** [cf. Corollary 3.8].

By contrast, the various "$\divideontimes$'s" in the portion of the diagram of Fig. 4.4 corresponding to ${}^{\dagger}\mathcal{D}^{\circledcirc}$ arise, ultimately, from various **primes** of an **[“arithmetic”!] number field**. These primes are permuted by the *multiplicative group* $\mathbb{F}_l^{\divideontimes} = \mathbb{F}_l^{\times}/\{\pm 1\}$, in a *cyclic* — i.e., *nonarchimedean* — fashion. Thus, the portion of the diagram of Fig. 4.4 corresponding to ${}^{\dagger}\mathcal{D}^{\circledcirc}$ may be described by the following terms:

> **arithmetic**, **multiplicative**, **nonarchimedean**, hence **étale-like** [cf. the discussion of Remark 4.3.2].

That is to say, the portions of the diagram of Fig. 4.4 corresponding to ${}^{\dagger}\mathfrak{D}_{>}$, ${}^{\dagger}\mathcal{D}^{\circledcirc}$ *differ quite fundamentally in structure*. In particular, it is not surprising that the only "common ground" of these two fundamentally structurally different portions consists of an *underlying set of cardinality* $l^{\divideontimes}$ [i.e., the portion of the diagram of Fig. 4.4 corresponding to ${}^{\dagger}\mathfrak{D}_J$].

(iii) The bijection ${}^{\dagger}\zeta_{\divideontimes}$ — or, perhaps more appropriately, its inverse

$$({}^{\dagger}\zeta_{\divideontimes})^{-1} : J \xrightarrow{\sim} \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

— may be thought of as relating *arithmetic* [i.e., if one thinks of the elements of the capsule index set J as *collections of primes of a number field*] to *geometry* [i.e., if one thinks of the elements of $\mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$ as corresponding to the [geometric!] *cusps* of the hyperbolic orbicurve]. From this point of view,

> $({}^{\dagger}\zeta_{\divideontimes})^{-1}$ may be thought of as a sort of **“combinatorial Kodaira-Spencer morphism”** [cf. the point of view of [HASurI], §1.4].

We refer to Remark 4.9.2, (iv), below, for another way to think about ${}^{\dagger}\zeta_{\divideontimes}$.

$$
\begin{array}{l}
[\, 1 < 2 < \ldots < j < \ldots < (l^{\divideontimes} - 1) < l^{\divideontimes} \,] \\
\qquad\qquad \mathfrak{D}_{>} \;=\; /^{\divideontimes} \\
\\
\qquad\qquad \Uparrow \;\; \phi^{\Theta}_{\divideontimes}
\end{array}
$$

$$
\begin{array}{ccccc}
\Bigg(1 & & & & \\
/^{\divideontimes} & 2 & & & \\
 & /^{\divideontimes} & \ddots & & \\
 & & & l^{\divideontimes} - 1 & \\
\mathfrak{D}_J & & & /^{\divideontimes} & l^{\divideontimes} \Bigg) \\
 & & & & /^{\divideontimes}
\end{array}
\quad
\begin{array}{c}
\phi^{\mathrm{NF}}_{\divideontimes} \\
\Rightarrow
\end{array}
\quad
\begin{array}{ccccccc}
 & & \divideontimes & \longrightarrow & \divideontimes & & \\
 & \nearrow & & & & \searrow & \\
\divideontimes & & & \mathbb{F}_l^{\divideontimes} \curvearrowright & & & \divideontimes \\
\uparrow & & & \mathcal{D}^{\circledcirc} = & & & \downarrow \\
\divideontimes & & & \mathcal{B}(\underline{C}_K)^0 & & & \divideontimes \\
 & \nwarrow & & & & \swarrow & \\
 & & \divideontimes & \ldots & \divideontimes & &
\end{array}
$$

Fig. 4.4: The combinatorial structure of a $\mathcal{D}$-ΘNF-Hodge theater

The following result follows immediately from the definitions.

Proposition 4.8. (First Properties of Base-NF-Bridges, Base-Θ-Bridges, and Base-ΘNF-Hodge Theaters) *Relative to a fixed collection of* **initial Θ-data***:*

(i) The **set of isomorphisms** *between* **two $\mathcal{D}$-NF-bridges** *forms an* **$\mathbb{F}_l^{\circledast}$-torsor***.*

(ii) The **set of isomorphisms** *between* **two $\mathcal{D}$-Θ-bridges** *(respectively,* **two $\mathcal{D}$-ΘNF-Hodge theaters***) is* **of cardinality one***.*

(iii) Given a $\mathcal{D}$-NF-bridge and a $\mathcal{D}$-Θ-bridge, the set of capsule-full poly-isomorphisms between the respective capsules of $\mathcal{D}$-prime-strips which allow one to **glue** *the given $\mathcal{D}$-NF- and $\mathcal{D}$-Θ-bridges together to form a $\mathcal{D}$-ΘNF-Hodge theater forms an* **$\mathbb{F}_l^{\circledast}$-torsor***.*

(iv) Given a $\mathcal{D}$-NF-bridge, there exists a [relatively simple — cf. the discussion of Examples 4.4, (i), (ii), (iii); 4.5, (i), (ii)] **functorial algorithm** *for constructing, up to an* **$\mathbb{F}_l^{\circledast}$-indeterminacy** *[cf. (i), (iii)], from the given $\mathcal{D}$-NF-bridge a* **$\mathcal{D}$-ΘNF-Hodge theater** *whose underlying $\mathcal{D}$-NF-bridge is the given $\mathcal{D}$-NF-bridge.*

Proposition 4.9. (Symmetries arising from Forgetful Functors) *Relative to a fixed collection of* **initial Θ-data***:*

(i) **(Base-NF-Bridges)** *The operation of associating to a $\mathcal{D}$-ΘNF-Hodge theater the underlying $\mathcal{D}$-NF-bridge of the $\mathcal{D}$-ΘNF-Hodge theater determines a* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-NF-bridges}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-NF-bridges}\end{array}}$$

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\text{NF}} \quad \mapsto \quad ({}^{\dagger}\mathcal{D}^{\circledcirc} \overset{{}^{\dagger}\phi^{\text{NF}}_{\circledast}}{\longleftarrow} {}^{\dagger}\mathfrak{D}_J)$$

whose output data admits an **$\mathbb{F}_l^{\circledast}$-symmetry** *which acts* **simply transitively** *on the index set [i.e., "J"] of the underlying capsule of $\mathcal{D}$-prime-strips [i.e., "${}^{\dagger}\mathfrak{D}_J$"] of this output data.*

(ii) **(Holomorphic Capsules)** *The operation of associating to a $\mathcal{D}$-ΘNF-Hodge theater the underlying capsule of $\mathcal{D}$-prime-strips of the $\mathcal{D}$-ΘNF-Hodge theater determines a* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of } l^{\circledast}\text{-capsules}\\ \text{of } \mathcal{D}\text{-prime-strips}\\ \text{and capsule-full poly-}\\ \text{isomorphisms of } l^{\circledast}\text{-capsules}\end{array}}$$

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\text{NF}} \quad \mapsto \quad {}^{\dagger}\mathfrak{D}_J$$

whose output data admits an $\mathfrak{S}_{l^*}$**-symmetry** *[where we write* $\mathfrak{S}_{l^*}$ *for the symmetric group on* l^* *letters] which acts* **transitively** *on the index set [i.e., "*J*"] of this output data. Thus, this functor may be thought of as an operation that consists of* **forgetting** *the* **labels** $\in \mathbb{F}_l^*$ *[i.e., forgetting the bijection* $J \xrightarrow{\sim} \mathbb{F}_l^*$ *of Proposition 4.7, (i)]. In particular, if one is only given this output data* ${}^{\dagger}\mathfrak{D}_J$ *up to isomorphism, then there is a total of precisely* l^* **possibilities** *for the element* $\in \mathbb{F}_l^*$ *to which a given index* $j \in J$ *corresponds [cf. Proposition 4.7, (i)], prior to the application of this functor.*

(iii) **(Mono-analytic Capsules)** *By composing the functor of (ii) with the* **mono-analyticization** *operation discussed in Definition 4.1, (iv), one obtains a* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of } l^*\text{-capsules}\\ \text{of } \mathcal{D}^{\vdash}\text{-prime-strips}\\ \text{and capsule-full poly-}\\ \text{isomorphisms of } l^*\text{-capsules}\end{array}}$$

$$ {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \quad \mapsto \quad {}^{\dagger}\mathfrak{D}^{\vdash}_J$$

whose output data satisfies the same symmetry properties with respect to labels as the output data of the functor of (ii).

Proof. Assertions (i), (ii), (iii) follow immediately from the definitions [cf. also Proposition 4.8, (i), in the case of assertion (i)]. ○

Remark 4.9.1.

(i) Ultimately, in the theory of the present series of papers [cf., especially, [IUTchII], §2], we shall be interested in

> **evaluating** the **étale theta function** of [EtTh] — i.e., in the spirit of the **Hodge-Arakelov theory** of [HASurI], [HASurII] — at the various $\mathcal{D}$-prime-strips of ${}^{\dagger}\mathfrak{D}_J$, in the fashion stipulated by the **labels** discussed in Proposition 4.7, (i).

These values of the étale theta function will be used to construct various *arithmetic line bundles*. We shall be interested in computing the *arithmetic degrees* — in the form of various *"log-volumes"* — of these arithmetic line bundles. In order to compute these *global* log-volumes, it is necessary to be able to *compare* the log-volumes that arise at $\mathcal{D}$-prime-strips with *different labels*. It is for this reason that the *non-labeled* output data of the functors of Proposition 4.9, (i), (ii), (iii) [cf. also Proposition 4.11, (i), (ii), below], are of *crucial importance* in the theory of the present series of papers. That is to say,

> the **non-labeled** output data of the functors of Proposition 4.9, (i), (ii), (iii) [cf. also Proposition 4.11, (i), (ii), below] — which allow one to consider **isomorphisms** between the $\mathcal{D}$-prime-strips that were originally

assigned **different labels** — make possible the **comparison** of objects [e.g., log-volumes] constructed relative to different labels.

In Proposition 4.11, (i), (ii), below, we shall see that by considering *"processions"*, one may perform such comparisons in a fashion that *minimizes the label indeterminacy* that arises.

(ii) Since the $\mathbb{F}_l^{\circledast}$-symmetry that appears in Proposition 4.9, (i), is *transitive*, it follows that one may use this action to perform **comparisons** as discussed in (i). This prompts the question:

> What is the difference between this $\mathbb{F}_l^{\circledast}$**-symmetry** and the $\mathfrak{S}_{l^{\circledast}}$**-symmetry** of the output data of the functors of Proposition 4.9, (ii), (iii)?

In a word, restricting to the $\mathbb{F}_l^{\circledast}$-symmetry of Proposition 4.9, (i), amounts to the *imposition of a "cyclic structure"* on the index set J [i.e., a structure of $\mathbb{F}_l^{\circledast}$-torsor on J]. Thus, relative to the issue of *comparability* raised in (i), this $\mathbb{F}_l^{\circledast}$-symmetry allows *comparison* between — i.e., involves *isomorphisms* between the non-labeled $\mathcal{D}$-prime-strips corresponding to — distinct members of this index set J, *without disturbing the cyclic structure* on J. This cyclic structure may be thought of as a sort of *combinatorial manifestation* of the link to the **global object** ${}^{\dagger}\mathcal{D}^{\circledcirc}$ that appears in a $\mathcal{D}$-NF-bridge. On the other hand,

> in order to **compare** these $\mathcal{D}$-prime-strips indexed by J **"in the absolute"** to $\mathcal{D}$-prime-strips that have nothing to do with J, it is necessary to **"forget the cyclic structure on** J**"**.

This is precisely what is achieved by considering the functors of Proposition 4.9, (ii), (iii), i.e., by working with the "full $\mathfrak{S}_{l^{\circledast}}$-symmetry".

Remark 4.9.2.

(i) The various elements of the index set of the capsule of $\mathcal{D}$-prime-strips of a $\mathcal{D}$-NF-bridge are *synchronized* in their correspondence with the labels "$1, 2, \ldots, l^{\circledast}$", in the sense that this correspondence is completely determined up to composition with the action of an element of $\mathbb{F}_l^{\circledast}$. In particular, this correspondence is always **bijective**.

> One may regard this phenomenon of **synchronization**, or *cohesion*, as an *important consequence* of the fact that the number field in question is represented in the $\mathcal{D}$-NF-bridge via a **single copy** [i.e., as opposed to a *capsule* whose index set is of cardinality ≥ 2] of $\mathcal{D}^{\circledcirc}$.

Indeed, consider a situation in which *each* $\mathcal{D}$-prime-strip in the capsule ${}^{\dagger}\mathfrak{D}_J$ is equipped with its own *"independent globalization"*, i.e., copy of $\mathcal{D}^{\circledcirc}$, to which it is related by a copy of "ϕ_j^{NF}", which [in order not to invalidate the *comparability of distinct labels* — cf. Remark 4.9.1, (i)] is regarded as being known *only up to composition with the action of an element of* $\mathbb{F}_l^{\circledast}$. Then if one thinks of the [manifestly *mutually disjoint* — cf. Definition 3.1, (f); Example 4.3, (i)] $\mathbb{F}_l^{\circledast}$-translates of $\underline{\mathbb{V}}^{\pm \mathrm{un}} \bigcap \mathbb{V}(K)^{\mathrm{bad}}$ [whose union is equal to $\underline{\mathbb{V}}^{\mathrm{Bor}} \bigcap \mathbb{V}(K)^{\mathrm{bad}}$] as being *labeled* by the elements of $\mathbb{F}_l^{\circledast}$, then *each* $\mathcal{D}$-prime-strip in the capsule ${}^{\dagger}\mathfrak{D}_J$ — i.e., each "$\bullet$" in Fig. 4.5 below — is subject, as depicted in Fig. 4.5, to an *independent indeterminacy* concerning the label $\in \mathbb{F}_l^{\circledast}$ to which it is associated. In particular, the *set of all possibilities for each association* includes correspondences between the index set J of the capsule ${}^{\dagger}\mathfrak{D}_J$ and the set of labels $\mathbb{F}_l^{\circledast}$ which **fail to be bijective**. Moreover, although $\mathbb{F}_l^{\circledast}$ arises essentially as a subquotient of a Galois group of extensions of number fields [cf. the *faithful poly-action* of $\mathbb{F}_l^{\circledast}$ on primes of $\mathbb{V}(K)$], the fact that it also acts *faithfully* on conjugates of the cusp $\underline{\epsilon}$ [cf. Example 4.3, (i)] implies that *"working with elements of* $\mathbb{V}(K)$ *up to* $\mathbb{F}_l^{\circledast}$*-indeterminacy"* may only be done *at the expense of "working with conjugates of the cusp* $\underline{\epsilon}$ *up to* $\mathbb{F}_l^{\circledast}$*-indeterminacy"*. That is to say, "working with nonsynchronized labels" is *inconsistent* with the construction of the *crucial bijection* ${}^{\dagger}\zeta_{\circledast}$ in Proposition 4.7, (iii).

$$
\begin{array}{cccccccc}
\bullet & \mapsto & 1? & 2? & 3? & \cdots & l^{\circledast}? \\
\bullet & \mapsto & 1? & 2? & 3? & \cdots & l^{\circledast}? \\
\vdots & & & & \vdots & & \\
\bullet & \mapsto & 1? & 2? & 3? & \cdots & l^{\circledast}?
\end{array}
$$

Fig. 4.5: Nonsynchronized labels

(ii) In the context of the discussion of (i), we observe that the *"single copy"* of $\mathcal{D}^{\circledcirc}$ may also be thought of as a **"single connected component"**, hence — from the point of view of *Galois categories* — as a **"single basepoint"**.

(iii) In the context of the discussion of (i), it is interesting to note that since the natural action of $\mathbb{F}_l^{\circledast}$ on $\mathbb{F}_l^{\circledast}$ is *transitive*, one obtains the same "set of all possibilities for each association", regardless of whether one considers independent $\mathbb{F}_l^{\circledast}$*-indeterminacies* at each index of J or independent $\mathfrak{S}_{l^{\circledast}}$*-indeterminacies* at each index of J [cf. the discussion of Remark 4.9.1, (ii)].

(iv) The **synchronized indeterminacy** [cf. (i)] exhibited by a $\mathcal{D}$-NF-bridge — i.e., at a more concrete level, the *crucial bijection* ${}^{\dagger}\zeta_{\circledast}$ of Proposition 4.7, (iii) — may be thought of as a sort of **combinatorial model** of the notion of a **"holomorphic structure"**. By contrast, the **nonsynchronized indeterminacies** discussed in (i) may be thought of as a sort of combinatorial model of the notion of a **"real analytic structure"**. Moreover, we observe that the theme of the above discussion — in which one considers

> *"how a given combinatorial holomorphic structure is* **'embedded'** *within its underlying combinatorial real analytic structure"*

— is very much in line with the *spirit of classical complex Teichmüller theory.*

(v) From the point of view discussed in (iv), the *main results* of the **"multiplicative combinatorial Teichmüller theory"** developed in the present §4 may be summarized as follows:

(a) *globalizability of labels*, in a fashion that is *independent of local structures* [cf. Remark 4.3.2, (b); Proposition 4.7, (iii)];
(b) *comparability of distinct labels* [cf. Proposition 4.9; Remark 4.9.1, (i)];
(c) *absolute comparability* [cf. Proposition 4.9, (ii), (iii); Remark 4.9.1, (ii)];

(d) *minimization of label indeterminacy — without sacrificing the symmetry necessary to perform comparisons! — via processions* [cf. Proposition 4.11, (i), (ii), below].

Remark 4.9.3.

(i) Ultimately, in the theory of the present series of papers [cf. [IUTchIII]], we would like to apply the *mono-anabelian theory* of [AbsTopIII] to the various local and global arithmetic fundamental groups [i.e., isomorphs of $\Pi_{\underline{C}_K}$, $\Pi_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$] that appear in a $\mathcal{D}$-ΘNF-Hodge theater [cf. the discussion of Remark 4.3.2]. To do this, it is of essential importance to have available not only the absolute Galois groups of the various local and global base fields involved, but also the *geometric fundamental groups* that lie inside the isomorphs of $\Pi_{\underline{C}_K}$, $\Pi_{\underline{v}}$ involved. Indeed, in the theory of [AbsTopIII], it is precisely the outer Galois action of the absolute Galois group of the base field on the geometric fundamental group that allows one to reconstruct the *ring structures group-theoretically* in a fashion that is compatible with *localization/globalization* operations as shown in Fig. 4.3. Here, we pause to recall that in [AbsTopIII], Remark 5.10.3, (i), one may find a discussion of the analogy between this phenomenon of **"entrusting of arithmetic moduli"** [to the outer Galois action on the geometric fundamental group] and the **Kodaira-Spencer isomorphism of an indigenous bundle** — an analogy that is reminiscent of the discussion of Remark 4.7.2, (iii).

(ii) Next, let us observe that the state of affairs discussed in (i) has important implications concerning the *circumstances that necessitate the use of "$\underline{X}_{\to_{\underline{v}}}$"* [i.e., as opposed to "$\underline{C}_{\underline{v}}$"] in the definition of "$\mathcal{D}_{\underline{v}}$" in Examples 3.3, 3.4 [cf. Remark 4.2.1]. Indeed, *localization/globalization* operations as shown in Fig. 4.3 give rise, when applied to the various geometric fundamental groups involved, to various *bijections* between local and global *sets of label classes of cusps*. Now suppose that *one uses "$\underline{C}_{\underline{v}}$" instead of "$\underline{X}_{\to_{\underline{v}}}$"* in the definition of "$\mathcal{D}_{\underline{v}}$" in Examples 3.3, 3.4. Then the existence of $\underline{v} \in \underline{\mathbb{V}}$ of the sort discussed in Remark 4.2.1, together with the condition of *compatiblity with localization/globalization* operations as shown in Fig. 4.3 — where we take, for instance,

$$\begin{array}{rcl} (v \text{ of Fig. 4.3}) & \stackrel{\mathrm{def}}{=} & (\underline{v} \text{ of Remark 4.2.1}) \\ (v' \text{ of Fig. 4.3}) & \stackrel{\mathrm{def}}{=} & (\underline{w} \text{ of Remark 4.2.1}) \end{array}$$

— imply that, at a combinatorial level, one is led, in effect, to a situation of the sort discussed in Remark 4.9.2, (i), i.e., a situation involving **nonsynchronized labels** [cf. Fig. 4.5], which, as discussed in Remark 4.9.2, (i), is *incompatible* with the construction of the *crucial bijection* ${}^{\dagger}\zeta$ of Proposition 4.7, (iii), an object which will play an important role in the theory of the present series of papers.

Definition 4.10. Let $\mathcal{C}$ be a *category*, n a positive integer. Then we shall refer to as a *procession of length n*, or *n-procession*, of $\mathcal{C}$ any diagram of the form

$$P_1 \hookrightarrow P_2 \hookrightarrow \ldots \hookrightarrow P_n$$

— where each P_j [for $j = 1, \ldots, n$] is a *j-capsule* [cf. §0] of objects of $\mathcal{C}$; each arrow $P_j \hookrightarrow P_{j+1}$ [for $j = 1, \ldots, n-1$] denotes the collection of *all capsule-full poly-morphisms* [cf. §0] from P_j to P_{j+1}. A *morphism* from an n-procession of $\mathcal{C}$ to an m-procession of $\mathcal{C}$

$$(P_1 \hookrightarrow \ldots \hookrightarrow P_n) \quad \to \quad (Q_1 \hookrightarrow \ldots \hookrightarrow Q_m)$$

consists of an *order-preserving injection* $\iota : \{1, \ldots, n\} \hookrightarrow \{1, \ldots, m\}$ [so $n \leq m$], together with a capsule-full poly-morphism $P_j \hookrightarrow Q_{\iota(j)}$ for each $j = 1, \ldots, n$.

$$/^{\circledast} \quad \hookrightarrow \quad /^{\circledast}/^{\circledast} \quad \hookrightarrow \quad /^{\circledast}/^{\circledast}/^{\circledast} \quad \hookrightarrow \quad \ldots \quad \hookrightarrow \quad (/^{\circledast} \ \ldots \ /^{\circledast})$$

Fig. 4.6: An $l^{\circledast}$-procession of $\mathcal{D}$-prime-strips

Proposition 4.11. **(Processions of Base-Prime-Strips)** *Relative to a fixed collection of* **initial Θ-data***:*

(i) **(Holomorphic Processions)** *Given a $\mathcal{D}$-Θ-bridge ${}^\dagger\phi^\Theta_{\circledast} : {}^\dagger\mathfrak{D}_J \to {}^\dagger\mathfrak{D}_>$ [cf. Definition 4.6, (ii)], with underlying capsule of $\mathcal{D}$-prime-strips ${}^\dagger\mathfrak{D}_J$, denote by* $\mathrm{Prc}({}^\dagger\mathfrak{D}_J)$ *the* **$l^{\circledast}$-procession of $\mathcal{D}$-prime-strips** *[cf. Fig. 4.6, where each "$/^{\circledast}$" denotes a $\mathcal{D}$-prime-strip] determined by considering the ["sub"]capsules of ${}^\dagger\mathfrak{D}_J$ corresponding to the subsets $\mathbb{S}^{\circledast}_1 \subseteq \ldots \subseteq \mathbb{S}^{\circledast}_j \stackrel{\mathrm{def}}{=} \{1, 2, \ldots, j\} \subseteq \ldots \subseteq \mathbb{S}^{\circledast}_{l^{\circledast}} \stackrel{\mathrm{def}}{=} \mathbb{F}^{\circledast}_l$ [where, by abuse of notation, we use the notation for positive integers to denote the images of these positive integers in $\mathbb{F}^{\circledast}_l$], relative to the bijection ${}^\dagger\chi : J \xrightarrow{\sim} \mathbb{F}^{\circledast}_l$ of Proposition 4.7, (i). Then the assignment ${}^\dagger\phi^\Theta_{\circledast} \mapsto \mathrm{Prc}({}^\dagger\mathfrak{D}_J)$ determines a* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta\text{-bridges}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta\text{-bridges}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of processions}\\ \text{of } \mathcal{D}\text{-prime-strips}\\ \text{and morphisms of}\\ \text{processions}\end{array}}$$

$${}^\dagger\phi^\Theta_{\circledast} \quad \mapsto \quad \mathrm{Prc}({}^\dagger\mathfrak{D}_J)$$

whose output data satisfies the following property: for each $n \in \{1, \ldots, l^{\circledast}\}$, there are precisely $\boldsymbol{n}$ **possibilities** *for the element $\in \mathbb{F}^{\circledast}_l$ to which a given index of the index set of the n-capsule that appears in the procession constituted by this output data corresponds, prior to the application of this functor. That is to say, by taking the product, over elements of $\mathbb{F}^{\circledast}_l$, of cardinalities of "sets of possibilies", one concludes that*

> *by considering* **processions** *— i.e., the functor discussed above, possibly pre-composed with the functor ${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \mapsto {}^\dagger\phi^\Theta_{\circledast}$ that associates to a $\mathcal{D}$-ΘNF-Hodge theater its associated $\mathcal{D}$-Θ-bridge — the indeterminacy consisting of $(l^{\circledast})^{(l^{\circledast})}$ possibilities that arises in Proposition 4.9, (ii), is* **reduced** *to an* **indeterminacy** *consisting of a total of* $l^{\circledast}!$ **possibilities**.

(ii) **(Mono-analytic Processions)** *By composing the functor of (i) with the* **mono-analyticization** *operation discussed in Definition 4.1, (iv), one obtains a*

natural functor

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta\text{-bridges}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta\text{-bridges}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of processions}\\ \text{of } \mathcal{D}^{\vdash}\text{-prime-strips}\\ \text{and morphisms of}\\ \text{processions}\end{array}}$$

$$^{\dagger}\phi^{\Theta}_{\divideontimes} \quad \mapsto \quad \mathrm{Prc}(^{\dagger}\mathfrak{D}^{\vdash}_{J})$$

whose output data satisfies the same indeterminacy properties with respect to labels as the output data of the functor of (i).

Proof. Assertions (i), (ii) follow immediately from the definitions. ◯

The following result is an immediate consequence of our discussion.

Corollary 4.12. **(Étale-pictures of Base-ΘNF-Hodge Theaters)** *Relative to a fixed collection of* **initial Θ-data***:*

(i) Consider the [composite] **functor**

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \quad \mapsto \quad ^{\dagger}\mathfrak{D}_{>} \quad \mapsto \quad ^{\dagger}\mathfrak{D}^{\vdash}_{>}$$

— from the category of $\mathcal{D}$-ΘNF-Hodge theaters and isomorphisms of $\mathcal{D}$-ΘNF-Hodge theaters [cf. Definition 4.6, (iii)] to the category of $\mathcal{D}^{\vdash}$-prime-strips and isomorphisms of $\mathcal{D}^{\vdash}$-prime-strips — obtained by assigning to the $\mathcal{D}$-ΘNF-Hodge theater $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ the **mono-analyticization** *[cf. Definition 4.1, (iv)] $^{\dagger}\mathfrak{D}^{\vdash}_{>}$ of the $\mathcal{D}$-prime-strip $^{\dagger}\mathfrak{D}_{>}$ that appears as the codomain of the* **underlying $\mathcal{D}$-Θ-bridge** *[cf. Definition 4.6, (ii)] of $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$. If $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$, $^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ are* **$\mathcal{D}$-ΘNF-Hodge theaters***, then we define the* **base-ΘNF-, or $\mathcal{D}$-ΘNF-, link**

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \quad \xrightarrow{\mathcal{D}} \quad ^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$$

from $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ to $^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ to be the **full poly-isomorphism**

$$^{\dagger}\mathfrak{D}^{\vdash}_{>} \ \stackrel{\sim}{\to} \ ^{\ddagger}\mathfrak{D}^{\vdash}_{>}$$

between the $\mathcal{D}^{\vdash}$-prime-strips obtained by applying the functor discussed above to $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$, $^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$.

(ii) If

$$\dots \quad \xrightarrow{\mathcal{D}} \quad ^{(n-1)}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \quad \xrightarrow{\mathcal{D}} \quad ^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \quad \xrightarrow{\mathcal{D}} \quad ^{(n+1)}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \quad \xrightarrow{\mathcal{D}} \quad \dots$$

[where $n \in \mathbb{Z}$] is an **infinite chain** *of* **$\mathcal{D}$-ΘNF-linked $\mathcal{D}$-ΘNF-Hodge theaters** *[cf. the situation discussed in Corollary 3.8], then we obtain a resulting* **chain of full poly-isomorphisms**

$$\dots \ \stackrel{\sim}{\to} \ ^{n}\mathfrak{D}^{\vdash}_{>} \ \stackrel{\sim}{\to} \ ^{(n+1)}\mathfrak{D}^{\vdash}_{>} \ \stackrel{\sim}{\to} \ \dots$$

[cf. the situation discussed in Remark 3.8.1, (ii)] between the $\mathcal{D}^{\vdash}$-prime-strips obtained by applying the functor of (i). That is to say, the output data of the functor of (i) forms a **constant invariant** *[cf. the discussion of Remark 3.8.1, (ii)] — i.e., a* **mono-analytic core** *[cf. the situation discussed in Remark 3.9.1] — of the above infinite chain.*

(iii) If we regard each of the $\mathcal{D}$-ΘNF-Hodge theaters of the chain of (ii) as a **spoke** *emanating from the mono-analytic core discussed in (ii), then we obtain a* **diagram** *— i.e., an* **étale-picture of $\mathcal{D}$-ΘNF-Hodge theaters** *— as in Fig. 4.7 below [cf. the situation discussed in Corollary 3.9, (i)]. In Fig. 4.7, "$>^{\vdash}$" denotes the mono-analytic core; "$/^{\circledast} \hookrightarrow /^{\circledast}/^{\circledast} \hookrightarrow \ldots$" denotes the "holomorphic" processions of Proposition 4.11, (i), together with the remaining ["holomorphic"] data of the corresponding $\mathcal{D}$-ΘNF-Hodge theater. Finally, [cf. the situation discussed in Corollary 3.9, (i)] this diagram satisfies the important property of admitting* **arbitrary permutation symmetries** *among the spokes [i.e., among the labels $n \in \mathbb{Z}$ of the $\mathcal{D}$-ΘNF-Hodge theaters].*

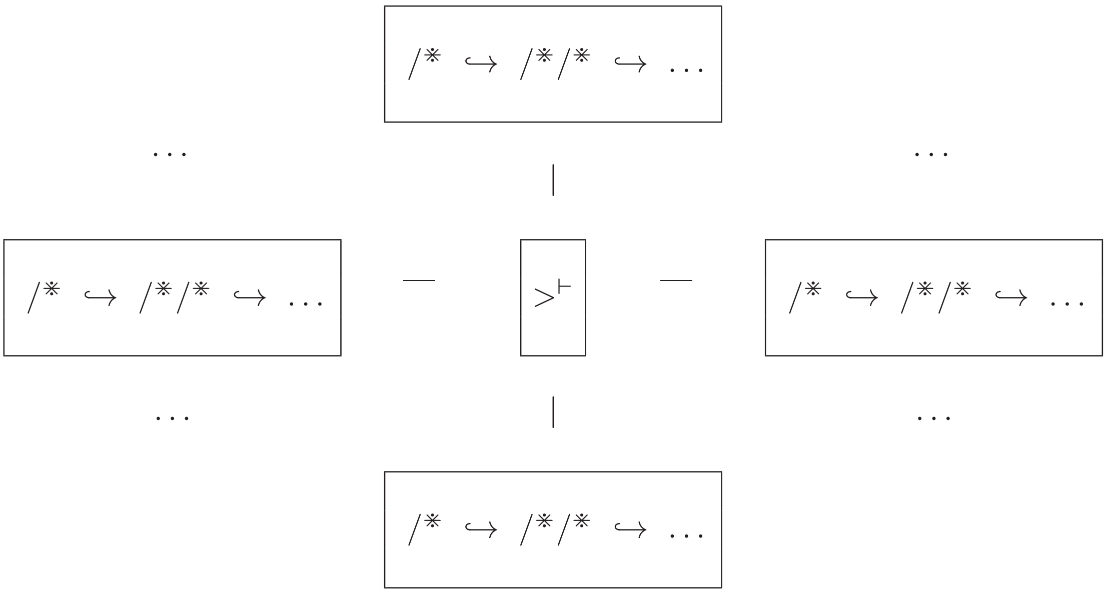

Fig. 4.7: Étale-picture of $\mathcal{D}$-ΘNF-Hodge theaters

Section 5: ΘNF-Hodge Theaters

In the present §5, we continue our discussion of various "enhancements" to the Θ-Hodge theaters of §3. Namely, we define the notion of a ΘNF**-Hodge theater** [cf. Definition 5.5, (iii)] and observe that these ΘNF-Hodge theaters satisfy the **same "functorial dynamics"** [cf. Corollary 5.6; Remark 5.6.1] as the base-ΘNF-Hodge theaters discussed in §4.

Let

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} = ({}^{\dagger}\mathcal{D}^{\circledcirc} \quad \overset{{}^{\dagger}\phi^{\mathrm{NF}}_{\circledast}}{\longleftarrow} \quad {}^{\dagger}\mathfrak{D}_J \quad \overset{{}^{\dagger}\phi^{\Theta}_{\circledast}}{\longrightarrow} \quad {}^{\dagger}\mathfrak{D}_{>})$$

be a $\mathcal{D}$-ΘNF*-Hodge theater* [cf. Definition 4.6], relative to a fixed collection of *initial* Θ*-data* $(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$ as in Definition 3.1.

Example 5.1. Global Frobenioids.

(i) By applying the *anabelian result* of [AbsTopIII], Theorem 1.9, via the *"*Θ*-approach"* discussed in Remark 3.1.2, to $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$, we may construct *group-theoretically* from $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$ an isomorph of "$\overline{F}^{\times}$" — which we shall denote

$$\mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

— equipped with its *natural* $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$*-action.* Here, we recall that this construction includes a reconstruction of the *field structure* on $\overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \overset{\mathrm{def}}{=} \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \bigcup \{0\}$. Next, let us recall [cf. Remark 3.1.7, (i)] the *unique model* $C_{F_{\mathrm{mod}}}$ of the F*-core* C_F over F_{mod}. Observe that one may construct *group-theoretically* from $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$, in a functorial fashion, a *profinite group* corresponding to "$C_{F_{\mathrm{mod}}}$" [cf. the algorithms of [AbsTopII], Corollary 3.3, (i), which are applicable in light of [AbsTopI], Example 4.8], which contains $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$ as an *open subgroup*; write ${}^{\dagger}\mathcal{D}^{\circledast}$ for $\mathcal{B}(-)^0$ of this profinite group, so we obtain a *natural morphism*

$${}^{\dagger}\mathcal{D}^{\circledcirc} \to {}^{\dagger}\mathcal{D}^{\circledast}$$

— i.e., a "category-theoretic version" of the natural morphism of hyperbolic orbicurves $\underline{C}_K \to C_{F_{\mathrm{mod}}}$ — together with a *natural extension* of the action of $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$ on $\mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$ to $\pi_1({}^{\dagger}\mathcal{D}^{\circledast})$. In particular, by taking $\pi_1({}^{\dagger}\mathcal{D}^{\circledast})$*-invariants*, we obtain a submonoid/subfield

$$\mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \subseteq \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}), \quad \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \subseteq \overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

corresponding to $F^{\times}_{\mathrm{mod}} \subseteq \overline{F}^{\times}$, $F_{\mathrm{mod}} \subseteq \overline{F}$. In a similar vein, by applying [AbsTopIII], Theorem 1.9 — cf., especially, the construction of the *Belyi cuspidalizations* of [AbsTopIII], Theorem 1.9, (a), and of the *field* "$K^{\times}_{Z_{\mathrm{NF}}} \cup \{0\}$" of [AbsTopIII], Theorem 1.9, (d), (e) — we conclude that we may construct *group-theoretically* from $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$, in a functorial fashion, an isomorph

$$\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \quad (\twoheadrightarrow \pi_1({}^{\dagger}\mathcal{D}^{\circledast}))$$

of the *absolute Galois group of the function field of* $C_{F_{\mathrm{mod}}}$ [i.e., equipped with its natural surjection to $\pi_1(^\dagger\mathcal{D}^\circledast)$ and well-defined up to inner automorphisms determined by elements of the kernel of this natural surjection], as well as isomorphs of the *pseudo-monoids of* κ*-,* ${}_\infty\kappa$*-, and* ${}_\infty\kappa\times$*-coric rational functions* associated to $C_{F_{\mathrm{mod}}}$ [cf. the discussion of Remark 3.1.7, (i), (ii); [AbsTopII], Corollary 3.3, (iii), which is applicable in light of [AbsTopI], Example 4.8] — which we shall denote

$$\mathbb{M}^\circledast_\kappa(^\dagger\mathcal{D}^\circledcirc), \quad \mathbb{M}^\circledast_{\infty\kappa}(^\dagger\mathcal{D}^\circledcirc), \quad \mathbb{M}^\circledast_{\infty\kappa\times}(^\dagger\mathcal{D}^\circledcirc)$$

— equipped with their *natural* $\pi_1^{\mathrm{rat}}(^\dagger\mathcal{D}^\circledast)$*-actions*. Thus, $\mathbb{M}^\circledast_\kappa(^\dagger\mathcal{D}^\circledcirc)$ may be identified with the subset of $\pi_1^{\mathrm{rat}}(^\dagger\mathcal{D}^\circledast)$*-invariants* of $\mathbb{M}^\circledast_{\infty\kappa}(^\dagger\mathcal{D}^\circledcirc)$, and $\mathbb{M}^\circledast(^\dagger\mathcal{D}^\circledcirc)$ may be identified with a certain subset [i.e., indeed, a certain "sub-pseudo-monoid"!] of $\mathbb{M}^\circledast_{\infty\kappa\times}(^\dagger\mathcal{D}^\circledcirc)$. Next, let us observe that it also follows from the *group-theoretic* constructions recalled above that one may reconstruct the quotients of $\pi_1(^\dagger\mathcal{D}^\circledcirc)$, $\pi_1(^\dagger\mathcal{D}^\circledast)$ that correspond, respectively, to the *absolute Galois groups* of K, F_{mod}. Thus, by forming the quotient of $\pi_1^{\mathrm{rat}}(^\dagger\mathcal{D}^\circledast)$ by the *intersection* of the *kernel* of the action of $\pi_1^{\mathrm{rat}}(^\dagger\mathcal{D}^\circledast)$ on $\mathbb{M}^\circledast_{\infty\kappa}(^\dagger\mathcal{D}^\circledcirc)$ with the inverse image in $\pi_1^{\mathrm{rat}}(^\dagger\mathcal{D}^\circledast)$ of the kernel of the *maximal solvable quotient* of [the quotient of $\pi_1^{\mathrm{rat}}(^\dagger\mathcal{D}^\circledast)$ that corresponds to] the absolute Galois group of F_{mod}, we obtain a *group-theoretic* construction for a *quotient*

$$\pi_1^{\mathrm{rat}}(^\dagger\mathcal{D}^\circledast) \twoheadrightarrow \pi_1^{\kappa\text{-sol}}(^\dagger\mathcal{D}^\circledast)$$

— whose *kernel* we denote by $\pi_1^{\mathrm{rat}/\kappa\text{-sol}}(^\dagger\mathcal{D}^\circledast)$ — that corresponds to the *quotient* "$\mathrm{Gal}(\overline{L}_C/L_C) \twoheadrightarrow \mathrm{Gal}(L_C(\kappa\text{-sol})/L_C)$" of Remark 3.1.7, (iv), as well as *pseudo-monoids* equipped with *natural* $\pi_1^{\kappa\text{-sol}}(^\dagger\mathcal{D}^\circledast)$*-actions*

$$\mathbb{M}^\circledast_{\infty\kappa}(^\dagger\mathcal{D}^\circledcirc), \quad \mathbb{M}^\circledast_{\kappa\text{-sol}}(^\dagger\mathcal{D}^\circledcirc), \quad \mathbb{M}^\circledast_{\mathrm{sol}}(^\dagger\mathcal{D}^\circledcirc)$$

— where $\mathbb{M}^\circledast_{\kappa\text{-sol}}(^\dagger\mathcal{D}^\circledcirc)$, $\mathbb{M}^\circledast_{\mathrm{sol}}(^\dagger\mathcal{D}^\circledcirc)$ denote the respective subsets of $\pi_1^{\mathrm{rat}/\kappa\text{-sol}}(^\dagger\mathcal{D}^\circledast)$*-invariants* of $\mathbb{M}^\circledast_{\infty\kappa\times}(^\dagger\mathcal{D}^\circledcirc)$, $\mathbb{M}^\circledast(^\dagger\mathcal{D}^\circledcirc)$. Moreover, by applying the *characterization of the subgroup* "$\mathrm{Gal}(K/F(\boldsymbol{\mu}_l)) \subseteq \mathrm{Gal}(K/F_{\mathrm{mod}})$" given in Remark 3.1.7, (iii), we obtain a *group-theoretic* construction for *subgroups*

$$\mathrm{Aut}^{SL}_{\underline{\epsilon}}(^\dagger\mathcal{D}^\circledcirc) \subseteq \mathrm{Aut}^{SL}(^\dagger\mathcal{D}^\circledcirc) \subseteq \mathrm{Aut}(^\dagger\mathcal{D}^\circledcirc)$$

that correspond to the *subgroups* "$\mathrm{Aut}^{SL}_{\underline{\epsilon}}(\underline{C}_K) \subseteq \mathrm{Aut}^{SL}(\underline{C}_K) \subseteq \mathrm{Aut}(\underline{C}_K)$" of Example 4.3, (i), hence induce *natural isomorphisms*

$$\mathrm{Aut}^{SL}(^\dagger\mathcal{D}^\circledcirc)/\mathrm{Aut}^{SL}_{\underline{\epsilon}}(^\dagger\mathcal{D}^\circledcirc) \xrightarrow{\sim} \mathrm{Aut}(^\dagger\mathcal{D}^\circledcirc)/\mathrm{Aut}_{\underline{\epsilon}}(^\dagger\mathcal{D}^\circledcirc) \xrightarrow{\sim} \mathbb{F}_l^{\circledast}$$

— i.e., which, in the spirit of Example 4.3, (iv), may be thought of as a *poly-action of* $\mathbb{F}_l^{\circledast}$ *on* $^\dagger\mathcal{D}^\circledcirc$. Finally, we observe that although this poly-action of $\mathbb{F}_l^{\circledast}$ on $\pi_1^{\mathrm{rat}}(^\dagger\mathcal{D}^\circledast)$ is *only well-defined up to conjugation* by elements of the subgroup

$$\pi_1^{\mathrm{rat}}(^\dagger\mathcal{D}^\circledcirc) \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{rat}}(^\dagger\mathcal{D}^\circledast) \times_{\pi_1(^\dagger\mathcal{D}^\circledast)} \pi_1(^\dagger\mathcal{D}^\circledcirc)$$

of $\pi_1^{\mathrm{rat}}(^\dagger\mathcal{D}^\circledast)$, it follows formally from the **linear disjointness** property discussed in Remark 3.1.7, (iii), that, by regarding this poly-action of $\mathbb{F}_l^{\circledast}$ as arising from the *action of elements of* $\mathrm{Aut}^{SL}(^\dagger\mathcal{D}^\circledcirc)$, one may conclude that, if we write $\pi_1^{\mathrm{rat}/\kappa\text{-sol}}(^\dagger\mathcal{D}^\circledcirc) \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{rat}/\kappa\text{-sol}}(^\dagger\mathcal{D}^\circledast) \bigcap \pi_1^{\mathrm{rat}}(^\dagger\mathcal{D}^\circledcirc)$, then the resulting **poly-action** of $\mathbb{F}_l^{\circledast}$ on $\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^{\circledast})$ is, in fact, **well-defined up to $\pi_1^{\mathrm{rat}/\kappa\text{-sol}}({}^\dagger\mathcal{D}^{\circledcirc})$-conjugacy indeterminacies**, hence, in particular, that the induced poly-action on [the *domain*, *codomain*, and *arrow* that constitute] the **"κ-sol-outer representation"**

$$\pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^{\circledast}) \quad \to \quad \mathrm{Out}^{\kappa\text{-sol}}(\pi_1^{\mathrm{rat}/\kappa\text{-sol}}({}^\dagger\mathcal{D}^{\circledast}))$$

— i.e., which may be associated to and is, in fact, *equivalent* to the exact sequence $1 \to \pi_1^{\mathrm{rat}/\kappa\text{-sol}}({}^\dagger\mathcal{D}^{\circledast}) \to \pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^{\circledast}) \to \pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^{\circledast}) \to 1$, regarded up to $\pi_1^{\mathrm{rat}/\kappa\text{-sol}}({}^\dagger\mathcal{D}^{\circledast})$-*conjugacy indeterminacies* [cf. the discussion of Remark 3.1.7, (iv)] — is, in fact, **well-defined without any conjugacy indeterminacies**, and, moreover, equal to the **trivial action**.

We shall refer to this phenomenon [cf. also Remark 5.1.5 below] as the phenomenon of κ-sol-**conjugate synchronization**.

(ii) Next, let us recall [cf. Definition 4.1, (v)] that the field structure on $\overline{\mathbb{M}}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})$ [i.e., "$\overline{F}$"] allows one to reconstruct *group-theoretically* from $\pi_1({}^\dagger\mathcal{D}^{\circledcirc})$ the *set of valuations* $\overline{\mathbb{V}}({}^\dagger\mathcal{D}^{\circledcirc})$ [i.e., "$\mathbb{V}(\overline{F})$"] on $\overline{\mathbb{M}}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})$ equipped with its natural $\pi_1({}^\dagger\mathcal{D}^{\circledast})$-action, hence also the *monoid* on ${}^\dagger\mathcal{D}^{\circledast}$ [i.e., in the sense of [FrdI], Definition 1.1, (ii)]

$$\Phi^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})(-)$$

that associates to an object $A \in \mathrm{Ob}({}^\dagger\mathcal{D}^{\circledast})$ the *monoid* $\Phi^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})(A)$ *of "stack-theoretic"* [cf. Remark 3.1.5] *arithmetic divisors* on the corresponding subfield $\overline{\mathbb{M}}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})^A \subseteq \overline{\mathbb{M}}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})$ [i.e., the monoid denoted "$\Phi(-)$" in [FrdI], Example 6.3; cf. also Remark 3.1.5 of the present paper], together with the natural morphism of monoids $\mathbb{M}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})^A \to \Phi^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})(A)^{\mathrm{gp}}$ [cf. the discussion of [FrdI], Example 6.3; Remark 3.1.5 of the present paper]. As discussed in [FrdI], Example 6.3 [cf. also Remark 3.1.5 of the present paper], this data determines, by applying [FrdI], Theorem 5.2, (ii), a *model Frobenioid*

$$\mathcal{F}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})$$

over the base category ${}^\dagger\mathcal{D}^{\circledast}$.

(iii) Let ${}^\dagger\mathcal{F}^{\circledast}$ be any *category* equivalent to $\mathcal{F}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})$. Thus, ${}^\dagger\mathcal{F}^{\circledast}$ is equipped with a *natural Frobenioid structure* [cf. [FrdI], Corollary 4.11; [FrdI], Theorem 6.4, (i); Remark 3.1.5 of the present paper]; write $\mathrm{Base}({}^\dagger\mathcal{F}^{\circledast})$ for the *base category* of this Frobenioid. Suppose further that we have been given a *morphism*

$${}^\dagger\mathcal{D}^{\circledcirc} \to \mathrm{Base}({}^\dagger\mathcal{F}^{\circledast})$$

which is *abstractly equivalent* [cf. §0] to the natural morphism ${}^\dagger\mathcal{D}^{\circledcirc} \to {}^\dagger\mathcal{D}^{\circledast}$ [cf. (i)]. In the following discussion, we shall use the resulting [*uniquely determined*, in light of the F-*coricity* of C_F, together with [AbsTopIII], Theorem 1.9!] isomorphism $\mathrm{Base}({}^\dagger\mathcal{F}^{\circledast}) \xrightarrow{\sim} {}^\dagger\mathcal{D}^{\circledast}$ to *identify* $\mathrm{Base}({}^\dagger\mathcal{F}^{\circledast})$ with ${}^\dagger\mathcal{D}^{\circledast}$. Let us denote by

$${}^\dagger\mathcal{F}^{\circledcirc} \quad \stackrel{\mathrm{def}}{=} \quad {}^\dagger\mathcal{F}^{\circledast}|_{{}^\dagger\mathcal{D}^{\circledcirc}} \quad (\to {}^\dagger\mathcal{F}^{\circledast})$$

the *restriction* of ${}^\dagger\mathcal{F}^\circledast$ to ${}^\dagger\mathcal{D}^\circledcirc$ via the natural morphism ${}^\dagger\mathcal{D}^\circledcirc \to {}^\dagger\mathcal{D}^\circledast$ and by

$$ {}^\dagger\mathcal{F}^\circledast_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} {}^\dagger\mathcal{F}^\circledast|_{\text{terminal objects}} \quad (\subseteq {}^\dagger\mathcal{F}^\circledast) $$

the *restriction* of ${}^\dagger\mathcal{F}^\circledast$ to the full subcategory of ${}^\dagger\mathcal{D}^\circledast$ determined by the terminal objects [i.e., "$C_{F_{\mathrm{mod}}}$"] of ${}^\dagger\mathcal{D}^\circledast$. Thus, when the data denoted here by the label "†" arises [in the evident sense] from data as discussed in Definition 3.1, the Frobenioid ${}^\dagger\mathcal{F}^\circledast_{\mathrm{mod}}$ may be thought of as the Frobenioid of *arithmetic line bundles on the stack* "S_{mod}" of Remark 3.1.5.

(iv) We continue to use the notation of (iii). We shall denote by a superscript "birat" the *birationalizations* [which are *category-theoretic* — cf. [FrdI], Corollary 4.10; [FrdI], Theorem 6.4, (i); Remark 3.1.5 of the present paper] of the Frobenioids ${}^\dagger\mathcal{F}^\circledcirc$, ${}^\dagger\mathcal{F}^\circledast$, ${}^\dagger\mathcal{F}^\circledast_{\mathrm{mod}}$; we shall also use this superscript to denote the images of objects and morphisms of these Frobenioids in their birationalizations. Thus, if $A \in \mathrm{Ob}({}^\dagger\mathcal{F}^\circledast)$, then $\mathcal{O}^\times(A^{\mathrm{birat}})$ may be naturally identified with the multiplicative group of nonzero elements of the number field [i.e., finite extension of F_{mod}] corresponding to A. In particular, by allowing A to *vary* among the *Frobenius-trivial objects* [a notion which is *category-theoretic* — cf. [FrdI], Definition 1.2, (iv); [FrdI], Corollary 4.11, (iv); [FrdI], Theorem 6.4, (i); Remark 3.1.5 of the present paper] of ${}^\dagger\mathcal{F}^\circledast$ that lie over *Galois objects* of ${}^\dagger\mathcal{D}^\circledast$, we obtain a *pair* [i.e., consisting of a topological group acting continuously on a discrete abelian group]

$$ \pi_1({}^\dagger\mathcal{D}^\circledast) \quad \curvearrowright \quad \widetilde{\mathcal{O}}^{\circledast\times} $$

— which we consider up to the action by the *"inner automorphisms of the pair"* arising from conjugation by $\pi_1({}^\dagger\mathcal{D}^\circledast)$. Write $\Phi_{{}^\dagger\mathcal{F}^\circledast}$ for the *divisor monoid* of the Frobenioid ${}^\dagger\mathcal{F}^\circledast$ [which is *category-theoretic* — cf. [FrdI], Corollary 4.11, (iii); [FrdI], Theorem 6.4, (i); Remark 3.1.5 of the present paper]. Thus, for each $\mathfrak{p} \in \mathrm{Prime}(\Phi_{{}^\dagger\mathcal{F}^\circledast}(A))$ [where we use the notation "Prime(−)" as in [FrdI], §0], the natural homomorphism $\mathcal{O}^\times(A^{\mathrm{birat}}) \to \Phi_{{}^\dagger\mathcal{F}^\circledast}(A)^{\mathrm{gp}}$ [cf. [FrdI], Proposition 4.4, (iii)] determines — i.e., by taking the inverse image via this homomorphism of [the union with $\{0\}$ of] the subset of $\Phi_{{}^\dagger\mathcal{F}^\circledast}(A)$ constituted by $\mathfrak{p}$ — a *submonoid* $\mathcal{O}^\triangleright_\mathfrak{p} \subseteq \mathcal{O}^\times(A^{\mathrm{birat}})$. That is to say, in more intuitive terms, this submonoid is the submonoid of integral elements of $\mathcal{O}^\times(A^{\mathrm{birat}})$ with respect to the valuation determined by $\mathfrak{p}$ of the number field corresponding to A. Write $\mathcal{O}^\times_\mathfrak{p} \subseteq \mathcal{O}^\triangleright_\mathfrak{p}$ for the submonoid of *invertible elements*. Thus, by allowing A to *vary* among the *Frobenius-trivial objects* of ${}^\dagger\mathcal{F}^\circledast$ that lie over *Galois objects* of ${}^\dagger\mathcal{D}^\circledast$ and considering the way in which the natural action of $\mathrm{Aut}_{{}^\dagger\mathcal{F}^\circledast}(A)$ on $\mathcal{O}^\times(A^{\mathrm{birat}})$ *permutes* the various submonoids $\mathcal{O}^\triangleright_\mathfrak{p}$, it follows that for each $\mathfrak{p}_0 \in \mathrm{Prime}(\Phi_{{}^\dagger\mathcal{F}^\circledast}(A_0))$, where $A_0 \in \mathrm{Ob}({}^\dagger\mathcal{F}^\circledast)$ lies over a *terminal object* of ${}^\dagger\mathcal{D}^\circledast$, we obtain a *closed subgroup* [well-defined up to conjugation]

$$ \Pi_{\mathfrak{p}_0} \subseteq \pi_1({}^\dagger\mathcal{D}^\circledast) $$

by considering the elements of $\mathrm{Aut}_{{}^\dagger\mathcal{F}^\circledast}(A)$ that *fix* the submonoid $\mathcal{O}^\triangleright_\mathfrak{p}$, for some system of $\mathfrak{p}$'s lying over $\mathfrak{p}_0$. That is to say, in more intuitive terms, the subgroup $\Pi_{\mathfrak{p}_0}$ is simply the *decomposition group* associated to some $v \in \mathbb{V}_{\mathrm{mod}}$. In particular, it follows that $\mathfrak{p}_0$ is *nonarchimedean* if and only if the p-cohomological dimension of $\Pi_{\mathfrak{p}_0}$ is equal to $2 + 1 = 3$ for infinitely many prime numbers p [cf., e.g., [NSW], Theorem 7.1.8, (i)].

(v) We continue to use the notation of (iv). Let us write

$$\pi_1({}^\dagger\mathcal{D}^\circledast) \quad \curvearrowright \quad {}^\dagger\mathbb{M}^\circledast, \qquad \pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^\circledast) \quad \curvearrowright \quad {}^\dagger\mathbb{M}^\circledast_{\mathrm{sol}}, \qquad {}^\dagger\mathbb{M}^\circledast_{\mathrm{mod}}$$

for the pair $\pi_1({}^\dagger\mathcal{D}^\circledast) \curvearrowright \widetilde{\mathcal{O}}^{\circledast\times}$ discussed in (iv) and its respective subsets [i.e., ${}^\dagger\mathbb{M}^\circledast_{\mathrm{sol}}$, ${}^\dagger\mathbb{M}^\circledast_{\mathrm{mod}}$] of $\pi_1^{\mathrm{rat}/\kappa\text{-sol}}({}^\dagger\mathcal{D}^\circledast)$-, $\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast)$-invariants. We shall refer to a pair [i.e., consisting of a *pseudo-monoid* equipped with a continuous action by $\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast)$]

$$\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast) \quad \curvearrowright \quad {}^\dagger\mathbb{M}^\circledast_{\infty\kappa} \qquad (\text{respectively, } \pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast) \quad \curvearrowright \quad {}^\dagger\mathbb{M}^\circledast_{\infty\kappa\times})$$

as an ${}_\infty\kappa$*-coric* (respectively, ${}_\infty\kappa\times$*-coric*) *structure* on ${}^\dagger\mathcal{F}^\circledast$ if it is isomorphic [i.e., as a pair consisting of a *pseudo-monoid* equipped with a continuous action by $\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast)$] to the pair

$$\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast) \quad \curvearrowright \quad \mathbb{M}^\circledast_{\infty\kappa}({}^\dagger\mathcal{D}^\circledcirc) \qquad (\text{respectively, } \pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast) \quad \curvearrowright \quad \mathbb{M}^\circledast_{\infty\kappa\times}({}^\dagger\mathcal{D}^\circledcirc))$$

of (i). Thus, the $\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast)$*-action* that appears in an ${}_\infty\kappa$-coric (respectively, ${}_\infty\kappa\times$-coric) structure necessarily *factors* (respectively, *does not factor*) through the natural surjection $\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast) \twoheadrightarrow \pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^\circledast)$ of (i). Suppose that we have been given an ${}_\infty\kappa$*-coric* (respectively, ${}_\infty\kappa\times$*-coric*) *structure* on ${}^\dagger\mathcal{F}^\circledast$. If "$(-)$" is a [commutative] *monoid*, then let us write

$$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}((-)) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, (-))$$

[cf. [AbsTopIII], Definition 3.1, (v); [AbsTopIII], Definition 5.1, (v)]; note that this notational convention also makes sense if "$(-)$" is a *cyclotomic pseudo-monoid* [cf. §0]. Also, let us write $\boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^\dagger\mathcal{D}^\circledcirc))$ for the *cyclotome* "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_{(-)})$" of [AbsTopIII], Theorem 1.9, which we think of as being applied *"via the* Θ*-approach"* [cf. Remark 3.1.2] to $\pi_1({}^\dagger\mathcal{D}^\circledcirc)$. Then let us *observe* that $\mathbb{M}^\circledast_{\infty\kappa}({}^\dagger\mathcal{D}^\circledcirc)$ (respectively, $\mathbb{M}^\circledast_{\infty\kappa\times}({}^\dagger\mathcal{D}^\circledcirc)$) is, in effect, *constructed* [cf. [AbsTopIII], Theorem 1.9, (d)] as a *subset* of

$$\varinjlim_{H} \; H^1(H, \boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^\dagger\mathcal{D}^\circledcirc)))$$

— where H ranges over the open subgroups of $\pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^\circledast)$ (respectively, $\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast)$). On the other hand, consideration of *Kummer classes* [i.e., of the action of $\pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^\circledast)$ (respectively, $\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast)$) on N-th roots of elements, for positive integers N] yields a *natural injection* of ${}^\dagger\mathbb{M}^\circledast_{\infty\kappa}$ (respectively, ${}^\dagger\mathbb{M}^\circledast_{\infty\kappa\times}$) into

$$\varinjlim_{H} \; H^1(H, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^\dagger\mathbb{M}^\circledast_{\infty\kappa})) \qquad (\text{respectively, } \varinjlim_{H} \; H^1(H, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^\dagger\mathbb{M}^\circledast_{\infty\kappa\times})))$$

— where H ranges over the open subgroups of $\pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^\circledast)$ (respectively, $\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast)$), and we observe that the asserted *injectivity* follows immediately from the corresponding injectivity in the case of $\mathbb{M}^\circledast_{\infty\kappa}({}^\dagger\mathcal{D}^\circledcirc)$ (respectively, $\mathbb{M}^\circledast_{\infty\kappa\times}({}^\dagger\mathcal{D}^\circledcirc)$). In particular, it follows immediately, by considering *divisors of zeroes and poles* [cf. the definition of a "κ*-coric function*" given in Remark 3.1.7, (i)] associated to Kummer classes of rational functions as in [AbsTopIII], Proposition 1.6, (iii), from the elementary observation that, relative to the natural inclusion $\mathbb{Q} \hookrightarrow \widehat{\mathbb{Z}} \otimes \mathbb{Q}$,

$$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^\times = \{1\}$$

that there exists a **unique isomorphism of cyclotomes**

$$\boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})) \ \stackrel{\sim}{\to}\ \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}) \qquad \text{(respectively, } \boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})) \ \stackrel{\sim}{\to}\ \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times}))$$

such that the resulting isomorphism between direct limits of cohomology modules as considered above induces an **isomorphism**

$$\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}) \ \stackrel{\sim}{\to}\ {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa} \qquad \text{(respectively, } \mathbb{M}^{\circledast}_{\infty\kappa\times}({}^{\dagger}\mathcal{D}^{\circledcirc}) \ \stackrel{\sim}{\to}\ {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times})$$

[i.e., of *pseudo-monoids* equipped with continuous actions by $\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})$]. In a similar vein, it follows immediately from the theory summarized in [AbsTopIII], Theorem 1.9, (d), that there exists a **unique isomorphism of cyclotomes**

$$\boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})) \ \stackrel{\sim}{\to}\ \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}^{\circledast})$$

such that the resulting isomorphism between direct limits of cohomology modules induces **isomorphisms**

$$\mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \ \stackrel{\sim}{\to}\ {}^{\dagger}\mathbb{M}^{\circledast}, \qquad \mathbb{M}^{\circledast}_{\mathrm{sol}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \ \stackrel{\sim}{\to}\ {}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{sol}}, \qquad \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \ \stackrel{\sim}{\to}\ {}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{mod}}$$

[i.e., of monoids equipped with continuous actions by $\pi_1({}^{\dagger}\mathcal{D}^{\circledast})$] in a fashion that is *compatible* with the *integral submonoids* "$\mathcal{O}^{\triangleright}_{\mathfrak{p}}$" [cf. the discussion of (iv)], relative to the *ring structure* constructed in [AbsTopIII], Theorem 1.9, (e), on the *domains* of these isomorphisms. In particular, it follows immediately from the above discussion that

> ${}^{\dagger}\mathcal{F}^{\circledast}$ always admits an ${}_{\infty}\kappa$*-coric* (respectively, ${}_{\infty}\kappa\times$*-coric*) *structure*, which is, moreover, *unique* up to a *uniquely determined isomorphism* [i.e., of *pseudo-monoids* equipped with continuous actions by $\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})$].

Thus, in the following, we shall regard, *without further notice*, this uniquely determined ${}_{\infty}\kappa$*-coric* (respectively, ${}_{\infty}\kappa\times$*-coric*) *structure* on ${}^{\dagger}\mathcal{F}^{\circledast}$ as a collection of data that is *naturally associated to* ${}^{\dagger}\mathcal{F}^{\circledast}$. Here, we observe that the various isomorphisms of the last few displays allow one to regard the pseudo-monoids ${}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}$, ${}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times}$ as being *related to the Frobenioid* ${}^{\dagger}\mathcal{F}^{\circledast}$ via ${}^{\dagger}\mathbb{M}^{\circledast}$ [cf. the definition of ${}^{\dagger}\mathbb{M}^{\circledast}$ at the beginning of the present (v)] and the morphisms

$${}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa} \ \hookrightarrow\ {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times}, \qquad ({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa})^{\times} \ \hookrightarrow\ ({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times})^{\times} \ \stackrel{\sim}{\to}\ {}^{\dagger}\mathbb{M}^{\circledast}$$

induced by the various isomorphisms of the last few displays, together with the corresponding inclusions/equalities

$$\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}) \ \subseteq\ \mathbb{M}^{\circledast}_{\infty\kappa\times}({}^{\dagger}\mathcal{D}^{\circledcirc}),$$

$$(\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}))^{\times} \ \subseteq\ (\mathbb{M}^{\circledast}_{\infty\kappa\times}({}^{\dagger}\mathcal{D}^{\circledcirc}))^{\times} \ =\ \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

— where we use the superscript "$\times$" to denote the subset of invertible elements of a pseudo-monoid [cf. the discussion of the initial portion of (i)]. Also, we shall write

$${}^{\dagger}\mathbb{M}^{\circledast}_{\kappa} \subseteq {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}, \quad {}^{\dagger}\mathbb{M}^{\circledast}_{\kappa\text{-sol}} \subseteq {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times}$$

for the respective "sub-pseudo-monoids" of $\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast)$-, $\pi_1^{\mathrm{rat}/\kappa\text{-sol}}({}^\dagger\mathcal{D}^\circledast)$-*invariants*. In this context, we observe further that it follows immediately from the discussion of Remark 3.1.7, (i), (ii), (iii) [cf. also [AbsTopII], Corollary 3.3, (iii), which is applicable in light of [AbsTopI], Example 4.8], and the theory summarized in [AbsTopIII], Theorem 1.9 [cf., especially, [AbsTopIII], Theorem 1.9, (a), (d), (e)], that

> any ${}_\infty\kappa\times$**-coric structure** $\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast) \curvearrowright {}^\dagger\mathbb{M}^\circledast_{\infty\kappa\times}$ on ${}^\dagger\mathcal{F}^\circledast$ **determines** an associated ${}_\infty\kappa$**-coric structure**
>
> $$\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast) \curvearrowright {}^\dagger\mathbb{M}^\circledast_{\infty\kappa} \subseteq {}^\dagger\mathbb{M}^\circledast_{\infty\kappa\times}$$
>
> by considering the subset of elements for which the **restriction** of the associated **Kummer class** [as in the above discussion] to some [or, equivalently, every] subgroup of $\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast)$ that corresponds to an open subgroup of the **decomposition group** of some **strictly critical** point of $C_{F_{\mathrm{mod}}}$ is a **torsion element** [i.e., corresponds to a root of unity],

and, moreover, that

> the operation of **restricting Kummer classes** [as in the above discussion] arising from ${}^\dagger\mathbb{M}^\circledast_\kappa \subseteq {}^\dagger\mathbb{M}^\circledast_{\infty\kappa}$ to subgroups of $\pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^\circledast)$ that correspond to **decomposition groups** of **non-critical** F_{mod}-, F_{sol}-*valued points* of $C_{F_{\mathrm{mod}}}$ yields a functorial algorithm for **reconstructing** the *monoids with* $\pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^\circledast)$*-action* ${}^\dagger\mathbb{M}^\circledast_{\mathrm{mod}}$, ${}^\dagger\mathbb{M}^\circledast_{\mathrm{sol}}$, together with the **field** structure — and hence, in particular, the topologies determined by the **valuations** — on the union of ${}^\dagger\mathbb{M}^\circledast_{\mathrm{mod}}$, ${}^\dagger\mathbb{M}^\circledast_{\mathrm{sol}}$ with $\{0\}$, from the ${}_\infty\kappa$**-coric structure** associated to ${}^\dagger\mathcal{F}^\circledast$.

A similar statement to the statement of the last display holds, if one makes the following substitutions:

$$\text{“}\pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^\circledast)\text{”} \rightsquigarrow \text{“}\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^\circledast)\text{”};$$

$$\text{“}F_{\mathrm{mod}}\text{-, } F_{\mathrm{sol}}\text{-”} \rightsquigarrow \text{“}\overline{F}\text{-”}; \qquad \text{“}{}^\dagger\mathbb{M}^\circledast_{\mathrm{mod}},\ {}^\dagger\mathbb{M}^\circledast_{\mathrm{sol}}\text{”} \rightsquigarrow \text{“}{}^\dagger\mathbb{M}^\circledast\text{”}.$$

In particular, we obtain a purely *category-theoretic* construction, from the *category* ${}^\dagger\mathcal{F}^\circledast$, of the *natural bijection*

$$\mathrm{Prime}({}^\dagger\mathcal{F}^\circledast_{\mathrm{mod}}) \quad \xrightarrow{\sim} \quad \mathbb{V}_{\mathrm{mod}}$$

— where we write $\mathrm{Prime}({}^\dagger\mathcal{F}^\circledast_{\mathrm{mod}})$ for the *set of primes* [cf. [FrdI], §0] of the divisor monoid of ${}^\dagger\mathcal{F}^\circledast_{\mathrm{mod}}$; we think of $\mathbb{V}_{\mathrm{mod}}$ as the set of $\pi_1({}^\dagger\mathcal{D}^\circledast)$-*orbits* of $\overline{\mathbb{V}}({}^\dagger\mathcal{D}^\circledcirc)$. Now, in the notation of the discussion of (iv), suppose that $\mathfrak{p}$ is *nonarchimedean* [i.e., lies over a nonarchimedean $\mathfrak{p}_0$]. Thus, $\mathfrak{p}$ determines a *valuation*, hence, in particular, a *topology* on the *ring* $\{0\} \cup \mathcal{O}^\times(A^{\mathrm{birat}})$. Write $\mathcal{O}^\times_{\widehat{\mathfrak{p}}}$, $\mathcal{O}^\triangleright_{\widehat{\mathfrak{p}}}$ for the respective *completions*, with respect to this topology, of the monoids $\mathcal{O}^\times_{\mathfrak{p}}$, $\mathcal{O}^\triangleright_{\mathfrak{p}}$. Then $\mathcal{O}^\triangleright_{\widehat{\mathfrak{p}}}$ may be identified with the multiplicative monoid of nonzero integral elements of the completion of the number field corresponding to A at the prime of this number field determined by $\mathfrak{p}$. Thus, again by allowing A to *vary* and considering the resulting *system of ind-topological monoids* "$\mathcal{O}^\triangleright_{\widehat{\mathfrak{p}}}$", we obtain a construction, for *nonarchimedean* $\mathfrak{p}_0$, of the *pair* [i.e., consisting of a topological group acting continuously on an ind-topological monoid]

$$\Pi_{\mathfrak{p}_0} \quad \curvearrowright \quad \widetilde{\mathcal{O}}^{\triangleright}_{\widehat{\mathfrak{p}_0}}$$

— which [since $\Pi_{\mathfrak{p}_0}$ is *commensurably terminal* in $\pi_1({}^\dagger\mathcal{D}^\circledast)$ — cf., e.g., [AbsAnab], Theorem 1.1.1, (i)] we consider up to the action by the *"inner automorphisms of the pair"* arising from conjugation by $\Pi_{\mathfrak{p}_0}$. In the language of [AbsTopIII], §3, this pair is an *"MLF-Galois $\mathbb{TM}$-pair of strictly Belyi type"* [cf. [AbsTopIII], Definition 3.1, (ii); [AbsTopIII], Remark 3.1.3].

(vi) Before proceeding, we observe that the discussion of (iv), (v) concerning ${}^\dagger\mathcal{F}^\circledast$, ${}^\dagger\mathcal{D}^\circledast$ may also be carried out for ${}^\dagger\mathcal{F}^\circledcirc$, ${}^\dagger\mathcal{D}^\circledcirc$. We leave the routine details to the reader.

(vii) Next, let us consider the *index set* J of the capsule of $\mathcal{D}$-prime-strips ${}^\dagger\mathfrak{D}_J$. For $j \in J$, write $\underline{\mathbb{V}}_j \stackrel{\text{def}}{=} \{\underline{v}_j\}_{\underline{v}\in\underline{\mathbb{V}}}$. Thus, we have a *natural bijection* $\underline{\mathbb{V}}_j \xrightarrow{\sim} \underline{\mathbb{V}}$, i.e., given by sending $\underline{v}_j \mapsto \underline{v}$. These bijections determine a *"diagonal subset"*

$$\underline{\mathbb{V}}_{\langle J\rangle} \quad \subseteq \quad \underline{\mathbb{V}}_J \quad \stackrel{\text{def}}{=} \quad \prod_{j\in J} \underline{\mathbb{V}}_j$$

— which also admits a *natural bijection* $\underline{\mathbb{V}}_{\langle J\rangle} \xrightarrow{\sim} \underline{\mathbb{V}}$. Thus, we obtain *natural bijections*

$$\underline{\mathbb{V}}_{\langle J\rangle} \xrightarrow{\sim} \underline{\mathbb{V}}_j \xrightarrow{\sim} \text{Prime}({}^\dagger\mathcal{F}^\circledast_{\text{mod}}) \xrightarrow{\sim} \mathbb{V}_{\text{mod}}$$

for $j \in J$. Write

$${}^\dagger\mathcal{F}^\circledast_{\langle J\rangle} \stackrel{\text{def}}{=} \{{}^\dagger\mathcal{F}^\circledast_{\text{mod}}, \underline{\mathbb{V}}_{\langle J\rangle} \xrightarrow{\sim} \text{Prime}({}^\dagger\mathcal{F}^\circledast_{\text{mod}})\}$$

$${}^\dagger\mathcal{F}^\circledast_{j} \stackrel{\text{def}}{=} \{{}^\dagger\mathcal{F}^\circledast_{\text{mod}}, \underline{\mathbb{V}}_{j} \xrightarrow{\sim} \text{Prime}({}^\dagger\mathcal{F}^\circledast_{\text{mod}})\}$$

for $j \in J$. That is to say, we think of ${}^\dagger\mathcal{F}^\circledast_j$ as a copy of ${}^\dagger\mathcal{F}^\circledast_{\text{mod}}$ *"situated on"* the constituent labeled j of the capsule ${}^\dagger\mathfrak{D}_J$; we think of ${}^\dagger\mathcal{F}^\circledast_{\langle J\rangle}$ as a copy of ${}^\dagger\mathcal{F}^\circledast_{\text{mod}}$ *"situated in a diagonal fashion on"* all the constituents of the capsule ${}^\dagger\mathfrak{D}_J$. Thus, we have a *natural embedding of categories*

$${}^\dagger\mathcal{F}^\circledast_{\langle J\rangle} \hookrightarrow {}^\dagger\mathcal{F}^\circledast_J \stackrel{\text{def}}{=} \prod_{j\in J} {}^\dagger\mathcal{F}^\circledast_j$$

— where, by abuse of notation, we write ${}^\dagger\mathcal{F}^\circledast_{\langle J\rangle}$ for the *underlying category* of [i.e., the first member of the pair] ${}^\dagger\mathcal{F}^\circledast_{\langle J\rangle}$. Here, we remark that we do *not* regard the category ${}^\dagger\mathcal{F}^\circledast_J$ as being *equipped with a Frobenioid structure*. Write

$${}^\dagger\mathcal{F}^{\circledast\mathbb{R}}_j; \quad {}^\dagger\mathcal{F}^{\circledast\mathbb{R}}_{\langle J\rangle}; \quad {}^\dagger\mathcal{F}^{\circledast\mathbb{R}}_J \stackrel{\text{def}}{=} \prod_{j\in J} {}^\dagger\mathcal{F}^{\circledast\mathbb{R}}_j$$

for the respective *realifications* [or product of the underlying categories of the realifications] of the corresponding Frobenioids whose notation does not contain a superscript "$\mathbb{R}$". [Here, we recall that the theory of realifications of Frobenioids is discussed in [FrdI], Proposition 5.3.]

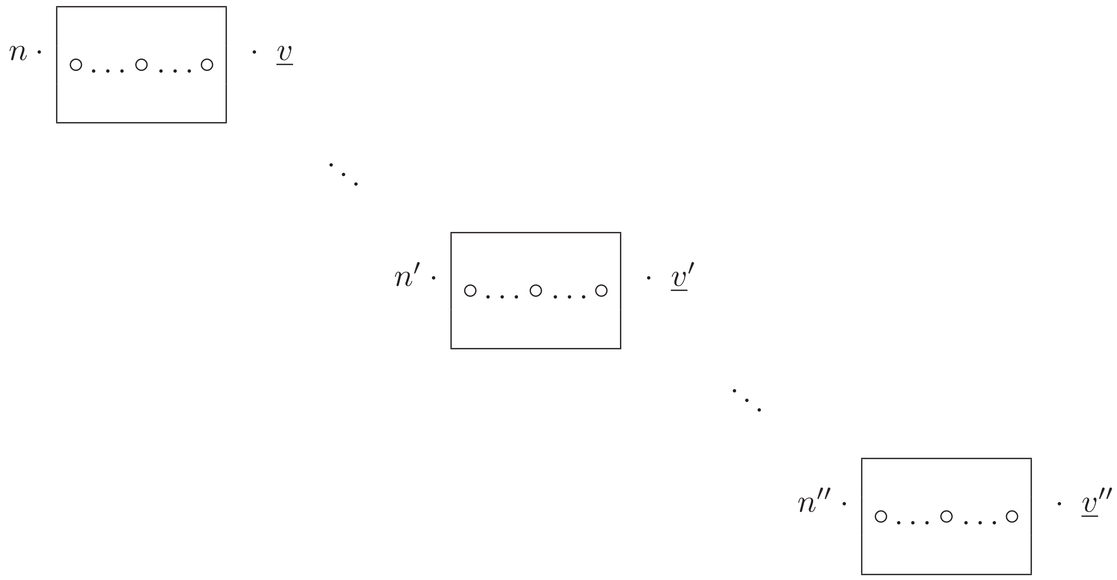

Fig. 5.1: Constant distribution

Remark 5.1.1. Thus, ${}^{\dagger}\mathcal{F}^{\circledast}_{\langle J\rangle}$ may be thought of as the Frobenioid associated to *divisors on* $\underline{\mathbb{V}}_J$ [i.e., finite formal sums of elements of this set with coefficients in $\mathbb{Z}$ or $\mathbb{R}$] whose dependence on $j \in J$ is *constant* — that is to say, divisors corresponding to *"constant distributions"* on $\underline{\mathbb{V}}_J$. Such constant distributions are depicted in Fig. 5.1 above. On the other hand, the product of [underlying categories of] Frobenioids ${}^{\dagger}\mathcal{F}^{\circledast}_J$ may be thought of as a sort of category of *"arbitrary distributions"* on $\underline{\mathbb{V}}_J$, i.e., divisors on $\underline{\mathbb{V}}_J$ whose dependence on $j \in J$ is *arbitrary.*

Remark 5.1.2. The constructions of Example 5.1 manifestly only require the $\mathcal{D}$*-NF-bridge* portion ${}^{\dagger}\phi^{\mathrm{NF}}_{\divideontimes}$ of the $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$.

Remark 5.1.3. In the context of the discussion of Example 5.1, (v), (vi), we note that unlike the case with ${}^{\dagger}\mathcal{F}^{\circledcirc}$, ${}^{\dagger}\mathcal{F}^{\circledast}$, ${}^{\dagger}\mathbb{M}^{\circledast}$, ${}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{sol}}$, ${}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}$, ${}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times}$, or ${}^{\dagger}\mathbb{M}^{\circledast}_{\kappa\text{-sol}}$, one cannot perform *Kummer theory* [cf. [FrdII], Definition 2.1, (ii)] with ${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}$, ${}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{mod}}$, or ${}^{\dagger}\mathbb{M}^{\circledast}_{\kappa}$. [That is to say, in more concrete terms, [unlike the case with $\overline{F}^{\times}$, $F^{\times}_{\mathrm{sol}}$, or ${}_{\infty}\kappa$-/${}_{\infty}\kappa\times$-coric rational functions] it is *not necessarily* the case that elements of $F^{\times}_{\mathrm{mod}}$ or κ-coric rational functions *admit* N*-th roots*, for N a nonnegative integer!] The fact that one *can* perform Kummer theory with ${}^{\dagger}\mathcal{F}^{\circledcirc}$, ${}^{\dagger}\mathcal{F}^{\circledast}$, or ${}^{\dagger}\mathbb{M}^{\circledast}$ implies that ${}^{\dagger}\mathbb{M}^{\circledast}$ equipped with its natural $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$-action, as well as the "birational monoid portions" of ${}^{\dagger}\mathcal{F}^{\circledcirc}$ or ${}^{\dagger}\mathcal{F}^{\circledast}$, satisfy various *strong rigidity properties* [cf. Corollary 5.3, (i), below]. For instance, these rigidity properties allow one to recover the *additive structure* on [the union with $\{0\}$ of] ${}^{\dagger}\mathbb{M}^{\circledast}$ equipped with its natural $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$-action [cf. the discussion of Example 5.1, (v), (vi)]. That is to say,

> the **additive structure** — or, equivalently, **ring/field structure** — on [the union with $\{0\}$ of] the "birational monoid portion" of ${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}$ may only be recovered if one is given the *additional datum consisting of the natural embedding* ${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}} \hookrightarrow {}^{\dagger}\mathcal{F}^{\circledast}$ [cf. Example 5.1, (iii)].

Put another way, if one only considers the category ${}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}}$ *without* the embedding ${}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}} \hookrightarrow {}^\dagger\mathcal{F}^{\circledast}$, then ${}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}}$ is subject to a **"Kummer black-out"** — one consequence of which is that there is no way to recover the additive structure on the "birational monoid portion" of ${}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}}$ [cf. also Remark 5.1.5 below]. In subsequent discussions, we shall refer to these observations concerning ${}^\dagger\mathcal{F}^{\circledcirc}$, ${}^\dagger\mathcal{F}^{\circledast}$, ${}^\dagger\mathbb{M}^{\circledast}$, ${}^\dagger\mathbb{M}^{\circledast}_{\mathrm{sol}}$, ${}^\dagger\mathbb{M}^{\circledast}_{\infty\kappa}$, ${}^\dagger\mathbb{M}^{\circledast}_{\infty\kappa\times}$, ${}^\dagger\mathbb{M}^{\circledast}_{\kappa\text{-sol}}$, ${}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}}$, ${}^\dagger\mathbb{M}^{\circledast}_{\mathrm{mod}}$, and ${}^\dagger\mathbb{M}^{\circledast}_{\kappa}$ by saying that ${}^\dagger\mathcal{F}^{\circledcirc}$, ${}^\dagger\mathcal{F}^{\circledast}$, ${}^\dagger\mathbb{M}^{\circledast}$, ${}^\dagger\mathbb{M}^{\circledast}_{\mathrm{sol}}$, ${}^\dagger\mathbb{M}^{\circledast}_{\infty\kappa}$, ${}^\dagger\mathbb{M}^{\circledast}_{\infty\kappa\times}$, and ${}^\dagger\mathbb{M}^{\circledast}_{\kappa\text{-sol}}$ are **Kummer-ready**, whereas ${}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}}$, ${}^\dagger\mathbb{M}^{\circledast}_{\mathrm{mod}}$, and ${}^\dagger\mathbb{M}^{\circledast}_{\kappa}$ are **Kummer-blind**. In particular, the various copies of [and products of copies of] ${}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}}$ — i.e., ${}^\dagger\mathcal{F}^{\circledast}_{j}$, ${}^\dagger\mathcal{F}^{\circledast}_{\langle J\rangle}$, ${}^\dagger\mathcal{F}^{\circledast}_{J}$ — considered in Example 5.1, (vii), are also *Kummer-blind*.

Remark 5.1.4. The various *functorial reconstruction algorithms for* **number fields** discussed in Example 5.1 are based on the technique of **Belyi cuspidalization**, as applied in the theory of [AbsTopIII], §1. At a more concrete level, this theory of [AbsTopIII], §1, may be thought of revolving around the point of view that

> *elements of* **number fields** *may be expressed* **geometrically** *by means of* **Belyi maps**.

Moreover, if one thinks of such elements of number fields as *"analytic functions"*, then the remainder of the theory of [AbsTopIII] [cf., especially, [AbsTopIII], §5] may be thought of as a sort of theory of

> **"analytic continuation"** *of the "analytic functions" constituted by elements of* **number fields** *in the context of the various* **logarithm** *maps at the various localizations of these number fields.*

This point of view is very much in line with the points of view discussed in Remarks 4.3.2, 4.3.3. Moreover, the geometric representation of elements of number fields via **Belyi maps** [i.e., whose existence may be regarded as a reflection of the *hyperbolic* nature of the projective line minus three points] is reminiscent of — indeed, may perhaps be regarded as an *arithmetic analogue* of — the "categories of **hyperbolic analytic continuations**", i.e., of copies of the **upper half-plane** regarded as equipped with their natural **hyperbolic metrics**, discussed in the "Motivating Example" given in the Introduction to [GeoAnbd]. Here, it is perhaps of interest to note that the *inequalities* "≤ 1" satisfied by the *derivatives* [i.e., with respect to the respective Poincaré metrics] of the holomorphic maps that appear in this "Motivating Example" in [GeoAnbd] are reminiscent of the *monotonically decreasing* nature of the various *"degrees"* — i.e., over $\mathbb{Q}$ of the *ramification locus* of the endomorphisms of the projective line over $\mathbb{Q}$ — that appear in the construction of Belyi maps [where we recall that this monotonic decreasing of degrees is the key observation that allows one to obtain Belyi maps which are *unramified* over the projective line minus three points].

Remark 5.1.5. Although the phenomenon of κ-sol**-conjugate synchronization** discussed in the final portion of Example 5.1, (i), will *not play as central a role* in the present series of papers as the *conjugate synchronization of local Galois groups* that will be discussed in [IUTchII], [IUTchIII], it has the following *interesting consequence*:

The **Kummer theory** of

$$\text{“}(\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \twoheadrightarrow)\ \pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}\text{”}, \quad \text{“}{}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{mod}}\text{”}, \quad \text{“}\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{sol}}\text{”}$$

— i.e., of the *abstract analogues* of *“$_{\infty}\kappa$-coric functions”*, *“$F^{\times}_{\mathrm{mod}}$”*, and *“$F^{\times}_{\mathrm{sol}}$”* as in Remark 3.1.7, (iii) — that was discussed in Example 5.1, (v), may be performed in a fashion that is **compatible without any conjugacy indeterminacies** with the **poly-action** of $(\mathrm{Aut}^{SL}({}^{\dagger}\mathcal{D}^{\circledcirc}) \twoheadrightarrow)$ $\mathbb{F}_l^{\circledast}$.

Here, we pause, however, to make the following *observation*: At first glance, it may appear as though the analogue obtained by Uchida of the Neukirch-Uchida theorem for *maximal solvable quotients* of absolute Galois groups of number fields [reviewed, for instance, in [GlSol], §3] — or, perhaps, some future *mono-anabelian* version of this result of Uchida — may be applied, in the context of the *“κ-sol-Kummer theory”* just discussed, to **reconstruct the ring structure** on the number fields involved [i.e., in the fashion of Example 5.1, (v)]. In fact, however,

> such a **“solvable-Uchida-type”** reconstruction is, in effect, **meaningless** from the point of view of the theory of the present series of papers since it is **fundamentally incompatible** with the **localization** operations that occur in the structure of a $\mathcal{D}$-ΘNF-Hodge theater — cf. the discussion of Remarks 4.3.1, 4.3.2.

Indeed, such a compatibility with localization would imply that the reconstruction of the ring structure may somehow be *“restricted”* to the absolute Galois groups of completions at nonarchimedean primes of a number field, i.e., in *contradiction* to the well-known fact that absolute Galois groups of such completions at nonarchimedean primes admit automorphisms that *do not arise from field automorphisms* [cf., e.g., [NSW], the Closing Remark preceding Theorem 12.2.7]. Finally, we note that this *incompatibility* of “solvable-Uchida-type” reconstructions of ring structures with the theory of the present series of papers is also interesting in the context of the point of view discussed in Remark 5.1.4: Suppose, for instance, that it was the case that the *outer action of the absolute Galois group of a number field* on the geometric fundamental group of a hyperbolic curve over the number field in fact *factors* through the *maximal solvable quotient* of the absolute Galois group. Then it is conceivable that some sort of version of the mono-anabelian theory of [AbsTopIII], §1, for *extensions* of such a maximal solvable quotient by the geometric fundamental group under consideration could be applied in the theory of the present series of papers to give a *reconstruction of the ring structure* of a number field that only requires the use of such *extensions* and is, moreover, *compatible* with the localization operations that occur in the various types of *“Hodge theaters”* that appear in the theory of the present series of papers — a state of affairs that would be **fundamentally at odds** with a *quite essential portion* of the **“spirit”** of the theory of the present series of papers, namely, the point of view of *“***dismantling** *the two underlying combinatorial dimensions of a* **ring***”*. In fact, however,

> the outer action referred to above does **not** admit such a **“solvable factorization”**.

Indeed, the nonexistence of such a “solvable factorization” is a formal consequence of the the *Galois injectivity* result discussed in [NodNon], Theorem C — a result that depends, in an essential way, on the theory of *Belyi maps*. Put another way,

Belyi maps not only play the role of allowing one to perform the sort of **"arithmetic analytic continuation via Belyi cuspidalizations"** [i.e., discussed in Remark 5.1.4] that is of central importance in the theory of the present series of papers, but also play the role of assuring one that such "arithmetic analytic continuations" **cannot be extended** to the case of *extensions* associated to **"solvable factorizations"** of outer actions of the sort just discussed.

Definition 5.2.

(i) We define a *holomorphic Frobenioid-prime-strip*, or *$\mathcal{F}$-prime-strip*, [relative to the given initial Θ-data] to be a collection of data

$$^{\ddagger}\mathfrak{F} = \{^{\ddagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

that satisfies the following conditions: (a) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then $^{\ddagger}\mathcal{F}_{\underline{v}}$ is a *category* $^{\ddagger}\mathcal{C}_{\underline{v}}$ which admits an equivalence of categories $^{\ddagger}\mathcal{C}_{\underline{v}} \xrightarrow{\sim} \mathcal{C}_{\underline{v}}$ [where $\mathcal{C}_{\underline{v}}$ is as in Examples 3.2, (iii); 3.3, (i)]; (b) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then $^{\ddagger}\mathcal{F}_{\underline{v}} = (^{\ddagger}\mathcal{C}_{\underline{v}}, {}^{\ddagger}\mathcal{D}_{\underline{v}}, {}^{\ddagger}\kappa_{\underline{v}})$ is a *collection of data* consisting of a category, an Aut-holomorphic orbispace, and a Kummer structure such that there exists an isomorphism of collections of data $^{\ddagger}\mathcal{F}_{\underline{v}} \xrightarrow{\sim} \underline{\underline{\mathcal{F}}}_{\underline{v}} = (\mathcal{C}_{\underline{v}}, \mathcal{D}_{\underline{v}}, \kappa_{\underline{v}})$ [where $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is as in Example 3.4, (i)].

(ii) We define a *mono-analytic Frobenioid-prime-strip*, or *$\mathcal{F}^{\vdash}$-prime-strip*, [relative to the given initial Θ-data] to be a collection of data

$$^{\ddagger}\mathfrak{F}^{\vdash} = \{^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

that satisfies the following conditions: (a) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then $^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}$ is a *split Frobenioid*, whose underlying Frobenioid we denote by $^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}$, which admits an isomorphism $^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{F}^{\vdash}_{\underline{v}}$ [where $\mathcal{F}^{\vdash}_{\underline{v}}$ is as in Examples 3.2, (v); 3.3, (i)]; (b) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then $^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}$ is a triple of data, consisting of a Frobenioid $^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}$, an object of $\mathbb{TM}^{\vdash}$, and a splitting of the Frobenioid, such that there exists an isomorphism of collections of data $^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{F}^{\vdash}_{\underline{v}}$ [where $\mathcal{F}^{\vdash}_{\underline{v}}$ is as in Example 3.4, (ii)].

(iii) A *morphism of $\mathcal{F}$- (respectively, $\mathcal{F}^{\vdash}$-) prime-strips* is defined to be a collection of isomorphisms, indexed by $\underline{\mathbb{V}}$, between the various constituent objects of the prime-strips. Following the conventions of §0, one thus has notions of *capsules of $\mathcal{F}$- (respectively, $\mathcal{F}^{\vdash}$-)* and *morphisms of capsules of $\mathcal{F}$- (respectively, $\mathcal{F}^{\vdash}$-) prime-strips.*

(iv) We define a *globally realified mono-analytic Frobenioid-prime-strip*, or *$\mathcal{F}^{\Vdash}$-prime-strip*, [relative to the given initial Θ-data] to be a collection of data

$$^{\ddagger}\mathfrak{F}^{\Vdash} = (^{\ddagger}\mathcal{C}^{\Vdash}, \ \mathrm{Prime}(^{\ddagger}\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}, \ ^{\ddagger}\mathfrak{F}^{\vdash}, \ \{^{\ddagger}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

that satisfies the following conditions: (a) $^{\ddagger}\mathcal{C}^{\Vdash}$ is a category [which is, in fact, equipped with a Frobenioid structure] that is isomorphic to the category $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ of

Example 3.5, (i); (b) "Prime($-$)" is defined as in the discussion of Example 3.5, (i); (c) $\mathrm{Prime}({}^{\ddagger}\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}$ is a bijection of sets; (d) ${}^{\ddagger}\mathfrak{F}^{\vdash} = \{{}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ is an $\mathcal{F}^{\vdash}$-prime-strip; (e) ${}^{\ddagger}\rho_{\underline{v}} : \Phi_{{}^{\ddagger}\mathcal{C}^{\Vdash},v} \xrightarrow{\sim} \Phi^{\mathrm{rlf}}_{{}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}}$, where "$\Phi_{{}^{\ddagger}\mathcal{C}^{\Vdash},v}$" and "$\Phi^{\mathrm{rlf}}_{{}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}}$" are defined as in the discussion of Example 3.5, (i), is an isomorphism of topological monoids [both of which are, in fact, isomorphic to $\mathbb{R}_{\geq 0}$]; (f) the collection of data in the above display is *isomorphic to the collection of data* $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ of Example 3.5, (ii). A *morphism of* $\mathcal{F}^{\Vdash}$*-prime-strips* is defined to be an isomorphism between collections of data as discussed above. Following the conventions of §0, one thus has notions of *capsules of* $\mathcal{F}^{\Vdash}$*-prime-strips* and *morphisms of capsules of* $\mathcal{F}^{\Vdash}$*-prime-strips*.

(v) Let ${}^{\ddagger}\mathfrak{D} = \{{}^{\ddagger}\mathcal{D}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$ be a $\mathcal{D}$*-prime-strip*, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$. Write $v \in \mathbb{V}_{\mathrm{mod}}$ for the valuation determined by $\underline{v}$. Then [cf. the discussion of Example 5.1, (i); Remark 3.1.7, (i)] one may construct *group-theoretically* from $\pi_1({}^{\ddagger}\mathcal{D}_{\underline{v}})$, in a functorial fashion, a *profinite group* corresponding to "C_v" [cf. the algorithms of [AbsTopII], Corollary 3.3, (i), which are applicable in light of [AbsTopI], Example 4.8; [AbsTopIII], Theorem 1.9], which contains $\pi_1({}^{\ddagger}\mathcal{D}_{\underline{v}})$ as an *open subgroup*; we write ${}^{\ddagger}\mathcal{D}_v$ for $\mathcal{B}(-)^0$ of this profinite group, so we obtain a *natural morphism*

$$ {}^{\ddagger}\mathcal{D}_{\underline{v}} \to {}^{\ddagger}\mathcal{D}_v $$

— i.e., a "category-theoretic version" of the natural morphism of hyperbolic orbicurves $\underline{\underline{X}}_{\underline{v}} = \underline{\underline{X}}_K \times_K K_{\underline{v}} \to C_v$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, or $\underline{X}_{\to\underline{v}} = \underline{X}_{\to K} \times_K K_{\underline{v}} \to C_v$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$. Next, let us observe [cf. Remark 3.1.7, (i); the construction of the *Belyi cuspidalizations* of [AbsTopIII], Theorem 1.9, (a), and of the *field* "$K^{\times}_{Z_{\mathrm{NF}}}$" of [AbsTopIII], Theorem 1.9, (d), (e)] that one may construct *group-theoretically* from $\pi_1({}^{\ddagger}\mathcal{D}_{\underline{v}})$, in a functorial fashion, an isomorph

$$ \pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \quad (\twoheadrightarrow \pi_1({}^{\ddagger}\mathcal{D}_v)) $$

of the *étale fundamental group* [i.e., equipped with its natural surjection to $\pi_1({}^{\ddagger}\mathcal{D}_v)$ and well-defined up to inner automorphisms determined by elements of the kernel of this natural surjection] of the scheme obtained by *base-changing to* $(F_{\mathrm{mod}})_v$ *the generic point of* $C_{F_{\mathrm{mod}}}$. Next, let us recall [cf. [AbsTopIII], Corollary 1.10, (b), (c), (d), (d′)] that one may construct *group-theoretically* from $\pi_1({}^{\ddagger}\mathcal{D}_{\underline{v}})$, in a functorial fashion, an *ind-topological monoid* [which is naturally isomorphic to $\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$]

$$ \mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}}) $$

equipped with its natural $\pi_1({}^{\ddagger}\mathcal{D}_v)$-action, as well as isomorphs of the *pseudo-monoids of* κ*-,* ${}_{\infty}\kappa$*-, and* ${}_{\infty}\kappa\times$*-coric rational functions* associated to C_v [cf. the discussion of Remark 3.1.7, (i), (ii); [AbsTopII], Corollary 3.3, (iii), which is applicable in light of [AbsTopI], Example 4.8; [AbsTopIII], Theorem 1.9, (a), (d), (e); [AbsTopIII], Corollary 1.10, (d), (d′)] — which we shall denote

$$ \mathbb{M}_{\kappa v}({}^{\ddagger}\mathcal{D}_{\underline{v}}), \quad \mathbb{M}_{\infty\kappa v}({}^{\ddagger}\mathcal{D}_{\underline{v}}), \quad \mathbb{M}_{\infty\kappa\times v}({}^{\ddagger}\mathcal{D}_{\underline{v}}) $$

— equipped with their *natural* $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$*-actions*. Thus, $\mathbb{M}_{\kappa v}({}^{\ddagger}\mathcal{D}_{\underline{v}})$ may be identified with the subset of $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$*-invariants* of $\mathbb{M}_{\infty\kappa v}({}^{\ddagger}\mathcal{D}_{\underline{v}})$, and [if we use the superscript "$\times$" to denote the subset of invertible elements of a pseudo-monoid, then] $\mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\times}$ may be identified with $\mathbb{M}_{\infty\kappa\times v}({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\times}$.

(vi) We continue to use the notation of (v). Suppose further that ${}^{\ddagger}\mathfrak{F} = \{{}^{\ddagger}\mathcal{F}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$ is an *$\mathcal{F}$-prime-strip* whose associated *$\mathcal{D}$-prime-strip* [cf. Remark 5.2.1, (i), below] is equal to ${}^{\ddagger}\mathfrak{D} = \{{}^{\ddagger}\mathcal{D}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$. Let

$$\pi_1({}^{\ddagger}\mathcal{D}_v) \quad \curvearrowright \quad {}^{\ddagger}\mathbb{M}_v$$

be an *ind-topological monoid* equipped with a continuous action by $\pi_1({}^{\ddagger}\mathcal{D}_v)$ that is *isomorphic* [i.e., as an ind-topological monoid equipped with a continuous action by $\pi_1({}^{\ddagger}\mathcal{D}_v)$] to the pair $\pi_1({}^{\ddagger}\mathcal{D}_v) \curvearrowright \mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})$ constructed in (v). One may regard such a pair $\pi_1({}^{\ddagger}\mathcal{D}_v) \curvearrowright {}^{\ddagger}\mathbb{M}_v$ as being *related to the Frobenioid* ${}^{\ddagger}\mathcal{F}_{\underline{v}}$ [cf. (i), (a)] via the *unique isomorphism* corresponding to the *identity automorphism* of ${}^{\ddagger}\mathfrak{D} = \{{}^{\ddagger}\mathcal{D}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$ [cf. Corollary 5.3, (ii), below] between ${}^{\ddagger}\mathcal{F}_{\underline{v}}$ and the *$p_{\underline{v}}$-adic Frobenioid* determined [cf. Remark 3.3.2] by the pair

$$\pi_1({}^{\ddagger}\mathcal{D}_{\underline{v}}) \curvearrowright {}^{\ddagger}\mathbb{M}_v$$

obtained by *restricting* the action of the pair $\pi_1({}^{\ddagger}\mathcal{D}_v) \curvearrowright {}^{\ddagger}\mathbb{M}_v$ to the *open subgroup* $\pi_1({}^{\ddagger}\mathcal{D}_{\underline{v}}) \subseteq \pi_1({}^{\ddagger}\mathcal{D}_v)$ [cf. (v)]. We shall refer to a pair [i.e., consisting of a *pseudo-monoid* equipped with a continuous action by $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$]

$$\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \quad \curvearrowright \quad {}^{\ddagger}\mathbb{M}_{\infty\kappa v} \qquad \text{(respectively, } \pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \quad \curvearrowright \quad {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}\text{)}$$

as an *${}_{\infty}\kappa$-coric* (respectively, *${}_{\infty}\kappa\times$-coric*) *structure* on ${}^{\ddagger}\mathcal{F}_{\underline{v}}$ if it is isomorphic [i.e., as a pair consisting of a *pseudo-monoid* equipped with a continuous action by $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$] to the pair

$$\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \quad \curvearrowright \quad \mathbb{M}_{\infty\kappa v}({}^{\ddagger}\mathcal{D}_{\underline{v}}) \qquad \text{(respectively, } \pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \quad \curvearrowright \quad \mathbb{M}_{\infty\kappa\times v}({}^{\ddagger}\mathcal{D}_{\underline{v}})\text{)}$$

of (v). Suppose that we have been given such an *${}_{\infty}\kappa$-coric* (respectively, *${}_{\infty}\kappa\times$-coric*) *structure* on ${}^{\ddagger}\mathcal{F}_{\underline{v}}$. In the following, we shall use the notational convention "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}((-))$" introduced in Example 5.1, (v). Also, let us write $\boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^{\ddagger}\mathcal{D}_{\underline{v}}))$ for the *cyclotome* "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_{(-)})$" of [AbsTopIII], Theorem 1.9, which we think of as being applied *"via the Θ-approach"* [cf. Remark 3.1.2] to $\pi_1({}^{\ddagger}\mathcal{D}_{\underline{v}})$. Then let us *observe* that $\mathbb{M}_{\infty\kappa v}({}^{\ddagger}\mathcal{D}_{\underline{v}})$ (respectively, $\mathbb{M}_{\infty\kappa\times v}({}^{\ddagger}\mathcal{D}_{\underline{v}})$) is, in effect, *constructed* [cf. [AbsTopIII], Theorem 1.9, (d)] as a *subset* of

$$\varinjlim_{H} \ H^1(H, \boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^{\ddagger}\mathcal{D}_{\underline{v}})))$$

— where H ranges over the open subgroups of $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$. On the other hand, consideration of *Kummer classes* [i.e., of the action of $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$ on N-th roots of elements, for positive integers N] yields a *natural injection* of ${}^{\ddagger}\mathbb{M}_{\infty\kappa v}$ (respectively, ${}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}$) into

$$\varinjlim_{H} \ H^1(H, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathbb{M}_{\infty\kappa v})) \qquad \text{(respectively, } \varinjlim_{H} \ H^1(H, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}))\text{)}$$

— where H ranges over the open subgroups of $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$, and we observe that the asserted *injectivity* follows immediately from the corresponding injectivity in the case of $\mathbb{M}_{\infty\kappa v}({}^{\ddagger}\mathcal{D}_{\underline{v}})$ (respectively, $\mathbb{M}_{\infty\kappa\times v}({}^{\ddagger}\mathcal{D}_{\underline{v}})$). In particular, it follows immediately [cf. the discussion of Example 5.1, (v)], by considering *divisors of zeroes and poles* [cf. the definition of a "*κ-coric function*" given in Remark 3.1.7, (i)] associated to Kummer classes of rational functions as in [AbsTopIII], Proposition 1.6, (iii), from the elementary observation that, relative to the natural inclusion $\mathbb{Q} \hookrightarrow \widehat{\mathbb{Z}} \otimes \mathbb{Q}$,

$$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$$

that there exists a **unique isomorphism of cyclotomes**

$$\boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^{\ddagger}\mathcal{D}_{\underline{v}})) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathbb{M}_{\infty\kappa v}) \qquad \text{(respectively, } \boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^{\ddagger}\mathcal{D}_{\underline{v}})) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}))$$

such that the resulting isomorphism between direct limits of cohomology modules as considered above induces an **isomorphism**

$$\mathbb{M}_{\infty\kappa v}({}^{\ddagger}\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} {}^{\ddagger}\mathbb{M}_{\infty\kappa v} \qquad \text{(respectively, } \mathbb{M}_{\infty\kappa\times v}({}^{\ddagger}\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v})$$

[i.e., of *pseudo-monoids* equipped with continuous actions by $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$]. In a similar vein, it follows immediately from the theory summarized in [AbsTopIII], Corollary 1.10, (d), (d′), that there exists a **unique isomorphism of cyclotomes**

$$\boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^{\ddagger}\mathcal{D}_{\underline{v}})) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathbb{M}_v)$$

such that the resulting isomorphism between direct limits of cohomology modules induces an **isomorphism**

$$\mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} {}^{\ddagger}\mathbb{M}_v$$

[i.e., of monoids equipped with continuous actions by $\pi_1({}^{\ddagger}\mathcal{D}_v)$]. In particular, it follows immediately from the above discussion that

> ${}^{\ddagger}\mathcal{F}_{\underline{v}}$ always admits an ${}_{\infty}\kappa$*-coric* (respectively, ${}_{\infty}\kappa\times$*-coric*) *structure*, which is, moreover, *unique* up to a *uniquely determined isomorphism* [i.e., of *pseudo-monoids* equipped with continuous actions by $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$].

Thus, in the following, we shall regard, *without further notice*, this uniquely determined ${}_{\infty}\kappa$*-coric* (respectively, ${}_{\infty}\kappa\times$*-coric*) *structure* on ${}^{\ddagger}\mathcal{F}_{\underline{v}}$ as a collection of data that is *naturally associated to* ${}^{\ddagger}\mathcal{F}_{\underline{v}}$. Here, we observe that the various isomorphisms of the last few displays allow one to regard the pseudo-monoids ${}^{\ddagger}\mathbb{M}_{\infty\kappa v}$, ${}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}$ as being *related to the Frobenioid* ${}^{\ddagger}\mathcal{F}_{\underline{v}}$ via ${}^{\ddagger}\mathbb{M}_v$ [cf. the discussion at the beginning of the present (vi) concerning the relationship between ${}^{\ddagger}\mathbb{M}_v$ and ${}^{\ddagger}\mathcal{F}_{\underline{v}}$] and the morphisms

$${}^{\ddagger}\mathbb{M}_{\infty\kappa v} \hookrightarrow {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}, \qquad {}^{\ddagger}\mathbb{M}^{\times}_{\infty\kappa v} \hookrightarrow {}^{\ddagger}\mathbb{M}^{\times}_{\infty\kappa\times v} \xrightarrow{\sim} {}^{\ddagger}\mathbb{M}^{\times}_v$$

induced by the various isomorphisms of the last few displays, together with the corresponding inclusions/equalities

$$\mathbb{M}_{\infty\kappa v}({}^{\ddagger}\mathcal{D}_{\underline{v}}) \subseteq \mathbb{M}_{\infty\kappa\times v}({}^{\ddagger}\mathcal{D}_{\underline{v}}),$$

$$\mathbb{M}_{\infty\kappa v}({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\times} \subseteq \mathbb{M}_{\infty\kappa\times v}({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\times} = \mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\times}$$

[cf. the discussion at the end of (v)]. Also, we shall write

$$^{\ddagger}\mathbb{M}_{\kappa v} \subseteq {}^{\ddagger}\mathbb{M}_{\infty\kappa v}$$

for the "sub-pseudo-monoid" of $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$-*invariants*. In this context, we observe further that it follows immediately from the discussion of Remark 3.1.7, (i), (ii) [cf. also [AbsTopII], Corollary 3.3, (iii), which is applicable in light of [AbsTopI], Example 4.8], and the theory summarized in [AbsTopIII], Theorem 1.9 [cf., especially, [AbsTopIII], Theorem 1.9, (a), (d), (e)], and [AbsTopIII], Corollary 1.10, (h), that

> any ${}_{\infty}\kappa\times$**-coric structure** $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \curvearrowright {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}$ on ${}^{\ddagger}\mathcal{F}_{\underline{v}}$ **determines** an associated ${}_{\infty}\kappa$**-coric structure**
>
> $$\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \curvearrowright {}^{\ddagger}\mathbb{M}_{\infty\kappa v} \subseteq {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}$$
>
> by considering the subset of elements for which the **restriction** of the associated **Kummer class** [as in the above discussion] to some [or, equivalently, every — cf. Remark 5.2.3 below] subgroup of $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$ that corresponds to an open subgroup of the **decomposition group** of some **strictly critical** point of C_v determines a **torsion element** $\in {}^{\ddagger}\mathbb{M}_v^{\times} \xrightarrow{\sim} {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}^{\times}$ [i.e., corresponds to a root of unity],

and, moreover, that

> the operation of **restricting Kummer classes** [as in the above discussion] arising from ${}^{\ddagger}\mathbb{M}_{\kappa v} \subseteq {}^{\ddagger}\mathbb{M}_{\infty\kappa v}$ to subgroups of $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$ that correspond to **decomposition groups** of **non-critical** $(F_{\mathrm{mod}})_v$-*valued points* of C_v yields a functorial algorithm for **reconstructing** the *submonoid of* $\pi_1({}^{\ddagger}\mathcal{D}_v)$-*invariants of* ${}^{\ddagger}\mathbb{M}_v^{\mathrm{gp}}$ [where the superscript "gp" denotes the groupification], together with the **ind-topological field** structure on the union of this monoid with $\{0\}$, from the ${}_{\infty}\kappa$**-coric structure** ${}^{\ddagger}\mathbb{M}_{\infty\kappa v}$ associated to ${}^{\ddagger}\mathcal{F}_{\underline{v}}$.

A similar statement to the statement of the last display holds, if one replaces the phrase "$(F_{\mathrm{mod}})_v$-*valued points*" by the phrase "$\overline{F}_{\underline{v}}$-*valued points*" and the phrase "*submonoid of* $\pi_1({}^{\ddagger}\mathcal{D}_v)$-*invariants* of ${}^{\ddagger}\mathbb{M}_v^{\mathrm{gp}}$" by the phrase "*pair* $\pi_1({}^{\ddagger}\mathcal{D}_v) \curvearrowright {}^{\ddagger}\mathbb{M}_v^{\mathrm{gp}}$".

(vii) Let ${}^{\ddagger}\mathfrak{D} = \{{}^{\ddagger}\mathcal{D}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$ be a $\mathcal{D}$-*prime-strip*, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Write $v \in \mathbb{V}_{\mathrm{mod}}$ for the valuation determined by $\underline{v}$. Then [cf. the discussion of Example 5.1, (i); Remark 3.1.7, (i)] one may construct *algorithmically* from the Aut-*holomorphic space* ${}^{\ddagger}\mathcal{D}_{\underline{v}}$, in a functorial fashion, an Aut-*holomorphic orbispace* ${}^{\ddagger}\mathcal{D}_v$ corresponding to "C_v" [cf. the algorithms of [AbsTopIII], Corollary 2.7, (a)], together with a *natural morphism*

$${}^{\ddagger}\mathcal{D}_{\underline{v}} \to {}^{\ddagger}\mathcal{D}_v$$

— i.e., an "Aut-holomorphic orbispace version" of the natural morphism of hyperbolic orbicurves $\underline{X}_{\rightarrow\underline{v}} \stackrel{\mathrm{def}}{=} \underline{X}_{\rightarrow K} \times_K K_{\underline{v}} \to C_v \times_{(F_{\mathrm{mod}})_v} K_{\underline{v}}$. Here, we observe [cf. the fact that C_K is a K-*core*; [AbsTopIII], Corollary 2.3] that one has a *natural isomorphism*

$$\mathrm{Aut}({}^{\ddagger}\mathcal{D}_v) \xrightarrow{\sim} \mathrm{Gal}(K_{\underline{v}}/(F_{\mathrm{mod}})_v) \ (\hookrightarrow \mathbb{Z}/2\mathbb{Z})$$

— i.e., obtained by considering whether an automorphism of ${}^\ddagger\mathcal{D}_v$ is *holomorphic* or *anti-holomorphic* — from the group of automorphisms of the Aut-holomorphic orbispace ${}^\ddagger\mathcal{D}_v$ onto the Galois group $\mathrm{Gal}(K_{\underline{v}}/(F_{\mathrm{mod}})_v)$. Write

$$ {}^\ddagger\mathcal{D}_v^{\mathrm{rat}} \ \to \ {}^\ddagger\mathcal{D}_v $$

for the projective system of Aut-holomorphic orbispaces that arise as *universal covering spaces* of *"co-finite"* open sub-orbispaces of ${}^\ddagger\mathcal{D}_v$ [i.e., open sub-orbispaces determined by forming complements of finite sets of points of the underlying topological orbispace of ${}^\ddagger\mathcal{D}_v$] that contain *every strictly critical point* [cf. Remark 3.1.7, (i)], as well as every point that is *not an NF-point* [cf. Remark 3.4.3, (ii)], of ${}^\ddagger\mathcal{D}_v$. Thus, ${}^\ddagger\mathcal{D}_v^{\mathrm{rat}}$ is *well-defined* up to the action of *deck transformations* over ${}^\ddagger\mathcal{D}_v$ [cf. the *countability* of the set of *NF-points* of ${}^\ddagger\mathcal{D}_v$; the discussion of *compatible systems of basepoints* at the end of Remark 2.5.3, (i)]. Next, let us recall the *complex archimedean topological field* $\overline{\mathcal{A}}_{{}^\ddagger\mathcal{D}_{\underline{v}}}$ [cf. the discussion of Example 3.4, (i), as well as Definition 3.6, (b); the discussion of (i) of the present Definition 5.2]. Write $\mathrm{Aut}(\overline{\mathcal{A}}_{{}^\ddagger\mathcal{D}_{\underline{v}}})$ for the group of automorphisms ($\cong \mathbb{Z}/2\mathbb{Z}$) of the topological field $\overline{\mathcal{A}}_{{}^\ddagger\mathcal{D}_{\underline{v}}}$. Observe that it follows immediately from the construction of $\overline{\mathcal{A}}_{{}^\ddagger\mathcal{D}_{\underline{v}}}$ in [AbsTopIII], Corollary 2.7, (e), that $\overline{\mathcal{A}}_{{}^\ddagger\mathcal{D}_{\underline{v}}}$ is equipped with a *natural* Aut-*holomorphic structure* [cf. [AbsTopIII], Definition 4.1, (i)], as well as with a *tautological co-holomorphicization* [cf. [AbsTopIII], Definition 2.1, (iv); [AbsTopIII], Proposition 2.6, (a)] with ${}^\ddagger\mathcal{D}_{\underline{v}}$. Write

$$ \mathbb{M}_v({}^\ddagger\mathcal{D}_{\underline{v}}) \ \subseteq \ \overline{\mathcal{A}}_{{}^\ddagger\mathcal{D}_{\underline{v}}} $$

for the topological submonoid consisting of nonzero elements of norm ≤ 1 [i.e., "$\mathcal{O}_{\mathbb{C}}^{\triangleright}$"]. Thus, $\overline{\mathcal{A}}_{{}^\ddagger\mathcal{D}_{\underline{v}}}$ may be identified with the union with $\{0\}$ of the groupification $\mathbb{M}_v({}^\ddagger\mathcal{D}_{\underline{v}})^{\mathrm{gp}}$. Moreover, the *pseudo-monoids of κ-, ${}_\infty\kappa$-, and ${}_\infty\kappa\times$-coric rational functions* associated to C_v [cf. the discussion of Remark 3.1.7, (i), (ii)] may be represented, via *algorithmic constructions* [cf. [AbsTopIII], Corollary 2.7, (b)], as pseudo-monoids of *"meromorphic functions"*

$$ \mathbb{M}_{\kappa v}({}^\ddagger\mathcal{D}_{\underline{v}}), \quad \mathbb{M}_{{}_\infty\kappa v}({}^\ddagger\mathcal{D}_{\underline{v}}), \quad \mathbb{M}_{{}_\infty\kappa\times v}({}^\ddagger\mathcal{D}_{\underline{v}}) $$

— i.e., as sets of morphisms of Aut-holomorphic orbispaces from [some constituent of the projective system] ${}^\ddagger\mathcal{D}_v^{\mathrm{rat}}$ to $\mathbb{M}_v({}^\ddagger\mathcal{D}_{\underline{v}})^{\mathrm{gp}}$ that are *compatible* with the *tautological co-holomorphicization* just discussed and, moreover, satisfy conditions corresponding to the conditions of the final display of Remark 3.1.7, (i). Here, $\mathbb{M}_{\kappa v}({}^\ddagger\mathcal{D}_{\underline{v}})$ may be identified with the subset of elements of $\mathbb{M}_{{}_\infty\kappa v}({}^\ddagger\mathcal{D}_{\underline{v}})$ that *descend* to some co-finite open sub-orbispace of ${}^\ddagger\mathcal{D}_v$ and, moreover, are *equivariant* with respect to the unique embedding $\mathrm{Aut}({}^\ddagger\mathcal{D}_v) \hookrightarrow \mathrm{Aut}(\overline{\mathcal{A}}_{{}^\ddagger\mathcal{D}_{\underline{v}}})$; [if we use the superscript "$\times$" to denote the subset of invertible elements of a pseudo-monoid, then] $\mathbb{M}_v({}^\ddagger\mathcal{D}_{\underline{v}})^\times$ may be identified with $\mathbb{M}_{{}_\infty\kappa\times v}({}^\ddagger\mathcal{D}_{\underline{v}})^\times$; we observe that both $\mathbb{M}_v({}^\ddagger\mathcal{D}_{\underline{v}})^\times$ and $\mathbb{M}_{{}_\infty\kappa\times v}({}^\ddagger\mathcal{D}_{\underline{v}})^\times$ are isomorphic, as abstract topological monoids, to $\mathbb{S}^1$ [i.e., "$\mathcal{O}_{\mathbb{C}}^\times$"].

(viii) We continue to use the notation of (vii). Suppose further that ${}^\ddagger\mathfrak{F} = \{{}^\ddagger\mathcal{F}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$ is an *$\mathcal{F}$-prime-strip* whose associated *$\mathcal{D}$-prime-strip* [cf. Remark 5.2.1, (i), below] is equal to ${}^\ddagger\mathfrak{D} = \{{}^\ddagger\mathcal{D}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$. Write

$$ {}^\ddagger\mathbb{M}_v $$

for the *topological monoid* [i.e., "$\mathcal{O}^{\rhd}({}^{\ddagger}\mathcal{C}_{\underline{v}})$" — cf. the discussion of Example 3.4, (i); Definition 3.6, (b)] that appears as the *domain* of the *Kummer structure* portion of the data that constitutes ${}^{\ddagger}\mathcal{F}_{\underline{v}}$ [cf. (i) of the present Definition 5.2]. Thus, the **Kummer structure** portion of ${}^{\ddagger}\mathcal{F}_{\underline{v}}$ may be regarded as an **isomorphism of topological monoids**

$$\mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}}) \stackrel{\sim}{\to} {}^{\ddagger}\mathbb{M}_v$$

[both of which are abstractly isomorphic to $\mathcal{O}^{\rhd}_{\mathbb{C}}$]. In particular, the Kummer structure determines an isomorphism of topological groups $\mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\mathrm{gp}} \stackrel{\sim}{\to} {}^{\ddagger}\mathbb{M}_v^{\mathrm{gp}}$ [both of which are abstractly isomorphic to $\mathbb{C}^{\times}$], hence also a *natural action* of $\mathrm{Aut}(\overline{\mathcal{A}}_{{}^{\ddagger}\mathcal{D}_{\underline{v}}})$ on ${}^{\ddagger}\mathbb{M}_v^{\mathrm{gp}}$. Next, let us observe that the *pseudo-monoid of* ${}_{\infty}\kappa$*- (respectively,* ${}_{\infty}\kappa\times$*-) coric rational functions* associated to C_v [cf. the discussion of Remark 3.1.7, (i), (ii)] may be represented, via *algorithmic constructions* [cf. [AbsTopIII], Corollary 2.7, (b)], as the pseudo-monoid of *"meromorphic functions"*

$${}^{\ddagger}\mathbb{M}_{\infty\kappa v} \quad (\text{respectively, } {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v})$$

by considering the set of maps from [some constituent of the projective system] ${}^{\ddagger}\mathcal{D}_v^{\mathrm{rat}}$ to

$${}^{\ddagger}\mathbb{M}_v^{\mathrm{gp}}$$

that satisfy the following condition: the map from [some constituent of the projective system] ${}^{\ddagger}\mathcal{D}_v^{\mathrm{rat}}$ to $\mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\mathrm{gp}}$ obtained by *composing* the given map with the inverse of [the result of applying "gp" to] the **Kummer structure isomorphism** $\mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}}) \stackrel{\sim}{\to} {}^{\ddagger}\mathbb{M}_v$ determines an element of the pseudo-monoid $\mathbb{M}_{\infty\kappa v}({}^{\ddagger}\mathcal{D}_{\underline{v}})$ (respectively, $\mathbb{M}_{\infty\kappa\times v}({}^{\ddagger}\mathcal{D}_{\underline{v}})$) discussed in (vii) above. We shall refer to

$${}^{\ddagger}\mathbb{M}_{\infty\kappa v} \quad (\text{respectively, } {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v})$$

as the [uniquely determined] ${}_{\infty}\kappa$*-coric* (respectively, ${}_{\infty}\kappa\times$*-coric*) *structure* on ${}^{\ddagger}\mathcal{F}_{\underline{v}}$ and write

$${}^{\ddagger}\mathbb{M}_{\kappa v} \subseteq {}^{\ddagger}\mathbb{M}_{\infty\kappa v}$$

for the subset of elements that *descend* to some co-finite open sub-orbispace of ${}^{\ddagger}\mathcal{D}_v$ and, moreover, are *equivariant* with respect to the unique embedding $\mathrm{Aut}({}^{\ddagger}\mathcal{D}_v) \hookrightarrow \mathrm{Aut}(\overline{\mathcal{A}}_{{}^{\ddagger}\mathcal{D}_{\underline{v}}})$. In the following, we shall use the notational convention "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}((-))$" introduced in Example 5.1, (v). Also, let us write

$$\begin{aligned}\boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathcal{D}_{\underline{v}}) &\stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\mathrm{gp}}) \\ &= \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\boldsymbol{\mu}}) = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\times})\end{aligned}$$

— where $\mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\times} \subseteq \mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})$ denotes the topological group of *units* of $\mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})$; $\mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\boldsymbol{\mu}} \subseteq \mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\times}$ denotes the subgroup of *torsion elements*; we observe that the *Kummer structure isomorphism* discussed above induces a natural **"Kummer structure cyclotomic isomorphism"** $\boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathcal{D}_{\underline{v}}) \stackrel{\sim}{\to} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathbb{M}_v)$ [of profinite groups abstractly isomorphic to $\widehat{\mathbb{Z}}$]; the superscript "Θ" may be thought of as expressing the fact that we wish to apply to "$\boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(-)$" the *interpretation via the archimedean version of the* **Θ-approach**, i.e., the interpretation in terms of **cuspidal inertia**

groups, discussed in Remark 3.4.3, (i). In this context, we observe that these cuspidal inertia groups may be interpreted as profinite completions of *subgroups* of the *group of deck transformations*

$$\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$$

determined, up to inner automorphism, by the projective system of covering spaces ${}^{\ddagger}\mathcal{D}_v^{\mathrm{rat}} \to {}^{\ddagger}\mathcal{D}_v$. Here, we observe that the pseudo-monoids $\mathbb{M}_{\infty\kappa v}({}^{\ddagger}\mathcal{D}_{\underline{v}})$, ${}^{\ddagger}\mathbb{M}_{\infty\kappa v}$ (respectively, $\mathbb{M}_{\infty\kappa\times v}({}^{\ddagger}\mathcal{D}_{\underline{v}})$, ${}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}$) admit *natural* $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$*-actions* in such a way that each of the *pairs*

$$\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \curvearrowright \mathbb{M}_{\infty\kappa v}({}^{\ddagger}\mathcal{D}_{\underline{v}}), \quad \pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \curvearrowright {}^{\ddagger}\mathbb{M}_{\infty\kappa v}$$

$$(\text{respectively, } \pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \curvearrowright \mathbb{M}_{\infty\kappa\times v}({}^{\ddagger}\mathcal{D}_{\underline{v}}), \quad \pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \curvearrowright {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v})$$

is well-defined up to $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$*-conjugacy*. Next, let us observe that by considering the action of the various *cuspidal inertia groups* just discussed on elements of the *pseudo-monoid* ${}^{\ddagger}\mathbb{M}_{\infty\kappa v}$ (respectively, ${}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}$) — i.e., in effect, by considering, in the fashion of [AbsTopIII], Proposition 1.6, (iii), *"local Kummer classes"* at the points that give rise to these cuspidal inertia groups — we obtain various $\mathbb{Q}$*-multiples* — i.e., corresponding to the *order of zeroes or poles* at the point that gives rise to the cuspidal inertia group under consideration — of the Kummer structure cyclotomic isomorphism $\boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathcal{D}_{\underline{v}}) \overset{\sim}{\to} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathbb{M}_v)$ discussed above. In particular, relative to the *natural identification* [cf. the various definitions involved!] of $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathbb{M}_v)$ with $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathbb{M}_{\infty\kappa v})$ (respectively, $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathbb{M}_{\infty\kappa\times v})$), it follows immediately [cf. the discussion of Example 5.1, (v)], by considering [i.e., in the fashion just discussed] *divisors of zeroes and poles* [cf. the definition of a "κ*-coric function*" given in Remark 3.1.7, (i)] of meromorphic functions, from the elementary observation that, relative to the natural inclusion $\mathbb{Q} \hookrightarrow \widehat{\mathbb{Z}} \otimes \mathbb{Q}$,

$$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$$

that one may *algorithmically reconstruct* the **Kummer structure cyclotomic isomorphism**

$$\boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathcal{D}_{\underline{v}}) \overset{\sim}{\to} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathbb{M}_{\infty\kappa v}) \quad (\text{respectively, } \boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathcal{D}_{\underline{v}}) \overset{\sim}{\to} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}({}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}))$$

— hence also the **Kummer structure isomorphism** $\mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\boldsymbol{\mu}} \overset{\sim}{\to} {}^{\ddagger}\mathbb{M}^{\boldsymbol{\mu}}_{\infty\kappa v}$ (respectively, $\mathbb{M}_v({}^{\ddagger}\mathcal{D}_{\underline{v}})^{\boldsymbol{\mu}} \overset{\sim}{\to} {}^{\ddagger}\mathbb{M}^{\boldsymbol{\mu}}_{\infty\kappa\times v}$) [where the superscript "$\boldsymbol{\mu}$'s" denote the subgroups of *torsion elements*] — from

> the **projective system of coverings of** Aut**-holomorphic orbispaces** ${}^{\ddagger}\mathcal{D}_v^{\mathrm{rat}} \to {}^{\ddagger}\mathcal{D}_v$, together with the **abstract pseudo-monoid** with $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v)$**-action** $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \curvearrowright {}^{\ddagger}\mathbb{M}_{\infty\kappa v}$ (respectively, $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \curvearrowright {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}$).

Since, moreover, a *rational algebraic function* is *completely determined* by its *divisor of zeroes and poles* together with its *value at a single point*, we thus conclude that one may *algorithmically reconstruct* the **isomorphism(s) of pseudo-monoids** determined by the **Kummer structure** on ${}^\ddagger\mathcal{F}_{\underline{v}}$ [i.e., by the *Kummer structure isomorphism* $\mathbb{M}_v({}^\ddagger\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} {}^\ddagger\mathbb{M}_v$ discussed above]

$$\mathbb{M}_{\kappa v}({}^\ddagger\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} {}^\ddagger\mathbb{M}_{\kappa v}, \quad \mathbb{M}_{\infty\kappa v}({}^\ddagger\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} {}^\ddagger\mathbb{M}_{\infty\kappa v}$$
$$(\text{respectively, } \mathbb{M}_{\infty\kappa\times v}({}^\ddagger\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} {}^\ddagger\mathbb{M}_{\infty\kappa\times v})$$

from

> the **projective system of coverings of** Aut**-holomorphic orbispaces** ${}^\ddagger\mathcal{D}_v^{\text{rat}} \to {}^\ddagger\mathcal{D}_v$, together with the **abstract pseudo-monoid** with $\pi_1^{\text{rat}}({}^\ddagger\mathcal{D}_v)$**-action** $\pi_1^{\text{rat}}({}^\ddagger\mathcal{D}_v) \curvearrowright {}^\ddagger\mathbb{M}_{\infty\kappa v}$ (respectively, $\pi_1^{\text{rat}}({}^\ddagger\mathcal{D}_v) \curvearrowright {}^\ddagger\mathbb{M}_{\infty\kappa\times v}$) and the collection of **splittings**
>
> $${}^\ddagger\mathbb{M}_{\infty\kappa v} \twoheadrightarrow {}^\ddagger\mathbb{M}^{\boldsymbol{\mu}}_{\infty\kappa v} \text{ (respectively, } {}^\ddagger\mathbb{M}_{\infty\kappa\times v} \twoheadrightarrow {}^\ddagger\mathbb{M}^{\times}_{\infty\kappa\times v})$$
>
> — where the superscript "$\boldsymbol{\mu}$" (respectively, "$\times$") denotes the subgroup of *torsion elements* (respectively, the topological group of *units*, which contains the subgroup of *torsion elements* as a *dense* subgroup) of ${}^\ddagger\mathbb{M}_{\infty\kappa v}$ (respectively, ${}^\ddagger\mathbb{M}_{\infty\kappa\times v}$) — determined [and parametrized], via the operation of **restriction**, by the collection of **systems of strictly critical points** of ${}^\ddagger\mathcal{D}_v^{\text{rat}} \to {}^\ddagger\mathcal{D}_v$ [i.e., systems of points lying over some strictly critical point of ${}^\ddagger\mathcal{D}_v$].

In this context, we observe further that it follows immediately from the discussion of Remark 3.1.7, (ii) [cf. also [AbsTopIII], Corollary 2.7, (b)], that

> the ${}_\infty\kappa$**-coric structure**
>
> $${}^\ddagger\mathbb{M}_{\infty\kappa v} \subseteq {}^\ddagger\mathbb{M}_{\infty\kappa\times v}$$
>
> on ${}^\ddagger\mathcal{F}_{\underline{v}}$ may be constructed from the ${}_\infty\kappa\times$**-coric structure** ${}^\ddagger\mathbb{M}_{\infty\kappa\times v}$ on ${}^\ddagger\mathcal{F}_{\underline{v}}$ by considering the subset of elements for which the **restriction** to some [or, equivalently, every] **system of strictly critical points** of ${}^\ddagger\mathcal{D}_v^{\text{rat}} \to {}^\ddagger\mathcal{D}_v$ is a **torsion element** $\in {}^\ddagger\mathbb{M}_v^{\times} \xrightarrow{\sim} {}^\ddagger\mathbb{M}^{\times}_{\infty\kappa\times v}$ [i.e., corresponds to a root of unity],

and, moreover, that

> the operation of **restricting** elements of ${}^\ddagger\mathbb{M}_{\kappa v} \subseteq {}^\ddagger\mathbb{M}_{\infty\kappa v}$ to systems of points of ${}^\ddagger\mathcal{D}_v^{\text{rat}}$ that lie over $\text{Aut}({}^\ddagger\mathcal{D}_v)$*-invariant* **non-critical** points of ${}^\ddagger\mathcal{D}_v$ yields a functorial algorithm for **reconstructing** the *submonoid of* $\text{Aut}({}^\ddagger\mathcal{D}_v)$*-invariants of* ${}^\ddagger\mathbb{M}_v^{\text{gp}}$ [where the superscript "gp" denotes the groupification], together with the **topological field** structure on the union of this monoid with $\{0\}$, from the ${}_\infty\kappa$**-coric structure** ${}^\ddagger\mathbb{M}_{\infty\kappa v}$ associated to ${}^\ddagger\mathcal{F}_{\underline{v}}$.

A similar statement to the statement of the last display holds if one replaces the phrase "$\text{Aut}({}^\ddagger\mathcal{D}_v)$*-invariant*" by the phrase "*arbitrary*" and the phrase "*submonoid of* $\text{Aut}({}^\ddagger\mathcal{D}_v)$*-invariants of* ${}^\ddagger\mathbb{M}_v^{\text{gp}}$" by the phrase "*monoid* ${}^\ddagger\mathbb{M}_v^{\text{gp}}$".

Remark 5.2.1.

(i) Note that it follows immediately from Definitions 4.1, (i), (iii); 5.2, (i), (ii); Examples 3.2, (vi), (c), (d); 3.3, (iii), (b), (c), that there exists a *functorial algorithm* for constructing $\mathcal{D}$- (respectively, $\mathcal{D}^{\vdash}$-) *prime-strips* from $\mathcal{F}$- (respectively, $\mathcal{F}^{\vdash}$-) *prime-strips.*

(ii) In a similar vein, it follows immediately from Definition 5.2, (i), (ii); Examples 3.2, (vi), (f); 3.3, (iii), (e); 3.4, (i), (ii), that there exists a *functorial algorithm* for constructing from an $\mathcal{F}$*-prime-strip* ${}^{\ddagger}\mathfrak{F} = \{{}^{\ddagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ an $\mathcal{F}^{\vdash}$*-prime-strip* ${}^{\ddagger}\mathfrak{F}^{\vdash}$

$$ {}^{\ddagger}\mathfrak{F} \ \mapsto \ {}^{\ddagger}\mathfrak{F}^{\vdash} = \{{}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} $$

— which we shall refer to as the *mono-analyticization* of ${}^{\ddagger}\mathfrak{F}$. Next, let us recall from the discussion of Example 3.5, (i), the relatively *simple structure* of the category "$\mathcal{C}^{\Vdash}_{\mathrm{mod}}$", i.e., which may be summarized, roughly speaking, as a collection, indexed by $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$, of copies of the topological monoid $\mathbb{R}_{\geq 0}$, which are related to one another by a "product formula". In particular, it follows immediately [cf. Definition 5.2, (i)] from the *rigidity* of the *divisor monoids* associated to the *Frobenioids* that appear at each of the components at $\underline{v} \in \underline{\mathbb{V}}$ of an $\mathcal{F}$*-prime-strip* [cf., especially, the *topological field structure* of the field "$\overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$" of Example 3.4, (i)!] that one may also construct from the $\mathcal{F}$*-prime-strip* ${}^{\ddagger}\mathfrak{F}$, via a *functorial algorithm* [cf. the constructions of Example 3.5, (i), (ii)], a collection of data

$$ {}^{\ddagger}\mathfrak{F} \ \mapsto \ {}^{\ddagger}\mathfrak{F}^{\Vdash} \ \stackrel{\mathrm{def}}{=} \ ({}^{\ddagger}\mathcal{C}^{\Vdash}, \ \mathrm{Prime}({}^{\ddagger}\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}, \ {}^{\ddagger}\mathfrak{F}^{\vdash}, \ \{{}^{\ddagger}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) $$

— i.e., consisting of a category [which is, in fact, equipped with a Frobenioid structure], a bijection, the $\mathcal{F}^{\vdash}$-prime-strip ${}^{\ddagger}\mathfrak{F}^{\vdash}$, and an isomorphism of topological monoids associated to ${}^{\ddagger}\mathcal{C}^{\Vdash}$ and ${}^{\ddagger}\mathfrak{F}^{\vdash}$, respectively, at each $\underline{v} \in \underline{\mathbb{V}}$ — which is isomorphic to the collection of data $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ of Example 3.5, (ii), i.e., which forms an $\mathcal{F}^{\Vdash}$*-prime-strip* [cf. Definition 5.2, (iv)].

Remark 5.2.2. Thus, from the point of view of the discussion of Remark 5.1.3, $\mathcal{F}$*-prime-strips* are *Kummer-ready* [i.e., at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ — cf. the theory of [FrdII], §2], whereas $\mathcal{F}^{\vdash}$*-prime-strips* are *Kummer-blind.*

Remark 5.2.3. In the context of the construction of ${}_{\infty}\kappa$*-coric structures* from ${}_{\infty}\kappa\times$*-coric structures* in Definition 5.2, (vi), we make the following observation. When $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, it is natural to take the *decomposition groups* corresponding to *strictly critical* points [i.e., to which one restricts the Kummer classes under consideration] to be decomposition groups that correspond to the point of C_v that arises as the *image* of the **zero-labeled evaluation points** [i.e., evaluation points corresponding to the label $0 \in |\mathbb{F}_l|$ — cf. the discussion of Example 4.4, (i)]. In the notation of Example 4.4, (i), this point of C_v may also be described simply as the point that arises as the image of the point "μ_{-}".

Corollary 5.3. (Isomorphisms of Global Frobenioids, Frobenioid-Prime-Strips, and Tempered Frobenioids) *Relative to a fixed collection of* **initial Θ-data***:*

(i) For $i = 1, 2$, let ${}^{i}\mathcal{F}^{\circledast}$ (respectively, ${}^{i}\mathcal{F}^{\circledcirc}$) be a **category** *which is equivalent to the category ${}^{\dagger}\mathcal{F}^{\circledast}$ (respectively, ${}^{\dagger}\mathcal{F}^{\circledcirc}$) of Example 5.1, (iii). Thus, ${}^{i}\mathcal{F}^{\circledast}$ (respectively, ${}^{i}\mathcal{F}^{\circledcirc}$) is equipped with a* **natural Frobenioid structure** *[cf. [FrdI], Corollary 4.11; [FrdI], Theorem 6.4, (i); Remark 3.1.5 of the present paper]. Write $\mathrm{Base}({}^{i}\mathcal{F}^{\circledast})$ (respectively, $\mathrm{Base}({}^{i}\mathcal{F}^{\circledcirc})$) for the base category of this Frobenioid. Then the natural map*

$$\mathrm{Isom}({}^{1}\mathcal{F}^{\circledast}, {}^{2}\mathcal{F}^{\circledast}) \to \mathrm{Isom}(\mathrm{Base}({}^{1}\mathcal{F}^{\circledast}), \mathrm{Base}({}^{2}\mathcal{F}^{\circledast}))$$
$$(\textit{respectively, } \mathrm{Isom}({}^{1}\mathcal{F}^{\circledcirc}, {}^{2}\mathcal{F}^{\circledcirc}) \to \mathrm{Isom}(\mathrm{Base}({}^{1}\mathcal{F}^{\circledcirc}), \mathrm{Base}({}^{2}\mathcal{F}^{\circledcirc})))$$

[cf. [FrdI], Corollary 4.11; [FrdI], Theorem 6.4, (i); Remark 3.1.5 of the present paper] is **bijective**.

*(ii) For $i = 1, 2$, let ${}^{i}\mathfrak{F}$ be an $\mathcal{F}$-***prime-strip***; ${}^{i}\mathfrak{D}$ the $\mathcal{D}$-***prime-strip** *associated to ${}^{i}\mathfrak{F}$ [cf. Remark 5.2.1, (i)]. Then the natural map*

$$\mathrm{Isom}({}^{1}\mathfrak{F}, {}^{2}\mathfrak{F}) \to \mathrm{Isom}({}^{1}\mathfrak{D}, {}^{2}\mathfrak{D})$$

[cf. Remark 5.2.1, (i)] is **bijective**.

*(iii) For $i = 1, 2$, let ${}^{i}\mathfrak{F}^{\vdash}$ be an $\mathcal{F}^{\vdash}$-***prime-strip***; ${}^{i}\mathfrak{D}^{\vdash}$ the $\mathcal{D}^{\vdash}$-***prime-strip** *associated to ${}^{i}\mathfrak{F}^{\vdash}$ [cf. Remark 5.2.1, (i)]. Then the natural map*

$$\mathrm{Isom}({}^{1}\mathfrak{F}^{\vdash}, {}^{2}\mathfrak{F}^{\vdash}) \to \mathrm{Isom}({}^{1}\mathfrak{D}^{\vdash}, {}^{2}\mathfrak{D}^{\vdash})$$

[cf. Remark 5.2.1, (i)] is **surjective**.

(iv) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Recall the category $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ of Example 3.2, (i). Thus, $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is equipped with a **natural Frobenioid structure** *[cf. [FrdI], Corollary 4.11; [EtTh], Proposition 5.1], with base category $\mathcal{D}_{\underline{v}}$. Then the natural homomorphism $\mathrm{Aut}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \to \mathrm{Aut}(\mathcal{D}_{\underline{v}})$ [cf. Example 3.2, (vi), (d)] is* **bijective**.

Proof. Assertion (i) follows immediately from the *category-theoreticity* of the *"isomorphism $\mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \ \xrightarrow{\sim}\ {}^{\dagger}\mathbb{M}^{\circledast}$"* of Example 5.1, (v) [cf. also the surrounding discussion; Example 5.1, (vi)]. [Here, we note in passing that this argument is entirely similar to the technique applied to the proof of the equivalence "$\mathfrak{Th}^{\circledcirc}_{\mathbb{T}} \xrightarrow{\sim} \mathbb{EA}^{\circledcirc}$" of [AbsTopIII], Corollary 5.2, (iv).] Assertion (ii) (respectively, (iii)) follows immediately from [AbsTopIII], Proposition 3.2, (iv); [AbsTopIII], Proposition 4.2, (i) [cf. also [AbsTopIII], Remarks 3.1.1, 4.1.1; the discussion of Definition 5.2, (vi), (viii), of the present paper] (respectively, [AbsTopIII], Proposition 5.8, (ii), (v)).

Finally, we consider assertion (iv). First, we recall that since automorphisms of $\mathcal{D}_{\underline{v}} = \mathcal{B}^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}})^{0}$ necessarily arise from *automorphisms of the scheme* $\underline{\underline{X}}_{\underline{v}}$ [cf. [AbsTopIII], Theorem 1.9; [AbsTopIII], Remark 1.9.1], *surjectivity* follows immediately from the construction of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. Thus, it remains to verify *injectivity*. To this end, let $\alpha \in \mathrm{Ker}(\mathrm{Aut}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \to \mathrm{Aut}(\mathcal{D}_{\underline{v}}))$. For simplicity, we suppose [without loss of generality] that α lies over the identity self-equivalence of $\mathcal{D}_{\underline{v}}$. Then I *claim* that to show that α is [isomorphic to — cf. §0] the *identity self-equivalence* of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, it suffices to verify that

> α induces [cf. [FrdI], Corollary 4.11; [EtTh], Proposition 5.1] the *identity* on the *rational function* and *divisor monoids* of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$.

Indeed, recall that since $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is a *Frobenioid of model type* [cf. [EtTh], Definition 3.6, (ii)], it follows from [FrdI], Corollary 5.7, (i), (iv), that α preserves *base-Frobenius pairs*. Thus, once one shows that α induces the *identity* on the *rational function* and *divisor monoids* of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, it follows, by arguing as in the construction of the equivalence of categories given in the proof of [FrdI], Theorem 5.2, (iv), that the various *units* obtained in [FrdI], Proposition 5.6, determine [cf. Remark 5.3.3 below; the argument of the first paragraph of the proof of [FrdI], Proposition 5.6] an *isomorphism* between α and the *identity self-equivalence* of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, as desired.

Thus, we proceed to show that α induces the *identity* on the *rational function* and *divisor monoids* of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, as follows. In light of the *category-theoreticity* [cf. [EtTh], Theorem 5.6] of the *cyclotomic rigidity isomorphism* of [EtTh], Proposition 5.5, the fact that α induces the *identity* on the rational function monoid follows immediately from the *naturality of the Kummer map* [cf. the discussion of Remark 3.2.4; [FrdII], Definition 2.1, (ii)], which is *injective* by [EtTh], Proposition 3.2, (iii) — cf. the argument of [EtTh], Theorem 5.7, applied to verify the category-theoreticity of the Frobenioid-theoretic theta function. Next, we consider the effect of α on the divisor monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. To this end, let us first recall that α preserves *cuspidal* and *non-cuspidal* elements of the monoids that appear in this divisor monoid [cf. Remark 3.2.4, (vi); [EtTh], Proposition 5.3, (i)]. In particular, by considering the non-cuspidal portion of the divisor of the Frobenioid-theoretic theta function and its conjugates [each of which is preserved by α, since α has already been shown to induce the identity on the rational function monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$], we conclude that α induces the *identity* on the *non-cuspidal* elements of the monoids that appear in the divisor monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [cf. [EtTh], Proposition 5.3, (v), (vi), for a discussion of closely related facts]. In a similar vein, since any divisor of degree zero on an elliptic curve that is supported on the *torsion points* of the elliptic curve admits a positive multiple which is *principal*, it follows by considering the cuspidal portions of divisors of appropriate rational functions [each of which is preserved by α, since α has already been shown to induce the identity on the rational function monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$] that α also induces the *identity* on the *cuspidal* elements of the monoids that appear in the divisor monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. This completes the proof of assertion (iv). ◯

Remark 5.3.1.

(i) In the situation of Corollary 5.3, (ii), let

$$\phi : {}^1\mathfrak{D} \to {}^2\mathfrak{D}$$

be a *morphism of $\mathcal{D}$-prime-strips* [i.e., which is not necessarily an isomorphism!] that induces an *isomorphism* between the respective collections of data indexed by $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$, as well as an *isomorphism* $\phi^{\vdash} : {}^1\mathfrak{D}^{\vdash} \xrightarrow{\sim} {}^2\mathfrak{D}^{\vdash}$ between the associated $\mathcal{D}^{\vdash}$*-prime-strips* [cf. Definition 4.1, (iv)]. Then let us *observe* that by applying Corollary 5.3, (ii), it follows that ϕ *lifts* to a *uniquely determined "arrow"*

$$\psi : {}^1\mathfrak{F} \to {}^2\mathfrak{F}$$

— which we think of as *"lying over"* ϕ — defined as follows: First, let us recall that, in light of our assumptions on ϕ, it follows immediately from the construction [cf. Examples 3.2, (iii); 3.3, (i); 3.4, (i)] of the various *p-adic* and *archimedean Frobenioids* [cf. [FrdII], Example 1.1, (ii); [FrdII], Example 3.3, (ii)] that appear in an $\mathcal{F}$-prime-strip that it makes sense to speak of the *"pull-back"* — i.e., by forming the *"categorical fiber product"* [cf. [FrdI], §0; [FrdI], Proposition 1.6] — of the Frobenioids that appear in the $\mathcal{F}$-prime-strip ${}^2\mathfrak{F}$ via the various morphisms at $\underline{v} \in \underline{\mathbb{V}}$ that constitute ϕ. That is to say, it follows from our assumptions on ϕ [cf. also [AbsTopIII], Proposition 3.2, (iv)] that ϕ determines a *pulled-back $\mathcal{F}$-prime-strip* "$\phi^*({}^2\mathfrak{F})$", whose associated $\mathcal{D}$-prime-strip [cf. Remark 5.2.1, (i)] is *tautologically equal* to ${}^1\mathfrak{D}$. On the other hand, by Corollary 5.3, (ii), it follows that this tautological equality of associated $\mathcal{D}$-prime-strips *uniquely determines* an isomorphism ${}^1\mathfrak{F} \xrightarrow{\sim} \phi^*({}^2\mathfrak{F})$. Then we *define* the arrow $\psi : {}^1\mathfrak{F} \to {}^2\mathfrak{F}$ to be the collection of data consisting of ϕ and this isomorphism ${}^1\mathfrak{F} \xrightarrow{\sim} \phi^*({}^2\mathfrak{F})$; we refer to ψ as the *"morphism uniquely determined by ϕ"* or the *"uniquely determined morphism that lies over ϕ"*. Also, we shall apply various terms used to *describe a morphism ϕ of $\mathcal{D}$-prime-strips* to the "arrow" of $\mathcal{F}$-prime-strips determined by ϕ.

(ii) The conventions discussed in (i) concerning liftings of morphisms of $\mathcal{D}$-prime-strips may also be applied to *poly-morphisms*. We leave the routine details to the reader.

Remark 5.3.2. Just as in the case of Corollary 5.3, (i), (ii), the *rigidity property* of Corollary 5.3, (iv), may be regarded as being essentially a consequence of the *"Kummer-readiness"* [cf. Remarks 5.1.3, 5.2.2] of the tempered Frobenioid $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ — cf. also the arguments applied in the proofs of [AbsTopIII], Proposition 3.2, (iv); [AbsTopIII], Corollary 5.2, (iv).

Remark 5.3.3. We take this opportunity to rectify a minor oversight in [FrdI]. The hypothesis that the Frobenioids under consideration be of *"unit-profinite type"* in [FrdI], Proposition 5.6 — hence also in [FrdI], Corollary 5.7, (iii) — may be *removed*. Indeed, if, in the notation of the proof of [FrdI], Proposition 5.6, one writes $\phi'_p = c_p \cdot \phi_p$, where $c_p \in \mathcal{O}^\times(A)$, for $p \in \mathfrak{Primes}$, then one has

$$\begin{aligned} c_2 \cdot c_p^2 \cdot \phi_2 \cdot \phi_p = c_2 \cdot \phi_2 \cdot c_p \cdot \phi_p = \phi'_2 \cdot \phi'_p = \phi'_p \cdot \phi'_2 \\ = c_p \cdot \phi_p \cdot c_2 \cdot \phi_2 = c_p \cdot c_2^p \cdot \phi_p \cdot \phi_2 = c_p \cdot c_2^p \cdot \phi_2 \cdot \phi_p \end{aligned}$$

— so $c_2 \cdot c_p^2 = c_p \cdot c_2^p$, i.e., $c_p = c_2^{p-1}$, for $p \in \mathfrak{Primes}$. Thus, $\phi'_p = c_2^{-1} \cdot \phi_p \cdot c_2$, so by taking $u \stackrel{\mathrm{def}}{=} c_2^{-1}$, one may *eliminate the final two paragraphs* of the proof of [FrdI], Proposition 5.6.

Let

$$^{\dagger}\mathcal{HT}^{\Theta} = (\{^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ ^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$$

be a Θ-*Hodge theater* [relative to the given initial Θ-data — cf. Definition 3.6] such that the associated *$\mathcal{D}$-prime-strip* $\{^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ is [for simplicity] *equal* to the $\mathcal{D}$-prime-strip ${}^{\dagger}\mathfrak{D}_{>}$ of the $\mathcal{D}$-ΘNF-*Hodge theater* in the discussion preceding Example 5.1. Write

$$^{\dagger}\mathfrak{F}_{>}$$

for the $\mathcal{F}$*-prime-strip* tautologically associated to this Θ-Hodge theater [cf. the data "$\{{}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$" of Definition 3.6; Definition 5.2, (i); Example 3.2, (iii); Example 3.3, (i)]. Thus, ${}^\dagger\mathfrak{D}_>$ may be identified with the $\mathcal{D}$-prime-strip associated [cf. Remark 5.2.1, (i)] to ${}^\dagger\mathfrak{F}_>$.

Example 5.4. Model Θ- and NF-Bridges.

(i) For $j \in J$, let

$$^\dagger\mathfrak{F}_j = \{{}^\dagger\mathcal{F}_{\underline{v}_j}\}_{\underline{v}_j\in\underline{\mathbb{V}}_j}$$

be an $\mathcal{F}$*-prime-strip* whose associated $\mathcal{D}$-prime-strip [cf. Remark 5.2.1, (i)] is equal to ${}^\dagger\mathfrak{D}_j$,

$$^\dagger\mathfrak{F}_{\langle J\rangle} = \{{}^\dagger\mathcal{F}_{\underline{v}_{\langle J\rangle}}\}_{\underline{v}_{\langle J\rangle}\in\underline{\mathbb{V}}_{\langle J\rangle}}$$

an $\mathcal{F}$*-prime-strip* whose associated $\mathcal{D}$-prime-strip we denote by ${}^\dagger\mathfrak{D}_{\langle J\rangle}$ [cf. Example 5.1, (vii)]. Write

$$^\dagger\mathfrak{F}_J \stackrel{\text{def}}{=} \prod_{j\in J} {}^\dagger\mathfrak{F}_j$$

— where the "formal product $\prod$" is to be understood as denoting the capsule with index set J for which the datum indexed by j is given by ${}^\dagger\mathfrak{F}_j$. Thus, ${}^\dagger\mathfrak{F}_{\langle J\rangle}$ may be related to ${}^\dagger\mathfrak{F}_>$, in a *natural fashion*, via the *full poly-isomorphism*

$$^\dagger\mathfrak{F}_{\langle J\rangle} \ \stackrel{\sim}{\to}\ {}^\dagger\mathfrak{F}_>$$

and to ${}^\dagger\mathfrak{F}_J$ via the *"diagonal arrow"*

$$^\dagger\mathfrak{F}_{\langle J\rangle} \ \to\ {}^\dagger\mathfrak{F}_J = \prod_{j\in J} {}^\dagger\mathfrak{F}_j$$

— i.e., the arrow defined as the collection of data indexed by J for which the datum indexed by j is given by the *full poly-isomorphism* ${}^\dagger\mathfrak{F}_{\langle J\rangle} \stackrel{\sim}{\to} {}^\dagger\mathfrak{F}_j$. Thus, we think of ${}^\dagger\mathfrak{F}_j$ as a copy of ${}^\dagger\mathfrak{F}_>$ *"situated on"* the constituent labeled j of the capsule ${}^\dagger\mathfrak{D}_J$; we think of ${}^\dagger\mathfrak{F}_{\langle J\rangle}$ as a copy of ${}^\dagger\mathfrak{F}_>$ *"situated in a diagonal fashion on"* all the constituents of the capsule ${}^\dagger\mathfrak{D}_J$.

(ii) Note that in addition to thinking of ${}^\dagger\mathfrak{F}_>$ as being related to ${}^\dagger\mathfrak{F}_j$ [for $j \in J$] via the *full poly-isomorphism* ${}^\dagger\mathfrak{F}_> \stackrel{\sim}{\to} {}^\dagger\mathfrak{F}_j$, we may also regard ${}^\dagger\mathfrak{F}_>$ as being related to ${}^\dagger\mathfrak{F}_j$ [for $j \in J$] via the *poly-morphism*

$$^\dagger\psi^\Theta_j : {}^\dagger\mathfrak{F}_j \to {}^\dagger\mathfrak{F}_>$$

uniquely determined by ${}^\dagger\phi^\Theta_j$ [i.e., as discussed in Remark 5.3.1]. Write

$$^\dagger\psi^\Theta_{\divideontimes} : {}^\dagger\mathfrak{F}_J \to {}^\dagger\mathfrak{F}_>$$

for the *collection of arrows* $\{{}^\dagger\psi^\Theta_j\}_{j\in J}$ — which we think of as *"lying over"* the collection of arrows ${}^\dagger\phi^\Theta_{\divideontimes} = \{{}^\dagger\phi^\Theta_j\}_{j\in J}$.

(iii) Next, let ${}^\dagger\mathcal{F}^\circledast$, ${}^\dagger\mathcal{F}^\circledcirc$ be as in Example 5.1, (iii); $\delta \in \mathrm{LabCusp}({}^\dagger\mathcal{D}^\circledcirc)$. Thus, [cf. the discussion of Example 4.3, (i)] there exists a *unique* $\mathrm{Aut}_{\underline{\epsilon}}({}^\dagger\mathcal{D}^\circledcirc)$-orbit of isomorphisms ${}^{\dagger}\mathcal{D}^{\circledcirc} \xrightarrow{\sim} \mathcal{D}^{\circledcirc}$ that maps $\delta \mapsto [\underline{\epsilon}] \in \mathrm{LabCusp}(\mathcal{D}^{\circledcirc})$. We shall refer to as a *δ-valuation* $\in \mathbb{V}({}^{\dagger}\mathcal{D}^{\circledcirc})$ [cf. Definition 4.1, (v)] any element that maps to an element of $\underline{\mathbb{V}}^{\pm\mathrm{un}}$ [cf. Example 4.3, (i)] via this $\mathrm{Aut}_{\underline{\epsilon}}({}^{\dagger}\mathcal{D}^{\circledcirc})$-orbit of isomorphisms. Note that the notion of a δ-valuation may also be defined *intrinsically* by means of the structure of $\mathcal{D}$-NF-bridge ${}^{\dagger}\phi^{\mathrm{NF}}_{\circledast}$. Indeed, [one verifies immediately that] a *δ-valuation* may be defined as a valuation $\in \mathbb{V}({}^{\dagger}\mathcal{D}^{\circledcirc})$ that lies in the "image" [in the evident sense] via ${}^{\dagger}\phi^{\mathrm{NF}}_{\circledast}$ of the unique $\mathcal{D}$-prime-strip ${}^{\dagger}\mathfrak{D}_j$ of the capsule ${}^{\dagger}\mathfrak{D}_J$ such that the bijection $\mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}({}^{\dagger}\mathfrak{D}_j)$ induced by ${}^{\dagger}\phi^{\mathrm{NF}}_{\circledast}$ [cf. the discussion of Example 4.5, (i)] maps δ to the element of $\mathrm{LabCusp}({}^{\dagger}\mathfrak{D}_j)$ that is *"labeled 1"*, relative to the bijection of the second display of Proposition 4.2.

(iv) We continue to use the notation of (iii). Then let us observe that by *localizing* at each of the δ-valuations $\in \mathbb{V}({}^{\dagger}\mathcal{D}^{\circledcirc})$, one may construct, in a natural way, an *$\mathcal{F}$-prime-strip*

$$ {}^{\dagger}\mathcal{F}^{\circledcirc}|_{\delta} $$

— which is well-defined *up to isomorphism* — from ${}^{\dagger}\mathcal{F}^{\circledcirc}$ [i.e., in the notation of Example 5.1, (iv), from $\widetilde{\mathcal{O}}^{\circledast\times}$, equipped with its natural $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$-action]. Indeed, at a *nonarchimedean* δ-valuation $\underline{v}$, this follows by considering the *$p_{\underline{v}}$-adic Frobenioids* [cf. Remark 3.3.2] associated to the restrictions to *suitable open subgroups* of $\Pi_{\mathfrak{p}_0} \bigcap \pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$ $(\subseteq \pi_1({}^{\dagger}\mathcal{D}^{\circledcirc}) \subseteq \pi_1({}^{\dagger}\mathcal{D}^{\circledast}))$ *determined by* $\delta \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$ [i.e., open subgroups corresponding to the coverings "$\underline{\underline{X}}$", "$\underset{\rightarrow}{X}$" discussed in Definition 3.1, (e), (f); cf. also Remark 3.1.2, (i)], where $\Pi_{\mathfrak{p}_0}$ is chosen [among its $\pi_1({}^{\dagger}\mathcal{D}^{\circledast})$-conjugates] so as to *correspond to* $\underline{v}$, of the *pairs*

$$ \text{“}\Pi_{\mathfrak{p}_0} \curvearrowright \widetilde{\mathcal{O}}^{\rhd}_{\widehat{\mathfrak{p}}_0}\text{”} $$

of Example 5.1, (v) [cf. also Example 5.1, (vi)]. [Here, we note that, when $\underline{v}$ lies over an element of $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$, one must replace these "*suitable open subgroups*" by their *tempered analogues*, i.e., by applying the *mono-anabelian algorithm* implicit in the proof of [SemiAnbd], Theorem 6.6.] On the other hand, at an *archimedean* δ-valuation $\underline{v}$, this follows by applying the *functorial algorithm* for reconstructing the corresponding Aut-*holomorphic orbispace at* $\underline{v}$ given in [AbsTopIII], Corollaries 2.8, 2.9, together with the discussion concerning the *"isomorphism* $\mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} {}^{\dagger}\mathbb{M}^{\circledast}$*"* in Example 5.1, (v) [cf. also Example 5.1, (vi)]. Here, we observe that since the natural projection map $\underline{\mathbb{V}}^{\pm\mathrm{un}} \to \mathbb{V}_{\mathrm{mod}}$ *fails to be injective*, in order to relate the restrictions obtained at different elements in a fiber of this map in a well-defined fashion, it is necessary to regard ${}^{\dagger}\mathcal{F}^{\circledcirc}|_{\delta}$ as being well-defined only up to isomorphism. Nevertheless, despite this indeterminacy inherent in the definition of ${}^{\dagger}\mathcal{F}^{\circledcirc}|_{\delta}$, it still makes sense to define, for an $\mathcal{F}$-prime-strip ${}^{\ddagger}\mathfrak{F}$ with underlying $\mathcal{D}$-prime-strip ${}^{\ddagger}\mathfrak{D}$ [cf. Remark 5.2.1, (i)], a *poly-morphism*

$$ {}^{\ddagger}\mathfrak{F} \to {}^{\dagger}\mathcal{F}^{\circledcirc} $$

to be a *full poly-isomorphism* ${}^{\ddagger}\mathfrak{F} \xrightarrow{\sim} {}^{\dagger}\mathcal{F}^{\circledcirc}|_{\delta}$ for some $\delta \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$ [cf. Definition 4.1, (vi)]. Moreover, it makes sense to *pre-compose* such poly-morphisms with isomorphisms of $\mathcal{F}$-prime-strips and to *post-compose* such poly-morphisms with isomorphisms between isomorphs of ${}^{\dagger}\mathcal{F}^{\circledcirc}$. Here, we note that such a poly-morphism ${}^{\ddagger}\mathfrak{F} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$ may be thought of as "lying over" an induced poly-morphism ${}^{\ddagger}\mathfrak{D} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$ [cf. Definition 4.1, (vi)], and that any poly-morphism ${}^{\ddagger}\mathfrak{F} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$ is *fixed* by pre-composition with automorphisms of ${}^{\ddagger}\mathfrak{F}$, as well as by post-composition with automorphisms $\in \mathrm{Aut}_{\underline{\epsilon}}({}^{\dagger}\mathcal{F}^{\circledcirc})$. Also, we observe that such a poly-morphism ${}^{\ddagger}\mathfrak{F} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$ is **compatible** with the local and global ${}_{\infty}\kappa$**-coric structures** [cf. Example 5.1, (v); Definition 5.2, (vi), (viii)] that appear in the *domain* and *codomain* of this poly-morphism in the following sense: the operation of **restriction** of associated **Kummer classes** [cf. the discussion of Example 5.1, (v); Definition 5.2, (vi), (viii); the constructions discussed in the present item (iv)] determines a collection, indexed by $\underline{v} \in \underline{\mathbb{V}}$, of poly-morphisms of **pseudo-monoids**

$$\left\{ \pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa} \quad \to \quad {}^{\ddagger}\mathbb{M}_{\infty\kappa v} \subseteq {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v} \right\}_{\underline{v}\in\underline{\mathbb{V}}}$$

— where the *global* data in the *domain* of the arrow that appears in the display is regarded as only being defined up to automorphisms induced by *inner automorphisms of* $\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})$ [cf. the discussion of Example 5.1, (i)]; the *local* data in the *codomain* of the arrow that appears in the display is regarded as only being defined up to automorphisms induced by *automorphisms of the* $\mathcal{F}$*-prime-strip* ${}^{\ddagger}\mathfrak{F}$ [cf. Definition 5.2, (vi), (viii); Corollary 5.3, (ii)]; the arrow of the display is *equivariant* with respect to the various *homomorphisms* $\pi_1^{\mathrm{rat}}({}^{\ddagger}\mathcal{D}_v) \to \pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})$ [i.e., relative to the respective actions of these groups on the pseudo-monoids in the domain and codomain of the arrow] *induced* [cf. the constructions discussed in the present item (iv), as well as the theory summarized in [AbsTopIII], Theorem 1.9, and [AbsTopIII], Corollaries 1.10, 2.8] by the given poly-morphism ${}^{\ddagger}\mathfrak{F} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$; when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, we regard the pseudo-monoids ${}^{\ddagger}\mathbb{M}_{\infty\kappa v} \subseteq {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v}$ as being equipped with the various *splittings* discussed in Definition 5.2, (viii). Finally, if $\{{}^{e}\mathfrak{F}\}_{e\in E}$ is a *capsule of* $\mathcal{F}$*-prime-strips* whose associated capsule of $\mathcal{D}$-prime-strips [cf. Remark 5.2.1, (i)] we denote by $\{{}^{e}\mathfrak{D}\}_{e\in E}$, then we define a *poly-morphism*

$$\{{}^{e}\mathfrak{F}\}_{e\in E} \to {}^{\dagger}\mathcal{F}^{\circledcirc} \text{ (respectively, } \{{}^{e}\mathfrak{F}\}_{e\in E} \to {}^{\dagger}\mathfrak{F})$$

to be a collection of poly-morphisms $\{{}^{e}\mathfrak{F} \to {}^{\dagger}\mathcal{F}^{\circledcirc}\}_{e\in E}$ (respectively, $\{{}^{e}\mathfrak{F} \to {}^{\dagger}\mathfrak{F}\}_{e\in E}$) [cf. Definition 4.1, (vi)]. Thus, a poly-morphism $\{{}^{e}\mathfrak{F}\}_{e\in E} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$ (respectively, $\{{}^{e}\mathfrak{F}\}_{e\in E} \to {}^{\dagger}\mathfrak{F}$) may be thought of as "lying over" an induced poly-morphism $\{{}^{e}\mathfrak{D}\}_{e\in E} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$ (respectively, $\{{}^{e}\mathfrak{D}\}_{e\in E} \to {}^{\dagger}\mathfrak{D}$) [cf. Definition 4.1, (vi)].

(v) We continue to use the notation of (iv). Now observe that by Corollary 5.3, (ii), there exists a *unique* poly-morphism

$${}^{\dagger}\psi^{\mathrm{NF}}_{\circledast} : {}^{\dagger}\mathfrak{F}_J \to {}^{\dagger}\mathcal{F}^{\circledcirc}$$

that lies over ${}^{\dagger}\phi^{\mathrm{NF}}_{\circledast}$.

(vi) We continue to use the notation of (v). Now observe that it follows from the definition of ${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}$ in terms of *terminal objects* of ${}^{\dagger}\mathcal{D}^{\circledast}$ [cf. Example 5.1, (iii)] that any poly-morphism ${}^{\dagger}\mathfrak{F}_{\langle J\rangle} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$ [cf. the notation of (i)] induces, via "restriction" [in the evident sense], an *isomorphism class of functors* [cf. Definition 5.2, (i); the notation of Example 5.1, (vii)]

$$({}^{\dagger}\mathcal{F}^{\circledcirc} \to {}^{\dagger}\mathcal{F}^{\circledast} \supseteq) \quad {}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} {}^{\dagger}\mathcal{F}^{\circledast}_{\langle J\rangle} \to {}^{\dagger}\mathcal{F}_{\underline{v}_{\langle J\rangle}}$$

for each $\underline{v}_{\langle J \rangle} \in \underline{\mathbb{V}}_{\langle J \rangle}$ — where, by abuse of notation when $\underline{v}_{\langle J \rangle} \in \underline{\mathbb{V}}^{\mathrm{arc}}_{\langle J \rangle}$, we write "${}^{\dagger}\mathcal{F}_{\underline{v}_{\langle J \rangle}}$" for the *category* portion of the "collection of data" that appears in Definition 5.2, (i), (b) — which is *independent* of the choice of the poly-morphism ${}^{\dagger}\mathfrak{F}_{\langle J \rangle} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$ [i.e., among its $\mathbb{F}_l^{\circledast}$-*conjugates*]. That is to say, the fact that ${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}$ is defined in terms of *terminal objects* of ${}^{\dagger}\mathcal{D}^{\circledast}$ [cf. also the definition of F_{mod} given in Definition 3.1, (b)!] implies that this particular isomorphism class of functors is *immune to* [i.e., fixed by] the various *indeterminacies* that appear in the choice of ${}^{\dagger}\mathfrak{F}_{\langle J \rangle} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$. Let us write

$$({}^{\dagger}\mathcal{F}^{\circledcirc} \to {}^{\dagger}\mathcal{F}^{\circledast} \supseteq) \quad {}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} {}^{\dagger}\mathcal{F}^{\circledast}_{\langle J \rangle} \ \to \ {}^{\dagger}\mathfrak{F}_{\langle J \rangle}$$

for the collection of isomorphism classes of restriction functors just defined, as $\underline{v}_{\langle J \rangle}$ ranges over the elements of $\underline{\mathbb{V}}_{\langle J \rangle}$. In a similar vein, we also obtain collections of *natural isomorphism classes of restriction functors*

$${}^{\dagger}\mathcal{F}^{\circledast}_{J} \ \to \ {}^{\dagger}\mathfrak{F}_{J}; \quad {}^{\dagger}\mathcal{F}^{\circledast}_{j} \ \to \ {}^{\dagger}\mathfrak{F}_{j}$$

for $j \in J$. Finally, just as in Example 5.1, (vii), we obtain natural *realifications*

$${}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\langle J \rangle} \ \to \ {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{\langle J \rangle}; \quad {}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{J} \ \to \ {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{J}; \quad {}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{j} \ \to \ {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{j}$$

of the various $\mathcal{F}$-prime-strips — i.e., *realifications* [cf. [FrdI], Corollary 5.4; [FrdII], Theorem 1.2, (i); [FrdII], Theorem 3.6, (i)] of each of the *Frobenioid* [that is to say, *category*] portions of the data of Definition 5.2, (i), (a), (b) — and isomorphism classes of restriction functors discussed so far.

(vii) We shall refer to as *"pivotal distributions"* the objects constructed in (vi)

$${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{pvt}} \ \to \ {}^{\dagger}\mathfrak{F}_{\mathrm{pvt}}; \quad {}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{pvt}} \ \to \ {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{\mathrm{pvt}}$$

in the case $j = 1$ — cf. Fig. 5.2 below.

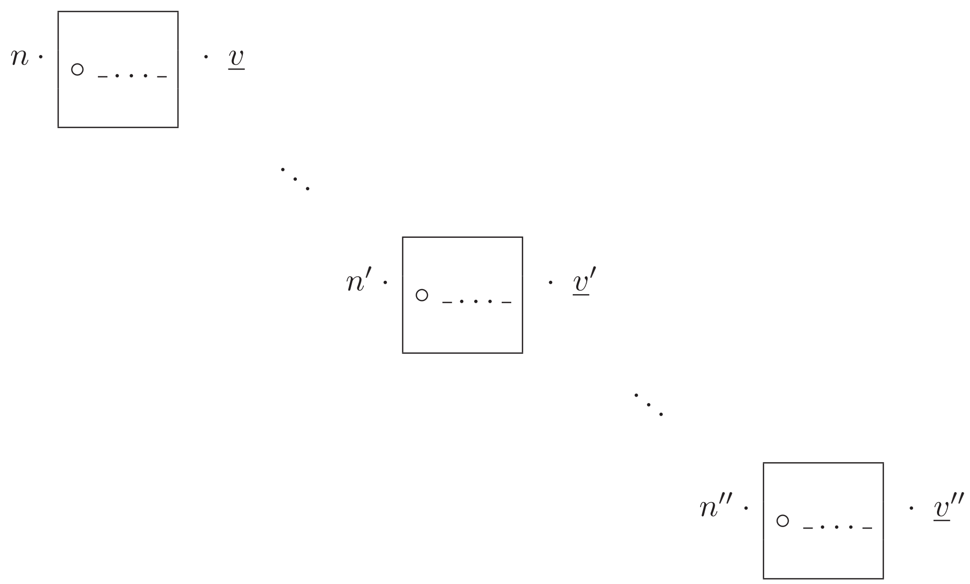

Fig. 5.2: Pivotal distribution

Remark 5.4.1. The constructions of Example 5.4, (i), (ii) (respectively, Example 5.4, (iii), (iv), (v), (vi), (vii)) manifestly only require the *$\mathcal{D}$-Θ-bridge* portion ${}^\dagger\phi_{\divideontimes}^{\Theta}$ (respectively, *$\mathcal{D}$-NF-bridge* portion ${}^\dagger\phi_{\divideontimes}^{\mathrm{NF}}$) of the $\mathcal{D}$-ΘNF-Hodge theater ${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ [cf. Remark 5.1.2].

Remark 5.4.2.

(i) At this point, it is useful to consider the various copies of ${}^\dagger\mathcal{F}_{\mathrm{mod}}^{\circledast}$ and its realifications introduced so far from the point of view of **"log-volumes"**, i.e., **arithmetic degrees** [cf., e.g., the discussion of [FrdI], Example 6.3; [FrdI], Theorem 6.4; Remark 3.1.5 of the present paper]. That is to say, since ${}^\dagger\mathcal{F}_j^{\circledast}$ may be thought of as a sort of *"section of ${}^\dagger\mathcal{F}_J^{\circledast}$ over ${}^\dagger\mathcal{F}_{\mathrm{mod}}^{\circledast}$"* — i.e., a sort of *"section of K over F_{mod}"* [cf. the discussion of *prime-strips* in Remark 4.3.1] — one way to think of log-volumes of ${}^\dagger\mathcal{F}_{\langle J\rangle}^{\circledast}$ is as quantities that *differ by a factor of $l^{\divideontimes}$* — i.e., corresponding, to the cardinality of $J \xrightarrow{\sim} \mathbb{F}_l^{\divideontimes}$ — from log-volumes of ${}^\dagger\mathcal{F}_j^{\circledast}$. Put another way, this amounts to thinking of arithmetic degrees that appear in the various factors of ${}^\dagger\mathcal{F}_J^{\circledast}$ as being

> **averaged** *over the elements of J* and hence of arithmetic degrees that appear in ${}^\dagger\mathcal{F}_{\langle J\rangle}^{\circledast}$ as the **"resulting averages"**.

This sort of averaging may be thought of as a sort of abstract, Frobenioid-theoretic analogue of the *normalization of arithmetic degrees* that is often used in the theory of heights [cf., e.g., [GenEll], Definition 1.2, (i)] that allows one to work with heights in such a way that the height of a point remains *invariant* with respect to change of the base field.

(ii) On the other hand, to work with the various *isomorphisms of Frobenioids* — such as ${}^\dagger\mathcal{F}_j^{\circledast} \xrightarrow{\sim} {}^\dagger\mathcal{F}_{\langle J\rangle}^{\circledast}$ — involved amounts [since the *arithmetic degree* is an *intrinsic invariant* of the Frobenioids involved — cf. [FrdI], Theorem 6.4, (iv); Remark 3.1.5 of the present paper] to thinking of arithmetic degrees that appear in the various factors of ${}^\dagger\mathcal{F}_J^{\circledast}$ as being

> **summed** *[i.e., without dividing by a factor of $l^{\divideontimes}$] over the elements of J* and hence of arithmetic degrees that appear in ${}^\dagger\mathcal{F}_{\langle J\rangle}^{\circledast}$ as the **"resulting sums"**.

This point of view may be thought of as a sort of abstract, Frobenioid-theoretic analogue of the *normalization of arithmetic degrees or heights* in which the height of a point is *multiplied by the degree of the field extension* when one executes a change of the base field.

The notions defined in the following *"Frobenioid-theoretic lifting"* of Definition 4.6 will play a *central role* in the theory of the present series of papers.

Definition 5.5.

(i) We define an *NF-bridge* [relative to the given initial Θ-data] to be a collection of data

$$({}^\ddagger\mathfrak{F}_J \quad \xrightarrow{{}^\ddagger\psi_{\divideontimes}^{\mathrm{NF}}} \quad {}^\ddagger\mathcal{F}^{\circledcirc} \quad \dashrightarrow \quad {}^\ddagger\mathcal{F}^{\circledast})$$

as follows:

(a) ${}^{\ddagger}\mathfrak{F}_J = \{{}^{\ddagger}\mathfrak{F}_j\}_{j \in J}$ is a *capsule of* $\mathcal{F}$*-prime-strips*, indexed by a finite index set J. Write ${}^{\ddagger}\mathfrak{D}_J = \{{}^{\ddagger}\mathfrak{D}_j\}_{j \in J}$ for the associated *capsule of* $\mathcal{D}$*-prime-strips* [cf. Remark 5.2.1, (i)].

(b) ${}^{\ddagger}\mathcal{F}^{\circledcirc}$, ${}^{\ddagger}\mathcal{F}^{\circledast}$ are *categories* equivalent to the categories ${}^{\dagger}\mathcal{F}^{\circledcirc}$, ${}^{\dagger}\mathcal{F}^{\circledast}$, respectively, of Example 5.1, (iii). Thus, each of ${}^{\ddagger}\mathcal{F}^{\circledcirc}$, ${}^{\ddagger}\mathcal{F}^{\circledast}$ is equipped with a *natural Frobenioid structure* [cf. [FrdI], Corollary 4.11; [FrdI], Theorem 6.4, (i); Remark 3.1.5 of the present paper]; write ${}^{\ddagger}\mathcal{D}^{\circledcirc}$, ${}^{\ddagger}\mathcal{D}^{\circledast}$ for the respective *base categories* of these Frobenioids.

(c) The arrow "$\dashrightarrow$" consists of the datum of a *morphism* ${}^{\ddagger}\mathcal{D}^{\circledcirc} \to {}^{\ddagger}\mathcal{D}^{\circledast}$ which is *abstractly equivalent* [cf. §0] to the natural morphism ${}^{\dagger}\mathcal{D}^{\circledcirc} \to {}^{\dagger}\mathcal{D}^{\circledast}$ of Example 5.1, (i), together with the datum of an *isomorphism* ${}^{\ddagger}\mathcal{F}^{\circledcirc} \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}^{\circledast}|_{{}^{\ddagger}\mathcal{D}^{\circledcirc}}$ [cf. the discussion of Example 5.1, (iii)].

(d) ${}^{\ddagger}\psi_{\divideontimes}^{\mathrm{NF}}$ is a *poly-morphism* that lifts [*uniquely*! — cf. Corollary 5.3, (ii)] a poly-morphism ${}^{\ddagger}\phi_{\divideontimes}^{\mathrm{NF}} : {}^{\ddagger}\mathfrak{D}_J \to {}^{\ddagger}\mathcal{D}^{\circledcirc}$ such that ${}^{\ddagger}\phi_{\divideontimes}^{\mathrm{NF}}$ forms a $\mathcal{D}$*-NF-bridge* [cf. Example 5.4, (v); Remark 5.4.1].

Thus, one verifies immediately that any NF-bridge as above determines an *associated* $\mathcal{D}$*-NF-bridge* $({}^{\ddagger}\phi_{\divideontimes}^{\mathrm{NF}} : {}^{\ddagger}\mathfrak{D}_J \to {}^{\ddagger}\mathcal{D}^{\circledcirc})$. We define a(n) *[iso]morphism of NF-bridges*

$$({}^1\mathfrak{F}_{J_1} \xrightarrow{{}^1\psi_{\divideontimes}^{\mathrm{NF}}} {}^1\mathcal{F}^{\circledcirc} \dashrightarrow {}^1\mathcal{F}^{\circledast}) \to ({}^2\mathfrak{F}_{J_2} \xrightarrow{{}^2\psi_{\divideontimes}^{\mathrm{NF}}} {}^2\mathcal{F}^{\circledcirc} \dashrightarrow {}^2\mathcal{F}^{\circledast})$$

to be a collection of arrows

$${}^1\mathfrak{F}_{J_1} \xrightarrow{\sim} {}^2\mathfrak{F}_{J_2}; \quad {}^1\mathcal{F}^{\circledcirc} \xrightarrow{\sim} {}^2\mathcal{F}^{\circledcirc}; \quad {}^1\mathcal{F}^{\circledast} \xrightarrow{\sim} {}^2\mathcal{F}^{\circledast}$$

— where ${}^1\mathfrak{F}_{J_1} \xrightarrow{\sim} {}^2\mathfrak{F}_{J_2}$ is a *capsule-full poly-isomorphism* [cf. §0], hence induces a poly-isomorphism ${}^1\mathfrak{D}_{J_1} \xrightarrow{\sim} {}^2\mathfrak{D}_{J_2}$; ${}^1\mathcal{F}^{\circledcirc} \xrightarrow{\sim} {}^2\mathcal{F}^{\circledcirc}$ is a *poly-isomorphism* which lifts a poly-isomorphism ${}^1\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^2\mathcal{D}^{\circledcirc}$ such that the pair of arrows ${}^1\mathfrak{D}_{J_1} \xrightarrow{\sim} {}^2\mathfrak{D}_{J_2}$, ${}^1\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^2\mathcal{D}^{\circledcirc}$ forms a morphism between the associated $\mathcal{D}$-NF-bridges; ${}^1\mathcal{F}^{\circledast} \xrightarrow{\sim} {}^2\mathcal{F}^{\circledast}$ is an *isomorphism* — which are *compatible* [in the evident sense] with the ${}^i\psi_{\divideontimes}^{\mathrm{NF}}$ [for $i = 1, 2$], as well as with the respective "$\dashrightarrow$'s". It is immediate that any morphism of NF-bridges induces a morphism between the associated $\mathcal{D}$-NF-bridges. There is an evident notion of composition of morphisms of NF-bridges.

(ii) We define a Θ-*bridge* [relative to the given initial Θ-data] to be a collection of data

$$({}^{\ddagger}\mathfrak{F}_J \xrightarrow{{}^{\ddagger}\psi_{\divideontimes}^{\Theta}} {}^{\ddagger}\mathfrak{F}_{>} \dashrightarrow {}^{\ddagger}\mathcal{HT}^{\Theta})$$

as follows:

(a) ${}^{\ddagger}\mathfrak{F}_J = \{{}^{\ddagger}\mathfrak{F}_j\}_{j \in J}$ is a *capsule of* $\mathcal{F}$*-prime-strips*, indexed by a finite index set J. Write ${}^{\ddagger}\mathfrak{D}_J = \{{}^{\ddagger}\mathfrak{D}_j\}_{j \in J}$ for the associated *capsule of* $\mathcal{D}$*-prime-strips* [cf. Remark 5.2.1, (i)].

(b) ${}^{\ddagger}\mathcal{HT}^{\Theta}$ is a Θ-*Hodge theater*.

(c) ${}^{\ddagger}\mathfrak{F}_{>}$ is the *$\mathcal{F}$-prime-strip* tautologically associated to ${}^{\ddagger}\mathcal{HT}^{\Theta}$ [cf. the discussion preceding Example 5.4]; we use the notation "$\dashrightarrow$" to denote this relationship between ${}^{\ddagger}\mathfrak{F}_{>}$ and ${}^{\ddagger}\mathcal{HT}^{\Theta}$. Write ${}^{\ddagger}\mathfrak{D}_{>}$ for the *$\mathcal{D}$-prime-strip* associated to ${}^{\ddagger}\mathfrak{F}_{>}$ [cf. Remark 5.2.1, (i)].

(d) ${}^{\ddagger}\psi^{\Theta}_{\circledast} = \{{}^{\ddagger}\psi^{\Theta}_{j}\}_{j\in J}$ is the *collection of poly-morphisms* ${}^{\ddagger}\psi^{\Theta}_{j} : {}^{\ddagger}\mathfrak{F}_{j} \to {}^{\ddagger}\mathfrak{F}_{>}$ determined [i.e., as discussed in Remark 5.3.1] by a $\mathcal{D}$-Θ-bridge ${}^{\ddagger}\phi^{\Theta}_{\circledast} = \{{}^{\ddagger}\phi^{\Theta}_{j} : {}^{\ddagger}\mathfrak{D}_{j} \to {}^{\ddagger}\mathfrak{D}_{>}\}_{j\in J}$.

Thus, one verifies immediately that any Θ-bridge as above determines an *associated $\mathcal{D}$-Θ-bridge* $({}^{\ddagger}\phi^{\Theta}_{\circledast} : {}^{\ddagger}\mathfrak{D}_{J} \ \to \ {}^{\ddagger}\mathfrak{D}_{>})$. We define a(n) *[iso]morphism of Θ-bridges*

$$({}^{1}\mathfrak{F}_{J_1} \ \stackrel{{}^{1}\psi^{\Theta}_{\circledast}}{\longrightarrow} \ {}^{1}\mathfrak{F}_{>} \ \dashrightarrow \ {}^{1}\mathcal{HT}^{\Theta}) \ \to \ ({}^{2}\mathfrak{F}_{J_2} \ \stackrel{{}^{2}\psi^{\Theta}_{\circledast}}{\longrightarrow} \ {}^{2}\mathfrak{F}_{>} \ \dashrightarrow \ {}^{2}\mathcal{HT}^{\Theta})$$

to be a collection of arrows

$${}^{1}\mathfrak{F}_{J_1} \stackrel{\sim}{\to} {}^{2}\mathfrak{F}_{J_2}; \quad {}^{1}\mathfrak{F}_{>} \stackrel{\sim}{\to} {}^{2}\mathfrak{F}_{>}; \quad {}^{1}\mathcal{HT}^{\Theta} \stackrel{\sim}{\to} {}^{2}\mathcal{HT}^{\Theta}$$

— where ${}^{1}\mathfrak{F}_{J_1} \stackrel{\sim}{\to} {}^{2}\mathfrak{F}_{J_2}$ is a *capsule-full poly-isomorphism* [cf. §0]; ${}^{1}\mathfrak{F}_{>} \stackrel{\sim}{\to} {}^{2}\mathfrak{F}_{>}$ is the *full poly-isomorphism*; ${}^{1}\mathcal{HT}^{\Theta} \stackrel{\sim}{\to} {}^{2}\mathcal{HT}^{\Theta}$ is an *isomorphism of Θ-Hodge theaters* [cf. Remark 3.6.2] — which are *compatible* [in the evident sense] with the ${}^{i}\psi^{\Theta}_{\circledast}$ [for $i = 1, 2$], as well as with the respective "$\dashrightarrow$'s" [cf. Corollary 5.6, (i), below]. It is immediate that any morphism of Θ-bridges induces a morphism between the associated $\mathcal{D}$-Θ-bridges. There is an evident notion of composition of morphisms of Θ-bridges.

(iii) We define a ΘNF-*Hodge theater* [relative to the given initial Θ-data] to be a collection of data

$${}^{\ddagger}\mathcal{HT}^{\Theta\mathrm{NF}} = ({}^{\ddagger}\mathcal{F}^{\circledast} \ \dashleftarrow \ {}^{\ddagger}\mathcal{F}^{\circledcirc} \ \stackrel{{}^{\ddagger}\psi^{\mathrm{NF}}_{\circledast}}{\longleftarrow} \ {}^{\ddagger}\mathfrak{F}_{J} \ \stackrel{{}^{\ddagger}\psi^{\Theta}_{\circledast}}{\longrightarrow} \ {}^{\ddagger}\mathfrak{F}_{>} \ \dashrightarrow \ {}^{\ddagger}\mathcal{HT}^{\Theta})$$

— where the data $({}^{\ddagger}\mathcal{F}^{\circledast} \ \dashleftarrow \ {}^{\ddagger}\mathcal{F}^{\circledcirc} \ \longleftarrow \ {}^{\ddagger}\mathfrak{F}_{J})$ forms an *NF-bridge*; the data $({}^{\ddagger}\mathfrak{F}_{J} \ \longrightarrow \ {}^{\ddagger}\mathfrak{F}_{>} \ \dashrightarrow \ {}^{\ddagger}\mathcal{HT}^{\Theta})$ forms a Θ-*bridge* — such that the *associated data* $\{{}^{\ddagger}\phi^{\mathrm{NF}}_{\circledast}, {}^{\ddagger}\phi^{\Theta}_{\circledast}\}$ [cf. (i), (ii)] forms a $\mathcal{D}$-ΘNF-Hodge theater. A(n) *[iso]morphism of* ΘNF-*Hodge theaters* is defined to be a pair of morphisms between the respective associated NF- and Θ-bridges that are *compatible* with one another in the sense that they induce the *same bijection* between the index sets of the respective capsules of $\mathcal{F}$-prime-strips. There is an evident notion of composition of morphisms of ΘNF-Hodge theaters.

Corollary 5.6. (Isomorphisms of Θ-Hodge Theaters, NF-Bridges, Θ-Bridges, and ΘNF-Hodge Theaters) *Relative to a fixed collection of* **initial Θ-data***:*

(i) The natural functorially induced map from the **set of isomorphisms** *between* **two Θ-Hodge theaters** *to the set of isomorphisms between the respective* **associated $\mathcal{D}$-prime-strips** *[cf. the discussion preceding Example 5.4; Remark 5.2.1, (i)] is* **bijective***.*

(ii) The natural functorially induced map from the **set of isomorphisms** *between* **two NF-bridges** *(respectively,* **two Θ-bridges***;* **two ΘNF-Hodge theaters***) to the set of isomorphisms between the respective* **associated $\mathcal{D}$-NF-bridges**

(respectively, **associated $\mathcal{D}$-Θ-bridges**; **associated $\mathcal{D}$-ΘNF-Hodge theaters***) is* **bijective**.

(iii) Given an NF-bridge and a Θ-bridge, the set of capsule-full poly-isomorphisms between the respective capsules of $\mathcal{F}$-prime-strips which allow one to **glue** *the given NF- and Θ-bridges together to form a ΘNF-Hodge theater forms an* $\mathbb{F}_l^{\circledast}$**-torsor**.

Proof. First, we consider assertion (i). Sorting through the data listed in the definition of a Θ-Hodge theater $^{\dagger}\mathcal{HT}^{\Theta}$ [cf. Definition 3.6], one verifies immediately that the only data that is not contained in the *associated $\mathcal{F}$-prime-strip* $^{\dagger}\mathfrak{F}_{>}$ [cf. the discussion preceding Example 5.4] is the *global data* of Definition 3.6, (c), and the *tempered Frobenioids* isomorphic to "$\underline{\underline{\mathcal{F}}}_{\underline{v}}$" [cf. Example 3.2, (i)] at the $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. That is to say, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, one verifies immediately that

$$^{\dagger}\mathcal{F}_{>,\underline{v}} \quad = \quad {}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$$

[cf. Example 3.3, (i); Example 3.4, (i); Definition 3.6; Definition 5.2, (i)]. On the other hand, one verifies immediately that this *global data* may be obtained by applying the functorial algorithm "$^{\ddagger}\mathfrak{F} \quad \mapsto \quad {}^{\ddagger}\mathfrak{F}^{\Vdash}$" summarized in the second display of Remark 5.2.1, (ii), to the *associated $\mathcal{F}$-prime-strips* that appear. Thus, assertion (i) follows by applying Corollary 5.3, (ii), to the *associated $\mathcal{F}$-prime-strips* and Corollary 5.3, (iv), to the various *tempered Frobenioids* at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. This completes the proof of assertion (i). In light of assertion (i), assertions (ii), (iii) follow immediately from the definitions and Corollary 5.3, (i), (ii) [cf. also Proposition 4.8, (iii), in the case of assertion (iii)]. ○

Remark 5.6.1. Observe that the various **"functorial dynamics"** studied in §4 — i.e., more precisely, analogues of Propositions 4.8, (i), (ii); 4.9; 4.11 — apply to the *NF-bridges*, *Θ-bridges*, and *ΘNF-Hodge theaters* studied in the present §5. Indeed, such analogues follow immediately from Corollaries 5.3, (ii), (iii); 5.6, (ii).

Section 6: Additive Combinatorial Teichmüller Theory

In the present §6, we discuss the **additive** analogue — i.e., which revolves around the **"functorial dynamics"** that arise from **labels**

$$\in \mathbb{F}_l$$

— of the *"multiplicative combinatorial Teichmüller theory"* developed in §4 for labels $\in \mathbb{F}_l^{\circledast}$. These considerations lead naturally to certain enhancements of the various *Hodge theaters* considered in §5. On the other hand, despite the resemblance of the theory of the present §6 to the theory of §4, §5, the theory of the present §6 is, in certain respects — especially those respects that form the analogue of the theory of §5 — substantially *technically simpler*.

In the following, we fix a collection of *initial* Θ*-data*

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$$

as in Definition 3.1; also, we shall use the various notations introduced in Definition 3.1 for various objects associated to this initial Θ-data.

Definition 6.1.

(i) We shall write

$$\mathbb{F}_l^{\rtimes\pm} \stackrel{\mathrm{def}}{=} \mathbb{F}_l \rtimes \{\pm 1\}$$

for the group determined by forming the semi-direct product with respect to the natural inclusion $\{\pm 1\} \hookrightarrow \mathbb{F}_l^{\times}$ and refer to an element of $\mathbb{F}_l^{\rtimes\pm}$ that maps to $+1$ (respectively, -1) via the natural surjection $\mathbb{F}_l^{\rtimes\pm} \twoheadrightarrow \{\pm 1\}$ as *positive* (respectively, *negative*). We shall refer to as an $\mathbb{F}_l^{\pm}$*-group* any set E equipped with a $\{\pm 1\}$-orbit of bijections $E \stackrel{\sim}{\rightarrow} \mathbb{F}_l$. Thus, any $\mathbb{F}_l^{\pm}$-group E is equipped with a natural $\mathbb{F}_l$*-module structure*. We shall refer to as an $\mathbb{F}_l^{\pm}$*-torsor* any set T equipped with an $\mathbb{F}_l^{\rtimes\pm}$-orbit of bijections $T \stackrel{\sim}{\rightarrow} \mathbb{F}_l$ [relative to the action of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathbb{F}_l$ by automorphisms of the form $\mathbb{F}_l \ni z \mapsto \pm z + \lambda \in \mathbb{F}_l$, for $\lambda \in \mathbb{F}_l$]. Thus, if T is an $\mathbb{F}_l^{\pm}$-torsor, then the abelian group of automorphisms of the underlying set of $\mathbb{F}_l$ given by the translations $\mathbb{F}_l \ni z \mapsto z + \lambda \in \mathbb{F}_l$, for $\lambda \in \mathbb{F}_l$, determines an abelian group

$$\mathrm{Aut}_+(T)$$

of *"positive automorphisms"* of the underlying set of T. Moreover, $\mathrm{Aut}_+(T)$ is equipped with a natural structure of $\mathbb{F}_l^{\pm}$-group [such that the abelian group structure of $\mathrm{Aut}_+(T)$ coincides with the $\mathbb{F}_l$-module structure of $\mathrm{Aut}_+(T)$ induced by this $\mathbb{F}_l^{\pm}$-group structure]. Finally, if T is an $\mathbb{F}_l^{\pm}$-torsor, then we shall write

$$\mathrm{Aut}_\pm(T)$$

for the group of automorphisms of the underlying set of T determined [relative to the $\mathbb{F}_l^{\pm}$-torsor structure on T] by the group of automorphisms of the underlying set of $\mathbb{F}_l$ given by $\mathbb{F}_l^{\rtimes\pm}$ [so $\mathrm{Aut}_+(T) \subseteq \mathrm{Aut}_\pm(T)$ is the unique subgroup of index 2].

(ii) Let

$$^{\dagger}\mathfrak{D} = \{^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

be a *$\mathcal{D}$-prime-strip* [relative to the given initial Θ-data]. Observe [cf. the discussion of Definition 4.1, (i)] that if $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, then $\pi_1(^{\dagger}\mathcal{D}_{\underline{v}})$ determines, in a functorial fashion, a *topological* [in fact, profinite if $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$] *group* corresponding to "$\underline{X}_{\underline{v}}$" [cf. Corollary 1.2 if $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$; [EtTh], Proposition 2.4, if $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$], which contains $\pi_1(^{\dagger}\mathcal{D}_{\underline{v}})$ as an open subgroup; thus, if we write $^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$ for $\mathcal{B}(-)^0$ of this topological group, then we obtain a *natural morphism* $^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$ [cf. §0]. In a similar vein, if $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, then since $\underline{X}_{\to\underline{v}}$ admits a $K_{\underline{v}}$-core, a routine translation into the "language of Aut-holomorphic orbispaces" of the argument given in the proof of Corollary 1.2 [cf. also [AbsTopIII], Corollary 2.4] reveals that $^{\dagger}\mathcal{D}_{\underline{v}}$ determines, in a functorial fashion, an Aut-holomorphic orbispace $^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$ corresponding to "$\underline{X}_{\underline{v}}$", together with a *natural morphism* $^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$ *of* Aut-*holomorphic orbispaces.* Thus, in summary, one obtains a collection of data

$$^{\dagger}\underline{\mathfrak{D}}^{\pm} = \{^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

completely determined by $^{\dagger}\mathfrak{D}$.

(iii) Suppose that we are in the situation of (ii). Then observe [cf. the discussion of Definition 4.1, (ii)] that by applying the *group-theoretic algorithm* of [AbsTopI], Lemma 4.5 [cf. also Remark 1.2.2, (ii), of the present paper], to the topological group $\pi_1(^{\dagger}\mathcal{D}_{\underline{v}})$ when $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, or by considering $\pi_0(-)$ of a cofinal collection of "neighborhoods of infinity" [i.e., complements of compact subsets] of the underlying topological space of $^{\dagger}\mathcal{D}_{\underline{v}}$ when $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, it makes sense to speak of the *set of cusps* of $^{\dagger}\mathcal{D}_{\underline{v}}$; a similar observation applies to $^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}$. If $\underline{v} \in \underline{\mathbb{V}}$, then we define a *$\pm$-label class of cusps* of $^{\dagger}\mathcal{D}_{\underline{v}}$ to be the set of cusps of $^{\dagger}\mathcal{D}_{\underline{v}}$ that lie over a single *cusp* [i.e., corresponding to an *arbitrary element* of the quotient "Q" that appears in the definition of a "hyperbolic orbicurve of type $(1, l\text{-tors})$" given in [EtTh], Definition 2.1] of $^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$; write

$$\text{LabCusp}^{\pm}(^{\dagger}\mathcal{D}_{\underline{v}})$$

for the *set of $\pm$-label classes of cusps* of $^{\dagger}\mathcal{D}_{\underline{v}}$. Thus, [for any $\underline{v} \in \underline{\mathbb{V}}$!] $\text{LabCusp}^{\pm}(^{\dagger}\mathcal{D}_{\underline{v}})$ admits a *natural action by* $\mathbb{F}_l^{\times}$ [cf. [EtTh], Definition 2.1], as well as a *zero element*

$$^{\dagger}\underline{\eta}^0_{\underline{v}} \in \text{LabCusp}^{\pm}(^{\dagger}\mathcal{D}_{\underline{v}})$$

and a *$\pm$-canonical element*

$$^{\dagger}\underline{\eta}^{\pm}_{\underline{v}} \in \text{LabCusp}^{\pm}(^{\dagger}\mathcal{D}_{\underline{v}})$$

— which is *well-defined up to multiplication by* ± 1, and which may be *constructed solely from* $^{\dagger}\mathcal{D}_{\underline{v}}$ [cf. Definition 4.1, (ii)] — such that, relative to the *natural bijection*

$$\left\{\text{LabCusp}^{\pm}(^{\dagger}\mathcal{D}_{\underline{v}}) \setminus \{^{\dagger}\underline{\eta}^0_{\underline{v}}\}\right\}/\{\pm 1\} \xrightarrow{\sim} \text{LabCusp}(^{\dagger}\mathcal{D}_{\underline{v}})$$

[cf. the notation of Definition 4.1, (ii)], we have $^{\dagger}\underline{\eta}^{\pm}_{\underline{v}} \mapsto {}^{\dagger}\underline{\eta}_{\underline{v}}$. In particular, we obtain a *natural bijection*

$$\text{LabCusp}^{\pm}(^{\dagger}\mathcal{D}_{\underline{v}}) \quad \xrightarrow{\sim} \quad \mathbb{F}_l$$

— which is *well-defined up to multiplication by* ± 1 and *compatible*, relative to the natural bijection to "LabCusp$(-)$" of the preceding display, with the natural bijection of the second display of Proposition 4.2. That is to say, in the terminology of (i), $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}})$ is equipped with a *natural* $\mathbb{F}_l^{\pm}$*-group structure*. This $\mathbb{F}_l^{\pm}$-group structure determines a *natural surjection*

$$\mathrm{Aut}({}^{\dagger}\mathcal{D}_{\underline{v}}) \twoheadrightarrow \{\pm 1\}$$

— i.e., by considering the induced automorphism of $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}})$. Write

$$\mathrm{Aut}_{+}({}^{\dagger}\mathcal{D}_{\underline{v}}) \subseteq \mathrm{Aut}({}^{\dagger}\mathcal{D}_{\underline{v}})$$

for the index two subgroup of *"positive automorphisms"* [i.e., the kernel of the above surjection] and $\mathrm{Aut}_{-}({}^{\dagger}\mathcal{D}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \mathrm{Aut}({}^{\dagger}\mathcal{D}_{\underline{v}}) \setminus \mathrm{Aut}_{+}({}^{\dagger}\mathcal{D}_{\underline{v}})$ [i.e., where "$\setminus$" denotes the *set-theoretic complement*] for the subset of *"negative automorphisms"*. In a similar vein, we shall write

$$\mathrm{Aut}_{+}({}^{\dagger}\mathfrak{D}) \subseteq \mathrm{Aut}({}^{\dagger}\mathfrak{D})$$

for the subgroup of *"positive automorphisms"* [i.e., automorphisms each of whose components, for $\underline{v} \in \underline{\mathbb{V}}$, is *positive*], and, if $\alpha \in \{\pm 1\}^{\underline{\mathbb{V}}}$ [i.e., where we write $\{\pm 1\}^{\underline{\mathbb{V}}}$ for the set of set-theoretic maps from $\underline{\mathbb{V}}$ to $\{\pm 1\}$],

$$\mathrm{Aut}_{\alpha}({}^{\dagger}\mathfrak{D}) \subseteq \mathrm{Aut}({}^{\dagger}\mathfrak{D})$$

for the subset of *"α-signed automorphisms"* [i.e., automorphisms each of whose components, for $\underline{v} \in \underline{\mathbb{V}}$, is *positive* if $\alpha(\underline{v}) = +1$ and *negative* if $\alpha(\underline{v}) = -1$].

(iv) Suppose that we are in the situation of (ii). Let

$${}^{\ddagger}\mathfrak{D} = \{{}^{\ddagger}\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$$

be another $\mathcal{D}$*-prime-strip* [relative to the given initial Θ-data]. Then for any $\underline{v} \in \underline{\mathbb{V}}$, we shall refer to as a $+$*-full poly-isomorphism* ${}^{\dagger}\mathcal{D}_{\underline{v}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{D}_{\underline{v}}$ any poly-isomorphism obtained as the $\mathrm{Aut}_{+}({}^{\dagger}\mathcal{D}_{\underline{v}})$- [or, equivalently, $\mathrm{Aut}_{+}({}^{\ddagger}\mathcal{D}_{\underline{v}})$-] *orbit* of an isomorphism ${}^{\dagger}\mathcal{D}_{\underline{v}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{D}_{\underline{v}}$. In particular, if ${}^{\dagger}\mathfrak{D} = {}^{\ddagger}\mathfrak{D}$, then there are precisely *two* $+$-full poly-isomorphisms ${}^{\dagger}\mathcal{D}_{\underline{v}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{D}_{\underline{v}}$, namely, the $+$-full poly-isomorphism determined by the identity isomorphism, which we shall refer to as *positive*, and the unique non-positive $+$-full poly-isomorphism, which we shall refer to as *negative*. In a similar vein, we shall refer to as a $+$*-full poly-isomorphism* ${}^{\dagger}\mathfrak{D} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}$ any poly-isomorphism obtained as the $\mathrm{Aut}_{+}({}^{\dagger}\mathfrak{D})$- [or, equivalently, $\mathrm{Aut}_{+}({}^{\ddagger}\mathfrak{D})$-] *orbit* of an isomorphism ${}^{\dagger}\mathfrak{D} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}$. In particular, if ${}^{\dagger}\mathfrak{D} = {}^{\ddagger}\mathfrak{D}$, then the set of $+$-full poly-isomorphisms ${}^{\dagger}\mathfrak{D} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}$ is in *natural bijective correspondence* [cf. the discussion of (iii) above] with the set $\{\pm 1\}^{\underline{\mathbb{V}}}$; we shall refer to the $+$-full poly-isomorphism ${}^{\dagger}\mathfrak{D} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}$ that corresponds to $\alpha \in \{\pm 1\}^{\underline{\mathbb{V}}}$ as the *α-signed $+$-full poly-isomorphism*. Finally, a *capsule-$+$-full poly-morphism* between capsules of $\mathcal{D}$-prime-strips

$$\{{}^{\dagger}\mathfrak{D}_t\}_{t \in T} \xrightarrow{\sim} \{{}^{\ddagger}\mathfrak{D}_{t'}\}_{t' \in T'}$$

is defined to be a poly-morphism between two capsules of $\mathcal{D}$-prime-strips determined by $+$*-full poly-isomorphisms* ${}^{\dagger}\mathfrak{D}_t \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{\iota(t)}$ [where $t \in T$] between the constituent objects indexed by corresponding indices, relative to some *injection* $\iota : T \hookrightarrow T'$.

(v) Write

$$\mathcal{D}^{\circledcirc\pm} \stackrel{\mathrm{def}}{=} \mathcal{B}(\underline{X}_K)^0$$

[cf. §0; the situation discussed in Definition 4.1, (v)]. Thus, we have a *finite étale double covering* $\mathcal{D}^{\circledcirc\pm} \to \mathcal{D}^{\circledcirc} = \mathcal{B}(\underline{C}_K)^0$. Just as in the case of $\mathcal{D}^{\circledcirc}$ [cf. Example 4.3, (i)], one may construct, in a *category-theoretic* fashion from $\mathcal{D}^{\circledcirc\pm}$, the *outer homomorphism*

$$\mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \to GL_2(\mathbb{F}_l)/\{\pm 1\}$$

arising from the *l-torsion points* of the elliptic curve $E_{\overline{F}}$ [i.e., from the Galois action on $\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l$]. Moreover, it follows from the construction of $\underline{X}_K$ that, relative to the *natural isomorphism* $\mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \stackrel{\sim}{\to} \mathrm{Aut}(\underline{X}_K)$ [cf., e.g., [AbsTopIII], Theorem 1.9], the *image* of the above outer homomorphism is equal to a subgroup of $GL_2(\mathbb{F}_l)/\{\pm 1\}$ that contains a *Borel subgroup* of $SL_2(\mathbb{F}_l)/\{\pm 1\}$ [cf. the discussion of Example 4.3, (i)] — i.e., the Borel subgroup corresponding to the *rank one quotient* of $\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l$ that gives rise to the covering $\underline{X}_K \to X_K$. In particular, this rank one quotient determines a *natural surjective homomorphism*

$$\mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \twoheadrightarrow \mathbb{F}_l^{\circledast}$$

[which may be *reconstructed category-theoretically* from $\mathcal{D}^{\circledcirc\pm}$!] — whose kernel we denote by $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm})$. One verifies immediately that the subgroup $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \stackrel{\sim}{\to} \mathrm{Aut}(\underline{X}_K)$ contains the subgroup $\mathrm{Aut}_K(\underline{X}_K) \subseteq \mathrm{Aut}(\underline{X}_K)$ of *K-linear automorphisms* and acts *transitively* on the cusps of $\underline{X}_K$. Next, let us write $\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})$ for the subgroup [which may be *reconstructed category-theoretically* from $\mathcal{D}^{\circledcirc\pm}$! — cf. [AbsTopI], Lemma 4.5, as well as Remark 1.2.2, (ii), of the present paper] of automorphisms that *fix the cusps* of $\underline{X}_K$. Then one obtains *natural outer isomorphisms*

$$\mathrm{Aut}_K(\underline{X}_K) \quad \stackrel{\sim}{\to} \quad \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \quad \stackrel{\sim}{\to} \quad \mathbb{F}_l^{\rtimes\pm}$$

[cf. the discussion preceding [EtTh], Definition 2.1] — where the second outer isomorphism depends, in an essential way, on the choice of the *cusp* $\underline{\epsilon}$ of $\underline{C}_K$ [cf. Definition 3.1, (f)]. Put another way, if we write $\mathrm{Aut}_{+}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})$ for the unique index two subgroup containing $\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm})$, then the cusp $\underline{\epsilon}$ determines a *natural* $\mathbb{F}_l^{\pm}$*-group structure* on the subgroup

$$\mathrm{Aut}_{+}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \quad \subseteq \quad \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm})$$

[which corresponds to the subgroup $\mathrm{Gal}(\underline{X}_K/X_K) \subseteq \mathrm{Aut}_K(\underline{X}_K)$ via the *natural outer isomorphisms* of the preceding display] and, in the notation of (vi) below, a *natural* $\mathbb{F}_l^{\pm}$*-torsor structure* on the set $\mathrm{LabCusp}^{\pm}(\mathcal{D}^{\circledcirc\pm})$. Write

$$\underline{\mathbb{V}}^{\pm} \quad \stackrel{\mathrm{def}}{=} \quad \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \cdot \underline{\mathbb{V}} \quad = \quad \mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \cdot \underline{\mathbb{V}} \quad \subseteq \quad \mathbb{V}(K)$$

[cf. the discussion of Example 4.3, (i); Remark 6.1.1 below] — where the "=" follows immediately from the *natural outer isomorphisms* discussed above. Then [by considering what happens at the elements of $\underline{\mathbb{V}}^{\pm} \bigcap \underline{\mathbb{V}}^{\mathrm{bad}}$] one verifies immediately that the subgroup $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \cong \mathrm{Aut}(\underline{X}_K)$ may be identified with the subgroup of $\mathrm{Aut}(\underline{X}_K)$ that *stabilizes* $\underline{\mathbb{V}}^{\pm}$.

(vi) Let

$$^{\dagger}\mathcal{D}^{\circledcirc\pm}$$

be any category isomorphic to $\mathcal{D}^{\circledcirc\pm}$. Then just as in the discussion of (iii) in the case of "$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$", it makes sense [cf. [AbsTopI], Lemma 4.5, as well as Remark 1.2.2, (ii), of the present paper] to speak of the *set of cusps* of $^{\dagger}\mathcal{D}^{\circledcirc\pm}$, as well as the *set of* $\pm$*-label classes of cusps*

$$\mathrm{LabCusp}^{\pm}(^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

— which, in this case, may be identified with the set of cusps of $^{\dagger}\mathcal{D}^{\circledcirc\pm}$.

(vii) Recall from [AbsTopIII], Theorem 1.9 [applied via the *"Θ-approach"* discussed in Remark 3.1.2], that [just as in the case of $\mathcal{D}^{\circledcirc}$ — cf. the discussion of Definition 4.1, (v)] there exists a *group-theoretic algorithm* for reconstructing, from $\pi_1(\mathcal{D}^{\circledcirc\pm})$ [cf. §0], the algebraic closure "$\overline{F}$" of the base field "K", hence also the *set of valuations* "$\mathbb{V}(\overline{F})$" from $\mathcal{D}^{\circledcirc\pm}$ [e.g., as a collection of topologies on $\overline{F}$ — cf., e.g., [AbsTopIII], Corollary 2.8]. Moreover, for $\underline{w} \in \mathbb{V}(K)^{\mathrm{arc}}$, let us recall [cf. Remark 3.1.2; [AbsTopIII], Corollaries 2.8, 2.9] that one may *reconstruct group-theoretically*, from $\pi_1(\mathcal{D}^{\circledcirc\pm})$, the Aut-*holomorphic orbispace* $\underline{\mathbb{X}}_{\underline{w}}$ associated to $\underline{X}_{\underline{w}}$. Let $^{\dagger}\mathcal{D}^{\circledcirc\pm}$ be as in (vi). Then let us write

$$\overline{\mathbb{V}}(^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

for the set of valuations [i.e., "$\mathbb{V}(\overline{F})$"], equipped with its natural $\pi_1(^{\dagger}\mathcal{D}^{\circledcirc\pm})$-action,

$$\mathbb{V}(^{\dagger}\mathcal{D}^{\circledcirc\pm}) \quad \stackrel{\mathrm{def}}{=} \quad \overline{\mathbb{V}}(^{\dagger}\mathcal{D}^{\circledcirc\pm})/\pi_1(^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

for the quotient of $\overline{\mathbb{V}}(^{\dagger}\mathcal{D}^{\circledcirc\pm})$ by $\pi_1(^{\dagger}\mathcal{D}^{\circledcirc\pm})$ [i.e., "$\mathbb{V}(K)$"], and, for $\underline{w} \in \mathbb{V}(^{\dagger}\mathcal{D}^{\circledcirc\pm})^{\mathrm{arc}}$,

$$\underline{\mathbb{X}}(^{\dagger}\mathcal{D}^{\circledcirc\pm}, \underline{w})$$

[i.e., "$\underline{\mathbb{X}}_{\underline{w}}$" — cf. the discussion of [AbsTopIII], Definition 5.1, (ii)] for the Aut-holomorphic orbispace obtained by applying these group-theoretic reconstruction algorithms to $\pi_1(^{\dagger}\mathcal{D}^{\circledcirc\pm})$. Now if $\mathbb{U}$ is an arbitrary Aut-*holomorphic orbispace*, then let us define a *morphism*

$$\mathbb{U} \to {}^{\dagger}\mathcal{D}^{\circledcirc\pm}$$

to be a morphism of Aut-holomorphic orbispaces [cf. [AbsTopIII], Definition 2.1, (ii)] $\mathbb{U} \to \underline{\mathbb{X}}(^{\dagger}\mathcal{D}^{\circledcirc\pm}, \underline{w})$ for some $\underline{w} \in \mathbb{V}(^{\dagger}\mathcal{D}^{\circledcirc\pm})^{\mathrm{arc}}$. Thus, it makes sense to speak of the pre-composite (respectively, post-composite) of such a morphism $\mathbb{U} \to {}^{\dagger}\mathcal{D}^{\circledcirc\pm}$ with a morphism of Aut-holomorphic orbispaces (respectively, with an isomorphism [cf. §0] $^{\dagger}\mathcal{D}^{\circledcirc\pm} \xrightarrow{\sim} {}^{\ddagger}\mathcal{D}^{\circledcirc\pm}$ [i.e., where $^{\ddagger}\mathcal{D}^{\circledcirc\pm}$ is a category equivalent to $\mathcal{D}^{\circledcirc\pm}$]).

Remark 6.1.1. In fact, in the notation of Example 4.3, (i); Definition 6.1, (v), it is not difficult to verify [cf. Remark 3.1.2, (i)] that $\underline{\mathbb{V}}^{\pm} = \underline{\mathbb{V}}^{\pm\mathrm{un}}$ ($\subseteq \mathbb{V}(K)$).

Example 6.2. Model Base-$\Theta^{\pm}$-Bridges.

(i) In the following, let us think of $\mathbb{F}_l$ as an $\mathbb{F}_l^{\pm}$*-group* [relative to the tautological $\mathbb{F}_l^{\pm}$-group structure]. Let

$$\mathfrak{D}_{\succ} = \{\mathcal{D}_{\succ,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}; \quad \mathfrak{D}_t = \{\mathcal{D}_{\underline{v}_t}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

— where $t \in \mathbb{F}_l$, and we use the notation $\underline{v}_t$ to denote the pair $(t, \underline{v})$ [cf. Example 4.3, (iv)] — be copies of the "tautological $\mathcal{D}$-prime-strip" $\{\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ [cf. Examples 4.3, (iv); 4.4, (ii)]. For each $t \in \mathbb{F}_l$, write

$$\phi^{\Theta^{\pm}}_{\underline{v}_t} : \mathcal{D}_{\underline{v}_t} \to \mathcal{D}_{\succ, \underline{v}}; \quad \phi^{\Theta^{\pm}}_t : \mathfrak{D}_t \to \mathfrak{D}_{\succ}$$

for the respective *positive* $+$*-full poly-isomorphisms*, i.e., relative to the respective identifications with the "tautological $\mathcal{D}$-prime-strip" $\{\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$. Write $\mathfrak{D}_{\pm}$ for the *capsule* $\{\mathfrak{D}_t\}_{t \in \mathbb{F}_l}$ [cf. the constructions of Example 4.4, (iv)] and

$$\phi^{\Theta^{\pm}}_{\pm} : \mathfrak{D}_{\pm} \to \mathfrak{D}_{\succ}$$

for the *collection of poly-morphisms* $\{\phi^{\Theta^{\pm}}_t\}_{t \in \mathbb{F}_l}$.

(ii) The collection of data

$$(\mathfrak{D}_{\pm}, \mathfrak{D}_{\succ}, \phi^{\Theta^{\pm}}_{\pm})$$

admits a *natural poly-automorphism of order two* $-1_{\mathbb{F}_l}$ defined as follows: the poly-automorphism $-1_{\mathbb{F}_l}$ acts on $\mathbb{F}_l$ as *multiplication by* -1 and induces the poly-isomorphisms $\mathfrak{D}_t \xrightarrow{\sim} \mathfrak{D}_{-t}$ [for $t \in \mathbb{F}_l$] and $\mathfrak{D}_{\succ} \xrightarrow{\sim} \mathfrak{D}_{\succ}$ determined [i.e., relative to the respective identifications with the "tautological $\mathcal{D}$-prime-strip" $\{\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$] by the $+$*-full poly-automorphism* whose *sign* at every $\underline{v} \in \underline{\mathbb{V}}$ is *negative*. One verifies immediately that $-1_{\mathbb{F}_l}$, defined in this way, is *compatible* [in the evident sense] with $\phi^{\Theta^{\pm}}_{\pm}$.

(iii) Let $\alpha \in \{\pm 1\}^{\underline{\mathbb{V}}}$. Then α determines a *natural poly-automorphism* $\alpha^{\Theta^{\pm}}$ *of order* $\in \{1, 2\}$ of the collection of data

$$(\mathfrak{D}_{\pm}, \mathfrak{D}_{\succ}, \phi^{\Theta^{\pm}}_{\pm})$$

as follows: the poly-automorphism $\alpha^{\Theta^{\pm}}$ acts on $\mathbb{F}_l$ as the *identity* and on $\mathfrak{D}_t$, for $t \in \mathbb{F}_l$, and $\mathfrak{D}_{\succ}$ as the α*-signed* $+$*-full poly-automorphism*. One verifies immediately that $\alpha^{\Theta^{\pm}}$, defined in this way, is *compatible* [in the evident sense] with $\phi^{\Theta^{\pm}}_{\pm}$.

Example 6.3. Model Base-Θ^{ell}-Bridges.

(i) In the following, let us think of $\mathbb{F}_l$ as an $\mathbb{F}_l^{\pm}$*-torsor* [relative to the tautological $\mathbb{F}_l^{\pm}$-torsor structure]. Let

$$\mathfrak{D}_t = \{\mathcal{D}_{\underline{v}_t}\}_{\underline{v} \in \underline{\mathbb{V}}}$$

[for $t \in \mathbb{F}_l$] and $\mathfrak{D}_{\pm}$ be as in Example 6.2, (i); $\mathcal{D}^{\circledcirc \pm}$ as in Definition 6.1, (v). In the following, let us *fix* an *isomorphism of* $\mathbb{F}_l^{\pm}$*-torsors*

$$\mathrm{LabCusp}^{\pm}(\mathcal{D}^{\circledcirc \pm}) \xrightarrow{\sim} \mathbb{F}_l$$

[cf. the discussion of Definition 6.1, (v)], which we shall use to *identify* $\mathrm{LabCusp}^{\pm}(\mathcal{D}^{\circledcirc \pm})$ with $\mathbb{F}_l$. Note that this identification induces an *isomorphism of groups*

$$\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc \pm}) / \mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc \pm}) \xrightarrow{\sim} \mathbb{F}_l^{\rtimes \pm}$$

[cf. the discussion of Definition 6.1, (v)], which we shall use to *identify* the group $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm})$ with the group $\mathbb{F}_l^{\rtimes\pm}$. If $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ (respectively, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$), then the natural restriction functor on finite étale coverings arising from the natural composite morphism $\underline{X}_{\underset{\to}{v}} \to \underline{X}_{\underline{v}} \to \underline{X}_K$ (respectively, $\underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}} \to \underline{X}_K$) determines [cf. Examples 3.2, (i); 3.3, (i)] a *natural morphism* $\phi^{\Theta^{\mathrm{ell}}}_{\bullet,\underline{v}} : \mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc\pm}$ [cf. the discussion of Example 4.3, (ii)]. If $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then [cf. Example 3.4, (i)] we have a *tautological morphism* $\mathcal{D}_{\underline{v}} = \underline{\mathbb{X}}_{\underset{\to}{v}} \to \underline{\mathbb{X}}_{\underline{v}} \xrightarrow{\sim} \underline{\mathbb{X}}(\mathcal{D}^{\circledcirc\pm}, \underline{v})$, hence a morphism $\phi^{\Theta^{\mathrm{ell}}}_{\bullet,\underline{v}} : \mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc\pm}$ [cf. the discussion of Example 4.3, (iii)]. For arbitrary $\underline{v} \in \underline{\mathbb{V}}$, write

$$\phi^{\Theta^{\mathrm{ell}}}_{\underline{v}_0} : \mathcal{D}_{\underline{v}_0} \to \mathcal{D}^{\circledcirc\pm}$$

for the *poly-morphism* given by the collection of morphisms $\mathcal{D}_{\underline{v}_0} \to \mathcal{D}^{\circledcirc\pm}$ of the form

$$\beta \circ \phi^{\Theta^{\mathrm{ell}}}_{\bullet,\underline{v}} \circ \alpha$$

— where $\alpha \in \mathrm{Aut}_{+}(\mathcal{D}_{\underline{v}_0})$; $\beta \in \mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm})$; we apply the tautological identification of $\mathcal{D}_{\underline{v}}$ with $\mathcal{D}_{\underline{v}_0}$ [cf. the discussion of Example 4.3, (ii), (iii), (iv)]. Write

$$\phi^{\Theta^{\mathrm{ell}}}_{0} : \mathfrak{D}_0 \to \mathcal{D}^{\circledcirc\pm}$$

for the *poly-morphism* determined by the collection $\{\phi^{\Theta^{\mathrm{ell}}}_{\underline{v}_0} : \mathcal{D}_{\underline{v}_0} \to \mathcal{D}^{\circledcirc\pm}\}_{\underline{v}\in\underline{\mathbb{V}}}$ [cf. the discussion of Example 4.3, (iv)]. Note that the presence of "β" in the definition of $\phi^{\Theta^{\mathrm{ell}}}_{\underline{v}_0}$ implies that it makes sense to *post-compose* $\phi^{\Theta^{\mathrm{ell}}}_{0}$ with an element of $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm}$. Thus, for any $t \in \mathbb{F}_l \subseteq \mathbb{F}_l^{\rtimes\pm}$, let us write

$$\phi^{\Theta^{\mathrm{ell}}}_{t} : \mathfrak{D}_t \to \mathcal{D}^{\circledcirc\pm}$$

for the result of *post-composing* $\phi^{\Theta^{\mathrm{ell}}}_{0}$ with the *"poly-action"* [i.e., action via poly-automorphisms] of t on $\mathcal{D}^{\circledcirc\pm}$ [and *pre-composing* with the tautological identification of $\mathfrak{D}_0$ with $\mathfrak{D}_t$] and

$$\phi^{\Theta^{\mathrm{ell}}}_{\pm} : \mathfrak{D}_{\pm} \to \mathcal{D}^{\circledcirc\pm}$$

for the collection of arrows $\{\phi^{\Theta^{\mathrm{ell}}}_{t}\}_{t\in\mathbb{F}_l}$.

(ii) Let $\gamma \in \mathbb{F}_l^{\rtimes\pm}$. Then γ determines a *natural poly-automorphism* $\gamma_{\pm}$ of $\mathfrak{D}_{\pm}$ as follows: the automorphism $\gamma_{\pm}$ acts on $\mathbb{F}_l$ via the usual action of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathbb{F}_l$ and, for $t \in \mathbb{F}_l$, induces the $+$-*full poly-isomorphism* $\mathfrak{D}_t \xrightarrow{\sim} \mathfrak{D}_{\gamma(t)}$ whose *sign* at every $\underline{v} \in \underline{\mathbb{V}}$ is equal to the sign of γ [cf. the construction of Example 6.2, (ii)]. Thus, we obtain a natural *poly-action* of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathfrak{D}_{\pm}$. On the other hand, the isomorphism $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm}$ of (i) determines a natural *poly-action* of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathcal{D}^{\circledcirc\pm}$. Moreover, one verifies immediately that $\phi^{\Theta^{\mathrm{ell}}}_{\pm}$ is *equivariant* with respect to these poly-actions of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathfrak{D}_{\pm}$ and $\mathcal{D}^{\circledcirc\pm}$; in particular, we obtain a *natural poly-action*

$$\mathbb{F}_l^{\rtimes\pm} \quad \curvearrowright \quad (\mathfrak{D}_{\pm}, \mathcal{D}^{\circledcirc\pm}, \phi^{\Theta^{\mathrm{ell}}}_{\pm})$$

of $\mathbb{F}_l^{\rtimes\pm}$ on the collection of data $(\mathfrak{D}_{\pm}, \mathcal{D}^{\circledcirc\pm}, \phi^{\Theta^{\mathrm{ell}}}_{\pm})$ [cf. the discussion of Example 4.3, (iv)].

Definition 6.4. In the following, we shall write $l^{\pm} \stackrel{\mathrm{def}}{=} l^{*}+1 = (l+1)/2$. [Here, we recall that the notation "l^{*}" was introduced at the beginning of §4.]

(i) We define a *base-*$\Theta^{\pm}$*-bridge*, or $\mathcal{D}$-$\Theta^{\pm}$*-bridge*, [relative to the given initial Θ-data] to be a poly-morphism

$$^{\dagger}\mathfrak{D}_T \quad \stackrel{^{\dagger}\phi_{\pm}^{\Theta^{\pm}}}{\longrightarrow} \quad {^{\dagger}\mathfrak{D}_{\succ}}$$

— where $^{\dagger}\mathfrak{D}_{\succ}$ is a $\mathcal{D}$*-prime-strip*; T is an $\mathbb{F}_l^{\pm}$*-group*; $^{\dagger}\mathfrak{D}_T = \{^{\dagger}\mathfrak{D}_t\}_{t\in T}$ is a *capsule of* $\mathcal{D}$*-prime-strips*, indexed by [the underlying set of] T — such that there exist isomorphisms

$$\mathfrak{D}_{\succ} \stackrel{\sim}{\rightarrow} {^{\dagger}\mathfrak{D}_{\succ}}, \quad \mathfrak{D}_{\pm} \stackrel{\sim}{\rightarrow} {^{\dagger}\mathfrak{D}_T}$$

— where we require that the bijection of index sets $\mathbb{F}_l \stackrel{\sim}{\rightarrow} T$ induced by the second isomorphism determine an *isomorphism of* $\mathbb{F}_l^{\pm}$*-groups* — conjugation by which maps $\phi_{\pm}^{\Theta^{\pm}} \mapsto {^{\dagger}\phi_{\pm}^{\Theta^{\pm}}}$. In this situation, we shall write

$$^{\dagger}\mathfrak{D}_{|T|}$$

for the $l^{\pm}$-capsule obtained from the l-capsule $^{\dagger}\mathfrak{D}_T$ by forming the quotient $|T|$ of the index set T of this underlying capsule by the action of $\{\pm 1\}$ and identifying the components of the capsule $^{\dagger}\mathfrak{D}_T$ indexed by the elements in the fibers of the quotient $T \twoheadrightarrow |T|$ via the constituent poly-morphisms of $^{\dagger}\phi_{\pm}^{\Theta^{\pm}} = \{^{\dagger}\phi_t^{\Theta^{\pm}}\}_{t\in T}$ [so each constituent $\mathcal{D}$-prime-strip of $^{\dagger}\mathfrak{D}_{|T|}$ is only well-defined up to a *positive automorphism*, but this *indeterminacy* will not affect applications of this construction — cf. Propositions 6.7; 6.8, (ii); 6.9, (i), below]. Also, we shall write

$$^{\dagger}\mathfrak{D}_{T^{*}}$$

for the l^{*}-capsule determined by the subset $T^{*} \stackrel{\mathrm{def}}{=} |T| \setminus \{0\}$ of nonzero elements of $|T|$. We define a(n) *[iso]morphism of* $\mathcal{D}$-$\Theta^{\pm}$*-bridges*

$$(^{\dagger}\mathfrak{D}_T \quad \stackrel{^{\dagger}\phi_{\pm}^{\Theta^{\pm}}}{\longrightarrow} \quad {^{\dagger}\mathfrak{D}_{\succ}}) \quad \rightarrow \quad (^{\ddagger}\mathfrak{D}_{T'} \quad \stackrel{^{\ddagger}\phi_{\pm}^{\Theta^{\pm}}}{\longrightarrow} \quad {^{\ddagger}\mathfrak{D}_{\succ}})$$

to be a pair of poly-morphisms

$$^{\dagger}\mathfrak{D}_T \stackrel{\sim}{\rightarrow} {^{\ddagger}\mathfrak{D}_{T'}}; \quad {^{\dagger}\mathfrak{D}_{\succ}} \stackrel{\sim}{\rightarrow} {^{\ddagger}\mathfrak{D}_{\succ}}$$

— where $^{\dagger}\mathfrak{D}_T \stackrel{\sim}{\rightarrow} {^{\ddagger}\mathfrak{D}_{T'}}$ is a *capsule-+-full poly-isomorphism* whose induced morphism on index sets $T \stackrel{\sim}{\rightarrow} T'$ is an *isomorphism of* $\mathbb{F}_l^{\pm}$*-groups*; $^{\dagger}\mathfrak{D}_{\succ} \stackrel{\sim}{\rightarrow} {^{\ddagger}\mathfrak{D}_{\succ}}$ is a *+-full poly-isomorphism* — which are *compatible* with $^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$, $^{\ddagger}\phi_{\pm}^{\Theta^{\pm}}$. There is an evident notion of composition of morphisms of $\mathcal{D}$-$\Theta^{\pm}$-bridges.

(ii) We define a *base-*Θ^{ell}*-bridge* [i.e., a *"base-*Θ*-elliptic-bridge"*], or $\mathcal{D}$-Θ^{ell}*-bridge*, [relative to the given initial Θ-data] to be a poly-morphism

$$^{\dagger}\mathfrak{D}_T \quad \stackrel{^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}}{\longrightarrow} \quad {^{\dagger}\mathcal{D}^{\circledcirc\pm}}$$

— where $^{\dagger}\mathcal{D}^{\circledcirc\pm}$ is a *category equivalent to* $\mathcal{D}^{\circledcirc\pm}$; T is an $\mathbb{F}_l^{\pm}$*-torsor*; $^{\dagger}\mathfrak{D}_T = \{^{\dagger}\mathfrak{D}_t\}_{t\in T}$ is a *capsule of* $\mathcal{D}$*-prime-strips*, indexed by [the underlying set of] T — such that there exist isomorphisms

$$\mathcal{D}^{\circledcirc\pm} \stackrel{\sim}{\rightarrow} {^{\dagger}\mathcal{D}^{\circledcirc\pm}}, \quad \mathfrak{D}_{\pm} \stackrel{\sim}{\rightarrow} {^{\dagger}\mathfrak{D}_T}$$

— where we require that the bijection of index sets $\mathbb{F}_l \xrightarrow{\sim} T$ induced by the second isomorphism determine an *isomorphism of* $\mathbb{F}_l^{\pm}$*-torsors* — conjugation by which maps $\phi_{\pm}^{\Theta^{\rm ell}} \mapsto {}^\dagger\phi_{\pm}^{\Theta^{\rm ell}}$. We define a(n) *[iso]morphism of* $\mathcal{D}$-$\Theta^{\rm ell}$*-bridges*

$$({}^\dagger\mathfrak{D}_T \quad \xrightarrow{{}^\dagger\phi_{\pm}^{\Theta^{\rm ell}}} \quad {}^\dagger\mathcal{D}^{\circledcirc\pm}) \quad \to \quad ({}^\ddagger\mathfrak{D}_{T'} \quad \xrightarrow{{}^\ddagger\phi_{\pm}^{\Theta^{\rm ell}}} \quad {}^\ddagger\mathcal{D}^{\circledcirc\pm})$$

to be a pair of poly-morphisms

$${}^\dagger\mathfrak{D}_T \xrightarrow{\sim} {}^\ddagger\mathfrak{D}_{T'}; \quad {}^\dagger\mathcal{D}^{\circledcirc\pm} \xrightarrow{\sim} {}^\ddagger\mathcal{D}^{\circledcirc\pm}$$

— where ${}^\dagger\mathfrak{D}_T \xrightarrow{\sim} {}^\ddagger\mathfrak{D}_{T'}$ is a *capsule-+-full poly-isomorphism* whose induced morphism on index sets $T \xrightarrow{\sim} T'$ is an *isomorphism of* $\mathbb{F}_l^{\pm}$*-torsors*; ${}^\dagger\mathcal{D}^{\circledcirc\pm} \to {}^\ddagger\mathcal{D}^{\circledcirc\pm}$ is a poly-morphism which is an $\mathrm{Aut}_{\rm csp}({}^\dagger\mathcal{D}^{\circledcirc\pm})$- [or, equivalently, $\mathrm{Aut}_{\rm csp}({}^\ddagger\mathcal{D}^{\circledcirc\pm})$-] *orbit* of isomorphisms — which are *compatible* with ${}^\dagger\phi_{\pm}^{\Theta^{\rm ell}}$, ${}^\ddagger\phi_{\pm}^{\Theta^{\rm ell}}$. There is an evident notion of composition of morphisms of $\mathcal{D}$-$\Theta^{\rm ell}$-bridges.

(iii) We define a *base-*$\Theta^{\pm{\rm ell}}$*-Hodge theater*, or $\mathcal{D}$-$\Theta^{\pm{\rm ell}}$*-Hodge theater*, [relative to the given initial Θ-data] to be a collection of data

$${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm{\rm ell}}} = ({}^\dagger\mathfrak{D}_{\succ} \quad \xleftarrow{{}^\dagger\phi_{\pm}^{\Theta^{\pm}}} \quad {}^\dagger\mathfrak{D}_T \quad \xrightarrow{{}^\dagger\phi_{\pm}^{\Theta^{\rm ell}}} \quad {}^\dagger\mathcal{D}^{\circledcirc\pm})$$

— where T is an $\mathbb{F}_l^{\pm}$*-group*; ${}^\dagger\phi_{\pm}^{\Theta^{\pm}}$ is a $\mathcal{D}$-$\Theta^{\pm}$-bridge; ${}^\dagger\phi_{\pm}^{\Theta^{\rm ell}}$ is a $\mathcal{D}$-$\Theta^{\rm ell}$-bridge [relative to the $\mathbb{F}_l^{\pm}$-torsor structure determined by the $\mathbb{F}_l^{\pm}$-group structure on T] — such that there exist isomorphisms

$$\mathfrak{D}_{\succ} \xrightarrow{\sim} {}^\dagger\mathfrak{D}_{\succ}; \quad \mathfrak{D}_{\pm} \xrightarrow{\sim} {}^\dagger\mathfrak{D}_T; \quad \mathcal{D}^{\circledcirc\pm} \xrightarrow{\sim} {}^\dagger\mathcal{D}^{\circledcirc\pm}$$

conjugation by which maps $\phi_{\pm}^{\Theta^{\pm}} \mapsto {}^\dagger\phi_{\pm}^{\Theta^{\pm}}$, $\phi_{\pm}^{\Theta^{\rm ell}} \mapsto {}^\dagger\phi_{\pm}^{\Theta^{\rm ell}}$. A(n) *[iso]morphism of* $\mathcal{D}$-$\Theta^{\pm{\rm ell}}$*-Hodge theaters* is defined to be a pair of morphisms between the respective associated $\mathcal{D}$-$\Theta^{\pm}$- and $\mathcal{D}$-$\Theta^{\rm ell}$-bridges that are *compatible* with one another in the sense that they induce the *same poly-isomorphism* between the respective capsules of $\mathcal{D}$-prime-strips. There is an evident notion of composition of morphisms of $\mathcal{D}$-$\Theta^{\pm{\rm ell}}$-Hodge theaters.

The following *additive* analogue of Proposition 4.7 follows immediately from the various definitions involved. Put another way, the content of Proposition 6.5 below may be thought of as a sort of *"intrinsic version"* of the constructions carried out in Examples 6.2, 6.3.

Proposition 6.5. **(Transport of ±-Label Classes of Cusps via Base-Bridges)** *Let*

$${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm{\rm ell}}} = ({}^\dagger\mathfrak{D}_{\succ} \quad \xleftarrow{{}^\dagger\phi_{\pm}^{\Theta^{\pm}}} \quad {}^\dagger\mathfrak{D}_T \quad \xrightarrow{{}^\dagger\phi_{\pm}^{\Theta^{\rm ell}}} \quad {}^\dagger\mathcal{D}^{\circledcirc\pm})$$

be a $\mathcal{D}$-$\Theta^{\pm{\rm ell}}$**-Hodge theater** *[relative to the given initial* Θ*-data]. Then:*

(i) For each $\underline{v} \in \underline{\mathbb{V}}$, $t \in T$, *the* $\mathcal{D}$-$\Theta^{\rm ell}$**-bridge** ${}^\dagger\phi_{\pm}^{\Theta^{\rm ell}}$ *induces a [single, well-defined!]* **bijection** *of sets of ±-label classes of cusps*

$${}^\dagger\zeta_{\underline{v}_t}^{\Theta^{\rm ell}} : \mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}_{\underline{v}_t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^\dagger\mathcal{D}^{\circledcirc\pm})$$

that is **compatible** *with the respective* $\mathbb{F}_l^{\pm}$**-torsor** *structures. Moreover, for* $\underline{w} \in \underline{\mathbb{V}}$, *the bijection*

$$^\dagger\xi^{\Theta^{\mathrm{ell}}}_{\underline{v}_t,\underline{w}_t} \stackrel{\mathrm{def}}{=} (^\dagger\zeta^{\Theta^{\mathrm{ell}}}_{\underline{w}_t})^{-1} \circ (^\dagger\zeta^{\Theta^{\mathrm{ell}}}_{\underline{v}_t}) : \mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}_{\underline{v}_t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}_{\underline{w}_t})$$

is **compatible** *with the respective* $\mathbb{F}_l^{\pm}$**-group** *structures. Write*

$$\mathrm{LabCusp}^{\pm}(^\dagger\mathfrak{D}_t)$$

for the $\mathbb{F}_l^{\pm}$*-group obtained by identifying the various* $\mathbb{F}_l^{\pm}$*-groups* $\mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}_{\underline{v}_t})$, *as* $\underline{v}$ *ranges over the elements of* $\underline{\mathbb{V}}$, *via the various* $^\dagger\xi^{\Theta^{\mathrm{ell}}}_{\underline{v}_t,\underline{w}_t}$. *Finally, the various* $^\dagger\zeta^{\Theta^{\mathrm{ell}}}_{\underline{v}_t}$ *determine a [single, well-defined!]* **bijection**

$$^\dagger\zeta^{\Theta^{\mathrm{ell}}}_t : \mathrm{LabCusp}^{\pm}(^\dagger\mathfrak{D}_t) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}^{\circledcirc\pm})$$

— which is compatible with the respective $\mathbb{F}_l^{\pm}$*-torsor structures.*

(ii) For each $\underline{v} \in \underline{\mathbb{V}}$, $t \in T$, *the* $\mathcal{D}$**-**$\Theta^{\pm}$**-bridge** $^\dagger\phi^{\Theta^{\pm}}_{\pm}$ *induces a [single, well-defined!]* **bijection** *of sets of* $\pm$*-label classes of cusps*

$$^\dagger\zeta^{\Theta^{\pm}}_{\underline{v}_t} : \mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}_{\underline{v}_t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}_{\succ,\underline{v}})$$

that is **compatible** *with the respective* $\mathbb{F}_l^{\pm}$**-group** *structures. Moreover, for* $\underline{w} \in \underline{\mathbb{V}}$, *the bijections*

$$^\dagger\xi^{\Theta^{\pm}}_{\succ,\underline{v},\underline{w}} \stackrel{\mathrm{def}}{=} (^\dagger\zeta^{\Theta^{\pm}}_{\underline{w}_0}) \circ {}^\dagger\xi^{\Theta^{\mathrm{ell}}}_{\underline{v}_0,\underline{w}_0} \circ (^\dagger\zeta^{\Theta^{\pm}}_{\underline{v}_0})^{-1} : \mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}_{\succ,\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}_{\succ,\underline{w}});$$
$$^\dagger\xi^{\Theta^{\pm}}_{\underline{v}_t,\underline{w}_t} \stackrel{\mathrm{def}}{=} (^\dagger\zeta^{\Theta^{\pm}}_{\underline{w}_t})^{-1} \circ {}^\dagger\xi^{\Theta^{\pm}}_{\succ,\underline{v},\underline{w}} \circ (^\dagger\zeta^{\Theta^{\pm}}_{\underline{v}_t}) : \mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}_{\underline{v}_t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}_{\underline{w}_t})$$

— where, by abuse of notation, we write "0" for the zero element of the $\mathbb{F}_l^{\pm}$*-group* T *— are* **compatible** *with the respective* $\mathbb{F}_l^{\pm}$**-group** *structures, and we have* $^\dagger\xi^{\Theta^{\pm}}_{\underline{v}_t,\underline{w}_t} = {}^\dagger\xi^{\Theta^{\mathrm{ell}}}_{\underline{v}_t,\underline{w}_t}$. *Write*

$$\mathrm{LabCusp}^{\pm}(^\dagger\mathfrak{D}_{\succ})$$

for the $\mathbb{F}_l^{\pm}$*-group obtained by identifying the various* $\mathbb{F}_l^{\pm}$*-groups* $\mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}_{\succ,\underline{v}})$, *as* $\underline{v}$ *ranges over the elements of* $\underline{\mathbb{V}}$, *via the various* $^\dagger\xi^{\Theta^{\pm}}_{\succ,\underline{v},\underline{w}}$. *Finally, for any* $t \in T$, *the various* $^\dagger\zeta^{\Theta^{\pm}}_{\underline{v}_t}$, $^\dagger\zeta^{\Theta^{\mathrm{ell}}}_{\underline{v}_t}$ *determine, respectively, a [single, well-defined!]* **bijection**

$$^\dagger\zeta^{\Theta^{\pm}}_t : \mathrm{LabCusp}^{\pm}(^\dagger\mathfrak{D}_t) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(^\dagger\mathfrak{D}_{\succ})$$

— which is compatible with the respective $\mathbb{F}_l^{\pm}$*-group structures.*

(iii) The assignment

$$T \ni t \mapsto {}^\dagger\zeta^{\Theta^{\mathrm{ell}}}_t(0) \in \mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}^{\circledcirc\pm})$$

— where, by abuse of notation, we write "0" for the zero element of the $\mathbb{F}_l^{\pm}$*-group* $\mathrm{LabCusp}^{\pm}(^\dagger\mathfrak{D}_t)$ *— determines a [single, well-defined!]* **bijection**

$$(^\dagger\zeta_{\pm})^{-1} : T \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(^\dagger\mathcal{D}^{\circledcirc\pm})$$

[i.e., whose inverse we denote by $^{\dagger}\zeta_{\pm}$*] — which is* **compatible** *with the respective* $\mathbb{F}_l^{\pm}$*-torsor structures. Moreover, for any* $t \in T$*, the composite bijection*

$$({}^{\dagger}\zeta_0^{\Theta^{\text{ell}}})^{-1} \circ ({}^{\dagger}\zeta_t^{\Theta^{\text{ell}}}) \circ ({}^{\dagger}\zeta_t^{\Theta^{\pm}})^{-1} \circ ({}^{\dagger}\zeta_0^{\Theta^{\pm}}) : \text{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_0) \xrightarrow{\sim} \text{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_0)$$

coincides with the automorphism of the set $\text{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_0)$ *determined, relative to the* $\mathbb{F}_l^{\pm}$*-group structure on this set, by the action of* $({}^{\dagger}\zeta_0^{\Theta^{\text{ell}}})^{-1}(({}^{\dagger}\zeta_{\pm})^{-1}(t))$*.*

(iv) Let $\alpha \in \text{Aut}_{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})/\text{Aut}_{\text{csp}}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})$*. Then if one replaces* ${}^{\dagger}\phi_{\pm}^{\Theta^{\text{ell}}}$ *by* $\alpha \circ {}^{\dagger}\phi_{\pm}^{\Theta^{\text{ell}}}$ *[cf. Proposition 6.6, (iv), below], then the resulting "*${}^{\dagger}\zeta_t^{\Theta^{\text{ell}}}$*" is related to the "*${}^{\dagger}\zeta_t^{\Theta^{\text{ell}}}$*" determined by the original* ${}^{\dagger}\phi_{\pm}^{\Theta^{\text{ell}}}$ *by post-composition with the image of* α *via the* **natural bijection** *[cf. the discussion of Definition 6.1, (v)]*

$$\text{Aut}_{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})/\text{Aut}_{\text{csp}}({}^{\dagger}\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \text{Aut}_{\pm}(\text{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})) \quad (\cong \mathbb{F}_l^{\rtimes\pm})$$

determined by the tautological action of $\text{Aut}_{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})/\text{Aut}_{\text{csp}}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})$ *on the set of* $\pm$*-label classes of cusps* $\text{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})$*.*

Next, let us observe that it follows immediately from the various definitions involved [cf. the discussion of Definition 6.1; Examples 6.2, 6.3], together with the explicit description of the various *poly-automorphisms* discussed in Examples 6.2, (ii), (iii); 6.3, (ii) [cf. also the various properties discussed in Proposition 6.5], that we have the following *additive* analogue of Proposition 4.8.

Proposition 6.6. (First Properties of Base-$\Theta^{\pm}$-Bridges, Base-Θ^{ell}-Bridges, and Base-$\Theta^{\pm\text{ell}}$-Hodge Theaters) *Relative to a fixed collection of* **initial Θ-data***:*

(i) The **set of isomorphisms** *between* **two $\mathcal{D}$-$\Theta^{\pm}$-bridges** *forms a* **torsor** *over the group*

$$\{\pm 1\} \times \left(\{\pm 1\}^{\underline{\mathbb{V}}}\right)$$

— where the first (respectively, second) factor corresponds to poly-automorphisms of the sort described in Example 6.2, (ii) (respectively, Example 6.2, (iii)). Moreover, the first factor may be thought of as corresponding to the induced isomorphisms of $\mathbb{F}_l^{\pm}$**-groups** *between the index sets of the capsules involved.*

(ii) The **set of isomorphisms** *between* **two $\mathcal{D}$-Θ^{ell}-bridges** *forms an* $\mathbb{F}_l^{\rtimes\pm}$**-torsor** *— i.e., more precisely, a torsor over a finite group that is equipped with a* **natural outer isomorphism** *to* $\mathbb{F}_l^{\rtimes\pm}$*. Moreover, this set of isomorphisms maps* **bijectively***, by considering the induced bijections, to the set of isomorphisms of* $\mathbb{F}_l^{\pm}$**-torsors** *between the index sets of the capsules involved.*

(iii) The **set of isomorphisms** *between* **two $\mathcal{D}$-$\Theta^{\pm\text{ell}}$-Hodge theaters** *forms a* $\{\pm 1\}$**-torsor***. Moreover, this set of isomorphisms maps* **bijectively***, by considering the induced bijections, to the set of isomorphisms of* $\mathbb{F}_l^{\pm}$**-groups** *between the index sets of the capsules involved.*

(iv) Given a $\mathcal{D}$-$\Theta^{\pm}$-bridge and a $\mathcal{D}$-Θ^{ell}-bridge, the set of capsule-+-full poly-isomorphisms between the respective capsules of $\mathcal{D}$-prime-strips which allow one to **glue** *the given $\mathcal{D}$-$\Theta^{\pm}$- and $\mathcal{D}$-Θ^{ell}-bridges together to form a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater forms a* **torsor** *over the group*

$$\mathbb{F}_l^{\rtimes\pm} \times \left(\{\pm 1\}^{\underline{\mathbb{V}}}\right)$$

— where the first factor corresponds to the $\mathbb{F}_l^{\rtimes\pm}$ of (ii); the subgroup $\{\pm 1\} \times \left(\{\pm 1\}^{\underline{\mathbb{V}}}\right)$ corresponds to the group of (i). Moreover, the first factor may be thought of as corresponding to the induced isomorphisms of $\mathbb{F}_l^{\pm}$**-torsors** *between the index sets of the capsules involved.*

(v) Given a $\mathcal{D}$-Θ^{ell}-bridge, there exists a [relatively simple — cf. the discussion of Example 6.2, (i)] **functorial algorithm** *for constructing, up to an* $\mathbb{F}_l^{\rtimes\pm}$**-indeterminacy** *[cf. (ii), (iv)], from the given $\mathcal{D}$-Θ^{ell}-bridge a* $\mathcal{D}$**-**$\Theta^{\pm\mathrm{ell}}$**-Hodge theater** *whose underlying $\mathcal{D}$-Θ^{ell}-bridge is the given $\mathcal{D}$-Θ^{ell}-bridge.*

$$[\,-l^* < \dots < -2 < -1 < 0 < 1 < 2 < \dots < l^*\,]$$

$$\mathfrak{D}_{\succ} = /^{\pm}$$

$$\Uparrow \ \phi_{\pm}^{\Theta^{\pm}}$$

$$\{\pm 1\} \curvearrowright \quad (\,-l^* < \dots < -2 < -1 < 0 < 1 < 2 < \dots < l^*\,)$$

$$(\,/^{\pm} \qquad\qquad /^{\pm} \quad /^{\pm} \quad /^{\pm} \quad /^{\pm} \quad /^{\pm} \qquad\qquad /^{\pm}\,)$$

$$\mathfrak{D}_T$$

$$\Downarrow \ \phi_{\pm}^{\Theta^{\mathrm{ell}}}$$

$$\begin{array}{ccccccc}
 & & \pm & \longrightarrow & \pm & & \\
 & \nearrow & & & & \searrow & \\
\pm & & & \mathbb{F}_l^{\rtimes\pm} \curvearrowright & & & \pm \\
\uparrow & & & \mathcal{D}^{\circledcirc\pm} = & & & \downarrow \\
\pm & & & \mathcal{B}(\underline{X}_K)^0 & & & \pm \\
 & \nwarrow & & & & \swarrow & \\
 & & \pm & \dots & \pm & &
\end{array}$$

Fig. 6.1: The combinatorial structure of a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater

Remark 6.6.1. The underlying combinatorial structure of a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater — or, essentially equivalently [cf. Definition 6.11, Corollary 6.12 below], of a $\Theta^{\pm\text{ell}}$-Hodge theater — is illustrated in Fig. 6.1 above. Thus, Fig. 6.1 may be thought of as a sort of *additive* analogue of the *multiplicative* situation illustrated in Fig. 4.4. In Fig. 6.1, the "$\Uparrow$" corresponds to the associated $[\mathcal{D}\text{-}]\Theta^{\pm}$*-bridge*, while the "$\Downarrow$" corresponds to the associated $[\mathcal{D}\text{-}]\Theta^{\text{ell}}$*-bridge*; the "$/^{\pm}$'s" denote $\mathcal{D}$-prime-strips.

Proposition 6.7. **(Base-Θ-Bridges Associated to Base-$\Theta^{\pm}$-Bridges)** *Relative to a fixed collection of* **initial Θ-data***, let*

$$^{\dagger}\mathfrak{D}_T \quad \xrightarrow{^{\dagger}\phi^{\Theta^{\pm}}_{\pm}} \quad ^{\dagger}\mathfrak{D}_{\succ}$$

be a **$\mathcal{D}$-$\Theta^{\pm}$-bridge***, as in Definition 6.4, (i). Then by replacing $^{\dagger}\mathfrak{D}_T$ by $^{\dagger}\mathfrak{D}_{T^*}$ [cf. Definition 6.4, (i)], identifying the $\mathcal{D}$-prime-strip $^{\dagger}\mathfrak{D}_{\succ}$ with the $\mathcal{D}$-prime-strip $^{\dagger}\mathfrak{D}_0$ via $^{\dagger}\phi^{\Theta^{\pm}}_0$ [cf. the discussion of Definition 6.4, (i)] to form a $\mathcal{D}$-prime-strip $^{\dagger}\mathfrak{D}_{>}$, replacing the various $+$-full poly-morphisms that occur in $^{\dagger}\phi^{\Theta^{\pm}}_{\pm}$ at the $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$ by the corresponding full poly-morphisms, and replacing the various $+$-full poly-morphisms that occur in $^{\dagger}\phi^{\Theta^{\pm}}_{\pm}$ at the $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ by the poly-morphisms described [via* **group-theoretic algorithms***!] in Example 4.4, (i), (ii), we obtain a* **functorial algorithm** *for constructing a [well-defined, up to a unique isomorphism!]* **$\mathcal{D}$-Θ-bridge**

$$^{\dagger}\mathfrak{D}_{T^*} \quad \xrightarrow{^{\dagger}\phi^{\Theta}_{*}} \quad ^{\dagger}\mathfrak{D}_{>}$$

as in Definition 4.6, (ii). Thus, the newly constructed $\mathcal{D}$-Θ-bridge is related to the given $\mathcal{D}$-$\Theta^{\pm}$-bridge via the following correspondences:

$$^{\dagger}\mathfrak{D}_T|_{(T\setminus\{0\})} \mapsto {}^{\dagger}\mathfrak{D}_{T^*}; \qquad ^{\dagger}\mathfrak{D}_0, {}^{\dagger}\mathfrak{D}_{\succ} \mapsto {}^{\dagger}\mathfrak{D}_{>}$$

— each of which maps **precisely two** *$\mathcal{D}$-prime-strips to a* **single** *$\mathcal{D}$-prime-strip.*

Proof. The various assertions of Proposition 6.7 follow immediately from the various definitions involved. ◯

Next, we consider *additive* analogues of Propositions 4.9, 4.11; Corollary 4.12.

Proposition 6.8. **(Symmetries arising from Forgetful Functors)** *Relative to a fixed collection of* **initial Θ-data***:*

(i) **(Base-Θ^{ell}-Bridges)** *The operation of associating to a $\mathcal{D}$-$\Theta^{\pm\text{ell}}$-Hodge theater the underlying $\mathcal{D}$-Θ^{ell}-bridge of the $\mathcal{D}$-$\Theta^{\pm\text{ell}}$-Hodge theater determines a* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{-Hodge theaters}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{-Hodge theaters}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta^{\text{ell}}\text{-bridges}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta^{\text{ell}}\text{-bridges}\end{array}}$$

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}} \qquad \mapsto \qquad (^{\dagger}\mathfrak{D}_T \quad \xrightarrow{^{\dagger}\phi^{\Theta^{\text{ell}}}_{\pm}} \quad ^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

whose output data admits an $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** *— i.e., more precisely, a symmetry given by the action of a finite group that is equipped with a* **natural outer isomorphism** *to* $\mathbb{F}_l^{\rtimes\pm}$ *— which acts* **doubly transitively** *[i.e., transitively with stabilizers of order two] on the index set [i.e., "T"] of the underlying capsule of* $\mathcal{D}$*-prime-strips [i.e., "$^{\dagger}\mathfrak{D}_T$"] of this output data.*

(ii) **(Holomorphic Capsules)** *The operation of associating to a* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*-Hodge theater* $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ *the* $l^{\pm}$*-capsule*

$$^{\dagger}\mathfrak{D}_{|T|}$$

associated to the underlying $\mathcal{D}$-$\Theta^{\pm}$*-bridge of* $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ *[cf. Definition 6.4, (i)] determines a* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-Hodge theaters}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-Hodge theaters}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of } l^{\pm}\text{-capsules}\\ \text{of } \mathcal{D}\text{-prime-strips}\\ \text{and capsule-full poly-}\\ \text{isomorphisms of } l^{\pm}\text{-capsules}\end{array}}$$

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \quad \mapsto \quad {}^{\dagger}\mathfrak{D}_{|T|}$$

whose output data admits an $\mathfrak{S}_{l^{\pm}}$**-symmetry** *[where we write* $\mathfrak{S}_{l^{\pm}}$ *for the symmetric group on* $l^{\pm}$ *letters] which acts* **transitively** *on the index set [i.e., "$|T|$"] of this output data. Thus, this functor may be thought of as an operation that consists of* **forgetting** *the* **labels** $\in |\mathbb{F}_l| = \mathbb{F}_l/\{\pm 1\}$ *[i.e., forgetting the bijection* $|T| \xrightarrow{\sim} |\mathbb{F}_l|$ *determined by the* $\mathbb{F}_l^{\pm}$*-group structure of* T *— cf. Definition 6.4, (i)]. In particular, if one is only given this output data* $^{\dagger}\mathfrak{D}_{|T|}$ *up to isomorphism, then there is a total of precisely* $l^{\pm}$ **possibilities** *for the element* $\in |\mathbb{F}_l|$ *to which a given index* $|t| \in |T|$ *corresponds, prior to the application of this functor.*

(iii) **(Mono-analytic Capsules)** *By composing the functor of (ii) with the* **mono-analyticization** *operation discussed in Definition 4.1, (iv), one obtains a* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-Hodge theaters}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-Hodge theaters}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of } l^{\pm}\text{-capsules}\\ \text{of } \mathcal{D}^{\vdash}\text{-prime-strips}\\ \text{and capsule-full poly-}\\ \text{isomorphisms of } l^{\pm}\text{-capsules}\end{array}}$$

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \quad \mapsto \quad {}^{\dagger}\mathfrak{D}^{\vdash}_{|T|}$$

whose output data satisfies the same symmetry properties with respect to labels as the output data of the functor of (ii).

Proof. Assertions (i), (ii), (iii) follow immediately from the definitions [cf. also Proposition 6.6, (ii), in the case of assertion (i)]. ◯

$$/^{\pm} \quad \hookrightarrow \quad /^{\pm}/^{\pm} \quad \hookrightarrow \quad /^{\pm}/^{\pm}/^{\pm} \quad \hookrightarrow \quad \dots \quad \hookrightarrow \quad /^{\pm}/^{\pm}/^{\pm} \ \dots \ /^{\pm}$$

Fig. 6.2: An $l^{\pm}$-procession of $\mathcal{D}$-prime-strips

Proposition 6.9. **(Processions of Base-Prime-Strips)** *Relative to a fixed collection of* **initial Θ-data***:*

(i) **(Holomorphic Processions)** *Given a $\mathcal{D}$-$\Theta^{\pm}$-bridge ${}^{\dagger}\phi^{\Theta^{\pm}}_{\pm} : {}^{\dagger}\mathfrak{D}_T \to {}^{\dagger}\mathfrak{D}_{\succ}$, with underlying capsule of $\mathcal{D}$-prime-strips ${}^{\dagger}\mathfrak{D}_T$ [cf. Definition 6.4, (i)], denote by* $\mathrm{Prc}({}^{\dagger}\mathfrak{D}_T)$ *the* **$l^{\pm}$-procession of $\mathcal{D}$-prime-strips** *[cf. Fig. 6.2, where each "$/^{\pm}$" denotes a $\mathcal{D}$-prime-strip] determined by considering the ["sub"]capsules of the capsule ${}^{\dagger}\mathfrak{D}_{|T|}$ of Definition 6.4, (i), corresponding to the subsets $\mathbb{S}^{\pm}_1 \subseteq \dots \subseteq \mathbb{S}^{\pm}_t \stackrel{\mathrm{def}}{=} \{0, 1, 2, \dots, t-1\} \subseteq \dots \subseteq \mathbb{S}^{\pm}_{l^{\pm}} = |\mathbb{F}_l|$ [where, by abuse of notation, we use the notation for nonnegative integers to denote the images of these nonnegative integers in $|\mathbb{F}_l|$], relative to the bijection $|T| \stackrel{\sim}{\to} |\mathbb{F}_l|$ determined by the $\mathbb{F}^{\pm}_l$-group structure of T [cf. Definition 6.4, (i)]. Then the assignment ${}^{\dagger}\phi^{\Theta^{\pm}}_{\pm} \mapsto \mathrm{Prc}({}^{\dagger}\mathfrak{D}_T)$ determines a* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta^{\pm}\text{-bridges}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta^{\pm}\text{-bridges}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of processions}\\ \text{of } \mathcal{D}\text{-prime-strips}\\ \text{and morphisms of}\\ \text{processions}\end{array}}$$

$$ {}^{\dagger}\phi^{\Theta^{\pm}}_{\pm} \qquad \mapsto \qquad \mathrm{Prc}({}^{\dagger}\mathfrak{D}_T)$$

whose output data satisfies the following property: for each $n \in \{1, \dots, l^{\pm}\}$, there are precisely ***n*** **possibilities** *for the element $\in |\mathbb{F}_l|$ to which a given index of the index set of the n-capsule that appears in the procession constituted by this output data corresponds, prior to the application of this functor. That is to say, by taking the product, over elements of $|\mathbb{F}_l|$, of cardinalities of "sets of possibilies", one concludes that*

> *by considering* **processions** *— i.e., the functor discussed above, possibly pre-composed with the functor ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \mapsto {}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}$ that associates to a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater its associated $\mathcal{D}$-$\Theta^{\pm}$-bridge — the indeterminacy consisting of $(l^{\pm})^{(l^{\pm})}$ possibilities that arises in Proposition 6.8, (ii), is* **reduced** *to an* **indeterminacy** *consisting of a total of* **$l^{\pm}$! possibilities***.*

(ii) **(Mono-analytic Processions)** *By composing the functor of (i) with the* **mono-analyticization** *operation discussed in Definition 4.1, (iv), one obtains a* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta^{\pm}\text{-bridges}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta^{\pm}\text{-bridges}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of processions}\\ \text{of } \mathcal{D}^{\vdash}\text{-prime-strips}\\ \text{and morphisms of}\\ \text{processions}\end{array}}$$

$$ {}^{\dagger}\phi^{\Theta^{\pm}}_{\pm} \qquad \mapsto \qquad \mathrm{Prc}({}^{\dagger}\mathfrak{D}^{\vdash}_T)$$

whose output data satisfies the same indeterminacy properties with respect to labels as the output data of the functor of (i).

(iii) The functors of (i), (ii) are **compatible**, *respectively, with the functors of Proposition 4.11, (i), (ii), relative to the functor [i.e., determined by the functorial algorithm] of Proposition 6.7, in the sense that the natural inclusions*

$$\mathbb{S}^{*}_{j} = \{1, \ldots, j\} \hookrightarrow \mathbb{S}^{\pm}_{t} = \{0, 1, \ldots, t-1\}$$

[cf. the notation of Proposition 4.11] — where $j \in \{1, \ldots, l^{*}\}$ and $t \stackrel{\text{def}}{=} j+1$ — determine **natural transformations**

$$\begin{array}{rcl} {}^{\dagger}\phi^{\Theta^{\pm}}_{\pm} & \mapsto & \Big(\mathrm{Prc}({}^{\dagger}\mathfrak{D}_{T^{*}}) \hookrightarrow \mathrm{Prc}({}^{\dagger}\mathfrak{D}_{T})\Big) \\ {}^{\dagger}\phi^{\Theta^{\pm}}_{\pm} & \mapsto & \Big(\mathrm{Prc}({}^{\dagger}\mathfrak{D}^{\vdash}_{T^{*}}) \hookrightarrow \mathrm{Prc}({}^{\dagger}\mathfrak{D}^{\vdash}_{T})\Big) \end{array}$$

from the respective composites of the functors of Proposition 4.11, (i), (ii), with the functor [determined by the functorial algorithm] of Proposition 6.7 to the functors of (i), (ii).

Proof. Assertions (i), (ii), (iii) follow immediately from the definitions. ◯

The following result is an immediate consequence of our discussion.

Corollary 6.10. **(Étale-pictures of Base-$\Theta^{\pm\mathrm{ell}}$-Hodge Theaters)** *Relative to a fixed collection of* **initial Θ-data***:*

(i) Consider the [composite] **functor**

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \quad \mapsto \quad {}^{\dagger}\mathfrak{D}_{>} \quad \mapsto \quad {}^{\dagger}\mathfrak{D}^{\vdash}_{>}$$

— from the category of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters and isomorphisms of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters [cf. Definition 6.4, (iii)] to the category of $\mathcal{D}^{\vdash}$-prime-strips and isomorphisms of $\mathcal{D}^{\vdash}$-prime-strips — obtained by assigning to the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ the **mono-analyticization** *[cf. Definition 4.1, (iv)] ${}^{\dagger}\mathfrak{D}^{\vdash}_{>}$ of the $\mathcal{D}$-prime-strip ${}^{\dagger}\mathfrak{D}_{>}$ associated, via the functorial algorithm of Proposition 6.7, to the* **underlying $\mathcal{D}$-$\Theta^{\pm}$-bridge** *of ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$. If ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$, ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ are* **$\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters***, then we define the* **base-$\Theta^{\pm\mathrm{ell}}$-, or $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-, link**

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \quad \stackrel{\mathcal{D}}{\longrightarrow} \quad {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$$

from ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ to ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ to be the **full poly-isomorphism**

$${}^{\dagger}\mathfrak{D}^{\vdash}_{>} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}^{\vdash}_{>}$$

between the $\mathcal{D}^{\vdash}$-prime-strips obtained by applying the functor discussed above to ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$, ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$.

(ii) If

$$\dots \xrightarrow{\mathcal{D}} {}^{(n-1)}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}} \xrightarrow{\mathcal{D}} {}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}} \xrightarrow{\mathcal{D}} {}^{(n+1)}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}} \xrightarrow{\mathcal{D}} \dots$$

[where $n \in \mathbb{Z}$] is an **infinite chain** *of* **$\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-linked $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-Hodge theaters** *[cf. the situation discussed in Corollary 3.8], then we obtain a resulting* **chain of full poly-isomorphisms**

$$\dots \stackrel{\sim}{\to} {}^{n}\mathfrak{D}^{\vdash}_{>} \stackrel{\sim}{\to} {}^{(n+1)}\mathfrak{D}^{\vdash}_{>} \stackrel{\sim}{\to} \dots$$

[cf. the situation discussed in Remark 3.8.1, (ii)] between the $\mathcal{D}^{\vdash}$-prime-strips obtained by applying the functor of (i). That is to say, the output data of the functor of (i) forms a **constant invariant** *[cf. the discussion of Remark 3.8.1, (ii)] — i.e., a* **mono-analytic core** *[cf. the situation discussed in Remark 3.9.1] — of the above infinite chain.*

$$\begin{array}{ccccc}
 & & \boxed{/^{\pm} \hookrightarrow /^{\pm}/^{\pm} \hookrightarrow \dots} & & \\
\dots & & & & \dots \\
 & & | & & \\
\boxed{/^{\pm} \hookrightarrow /^{\pm}/^{\pm} \hookrightarrow \dots} & — & \boxed{>^{\vdash} = \{0, \succ\}^{\vdash}} & — & \boxed{/^{\pm} \hookrightarrow /^{\pm}/^{\pm} \hookrightarrow \dots} \\
\dots & & | & & \dots \\
 & & \boxed{/^{\pm} \hookrightarrow /^{\pm}/^{\pm} \hookrightarrow \dots} & &
\end{array}$$

Fig. 6.3: Étale-picture of $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-Hodge theaters

(iii) If we regard each of the $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-Hodge theaters of the chain of (ii) as a **spoke** *emanating from the mono-analytic core discussed in (ii), then we obtain a* **diagram** *— i.e., an* **étale-picture of $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-Hodge theaters** *— as in Fig. 6.3 [cf. the situation discussed in Corollary 3.9, (i)]. In Fig. 6.3, "$>^{\vdash}$" denotes the mono-analytic core, obtained [cf. (i); Proposition 6.7] by* **identifying** *the mono-analyticized $\mathcal{D}$-prime-strips of the $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-Hodge theater labeled "0" and "$\succ$"; "$/^{\pm} \hookrightarrow /^{\pm}/^{\pm} \hookrightarrow \dots$" denotes the "holomorphic" processions of Proposition 6.9, (i), together with the remaining ["holomorphic"] data of the corresponding $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-Hodge theater. In particular, the mono-analyticizations of the* **zero-labeled** *$\mathcal{D}$-prime-strips — i.e., the $\mathcal{D}$-prime-strips corresponding to the first "$/^{\pm}$" in the processions just discussed — in the various spokes are* **identified with one another***. Put another way, the coric $\mathcal{D}^{\vdash}$-prime-strip "$>^{\vdash}$" may be thought of as being equipped with* **various distinct "holomorphic structures"** *— i.e., $\mathcal{D}$-prime-strip structures that give rise to the $\mathcal{D}^{\vdash}$-prime-strip structure — corresponding to the various*

spokes. Finally, [cf. the situation discussed in Corollary 3.9, (i)] this diagram satisfies the important property of admitting **arbitrary permutation symmetries** *among the spokes [i.e., among the labels $n \in \mathbb{Z}$ of the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters].*

(iv) The constructions of (i), (ii), (iii) are **compatible***, respectively, with the constructions of Corollary 4.12, (i), (ii), (iii), relative to the functor [i.e., determined by the functorial algorithm] of Proposition 6.7, in the evident sense [cf. the compatibility discussed in Proposition 6.9, (iii)].*

Finally, we conclude with *additive* analogues of Definition 5.5, Corollary 5.6.

Definition 6.11.

(i) We define a $\Theta^{\pm}$-*bridge* [relative to the given initial Θ-data] to be a poly-morphism

$$^{\dagger}\mathfrak{F}_T \quad \xrightarrow{^{\dagger}\psi_{\pm}^{\Theta^{\pm}}} \quad ^{\dagger}\mathfrak{F}_{\succ}$$

— where $^{\dagger}\mathfrak{F}_{\succ}$ is an $\mathcal{F}$-*prime-strip*; T is an $\mathbb{F}_l^{\pm}$-*group*; $^{\dagger}\mathfrak{F}_T = \{^{\dagger}\mathfrak{F}_t\}_{t \in T}$ is a *capsule of $\mathcal{F}$-prime-strips*, indexed by [the underlying set of] T — that lifts a $\mathcal{D}$-$\Theta^{\pm}$-bridge $^{\dagger}\phi_{\pm}^{\Theta^{\pm}} : {}^{\dagger}\mathfrak{D}_T \to {}^{\dagger}\mathfrak{D}_{\succ}$ [cf. Corollary 5.3, (ii)]. In this situation, we shall write

$$^{\dagger}\mathfrak{F}_{|T|}$$

for the $l^{\pm}$-capsule obtained from the l-capsule $^{\dagger}\mathfrak{F}_T$ by forming the quotient $|T|$ of the index set T of this underlying capsule by the action of $\{\pm 1\}$ and identifying the components of the capsule $^{\dagger}\mathfrak{F}_T$ indexed by the elements in the fibers of the quotient $T \twoheadrightarrow |T|$ via the constituent poly-morphisms of $^{\dagger}\psi_{\pm}^{\Theta^{\pm}} = \{^{\dagger}\psi_t^{\Theta^{\pm}}\}_{t \in T}$ [so each constituent $\mathcal{F}$-prime-strip of $^{\dagger}\mathfrak{F}_{|T|}$ is only well-defined up to a *positive automorphism* [i.e., up to an automorphism such that the induced automorphism of the associated $\mathcal{D}$-prime-strip is *positive*], but this *indeterminacy* will not affect applications of this construction — cf. the discussion of Definition 6.4, (i)]. Also, we shall write

$$^{\dagger}\mathfrak{F}_{T^{*}}$$

for the l^{*}-capsule determined by the subset $T^{*} \stackrel{\mathrm{def}}{=} |T| \setminus \{0\}$ of nonzero elements of $|T|$. We define a(n) *[iso]morphism of* $\Theta^{\pm}$-*bridges*

$$(^{\dagger}\mathfrak{F}_T \quad \xrightarrow{^{\dagger}\psi_{\pm}^{\Theta^{\pm}}} \quad ^{\dagger}\mathfrak{F}_{\succ}) \quad \to \quad (^{\ddagger}\mathfrak{F}_{T'} \quad \xrightarrow{^{\ddagger}\psi_{\pm}^{\Theta^{\pm}}} \quad ^{\ddagger}\mathfrak{F}_{\succ})$$

to be a pair of poly-isomorphisms

$$^{\dagger}\mathfrak{F}_T \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}_{T'}; \quad ^{\dagger}\mathfrak{F}_{\succ} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}_{\succ}$$

that lifts a morphism between the associated $\mathcal{D}$-$\Theta^{\pm}$-bridges $^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$, $^{\ddagger}\phi_{\pm}^{\Theta^{\pm}}$. There is an evident notion of composition of morphisms of $\Theta^{\pm}$-bridges.

(ii) We define a Θ^{ell}-*bridge* [relative to the given initial Θ-data]

$$^{\dagger}\mathfrak{F}_T \quad \xrightarrow{^{\dagger}\psi_{\pm}^{\Theta^{\mathrm{ell}}}} \quad ^{\dagger}\mathcal{D}^{\circledcirc\pm}$$

— where ${}^{\dagger}\mathcal{D}^{\circledcirc\pm}$ is a *category equivalent to* $\mathcal{D}^{\circledcirc\pm}$; T is an $\mathbb{F}_l^{\pm}$*-torsor*; ${}^{\dagger}\mathfrak{F}_T = \{{}^{\dagger}\mathfrak{F}_t\}_{t\in T}$ is a *capsule of* $\mathcal{F}$*-prime-strips*, indexed by [the underlying set of] T — to be a $\mathcal{D}$-Θ^{ell}-bridge ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}} : {}^{\dagger}\mathfrak{D}_T \to {}^{\dagger}\mathcal{D}^{\circledcirc\pm}$ — where we write ${}^{\dagger}\mathfrak{D}_T$ for the capsule of $\mathcal{D}$-prime-strips associated to ${}^{\dagger}\mathfrak{F}_T$ [cf. Remark 5.2.1, (i)]. We define a(n) *[iso]morphism of* Θ^{ell}*-bridges*

$$({}^{\dagger}\mathfrak{F}_T \quad \xrightarrow{{}^{\dagger}\psi_{\pm}^{\Theta^{\mathrm{ell}}}} \quad {}^{\dagger}\mathcal{D}^{\circledcirc\pm}) \quad \to \quad ({}^{\ddagger}\mathfrak{F}_{T'} \quad \xrightarrow{{}^{\ddagger}\psi_{\pm}^{\Theta^{\mathrm{ell}}}} \quad {}^{\ddagger}\mathcal{D}^{\circledcirc\pm})$$

to be a pair of poly-isomorphisms

$${}^{\dagger}\mathfrak{F}_T \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}_{T'}; \quad {}^{\dagger}\mathcal{D}^{\circledcirc\pm} \xrightarrow{\sim} {}^{\ddagger}\mathcal{D}^{\circledcirc\pm}$$

that determines a morphism between the associated $\mathcal{D}$-Θ^{ell}-bridges ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$, ${}^{\ddagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$. There is an evident notion of composition of morphisms of Θ^{ell}-bridges.

(iii) We define a $\Theta^{\pm\mathrm{ell}}$*-Hodge theater* [relative to the given initial Θ-data] to be a collection of data

$${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}} = ({}^{\dagger}\mathfrak{F}_{\succ} \quad \xleftarrow{{}^{\dagger}\psi_{\pm}^{\Theta^{\pm}}} \quad {}^{\dagger}\mathfrak{F}_T \quad \xrightarrow{{}^{\dagger}\psi_{\pm}^{\Theta^{\mathrm{ell}}}} \quad {}^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

— where ${}^{\dagger}\psi_{\pm}^{\Theta^{\pm}}$ is a $\Theta^{\pm}$*-bridge*; ${}^{\dagger}\psi_{\pm}^{\Theta^{\mathrm{ell}}}$ is a Θ^{ell}*-bridge* — such that the *associated data* $\{{}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}, {}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}\}$ [cf. (i), (ii)] forms a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater. A(n) *[iso]morphism of* $\Theta^{\pm\mathrm{ell}}$*-Hodge theaters* is defined to be a pair of morphisms between the respective associated $\Theta^{\pm}$- and Θ^{ell}-bridges that are *compatible* with one another in the sense that they induce the *same poly-isomorphism* between the respective capsules of $\mathcal{F}$-prime-strips. There is an evident notion of composition of morphisms of $\Theta^{\pm\mathrm{ell}}$-Hodge theaters.

Corollary 6.12. (Isomorphisms of $\Theta^{\pm}$-Bridges, Θ^{ell}-Bridges, and $\Theta^{\pm\mathrm{ell}}$-Hodge Theaters) *Relative to a fixed collection of* **initial Θ-data***:*

(i) The natural functorially induced map from the **set of isomorphisms** *between* **two $\Theta^{\pm}$-bridges** *(respectively,* **two Θ^{ell}-bridges***;* **two $\Theta^{\pm\mathrm{ell}}$-Hodge theaters***) to the set of isomorphisms between the respective* **associated $\mathcal{D}$-$\Theta^{\pm}$-bridges** *(respectively,* **associated $\mathcal{D}$-Θ^{ell}-bridges***;* **associated $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters***) is* **bijective***.*

(ii) Given a $\Theta^{\pm}$-bridge and a Θ^{ell}-bridge, the set of capsule-+-full poly-isomorphisms between the respective capsules of $\mathcal{F}$-prime-strips which allow one to **glue** *the given $\Theta^{\pm}$- and Θ^{ell}-bridges together to form a $\Theta^{\pm\mathrm{ell}}$-Hodge theater forms a* **torsor** *over the group*

$$\mathbb{F}_l^{\rtimes\pm} \ \times \ \left(\{\pm 1\}^{\underline{\mathbb{V}}}\right)$$

[cf. Proposition 6.6, (iv)]. Moreover, the first factor may be thought of as corresponding to the induced isomorphisms of $\mathbb{F}_l^{\pm}$**-torsors** *between the index sets of the capsules involved.*

Proof. Assertions (i), (ii) follow immediately from Definition 6.11; Corollary 5.3, (ii) [cf. also Proposition 6.6, (iv), in the case of assertion (ii)]. ○

Remark 6.12.1. By applying Corollary 6.12, a similar remark to Remark 5.6.1 may be made concerning the $\Theta^{\pm}$-*bridges*, Θ^{ell}-*bridges*, and $\Theta^{\pm\mathrm{ell}}$-*Hodge theaters* studied in the present §6. We leave the routine details to the reader.

Remark 6.12.2. Relative to a fixed collection of *initial* Θ-*data*:

(i) Suppose that $({}^{\dagger}\mathfrak{F}_T \ \rightarrow \ {}^{\dagger}\mathfrak{F}_{\succ})$ is a $\Theta^{\pm}$-*bridge*; write $({}^{\dagger}\mathfrak{D}_T \ \rightarrow \ {}^{\dagger}\mathfrak{D}_{\succ})$ for the *associated* $\mathcal{D}$-$\Theta^{\pm}$-*bridge* [cf. Definition 6.11, (i)]. Then Proposition 6.7 gives a *functorial algorithm* for constructing a $\mathcal{D}$-Θ-*bridge* $({}^{\dagger}\mathfrak{D}_{T^{*}} \ \rightarrow \ {}^{\dagger}\mathfrak{D}_{>})$ from this $\mathcal{D}$-$\Theta^{\pm}$-*bridge* $({}^{\dagger}\mathfrak{D}_T \ \rightarrow \ {}^{\dagger}\mathfrak{D}_{\succ})$. Suppose that this $\mathcal{D}$-Θ-bridge $({}^{\dagger}\mathfrak{D}_{T^{*}} \ \rightarrow \ {}^{\dagger}\mathfrak{D}_{>})$ arises as the $\mathcal{D}$-Θ-bridge associated to a Θ-*bridge* $({}^{\ddagger}\mathfrak{F}_J \ \rightarrow \ {}^{\ddagger}\mathfrak{F}_{>} \ \dashrightarrow \ {}^{\ddagger}\mathcal{HT}^{\Theta})$ [so $J = T^{*}$ — cf. Definition 5.5, (ii)]. Then since the portion "${}^{\ddagger}\mathfrak{F}_J \rightarrow {}^{\ddagger}\mathfrak{F}_{>}$" of this Θ-bridge is *completely determined* [cf. Definition 5.5, (ii), (d)] by the associated $\mathcal{D}$-Θ-bridge, one verifies immediately that

> one may regard this portion "${}^{\ddagger}\mathfrak{F}_J \rightarrow {}^{\ddagger}\mathfrak{F}_{>}$" of the Θ-bridge as having been constructed via a *functorial algorithm* similar to the functorial algorithm of Proposition 6.7 [cf. also Definition 5.5, (ii), (d); the discussion of Remark 5.3.1] from the $\Theta^{\pm}$-*bridge* $({}^{\dagger}\mathfrak{F}_T \ \rightarrow \ {}^{\dagger}\mathfrak{F}_{\succ})$.

Since, moreover, *isomorphisms between* Θ-*bridges* are in natural bijective correspondence with isomorphisms between the *associated* $\mathcal{D}$-Θ-*bridges* [cf. Corollary 5.6, (ii)], it thus follows immediately [cf. Corollary 5.3, (ii)] that *isomorphisms between* Θ-*bridges* are in natural bijective correspondence with isomorphisms between the *portions of* Θ-*bridges* [i.e., "${}^{\ddagger}\mathfrak{F}_J \rightarrow {}^{\ddagger}\mathfrak{F}_{>}$"] considered above. Thus, in summary, if $({}^{\ddagger}\mathfrak{F}_J \ \rightarrow \ {}^{\ddagger}\mathfrak{F}_{>} \ \dashrightarrow \ {}^{\ddagger}\mathcal{HT}^{\Theta})$ is a Θ-*bridge* for which the portion "${}^{\ddagger}\mathfrak{F}_J \rightarrow {}^{\ddagger}\mathfrak{F}_{>}$" is obtained via the functorial algorithm discussed above from the $\Theta^{\pm}$-*bridge* $({}^{\dagger}\mathfrak{F}_T \ \rightarrow \ {}^{\dagger}\mathfrak{F}_{\succ})$, then, for simplicity, we shall describe this state of affairs by saying that

> the **Θ-bridge** $({}^{\ddagger}\mathfrak{F}_J \ \rightarrow \ {}^{\ddagger}\mathfrak{F}_{>} \ \dashrightarrow \ {}^{\ddagger}\mathcal{HT}^{\Theta})$ is **glued to the $\Theta^{\pm}$-bridge** $({}^{\dagger}\mathfrak{F}_T \ \rightarrow \ {}^{\dagger}\mathfrak{F}_{\succ})$ via the **functorial algorithm** of Proposition 6.7.

We leave the routine details of giving a more explicit description [say, in the style of the statement of Proposition 6.7] of such functorial algorithms to the reader. A similar [but easier!] construction may be given for $\mathcal{D}$-Θ-*bridges* and $\mathcal{D}$-$\Theta^{\pm}$-*bridges*.

(ii) Now observe that

> by **gluing** a **$\Theta^{\pm\mathrm{ell}}$-Hodge theater** [cf. Definition 6.11, (iii)] to a **ΘNF-Hodge theater** [cf. Definition 5.5, (iii)] along the respective associated $\Theta^{\pm}$- *and* Θ-*bridges* via the functorial algorithm of Proposition 6.7 [cf. (i)], one obtains the notion of a
>
> **"$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater"**

— cf. Definition 6.13, (i), below. Here, we note that by Proposition 4.8, (ii); Corollary 5.6, (ii), the *gluing isomorphism* that occurs in such a gluing operation is *unique*. Then by applying Propositions 4.8, 6.6, and Corollaries 5.6, 6.12, one may verify *analogues* of these results for such $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters. In a similar vein, one may glue a $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-*Hodge theater* to a $\mathcal{D}$-ΘNF-*Hodge theater* to obtain a *"$\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theater"* [cf. Definition 6.13, (ii), below]. We leave the routine details to the reader.

Remark 6.12.3.

(i) One way to think of the notion of a ΘNF-*Hodge theater* studied in §4 is as a sort of

total space *of a local system of* $\mathbb{F}_l^{\circledast}$*-torsors*

over a *"base space"* that represents a sort of *"homotopy"* between a *number field* and a *Tate curve* [i.e., the elliptic curve under consideration at the $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$]. From this point of view, the notion of a $\Theta^{\pm \mathrm{ell}}$-*Hodge theater* studied in the present §6 may be thought of as a sort of

total space *of a local system of* $\mathbb{F}_l^{\rtimes \pm}$*-torsors*

over a similar "base space". Here, it is interesting to note that these $\mathbb{F}_l^{\circledast}$- and $\mathbb{F}_l^{\rtimes \pm}$-torsors arise, on the one hand, from the *l-torsion points* of the elliptic curve under consideration, hence may be thought of as

discrete approximations *of*
[the geometric portion of] this **elliptic curve** *over a number field*

[cf. the point of view of scheme-theoretic Hodge-Arakelov theory discussed in [HASurI], §1.3.4]. On the other hand, if one thinks in terms of the *tempered fundamental groups* of the Tate curves that occur at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, then these $\mathbb{F}_l^{\circledast}$- and $\mathbb{F}_l^{\rtimes \pm}$-torsors may be thought of as

finite approximations *of the copy of "$\mathbb{Z}$"*

that occurs as the *Galois group* of a well-known tempered covering of the Tate curve [cf. the discussion of [EtTh], Remark 2.16.2]. Note, moreover, that if one works with $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters* [cf. Remark 6.12.2, (ii)], then one is, in effect, working with both the **additive** and the **multiplicative** structures of this copy of $\mathbb{Z}$ — although, unlike the situation that occurs when one works with **rings**, i.e., in which the additive and multiplicative structures are **"entangled"** with one another in some sort of complicated fashion [cf. the discussion of [AbsTopIII], Remark 5.6.1], if one works with $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters, then each of the additive and multiplicative structures occurs in an *independent* fashion [i.e., in the form of $\Theta^{\pm \mathrm{ell}}$- and ΘNF-Hodge theaters], i.e., **"extracted"** *from this entanglement.*

(ii) At this point, it is useful to recall that the idea of a *distinct* [i.e., from the copy of $\mathbb{Z}$ implicit in the "base space"] *"local system-theoretic" copy of* $\mathbb{Z}$ occurring over a "base space" that represents a number field is reminiscent not only of the discussion of [EtTh], Remark 2.16.2, but also of the *Teichmüller-theoretic point of view* discussed in [AbsTopIII], §I5. That is to say, relative to the analogy with *p-adic Teichmüller theory*, the "base space" that represents a number field corresponds to a hyperbolic curve in positive characteristic, while the "local system-theoretic" copy of $\mathbb{Z}$ — which, as discussed in (i), also serves as a discrete approximation of the [geometric portion of the] elliptic curve under consideration — corresponds to a *nilpotent ordinary indigenous bundle* over the positive characteristic hyperbolic curve.

(iii) Relative to the analogy discussed in (ii) between the "local system-theoretic" copy of $\mathbb{Z}$ of (i) and the indigenous bundles that occur in p-adic Teichmüller theory, it is interesting to note that the *two combinatorial dimensions* [cf. [AbsTopIII], Remark 5.6.1] corresponding to the **additive** and **multiplicative** [i.e., "$\mathbb{F}_l^{\rtimes\pm}$-" and "$\mathbb{F}_l^{*}$-"] **symmetries** of $\Theta^{\pm\text{ell}}$-, ΘNF-Hodge theaters may be thought of as corresponding, respectively, to the **two real dimensions**

$$\cdot\ z \mapsto z + a, \qquad z \mapsto -\overline{z} + a;$$

$$\cdot\ z \mapsto \frac{z \cdot \cos(t) - \sin(t)}{z \cdot \sin(t) + \cos(t)}, \qquad z \mapsto \frac{\overline{z} \cdot \cos(t) + \sin(t)}{\overline{z} \cdot \sin(t) - \cos(t)}$$

— where $a, t \in \mathbb{R}$; z denotes the standard coordinate on $\mathfrak{H}$ — of transformations of the **upper half-plane** $\mathfrak{H}$, i.e., an object that is very closely related to the *canonical indigenous bundles* that occur in the classical complex uniformization theory of hyperbolic Riemann surfaces [cf. the discussions of Remarks 4.3.3, 5.1.4]. Here, it is also of interest to observe that the above **additive symmetry** of the upper half-plane is closely related to the coordinate on the upper half-plane determined by the **"classical *q*-parameter"**

$$q \quad \stackrel{\text{def}}{=} \quad e^{2\pi i z}$$

— a situation that is reminiscent of the close relationship, in the theory of the present series of papers, between the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** and the **Kummer theory** surrounding the *Hodge-Arakelov-theoretic evaluation of the* **theta function** *on the* ***l*-torsion points** at bad primes [cf. Remark 6.12.6, (ii), below; the theory of [IUTchII]]. Moreover, the *fixed* basepoint "$\underline{\mathbb{V}}^{\pm}$" [cf. Definition 6.1, (v)] with respect to which one considers l-torsion points in the context of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is reminiscent of the fact that the above additive symmetries of the upper half-plane *fix* the *cusp at infinity*. Indeed, taken as a whole, the geometry and coordinate naturally associated to this additive symmetry of the upper half-plane may be thought of, at the level of **"combinatorial prototypes"**, as the geometric apparatus associated to a **cusp** [i.e., as opposed to a node — cf. the discussion of [NodNon], Introduction]. By contrast, the **"toral" multiplicative symmetry** of the upper half-plane recalled above is closely related to the coordinate on the upper half-plane that determines a biholomorphic isomorphism with the **unit disc**

$$w \quad \stackrel{\text{def}}{=} \quad \tfrac{z-i}{z+i}$$

— a situation that is reminiscent of the close relationship, in the theory of the present series of papers, between the $\mathbb{F}_l^{*}$**-symmetry** and the **Kummer theory** surrounding the **number field** F_{mod} [cf. Remark 6.12.6, (iii), below; the theory of §5 of the present paper]. Moreover, the *action* of $\mathbb{F}_l^{*}$ on the "collection of basepoints for the l-torsion points" $\underline{\mathbb{V}}^{\text{Bor}} = \mathbb{F}_l^{*} \cdot \underline{\mathbb{V}}^{\pm\text{un}}$ [cf. Example 4.3, (i)] in the context of the $\mathbb{F}_l^{\circledast}$-symmetry is reminiscent of the fact that the multiplicative symmetries of the upper half-plane recalled above *act transitively* on the *entire boundary of the upper half-plane*. That is to say, taken as a whole, the geometry and coordinate naturally associated to this multiplicative symmetry of the upper half-plane may be thought of, at the level of **"combinatorial prototypes"**, as the geometric apparatus associated to a **node**, i.e., of the sort that occurs in the reduction modulo p of a *Hecke correspondence* [cf. the discussion of [IUTchII], Remark 4.11.4, (iii), (c); [NodNon], Introduction]. Finally, we note that, just as in the case of the $\mathbb{F}_l^{\rtimes\pm}$-, $\mathbb{F}_l^{\circledast}$-symmetries discussed in the present paper, the only "coric" symmetries, i.e., symmetries common to *both* the additive and multiplicative symmetries of the upper half-plane recalled above, are the symmetries "$\{\pm 1\}$" [i.e., the symmetries $z \mapsto z, -\overline{z}$ in the case of the upper half-plane]. The observations of the above discussion are summarized in Fig. 6.4 below.

Remark 6.12.4.

(i) Just as in the case of the $\mathbb{F}_l^{\circledast}$-*symmetry* of Proposition 4.9, (i), the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* of Proposition 6.8, (i), will eventually be applied, in the theory of the present series of papers [cf. theory of [IUTchII], [IUTchIII]], to establish an

explicit **network of comparison isomorphisms**

relating various objects — such as **log-volumes** — associated to the non-labeled prime-strips that are permuted by this symmetry [cf. the discussion of Remark 4.9.1, (i)]. Moreover, just as in the case of the $\mathbb{F}_l^{\circledast}$-symmetry studied in §4 [cf. the discussion of Remark 4.9.2], one important property of this "network of comparison isomorphisms" is that it operates *without* **"label crushing"** [cf. Remark 4.9.2, (i)] — i.e., without disturbing the **bijective** *relationship* between the set of indices of the symmetrized collection of prime-strips and the set of labels $\in T \overset{\sim}{\to} \mathbb{F}_l$ under consideration. Finally, just as in the situation studied in §4,

> *this crucial* **synchronization of labels** *is essentially a consequence of the* **single connected component**

— or, at a more abstract level, the **single basepoint** — of the global object [i.e., "${}^{\dagger}\mathcal{D}^{\circledcirc\pm}$" in the present §6; "${}^{\dagger}\mathcal{D}^{\circledcirc}$" in §4] that appears in the [$\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$- or $\mathcal{D}$-ΘNF-] Hodge theater under consideration [cf. Remark 4.9.2, (ii)].

(ii) At a more concrete level, the "synchronization of labels" discussed in (i) is realized by means of the *crucial bijections*

$$ {}^{\dagger}\zeta_{\circledast} : \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc}) \overset{\sim}{\to} J; \qquad {}^{\dagger}\zeta_{\pm} : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm}) \overset{\sim}{\to} T $$

of Propositions 4.7, (iii); 6.5, (iii). Here, we pause to observe that it is precisely the existence of these

> bijections relating **index sets** *of capsules of* $\mathcal{D}$*-prime-strips* to *sets of* **global** [$\pm$-]**label classes** *of cusps*

	Classical *upper half-plane*	$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters* *in inter-universal* *Teichmüller theory*
Additive **symmetry**	$z \mapsto \quad z + a,$ $z \mapsto -\overline{z} + a \quad (a \in \mathbb{R})$	$\mathbb{F}_l^{\rtimes\pm}$- **symmetry**
"Functions" assoc'd to *add. symm.*	$q \overset{\mathrm{def}}{=} e^{2\pi i z}$	**theta fn.** evaluated at ***l*-tors.** [cf. I, 6.12.6, (ii)]
Basepoint assoc'd to *add. symm.*	*single* cusp at infinity	$\underline{\mathbb{V}}^{\pm}$ [cf. I, 6.1, (v)]
Combinatorial **prototype** assoc'd to *add. symm.*	**cusp**	**cusp**
Multiplicative **symmetry**	$z \mapsto \frac{z\cdot\cos(t)-\sin(t)}{z\cdot\sin(t)+\cos(t)},$ $z \mapsto \frac{\overline{z}\cdot\cos(t)+\sin(t)}{\overline{z}\cdot\sin(t)-\cos(t)} \quad (t \in \mathbb{R})$	$\mathbb{F}_l^{\circledast}$- **symmetry**
"Functions" assoc'd to *mult. symm.*	$w \overset{\mathrm{def}}{=} \frac{z-i}{z+i}$	elements of the **number field** F_{mod} [cf. I, 6.12.6, (iii)]
Basepoints assoc'd to *mult. symm.*	$\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}, \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix}$ $\curvearrowright$ {*entire* boundary of $\mathfrak{H}$ }	$\mathbb{F}_l^{\circledast} \curvearrowright \underline{\mathbb{V}}^{\mathrm{Bor}} = \mathbb{F}_l^{\circledast} \cdot \underline{\mathbb{V}}^{\pm\mathrm{un}}$ [cf. I, 4.3, (i)]
Combinatorial **prototype** assoc'd to *mult. symm.*	**nodes** of mod p Hecke correspondence [cf. II, 4.11.4, (iii), (c)]	**nodes** of mod p Hecke correspondence [cf. II, 4.11.4, (iii), (c)]
Coric symmetries	$z \mapsto z, -\overline{z}$	$\{\pm 1\}$

Fig. 6.4: Comparison of $\mathbb{F}_l^{\rtimes\pm}$-, $\mathbb{F}_l^{\circledast}$-symmetries
with the geometry of the upper half-plane

that distinguishes the finer **"combinatorially holomorphic"** [cf. Remarks 4.9.1, (ii); 4.9.2, (iv)] $\mathbb{F}_l^{\circledast}$- and $\mathbb{F}_l^{\rtimes\pm}$-**symmetries** of Propositions 4.9, (i); 6.8, (i), from the coarser **"combinatorially real analytic"** [cf. Remarks 4.9.1, (ii); 4.9.2, (iv)] $\mathfrak{S}_{l^{\circledast}}$- and $\mathfrak{S}_{l^{\pm}}$-**symmetries** of Propositions 4.9, (ii), (iii); 6.8, (ii), (iii) — i.e., which do *not* admit a *compatible* bijection between the index sets of the capsules involved and some sort of *set of [±-]label classes of cusps* [cf. the discussion of Remark 4.9.2, (i)]. This relationship with a set of [±-]label classes of cusps will play a *crucial role* in the theory of the *Hodge-Arakelov-theoretic evaluation of the étale theta function* that will be developed in [IUTchII].

(iii) On the other hand, one significant feature of the additive theory of the present §6 which does not appear in the multiplicative theory of §4 is the phenomenon of **"global ±-synchronization"** — i.e., at a more concrete level, the various isomorphisms "$^{\dagger}\xi$" that appear in Proposition 6.5, (i), (ii) — between the ±-indeterminacies that occur at the various $\underline{v} \in \underline{\mathbb{V}}$. Note that this global ±-synchronization is a **necessary "pre-condition"** [i.e., since the natural additive action of $\mathbb{F}_l$ on $\mathbb{F}_l$ is *not compatible* with the natural surjection $\mathbb{F}_l \twoheadrightarrow |\mathbb{F}_l|$] for the **additive portion** [i.e., corresponding to $\mathbb{F}_l \subseteq \mathbb{F}_l^{\rtimes\pm}$] of the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* of Proposition 6.8, (i). This "additive portion" of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry plays the *crucial* role of allowing one to relate the *zero* and *nonzero* elements of $\mathbb{F}_l$ [cf. the discussion of Remark 6.12.5 below].

(iv) One important property of both the "$^{\dagger}\zeta$'s" discussed in (ii) and the "$^{\dagger}\xi$'s" discussed in (iii) is that they are constructed by means of **functorial algorithms** from the *intrinsic structure* of a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$- or $\mathcal{D}$-ΘNF-Hodge theater [cf. Propositions 4.7, (iii); 6.5, (i), (ii), (iii)] — i.e., not by means of comparison with some **fixed reference model** [cf. the discussion of [AbsTopIII], §I4], such as the objects constructed in Examples 4.3, 4.4, 4.5, 6.2, 6.3. This property will be of *crucial importance* when, in the theory of [IUTchIII], we combine the theory developed in the present series of papers with the theory of *log-shells* developed in [AbsTopIII].

Remark 6.12.5.

(i) One fundamental difference between the $\mathbb{F}_l^{\circledast}$-symmetry of §4 and the $\mathbb{F}_l^{\rtimes\pm}$-symmetry of the present §6 lies in the *inclusion of the zero element* $\in \mathbb{F}_l$ in the symmetry under consideration. This inclusion of the zero element $\in \mathbb{F}_l$ means, in particular, that the resulting *network of comparison isomorphisms* [cf. Remark 6.12.4, (i)]

> allows one to relate the **"zero-labeled"** *prime-strip* to the various **"nonzero-labeled"** *prime-strips*, i.e., the prime-strips labeled by nonzero elements $\in \mathbb{F}_l$ [or, essentially equivalently, $\in \mathbb{F}_l^{\circledast}$].

Moreover, as reviewed in Remark 6.12.4, (ii), the $\mathbb{F}_l^{\rtimes\pm}$-symmetry allows one to relate the zero-labeled and non-zero-labeled prime-strips to one another in a *"combinatorially holomorphic"* fashion, i.e., in a fashion that is compatible with the various natural bijections [i.e., "$^{\dagger}\zeta$"] with various *sets of global ±-label classes of cusps*. Here, it is useful to recall that *evaluation at* [torsion points closely related to] *the zero-labeled cusps* [cf. the discussion of "evaluation points" in Example 4.4, (i)] plays an important role in the theory of *normalization of the étale theta function*

— cf. the theory of étale theta functions *"of standard type"*, as discussed in [EtTh], Theorem 1.10; the theory to be developed in [IUTchII].

(ii) Whereas the $\mathbb{F}_l^{\rtimes\pm}$-symmetry of the theory of the present §6 has the *advantage* that it allows one to relate zero-labeled and non-zero-labeled prime-strips, it has the [tautological!] *disadvantage* that it does not allow one to **"insulate"** *the non-zero-labeled prime-strips from* **confusion** *with the zero-labeled prime-strip.* This issue will be of substantial importance in the theory of **Gaussian Frobenioids** [to be developed in [IUTchII]], i.e., Frobenioids that, roughly speaking, arise from the *theta values*

$$\{ \underline{\underline{q}}_{\underline{v}}^{\underline{\underline{j}}^2} \}_{\underline{\underline{j}}}$$

[cf. the discussion of Example 4.4, (i)] *at the non-zero-labeled evaluation points.* Moreover, ultimately, in [IUTchII], [IUTchIII], we shall relate these Gaussian Frobenioids to various *global arithmetic line bundles* on the number field F. This will require the use of both the *additive* and the *multiplicative* structures on the number field; in particular, it will require the use of the theory developed in §5.

(iii) By contrast, since, in the theory of the present series of papers, we shall not be interested in analogues of the Gaussian Frobenioids that involve the zero-labeled evaluation points, we shall not require an "additive analogue" of the portion [cf. Example 5.1] of the theory developed in §5 concerning global Frobenioids.

Remark 6.12.6.

(i) Another fundamental difference between the $\mathbb{F}_l^{\circledast}$-symmetry of §4 and the $\mathbb{F}_l^{\rtimes\pm}$-symmetry of the present §6 lies in the **geometric** *nature of the "single basepoint"* [cf. the discussion of Remark 6.12.4] that underlies the $\mathbb{F}_l^{\rtimes\pm}$-symmetry. That is to say, the various *labels* $\in T \xrightarrow{\sim} \mathbb{F}_l$ that appear in a [$\mathcal{D}$-]$\Theta^{\pm\mathrm{ell}}$-Hodge theater correspond — *throughout* the various portions [e.g., bridges] of the [$\mathcal{D}$-]$\Theta^{\pm\mathrm{ell}}$-Hodge theater — to collections of cusps in a **single copy** [i.e., connected component] of "$\mathcal{D}_{\underline{v}}$" at each $\underline{v} \in \underline{\mathbb{V}}$; these collections of cusps are *permuted* by the $\mathbb{F}_l^{\rtimes\pm}$-symmetry of the [$\mathcal{D}$-]$\Theta^{\mathrm{ell}}$-bridge [cf. Proposition 6.8, (i)] *without permuting* the collection of valuations $\underline{\mathbb{V}}^{\pm}$ ($\subseteq \mathbb{V}(K)$) [cf. the discussion of Definition 6.1, (v)]. This contrasts sharply with the **arithmetic** *nature of the "single basepoint"* [cf. the discussion of Remark 6.12.4] that underlies the $\mathbb{F}_l^{\circledast}$-symmetry of §4, i.e., in the sense that the $\mathbb{F}_l^{\circledast}$-symmetry [cf. Proposition 4.9, (i)] *permutes* the various $\mathbb{F}_l^{\circledast}$-translates of $\underline{\mathbb{V}}^{\pm} = \underline{\mathbb{V}}^{\pm\mathrm{un}} \subseteq \underline{\mathbb{V}}^{\mathrm{Bor}}$ ($\subseteq \mathbb{V}(K)$) [cf. Example 4.3, (i); Remark 6.1.1].

(ii) The **geometric** *nature of the "single basepoint"* of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry of a [$\mathcal{D}$-]$\Theta^{\pm\mathrm{ell}}$-Hodge theater [cf. (i)] is more suited to the theory of the

Hodge-Arakelov-theoretic evaluation *of the* **étale theta function**

to be developed in [IUTchII], in which the *existence of a "single basepoint"* corresponding to a single connected component of "$\mathcal{D}_{\underline{v}}$" for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ plays a *central role.*

(iii) By contrast, the **arithmetic** *nature of the "single basepoint"* of the $\mathbb{F}_l^{\circledast}$-symmetry of a [$\mathcal{D}$-]$\Theta$NF-Hodge theater [cf. (i)] is more suited to the

explicit construction *of the* **number field** F_{mod} *[cf. Example 5.1]*

— i.e., to the construction of an object which is invariant with respect to the $\mathrm{Aut}(\underline{C}_K)/\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \xrightarrow{\sim} \mathbb{F}_l^{\circledast}$-symmetries that appear in the discussion of Example 4.3, (iv). That is to say, if one attempts to carry out a similar construction to the construction of Example 5.1 with respect to the copy of $\mathcal{D}^{\circledcirc\pm}$ that appears in a $[\mathcal{D}\text{-}]\Theta^{\mathrm{ell}}$-bridge, then one must *sacrifice the crucial* **ridigity** *with respect to* $\mathrm{Aut}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_l^{\circledast}$ [cf. Definition 6.1, (v)] that arises from the structure [i.e., definition] of a $[\mathcal{D}\text{-}]\Theta^{\mathrm{ell}}$-bridge [cf. Example 6.3; Definition 6.4, (ii)]. Moreover, if one sacrifices this $\mathbb{F}_l^{\circledast}$-rigidity, then one no longer has a situation in which the symmetry under consideration is defined relative to a *single copy of "$\mathcal{D}_{\underline{v}}$"* at each $\underline{v} \in \underline{\mathbb{V}}$, i.e., defined with respect to a *"single geometric basepoint"*. In particular, once one sacrifices this $\mathbb{F}_l^{\circledast}$-rigidity, the resulting symmetries are *no longer compatible* with the theory of the *Hodge-Arakelov-theoretic evaluation of the étale theta function* to be developed in [IUTchII] [cf. (ii)].

(iv) One way to understand the difference discussed in (iii) between the *global portions* [i.e., the portions involving copies of $\mathcal{D}^{\circledcirc}$, $\mathcal{D}^{\circledcirc\pm}$] of a $[\mathcal{D}\text{-}]\Theta$NF-Hodge theater and a $[\mathcal{D}\text{-}]\Theta^{\pm\mathrm{ell}}$-Hodge theater is as a reflection of the fact that whereas the *Borel subgroup*

$$\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \quad \subseteq \quad SL_2(\mathbb{F}_l)$$

is *normally terminal* in $SL_2(\mathbb{F}_l)$ [cf. the discussion of Example 4.3], the *"semi-unipotent" subgroup*

$$\left\{\begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}\right\} \quad \subseteq \quad SL_2(\mathbb{F}_l)$$

[which corresponds to the subgroup $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm})$ — cf. the discussion of Definition 6.1, (v)] *fails to be normally terminal* in $SL_2(\mathbb{F}_l)$.

(v) In summary, taken as a whole, a $[\mathcal{D}\text{-}]\Theta^{\pm\mathrm{ell}}$NF-Hodge theater [cf. Remark 6.12.2, (ii)] may be thought of as a sort of

"intricate relay between geometric and arithmetic basepoints"

that allows one to carry out, in a consistent fashion, both

(a) the theory of the *Hodge-Arakelov-theoretic evaluation of the étale theta function* to be developed in [IUTchII] [cf. (ii)] and
(b) the *explicit construction of the number field* F_{mod} in Example 5.1 [cf. (iii)].

Moreover, if one thinks of $\mathbb{F}_l$ as a *finite approximation of* $\mathbb{Z}$ [cf. Remark 6.12.3], then this intricate relay between geometric and arithmetic — or, alternatively, $\mathbb{F}_l^{\rtimes\pm}$ [i.e., *additive!*]- and $\mathbb{F}_l^{\circledast}$ [i.e., *multiplicative!*]- basepoints — may be thought of as a sort of

global combinatorial resolution of the **two combinatorial dimensions** — i.e., **additive** and **multiplicative** [cf. [AbsTopIII], Remark 5.6.1] — of the **ring** $\mathbb{Z}$.

Finally, we observe in passing that — from a *computational point of view* [cf. the theory of [IUTchIV]!] — it is especially natural to regard $\mathbb{F}_l$ as a "good approximation" of $\mathbb{Z}$ when l is **"sufficiently large"**, as is indeed the case in the situations discussed in [GenEll], §4 [cf. also Remark 3.1.2, (iv)].

$$
\begin{array}{ccccc}
 & \begin{bmatrix} -l^{*} & < \ \dots \ < & -1 & < 0 \\ & < \ 1 \ < & \dots & < \ l^{*} \end{bmatrix} & \{ \underline{\underline{\mathcal{F}}}_{\underline{v}} \}_{\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}} & & \begin{bmatrix} 1 & < \ \dots \\ & < \ l^{*} \end{bmatrix} \\
 & & \ddots & & \\
 & \mathfrak{D}_{\succ} \ = \ /^{\pm} & & & \mathfrak{D}_{>} \ = \ /^{*} \\
 & \Uparrow \ \phi_{\pm}^{\Theta^{\pm}} & \overset{\mathrm{glue}}{\Rightarrow} \quad \{0, \succ\} \ = > \quad \overset{\mathrm{glue}}{\Leftarrow} & & \Uparrow \ \phi_{*}^{\Theta} \\
\overset{\{\pm 1\}}{\curvearrowright} & \begin{pmatrix} -l^{*} & < \ \dots \ < & -1 & < 0 \\ & < \ 1 \ < & \dots & < \ l^{*} \end{pmatrix} & & & \begin{pmatrix} 1 & < \ \dots \\ & < \ l^{*} \end{pmatrix} \\
 & /^{\pm} \ \dots \ /^{\pm} \ /^{\pm} \ /^{\pm} \ \dots \ /^{\pm} & & & /^{*} \ \dots \ /^{*} \\
 & \mathfrak{D}_{T} & & & \mathfrak{D}_{J} \\
 & \Downarrow \ \phi_{\pm}^{\Theta^{\mathrm{ell}}} & & & \Downarrow \ \phi_{*}^{\mathrm{NF}} \\
 & & \mathcal{F}_{\mathrm{mod}}^{\circledast} & & \\
 & \begin{array}{ccc} \pm & \rightarrow & \pm \\ \uparrow & \overset{\mathbb{F}_l^{\rtimes \pm}}{\curvearrowright} & \downarrow \\ \pm & \leftarrow & \pm \end{array} & \begin{array}{ccc} \cap & & \\ \mathcal{F}^{\circledast} & \dashleftarrow & \mathcal{F}^{\circledcirc} \\ & & \ddots \end{array} & & \begin{array}{ccc} * & \rightarrow & * \\ \uparrow & \overset{\mathbb{F}_l^{*}}{\curvearrowright} & \downarrow \\ * & \leftarrow & * \end{array} \\
 & \mathcal{D}^{\circledcirc \pm} \ = \ \mathcal{B}(\underline{X}_K)^0 & & & \mathcal{D}^{\circledcirc} \ = \ \mathcal{B}(\underline{C}_K)^0
\end{array}
$$

Fig. 6.5: The combinatorial structure of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater [cf. also Figs. 4.4, 4.7, 6.1, 6.3, 6.6]

Definition 6.13.

(i) We define a $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater* [relative to the given initial Θ-data]

$$
{}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}
$$

to be a *triple*, consisting of the following data: (a) a $\Theta^{\pm \mathrm{ell}}$-*Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}}$ [cf. Definition 6.11, (iii)]; (b) a ΘNF-*Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}}$ [cf. Definition 5.5, (iii)]; (c) the [necessarily unique!] *gluing isomorphism* between ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}}$ and ${}^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}}$ [cf. the discussion of Remark 6.12.2, (i), (ii)]. An illustration of the combinatorial structure of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater is given in Fig. 6.5 above [cf. also Fig. 6.6 below].

(ii) We define a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* [relative to the given initial Θ-data]

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

to be a *triple*, consisting of the following data: (a) a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-*Hodge theater* $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ [cf. Definition 6.4, (iii)]; (b) a $\mathcal{D}$-ΘNF-*Hodge theater* $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ [cf. Definition 4.6, (iii)]; (c) the [necessarily unique!] *gluing isomorphism* between $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ and $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ [cf. the discussion of Remark 6.12.2, (i), (ii)].

Frobenioid that appears in a $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater*	*Brief description*	*Reference*
Data at $\underline{v} \in \underline{\mathbb{V}}$ of $\mathcal{F}$*-prime-strip* corresponding to each $/^{\pm}$, $/^{\circledast}$	When $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, corresponds to $\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$	I, 5.2, (i)
$\underline{\underline{\mathcal{F}}}_{\underline{v}}$ at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$	*tempered Frobenioid* over the portion of $\mathfrak{D}_{>}$ at $\underline{v}$	I, 5.5, (ii), (iii); I, 3.6, (a); discussion preceding I, 5.4
$\mathcal{F}^{\circledast}_{\mathrm{mod}}$	*[non-realified] global Frobenioid* corresponding to F_{mod}	I, 5.5, (i), (iii); I, 5.1, (iii)
$\mathcal{F}^{\circledast}$	*[non-realified] global Frobenioid* corresponding to $\pi_1(\mathcal{D}^{\circledast}) \curvearrowright \overline{F}$	I, 5.5, (i), (iii); I, 5.1, (ii), (iii)
$\mathcal{F}^{\circledcirc}$	*[non-realified] global Frobenioid* corresponding to $\pi_1(\mathcal{D}^{\circledcirc}) \curvearrowright \overline{F}$	I, 5.5, (i), (iii); I, 5.1, (iii)

Fig. 6.6: The Frobenioids that appear in a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater

Bibliography

[André] Y. André, On a Geometric Description of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and a p-adic Avatar of $\widehat{GT}$, *Duke Math. J.* **119** (2003), pp. 1-39.

[Asada] M. Asada, The faithfulness of the monodromy representations associated with certain families of algebraic curves, *J. Pure Appl. Algebra* **159** (2001), pp. 123-147.

[Falt] G. Faltings, Endlichkeitssätze für Abelschen Varietäten über Zahlkörpern, *Invent. Math.* **73** (1983), pp. 349-366.

[FRS] B. Fine, G. Rosenberger, and M. Stille, Conjugacy pinched and cyclically pinched one-relator groups, *Rev. Mat. Univ. Complut. Madrid* **10** (1997), pp. 207-227.

[Groth] A. Grothendieck, *letter to G. Faltings* (June 1983) in Lochak, L. Schneps, *Geometric Galois Actions; 1. Around Grothendieck's Esquisse d'un Programme*, *London Math. Soc. Lect. Note Ser.* **242**, Cambridge Univ. Press (1997).

[NodNon] Y. Hoshi, S. Mochizuki, On the Combinatorial Anabelian Geometry of Nodally Nondegenerate Outer Representations, *Hiroshima Math. J.* **41** (2011), pp. 275-342.

[CbTpII] Y. Hoshi, S. Mochizuki, *Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves II: Tripods and Combinatorial Cuspidalization*, RIMS Preprint **1762** (November 2012).

[JP] K. Joshi and C. Pauly, Hitchin-Mochizuki morphism, Opers and Frobenius-destabilized vector bundles over curves, *Adv. Math.* **274** (2015), pp. 39-75.

[Kim] M. Kim, The motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel, *Invent. Math.* **161** (2005), pp. 629-656.

[Lang] S. Lang, *Algebraic number theory*, Addison-Wesley Publishing Co. (1970).

[Lehto] O. Lehto, *Univalent Functions and Teichmüller Spaces*, *Graduate Texts in Mathematics* **109**, Springer-Verlag (1987).

[PrfGC] S. Mochizuki, The Profinite Grothendieck Conjecture for Closed Hyperbolic Curves over Number Fields, *J. Math. Sci. Univ. Tokyo* **3** (1996), pp. 571-627.

[pOrd] S. Mochizuki, A Theory of Ordinary p-adic Curves, *Publ. Res. Inst. Math. Sci.* **32** (1996), pp. 957-1151.

[pTeich] S. Mochizuki, *Foundations of p-adic Teichmüller Theory*, AMS/IP Studies in Advanced Mathematics **11**, American Mathematical Society/International Press (1999).

[pGC] S. Mochizuki, The Local Pro-p Anabelian Geometry of Curves, *Invent. Math.* **138** (1999), pp. 319-423.

[HASurI] S. Mochizuki, A Survey of the Hodge-Arakelov Theory of Elliptic Curves I, *Arithmetic Fundamental Groups and Noncommutative Algebra*, *Proceedings of*

Symposia in Pure Mathematics **70**, American Mathematical Society (2002), pp. 533-569.

[HASurII] S. Mochizuki, A Survey of the Hodge-Arakelov Theory of Elliptic Curves II, *Algebraic Geometry 2000, Azumino*, *Adv. Stud. Pure Math.* **36**, Math. Soc. Japan (2002), pp. 81-114.

[AbsAnab] S. Mochizuki, The Absolute Anabelian Geometry of Hyperbolic Curves, *Galois Theory and Modular Forms*, Kluwer Academic Publishers (2004), pp. 77-122.

[CanLift] S. Mochizuki, The Absolute Anabelian Geometry of Canonical Curves, *Kazuya Kato's fiftieth birthday*, *Doc. Math. 2003, Extra Vol.*, pp. 609-640.

[GeoAnbd] S. Mochizuki, The Geometry of Anabelioids, *Publ. Res. Inst. Math. Sci.* **40** (2004), pp. 819-881.

[SemiAnbd] S. Mochizuki, Semi-graphs of Anabelioids, *Publ. Res. Inst. Math. Sci.* **42** (2006), pp. 221-322.

[QuCnf] S. Mochizuki, Conformal and quasiconformal categorical representation of hyperbolic Riemann surfaces, *Hiroshima Math. J.* **36** (2006), pp. 405-441.

[GlSol] S. Mochizuki, Global Solvably Closed Anabelian Geometry, *Math. J. Okayama Univ.* **48** (2006), pp. 57-71.

[CombGC] S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math. J.* **59** (2007), pp. 455-479.

[Cusp] S. Mochizuki, Absolute anabelian cuspidalizations of proper hyperbolic curves, *J. Math. Kyoto Univ.* **47** (2007), pp. 451-539.

[FrdI] S. Mochizuki, The Geometry of Frobenioids I: The General Theory, *Kyushu J. Math.* **62** (2008), pp. 293-400.

[FrdII] S. Mochizuki, The Geometry of Frobenioids II: Poly-Frobenioids, *Kyushu J. Math.* **62** (2008), pp. 401-460.

[EtTh] S. Mochizuki, The Étale Theta Function and its Frobenioid-theoretic Manifestations, *Publ. Res. Inst. Math. Sci.* **45** (2009), pp. 227-349.

[AbsTopI] S. Mochizuki, Topics in Absolute Anabelian Geometry I: Generalities, *J. Math. Sci. Univ. Tokyo* **19** (2012), pp. 139-242.

[AbsTopII] S. Mochizuki, Topics in Absolute Anabelian Geometry II: Decomposition Groups and Endomorphisms, *J. Math. Sci. Univ. Tokyo* **20** (2013), pp. 171-269.

[AbsTopIII] S. Mochizuki, Topics in Absolute Anabelian Geometry III: Global Reconstruction Algorithms, *J. Math. Sci. Univ. Tokyo* **22** (2015), pp. 939-1156.

[GenEll] S. Mochizuki, Arithmetic Elliptic Curves in General Position, *Math. J. Okayama Univ.* **52** (2010), pp. 1-28.

[CombCusp] S. Mochizuki, On the Combinatorial Cuspidalization of Hyperbolic Curves, *Osaka J. Math.* **47** (2010), pp. 651-715.

[IUTchII] S. Mochizuki, *Inter-universal Teichmüller Theory II: Hodge-Arakelov-theoretic Evaluation*, RIMS Preprint **1757** (August 2012).

[IUTchIII] S. Mochizuki, *Inter-universal Teichmüller Theory III: Canonical Splittings of the Log-theta-lattice*, RIMS Preprint **1758** (August 2012).

[IUTchIV] S. Mochizuki, *Inter-universal Teichmüller Theory IV: Log-volume Computations and Set-theoretic Foundations*, RIMS Preprint **1759** (August 2012).

[MNT] S. Mochizuki, H. Nakamura, A. Tamagawa, The Grothendieck conjecture on the fundamental groups of algebraic curves, *Sugaku Expositions* **14** (2001), pp. 31-53.

[Config] S. Mochizuki, A. Tamagawa, The algebraic and anabelian geometry of configuration spaces, *Hokkaido Math. J.* **37** (2008), pp. 75-131.

[NSW] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*, *Grundlehren der Mathematischen Wissenschaften* **323**, Springer-Verlag (2000).

[NS] N. Nikolov and D. Segal, Finite index subgroups in profinite groups, *C. R. Math. Acad. Sci. Paris* **337** (2003), pp. 303-308.

[RZ] Ribes and Zaleskii, *Profinite Groups*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **3**, Springer-Verlag (2000).

[Stb1] P. F. Stebe, A residual property of certain groups, *Proc. Amer. Math. Soc.* **26** (1970), pp. 37-42.

[Stb2] P. F. Stebe, Conjugacy separability of certain Fuchsian groups, *Trans. Amer. Math. Soc.* **163** (1972), pp. 173-188.

[Stl] J. Stillwell, *Classical topology and combinatorial group theory. Second edition*, *Graduate Texts in Mathematics* **72**, Springer-Verlag (1993).

[Tama1] A. Tamagawa, The Grothendieck Conjecture for Affine Curves, *Compositio Math.* **109** (1997), pp. 135-194.

[Tama2] A. Tamagawa, Resolution of nonsingularities of families of curves, *Publ. Res. Inst. Math. Sci.* **40** (2004), pp. 1291-1336.

[Wiles] A. Wiles, Modular elliptic curves and Fermat's last theorem, *Ann. of Math.* **141** (1995), pp. 443-551.

Updated versions of preprints are available at the following webpage:

`http://www.kurims.kyoto-u.ac.jp/~motizuki/papers-english.html`

INTER-UNIVERSAL TEICHMÜLLER THEORY II: HODGE-ARAKELOV-THEORETIC EVALUATION

Shinichi Mochizuki

April 2020

Abstract. In the present paper, which is the second in a series of four papers, we study the **Kummer theory** surrounding the Hodge-Arakelov-theoretic evaluation — i.e., evaluation in the style of the **scheme-theoretic Hodge-Arakelov theory** established by the author in previous papers — of the [reciprocal of the l-th root of the] **theta function** at **l-torsion points** [strictly speaking, shifted by a suitable 2-torsion point], for $l \geq 5$ a prime number. In the first paper of the series, we studied *"miniature models of conventional scheme theory"*, which we referred to as $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters*, that were associated to certain data, called *initial* Θ*-data*, that includes an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$. The underlying Θ-Hodge theaters of these $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters were *glued* to one another by means of *"*Θ*-links"*, that identify the [reciprocal of the l-th root of the] *theta function* at primes of bad reduction of E_F in one $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater with [$2l$-th roots of] the *$q$-parameter* at primes of bad reduction of E_F in another $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater. The theory developed in the present paper allows one to construct certain new versions of this "Θ-link". One such new version is the **$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-link**, which is similar to the Θ-link, but involves the *theta values at l-torsion points*, rather than the theta function itself. One important aspect of the constructions that underlie the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-link is the study of **multiradiality** properties, i.e., properties of the *"arithmetic holomorphic structure"* — or, more concretely, the *ring/scheme* structure — arising from *one* $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater that may be formulated in such a way as to make sense from the point of the arithmetic holomorphic structure of *another* $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater which is related to the original $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater by means of the [*non-scheme-theoretic*!] $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-link. For instance, certain of the various *rigidity* properties of the *étale theta function* studied in an earlier paper by the author may be intepreted as *multiradiality* properties in the context of the theory of the present series of papers. Another important aspect of the constructions that underlie the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-link is the study of **"conjugate synchronization"** via the **$\mathbb{F}_l^{\rtimes \pm}$-symmetry** of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater. Conjugate synchronization refers to a certain system of isomorphisms — which are *free* of any *conjugacy indeterminacies*! — between copies of local absolute Galois groups at the various l-torsion points at which the theta function is evaluated. Conjugate synchronization plays an important role in the Kummer theory surrounding the evaluation of the theta function at l-torsion points and is applied in the study of **coricity** properties of [i.e., the study of objects left *invariant* by] the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-link. Global aspects of conjugate synchronization require the resolution, via results obtained in the first paper of the series, of certain technicalities involving **profinite conjugates** of tempered cuspidal inertia groups.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Contents:

Introduction

In the following discussion, we shall continue to use the notation of the Introduction to the first paper of the present series of papers [cf. [IUTchI], §I1]. In particular, we assume that are given an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$. In the present paper, which forms the second paper of the series, we study the **Kummer theory** surrounding the **Hodge-Arakelov-theoretic evaluation** [cf. Fig. I.1 below] — i.e., evaluation in the style of the *scheme-theoretic Hodge-Arakelov theory* of [HASurI], [HASurII] — of the *reciprocal of the* l*-th root of the* **theta function**

$$\underline{\underline{\Theta}}_{\underline{v}} \quad \stackrel{\mathrm{def}}{=} \quad \left\{ \left(\sqrt{-1} \cdot \sum_{m \in \mathbb{Z}} q_{\underline{v}}^{\frac{1}{2}(m+\frac{1}{2})^2} \right)^{-1} \cdot \left(\sum_{n \in \mathbb{Z}} (-1)^n \cdot q_{\underline{v}}^{\frac{1}{2}(n+\frac{1}{2})^2} \cdot U_{\underline{v}}^{n+\frac{1}{2}} \right) \right\}^{-\frac{1}{l}}$$

[cf. [EtTh], Proposition 1.4; [IUTchI], Example 3.2, (ii)] at ***l*-torsion points** [strictly speaking, shifted by a suitable 2-torsion point] in the context of the theory of $\Theta^{\pm \mathrm{ell}}$**NF-Hodge theaters** developed in [IUTchI]. Here, relative to the notation of [IUTchI], §I1, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$; $q_{\underline{v}}$ denotes the *q-parameter* at $\underline{v}$ of the given *elliptic curve* E_F over a *number field* F; $U_{\underline{v}}$ denotes the standard multiplicative coordinate on the Tate curve obtained by localizing E_F at $\underline{v}$. Let $\underline{\underline{q}}_{\underline{v}}$ be a $2l$-th root of $q_{\underline{v}}$. Then these *"***theta values** *at* l*-torsion points"* will, up to a factor given by a $2l$*-th root of unity*, turn out to be of the form [cf. Remark 2.5.1, (i)]

$$\underline{\underline{q}}_{\underline{v}}^{\,j^2}$$

[*Frobenius-like!*] *Frobenioid-theoretic* **theta function**	**Kummer**	[*étale-like!*] *Galois-theoretic étale* **theta function**
evalu- $\Downarrow$ **ation**	$\curvearrowright$	**evalu-** $\Downarrow$ **ation**
[*Frobenius-like!*] *Frobenioid-theoretic* **theta values**	**Kummer**	[*étale-like!*] *Galois-theoretic* **theta values**

Fig. I.1: The Kummer theory surrounding Hodge-Arakelov-theoretic evaluation

— where $\underline{\underline{j}} \in \{0, 1, \ldots, l^{*} \stackrel{\mathrm{def}}{=} (l-1)/2\}$, so $\underline{\underline{j}}$ is uniquely determined by its image $j \in |\mathbb{F}_l| \stackrel{\mathrm{def}}{=} \mathbb{F}_l/\{\pm 1\} = \{0\} \bigcup \mathbb{F}_l^{*}$ [cf. the notation of [IUTchI], §I1].

In order to understand the significance of *Kummer theory* in the context of Hodge-Arakelov-theoretic evaluation, it is important to recall the notions of *"Frobenius-like"* and *"étale-like"* mathematical structures [cf. the discussion of [IUTchI], §I1]. In the present series of papers, the Frobenius-like structures constituted by [the monoidal portions of] *Frobenioids* — i.e., more concretely, by various *monoids* — play the important role of allowing one to construct **gluing isomorphisms** such as the **Θ-link** which lie **outside** the framework of **conventional scheme/ring theory** [cf. the discussion of [IUTchI], §I2]. Such gluing isomorphisms give rise to *Frobenius-pictures* [cf. the discussion of [IUTchI], §I1]. On the other hand, the étale-like structures constituted by various *Galois* and *arithmetic fundamental groups* give rise to the **canonical splittings** of such Frobenius-pictures furnished by corresponding *étale-pictures* [cf. the discussion of [IUTchI], §I1]. In [IUTchIII], **absolute anabelian geometry** will be applied to these Galois and arithmetic fundamental groups to obtain **descriptions** of **alien arithmetic holomorphic structures**, i.e., arithmetic holomorphic structures that lie on the opposite side of a Θ-link from a given arithmetic holomorphic structure [cf. the discussion of [IUTchI], §I3]. Thus, in light of the *equally crucial* but *substantially different* roles played by Frobenius-like and étale-like structures in the present series of papers, it is of crucial importance to be able

> to **relate** corresponding **Frobenius-like** and **étale-like** versions of various objects to one another.

This is the role played by **Kummer theory**. In particular, in the present paper, we shall study in detail the Kummer theory that relates Frobenius-like and étale-like versions of the **theta function** and its **theta values** at l-torsion points to one another [cf. Fig. I.1].

One important notion in the theory of the present paper is the notion of *multiradiality*. To understand this notion, let us recall the **étale-picture** discussed in [IUTchI], §I1 [cf. [IUTchI], Fig. I1.6]. In the context of the present paper, we shall be especially interested in the *étale-like version* of the *theta function* and its *theta values* constructed in each $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{(-)}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$; thus, one can think of the étale-picture under consideration as consisting of the diagram given in Fig. I.2 below. As discussed earlier, we shall ultimately be interested in applying various *absolute anabelian reconstruction algorithms* to the various arithmetic fundamental groups that [implicitly] appear in such étale-pictures in order to obtain *descriptions of alien holomorphic structures*, i.e., descriptions of objects that arise on one *"spoke"* [i.e., "arrow emanating from the core"] that make sense from the point of view of *another spoke*. In this context, it is natural to classify the various *algorithms* applied to the arithmetic fundamental groups lying in a *given spoke* as follows [cf. Example 1.7]:

· We shall refer to an algorithm as **coric** if it in fact only depends on input data arising from the *mono-analytic core* of the étale-picture, i.e., the data that is *common to all spokes*.

· We shall refer to an algorithm as **uniradial** if it expresses the objects constructed from the given spoke in terms that only make sense within the given spoke.

· We shall refer to an algorithm as **multiradial** if it expresses the objects constructed from the given spoke in terms of *corically constructed* objects, i.e., objects that make sense from the point of view of *other spokes*.

Thus, *multiradial* algorithms are compatible with **simultaneous execution** *at multiple spokes* [cf. Example 1.7, (v); Remark 1.9.1], while *uniradial* algorithms may only be consistently executed at a *single* spoke. Ultimately, in the present series of papers, we shall be interested — relative to the goal of obtaining *"descriptions of alien holomorphic structures"* — in the establishment of *multiradial* algorithms for constructing the *objects of interest*, e.g., [in the context of the present paper] the étale-like versions of the **theta functions** and the corresponding **theta values** discussed above. Typically, in order to obtain such multiradial algorithms, i.e., algorithms that make sense from the point of view of other spokes, it is necessary to allow for some sort of **"indeterminacy"** in the descriptions that appear in the algorithms of the objects constructed from the given spoke.

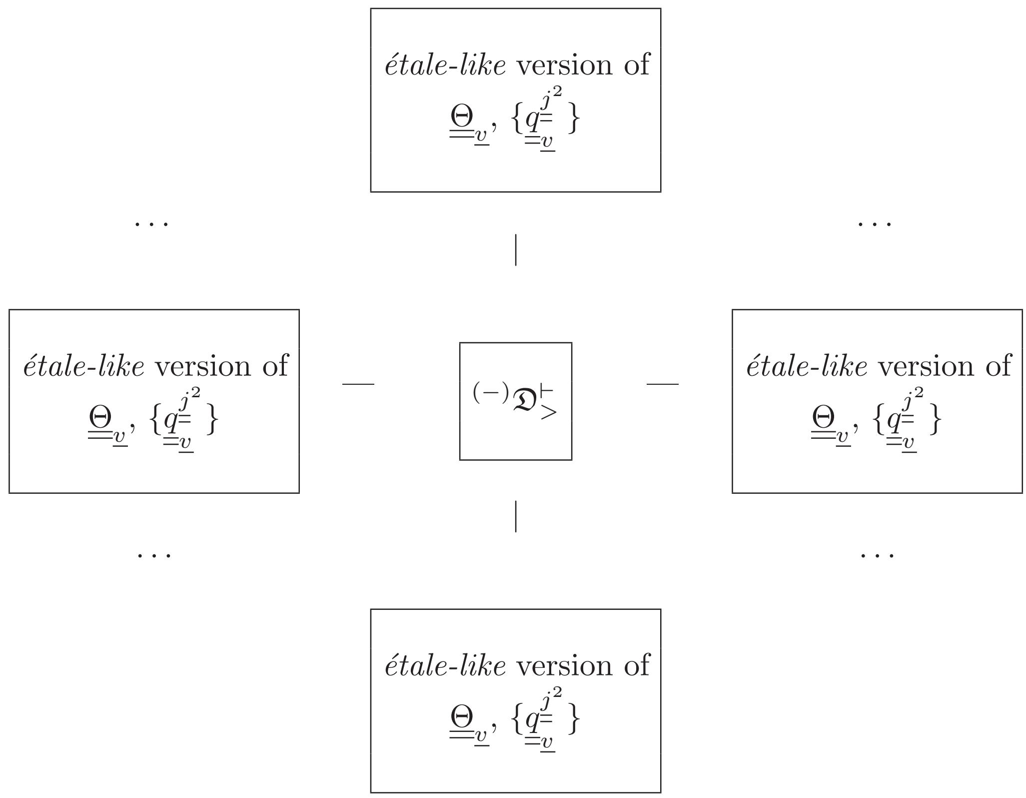

Fig. I.2: Étale-picture of étale-like versions of theta functions, theta values

Relative to the analogy between the inter-universal Teichmüller theory of the present series of papers and the classical theory of **holomorphic** *structures on Riemann surfaces* [cf. the discussion of [IUTchI], §I4], one may think of **coric** algorithms as corresponding to constructions that depend only on the *underlying* **real analytic** *structure* on the Riemann surface. Then **uniradial** algorithms correspond to constructions that *depend*, in an essential way, on the **holomorphic** *structure* of the given Riemann surface, while **multiradial** algorithms correspond to constructions of **holomorphic** objects associated to the Riemann surface which are *expressed* [perhaps by allowing for certain *indeterminacies*!] solely in terms of the *underlying* **real analytic** *structure of the Riemann surface* — cf. Fig. I.3 below; the discussion of Remark 1.9.2. Perhaps the most *fundamental motivating example* in this context is the description of "alien holomorphic structures" by means of the **Teichmüller deformations** reviewed at the beginning of [IUTchI], §I4, relative to **"unspecified/indeterminate"** *deformation data* [i.e., consisting of a nonzero square differential and a dilation factor]. Indeed, for instance, in the case of once-punctured elliptic curves, by applying well-known facts concerning Teichmüller mappings [cf., e.g., [Lehto], Chapter V, Theorem 6.3], it is not difficult to formulate the classical result that

> *"the homotopy class of every orientation-preserving homeomorphism between pointed compact Riemann surfaces of genus one 'lifts' to a unique Teichmüller mapping"*

in terms of the *"multiradial formalism"* discussed in the present paper [cf. Example 1.7]. [We leave the routine details to the reader.]

abstract algorithms	*inter-universal Teichmüller theory*	*classical complex Teichmüller theory*
uniradial algorithms	arithmetic holomorphic structures	holomorphic structures
multiradial algorithms	arithmetic holomorphic structures described in terms of underlying mono-analytic structures	holomorphic structures described in terms of underlying real analytic structures
coric algorithms	underlying mono-analytic structures	underlying real analytic structures

Fig. I.3: Uniradiality, Multiradiality, and Coricity

One interesting aspect of the theory of the present series of papers may be seen in the set-theoretic function arising from the **theta values** considered above

$$\underline{\underline{j}} \quad \mapsto \quad \underline{\underline{q}}_v^{\underline{\underline{j}}^2}$$

— a function that is reminiscent of the **Gaussian distribution** $(\mathbb{R} \ni)\ x \mapsto e^{-x^2}$ on the real line. From this point of view, the passage from the *Frobenius-picture* to the canonical splittings of the *étale-picture* [cf. the discussion of [IUTchI],

§I1], i.e., in effect, the *computation* of the Θ-*links* that occur in the Frobenius-picture by means of the various *multiradial* algorithms that will be established in the present series of papers, may be thought of [cf. the diagram of Fig. I.2!] as a sort of **global arithmetic/Galois-theoretic** analogue of the computation of the **classical Gaussian integral**

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx \;=\; \sqrt{\pi}$$

via the passage from **cartesian** coordinates, i.e., which correspond to the **Frobenius-picture**, to **polar** coordinates, i.e., which correspond to the **étale-picture** — cf. the discussion of Remark 1.12.5.

One way to understand the difference between coricity, multiradiality, and uniradiality at a *purely combinatorial level* is by considering the $\mathbb{F}_l^{\circledast}$**-** and $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** discussed in [IUTchI], §I1 [cf. the discussion of Remark 4.7.4 of the present paper]. Indeed, at a purely combinatorial level, the $\mathbb{F}_l^{\circledast}$-*symmetry* may be thought of as consisting of the natural action of $\mathbb{F}_l^{\circledast}$ on the set of *labels* $|\mathbb{F}_l| = \{0\} \bigcup \mathbb{F}_l^{\circledast}$ [cf. the discussion of [IUTchI], §I1]. Here, the *label* 0 corresponds to the *[mono-analytic] core*. Thus, the corresponding étale-picture consists of various *copies of* $|\mathbb{F}_l|$ glued together along the *coric label* 0 [cf. Fig. I.4 below]. In particular, the various actions of copies of $\mathbb{F}_l^{\circledast}$ on corresponding copies of $|\mathbb{F}_l|$ are *"compatible with simultaneous execution"* in the sense that they *commute with one another*. That is to say, at least at the level of labels, the $\mathbb{F}_l^{\circledast}$**-symmetry** is **multiradial**.

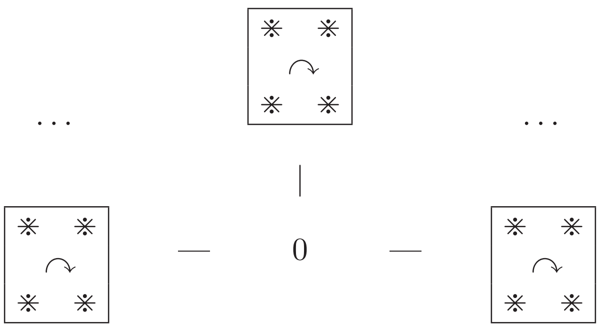

Fig. I.4: Étale-picture of $\mathbb{F}_l^{\circledast}$-symmetries

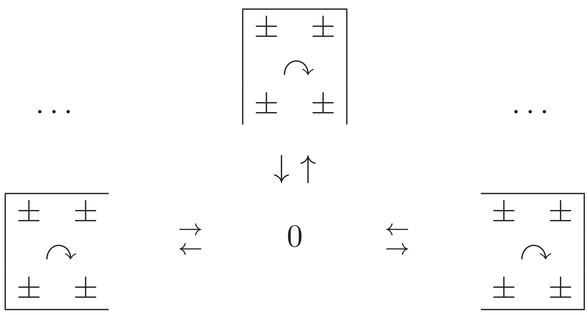

Fig. I.5: Étale-picture of $\mathbb{F}_l^{\rtimes\pm}$-symmetries

In a similar vein, at a purely combinatorial level, the $\mathbb{F}_l^{\rtimes\pm}$*-symmetry* may be thought of as consisting of the natural action of $\mathbb{F}_l^{\rtimes\pm}$ on the set of *labels* $\mathbb{F}_l$ [cf. the discussion of [IUTchI], §I1]. Here again, the *label* 0 corresponds to the *[mono-analytic] core*. Thus, the corresponding étale-picture consists of various *copies of* $\mathbb{F}_l$ glued together along the *coric label* 0 [cf. Fig. I.5 above]. In particular, the various actions of copies of $\mathbb{F}_l^{\rtimes\pm}$ on corresponding copies of $\mathbb{F}_l$ are *"incompatible with simultaneous execution"* in the sense that they clearly *fail to commute with one another*. That is to say, at least at the level of labels, the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** is **uniradial**.

Since, ultimately, in the present series of papers, we shall be interested in the establishment of *multiradial algorithms*, "special care" will be necessary in order to obtain multiradial algorithms for constructing objects related to the *a priori* **uniradial** $\mathbb{F}_l^{\rtimes\pm}$*-symmetry* [cf. the discussion of Remark 4.7.3 of the present paper; [IUTchIII], Remark 3.11.2, (i), (ii)]. The multiradiality of such algorithms will be closely related to the fact that the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is applied to relate the various *copies of* **local units** *modulo torsion, i.e., "*$\mathcal{O}^{\times\boldsymbol{\mu}}$*"* [cf. the notation of [IUTchI], §1] at various labels $\in \mathbb{F}_l$ that lie in various spokes of the étale-picture [cf. the discussion of Remark 4.7.3, (ii)]. This contrasts with the way in which the *a priori* **multiradial** $\mathbb{F}_l^{\circledast}$*-symmetry* will be applied, namely to treat various **"weighted volumes"** corresponding to the *local value groups* and *global realified Frobenioids* at various labels $\in \mathbb{F}_l^{\circledast}$ that lie in various spokes of the étale-picture [cf. the discussion of Remark 4.7.3, (iii)]. Relative to the analogy between the theory of the present series of papers and *p-adic Teichmüller theory* [cf. [IUTchI], §I4], various aspects of the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** are reminiscent of the **additive** *monodromy over the* **ordinary locus** of the canonical curves that occur in p-adic Teichmüller theory; in a similar vein, various aspects of the $\mathbb{F}_l^{\circledast}$**-symmetry** may be thought of as corresponding to the **multiplicative** *monodromy at the* **supersingular points** of the canonical curves that occur in p-adic Teichmüller theory — cf. the discussion of Remark 4.11.4, (iii); Fig. I.7 below.

Before discussing the theory of *multiradiality* in the context of the theory of *Hodge-Arakelov-theoretic evaluation theory* developed in the present paper, we pause to review the theory of **mono-theta environments** developed in [EtTh]. One starts with a *Tate curve* over a mixed-characteristic nonarchimedean local field. The *mono-theta environment* associated to such a curve is, roughly speaking, the *Kummer-theoretic data* that arises by extracting *N-th roots of the* **theta trivialization** of the ample line bundle associated to the origin over suitable tempered coverings of the curve [cf. [EtTh], Definition 2.13, (ii)]. Such mono-theta environments may be constructed *purely group-theoretically* from the *[arithmetic] tempered fundamental group* of the once-punctured elliptic curve determined by the given Tate curve [cf. [EtTh], Corollary 2.18], or, alternatively, *purely category-theoretically* from the *tempered Frobenioid* determined by the theory of line bundles and divisors over tempered coverings of the Tate curve [cf. [EtTh], Theorem 5.10, (iii)]. Indeed, the *isomorphism of mono-theta environments* between the mono-theta environments arising from these two constructions of mono-theta environments — i.e., from tempered fundamental groups, on the one hand, and from tempered Frobenioids, on the other [cf. Proposition 1.2 of the present paper] — may be thought of as a sort of **Kummer isomorphism** for **mono-theta environments** [cf. Proposition 3.4 of the present paper, as well as [IUTchIII], Proposition 2.1, (iii)]. One important consequence of the theory of [EtTh] asserts that mono-theta environments satisfy the following three **rigidity** properties:

(a) **cyclotomic rigidity**,
(b) **discrete rigidity**, and
(c) **constant multiple rigidity**

— cf. the Introduction to [EtTh].

Discrete rigidity assures one that one may work with **$\underline{\mathbb{Z}}$-translates** [where we write $\underline{\mathbb{Z}}$ for the copy of "$\mathbb{Z}$" that acts as a group of covering transformations on the tempered coverings involved], as opposed to $\widehat{\underline{\mathbb{Z}}}$-translates [i.e., where $\widehat{\underline{\mathbb{Z}}} \cong \widehat{\mathbb{Z}}$ denotes the profinite completion of $\underline{\mathbb{Z}}$], of the *theta function*, i.e., one need not contend with *$\widehat{\mathbb{Z}}$-powers of canonical multiplicative coordinates [i.e., "$\mathcal{U}$"], or q-parameters* [cf. Remark 3.6.5, (iii); [IUTchIII], Remark 2.1.1, (v)]. Although we will certainly *"use"* this discrete rigidity throughout the theory of the present series of papers, this property of mono-theta environments will not play a particularly prominent role in the theory of the present series of papers. The $\widehat{\mathbb{Z}}$-powers of "U" and "q" that would occur if one does not have discrete rigidity may be compared to the **PD-formal series** that are obtained, *a priori*, if one attempts to construct the *canonical parameters of p-adic Teichmüller theory* via formal integration. Indeed, PD-formal power series become necessary if one attempts to treat such canonical parameters as objects which admit *arbitrary $\widehat{\mathcal{O}}$-powers*, where $\widehat{\mathcal{O}}$ denotes the completion of the local ring to which the canonical parameter belongs [cf. the discussion of Remark 3.6.5, (iii); Fig. I.6 below].

Constant multiple rigidity plays a somewhat more central role in the present series of papers, in particular in relation to the theory of the **$\mathfrak{log}$-link**, which we shall discuss in [IUTchIII] [cf. the discussion of Remark 1.12.2 of the present paper; [IUTchIII], Remark 1.2.3, (i); [IUTchIII], Proposition 3.5, (ii); [IUTchIII], Remark 3.11.2, (iii)]. Constant multiple rigidity asserts that the multiplicative monoid

$$\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}$$

— which we shall refer to as the **theta monoid** — generated by the *reciprocal of the l-th root of the* **theta function** and the *group of units* of the ring of integers of the base field $\overline{F}_{\underline{v}}$ [cf. the notation of [IUTchI], §I1] admits a **canonical splitting**, up to $2l$-th roots of unity, that arises from **evaluation** at the *[2-]torsion point* corresponding to the **label** $0 \in \mathbb{F}_l$ [cf. Corollary 1.12, (ii); Proposition 3.1, (i); Proposition 3.3, (i)]. Put another way, this canonical splitting is the splitting determined, up to $2l$-th roots of unity, by $\underline{\underline{\Theta}}_{\underline{v}} \in \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}$. The theta monoid of the above display, as well as the associated canonical splitting, may be *constructed algorithmically* from the mono-theta environment [cf. Proposition 3.1, (i)]. Relative to the analogy between the theory of the present series of papers and p-adic Teichmüller theory, these canonical splittings may be thought of as corresponding to the **canonical coordinates** *of p-adic Teichmüller theory*, i.e., more precisely, to the fact that such canonical coordinates are also completely determined *without any constant multiple indeterminacies* — cf. Fig. I.6 below; Remark 3.6.5, (iii); [IUTchIII], Remark 3.12.4, (i).

Mono-theta-theoretic rigidity property in inter-universal Teichmüller theory	*Corresponding phenomenon in p-adic Teichmüller theory*
mono-theta-theoretic **constant multiple** *rigidity*	*lack of constant multiple indeterminacy of* **canonical coordinates** *on canonical curves*
mono-theta-theoretic **cyclotomic** *rigidity*	*lack of* $\widehat{\mathbb{Z}}^\times$*-power indeterminacy of* **canonical coordinates** *on canonical curves,* **Kodaira-Spencer isomorphism**
multiradiality *of mono-theta-theoretic* **constant multiple**, **cyclotomic** *rigidity*	**Frobenius-invariant** *nature of canonical coordinates*
mono-theta-theoretic **discrete** *rigidity*	*formal* = **"non-PD-formal"** *nature of canonical coordinates on canonical curves*

Fig. I.6: Mono-theta-theoretic rigidity properties in inter-universal Teichmüller theory and corresponding phenomena in p-adic Teichmüller theory

Cyclotomic rigidity consists of a *rigidity isomorphism*, which may be *constructed algorithmically* from the mono-theta environment, between

· the portion of the mono-theta environment — which we refer to as the **exterior cyclotome** — that arises from the roots of unity of the *base field* and

· a certain copy of the once-Tate-twisted Galois module "$\widehat{\mathbb{Z}}(1)$" — which we refer to as the **interior cyclotome** — that appears as a subquotient of the *geometric* tempered fundamental group

[cf. Definition 1.1, (ii); Remark 1.1.1; Proposition 1.3, (i)]. This rigidity is *remarkable* — as we shall see in our discussion below of the corresponding *multiradiality* property — in that unlike the *"conventional"* construction of such cyclotomic rigidity isomorphisms via *local class field theory* [cf. Proposition 1.3, (ii)], which requires one to use the *entire* monoid with Galois action $G_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$, the only portion of the monoid $\mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$ that appears in this construction is the portion [i.e., the "exterior cyclotome"] corresponding to the *torsion subgroup* $\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \subseteq \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$ [cf. the notation of [IUTchI], §I1]. This construction depends, in an essential way, on the *commutator structure of theta groups*, but constitutes a somewhat different approach to utilizing this commutator structure from the "classical approach" involving *irreducibility* of representations of theta groups [cf. Remark 3.6.5, (ii); the Introduction to [EtTh]]. One important aspect of this dependence on the commutator structure of the theta group is that the theory of cyclotomic rigidity yields an explanation for the importance of the special role played by the **first power** *of [the reciprocal of the l-th root of] the* **theta function** in the present series of papers [cf. Remark 3.6.4, (iii), (iv), (v); the Introduction to [EtTh]]. Relative to the analogy between the theory of the present series of papers and p-adic Teichmüller theory, mono-theta-theoretic cyclotomic rigidity may be thought of as corresponding either to the fact that the *canonical coordinates of p-adic Teichmüller theory* are completely determined *without any* $\widehat{\mathbb{Z}}^{\times}$*-power indeterminacies* or [roughly equivalently] to the **Kodaira-Spencer isomorphism** of the canonical indigenous bundle — cf. Fig. I.6; Remark 3.6.5, (iii); Remark 4.11.4, (iii), (b).

The **theta monoid**

$$\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}$$

discussed above admits both **étale-like** and **Frobenius-like** [i.e., *Frobenioid-theoretic*] versions, which may be related to one another via a **Kummer isomorphism** [cf. Proposition 3.3, (i)]. The unit portion, together with its natural Galois action, of the Frobenioid-theoretic version of the theta monoid

$$G_{\underline{v}} \curvearrowright \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}$$

forms the portion at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ of the $\mathcal{F}^{\vdash\times}$*-prime-strip* "$\mathfrak{F}^{\vdash\times}_{\mathrm{mod}}$" that is *preserved, up to isomorphism, by the* Θ*-link* [cf. the discussion of [IUTchI], §I1; [IUTchI], Theorem A, (ii)]. In the theory of the present paper, we shall introduce *modified versions of the* Θ*-link of [IUTchI]* [cf. the discussion of the "$\Theta^{\times\boldsymbol{\mu}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links" below], which, unlike the Θ-link of [IUTchI], *only preserve* [up to isomorphism] the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$**-prime-strips** — i.e., which consist of the data

$$G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} = \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}/\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$$

[cf. the notation of [IUTchI], §I1] at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ — associated to the $\mathcal{F}^{\vdash\times}$-prime-strips preserved [up to isomorphism] by the Θ-link of [IUTchI]. Since this data is only preserved up to isomorphism, it follows that the topological group "$G_{\underline{v}}$" must be regarded as being *only known up to isomorphism*, while the monoid $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$ must be regarded as being *only known up to* [the automorphisms of this monoid determined by the natural action of] $\widehat{\mathbb{Z}}^{\times}$. That is to say, one must regard

the data $G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$ as *subject to* $\mathrm{Aut}(G_{\underline{v}})$**-,** $\widehat{\mathbb{Z}}^{\times}$**-indetermnacies**.

These indeterminacies will play an important role in the theory of the present series of papers — cf. the indeterminacies "(Ind1)", "(Ind2)" of [IUTchIII], Theorem 3.11, (i).

Now let us return to our discussion of the various mono-theta-theoretic rigidity properties. The *key observation* concerning these rigidity properties, as reviewed above, in the context of the $\mathrm{Aut}(G_{\underline{v}})$-, $\widehat{\mathbb{Z}}^{\times}$-indeterminacies just discussed, is the following:

> the **canonical splittings**, via *"evaluation at the zero section"*, of the *theta monoids*, together with the construction of the **mono-theta-theoretic cyclotomic rigidity isomorphism**, are **compatible** with, in the sense that they are *left unchanged* by, the $\mathrm{Aut}(G_{\underline{v}})$**-**, $\widehat{\mathbb{Z}}^{\times}$**-indeterminacies** discussed above

— cf. Corollaries 1.10, 1.12; Proposition 3.4, (i). Indeed, this observation constitutes the *substantive content* of the **multiradiality** of mono-theta-theoretic constant multiple/cyclotomic rigidity [cf. Fig. I.6] and will play an important role in the statements and proofs of the *main results* of the present series of papers [cf. [IUTchIII], Theorem 2.2; [IUTchIII], Corollary 2.3; [IUTchIII], Theorem 3.11, (iii), (c); Step (ii) of the proof of [IUTchIII], Corollary 3.12]. At a *technical level*, this "key observation" simply amounts to the observation that the only portion of the monoid $\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}$ that is *relevant* to the construction of the *canonical splittings* and *cyclotomic rigidity isomorphism* under consideration is the *torsion subgroup* $\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$, which [by definition!] *maps to the* **identity** *element* of $\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$, hence is *immune* to the various *indeterminacies* under consideration. That is to say, the multiradiality of mono-theta-theoretic constant multiple/cyclotomic rigidity may be regarded as an essentially formal consequence of the **triviality** of the natural homomorphism

$$\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \quad\to\quad \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$$

[cf. Remark 1.10.2].

After discussing, in §1, the multiradiality theory surrounding the various rigidity properties of the mono-theta environment, we take up the task, in §2 and §3, of establishing the theory of **Hodge-Arakelov-theoretic evaluation**, i.e., of passing [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$]

$$\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}} \quad\rightsquigarrow\quad \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \cdot \{\underline{\underline{q}}^{j^2}_{\underline{v}}\}^{\mathbb{N}}_{\underline{j}=1,\dots,l^*}$$

from **theta monoids** as discussed above [i.e., the monoids on the left-hand side of the above display] to **Gaussian monoids** [i.e., the monoids on the right-hand side of the above display] by means of the operation of *"evaluation" at* ***l*-torsion points**. Just as in the case of theta monoids, Gaussian monoids admit both *étale-like* versions, which constitute the main topic of §2, and *Frobenius-like* [i.e., *Frobenioid-theoretic*] versions, which constitute the main topic of §3. Moreover, as discussed at the beginning of the present Introduction, it is of crucial importance in the theory of the present series of papers to be able to *relate* these *étale-like* and *Frobenius-like* versions to one another via **Kummer theory**. One important observation in this context — which we shall refer to as the *"principle of* **Galois evaluation"** — is the following: it is essentially a *tautology* that

> this requirement of **compatibility** with **Kummer theory** *forces* any sort of *"evaluation operation"* to arise from **restriction** *to* **Galois sections** of the [arithmetic] tempered fundamental groups involved

[i.e., Galois sections of the sort that arise from *rational points* such as l-torsion points!] — cf. the discussion of Remarks 1.12.4, 3.6.2. This tautology is interesting both in light of the history surrounding the *Section Conjecture* in anabelian geometry [cf. [IUTchI], §I5] and in light of the fact that the theory of [SemiAnbd] that is applied to prove [IUTchI], Theorem B — a result which plays an important role in the theory of §2 of the present paper! [cf. the discussion below] — may be thought of as a sort of **"Combinatorial Section Conjecture"**.

At this point, we remark that, unlike the theory of theta monoids discussed above, the theory of Gaussian monoids developed in the present paper does *not*, by itself, admit a *multiradial formulation* [cf. Remarks 2.9.1, (iii); 3.4.1, (ii); 3.7.1]. In order to obtain a multiradial formulation of the theory of Gaussian monoids — which is, in some sense, the *ultimate goal* of the present series of papers! — it will be necessary to *combine* the theory of the present paper with the theory of the $\mathfrak{log}$**-link** developed in [IUTchIII]. This will allow us to obtain a **multiradial formulation** of the theory of **Gaussian monoids** in [IUTchIII], Theorem 3.11.

One important aspect of the theory of Hodge-Arakelov-theoretic evaluation is the notion of **conjugate synchronization**. Conjugate synchronization refers to a collection of *"symmetrizing isomorphisms"* between the various *copies of the local absolute Galois group* $G_{\underline{v}}$ associated to the *labels* $\in \mathbb{F}_l$ at which one evaluates the theta function [cf. Corollaries 3.5, (i); 3.6, (i); 4.5, (iii); 4.6, (iii)]. We shall also use the term "conjugate synchronization" to refer to similar collections of "symmetrizing isomorphisms" for copies of various objects [such as the monoid $\mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$] closely related to the absolute Galois group $G_{\underline{v}}$. With regard to the collections of isomorphisms between copies of $G_{\underline{v}}$, it is of *crucial importance* that these isomorphisms be *completely well-defined*, i.e., *without any* **conjugacy indeterminacies**! Indeed, if one allows conjugacy indeterminacies [i.e., put another way, if one allows oneself to work with *outer* isomorphisms, as opposed to isomorphisms], then one must sacrifice *either*

- the distinct nature of **distinct labels** $\in |\mathbb{F}_l|$ — which is necessary in order to keep track of the *distinct theta values* "$\underline{\underline{q}}^{\underline{\underline{j}}^2}$" for distinct $\underline{\underline{j}}$ — *or*

- the crucial **compatibility** of étale-like and Frobenius-like versions of the symmetrizing isomorphisms with **Kummer theory**

— cf. the discussion of Remark 3.8.3, (ii); [IUTchIII], Remark 1.5.1; Step (vii) of the proof of [IUTchIII], Corollary 3.12. In this context, it is also of interest to observe that it follows from certain *elementary combinatorial considerations* that one must require that

· these symmetrizing isomorphisms arise from a **group action**, i.e., such as the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**

— cf. the discussion of Remark 3.5.2. Moreover, since it will be of crucial importance to apply these symmetrizing isomorphisms, in [IUTchIII], §1 [cf., especially, [IUTchIII], Remark 1.3.2], in the context of the $\mathfrak{log}$**-link** — whose definition depends on the *local* **ring** *structures* at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. the discussion of [AbsTopIII], §I3] — it will be necessary to invoke the fact that

· the symmetrizing isomorphisms at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ arise from conjugation operations within a certain **[arithmetic] tempered fundamental group** — namely, the tempered fundamental group of $\underline{X}_{\underline{v}}$ [cf. the notation of [IUTchI], §I1] — that contains $\Pi_{\underline{v}}$ as an open subgroup of finite index

— cf. the discussion of Remark 3.8.3, (ii). Here, we note that these "conjugation operations" related to the $\mathbb{F}_l^{\rtimes\pm}$-symmetry may be applied to establish *conjugate synchronization* precisely because they arise from conjugation by elements of the *geometric* tempered fundamental group [cf. Remark 3.5.2, (iii)].

The *significance* of establishing **conjugate synchronization** — i.e., subject to the various requirements discussed above! — lies in the fact that the resulting symmetrizing isomorphisms allow one to

construct the crucial **coric** $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$**-prime-strips**

— i.e., the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips that are preserved, up to isomorphism, by the *modified versions of the* Θ*-link of [IUTchI]* [cf. the discussion of the "$\Theta^{\times\boldsymbol{\mu}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links" below] that are introduced in §4 of the present paper [cf. Corollary 4.10, (i), (iv); [IUTchIII], Theorem 1.5, (iii); the discussion of [IUTchIII], Remark 1.5.1, (i)].

In §4, the theory of conjugate synchronization established in §3 [cf. Corollaries 3.5, (i); 3.6, (i)] is extended so as to apply to *arbitrary* $\underline{v} \in \underline{\mathbb{V}}$, i.e., not just $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. Corollaries 4.5, (iii); 4.6, (iii)]. In particular, in order to work with the theta value **labels** $\in \mathbb{F}_l$ in the context of the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, i.e., which involves the action

$$\mathbb{F}_l^{\rtimes\pm} \quad \curvearrowright \quad \mathbb{F}_l$$

on the labels $\in \mathbb{F}_l$, one must avail oneself of the **global portion** *of the* $\Theta^{\pm\mathrm{ell}}$*-Hodge theaters* that appear. Indeed, this global portion allows one to *synchronize* the *a priori* **independent** *indeterminacies* with respect to the action of $\{\pm 1\}$ on the various $\underline{\underline{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], $\underset{\rightarrow}{X}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$] — cf. the discussion of Remark 4.5.3, (iii). On the other hand, the copy of the arithmetic fundamental group of $\underline{X}_K$ that constitutes this global portion of the $\Theta^{\pm\mathrm{ell}}$-Hodge theater is *profinite*, i.e., it does not admit a "globally tempered version" whose localization at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ is naturally isomorphic to the corresponding tempered fundamental group at $\underline{v}$. One important consequence of this state of affairs is that

in order to apply the **global** $\pm$**-synchronization** afforded by the $\Theta^{\pm\mathrm{ell}}$-Hodge theater in the context of the theory of *Hodge-Arakelov-theoretic evaluation* at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ relative to *labels* $\in \mathbb{F}_l$ that correspond to conjugacy classes of cuspidal inertia groups of *tempered* fundamental groups at

> $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, it is necessary to compute the **profinite conjugates** *of such tempered cuspidal inertia groups*

— cf. the discussion of [IUTchI], Remark 4.5.1, as well as Remarks 2.5.2 and 4.5.3, (iii), of the present paper, for more details. This is precisely what is achieved by the application of [IUTchI], Theorem B [i.e., in the form of [IUTchI], Corollary 2.5; cf. also [IUTchI], Remark 2.5.2] in §2 of the present paper.

As discussed above, the theory of Hodge-Arakelov-theoretic evaluation developed in §1, §2, §3 is *strictly local* [at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] in nature. Thus, in §4, we discuss the *essentially routine* extensions of this theory, e.g., of the theory of *Gaussian monoids*, to the "remaining portion" of the $\Theta^{\pm \mathrm{ell}}$-*Hodge theater*, i.e., to $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, as well as to the case of *global realified Frobenioids* [cf. Corollaries 4.5, (iv), (v); 4.6, (iv), (v)]. We also discuss the corresponding complements, involving the theory of [IUTchI], §5, for ΘNF-*Hodge theaters* [cf. Corollaries 4.7, 4.8]. This leads naturally to the construction of *modified versions of the* Θ-*link of [IUTchI]* [cf. Corollary 4.10, (iii)]. These modified versions may be described as follows:

- The $\Theta^{\times\boldsymbol{\mu}}$**-link** is essentially the same as the Θ-link of [IUTchI], Theorem A, except that $\mathcal{F}^{\Vdash}$-prime-strips are replaced by $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strips** [cf. [IUTchI], Fig. I1.2] — i.e., roughly speaking, the various local "$\mathcal{O}^{\times}$" are replaced by "$\mathcal{O}^{\times\boldsymbol{\mu}} = \mathcal{O}^{\times}/\mathcal{O}^{\boldsymbol{\mu}}$".

- The $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$**-link** is essentially the same as the $\Theta^{\times\boldsymbol{\mu}}$-link, except that the theta monoids that give rise to the $\Theta^{\times\boldsymbol{\mu}}$-link are replaced, via composition with a certain isomorphism that arises from *Hodge-Arakelov-theoretic evaluation*, by **Gaussian monoids** [cf. the above discussion!] — i.e., roughly speaking, the various "$\underline{\underline{\Theta}}_{\underline{v}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ are replaced by "$\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{\underline{j}=1,\ldots,l^*}$".

The basic properties of the $\Theta^{\times\boldsymbol{\mu}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links, including the corresponding *Frobenius-* and *étale-pictures*, are summarized in Theorems A, B below [cf. Corollaries 4.10, 4.11 for more details]. Relative to the analogy between the theory of the present series of papers and p-adic Teichmüller theory, the *passage from the* $\Theta^{\times\boldsymbol{\mu}}$-*link to the* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-*link* via **Hodge-Arakelov-theoretic evaluation** may be thought of as corresponding to the passage

$$\mathcal{MF}^{\nabla}\textbf{-objects} \quad \rightsquigarrow \quad \textbf{Galois representations}$$

in the case of the **canonical indigenous bundles** that occur in p-adic Teichmüller theory — cf. the discussion of Remark 4.11.4, (ii), (iii). In particular, the corresponding passage from the *Frobenius-picture* associated to the $\Theta^{\times\boldsymbol{\mu}}$-link to the Frobenius-picture associated to the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link — or, more properly, relative to the point of view of [IUTchIII] [cf. also the discussion of [IUTchI], §I4], from the *log-theta-lattice* arising from the $\Theta^{\times\boldsymbol{\mu}}$-link to the log-theta-lattice arising from the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link — corresponds [i.e.., relative to the analogy with p-adic Teichmüller theory] to the passage

> from thinking of **canonical liftings** as being determined by **canonical** $\mathcal{MF}^{\nabla}$**-objects** to thinking of canonical liftings as being determined by **canonical Galois representations** [cf. Fig. I.7 below].

In this context, it is of interest to note that this point of view is precisely the point of view taken in the absolute anabelian reconstruction theory developed in [CanLift], §3 [cf. Remark 4.11.4, (iii), (a)]. Finally, we observe that from this point of view, the important theory of **conjugate synchronization** *via the* $\mathbb{F}_l^{\rtimes\pm}$*-symmetry* may be thought of as corresponding to the theory of the *deformation of the canonical Galois representation from "mod* p^n*" to "mod* p^{n+1}*"* [cf. Fig. I.7 below; the discussion of Remark 4.11.4, (iii), (d)].

Property related to Hodge-Arakelov-theoretic evaluation in inter-universal Teichmüller theory	*Corresponding phenomenon in p-adic Teichmüller theory*
passage from $\Theta^{\times\boldsymbol{\mu}}$**-link** *to* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$**-link**	*passage from canonicality via* $\mathcal{MF}^{\nabla}$**-objects** *to canonicality via* **crystalline Galois representations**
$\mathbb{F}_l^{\rtimes\pm}$-, $\mathbb{F}_l^{\circledast}$- **symmetries**	**ordinary, supersingular monodromy** *of canonical Galois representation*
conjugate synchronization *via* $\mathbb{F}_l^{\rtimes\pm}$*-symmetry*	**deformation** *of* **canonical Galois representation** *from "mod* p^n*" to "mod* p^{n+1}*"*

Fig. I.7: Properties related to Hodge-Arakelov-theoretic evaluation in inter-universal Teichmüller theory and corresponding phenomena in p-adic Teichmüller theory

Certain aspects of the various constructions discussed above are summarized in the following two results, i.e., *Theorems A, B*, which are abbreviated versions of Corollaries 4.10, 4.11, respectively. On the other hand, many important aspects — such as **multiradiality**! — of these constructions do not appear explicitly in Theorems A, B. The main reason for this is that it is difficult to formulate *"final results"* concerning such aspects as multiradiality in the absence of the framework that is to be developed in [IUTchIII].

Theorem A. **(Frobenius-pictures of** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge Theaters)** *Fix a collection of* **initial** Θ**-data** $(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$ *as in [IUTchI], Definition 3.1. Let* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$; ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *be* $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *[relative to the given initial* Θ*-data] — cf. [IUTchI], Definition 6.13, (i). Write* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$,

$^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *for the associated* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *— cf. [IUTchI], Definition 6.13, (ii). Then:*

(i) **(Constant Prime-Strips)** *By applying the* **symmetrizing isomorphisms***, with respect to the* $\mathbb{F}_l^{\rtimes\pm}$**-symmetry***, of Corollary 4.6, (iii), to the data of the underlying* $\Theta^{\pm\mathrm{ell}}$*-Hodge theater of* $^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *that is labeled by* $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ})$*, one may construct, in a natural fashion, an* $\mathcal{F}^{\Vdash}$**-prime-strip**

$$^{\dagger}\mathfrak{F}^{\Vdash}_{\triangle} \ = \ ({}^{\dagger}\mathcal{C}^{\Vdash}_{\triangle}, \ \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\triangle}) \xrightarrow{\sim} \underline{\mathbb{V}}, \ {}^{\dagger}\mathfrak{F}^{\vdash}_{\triangle}, \ \{{}^{\dagger}\rho_{\triangle,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

that is equipped with a **natural identification isomorphism** *of* $\mathcal{F}^{\Vdash}$*-prime-strips* $^{\dagger}\mathfrak{F}^{\Vdash}_{\triangle} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ *between* $^{\dagger}\mathfrak{F}^{\Vdash}_{\triangle}$ *and the* $\mathcal{F}^{\Vdash}$*-prime-strip* $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ *of [IUTchI], Theorem A, (ii); this isomorphism induces a* **natural identification isomorphism** *of* $\mathcal{D}^{\vdash}$*-prime-strips* $^{\dagger}\mathfrak{D}^{\vdash}_{\triangle} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}^{\vdash}_{>}$ *between the* $\mathcal{D}^{\vdash}$*-prime-strip* $^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}$ *associated to* $^{\dagger}\mathfrak{F}^{\Vdash}_{\triangle}$ *and the* $\mathcal{D}^{\vdash}$*-prime-strip* $^{\dagger}\mathfrak{D}^{\vdash}_{>}$ *of [IUTchI], Theorem A, (iii).*

(ii) **(Theta and Gaussian Prime-Strips)** *By applying the constructions of Corollary 4.6, (iv), (v), to the underlying* Θ*-bridge and* $\Theta^{\pm\mathrm{ell}}$*-Hodge theater of* $^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*, one may construct, in a natural fashion,* $\mathcal{F}^{\Vdash}$**-prime-strips**

$$^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \ = \ ({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{env}}, \ \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{env}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \ {}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}}, \ \{{}^{\dagger}\rho_{\mathrm{env},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

$$^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}} \ = \ ({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{gau}}, \ \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{gau}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \ {}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{gau}}, \ \{{}^{\dagger}\rho_{\mathrm{gau},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

that are equipped with a **natural identification isomorphism** *of* $\mathcal{F}^{\Vdash}$*-prime-strips* $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$ *between* $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}$ *and the* $\mathcal{F}^{\Vdash}$*-prime-strip* $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$ *of [IUTchI], Theorem A, (ii), as well as an* **evaluation isomorphism**

$$^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \quad \xrightarrow{\sim} \quad {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$$

of $\mathcal{F}^{\Vdash}$*-prime-strips.*

(iii) **($\Theta^{\times\boldsymbol{\mu}}$- and $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-Links)** *Write* $^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle}$ *(respectively,* $^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}}$*;* $^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{gau}}$*) for the* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip** *associated to the* $\mathcal{F}^{\Vdash}$*-prime-strip* $^{\ddagger}\mathfrak{F}^{\Vdash}_{\triangle}$ *(respectively,* $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}$*;* $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$*). We shall refer to the full poly-isomorphism* $^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle}$ *as the* $\Theta^{\times\boldsymbol{\mu}}$**-link**

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta^{\times\boldsymbol{\mu}}} \quad {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

[cf. the "Θ*-link" of [IUTchI], Theorem A, (ii)] from* $^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *to* $^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*, and to the full poly-isomorphism* $^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{gau}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle}$ *— which may be regarded as being obtained from the full poly-isomorphism* $^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle}$ *by composition with the inverse of the* **evaluation isomorphism** *of (ii) — as the* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$**-link**

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} \quad {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

from $^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *to* $^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*.*

(iv) **(Coric $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-Prime-Strips)** *The definition of the* **unit portion** *of the* **theta** *and* **Gaussian monoids** *that appear in the construction of the* $\mathcal{F}^{\Vdash}$*-prime-strips* $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}$*,* $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$ *of (ii) gives rise to* **natural isomorphisms**

$$^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \quad \xrightarrow{\sim} \quad {}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}} \quad \xrightarrow{\sim} \quad {}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{gau}}$$

of the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips associated to the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\triangle}$, ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}$, ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$. Moreover, by composing these natural isomorphisms with the poly-isomorphisms induced on the respective $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips by the $\Theta^{\times\boldsymbol{\mu}}$- and $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links of (iii), one obtains a poly-isomorphism

$$ {}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \quad \stackrel{\sim}{\rightarrow} \quad {}^{\ddagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} $$

which coincides with the **full** *poly-isomorphism between these two $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips — that is to say, "${}^{(-)}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}$" is an* **invariant** *of both the $\Theta^{\times\boldsymbol{\mu}}$- and $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links. Finally, this full poly-isomorphism induces the* **full** *poly-isomorphism*

$$ {}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle} \quad \stackrel{\sim}{\rightarrow} \quad {}^{\ddagger}\mathfrak{D}^{\vdash}_{\triangle} $$

between the associated $\mathcal{D}^{\vdash}$-prime-strips; we shall refer to this poly-isomorphism as the **$\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-link** *from ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ to ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.*

(v) **(Frobenius-pictures)** *Let $\{{}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n\in\mathbb{Z}}$ be a* **collection of distinct $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** *indexed by the integers. Then by applying the* **$\Theta^{\times\boldsymbol{\mu}}$- and $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links** *of (iii), we obtain* **infinite chains**

$$ \dots \stackrel{\Theta^{\times\boldsymbol{\mu}}}{\longrightarrow} {}^{(n-1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\Theta^{\times\boldsymbol{\mu}}}{\longrightarrow} {}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\Theta^{\times\boldsymbol{\mu}}}{\longrightarrow} {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\Theta^{\times\boldsymbol{\mu}}}{\longrightarrow} \dots $$

$$ \dots \stackrel{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}}{\longrightarrow} {}^{(n-1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}}{\longrightarrow} {}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}}{\longrightarrow} {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}}{\longrightarrow} \dots $$

of **$\Theta^{\times\boldsymbol{\mu}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-linked $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** *— cf. Fig. I.8 below, in the case of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link. Either of these infinite chains may be represented symbolically as an* **oriented graph $\vec{\Gamma}$**

$$ \dots \quad \rightarrow \quad \bullet \quad \rightarrow \quad \bullet \quad \rightarrow \quad \bullet \quad \rightarrow \quad \dots $$

— i.e., where the arrows correspond to either the "$\stackrel{\Theta^{\times\boldsymbol{\mu}}}{\longrightarrow}$'s" or the "$\stackrel{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}}{\longrightarrow}$'s", and the "$\bullet$'s" correspond to the "${}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$". This oriented graph $\vec{\Gamma}$ admits a natural action by $\mathbb{Z}$ — i.e., a **translation symmetry** *— but it does* **not admit arbitrary permutation symmetries**. *For instance, $\vec{\Gamma}$ does not admit an automorphism that switches two adjacent vertices, but leaves the remaining vertices fixed.*

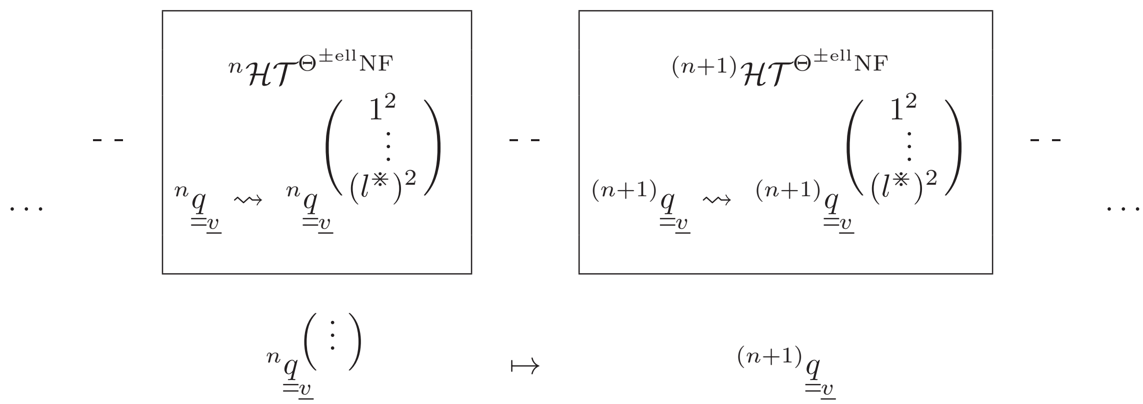

Fig. I.8: Frobenius-picture associated to the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link

Theorem B. (Étale-pictures of Base-$\Theta^{\pm\text{ell}}$NF-Hodge Theaters) *Suppose that we are in the situation of Theorem A, (v).*

(i) Write

$$\dots \xrightarrow{\mathcal{D}} {}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}} \xrightarrow{\mathcal{D}} {}^{(n+1)}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}} \xrightarrow{\mathcal{D}} \dots$$

— where $n \in \mathbb{Z}$ — for the **infinite chain** *of* $\mathcal{D}$**-**$\Theta^{\pm\text{ell}}$**NF-linked** $\mathcal{D}$**-**$\Theta^{\pm\text{ell}}$**NF-Hodge theaters** *[cf. Theorem A, (iv), (v)] induced by either of the infinite chains of Theorem A, (v). Then this infinite chain induces a* **chain of full poly-isomorphisms**

$$\dots \overset{\sim}{\to} {}^{n}\mathfrak{D}^{\vdash}_{\triangle} \overset{\sim}{\to} {}^{(n+1)}\mathfrak{D}^{\vdash}_{\triangle} \overset{\sim}{\to} \dots$$

[cf. Theorem A, (iv)]. That is to say, "${}^{(-)}\mathfrak{D}^{\vdash}_{\triangle}$" forms a **constant invariant** *— i.e., a* **"mono-analytic core"** *[cf. the discussion of [IUTchI], §I1] — of the above infinite chain.*

(ii) If we regard each of the $\mathcal{D}$-$\Theta^{\pm\text{ell}}$NF-Hodge theaters of the chain of (i) as a **spoke** *emanating from the mono-analytic core "${}^{(-)}\mathfrak{D}^{\vdash}_{\triangle}$" discussed in (i), then we obtain a* **diagram** *— i.e., an* **étale-picture of** $\mathcal{D}$**-**$\Theta^{\pm\text{ell}}$**NF-Hodge theaters** *— as in Fig. I.9 below [cf. the situation discussed in [IUTchI], Theorem A, (iii)]. Thus, each spoke may be thought of as a* **distinct "arithmetic holomorphic structure"** *on the mono-analytic core. Finally, [cf. the situation discussed in [IUTchI], Theorem A, (iii)] this diagram satisfies the important property of admitting* **arbitrary permutation symmetries** *among the spokes [i.e., the labels $n \in \mathbb{Z}$ of the $\mathcal{D}$-$\Theta^{\pm\text{ell}}$NF-Hodge theaters].*

(iii) The constructions of (i) and (ii) are **compatible***, in the evident sense, with the constructions of [IUTchI], Theorem A, (iii), relative to the* **natural identification isomorphisms** ${}^{(-)}\mathfrak{D}^{\vdash}_{\triangle} \overset{\sim}{\to} {}^{(-)}\mathfrak{D}^{\vdash}_{>}$ *[cf. Theorem A, (i)].*

$$\begin{array}{ccccc}
 & & \boxed{{}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}}} & & \\
\dots & & & & \dots \\
 & & | & & \\
\boxed{{}^{n'}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}}} & — & \boxed{{}^{(-)}\mathfrak{D}^{\vdash}_{\triangle}} & — & \boxed{{}^{n''}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}}} \\
 & & | & & \\
\dots & & & & \dots \\
 & & \boxed{{}^{n'''}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}}} & &
\end{array}$$

Fig. I.9: Étale-picture of $\mathcal{D}$-$\Theta^{\pm\text{ell}}$NF-Hodge theaters

Acknowledgements:

The research discussed in the present paper profited enormously from the generous support that the author received from the *Research Institute for Mathematical Sciences*, a Joint Usage/Research Center located in Kyoto University. At a personal level, I would like to thank *Fumiharu Kato*, *Akio Tamagawa*, *Go Yamashita*, *Mohamed Saïdi*, *Yuichiro Hoshi*, *Ivan Fesenko*, *Fucheng Tan*, *Emmanuel Lepage*, *Arata Minamide*, and *Wojciech Porowski* for many stimulating discussions concerning the material presented in this paper. Also, I feel deeply indebted to *Go Yamashita*, *Mohamed Saïdi*, and *Yuichiro Hoshi* for their meticulous reading of and numerous comments concerning the present paper. Finally, I would like to express my deep gratitude to *Ivan Fesenko* for his quite substantial efforts to disseminate — for instance, in the form of a survey that he wrote — the theory discussed in the present series of papers.

Notations and Conventions:

We shall continue to use the "Notations and Conventions" of [IUTchI], §0.

Section 1: Multiradial Mono-theta Environments

In the present §1, we review the theory of **mono-theta environments** developed in [EtTh] and give a **"multiradial"** interpretation of this theory, which will be of substantial importance in the present series of papers. Roughly speaking, in the language of [AbsTopIII], §I3, this interpretation consists of the computation of which portion of the various objects constructed from the *"arithmetic holomorphic structures"* of various $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters may be **glued** together, in a fashion consistent with the constructions of the objects of interest, via a **"mono-analytic"** [i.e., "arithmetic real analytic"] **core**. Put another way, this computation may be thought of as the computation of

> what **one** *arithmetic holomorphic structure* looks like from the point of view of a **distinct** *arithmetic holomorphic structure* that is only related to the original arithmetic holomorphic structure via the mono-analytic core.

In fact, this sort of computation forms one of the *central themes* of the present series of papers.

Let $N \in \mathbb{N}_{\geq 1}$ be a positive integer; l an *odd* prime number; k an MLF of *odd residue characteristic* $p \neq l$ that contains a *primitive* $4l$*-th root of unity*; $\overline{k}$ an algebraic closure of k;

$$\underline{\underline{X}}_k$$

a *hyperbolic curve of type* $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$ [cf. [EtTh], Definition 2.5, (i)] over k that admits a *stable model* over the ring of integers $\mathcal{O}_k$ of k; $\underline{\underline{X}}_k \to C_k$ the k*-core* determined by $\underline{\underline{X}}_k$ [cf. the discussion at the beginning of [EtTh], §2]. Write $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ for the *tempered fundamental group* of $\underline{\underline{X}}_k$; $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$; $\Delta^{\mathrm{tp}}_{\underline{\underline{X}}_k} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k} \twoheadrightarrow G_k) \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ for the *geometric tempered fundamental group* of $\underline{\underline{X}}_k$. We shall use similar notation for objects associated to C_k.

Definition 1.1. Let

$$\mathbb{M}^{\Theta}$$

be a *mod* N *mono-theta environment* [cf. [EtTh], Definition 2.13, (ii)] which is isomorphic to the mod N *model* mono-theta environment determined by $\underline{\underline{X}}_k$; write

$$\Pi_{\mathbb{M}^{\Theta}}$$

for the *underlying topological group* of $\mathbb{M}^{\Theta}$ [cf. [EtTh], Definition 2.13, (ii), (a)]. Then:

(i) There exist *functorial algorithms*

$$\mathbb{M}^{\Theta} \mapsto \Pi_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta}); \quad \mathbb{M}^{\Theta} \mapsto \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}); \quad \mathbb{M}^{\Theta} \mapsto G(\mathbb{M}^{\Theta}); \quad \mathbb{M}^{\Theta} \mapsto \Delta_{\mathbb{M}^{\Theta}};$$

$$\mathbb{M}^{\Theta} \mapsto \Delta_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta}); \quad \mathbb{M}^{\Theta} \mapsto \Delta_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}); \quad \mathbb{M}^{\Theta} \mapsto (l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}); \quad \mathbb{M}^{\Theta} \mapsto \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta})$$

for constructing from $\mathbb{M}^{\Theta}$ a quotient $\Pi_{\mathbb{M}^{\Theta}} \twoheadrightarrow \Pi_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta})$ [cf. [EtTh], Corollary 2.18, (iii)]; a topological group $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta})$ which is isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ and contains $\Pi_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta})$ as a normal open subgroup [cf. [EtTh], Corollary 2.18, (iii)]; a quotient $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}) \twoheadrightarrow G(\mathbb{M}^{\Theta})$ corresponding to G_k [cf. [EtTh], Corollary 2.18, (i)], which may also be thought of as a quotient $\Pi_{\mathbb{M}^{\Theta}} \twoheadrightarrow \Pi_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta}) \twoheadrightarrow G(\mathbb{M}^{\Theta})$; a closed normal subgroup $\Delta_{\mathbb{M}^{\Theta}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{\mathbb{M}^{\Theta}} \twoheadrightarrow G(\mathbb{M}^{\Theta})) \subseteq \Pi_{\mathbb{M}^{\Theta}}$; a closed normal subgroup $\Delta_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta}) \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta}) \twoheadrightarrow G(\mathbb{M}^{\Theta})) \subseteq \Pi_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta})$; a closed normal subgroup $\Delta_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}) \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}) \twoheadrightarrow G(\mathbb{M}^{\Theta})) \subseteq \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta})$ corresponding to $\Delta^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ [cf. [EtTh], Corollary 2.18, (i)]; a subquotient $(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta})$ of $\Pi_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta})$ which admits a natural $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta})$-action [hence also a $\Pi_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta})$-action, as well as, by composition, a $\Pi_{\mathbb{M}^{\Theta}}$-action] relative to which it is abstractly isomorphic to $\widehat{\mathbb{Z}}(1)$ [cf. [EtTh], Corollary 2.18, (i)]; a closed normal subgroup $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}) \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{\mathbb{M}^{\Theta}} \twoheadrightarrow \Pi_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta})) \subseteq \Pi_{\mathbb{M}^{\Theta}}$ [cf. [EtTh], Corollary 2.19, (i)] which admits a natural $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta})$-action [hence also a $\Pi_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta})$-action, as well as, by composition, a $\Pi_{\mathbb{M}^{\Theta}}$-action] relative to which it is abstractly isomorphic to $(\mathbb{Z}/N\mathbb{Z})(1)$. Also, we recall that the structure of $\mathbb{M}^{\Theta}$ determines a *lifting* of the natural outer action of

$$(l \cdot \underline{\mathbb{Z}})(\mathbb{M}^{\Theta}) \stackrel{\mathrm{def}}{=} \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta})/\Pi_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta}) \cong \Delta_{\underline{\underline{X}}}(\mathbb{M}^{\Theta})/\Delta_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta})$$

on $\Delta_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta})$ to an outer action of $(l \cdot \underline{\mathbb{Z}})(\mathbb{M}^{\Theta})$ on $\Delta_{\mathbb{M}^{\Theta}}$ [cf. [EtTh], Definition 2.13, (i), (ii), and the preceding discussion; [EtTh], Proposition 2.14, (i)].

(ii) We shall refer to $(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta})$ (respectively, $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta})$) as the **interior** (respectively, **exterior**) **cyclotome** associated to $\mathbb{M}^{\Theta}$. By [EtTh], Corollary 2.19, (i), there is a *functorial algorithm* for constructing from $\mathbb{M}^{\Theta}$ a **cyclotomic rigidity isomorphism**

$$(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}) \otimes (\mathbb{Z}/N\mathbb{Z}) \ \stackrel{\sim}{\to}\ \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta})$$

between the reductions modulo N of the interior and exterior cyclotomes associated to $\mathbb{M}^{\Theta}$.

Remark 1.1.1. In light of its importance in the present series of papers, we pause to review the **mono-theta-theoretic cyclotomic rigidity isomorphism** of Definition 1.1, (ii), in more detail, as follows.

(i) First, we recall from [EtTh], Proposition 2.4 [cf. also the construction of the covering "$Y^{\log} \to X^{\log}$" at the beginning of [EtTh], §1], that the topological group $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta})$ determines topological groups $\Pi_Y(\mathbb{M}^{\Theta})$, $\Pi_{\underline{X}}(\mathbb{M}^{\Theta})$, and $\Pi_C(\mathbb{M}^{\Theta})$ — i.e., corresponding to the coverings "$Y^{\log} \to \underline{X}^{\log} \to C^{\log}$" of the discussion preceding [EtTh], Definition 2.7 — all of which [together with $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta})$] may be regarded as *open subgroups* of $\Pi_C(\mathbb{M}^{\Theta})$

$$\Pi_Y(\mathbb{M}^{\Theta}) \ \subseteq\ \Pi_{\underline{X}}(\mathbb{M}^{\Theta}) \ \subseteq\ \Pi_C(\mathbb{M}^{\Theta}) \ \ (\supseteq\ \Pi_{\underline{X}}(\mathbb{M}^{\Theta}) \ \supseteq\ \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}))$$

that are equipped with compatible surjections to $G(\mathbb{M}^{\Theta})$. Write

$$\Delta_Y(\mathbb{M}^{\Theta}) \ \subseteq\ \Delta_{\underline{X}}(\mathbb{M}^{\Theta}) \ \subseteq\ \Delta_C(\mathbb{M}^{\Theta}) \ \ (\supseteq\ \Delta_{\underline{X}}(\mathbb{M}^{\Theta}) \ \supseteq\ \Delta_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}))$$

for the respective kernels of these surjections. Moreover, the various topological groups of the above two displays are equipped with *subquotients* denoted by means of a *superscript* "Θ" or a *superscript* "ell" [cf. the discussion at the beginning of [EtTh], §1]. These subquotients are completely determined by the topological group structure of $\Pi_C(\mathbb{M}^\Theta)$ [cf. the discussion at the beginning of [EtTh], §1; the proof of [EtTh], Proposition 1.8]. For instance, we observe that one may reconstruct from the topological group $\Pi_{\underline{\underline{X}}}(\mathbb{M}^\Theta)$ [cf. [EtTh], Corollary 2.18, (i)] the *quotient*

$$\Pi_{\mathbb{M}^\Theta} \twoheadrightarrow \Pi_{\underline{\underline{Y}}}(\mathbb{M}^\Theta) \twoheadrightarrow \Pi_Y^{\mathrm{ell}}(\mathbb{M}^\Theta)$$

[which isomorphic to $\widehat{\mathbb{Z}}(1) \rtimes G_k$, relative to the natural cyclotomic action of G_k on $\widehat{\mathbb{Z}}(1)$] corresponding to the quotient "$\underline{\Pi}_Y^{\mathrm{tp}} \twoheadrightarrow (\underline{\Pi}_Y^{\mathrm{tp}})^{\mathrm{ell}}$" of the discussion at the beginning of [EtTh], §1.

(ii) Observe that any closed subgroup $H \subseteq \Pi_{\underline{\underline{Y}}}(\mathbb{M}^\Theta)$ determines, by forming the inverse image via the quotient $\Pi_{\mathbb{M}^\Theta} \twoheadrightarrow \Pi_{\underline{\underline{Y}}}(\mathbb{M}^\Theta)$, a closed subgroup $\Pi_{\mathbb{M}^\Theta}|_H \subseteq \Pi_{\mathbb{M}^\Theta}$. On the other hand, by forming the quotient of $\Pi_{\mathbb{M}^\Theta}$ by the restriction of the *"theta section portion"* of the *mono-theta environment* $\mathbb{M}^\Theta$ [cf. [EtTh], Definition 2.13, (ii), (c)] to the subgroup $\mathrm{Ker}(\Pi_{\underline{\underline{Y}}}(\mathbb{M}^\Theta) \twoheadrightarrow \Pi_{\underline{\underline{Y}}}^\Theta(\mathbb{M}^\Theta)) \subseteq \Pi_{\underline{\underline{Y}}}(\mathbb{M}^\Theta)$, it makes sense to speak of the *quotient of* $\Pi_{\mathbb{M}^\Theta}$

$$(\Pi_{\mathbb{M}^\Theta} \twoheadrightarrow) \quad \Pi_{\mathbb{M}^\Theta}|_{\Pi_{\underline{\underline{Y}}}^\Theta(\mathbb{M}^\Theta)} \quad (\twoheadrightarrow \Pi_{\underline{\underline{Y}}}^\Theta(\mathbb{M}^\Theta))$$

determined by the quotient $\Pi_{\underline{\underline{Y}}}(\mathbb{M}^\Theta) \twoheadrightarrow \Pi_{\underline{\underline{Y}}}^\Theta(\mathbb{M}^\Theta)$ — cf. the discussion at the beginning of the proof of [EtTh], Corollary 2.19, (i). In particular, it makes sense to speak of the *subquotient of* $\Pi_{\mathbb{M}^\Theta}$ determined by any closed subgroup — i.e., such as $(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta) \subseteq \Pi_{\underline{\underline{Y}}}^\Theta(\mathbb{M}^\Theta)$ — of $\Pi_{\underline{\underline{Y}}}^\Theta(\mathbb{M}^\Theta)$.

(iii) In addition to the *subgroup*

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta) \hookrightarrow \Pi_{\mathbb{M}^\Theta}|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)}$$

determined by the subgroup $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta) \subseteq \Pi_{\mathbb{M}^\Theta}$ of Definition 1.1, (i), the *"theta section portion"* of the *mono-theta environment* $\mathbb{M}^\Theta$ [cf. [EtTh], Definition 2.13, (ii), (c)] determines, by restriction, a *subgroup*

$$s^\Theta(\mathbb{M}^\Theta)|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)} \subseteq \Pi_{\mathbb{M}^\Theta}|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)}$$

that maps isomorphically to $(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)$ via the natural projection $\Pi_{\mathbb{M}^\Theta}|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)} \twoheadrightarrow (l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)$ [cf. the proof of [EtTh], Corollary 2.19, (i)]. On the other hand, by considering *liftings* γ of automorphisms of $\Delta_{\underline{\underline{Y}}}(\mathbb{M}^\Theta)$ determined by conjugation by elements of $\Delta_{\underline{\underline{X}}}(\mathbb{M}^\Theta)$ to automorphisms of $\Pi_{\mathbb{M}^\Theta}$ that determine outer automorphisms of the sort that appear in the definition of a *mono-theta environment* [cf. [EtTh], Definition 2.13, (ii), (b)] and then forming the *"commutator $\gamma(\beta) \cdot \beta^{-1}$"* of such liftings with arbitrary elements $\beta \in \Delta_{\underline{\underline{Y}}}(\mathbb{M}^\Theta)$ [cf. [EtTh], Proposition 2.14, (i)], one obtains a natural bilinear **"commutator map"**

$$[-,-]: \ (\Delta_{\underline{\underline{X}}}(\mathbb{M}^\Theta)/\Delta_{\underline{\underline{Y}}}(\mathbb{M}^\Theta)) \times \Delta_{\underline{\underline{Y}}}^{\mathrm{ell}}(\mathbb{M}^\Theta) \to \Pi_{\mathbb{M}^\Theta}|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)}$$

— where we recall that $(l \cdot \underline{\mathbb{Z}}) \stackrel{\sim}{\to} \Delta_{\underline{\underline{X}}}(\mathbb{M}^\Theta)/\Delta_{\underline{\underline{Y}}}(\mathbb{M}^\Theta)$ is abstractly isomorphic to $\mathbb{Z}$, while $\Delta_{\underline{\underline{Y}}}^{\mathrm{ell}}(\mathbb{M}^\Theta)$ is abstractly isomorphic to $\widehat{\mathbb{Z}}$ — whose *image* determines a *subgroup*

$$s^{\mathrm{alg}}(\mathbb{M}^\Theta)|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)} \subseteq \Pi_{\mathbb{M}^\Theta}|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)}$$

that maps isomorphically to $(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)$ via the natural projection $\Pi_{\mathbb{M}^\Theta}|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)} \twoheadrightarrow (l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)$ [cf. the proof of [EtTh], Corollary 2.19, (i)]. The **mono-theta-theoretic cyclotomic rigidity isomorphism** of Definition 1.1, (ii), is then reconstructed [cf. [EtTh], Corollary 2.19, (i)] by forming the *difference* of the two sections $s^\Theta(\mathbb{M}^\Theta)|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)}$, $s^{\mathrm{alg}}(\mathbb{M}^\Theta)|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)}$.

(iv) Next, we observe that the mono-theta-theoretic cyclotomic rigidity isomorphism of Definition 1.1, (ii), admits a certain **symmetry** with respect to the group $\Delta_C(\mathbb{M}^\Theta)/\Delta_{\underline{X}}(\mathbb{M}^\Theta) \cong \mathbb{F}_l^{\rtimes\pm}$ [cf. [IUTchI], Definition 6.1, (v)], as follows. First of all, let us observe that the *natural conjugation action* of $\Pi_{\underline{\underline{Y}}}(\mathbb{M}^\Theta)$ on $\Pi_{\mathbb{M}^\Theta}|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)}$ *factors* through the natural surjection $\Pi_{\underline{\underline{Y}}}(\mathbb{M}^\Theta) \twoheadrightarrow G(\mathbb{M}^\Theta)$. In particular, by applying the natural surjection $\Pi_C(\mathbb{M}^\Theta) \twoheadrightarrow G(\mathbb{M}^\Theta)$, one may regard $\Pi_{\mathbb{M}^\Theta}|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)}$ as being equipped with a *"naively defined" action by* $\Pi_C(\mathbb{M}^\Theta)$. On the other hand, let us recall from the discussion preceding [EtTh], Definition 2.13, that the *"model"* for $\Pi_{\mathbb{M}^\Theta}$ is *originally constructed* as the subgroup

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta) \rtimes \Pi_{\underline{\underline{Y}}}(\mathbb{M}^\Theta) \subseteq \Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta) \rtimes \Pi_C(\mathbb{M}^\Theta)$$

— where the semi-direct products are formed relative to the natural *cyclotomic action* of $\Pi_C(\mathbb{M}^\Theta)$. Here, the evident *subquotient* $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta) \rtimes (l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)$ of $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta) \rtimes \Pi_C(\mathbb{M}^\Theta)$ — i.e., which corresponds to the *subquotient* $\Pi_{\mathbb{M}^\Theta}|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)}$ of $\Pi_{\mathbb{M}^\Theta}$ — is easily verified to be *stabilized* by the action via conjugation of $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta) \rtimes \Pi_C(\mathbb{M}^\Theta)$. Moreover, one verifies easily that this conjugation action of $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta) \rtimes \Pi_C(\mathbb{M}^\Theta)$ *factors* through the natural quotient $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta) \rtimes \Pi_C(\mathbb{M}^\Theta) \twoheadrightarrow \Pi_C(\mathbb{M}^\Theta) \twoheadrightarrow G(\mathbb{M}^\Theta)$ and coincides with the action of $G(\mathbb{M}^\Theta)$ via the *cyclotomic character* $G(\mathbb{M}^\Theta) \to \widehat{\mathbb{Z}}^\times$ on the *abelian profinite group* $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta) \rtimes (l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)$ [where we recall that $\widehat{\mathbb{Z}}^\times$ acts *tautologically* on any abelian profinite group]. That is to say, in summary, even if one is not equipped with the *"model embedding"* $\Pi_{\mathbb{M}^\Theta} \hookrightarrow \Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta) \rtimes \Pi_C(\mathbb{M}^\Theta)$,

> the *"naively defined" action* of $\Pi_C(\mathbb{M}^\Theta)$ on $\Pi_{\mathbb{M}^\Theta}|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)}$ is in fact a *"natural action"* in the sense that it necessarily **coincides** with the natural conjugation action arising from this *"model embedding"*.

Next, let us observe that the *inclusion* $\Delta_{\underline{\underline{X}}}(\mathbb{M}^\Theta) \subseteq \Delta_{\underline{X}}(\mathbb{M}^\Theta)$ induces *natural isomorphisms*

$$\Delta_{\underline{\underline{X}}}(\mathbb{M}^\Theta)/\Delta_{\underline{\underline{Y}}}(\mathbb{M}^\Theta) \xrightarrow{\sim} \Delta_{\underline{X}}(\mathbb{M}^\Theta)/\Delta_Y(\mathbb{M}^\Theta), \quad \Delta^{\mathrm{ell}}_{\underline{\underline{Y}}}(\mathbb{M}^\Theta) \xrightarrow{\sim} \Delta^{\mathrm{ell}}_Y(\mathbb{M}^\Theta)$$

of subquotients of $\Pi_C(\mathbb{M}^\Theta)$, whose *codomains* are [unlike the *domains* of these isomorphisms!] *stabilized* by the conjugation action of $\Pi_C(\mathbb{M}^\Theta)$. In particular, by applying these natural isomorphisms, one may regard the **"commutator map"** of (iii) as a map

$$[-,-]: (\Delta_{\underline{X}}(\mathbb{M}^\Theta)/\Delta_Y(\mathbb{M}^\Theta)) \times \Delta^{\mathrm{ell}}_Y(\mathbb{M}^\Theta) \to \Pi_{\mathbb{M}^\Theta}|_{(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)}$$

— i.e., a map for which both the *domain* and the *codomain* are equipped with **natural actions** by $\Pi_C(\mathbb{M}^\Theta)$. Now one verifies easily that this "commutator map" is **equivariant** with respect to these natural actions by $\Pi_C(\mathbb{M}^\Theta)$, and, moreover, that the various *subgroups* of $\Pi_{\mathbb{M}^\Theta}|_{(l\cdot\Delta_\Theta)(\mathbb{M}^\Theta)}$ constructed in (iii) are **stabilized** by the natural action by $\Pi_C(\mathbb{M}^\Theta)$. In this context, it is also of interest to note that, in fact, it follows immediately from a similar argument to the argument concerning the *automorphisms* of a *mono-theta environment* given in the proof of [EtTh], Corollary 2.18, (iv), that up to *composition with automorphisms* of $\Pi_{\mathbb{M}^\Theta}$ that differ from the *identity* automorphism by a *twisted homomorphism* $\Pi_{\mathbb{M}^\Theta} \twoheadrightarrow \Pi_{\underline{\underline{Y}}}(\mathbb{M}^\Theta) \twoheadrightarrow \Pi_Y^{\mathrm{ell}}(\mathbb{M}^\Theta) \to \Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta)$ that arises from a *Kummer class* of a product of integral powers of "$(\ddot{U})^2$" and "$q_X^{\frac{l}{2}}$" [cf. [EtTh], Proposition 1.4, (ii)] — i.e., automorphisms that have *no effect* on the *construction* of the **"commutator map"** of the above display! — the **"model embedding"** $\Pi_{\mathbb{M}^\Theta} \hookrightarrow \Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta) \rtimes \Pi_C(\mathbb{M}^\Theta)$ may be **reconstructed algorithmically** from the mono-theta environment $\mathbb{M}^\Theta$. Thus, in summary,

> the various constructions discussed in (iii) that underlie the **mono-theta-theoretic cyclotomic rigidity isomorphism** of Definition 1.1, (ii), are **stabilized** by the natural action by $\Pi_C(\mathbb{M}^\Theta)$, hence, in particular, by the natural action by $(\Pi_C(\mathbb{M}^\Theta) \supseteq)\ \Delta_C(\mathbb{M}^\Theta) \twoheadrightarrow \Delta_C(\mathbb{M}^\Theta)/\Delta_{\underline{X}}(\mathbb{M}^\Theta) \cong \mathbb{F}_l^{\rtimes\pm}$.

Here, we remark that the fact that these constructions are stabilized by the action of $\Delta_{\underline{X}}(\mathbb{M}^\Theta)$ is *"less interesting"* in the sense that the automorphisms of $\Pi_{\underline{\underline{X}}}(\mathbb{M}^\Theta)$ that arise from the conjugation action by $\Delta_{\underline{X}}(\mathbb{M}^\Theta)$ *lift* [indeed, *"almost uniquely"*! — cf. [EtTh], Corollary 2.18, (iv)] to automorphisms of $\mathbb{M}^\Theta$, hence stabilize the constructions under consideration as a *consequence* of the *functoriality* of these constructions with respect to *automorphisms* [cf. [EtTh], Corollary 2.19, (i)]. It is for this reason that, in the present context, it is natural to regard the *symmetry properties of interest* as being symmetries with respect to the *quotient* $\Delta_C(\mathbb{M}^\Theta) \twoheadrightarrow \Delta_C(\mathbb{M}^\Theta)/\Delta_{\underline{X}}(\mathbb{M}^\Theta) \cong \mathbb{F}_l^{\rtimes\pm}$. On the other hand, the approach of the above discussion via *model embeddings* to this **full symmetry** with respect to $\mathbb{F}_l^{\rtimes\pm}$ may also be regarded as being simply an *explicit computation*, in the case of this $\mathbb{F}_l^{\rtimes\pm}$*-symmetry*, of the **functoriality** of the constructions under consideration with respect to **isomorphisms** [cf. [EtTh], Corollary 2.19, (i)].

(v) In the context of the discussion following the final display of (iv), it is perhaps of interest to recall that the **symmetries** of **mono-theta environments** relative to the conjugation action by $\Delta_{\underline{X}}(\mathbb{M}^\Theta)$ are a consequence of the *"shifting automorphisms"* discussed in [EtTh], Proposition 2.14, (ii) [cf. the discussion of [EtTh], Remark 2.14.3]. That is to say, despite the fact that *the meromorphic function constituted by the* **theta function** *does* **not** *admit such symmetries*, the corresponding **mono-theta** environment *does* admit such symmetries. This is one important difference between the theory of *mono-theta* environments and the theory of *bi-theta environments* [cf. the discussion of [EtTh], Remark 2.14.3]. Alternatively, the existence of such symmetries may be regarded as

> one of the **fundamental** differences between the **mono-theta-theoretic** approach to **cyclotomic rigidity** taken in [EtTh] and the approach to cyclotomic rigidity taken in [IUTchI], Example 5.1, (v), via **Kummer classes** of **rational functions**.

Put another way, this fundamental difference may be thought of as the difference between constructing a cyclotomic rigidity isomorphism from a **line bundle** —

i.e., which, in general, admits *more symmetries* than a rational function — and constructing a cyclotomic rigidity isomorphism from a **rational function**. On the other hand, if one attempts to mimick the approach of [EtTh] [i.e., of constructing *"shifting automorphisms"* as in [EtTh], Proposition 2.14, (ii)] in the case of symmetries with respect to the *quotient* $\Delta_C(\mathbb{M}^\Theta) \twoheadrightarrow \Delta_C(\mathbb{M}^\Theta)/\Delta_{\underline{X}}(\mathbb{M}^\Theta) \cong \mathbb{F}_l^{\rtimes\pm}$, then it is necessary to allow *"denominators of the form $\frac{1}{l}$"* when one works with the module $\Pi_{\mathbb{M}^\Theta}|_{(l\cdot\Delta_\Theta)(\mathbb{M}^\Theta)}$. In fact, however, when one computes the *commutator map* $[-,-]$ considered in (iv), such terms with denominators *vanish*, as a consequence of the fact that $\Pi_{\mathbb{M}^\Theta}|_{(l\cdot\Delta_\Theta)(\mathbb{M}^\Theta)}$ *commutes* with the elements of interest in the computation of this commutator map. It is precisely this state of affairs that allows one to construct an $\mathbb{F}_l^{\rtimes\pm}$**-symmetric cyclotomic rigidity isomorphism** as discussed in (iv), that is to say, which, by itself, is somewhat *weaker* than the "full mono-theta environment" [i.e., which does *not* admit $\mathbb{F}_l^{\rtimes\pm}$*-symmetries* unless one allows for *denominators* as discussed above!]. Thus, in summary, by comparison to the approach to cyclotomic rigidity taken in [EtTh], the slightly weaker approach discussed in (iv) may be thought of as corresponding to the difference between constructing a cyclotomic rigidity isomorphism from a **line bundle** and constructing a cyclotomic rigidity isomorphism from the **curvature**, or **first Chern class**, of the line bundle [cf. the discussion of Remark 3.6.5 below].

One key property of mono-theta environments is that they may be constructed either *group-theoretically* from $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ or *category-theoretically* from certain tempered Frobenioids related to $\underline{\underline{X}}_k$.

Proposition 1.2. (Group- and Frobenioid-theoretic Constructions of Mono-theta Environments)

(i) Let Π be a **topological group** *isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$. Then there exists a* **functorial group-theoretic algorithm**

$$\Pi \mapsto \mathbb{M}^\Theta(\Pi)$$

for constructing from the topological group Π a **mod N mono-theta environment "up to isomorphism"** *[cf. [EtTh], Corollary 2.18, (ii)] such that the composite of this algorithm with the algorithm $\mathbb{M}^\Theta(\Pi) \mapsto \Pi_{\underline{\underline{X}}}(\mathbb{M}^\Theta(\Pi))$ discussed in Definition 1.1, (i), admits a* **functorial isomorphism** *$\Pi \xrightarrow{\sim} \Pi_{\underline{\underline{X}}}(\mathbb{M}^\Theta(\Pi))$. Here, the "isomorphism indeterminacy" of $\mathbb{M}^\Theta(\Pi)$ is with respect to a group of "$\boldsymbol{\mu}_N$-conjugacy classes" of automorphisms which is of order 1 (respectively, 2) if N is* **odd** *(respectively,* **even***) [cf. [EtTh], Corollary 2.18, (iv)].*

(ii) Let $\mathcal{C}$ be a **category** *equivalent to the* **tempered Frobenioid** *determined by $\underline{\underline{X}}_k$ [i.e., the Frobenioid denoted "$\mathcal{C}$" in the discussion at the beginning of [EtTh], §5; the Frobenioid denoted "$\underline{\underline{\mathcal{F}}}_v$" in the discussion of [IUTchI], Example 3.2, (i)]. Thus, $\mathcal{C}$ admits a natural Frobenioid structure over a* **base category** *$\mathcal{D}$ equivalent to $\mathcal{B}^{\mathrm{temp}}(\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k})^0$ [cf. [FrdI], Corollary 4.11, (ii), (iv); [EtTh], Proposition 5.1]. Then there exists a* **functorial algorithm**

$$\mathcal{C} \mapsto \mathbb{M}^\Theta(\mathcal{C})$$

for constructing from the category $\mathcal{C}$ *a* **mod** N **mono-theta environment** *[cf. [EtTh], Theorem 5.10, (iii)] such that the composite of this algorithm with the algorithm* $\mathbb{M}^{\Theta}(\mathcal{C}) \mapsto \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}(\mathcal{C}))$ *discussed in Definition 1.1, (i), admits a* **functorial isomorphism** $\mathcal{D} \xrightarrow{\sim} \mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}(\mathcal{C})))^{0}$.

Proof. The assertions of Proposition 1.2 follow immediately from the results of [EtTh] that are quoted in the statements of these assertions. ◯

The *cyclotomic rigidity isomorphism* of Definition 1.1, (ii), that arises in the case of the mono-theta environment $\mathbb{M}^{\Theta}(\mathcal{C})$ constructed from the tempered Frobenioid $\mathcal{C}$ [cf. Proposition 1.2, (ii)] is *compatible* with a certain cyclotomic rigidity isomorphism that arises in the theory of [AbsTopIII] [cf. also [FrdII], Theorem 2.4, (ii)] in the following sense.

Proposition 1.3. (Compatibility of Cyclotomic Rigidity Isomorphisms) *In the situation of Proposition 1.2, (ii):*

(i) **(Mono-theta Environments Associated to Tempered Frobenioids)** *For a suitable object* $S \in \mathrm{Ob}(\mathcal{C})$ *[cf. [EtTh], Lemma 5.9, (v)], whose image in* $\mathcal{D}$ *we denote by* $S^{\mathrm{bs}} \in \mathrm{Ob}(\mathcal{D})$*, the* **interior cyclotome** $(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}(\mathcal{C})) \otimes (\mathbb{Z}/N\mathbb{Z})$ *corresponds to a certain* **subquotient** *of* $\mathrm{Aut}(S^{\mathrm{bs}})$*, which we denote by* $(l \cdot \Delta_{\Theta})_S \otimes (\mathbb{Z}/N\mathbb{Z})$*, while the* **exterior cyclotome** $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}(\mathcal{C}))$ *corresponds to the subgroup* $\boldsymbol{\mu}_N(S) \subseteq \mathcal{O}^{\times}(S) \subseteq \mathrm{Aut}(S)$*. In particular, the* **cyclotomic rigidity isomorphism** *of Definition 1.1, (ii), takes the form of an isomorphism*

$$(l \cdot \Delta_{\Theta})_S \otimes (\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} \boldsymbol{\mu}_N(S) \qquad (*^{\text{mono-}\Theta})$$

[cf. [EtTh], Proposition 5.5; [EtTh], Lemma 5.9, (v)].

(ii) **(MLF-Galois Pairs)** *Relative to the formal correspondence between p-adic Frobenioids [such as the* **base-field-theoretic hull** $\mathcal{C}^{\text{bs-fld}}$ *associated to* $\mathcal{C}$ *— cf. [EtTh], Definition 3.6, (iv)] and* **"MLF-Galois** $\mathbb{TM}$**-pairs"** *in the theory of [AbsTopIII] [cf. [AbsTopIII], Remark 3.1.1], "*$\boldsymbol{\mu}_N(S)$*" [cf. (i)] corresponds to "*$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(M_{\mathbb{TM}}) \otimes (\mathbb{Z}/N\mathbb{Z})$*" in the theory of [AbsTopIII], §3 [cf. [AbsTopIII], Definition 3.1, (v)], while "*$(l \cdot \Delta_{\Theta})_S \otimes (\mathbb{Z}/N\mathbb{Z})$*" [cf. (i)] corresponds to "*$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_X) \otimes (\mathbb{Z}/N\mathbb{Z})$*" in the theory of [AbsTopIII], §1 [cf. [AbsTopIII], Theorem 1.9, (b); [AbsTopIII], Remark 1.10.1, (ii); [IUTchI], Remark 3.1.2, (iii)]. In particular, by composing the inverse of the natural isomorphism "*$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_X)$*" of [AbsTopIII], Corollary 1.10, (c), with the inverse of the natural isomorphism "*$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(M_{\mathbb{TM}}) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G)$*" of [AbsTopIII], Remark 3.2.1, we obtain* **another cyclotomic rigidity isomorphism**

$$(l \cdot \Delta_{\Theta})_S \otimes (\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} \boldsymbol{\mu}_N(S) \qquad (*^{\text{bs-Gal}})$$

[cf. the various identifications/correspondences of notation discussed above].

(iii) **(Compatibility)** *The cyclotomic rigidity isomorphisms* $(*^{\text{mono-}\Theta})$*,* $(*^{\text{bs-Gal}})$ *of [EtTh], [AbsTopIII] [cf. (i), (ii)]* **coincide***.*

Proof. Assertions (i), (ii) follow immediately from the results and definitions of [EtTh], [AbsTopIII] that are quoted in the statements of these assertions. Assertion (iii) follows immediately from the fact that in the situation where the Frobenioid $\mathcal{C}$ involved is not just "some abstract category", but rather arises from familiar objects of scheme theory [cf. the theory of [EtTh], §1!], both isomorphisms ($*^{\text{mono-}\Theta}$), ($*^{\text{bs-Gal}}$) *coincide with the conventional identification* between the cyclotomes involved that arises from *conventional scheme theory.* ◯

Proposition 1.4. **(Étale Theta Functions of Standard Type)** *Let* Π *be as in Proposition 1.2, (i). Then there are* **functorial group-theoretic algorithms** *[cf. [EtTh], Corollary 2.18, (i)]*

$$\Pi \mapsto \Pi_{\underline{\underline{\ddot{Y}}}}(\Pi); \quad \Pi \mapsto (l \cdot \Delta_\Theta)(\Pi)$$

for constructing from Π *the open subgroup* $\Pi_{\underline{\underline{\ddot{Y}}}}(\Pi) \subseteq \Pi$ *corresponding to the tempered covering "*$\underline{\underline{\ddot{Y}}}$*" [cf. the discussion preceding [EtTh], Definition 2.7] and a certain subquotient* $(l \cdot \Delta_\Theta)(\Pi)$ *of* Π *[cf. the subquotient "*$(l \cdot \Delta_\Theta)(\mathbb{M}^\Theta)$*" of Definition 1.1, (i)], as well as a functorial group-theoretic algorithm*

$$\Pi \quad \mapsto \quad \underline{\underline{\theta}}(\Pi) \quad \subseteq \quad H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\Pi), (l \cdot \Delta_\Theta)(\Pi))$$

— cf. the **constant multiple rigidity** *property of [EtTh], Corollary 2.19, (iii) — for constructing from* Π *the set* $\underline{\underline{\theta}}(\Pi)$ *of* $\boldsymbol{\mu}_l$*-multiples [i.e., where* $\boldsymbol{\mu}_l$ *denotes the group of* l*-th roots of unity] of the reciprocal of the "*$(l \cdot \underline{\mathbb{Z}} \times \boldsymbol{\mu}_2)$*-orbit* $\underline{\underline{\ddot{\eta}}}^{\Theta, l \cdot \underline{\mathbb{Z}} \times \boldsymbol{\mu}_2}$ *of an* l*-th root of the* **étale theta function of standard type***" of [EtTh], Definition 2.7. In this context, we shall write*

$$_{\infty}\underline{\underline{\theta}}(\Pi) \quad \subseteq \quad \varinjlim_J \; H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\Pi)|_J, (l \cdot \Delta_\Theta)(\Pi))$$

— where $_{\infty}\underline{\underline{\theta}}(\Pi)$ *denotes the subset of elements of the direct limit of cohomology modules in the display for which some [positive integer]* **multiple** *[i.e., some [positive integer]* **power***, if one writes these modules "multiplicatively"] coincides,* **up to torsion***, with an element of* $\underline{\underline{\theta}}(\Pi)$*;* J *ranges over the finite index open subgroups of* Π*; the notation "*$|_J$*" denotes the fiber product "*$\times_\Pi J$*".*

Proof. The assertions of Proposition 1.4 follow immediately from the results and definitions of [EtTh] that are quoted in the statements of these assertions. ◯

Remark 1.4.1. Before proceeding, let us recall from [EtTh], §1, §2, the theory surrounding the *"étale theta functions of standard type"* that appeared in Proposition 1.4.

(i) Write

$$\underline{\underline{X}}_k \to \underline{X}_k \to \underline{C}_k$$

for the *hyperbolic orbicurves of type* $(1, l\text{-tors})$, $(1, l\text{-tors})_\pm$ determined by $\underline{\underline{X}}_k$ [cf. [EtTh], Proposition 2.4]. Thus, $\underline{X}_k$ has a unique *zero cusp* [i.e., the unique cusp fixed by the action of the Galois group $\mathrm{Gal}(\underline{X}_k/\underline{C}_k)$]. Write

$$\mu_- \in \underline{X}_k(k)$$

for the *unique torsion point of order* 2 whose closure in any stable model of $\underline{\underline{X}}_k$ over $\mathcal{O}_k$ intersects the same irreducible component of the special fiber of the stable model as the zero cusp [cf. the discussion of [IUTchI], Example 4.4, (i)].

(ii) The *unique order two automorphism* $\iota_{\underline{\underline{X}}}$ of $\underline{\underline{X}}_k$ over k [cf. [EtTh], Remark 2.6.1] lies over an *order two automorphism* $\iota_{\underline{X}}$ [cf. [EtTh], Remark 2.6.1] and corresponds at the level of *tempered fundamental groups* [cf., e.g., [SemiAnbd], Theorem 6.4] to the unique order two $\Delta^{\mathrm{tp}}_{\underline{\underline{X}}_k}$-outer automorphism of $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ over G_k, which, by abuse of notation, we shall also denote by $\iota_{\underline{\underline{X}}}$. Write

$$\underline{\underline{\ddot{Y}}}_k \to \underline{\underline{Y}}_k \to \underline{\underline{X}}_k$$

for the tempered coverings of $\underline{\underline{X}}_k$ that correspond, respectively, to the open subgroups $\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}_k} \stackrel{\mathrm{def}}{=} \Pi_{\underline{\underline{\ddot{Y}}}}(\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}) \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ [cf. Proposition 1.4], $\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}_k} \stackrel{\mathrm{def}}{=} \Pi_{\underline{\underline{Y}}}(\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}) \stackrel{\mathrm{def}}{=} \Pi_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta}(\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k})) \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ [cf. Definition 1.1, (i); Proposition 1.2, (i)]. Since k contains a *primitive* $4l$*-th root of unity*, it follows from the definition of an "étale theta function of standard type" [cf. [EtTh], Definition 1.9, (ii); [EtTh], Definition 2.7] that *there exist rational points*

$$(\mu_-)_{\underline{\underline{\ddot{Y}}}} \in \underline{\underline{\ddot{Y}}}_k(k), \quad (\mu_-)_{\underline{\underline{X}}} \in \underline{\underline{X}}_k(k)$$

such that $(\mu_-)_{\underline{\underline{\ddot{Y}}}} \mapsto (\mu_-)_{\underline{\underline{X}}} \mapsto \mu_-$. Since $\iota_{\underline{X}}$ *fixes* μ_-, it follows immediately that $\iota_{\underline{\underline{X}}}$ *fixes the* $\mathrm{Gal}(\underline{\underline{X}}_k/\underline{X}_k)$*-orbit* of $(\mu_-)_{\underline{\underline{X}}}$, hence [since $\mathrm{Aut}(\underline{\underline{X}}_k) \cong \mathbb{Z}/2l\mathbb{Z}$, where we recall that $l \neq 2$ — cf. [EtTh], Remark 2.6.1] that $\iota_{\underline{\underline{X}}}$ *fixes* $(\mu_-)_{\underline{\underline{X}}}$. One verifies immediately that this implies that there exists an *order two automorphism* $\iota_{\underline{\underline{\ddot{Y}}}}$ of $\underline{\underline{\ddot{Y}}}_k$ lifting $\iota_{\underline{\underline{X}}}$ which is *uniquely determined up to* $l \cdot \underline{\mathbb{Z}}$*-conjugacy and composition with an element* $\in \mathrm{Gal}(\underline{\underline{\ddot{Y}}}_k/\underline{\underline{Y}}_k)$ by the condition that it *fix* the $\mathrm{Gal}(\underline{\underline{\ddot{Y}}}_k/\underline{\underline{Y}}_k)$-orbit of *some element* [which, by abuse of notation, we shall continue to denote by "$(\mu_-)_{\underline{\underline{\ddot{Y}}}}$"] of the $\mathrm{Gal}(\underline{\underline{\ddot{Y}}}_k/\underline{\underline{X}}_k)$*-orbit* of $(\mu_-)_{\underline{\underline{\ddot{Y}}}}$. Here, we think of $l \cdot \underline{\mathbb{Z}}$, $\mathrm{Gal}(\underline{\underline{\ddot{Y}}}_k/\underline{\underline{Y}}_k)$ $(\cong \mathbb{Z}/2\mathbb{Z})$ as the subquotients appearing in the *natural exact sequence*

$$1 \to \mathrm{Gal}(\underline{\underline{\ddot{Y}}}_k/\underline{\underline{Y}}_k) \to \mathrm{Gal}(\underline{\underline{\ddot{Y}}}_k/\underline{\underline{X}}_k) \to l \cdot \underline{\mathbb{Z}} \to 1$$

determined by the coverings $\underline{\underline{\ddot{Y}}}_k \to \underline{\underline{Y}}_k \to \underline{\underline{X}}_k$. Again, by abuse of notation, we shall also denote by $\iota_{\underline{\underline{\ddot{Y}}}}$ the corresponding $\Delta^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}_k}$ $(= \Delta^{\mathrm{tp}}_{\underline{\underline{X}}_k} \bigcap \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}_k})$-outer automorphism of $\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}_k}$. We shall refer to the various automorphisms $\iota_{\underline{\underline{X}}}$, $\iota_{\underline{\underline{\ddot{Y}}}}$ as **inversion automorphisms** [cf. [EtTh], Proposition 1.5, (iii)]. Write

$$D_{\mu_-} \subseteq \Pi_{\underline{\underline{\ddot{Y}}}_k}$$

for the *decomposition group* of $(\mu_-)_{\underline{\underline{\ddot{Y}}}}$ [which is well-defined up to $\Delta^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}_k}$-conjugacy] — so D_{μ_-} is determined by $\iota_{\underline{\underline{\ddot{Y}}}}$ up to $\Delta^{\mathrm{tp}}_{Y_k}$ $(\stackrel{\mathrm{def}}{=} \Delta_Y(\mathbb{M}^{\Theta}(\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k})))$*-conjugacy* [cf. the notation of Remark 1.1.1, (i)]. We shall refer to either of the pairs

$$(\iota_{\underline{\underline{\ddot{Y}}}} \in \mathrm{Aut}(\underline{\underline{\ddot{Y}}}_k), (\mu_-)_{\underline{\underline{\ddot{Y}}}}); \quad (\iota_{\underline{\underline{\ddot{Y}}}} \in \mathrm{Aut}(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}_k})/\mathrm{Inn}(\Delta^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}_k}), D_{\mu_-})$$

as a **pointed inversion automorphism**. Again, we recall from [EtTh], Definition 1.9, (ii); [EtTh], Definition 2.7, that

> *an "étale theta function of standard type" is defined precisely by the condition that its restriction to* D_{μ_-} *be a* **2*l*-th root of unity**.

Proposition 1.5. **(Projective Systems of Mono-theta Environments)** *In the notation of the above discussion, let*

$$\mathbb{M}_*^{\Theta} \quad = \quad \{\dots \ \to \ \mathbb{M}_{M'}^{\Theta} \ \to \mathbb{M}_M^{\Theta} \ \to \ \dots\}$$

be a **projective system of mono-theta environments** *— where* $\mathbb{M}_M^{\Theta}$ *is a mod* M *mono-theta environment [which is isomorphic to the mod* M *model mono-theta environment determined by* $\underline{\underline{X}}_k$*], and the index* M *of the projective system varies* **multiplicatively** *among the elements of* $\mathbb{N}_{\geq 1}$ *[cf. [EtTh], Corollary 2.19, (ii), (iii)]. Then:*

(i) Such a projective system is **uniquely determined, up to isomorphism,** *by* $\underline{\underline{X}}_k$ *[cf. Remark 1.5.1 below; the* **discrete rigidity** *property of [EtTh], Corollary 2.19, (ii)].*

(ii) The transition morphisms of the resulting projective system of topological groups $\{\dots \ \to \ \Pi_{\underline{\underline{X}}}(\mathbb{M}_{M'}^{\Theta}) \ \to \Pi_{\underline{\underline{X}}}(\mathbb{M}_M^{\Theta}) \ \to \ \dots\}$ *[cf. the notation of Definition 1.1, (i)] are all isomorphisms. Moreover, any isomorphism of topological groups* $\Pi_{\underline{\underline{X}}}(\mathbb{M}_{M'}^{\Theta}) \xrightarrow{\sim} \Pi_{\underline{\underline{X}}}(\mathbb{M}_M^{\Theta})$*, where* M *divides* M'*,* **lifts** *to a morphism of mono-theta environments* $\mathbb{M}_{M'}^{\Theta} \to \mathbb{M}_M^{\Theta}$ *[cf. [EtTh], Corollary 2.18, (iv)]. Thus, to simplify the notation, we shall* **identify** *these topological groups via these transition morphisms and denote the resulting topological group by the notation* $\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta})$*. In particular, we have an* **open subgroup** $\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}_*^{\Theta}) \subseteq \Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta})$*, a* **subquotient** $(l \cdot \Delta_{\Theta})(\mathbb{M}_*^{\Theta})$ *of* $\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta})$*, and a* **quotient** $\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta}) \twoheadrightarrow G(\mathbb{M}_*^{\Theta})$ *[cf. Definition 1.1, (i); Proposition 1.4].*

(iii) The projective system of exterior cyclotomes $\{\dots \ \to \ \Pi_{\boldsymbol{\mu}}(\mathbb{M}_{M'}^{\Theta}) \ \to \ \Pi_{\boldsymbol{\mu}}(\mathbb{M}_M^{\Theta}) \ \to \ \dots\}$ *[cf. the notation of Definition 1.1, (i)] determines a projective limit* **exterior cyclotome** $\Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^{\Theta})$ *which is equipped with a uniquely determined* **cyclotomic rigidity isomorphism**

$$(l \cdot \Delta_{\Theta})(\mathbb{M}_*^{\Theta}) \xrightarrow{\sim} \Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^{\Theta})$$

[i.e., obtained by applying the cyclotomic rigidity isomorphisms of Definition 1.1, (ii), to the various members of the projective system $\mathbb{M}_*^{\Theta}$*]. In particular, [cf. Proposition 1.4] we obtain a* **functorial algorithm**

$$\mathbb{M}_*^{\Theta} \quad \mapsto \quad \underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}_*^{\Theta}) \quad \subseteq \quad H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}_*^{\Theta}), \Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^{\Theta}))$$

*— where one may think of the "*env*" as an abbreviation of the term "[mono-theta]* **environment***" — for constructing from* $\mathbb{M}_*^{\Theta}$ *an* **exterior cyclotome** *version*

$\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$ *of* $\underline{\underline{\theta}}(\Pi)$ *[i.e., by transporting* $\underline{\underline{\theta}}(\Pi)$ *via the above cyclotomic rigidity isomorphism] — cf. [EtTh], Corollary 2.19, (iii). In this context, we shall write*

$$_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) \quad \subseteq \quad \varinjlim_{J} \ H^{1}(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_{*})|_{J}, \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}))$$

— where $_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$ *denotes the subset of elements of the direct limit of cohomology modules in the display for which some [positive integer]* **multiple** *[i.e., some [positive integer]* **power***, if one writes these modules "multiplicatively"] coincides,* **up to torsion***, with an element of* $\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$*; J ranges over the finite index open subgroups of* $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$*.*

(iv) Suppose that $\mathbb{M}^{\Theta}_{*}$ *arises from a* **tempered Frobenioid** $\mathcal{C}$ *[cf. Propositions 1.2, (ii); 1.3]. Then this construction of* $\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$ *[cf. (iii)] is* **compatible** *with the* **Kummer-theoretic construction** *of the étale theta function — i.e., by considering Galois actions on roots of the* **Frobenioid-theoretic theta function** *[cf. the theory of [EtTh], §5]. In particular, it is compatible with the* **Kummer theory of the base-field-theoretic hull** $\mathcal{C}^{\text{bs-fld}}$ *[cf. [FrdII], Theorem 2.4; [AbsTopIII], Proposition 3.2, (ii); [AbsTopIII], Remark 3.1.1].*

Proof. The assertions of Proposition 1.5 follow immediately from the results and definitions of [EtTh] [as well as [FrdII], [AbsTopIII]] that are quoted in the statements of these assertions. ◯

Remark 1.5.1. We recall in passing that one important consequence of the *discrete rigidity* property established in [EtTh], Corollary 2.19, (ii) — which, in effect, allows one to restrict one's attention to $l \cdot \underline{\mathbb{Z}}$*-translates* [i.e., as opposed to $l \cdot \underline{\widehat{\mathbb{Z}}}$-translates] of the usual theta function — is the resulting *compatibility* of projective systems of mono-theta environments [as in Proposition 1.5] with the **discrete structure** inherent in the various isomorphs of the monoid $\mathbb{N}$ that appear in the structure of the *tempered Frobenioids* that arise in the theory [cf. [EtTh], Remark 2.19.4; [EtTh], Remark 5.10.4, (i), (ii)].

Remark 1.5.2. Note that, in the notation of Proposition 1.5, (iii), by considering *"tautological Kummer classes"* of elements of $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*})$, one obtains a natural $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$*-equivariant injection*

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}) \ \otimes \ \mathbb{Q}/\mathbb{Z} \ \hookrightarrow \ \varinjlim_{J} \ H^{1}(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_{*})|_{J}, \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}))$$

whose image is equal to the *torsion subgroup* of the codomain of the injection. Indeed, it follows immediately from the fact that $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*})$ is torsion-free that the torsion subgroup of the codomain of the displayed injection may be identified with the torsion subgroup of

$$\varinjlim_{J} \ H^{1}(J_{G}, \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}))$$

— where J ranges over the finite index open subgroups of $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$; we write J_{G} for the image of J in $G(\mathbb{M}^{\Theta}_{*})$. The desired conclusion thus follows immediately from the well-known Kummer theory of MLF's, i.e., the fact that the Kummer map $(\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}) \otimes \mathbb{Q}/\mathbb{Z})^{J} \to H^1(J_G, \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}))$ [where the superscript "J" denotes the submodule of J-invariants] is injective with image equal to the torsion subgroup of the codomain.

Before proceeding, we review a certain portion of the theory of [AbsTopII] that is relevant to the content of the present §1.

Proposition 1.6. **(Cores and Cuspidalizations)** *Let* Π *be as in Proposition 1.2, (i). Write* $\Delta \subseteq \Pi$ *for the [group-theoretic! — cf., e.g., [AbsAnab], Lemma 1.3.8] subgroup corresponding to* $\Delta^{\mathrm{tp}}_{\underline{\underline{X}}_k}$. *Then:*

(i) **(Cores)** *There exists a* **functorial group-theoretic algorithm** *[cf. [AbsTopII], Corollary 3.3, (i); [AbsTopII], Remark 3.3.3]*

$$\Pi \quad \mapsto \quad \left\{(\Pi \subseteq)\ \Pi_C(\Pi) \twoheadrightarrow \Pi/\Delta\right\}$$

for constructing from Π *a topological group* $\Pi_C(\Pi)$ *equipped with an augmentation [i.e., a surjection]* $\Pi_C(\Pi) \twoheadrightarrow \Pi/\Delta$ *— whose kernel we denote by* $\Delta_C(\Pi)$ *— that contains* Π *as an open subgroup in a fashion that is compatible with the respective surjections to* Π/Δ *and which satisfies the property that when* $\Pi = \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$, *the inclusion* $\Pi \subseteq \Pi_C(\Pi)$ *may be naturally identified with the inclusion* $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k} \subseteq \Pi^{\mathrm{tp}}_{C_k}$.

(ii) **(Elliptic Cuspidalizations)** *Let* N *be a positive integer. Then there exists a* **functorial group-theoretic algorithm** *[cf. [AbsTopII], Corollary 3.3, (iii); [AbsTopII], Remark 3.3.3]*

$$\Pi \quad \mapsto \quad \left\{\Pi_{U_N}(\Pi) \twoheadrightarrow \Pi\right\}$$

for constructing from Π *a topological group* $\Pi_{U_N}(\Pi)$ *equipped with a surjection* $\Pi_{U_N}(\Pi) \twoheadrightarrow \Pi$ *[so the augmentation* $\Pi \twoheadrightarrow \Pi/\Delta$ *determines, by composition, an augmentation* $\Pi_{U_N}(\Pi) \twoheadrightarrow \Pi/\Delta$*] such that when* $\Pi = \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$, *the surjection* $\Pi_{U_N}(\Pi) \twoheadrightarrow \Pi$ *may be naturally identified with a certain surjection — i.e., "elliptic cuspidalization" — that arises from a certain open immersion determined by the* N*-torsion points of a once-punctured elliptic curve that forms a double covering of* C_k *[cf. [AbsTopII], Corollary 3.3, (iii)].*

Proof. The assertions of Proposition 1.6 follow immediately from the results of [AbsTopII] that are quoted in the statements of these assertions [cf. also Remark 1.6.1 below]. ◯

Remark 1.6.1. We recall in passing that the construction of Proposition 1.6, (i), amounts, in effect, to the computation of various *centralizers of the image of various open subgroups of* Π/Δ *in the outer automorphism groups of various open subgroups of* Δ. In a similar vein, the construction of Proposition 1.6, (ii), amounts to the computation of various *outer isomorphisms between various subquotients of*

Δ *that are compatible with the outer actions of various open subgroups of* Π/Δ. More generally, although in Proposition 1.6, we restricted our attention to the construction of *cores* and *elliptic cuspidalizations*, an analogous result may be obtained for more general functorial group-theoretic algorithms involving *"chains of elementary operations"*, as discussed in [AbsTopI], §4 — e.g., for *Belyi cuspidalizations*, as discussed in [AbsTopII], Corollary 3.7.

Next, we proceed to discuss the *"multiradial"* interpretation of the theory of [EtTh] that is of interest in the context of the present series of papers. We begin by examining various examples of the sort of situation that gives rise to such an interpretation.

Example 1.7. Radial and Coric Data I: Generalities.

(i) In the following discussion, we would like to consider a certain *"type of mathematical data"*, which we shall refer to as **radial data**. This notion of a "type of mathematical data" may be *formalized* — cf. [IUTchIV], §3, for more details. From the point of view of the present discussion, one may think of a "type of mathematical data" as the input or output data of a "functorial algorithm" [cf. the discussion of [IUTchI], Remark 3.2.1]. At a more concrete level, we shall assume that this "type of mathematical data" gives rise to a *category*

$$\mathcal{R}$$

— i.e., each of whose *objects* is a specific collection of radial data, and each of whose *morphisms* is an isomorphism. In the following discussion, we shall also consider another "type of mathematical data", which we shall refer to as **coric data**. Write

$$\mathcal{C}$$

for the category obtained by considering specific collections of coric data and isomorphisms of collections of coric data. In addition, we shall assume that we are given a *functorial algorithm* — which we shall refer to as **radial** — whose *input data* consists of a collection of radial data, and whose *output data* consists of a collection of coric data. Thus, this functorial algorithm gives rise to a *functor* $\Phi : \mathcal{R} \to \mathcal{C}$. In the following discussion, we shall assume that this functor is *essentially surjective.* We shall refer to the category $\mathcal{R}$ and the functor Φ as *radial* and to the category $\mathcal{C}$ as *coric*. Finally, if I is some *nonempty index set*, then we shall often consider collections

$$\{\Phi_i : \mathcal{R}_i \to \mathcal{C}\}_{i \in I}$$

of copies of Φ and $\mathcal{R}$, such that the various copies of Φ have the *same codomain* $\mathcal{C}$ — cf. Fig. 1.1 below. Thus, one may think of each $\mathcal{R}_i$ as the category of radial data *equipped with a label* $i \in I$, and isomorphisms of such data.

(ii) We shall refer to a triple $(\mathcal{R}, \mathcal{C}, \Phi : \mathcal{R} \to \mathcal{C})$ [or to the triple consisting of the corresponding "types of mathematical objects" and "functorial algorithm"] of the sort discussed in (i) as a **radial environment**. If Φ is *full*, then we shall refer to the radial environment under consideration as **multiradial**. We shall refer to a radial environment which is not multiradial as **uniradial**. Suppose that the radial environment $(\mathcal{R}, \mathcal{C}, \Phi : \mathcal{R} \to \mathcal{C})$ under consideration is *uniradial.* Then an object of $\mathcal{R}$ may, in general, *lose* a certain portion of its *rigidity* — i.e., may be subject to a certain **additional indeterminacy** — when it is mapped to $\mathcal{C}$. Put another way, in general, an object of $\mathcal{C}$ is *imparted* with a certain **additional rigidity** — i.e., *loses* a certain portion of its *indeterminacy* — when one fixes a *lifting* of the object to $\mathcal{R}$. Thus, in summary,

> the condition that $(\mathcal{R}, \mathcal{C}, \Phi : \mathcal{R} \to \mathcal{C})$ be multiradial may be thought of as a condition to the effect that the *application of the radial algorithm does not result in any loss of rigidity.*

Finally, we observe that, if $(\mathcal{R}, \mathcal{C}, \Phi : \mathcal{R} \to \mathcal{C})$ is an *arbitrary radial environment* such that *any two collections of radial data are isomorphic*, then one may define the associated **[tautological] multiradialization**

$$(\mathcal{R}^{\mathrm{mtz}}, \mathcal{C}, \Phi^{\mathrm{mtz}} : \mathcal{R}^{\mathrm{mtz}} \to \mathcal{C})$$

of this radial environment as follows: A collection of *radial data*

$$(R, C, \alpha)$$

of this *multiradialization* consists of an object R of $\mathcal{R}$, an object C of $\mathcal{C}$, and the *full poly-isomorphism* [cf. [IUTchI], §0] $\alpha : \Phi(R) \xrightarrow{\sim} C$. An *isomorphism of collections of radial data* $(R, C, \alpha) \xrightarrow{\sim} (R^*, C^*, \alpha^*)$ of the *multiradialization* consists of a pair of isomorphisms $R \xrightarrow{\sim} R^*$, $C \xrightarrow{\sim} C^*$ [which are necessarily compatible with α, α^*]. The *coric data* of the *multiradialization* is taken to be the coric data of the original radial environment $(\mathcal{R}, \mathcal{C}, \Phi : \mathcal{R} \to \mathcal{C})$. The **radial algorithm** of the *multiradialization* is taken to be the assignment

$$(R, C, \alpha) \mapsto C$$

— whose associated radial functor is clearly *full* [cf. our assumption that any two collections of radial data are *isomorphic*!] and *essentially surjective*, hence determines a [*tautologically!*] *multiradial environment* $(\mathcal{R}^{\mathrm{mtz}}, \mathcal{C}, \Phi^{\mathrm{mtz}} : \mathcal{R}^{\mathrm{mtz}} \to \mathcal{C})$, together with a *natural functor* $\mathcal{R} \to \mathcal{R}^{\mathrm{mtz}}$ [i.e., given by the assignment $R \mapsto (R, \Phi(R), \Phi(R) \xrightarrow{\sim} \Phi(R))$]. Indeed,

> the **tautological multiradialization** of the given radial environment may be thought of as the result of *"forgetting, in a minimal possible fashion, the uniradiality"* of the original radial environment $(\mathcal{R}, \mathcal{C}, \Phi : \mathcal{R} \to \mathcal{C})$.

$$\begin{array}{ccccc}
 & & \mathcal{R}_i & & \\
 & \dots & \downarrow & \dots & \\
\mathcal{R}_{i'} & \longrightarrow & \boxed{\mathcal{C}} & \longleftarrow & \mathcal{R}_{i''} \\
 & \dots & \uparrow & \dots & \\
 & & \mathcal{R}_{i'''} & &
\end{array}$$

Fig. 1.1: Radial functors valued in a single coric category

(iii) In passing, we pause to observe that one way to think of the significance of the *multiradiality* of a radial environment $(\mathcal{R}, \mathcal{C}, \Phi : \mathcal{R} \to \mathcal{C})$ is as follows: Write

$$\mathcal{R} \times_{\mathcal{C}} \mathcal{R}$$

for the category whose *objects* are triples (R_1, R_2, α) consisting of a pair of objects R_1, R_2 of $\mathcal{R}$ and an isomorphism $\alpha : \Phi(R_1) \xrightarrow{\sim} \Phi(R_2)$ between the images of R_1, R_2 via Φ, and whose *morphisms* are the morphisms [in the evident sense] between such triples [cf. the discussion of the *"categorical fiber product"* given in [FrdI], §0]. Write $\mathfrak{sw} : \mathcal{R} \times_{\mathcal{C}} \mathcal{R} \xrightarrow{\sim} \mathcal{R} \times_{\mathcal{C}} \mathcal{R}$ for the functor $(R_1, R_2, \alpha) \mapsto (R_2, R_1, \alpha^{-1})$ obtained by *switching* the two factors of $\mathcal{R}$. Then

> one formal consequence of the **multiradiality** of a radial environment $(\mathcal{R}, \mathcal{C}, \Phi : \mathcal{R} \to \mathcal{C})$ is the property that the **switching functor** $\mathfrak{sw}$: $\mathcal{R} \times_{\mathcal{C}} \mathcal{R} \xrightarrow{\sim} \mathcal{R} \times_{\mathcal{C}} \mathcal{R}$ *preserves the* **isomorphism class** of objects of $\mathcal{R} \times_{\mathcal{C}} \mathcal{R}$.

Indeed, one verifies immediately that this multiradiality is, in fact, *equivalent* to the condition that every object (R_1, R_2, α) of $\mathcal{R} \times_{\mathcal{C}} \mathcal{R}$ be isomorphic to the object $(R_1, R_1, \mathrm{id} : \Phi(R_1) \xrightarrow{\sim} \Phi(R_1))$ [which is manifestly left unchanged by the switching functor].

(iv) Next, suppose that we are given another radial environment $(\mathcal{R}^\dagger, \mathcal{C}^\dagger, \Phi^\dagger : \mathcal{R}^\dagger \to \mathcal{C}^\dagger)$. We shall refer to the "type of mathematical object"/"functorial algorithm" that gives rise to $\mathcal{R}^\dagger$ (respectively, $\mathcal{C}^\dagger$; $\Phi^\dagger$) as *daggered radial data* (respectively, *daggered coric data*; the *daggered radial functorial algorithm*). Also, let us suppose that we are given a 1-*commutative diagram*

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\Psi_{\mathcal{R}}} & \mathcal{R}^\dagger \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi^\dagger} \\ \mathcal{C} & \xrightarrow{\Psi_{\mathcal{C}}} & \mathcal{C}^\dagger \end{array}$$

— where $\Psi_{\mathcal{R}}$ and $\Psi_{\mathcal{C}}$ arise from "functorial algorithms". If $(\mathcal{R}, \mathcal{C}, \Phi : \mathcal{R} \to \mathcal{C})$ is *multiradial* (respectively, *uniradial*), then we shall refer to $\Psi_{\mathcal{R}}$ as **multiradially defined** (respectively, **uniradially defined**), or [when there is no fear of confusion between Φ and $\Psi_{\mathcal{R}}$] as **multiradial** (respectively, **uniradial**). If $\Psi_{\mathcal{R}}$ admits a 1-factorization $\Xi_{\mathcal{R}} \circ \Phi$ for some $\Xi_{\mathcal{R}} : \mathcal{C} \to \mathcal{R}^\dagger$ that arises from a functorial algorithm, then we shall say that $\Psi_{\mathcal{R}}$ is **corically defined**, or [when there is no fear of confusion] **coric**. Thus, by considering the case where $\mathcal{R} = \mathcal{C}$, $\Phi = \mathrm{id}_{\mathcal{R}}$, *one may think of the notion of a corically defined* $\Psi_{\mathcal{R}}$ *as a sort of* **special case** *of the notion of a multiradial* $\Psi_{\mathcal{R}}$.

(v) Suppose that we are in the situation of (iv), and that $\Psi_{\mathcal{R}}$ is **multiradially defined**. Then one way to think of the significance of the multiradiality of $\Psi_{\mathcal{R}}$ is as follows:

> The multiradiality of $\Psi_{\mathcal{R}}$ renders it possible to consider the **simultaneous execution** of the functorial algorithm corresponding to $\Psi_{\mathcal{R}}$ relative to various collections of radial input data indexed by the set I [cf. Fig. 1.1] in a fashion that is compatible with the **identification** of the **coric** portions [i.e., corresponding to Φ] of these collections of radial input data

— cf. Remark 1.9.1 below for more on this point of view. That is to say, at a more technical level, if one implements this identification of the various coric portions by means of various *gluing isomorphisms in* $\mathcal{C}$, then the multiradiality of $\Psi_{\mathcal{R}}$ implies that one may *lift* these gluing isomorphisms in $\mathcal{C}$ to gluing isomorphisms in $\mathcal{R}$; one may then apply $\Psi_{\mathcal{R}}$ to these gluing isomorphisms in $\mathcal{R}$ to obtain gluing isomorphisms of the output data of $\Psi_{\mathcal{R}}$. Put another way, if one assumes instead that $\Psi_{\mathcal{R}}$ is *uniradial*, then the output data of $\Psi_{\mathcal{R}}$ depends, a priori, on the *"additional rigidity"* [cf. (ii)] of objects of $\mathcal{R}$ relative to these images in $\mathcal{C}$; thus, if one attempts to identify these images in $\mathcal{C}$ via *arbitrary* gluing isomorphisms in $\mathcal{C}$, then one does not have any way to compute the effect of such gluing isomorphisms on the output data of $\Psi_{\mathcal{R}}$.

Remark 1.7.1. One way to understand the significance of the fullness condition in the definition of a *multiradial environment* is as a condition that allows one to execute a sort of **parallel transport** operation between *"fibers"* of the radial functor $\Phi : \mathcal{R} \to \mathcal{C}$ [cf. the notation of Example 1.7, (iv)] — i.e., by *lifting* isomorphisms in $\mathcal{C}$ to isomorphisms in $\mathcal{R}$ [cf. the discussion of Example 1.7, (v)]. Here, it is perhaps of interest to make the tautological observation that, up to an **indeterminacy** arising from the extent that Φ *fails to be faithful*, such liftings are unique. That is to say, whereas a **uniradial environment** may be thought of as a sort of abstraction of the geometric notion of a "**fibration** *that is not equipped with a connection*",

> a **multiradial environment** may be thought of as a sort of abstraction of the geometric notion of a *"fibration equipped with a* **connection***"* — i.e., that allows one to execute parallel transport operations between the "fibers".

Relative to this point of view, one may think of the **coric data** as the portion of the radial data of a multiradial environment that is **horizontal** with respect to the "connection structure". We refer to Remarks 1.9.1, 1.9.2 below for more on the *significance of multiradiality.*

Example 1.8. Radial and Coric Data II: Concrete Examples. In this following, we consider various concrete examples of *multiradial environments*, many of which may, in fact, be understood as *special cases* of the notion of the *tautological multiradialization* associated to a suitable choice of radial environment, i.e., as discussed in Example 1.7, (ii).

(i) From the point of view of the theory to be developed in the remainder of the present §1, perhaps the most basic example of a radial environment is the following. We define a collection of *radial data*

$$(\Pi, G, \alpha)$$

to consist of a *topological group* Π isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$, a *topological group* G isomorphic to G_k, and the *full poly-isomorphism* [cf. [IUTchI], §0] of topological groups $\alpha : \Pi/\Delta \stackrel{\sim}{\to} G$, where we write $\Delta \subseteq \Pi$ for the [group-theoretic! — cf., e.g., [AbsAnab], Lemma 1.3.8] subgroup corresponding to $\Delta^{\mathrm{tp}}_{\underline{\underline{X}}_k}$. An *isomorphism* *of collections of radial data* $(\Pi, G, \alpha) \xrightarrow{\sim} (\Pi^*, G^*, \alpha^*)$ is defined to be a pair of isomorphisms of topological groups $\Pi \xrightarrow{\sim} \Pi^*$, $G \xrightarrow{\sim} G^*$ [which are necessarily compatible with α, α^*!]. A collection of *coric data* is defined to be a topological group isomorphic to G_k; an *isomorphism of collections of coric data* is defined to be an isomorphism of topological groups. The **radial algorithm** is the algorithm given by the assignment

$$(\Pi, G, \alpha) \mapsto G$$

— whose associated radial functor is *full* and *essentially surjective*, hence determines a *multiradial environment*. Note that this example may be thought of as a sort of formalization in the present context of the situation depicted in [IUTchI], Fig. 3.2, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ — cf. Fig. 1.2 below. Here, we recall that the topological group "G" [which is isomorphic to G_k] that appears in the center of Fig. 1.2 is regarded as being known *only up to isomorphism*, and that the various isomorphs of $\Pi_{\underline{\underline{X}}_k}$ that appear in the *"spokes"* of Fig. 1.2 may be regarded as various **"arithmetic holomorphic structures"** on "G" [cf. [IUTchI], Remark 3.8.1, (iii)].

$$\begin{tikzcd}
 & {}^{i}\Pi \arrow[d] & \\
\ldots & & \ldots \\
{}^{i'}\Pi \arrow[r] & \boxed{G} & {}^{i''}\Pi \arrow[l] \\
\ldots & & \ldots \\
 & {}^{i'''}\Pi \arrow[u] &
\end{tikzcd}$$

Fig. 1.2: Different arithmetic holomorphic structures on a single coric G

(ii) Recall the *functorial group-theoretic algorithm*

$$\Pi \quad \mapsto \quad (\Pi \curvearrowright M_{\mathbb{TM}}(\Pi)) \qquad (*_{\mathbb{TM}})$$

of [AbsTopIII], §3 [cf., especially, the functors $\kappa_{\mathfrak{An}}$, $\phi_{\mathfrak{An}}$ of [AbsTopIII], Definition 3.1, (vi); [AbsTopIII], Corollary 3.6, (ii); [IUTchI], Remark 3.1.2] that assigns to a topological group Π isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ an MLF-Galois $\mathbb{TM}$-pair, which we shall denote $\Pi \curvearrowright M_{\mathbb{TM}}(\Pi)$, and which is isomorphic to *the "model" MLF-Galois* $\mathbb{TM}$-*pair determined by the natural action of* $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ *on the ind-topological monoid* $\mathcal{O}^{\triangleright}_{\overline{k}}$. In fact, [the union with $\{0\}$ of] the underlying ind-topological monoid $M_{\mathbb{TM}}(\Pi)$ is also equipped with a *natural ring structure* [cf. [AbsTopIII], Proposition 3.2, (iii)]. On the other hand, if one is willing to sacrifice this ring structure, then there exists a *functorial group-theoretic algorithm*

$$G \quad \mapsto \quad (G \curvearrowright \mathcal{O}^{\triangleright}(G)) \qquad (*_{\triangleright})$$

[cf. [AbsTopIII], Proposition 5.8, (i)] that assigns to a topological group G isomorphic to G_k an MLF-Galois $\mathbb{TM}$-pair, which we shall denote $G \curvearrowright \mathcal{O}^{\triangleright}(G)$, and which is isomorphic to the MLF-Galois $\mathbb{TM}$-pair determined by the natural action of G_k on the ind-topological monoid $\mathcal{O}^{\triangleright}_{\overline{k}}$. Moreover, by [AbsTopIII], Proposition 3.2, (iv) [cf. also Remark 1.11.1, (i), (a), below], there is a [uniquely determined] *functorial tautological isomorphism of MLF-Galois* $\mathbb{TM}$*-pairs*

$$(\Pi \curvearrowright M_{\mathbb{TM}}(\Pi)) \quad \stackrel{\sim}{\to} \quad (\Pi/\Delta \curvearrowright \mathcal{O}^{\triangleright}(\Pi/\Delta))|_{\Pi} \qquad (*_{\mathbb{TM}\triangleright})$$

— where $\Delta \subseteq \Pi$ is as in (i), and the notation "$|_{\Pi}$" denotes the restriction of the action of Π/Δ to an action of Π. Then another important example of a radial environment is the following. We define a collection of *radial data*

$$(\Pi \curvearrowright M_{\mathbb{TM}}(\Pi), G \curvearrowright \mathcal{O}^{\triangleright}(G), \alpha_{\triangleright})$$

to consist of the *output data of the algorithm* $(*_{\mathbb{TM}})$ associated to a *topological group* Π isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$, the *output data of the algorithm* $(*_{\triangleright})$ associated to a *topological group* G isomorphic to G_k, and the *poly-isomorphism* [cf. [IUTchI], §0] of MLF-Galois $\mathbb{TM}$-pairs

$$\alpha_{\triangleright} : (\Pi \curvearrowright M_{\mathbb{TM}}(\Pi)) \quad \stackrel{\sim}{\to} \quad (G \curvearrowright \mathcal{O}^{\triangleright}(G))|_{\Pi}$$

determined [in light of [AbsTopIII], Proposition 3.2, (iv)] by the composite of the natural surjection $\Pi \twoheadrightarrow \Pi/\Delta$ with the full poly-isomorphism of topological groups $\Pi/\Delta \stackrel{\sim}{\to} G$ [where $\Delta \subseteq \Pi$ is as in (i)]. An *isomorphism of collections of radial data* $(\Pi \curvearrowright M_{\mathbb{TM}}(\Pi), G \curvearrowright \mathcal{O}^{\triangleright}(G), \alpha_{\triangleright}) \stackrel{\sim}{\to} (\Pi^* \curvearrowright M_{\mathbb{TM}}(\Pi^*), G^* \curvearrowright \mathcal{O}^{\triangleright}(G^*), \alpha^*_{\triangleright})$ is defined to be a pair of isomorphisms of MLF-Galois $\mathbb{TM}$-pairs $(\Pi \curvearrowright M_{\mathbb{TM}}(\Pi)) \stackrel{\sim}{\to} (\Pi^* \curvearrowright M_{\mathbb{TM}}(\Pi^*))$, $(G \curvearrowright \mathcal{O}^{\triangleright}(G)) \stackrel{\sim}{\to} (G^* \curvearrowright \mathcal{O}^{\triangleright}(G^*))$ [which are necessarily compatible with $\alpha_{\triangleright}$, $\alpha^*_{\triangleright}$!]. A collection of *coric data* is defined to be the *output data of the algorithm* $(*_{\triangleright})$ for some topological group isomorphic to G_k; an *isomorphism of collections of coric data* is defined to be the isomorphism between collections of output data of $(*_{\triangleright})$ associated to an isomorphism of topological groups. The **radial algorithm** is the algorithm given by the assignment

$$(\Pi \curvearrowright M_{\mathbb{TM}}(\Pi), G \curvearrowright \mathcal{O}^{\triangleright}(G), \alpha_{\triangleright}) \mapsto (G \curvearrowright \mathcal{O}^{\triangleright}(G))$$

— whose associated radial functor is *full* and *essentially surjective*, hence determines a *multiradial environment.*

(iii) Let

$$\Gamma \quad \subseteq \quad \widehat{\mathbb{Z}}^{\times}$$

be a *closed subgroup* [cf. Remark 1.11.1, (i), (ii), below, for more on the significance of Γ]. Then by considering the *subgroups of invertible elements* of the various ind-topological monoids that appeared in (ii), one obtains *functorial group-theoretic algorithms*

$$\Pi \quad \mapsto \quad (\Pi \curvearrowright M^{\times}_{\mathbb{TM}}(\Pi)); \qquad G \quad \mapsto \quad (G \curvearrowright \mathcal{O}^{\times}(G)) \qquad (*_{\times})$$

defined, respectively, on topological groups Π isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ and G isomorphic to G_k. Here, we note that we may think of Γ as acting on the output data of the second algorithm of $(*_\times)$ by means of the trivial action on G and the natural action of $\widehat{\mathbb{Z}}^\times$ on $\mathcal{O}^\times(G)$. Then one obtains another example of a radial environment as follows. We define a collection of *radial data*

$$(\Pi \curvearrowright M^\times_{\mathbb{TM}}(\Pi), G \curvearrowright \mathcal{O}^\times(G), \alpha_\times)$$

to consist of the *output data of the first algorithm of* $(*_\times)$ associated to a *topological group* Π isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$, the *output data of the second algorithm of* $(*_\times)$ associated to a *topological group* G isomorphic to G_k, and the *poly-isomorphism* [cf. [IUTchI], §0] of ind-topological modules equipped with topological group actions

$$\alpha_\times : (\Pi \curvearrowright M^\times_{\mathbb{TM}}(\Pi)) \quad \overset{\sim}{\to} \quad (G \curvearrowright \mathcal{O}^\times(G))|_\Pi$$

determined by the Γ-*orbit* of the poly-isomorphism "$\alpha_\triangleright|_\times$" induced by the poly-isomorphism $\alpha_\triangleright$ of (ii). An *isomorphism of collections of radial data* $(\Pi \curvearrowright M^\times_{\mathbb{TM}}(\Pi), G \curvearrowright \mathcal{O}^\times(G), \alpha_\times) \overset{\sim}{\to} (\Pi^* \curvearrowright M^\times_{\mathbb{TM}}(\Pi^*), G^* \curvearrowright \mathcal{O}^\times(G^*), \alpha^*_\times)$ is defined to consist of the isomorphism of ind-topological modules equipped with topological group actions $(\Pi \curvearrowright M^\times_{\mathbb{TM}}(\Pi)) \overset{\sim}{\to} (\Pi^* \curvearrowright M^\times_{\mathbb{TM}}(\Pi^*))$ induced by an isomorphism of topological groups $\Pi \overset{\sim}{\to} \Pi^*$, together with a Γ-*multiple* of the isomorphism of ind-topological modules equipped with topological group actions $(G \curvearrowright \mathcal{O}^\times(G)) \overset{\sim}{\to} (G^* \curvearrowright \mathcal{O}^\times(G^*))$ induced by an isomorphism of topological groups $G \overset{\sim}{\to} G^*$ [so one verifies immediately that these isomorphisms are compatible with $\alpha_\times$, $\alpha^*_\times$ in the evident sense]. A collection of *coric data* is defined to be the *output data of the second algorithm of* $(*_\times)$ for some topological group isomorphic to G_k; an *isomorphism of collections of coric data* is defined to be a Γ-*multiple* of the isomorphism between collections of output data of $(*_\times)$ associated to an isomorphism of topological groups. The **radial algorithm** is the algorithm given by the assignment

$$(\Pi \curvearrowright M^\times_{\mathbb{TM}}(\Pi), G \curvearrowright \mathcal{O}^\times(G), \alpha_\times) \mapsto (G \curvearrowright \mathcal{O}^\times(G))$$

— whose associated radial functor is *full* and *essentially surjective*, hence determines a *multiradial environment*.

(iv) By considering the *subgroups of torsion elements* of the various ind-topological monoids that appeared in (ii) and (iii), one obtains *functorial group-theoretic algorithms*

$$\Pi \quad \mapsto \quad (\Pi \curvearrowright M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi)); \qquad G \quad \mapsto \quad (G \curvearrowright \mathcal{O}^{\boldsymbol{\mu}}(G)) \qquad (*_{\boldsymbol{\mu}})$$

defined, respectively, on topological groups Π isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ and G isomorphic to G_k — i.e., a *"cyclotomic version"* of the algorithms of $(*_\times)$ [cf. (iii)]. Moreover, by forming the quotients $M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(-) \overset{\mathrm{def}}{=} M^\times_{\mathbb{TM}}(-)/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(-)$, $\mathcal{O}^{\times\boldsymbol{\mu}}(-) \overset{\mathrm{def}}{=} \mathcal{O}^\times(-)/\mathcal{O}^{\boldsymbol{\mu}}(-)$, one obtains *functorial group-theoretic algorithms*

$$\Pi \quad \mapsto \quad (\Pi \curvearrowright M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi)); \qquad G \quad \mapsto \quad (G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)) \qquad (*_{\times\boldsymbol{\mu}})$$

defined, respectively, on topological groups Π isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ and G isomorphic to G_k — i.e., a *"co-cyclotomic version"* of the algorithms of $(*_\times)$ [cf. (iii)]. Now one verifies easily that by replacing the symbol "$\times$" in (iii) by the symbol "$\boldsymbol{\mu}$" or, alternatively, by the symbol "$\times\boldsymbol{\mu}$",

one obtains, respectively, *"cyclotomic"* and *"co-cyclotomic"* versions of the example treated in (iii). In the case of "$\times\boldsymbol{\mu}$", let us write

$$\mathrm{Ism}(G)$$

for the **compact** *topological group* of G**-isometries** of $\mathcal{O}^{\times\boldsymbol{\mu}}(G)$, i.e., G-equivariant automorphisms of the ind-topological module $\mathcal{O}^{\times\boldsymbol{\mu}}(G)$ that, for each open subgroup $H \subseteq G$, *preserve* the "lattice" in $\mathcal{O}^{\times\boldsymbol{\mu}}(G)^H$ determined by the *image* of $\mathcal{O}^{\times}(G)^H$ [i.e., where the superscript "H" denotes the submodule of H-invariants]. Let

$$\Gamma^{\times\boldsymbol{\mu}} \quad \subseteq \quad \mathrm{Ism}(-)$$

be a *closed subgroup*, i.e., a collection of closed subgroups of each $\mathrm{Ism}(G)$ that is preserved by arbitrary isomorphisms of topological groups $G_1 \overset{\sim}{\rightarrow} G_2$. Then one verifies easily that, in the *"co-cyclotomic"* version discussed above of the example treated in (iii),

one may replace the "Γ" in (iii) by such a "$\Gamma^{\times\boldsymbol{\mu}}$".

Finally, we observe that one example of such a "$\Gamma^{\times\boldsymbol{\mu}}$" — which we shall denote by means of the notation

$$\mathrm{Ism}$$

— is the case where one takes $\Gamma^{\times\boldsymbol{\mu}}$ to be the entire group "$\mathrm{Ism}(-)$"; another example of such a "$\Gamma^{\times\boldsymbol{\mu}}$" is the image $\mathrm{Im}(\widehat{\mathbb{Z}}^{\times})$ of the natural homomorphism $\widehat{\mathbb{Z}}^{\times} \twoheadrightarrow \mathbb{Z}_p^{\times} \hookrightarrow \mathrm{Ism}$.

(v) Another example of a radial environment may be obtained as follows. We define a collection of *radial data*

$$(\Pi \curvearrowright M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi), G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G), \alpha_{\boldsymbol{\mu},\times\boldsymbol{\mu}})$$

to consist of the *output data of the first algorithm of* $(*_{\boldsymbol{\mu}})$ associated to a *topological group* Π isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$, the *output data of the second algorithm of* $(*_{\times\boldsymbol{\mu}})$ associated to a *topological group* G isomorphic to G_k, and the *poly-morphism* [cf. [IUTchI], §0] of ind-topological modules equipped with topological group actions

$$\alpha_{\boldsymbol{\mu},\times\boldsymbol{\mu}} : (\Pi \curvearrowright M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi)) \quad \to \quad (G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G))|_{\Pi}$$

determined by the full poly-isomorphism $\Pi/\Delta \overset{\sim}{\rightarrow} G$ [cf. (i)] and the **trivial** homomorphism $M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi) \to \mathcal{O}^{\times\boldsymbol{\mu}}(G)$ — i.e., the composite of the natural homomorphisms $M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi) \subseteq M^{\times}_{\mathbb{TM}}(\Pi) \overset{\sim}{\rightarrow} \mathcal{O}^{\times}(G) \twoheadrightarrow \mathcal{O}^{\times\boldsymbol{\mu}}(G)$ [where the "$\overset{\sim}{\rightarrow}$" arises from the poly-isomorphism $\alpha_{\times}$ of (iii)]. An *isomorphism of collections of radial data* $(\Pi \curvearrowright M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi), G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G), \alpha_{\boldsymbol{\mu},\times\boldsymbol{\mu}}) \overset{\sim}{\rightarrow} (\Pi^* \curvearrowright M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^*), G^* \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G^*), \alpha^*_{\boldsymbol{\mu},\times\boldsymbol{\mu}})$ is defined to consist of the isomorphism of ind-topological modules equipped with topological group actions $(\Pi \curvearrowright M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi)) \overset{\sim}{\rightarrow} (\Pi^* \curvearrowright M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^*))$ induced by an isomorphism of topological groups $\Pi \overset{\sim}{\rightarrow} \Pi^*$, together with a $\Gamma^{\times\boldsymbol{\mu}}$-*multiple* of the isomorphism of ind-topological modules equipped with topological group actions

$(G \curvearrowright \mathcal{O}^{\times \boldsymbol{\mu}}(G)) \xrightarrow{\sim} (G^* \curvearrowright \mathcal{O}^{\times \boldsymbol{\mu}}(G^*))$ induced by an isomorphism of topological groups $G \xrightarrow{\sim} G^*$ [so one verifies immediately that these isomorphisms are compatible with $\alpha_{\boldsymbol{\mu}, \times \boldsymbol{\mu}}$, $\alpha^*_{\boldsymbol{\mu}, \times \boldsymbol{\mu}}$ in the evident sense]. A collection of *coric data* is defined to be the *output data of the second algorithm of* $(*_{\times \boldsymbol{\mu}})$ for some topological group isomorphic to G_k; an *isomorphism of collections of coric data* is defined to be a $\Gamma^{\times \boldsymbol{\mu}}$*-multiple* of the isomorphism between collections of output data of $(*_{\times \boldsymbol{\mu}})$ associated to an isomorphism of topological groups. [That is to say, the definition of the coric data is the same as in the *"co-cyclotomic"* version discussed in (iv).] The **radial algorithm** is the algorithm given by the assignment

$$(\Pi \curvearrowright M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi), G \curvearrowright \mathcal{O}^{\times \boldsymbol{\mu}}(G), \alpha_{\boldsymbol{\mu}, \times \boldsymbol{\mu}}) \mapsto (G \curvearrowright \mathcal{O}^{\times \boldsymbol{\mu}}(G))$$

— whose associated radial functor is *full* and *essentially surjective*, hence determines a *multiradial environment.*

(vi) By replacing the notation "$M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi)$" in the discussion of (v) by the notation "$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z}$" [cf. Propositions 1.2, (i); 1.5, (i), (iii)], one verifies immediately that one obtains an *"exterior-cyclotomic version"* of the *multiradial environment* constructed in (v).

(vii) In the discussion to follow, we shall also consider the *functorial group-theoretic algorithms*

$$\Pi \quad \mapsto \quad (\Pi \curvearrowright M^{\mathrm{gp}}_{\mathbb{TM}}(\Pi)); \qquad G \quad \mapsto \quad (G \curvearrowright \mathcal{O}^{\mathrm{gp}}(G)) \qquad (*_{\mathrm{gp}})$$

obtained by passing to the respective *groupifications* of the monoids $M_{\mathbb{TM}}(\Pi)$, $\mathcal{O}^{\triangleright}(G)$, as well as the *functorial group-theoretic algorithms*

$$\Pi \quad \mapsto \quad (\Pi \curvearrowright M^{\widehat{\mathrm{gp}}}_{\mathbb{TM}}(\Pi)); \qquad G \quad \mapsto \quad (G \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G)) \qquad (*_{\widehat{\mathrm{gp}}})$$

obtained by passing to the respective inductive limits of the *profinite completions* of $M^{\mathrm{gp}}_{\mathbb{TM}}(\Pi)^J$, $\mathcal{O}^{\mathrm{gp}}(G)^J$ [i.e., where the superscript "J" denotes the submodule of J-invariants], as J ranges over the open subgroups of Π or G. Thus, there is a *natural action* of Γ on the underlying ind-topological modules of $M^{\widehat{\mathrm{gp}}}_{\mathbb{TM}}(\Pi)$, $\mathcal{O}^{\widehat{\mathrm{gp}}}(G)$; by considering the Γ*-orbit* of the poly-isomorphism induced by the poly-isomorphism $\alpha_{\triangleright}$ of (ii), one obtains a *poly-isomorphism*

$$\alpha_{\widehat{\mathrm{gp}}} : (\Pi \curvearrowright M^{\widehat{\mathrm{gp}}}_{\mathbb{TM}}(\Pi)) \quad \xrightarrow{\sim} \quad (G \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G))|_{\Pi}$$

that is *compatible* [in the evident sense] with the poly-isomorphism $\alpha_{\times}$ of (iii).

(viii) The following example of a radial environment is another variant of the example of (iii). We define a collection of *radial data*

$$(\Pi \curvearrowright M_{\mathbb{TM}}(\Pi), G \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G), \alpha_{\triangleright, \times \boldsymbol{\mu}})$$

to consist of the *output data of the algorithm of* $(*_{\mathbb{TM}})$ associated to a *topological group* Π isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$, the *output data of the second algorithm of* $(*_{\widehat{\mathrm{gp}}})$

[cf. (vii)] associated to a *topological group* G isomorphic to G_k, and the following *diagram* $\alpha_{\rhd,\times\boldsymbol{\mu}}$ *of poly-morphisms* of ind-topological monoids equipped with topological group actions

$$\begin{array}{ccccc} (\Pi \curvearrowright M_{\mathbb{TM}}(\Pi)) & \hookrightarrow & (\Pi \curvearrowright M_{\mathbb{TM}}^{\widehat{\mathrm{gp}}}(\Pi)) & & \\ & \stackrel{\sim}{\to} & (G \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G))|_{\Pi} & \hookleftarrow & (G \curvearrowright \mathcal{O}^{\times}(G))|_{\Pi} \\ & & & \twoheadrightarrow & (G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G))|_{\Pi} \end{array}$$

— where the " $\hookrightarrow$ " denotes the natural inclusion; the " $\stackrel{\sim}{\to}$ " denotes the poly-isomorphism $\alpha_{\widehat{\mathrm{gp}}}$ of (vii); the " $\hookleftarrow$ " denotes the natural inclusion; the " $\twoheadrightarrow$ " denotes the natural surjection. An *isomorphism of collections of radial data* $(\Pi \curvearrowright M_{\mathbb{TM}}(\Pi), G \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G), \alpha_{\rhd,\times\boldsymbol{\mu}}) \stackrel{\sim}{\to} (\Pi^* \curvearrowright M_{\mathbb{TM}}(\Pi^*), G^* \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G^*), \alpha^*_{\rhd,\times\boldsymbol{\mu}})$ is defined to consist of the isomorphism of ind-topological monoids equipped with topological group actions $(\Pi \curvearrowright M_{\mathbb{TM}}(\Pi)) \stackrel{\sim}{\to} (\Pi^* \curvearrowright M_{\mathbb{TM}}(\Pi^*))$ induced by an isomorphism of topological groups $\Pi \stackrel{\sim}{\to} \Pi^*$, together with a Γ-*multiple* of the isomorphism of ind-topological modules equipped with topological group actions $(G \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G)) \stackrel{\sim}{\to} (G^* \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G^*))$ induced by an isomorphism of topological groups $G \stackrel{\sim}{\to} G^*$ [so one verifies immediately that these isomorphisms are compatible with $\alpha_{\rhd,\times\boldsymbol{\mu}}$, $\alpha^*_{\rhd,\times\boldsymbol{\mu}}$ in the evident sense]; here, we note that any such isomorphism $(G \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G)) \stackrel{\sim}{\to} (G^* \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G^*))$ induces isomorphisms $(G \curvearrowright \mathcal{O}^{\times}(G)) \stackrel{\sim}{\to} (G^* \curvearrowright \mathcal{O}^{\times}(G^*))$, $(G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)) \stackrel{\sim}{\to} (G^* \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G^*))$ in a fashion compatible with $\alpha_{\rhd,\times\boldsymbol{\mu}}$, $\alpha^*_{\rhd,\times\boldsymbol{\mu}}$. The definition of *coric data* and *isomorphisms of collections of coric data* is the same as in (v) [i.e., where one takes "$\Gamma^{\times\boldsymbol{\mu}}$" to be the image $\mathrm{Im}(\Gamma)$ of $\Gamma \subseteq \widehat{\mathbb{Z}}^{\times}$]. The **radial algorithm** is the algorithm given by the assignment

$$(\Pi \curvearrowright M_{\mathbb{TM}}(\Pi), G \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G), \alpha_{\rhd,\times\boldsymbol{\mu}}) \mapsto (G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G))$$

— whose associated radial functor is *full* and *essentially surjective*, hence determines a *multiradial environment*.

(ix) Note that if G is a topological group isomorphic to G_k, then, in addition to $G \curvearrowright \mathcal{O}^{\times}(G)$, $G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)$, one may also construct the *log-shell* $\mathcal{I}(G) \subseteq \mathcal{O}^{\times\boldsymbol{\mu}}(G)$ [i.e., p^{-1} times the image of the G-invariants of $\mathcal{O}^{\times}(G)$ in $\mathcal{O}^{\times\boldsymbol{\mu}}(G)$ — cf. [AbsTopIII], Proposition 5.8, (ii)]. In particular, if one replaces the notation "$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)$" in the discussion of (v), (vi), and (viii) by the notation "$(G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G), \mathcal{I}(G) \subseteq \mathcal{O}^{\times\boldsymbol{\mu}}(G))$" [i.e., "$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)$ equipped with its associated log-shell"], then one verifies immediately that one obtains a *"log-shell version"* of the *multiradial environments* constructed in (v), (vi), and (viii).

Remark 1.8.1. In the context of the various examples given in Example 1.8, (iii), (iv), (v), (vi), (vii), (viii), and (ix), it is useful to note that

> *no automorphism of* $\mathcal{O}^{\times\boldsymbol{\mu}}(G)$ induced by *an element of* $\mathrm{Aut}(G)$ [e.g., an element of G, regarded as an inner automorphism of G] *coincides* with an automorphism of $\mathcal{O}^{\times\boldsymbol{\mu}}(G)$ induced by an *element of* Γ *that has nontrivial image in* $\mathbb{Z}_p^{\times}$.

Indeed, this follows immediately by observing that the composite with the p-adic logarithm of the *cyclotomic character* of G determines [in light of the definition of $\mathcal{O}^\times(G)$, in terms of abelianizations of open subgroups of G — cf. [AbsTopIII], Proposition 5.8, (i)] a natural surjection $\mathcal{O}^{\times\boldsymbol{\mu}}(G) \twoheadrightarrow \mathbb{Q}_p$, which [cf., e.g., [AbsAnab], Proposition 1.2.1, (vi)] is $\mathrm{Aut}(G)$-*equivariant*, relative to the *trivial* action of $\mathrm{Aut}(G)$ on $\mathbb{Q}_p$, and Γ-*equivariant*, relative to the *natural* action of $\Gamma \subseteq \widehat{\mathbb{Z}}^\times$ [via the natural surjection $\widehat{\mathbb{Z}}^\times \twoheadrightarrow \mathbb{Z}_p^\times$] on $\mathbb{Q}_p$.

Example 1.9. Radial and Coric Data III: Graphs of Functorial Group-theoretic Algorithms.

(i) Let $\mathcal{E}$ and $\mathcal{F}$ be *categories* that arise from "types of mathematical data" [cf. the discussion of Example 1.7, (i)]; $\Xi : \mathcal{E} \to \mathcal{F}$ a *functor* that arises from a "functorial algorithm" [cf. the discussion of Example 1.7, (i)]. Then one may define a new *category* $\mathcal{G}$ — that also arises from a "type of mathematical data" — as follows: the *objects* of $\mathcal{G}$ are pairs $(E, \Xi(E))$, where $E \in \mathrm{Ob}(\mathcal{E})$, and $\Xi(E) \in \mathrm{Ob}(\mathcal{F})$ is the image of E via Ξ; the *morphisms* of $\mathcal{G}$ are the pairs of arrows $(f : E \to E', \Xi(f) : \Xi(E) \to \Xi(E'))$. We shall refer to $\mathcal{G}$ [or the "type of mathematical data" that gives rise to $\mathcal{G}$] as the *graph of* Ξ. Note that this construction was applied, in effect, in the discussion of the various radial environments constructed in Example 1.8. Finally, we observe that we have *natural functors* $\mathcal{E} \to \mathcal{G}$ [given by $E \mapsto (E, \Xi(E))$], $\mathcal{G} \to \mathcal{E}$ [given by $(E, \Xi(E)) \mapsto E$], $\mathcal{G} \to \mathcal{F}$ [given by $(E, \Xi(E)) \mapsto \Xi(E)$].

(ii) In the notation of (i), suppose that $\mathcal{E}$ is the category of topological groups isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ and isomorphisms of topological groups, and that Ξ is some *"functorial group-theoretic algorithm"* [whose input data consists of a topological group isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$]. Let $(\mathcal{R}, \mathcal{C}, \Phi)$ be the *radial environment* of Example 1.8, (i). Then composing the functor $\mathcal{R} \to \mathcal{E}$ given by the assignment $(\Pi, G, \alpha) \mapsto \Pi$ with $\Xi : \mathcal{E} \to \mathcal{F}$ yields a functor $\mathcal{R} \to \mathcal{F}$, whose *graph* we denote by $\mathcal{R}^\dagger$. Thus, by considering the natural functors $\Psi_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}^\dagger$ [cf. (i)], $\mathcal{R}^\dagger \to \mathcal{R} \to \mathcal{C}$, and taking $\mathcal{C}^\dagger \stackrel{\mathrm{def}}{=} \mathcal{C}$, we obtain a *diagram* as in the display of Example 1.7, (iv). Since $(\mathcal{R}, \mathcal{C}, \Phi)$ is a *multiradial environment*, it thus follows that $\Psi_{\mathcal{R}}$ is *multiradially defined* [cf. Example 1.7, (iv)]. That is to say, by using the radial environment of Example 1.8, (i), one concludes that

> any *"functorial group-theoretic algorithm"* whose input data consists of a topological group isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ gives rise — in a *tautological* fashion [cf. the discussion of *tautological multiradializations* in Example 1.7, (ii)] — to a *multiradially defined* functor.

This approach will be discussed further in Remark 1.9.1 below.

(iii) On the other hand, one may also construct a *radial environment* as follows. We define a collection of *radial data* to be a topological group Π isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$, and an *isomorphism of collections of radial data* to be an isomorphism of topological groups. The definitions of *coric data* and *isomorphisms of collections of coric data* are the same as in Example 1.8, (i). The *radial functor* $\Phi : \mathcal{R} \to \mathcal{C}$ is defined via the assignment $\Pi \mapsto \Pi/\Delta$ [cf. the notation of Example 1.8, (i)]. Thus, Φ *fails to be full* [cf., e.g., [AbsTopIII], §I3; [AbsTopIII], Remark 1.9.1]. That is to say, $(\mathcal{R}, \mathcal{C}, \Phi)$ is a *uniradial environment*. Now suppose that $\Xi : \mathcal{E} \to \mathcal{F}$ is as in (ii). Then since $\mathcal{R}$ may be identified with $\mathcal{E}$, the graph of $\Xi : \mathcal{R} = \mathcal{E} \to \mathcal{F}$ yields a category $\mathcal{R}^\dagger$ equipped with natural functors $\Psi_\mathcal{R} : \mathcal{R} \to \mathcal{R}^\dagger$, $\Phi^\dagger : \mathcal{R}^\dagger \to \mathcal{R} \to \mathcal{C}^\dagger \stackrel{\text{def}}{=} \mathcal{C}$. In particular, we obtain a *diagram* as in the display of Example 1.7, (iv). Since $(\mathcal{R}, \mathcal{C}, \Phi)$ is a *uniradial environment*, it thus follows that $\Psi_\mathcal{R}$ is *uniradially defined* [cf. Example 1.7, (iv)]. That is to say, by using the radial environment just defined, one concludes that

> any *"functorial group-theoretic algorithm"* whose input data consists of a topological group isomorphic to $\Pi^{\text{tp}}_{\underline{\underline{X}}_k}$ also gives rise — in a *tautological* fashion — to a *uniradially defined* functor.

This approach will be discussed further in Remark 1.9.1 below.

(iv) Let Π be a topological group isomorphic to $\Pi^{\text{tp}}_{\underline{\underline{X}}_k}$; $\Delta \subseteq \Pi$ the subgroup of Example 1.8, (i). Recall the isomorphism "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_X)$" of [AbsTopIII], Corollary 1.10, (c), which is constructed by means of a *"functorial group-theoretic algorithm"*. The inverse of this isomorphism yields a *cyclotomic rigidity isomorphism*

$$(l \cdot \Delta_\Theta)(\Pi) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi/\Delta)$$

[cf. the discussion of Proposition 1.3, (ii)] — where we write "$(l \cdot \Delta_\Theta)(\Pi)$" for the [group-theoretic!] subquotient of Π discussed in [EtTh], Corollary 2.18, (i). Thus, in summary, one has a *"functorial group-theoretic algorithm"* whose *input data* consists of the topological group Π, and whose *output data* may be thought of as consisting of Π, the two topological Π-modules "$(l \cdot \Delta_\Theta)(\Pi)$", "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi/\Delta)$", and the above isomorphism of Π-modules $(l \cdot \Delta_\Theta)(\Pi) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi/\Delta)$. Thus, if one takes this "functorial group-theoretic algorithm" to be the algorithm that gives rise to the functor Ξ in the discussion of (ii) and (iii), then one concludes that the above cyclotomic rigidity isomorphism $(l \cdot \Delta_\Theta)(\Pi) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi/\Delta)$ may be thought of as giving rise to either

(a) a *multiradially defined* functor, via the approach of (ii), or
(b) a *uniradially defined* functor, via the approach of (iii).

On the other hand, there is also another way to obtain a *multiradially defined* functor from this cyclotomic rigidity isomorphism, as follows. Let $(\mathcal{R}, \mathcal{C}, \Phi)$ be the *multiradial environment* of Example 1.8, (i). Now define a collection of *daggered radial data*

$$(\Pi, G, \alpha, (l \cdot \Delta_\Theta)(\Pi) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G))$$

to consist of *radial data* (Π, G, α) as in Example 1.8, (i), together with the *poly-isomorphism* $(l \cdot \Delta_\Theta)(\Pi) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G)$ obtained by composing the above cyclotomic rigidity isomorphism "$(l \cdot \Delta_\Theta)(\Pi) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi/\Delta)$" with the poly-isomorphism $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi/\Delta) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G)$ induced by the poly-isomorphism $\alpha : \Pi/\Delta \xrightarrow{\sim} G$. Thus, the poly-isomorphism $(l \cdot \Delta_\Theta)(\Pi) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G)$ consists not of a single isomorphism of topological modules, but rather of an $\text{Aut}(G)$-*orbit* — or, more precisely, a Γ-*orbit*, where $\Gamma \subseteq \widehat{\mathbb{Z}}^\times$ is the image of $\text{Aut}(G)$ via the cyclotomic character on $\text{Aut}(G)$ [cf. [AbsAnab], Proposition 1.2.1, (vi)] — of isomorphisms of topological modules. An *isomorphism of collections of daggered radial data* is defined to be an isomorphism between the underlying collections of radial data [which is necessarily compatible with the poly-isomorphism of topological modules that constitutes the final member of the collections of daggered radial data in question]. Thus, if we take $\mathcal{C}^\dagger \stackrel{\mathrm{def}}{=} \mathcal{C}$, then the "functorial group-theoretic algorithm" that gives rise to the cyclotomic rigidity isomorphism "$(l \cdot \Delta_\Theta)(\Pi) \stackrel{\sim}{\to} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi/\Delta)$" yields a *functor* $\Psi_\mathcal{R} : \mathcal{R} \to \mathcal{R}^\dagger$ [that arises from a "functorial algorithm"], together with a *diagram* as in the display of Example 1.7, (iv). That is to say,

(c) this *multiradially defined* functor $\Psi_\mathcal{R} : \mathcal{R} \to \mathcal{R}^\dagger$ yields an alternative [i.e., relative to (a)] multiradial approach to representing the "functorial group-theoretic algorithm" that gives rise to the cyclotomic rigidity isomorphism "$(l \cdot \Delta_\Theta)(\Pi) \stackrel{\sim}{\to} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi/\Delta)$".

This is the approach taken in Corollary 1.11, (b), below.

Remark 1.9.1. In general, the portion of the "functorial group-theoretic algorithm" that appears in the discussion of Example 1.9, (ii), (iii), and (iv), which involves the quotient Π/Δ of Π will depend not only on the structure of the abstract topological group underlying Π/Δ, but also on the structure of Π/Δ as a *quotient* of Π — i.e., from the point of view of the discussion of Example 1.8, (i), on the *"arithmetic holomorphic structure"* on the topological group Π/Δ determined by this quotient structure. In fact, the *original motivation* for the introduction of the *"multiradial terminology"* of Example 1.7 was precisely to study the extent to which such "functorial group-theoretic algorithms" could be formulated in such a way as to **compute**

> *which* portions of the output data of such algorithms *do indeed depend* in an essential way on the **"arithmetic holomorphic structure"** and *which portions* are **"mono-analytic"** [cf. [AbsTopIII], §I3], i.e., depend only on the structure of the topological group Π/Δ [which one thinks of as a sort of "underlying arithmetic real analytic structure" of the "arithmetic holomorphic structures" involved].

From this point of view, the *tautological* approach of Example 1.9, (ii) [i.e., Example 1.9, (iv), (a)], may be thought of as expressing the idea that if one thinks of each of the quotients "Π/Δ" in the *"spokes"* of Fig. 1.2 as being equipped with a *fixed "arithmetic holomorphic structure"* and hence only related to the coric "G" *via some indeterminate isomorphism of topological groups*, then one obtains a *multiradially defined functor*, i.e., a functor that is *tautologically compatible* with mono-analytic deformations of the various "arithmetic holomorphic structures" that one might impose on the coric "G". Put another way, this multiradially defined algorithm is an algorithm that is *tautologically compatible* with **simultaneous execution on multiple spokes** of Fig. 1.2. By contrast, the *tautological* approach of Example 1.9, (iii) [i.e., Example 1.9, (iv), (b)], may be thought of as expressing the idea that if one tries to *identify* the various quotients "Π/Δ" in the *"spokes"* of Fig. 1.2 via arbitrary mono-analytic isomorphisms, then one only obtains a *uniradially defined functor*, i.e., a functor that *fails to be compatible* with mono-analytic identifications [i.e., gluing isomorphisms] of the various "arithmetic holomorphic structures" on the coric "G". Put another way, this uniradially defined algorithm is an algorithm that *can only be consistently executed on* **one spoke at a time**. Finally, the approach of Example 1.9, (iv), (c), expresses the idea that, in the case of the particular cyclotomic rigidity isomorphism under consideration, if one *weakens the rigidity* of this isomorphism by working with this isomorphism up to a *certain indeterminacy*, then one may construct a *multiradially defined functor*, i.e., a functor that is indeed *compatible* with mono-analytic identifications [i.e., gluing isomorphisms] of the various "arithmetic holomorphic structures" on the coric "G", albeit up to a certain *specified indeterminacy*. Thus, the multiradiality obtained in Example 1.9, (iv), (c), depends, in an essential way, on the *content* of the "functorial group-theoretic algorithm" involved. This approach taken in Example 1.9, (iv), (c), is *representative of the approach taken in Corollaries 1.10, 1.11, and 1.12 below*, which may be thought of as **"computations"** of the **"certain indeterminacy"** that one must allow in order to construct a multiradially defined functor as in Example 1.9, (iv), (c).

Remark 1.9.2.

(i) One way to summarize the discussion of Remark 1.9.1 is as follows. If **uniradially defined** functors correspond to constructions that depend, in a strict sense, on the *"arithmetic holomorphic structure"*, while **corically defined** functors correspond to constructions that only depend on the underlying *mono-analytic structure* [i.e., "arithmetic real analytic structure"], then **multiradially defined** functors correspond to constructions that depend on the *"arithmetic holomorphic structure"*, but only in a fashion that is

> **compatible** *with a given description of how this arithmetic holomorphic structure is* **related** *to — e.g.,* **"embedded in"** *— the underlying mono-analytic structure.*

For instance, in the various *multiradial environments* of Example 1.8, this description of the relation to the underlying mono-analytic structure is given, at a concrete level, by the various poly-morphisms [or diagrams of poly-morphisms] "$\alpha_{(-)}$" that appear in the radial data of these multiradial environments. This point of view is summarized in Fig. 1.3 below.

(ii) From the point of view of the analogy with *connections* discussed in Remark 1.7.1, one may think of a *multiradial environment* as a structure that allows one to execute **parallel transport** operations between **distinct** *arithmetic holomorphic stuctures*, i.e., to describe what **one** *arithmetic holomorphic structure* looks like from the point of view of a **distinct** *arithmetic holomorphic structure* that is only related to the original arithmetic holomorphic structure via the mono-analytic core.

(iii) From the point of view of the analogy with *connections* discussed in Remark 1.7.1, it is also interesting to observe that one may think of the different approaches to multiradiality discussed in Example 1.9, (iv), (a), (c), as being roughly analogous to the phenomenon of **distinct connection structures** *on a* **single fibration**. Moreover, of these different approaches, the *tautological, "general nonsense" approach* of Example 1.9, (iv), (a), is, in some sense, [not surprisingly!] the *"least interesting"* [although it will at times be of use in the theory of the present series of papers!]. This sort of "general nonsense" approach is reminiscent of the *tautological* approach to constructing connections that occurs in the *p-adic theory of the crystalline site*, i.e., by simply forming the tensor product with

> the ring of functions of the PD-envelope along the diagonal of the fiber product of two copies of the space under consideration.

From the point of view of the issue of "describing what one arithmetic holomorphic structure looks like from the point of view of another" [cf. (ii)], the "tautological" approach is not very interesting precisely because it involves working, in effect, with

> the "tautological" collection of *"***labels** *of all possible arithmetic holomorphic structures"* — i.e., corresponding to the various choices of **one particular arrow** among the arrows that constitute the poly-morphism denoted "α" in Example 1.8, (i) — without describing in *further, more explicit terms* what these various "alien" arithmetic holomorphic structures look like relative to structures determined by a given arithmetic holomorphic structure.

By contrast, the *"non-tautological"* approach to multiradiality of Example 1.9, (iv), (c), by means of the **explicit computation of indeterminacies** is much more interesting in that it yields a *more detailed, explicit description* of a structure [e.g., a cyclotomic rigidity isomorphism] associated to an "alien" arithmetic holomorphic structure in terms of the structure associated to a given arithmetic holomorphic structure.

<u>*abstract general nonsense*</u>	<u>*inter-universal Teichmüller theory*</u>	<u>*classical complex Teichmüller theory*</u>
uniradially defined functors	arithmetic holomorphic structures	holomorphic structures
multiradially defined functors	arithmetic holomorphic structures described in terms of underlying mono-analytic structures	holomorphic structures described in terms of underlying real analytic structures
corically defined functors	underlying mono-analytic structures	underlying real analytic structures

Fig. 1.3: Uniradiality, Multiradiality, and Coricity

We now proceed to discuss our *main results* concerning *multiradiality.*

Corollary 1.10. (Multiradial Mono-theta Cyclotomic Rigidity Isomorphisms) *Write* $(\mathcal{R}, \mathcal{C}, \Phi : \mathcal{R} \to \mathcal{C})$ *— i.e., in the notation of Example 1.8, (v),* *(vi),*

$$(\Pi \curvearrowright \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z}, G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G), \alpha_{\boldsymbol{\mu},\times\boldsymbol{\mu}}) \mapsto (G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G))$$

— for the **multiradial environment** *constituted by the* **exterior-cyclotomic version** *[cf. Example 1.8, (vi)] of the multiradial environment discussed in Example 1.8, (v). Consider the* **cyclotomic rigidity isomorphism**

$$(l \cdot \Delta_{\Theta})(\Pi) \stackrel{\sim}{\to} \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \qquad (*^{\text{mono-}\Theta}_{\Pi})$$

[where we identify $(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{*}(\Pi))$ with $(l \cdot \Delta_{\Theta})(\Pi)$ — cf. Proposition 1.4] obtained by composing the functorial algorithm $\Pi \mapsto \mathbb{M}^{\Theta}_{*}(\Pi)$ of Proposition 1.2, (i) [cf. also Proposition 1.5, (i)], with the functorial algorithm for constructing a cyclotomic rigidity isomorphism of Proposition 1.5, (iii). Then the data consisting of the topological group Π, the topological Π-modules constituted by the domain and codomain of $(*^{\text{mono-}\Theta}_{\Pi})$, and the isomorphism $(*^{\text{mono-}\Theta}_{\Pi})$ determines a functor $\mathcal{R} \to \mathcal{F}$ [i.e., where $\mathcal{F}$ denotes the category defined in the evident way so as to accommodate the data just listed] which arises from a functorial algorithm in the topological group Π; denote the corresponding **graph** *[cf. Example 1.9, (i)] by $\mathcal{R}^{\dagger}$. In particular, the resulting natural functor $\Psi_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}^{\dagger}$ [cf. Example 1.9, (i)] is* **multiradially defined**.

Proof. The various assertions of Corollary 1.10 follow immediately from the definitions involved. ○

Remark 1.10.1. We recall in passing that the domain and codomain of the isomorphism $(*^{\text{mono-}\Theta}_{\Pi})$ of Corollary 1.10, as well as the isomorphism $(*^{\text{mono-}\Theta}_{\Pi})$ itself, are constructed from various *subquotients of* [the projective system of topological groups] $\Delta_{\mathbb{M}^{\Theta}_{*}(\Pi)}$ which are completely determined by the structure of $\Delta_{\mathbb{M}^{\Theta}_{*}(\Pi)}$ as a projective system of *topological groups*, the subgroups of $\Delta_{\mathbb{M}^{\Theta}_{*}(\Pi)}$ determined by the *images of the "theta section" portions* of the system of mono-theta environments $\mathbb{M}^{\Theta}_{*}(\Pi)$, and the *images* [arising from the natural outer actions involved — cf. Definition 1.1, (i)] of $(l \cdot \underline{\underline{\mathbb{Z}}})(\mathbb{M}^{\Theta}_{*}(\Pi))$ and $G(\mathbb{M}^{\Theta}_{*}(\Pi))$ in $\mathrm{Out}(\Delta_{\mathbb{M}^{\Theta}_{*}(\Pi)})$. Indeed, the algorithms described in the proofs of [EtTh], Corollary 2.18, (i), (iii); [EtTh], Corollary 2.19, (i), for constructing the various subquotients of $\Delta_{\mathbb{M}^{\Theta}_{*}(\Pi)}$ corresponding to the domain and codomain of $(*^{\text{mono-}\Theta}_{\Pi})$, as well as to the graph of the isomorphism $(*^{\text{mono-}\Theta}_{\Pi})$ itself, depend only on the structure of the projective system of topological groups $\Delta_{\mathbb{M}^{\Theta}_{*}(\Pi)}$ [cf., e.g., [EtTh], Proposition 2.11, (i)], the subgroups of $\Delta_{\mathbb{M}^{\Theta}_{*}(\Pi)}$ determined by the images of the *"theta section" portions* of the system of mono-theta environments $\mathbb{M}^{\Theta}_{*}(\Pi)$ [cf. [EtTh], Definition 2.13, (ii), (c)], and the construction of the group $\Delta_{C}(\Pi)$ [which was reviewed in Proposition 1.6, (i)] containing $(\Delta_{\mathbb{M}^{\Theta}_{*}(\Pi)} \twoheadrightarrow)\ \Delta_{\underline{\underline{Y}}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \subseteq \Delta_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \cong \Delta$, which is used to construct the various subquotients that appear in the crucial [EtTh], Proposition 2.12; [EtTh], Proposition 2.14, (i).

Remark 1.10.2. In words, the content of Corollary 1.10 may be understood as follows:

Since $\mathcal{O}^{\times\boldsymbol{\mu}}(G)$ is constructed by forming the quotient of $\mathcal{O}^{\times}(G)$ by the roots of unity [i.e., $\mathcal{O}^{\boldsymbol{\mu}}(G)$] — recall the **triviality** of the homomorphism $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z} \to \mathcal{O}^{\times\boldsymbol{\mu}}(G)$ [cf. Example 1.8, (v), (vi)]! — any rigidification of the **cyclotome** $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(\Pi))$ that depends *only* on the structure of the **mono-theta-environment** $\mathbb{M}^{\Theta}_{*}(\Pi)$" will be **tautologically compatible** with the **coricity** of $\mathcal{O}^{\times\boldsymbol{\mu}}(G)$, i.e., with the *"sharing of $\mathcal{O}^{\times\boldsymbol{\mu}}(G)$" by distinct arithmetic holomorphic structures* [cf. the discussion of Remark 1.9.1; Fig. 1.4 below].

This contrasts sharply with the situation to be considered in Corollary 1.11 below — cf. Remarks 1.11.3, 1.11.4, below. A similar statement may be made concerning the **subquotient** $(l \cdot \Delta_{\Theta})(\Pi)$ of $\Delta \subseteq \Pi$, which maps **trivially** to $\Pi/\Delta \xrightarrow{\sim} G$.

$$
\begin{tikzcd}
\dots & {}^{i}\Pi \arrow[d] & \dots \\
{}^{i'}\Pi \arrow[r] & \boxed{G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G) \curvearrowleft \Gamma^{\times\boldsymbol{\mu}}} & {}^{i''}\Pi \arrow[l] \\
\dots & {}^{i'''}\Pi \arrow[u] & \dots
\end{tikzcd}
$$

Fig. 1.4: A single coric pair $G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)$, regarded up to the action of $\Gamma^{\times\boldsymbol{\mu}}$

Remark 1.10.3. In the context of Corollary 1.10, it is useful to recall the following [cf. the discussion of [EtTh], Remark 1.10.4, (ii)]. Although at first glance, it might appear as though it might be possible to develop a similar theory to the theory developed in the present series of papers based on a **more general** *sort of* **meromorphic function** *than the theta function*, it is by no means clear that such a more general meromorphic function satisfies the crucial **cyclotomic rigidity**, **discrete rigidity**, and **constant multiple rigidity** properties studied in [EtTh]. Of these properties, the cyclotomic rigidity property, which forms the basis of Corollary 1.10, depends most explicitly [cf. [EtTh], Remark 2.19.2] on the structure of the *theta quotient* $1 \to \Delta_{\Theta} \to \Delta^{\Theta}_{X} \to \Delta^{\mathrm{ell}}_{X} \to 1$ reviewed in [IUTchI], Remark 3.1.2, (iii) [cf. also the discussion of Remark 1.1.1 of the present paper], i.e., which corresponds to the *"theta group"* in more classical treatments of the theta function. Since the theta function is, roughly speaking, essentially characterized among meromorphic functions by the property that it satisfies the **"theta symmetries"** arising from the theta group, it is thus difficult to see how to generalize the theory of the present series of papers so as to treat more general meromorphic functions than the theta function [cf. Remark 1.1.1, (v); [IUTchIII], Remark 2.3.3, for a more detailed discussion of related issues]. Also, in this context, it is useful to recall that unlike the theta function itself, which is strictly *local* in nature [i.e., in the sense that it is only defined, *a priori*, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], the theta quotient Δ^{Θ}_{X}, hence, in particular, the subquotient Δ_{Θ}, is defined *globally* [cf. the discussion of [IUTchI], Remark 3.1.2] over the various number fields involved, hence may be applied to the execution of various global anabelian reconstruction algorithms via the *"Θ-approach"* [cf. [IUTchI], Remark 3.1.2].

Corollary 1.11. (Multiradial MLF-Galois Pair Cyclotomic Rigidity Isomorphisms with Indeterminacies) *Write* $(\mathcal{R}, \mathcal{C}, \Phi : \mathcal{R} \to \mathcal{C})$ *— i.e., in the notation of Example 1.8, (viii),*

$$(\Pi \curvearrowright M_{\mathbb{TM}}(\Pi), G \curvearrowright \widehat{\mathcal{O}^{\mathrm{gp}}}(G), \alpha_{\rhd, \times\boldsymbol{\mu}}) \mapsto (G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G))$$

— for the **multiradial environment** *discussed in Example 1.8, (viii). Consider*

(a) the Γ**-orbit** *[where we recall that* $\Gamma \subseteq \widehat{\mathbb{Z}}^{\times}$ *is a closed subgroup]*

$$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\mathcal{O}^{\times}(G)) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathcal{O}^{\times}(G)) \qquad (*^{\text{bs-Gal}}_{G,\rhd})$$

of the **cyclotomic rigidity isomorphism** *obtained by applying to the MLF-Galois pair determined by* $G \curvearrowright \mathcal{O}^{\rhd}(G)$ *the algorithm applied to construct [the inverse of] the isomorphism "*$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(M_{\mathbb{TM}}) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G)$*" in [AbsTopIII], Remark 3.2.1 [cf. the discussion of Proposition 1.3, (ii)]; and*

(b) the $\mathrm{Aut}(G)$**-orbit** *[where we recall from [AbsAnab], Proposition 1.2.1, (vi), that* $\mathrm{Aut}(G)$ *admits a natural cyclotomic character] of isomorphisms*

$$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G) \xrightarrow{\sim} (l \cdot \Delta_{\Theta})(\Pi) \qquad (*^{\text{bs-Gal}}_{G,\Pi})$$

obtained by composing the poly-isomorphism induced by applying "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(-)$*" to the [inverse of the]* **full poly-isomorphism** *of topological groups* $\alpha : \Pi/\Delta \xrightarrow{\sim} G$ *[cf. Example 1.8, (i)] with the* **natural isomorphism** *"*$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_X)$*" of [AbsTopIII], Corollary 1.10, (c) [cf. the discussion of Proposition 1.3, (ii)].*

Then the data consisting of the triple (Π, G, α) *[cf. Example 1.8, (i)], the topological* G*-modules constituted by the domain and codomain of* $(*^{\text{bs-Gal}}_{G,\rhd})$*, the topological* Π*-module constituted by the codomain of* $(*^{\text{bs-Gal}}_{G,\Pi})$*, and the poly-isomorphisms* $(*^{\text{bs-Gal}}_{G,\rhd})$ *and* $(*^{\text{bs-Gal}}_{G,\Pi})$ *determines a functor* $\mathcal{R} \to \mathcal{F}$ *which arises from a functorial algorithm in the triple* (Π, G, α)*; denote the corresponding* **graph** *[cf. Example 1.9, (i)] by* $\mathcal{R}^{\dagger}$*. In particular, the resulting natural functor* $\Psi_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}^{\dagger}$ *[cf. Example 1.9, (i)] is* **multiradially defined***.*

Proof. The various assertions of Corollary 1.11 follow immediately from the definitions involved. ○

Remark 1.11.1.

(i) In the context of Corollary 1.11, it is useful to recall that:

(a) the group of automorphisms of the underlying ind-topological monoid equipped with a topological group action — i.e., in the terminology of [AbsTopIII], Definition 3.1, (ii), *MLF-Galois* $\mathbb{TM}$*-pair* — of

$$G \curvearrowright \mathcal{O}^{\rhd}(G)$$

maps *bijectively* [i.e., by forgetting "$\mathcal{O}^{\triangleright}(G)$"] onto the group of automorphisms of the topological group G [cf. [AbsTopIII], Proposition 3.2, (iv)];

(b) the group of automorphisms of the underlying ind-topological module equipped with a topological group action — i.e., in the terminology of [AbsTopIII], Definition 3.1, (ii), *MLF-Galois* $\mathbb{TCG}$*-pair* — of

$$G \curvearrowright \mathcal{O}^{\times}(G)$$

maps *surjectively* [i.e., by forgetting "$\mathcal{O}^{\times}(G)$"] onto the group of automorphisms of the topological group G, with *kernel* given by the [G-linear] automorphisms of [the underlying ind-topological module of] $\mathcal{O}^{\times}(G)$ determined by the *natural action of* $\widehat{\mathbb{Z}}^{\times}$ [cf. [AbsTopIII], Proposition 3.3, (ii)].

Also, we observe that by the same proof involving the Kummer map as that given for (b) in [AbsTopIII], Proposition 3.3, (ii), it follows that

(c) the group of automorphisms of the underlying ind-topological module equipped with a topological group action of

$$G \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G)$$

maps *surjectively* [i.e., by forgetting "$\mathcal{O}^{\widehat{\mathrm{gp}}}(G)$"] onto the group of automorphisms of the topological group G, with *kernel* given by the [G-linear] automorphisms of [the underlying ind-topological module of] $\mathcal{O}^{\widehat{\mathrm{gp}}}(G)$ determined by the *natural action of* $\widehat{\mathbb{Z}}^{\times}$ [or, equivalently, maps *bijectively* onto the group of automorphisms of the underlying ind-topological module equipped with a topological group action of $G \curvearrowright \mathcal{O}^{\times}(G)$ — cf. (b)].

On the other hand, one verifies immediately that

(d) the underlying ind-topological module of $\mathcal{O}^{\times\boldsymbol{\mu}}(G)$ is *divisible*, hence admits a *natural action by* $\mathbb{Q}_p$.

In particular, if, in (b), one replaces "$\mathcal{O}^{\times}$" by "$\mathcal{O}^{\times\boldsymbol{\mu}}$", then the resulting description of the kernel is *false*.

(ii) In the present series of papers, we shall primarily be interested in Corollary 1.11 in the case where

$$\Gamma = \widehat{\mathbb{Z}}^{\times}.$$

That is to say, allowing for a Γ $(= \widehat{\mathbb{Z}}^{\times})$*-multiple indeterminacy* corresponds precisely to working, in the case of $G \curvearrowright \mathcal{O}^{\times}(G)$, with the *underlying ind-topological module equipped with topological group action* [cf. (i), (b)].

Remark 1.11.2.

(i) Observe that, in the context of the discussion of Remark 1.11.1, (i), (b), the natural action of $\widehat{\mathbb{Z}}^{\times}$ on [the underlying ind-topological module equipped with a topological group action of] $G \curvearrowright \mathcal{O}^{\times}(G)$ is compatible with pull-back via the composite of the natural surjection $\Pi \twoheadrightarrow \Pi/\Delta$ with any isomorphism $\Pi/\Delta \xrightarrow{\sim} G$ [cf.

the notation of Example 1.8]. That is to say, one has a *natural action of* $\widehat{\mathbb{Z}}^\times$ *on* [the underlying ind-topological module equipped with a topological group action of] *the resulting pair* $\Pi \curvearrowright \mathcal{O}^\times(G)$. Observe, moreover, that this action of $\widehat{\mathbb{Z}}^\times$ *fails to be compatible with the ring structure* on $\mathcal{O}^\times(G) \otimes \mathbb{Q}$ [i.e., the ring structure determined by applying the p-adic logarithm]. That is to say, even though this ring structure on "$\mathcal{O}^\times$" may [unlike the case with G!] be reconstructed from the topological group Π [cf. [AbsTopIII], Theorem 1.9], *the natural action of* $\widehat{\mathbb{Z}}^\times$ *on* $\Pi \curvearrowright \mathcal{O}^\times(G)$ *fails to preserve the ring structure reconstructed from* Π.

(ii) The observations of (i) are of interest in the context of understanding our adoption of **"G" as opposed to "Π"** in the construction of the **"Θ-link"** *between distinct* Θ*-Hodge theaters* given in [IUTchI], Corollary 3.7. That is to say, even if one tries to *"force a preservation of arithmetic holomorphic structures"* between distinct Θ-Hodge theaters by working with "$\Pi \curvearrowright \mathcal{O}^\times(G)$" instead of "$G \curvearrowright \mathcal{O}^\times(G)$", this does not result in the establishment of a consistent common arithmetic holomorphic structure for distinct Θ-Hodge theaters, since the establishment of such a consistent common arithmetic holomorphic structure is already *obstructed* by the fact that distinct Θ-Hodge theaters only share a common "$\mathcal{O}^\times$" [cf. [IUTchI], Corollary 3.7, (iii)] — on which $\widehat{\mathbb{Z}}^\times$ acts [cf. (i)] — i.e., as opposed to a common "$\mathcal{O}^\rhd$". Here, we recall that *the establishment of a common "$\mathcal{O}^\rhd$" is obstructed, in a quite essential manner, by the "valuative portion* ${}^\dagger\underline{\underline{\Theta}}_{\underline{v}} \mapsto {}^\ddagger\underline{\underline{q}}_{\underline{v}}$*" of the* Θ*-link* [cf. [IUTchI], Remark 3.8.1, (i)].

Remark 1.11.3.

(i) In some sense, the starting point of any discussion of radial environments is the description of the *radial functor*, i.e., the *specification of "which portion of the radial data one takes for one's* **coric data***"*. From the point of view of the theory of [IUTchI], §3 [cf., especially, the portion at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ of [IUTchI], Corollaries 3.7, 3.9], the coric data should, in particular, include the *quotient* $\Pi \twoheadrightarrow \Pi/\Delta \cong G$ of the topological group Π isomorphic to $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$ that appears in a Θ-Hodge theater. On the other hand, in [IUTchIII], we shall ultimately be interested in applying the theory of [AbsTopIII], §3, §5, in which various objects [such as "$\Pi \curvearrowright M_{\mathbb{TM}}(\Pi)$", "$G \curvearrowright \mathcal{O}^\rhd(G)$", "$G \curvearrowright \mathcal{O}^\times(G)$", etc.] are *constructed group-theoretically* from Π or G. One important aspect of the theory of [AbsTopIII], §3, §5, is that after these objects are constructed group-theoretically from Π or G, one then proceeds to **forget the "anabelian structure"** of these objects, i.e., one forgets the data consisting of the way in which these objects [such as MLF-Galois $\mathbb{TM}$-pairs, MLF-Galois $\mathbb{TCG}$-pairs, etc.] are constructed from Π or G. From the point of view of the issue of "specification of coric data", if one takes, for instance, "G" to be a part of one's coric data, then any objects constructed group-theoretically from G may also be regarded naturally as constituting a portion of the coric data — so long as one regards these objects as *being equipped with the corresponding "anabelian structures"* [i.e., the data that specifies the way in which they were constructed group-theoretically from G]. On the other hand, once one *forgets* these anabelian structures, it is no longer the case that such an object may also be regarded naturally as constituting a portion of the coric data. That is to say, once one forgets the anabelian structure of such an object, it is necessary to **specify explicitly** that such an object is to be regarded as a portion of the coric data, i.e., as a portion of the radial data that is to be *subject to the* **"gluing"**, *or* **"identification"**, discussed in Example 1.7, (v).

(ii) In light of the *"coricity of $\mathcal{O}^\times$"* given in [IUTchI], Corollary 3.7, (iii), in addition to "G", it is possible to take the underlying MLF-Galois $\mathbb{TCG}$-pair of "$G \curvearrowright \mathcal{O}^\times(G)$" to be part of one's coric data. By applying Remark 1.11.1, (i), (b), it follows that this amounts to working with "$G \curvearrowright \mathcal{O}^\times(G)$" *up to an* $(\mathrm{Aut}(G), \Gamma\ (= \widehat{\mathbb{Z}}^\times))$*-indeterminacy* — where we recall from Remark 1.8.1 that the p-adic portion of the Γ-indeterminacy *cannot be subsumed* into the $\mathrm{Aut}(G)$-indeterminacy [i.e., which arises from the fact that G is only known up to isomorphism as a topological group]. This situation is precisely the situation formulated in Example 1.8, (iii). On the other hand, as we saw already in Corollary 1.10 [cf. Remark 1.10.2], and as we shall see again in Corollary 1.12 below, in order to construct certain multiradially defined functors that will be of substantial importance in the development of the theory of the present series of papers, it is necessary to *form the quotient of "$\mathcal{O}^\times(-)$" by its torsion subgroup "$\mathcal{O}^{\boldsymbol{\mu}}(-)$"*, i.e., to work with "$\mathcal{O}^{\times\boldsymbol{\mu}}(-)$", rather than "$\mathcal{O}^\times(-)$". Here, we note [cf. Example 1.8, (ix); Remark 1.11.1, (i), (d)] that one does *not* wish here to work *solely* with the underlying ind-topological module equipped with topological group action determined by "$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)$". On the other hand, by applying [IUTchI], Corollary 3.7, (iii), together with Remark 1.11.1, (i), (b), one concludes that it is possible

> *to glue together, in a consistent fashion, the various "$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)$" [cf. Fig. 1.4] arising from distinct Θ-Hodge theaters up to an* $(\mathrm{Aut}(G), \Gamma\ (= \widehat{\mathbb{Z}}^\times))$**-indeterminacy**

[where again we recall from Remark 1.8.1 that the p-adic portion of the Γ-indeterminacy *cannot be subsumed* into the $\mathrm{Aut}(G)$-indeterminacy]. This sort of situation is formulated in the radial environments of Example 1.8, (v), (vi), (viii), (ix) [i.e., where one takes "$\Gamma^{\times\boldsymbol{\mu}}$" to be the image $\mathrm{Im}(\Gamma)$ of Γ]. One important point in this context is that even if one takes "$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)$" [i.e., as opposed to "$G \curvearrowright \mathcal{O}^{\rhd}(G)$", "$G \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G)$", or "$G \curvearrowright \mathcal{O}^\times(G)$"] as one's coric data, the condition of *compatibility* with respect to the natural maps

$$\mathcal{O}^{\widehat{\mathrm{gp}}}(G) \hookleftarrow \mathcal{O}^\times(G) \twoheadrightarrow \mathcal{O}^{\times\boldsymbol{\mu}}(G)$$

[cf. Example 1.8, (viii)] implies that

> *the* $(\mathrm{Aut}(G), \Gamma\ (= \widehat{\mathbb{Z}}^\times))$*-indeterminacy on "$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)$"* **induces** *a* $(\mathrm{Aut}(G), \Gamma\ (= \widehat{\mathbb{Z}}^\times))$*-indeterminacy on "$G \curvearrowright \mathcal{O}^\times(G)$" and "$G \curvearrowright \mathcal{O}^{\widehat{\mathrm{gp}}}(G)$"*

— where one may think of the "Γ-indeterminacy on $\mathcal{O}^{\widehat{\mathrm{gp}}}(G)$" as representing the *"$\Gamma$-indeterminacy in the specification of the submonoid $\mathcal{O}^{\rhd}(G) \subseteq \mathcal{O}^{\widehat{\mathrm{gp}}}(G)$"*. It is precisely these indeterminacies that induce the indeterminacies — i.e., *"orbits"* — that appear in Corollary 1.11, (a), (b), in sharp contrast to the *"strict cyclotomic rigidity"* [i.e., without any indeterminacies] of Corollary 1.10 [cf. Remark 1.10.2].

(iii) Note that the algorithms applied to construct cyclotomic rigidity isomorphisms in Corollaries 1.10 and 1.11, (a), are obtained by *composing* with a group-theoretic construction algorithm an algorithm whose *input data* is **"post-anabelian"** — i.e., consists of a type of mathematical object that arises *upon forgetting the anabelian structure* determined by the group-theoretic construction algorithm. More concretely, this post-anabelian input data consists of a *system of mono-theta environments* in the case of Corollary 1.10 and of an *MLF-Galois* $\mathbb{TM}$-*pair* in the case of Corollary 1.11, (a). As discussed in (ii), the indeterminacies that act on this post-anabelian input data induce the indeterminacies — i.e., *"orbits"* — that appear in Corollary 1.11, (a), (b). Put another way,

(a) the indeterminacies — i.e., *"orbits"* — that appear in Corollary 1.11, (a), (b), are a consequence of the highly **nontrivial** relationship [cf. the discussion of (ii)] between the *input data "*$\mathcal{O}^{\rhd}(-)$*"* of the cyclotomic rigidity algorithm involved and the *coric data "*$\mathcal{O}^{\times\boldsymbol{\mu}}(-)$*"*.

By contrast,

(b) the "strict cyclotomic rigidity" asserted in Corollary 1.10 is a consequence [cf. Remark 1.10.2] of the **triviality** of the homomorphism that relates the **cyclotomic portion** of "$\mathcal{O}^{\rhd}(-)$" — which is the **only portion** of "$\mathcal{O}^{\rhd}(-)$" that appears in a **mono-theta environment** — to the *coric data "*$\mathcal{O}^{\times\boldsymbol{\mu}}(-)$*"*.

Here, it is important to note that although frequently in discussions of various "reconstruction algorithms" [such as group-theoretic reconstruction algorithms], emphasis is placed on the *existence of "some" reconstruction algorithm*, the present discussion of the **multiradiality** of cyclotomic rigidity isomorphisms in the context of Corollaries 1.10, 1.11 yields an important example of the phenomenon that sometimes not only the *existence of "some" reconstruction algorithm*, but also the **content** *of the reconstruction algorithm* [cf. the discussion of [IUTchIV], Example 3.5] is of crucial importance in the development of the theory.

(iv) Here, we note in passing that one may *eliminate the* $(\mathrm{Aut}(G), \Gamma)$-*indeterminacy* of Corollary 1.11, (a), (b), by working, in the fashion of Example 1.9, (iv), (b), with **uniradially defined** functors [that is to say, in the case of Corollary 1.11, (a), (b), replacing "$G \curvearrowright \mathcal{O}^{\rhd}(G)$" by "$\Pi \curvearrowright M_{\mathbb{TM}}(\Pi)$" and "$G$" by "$\Pi/\Delta$" and working with the *uniradial environment* corresponding to the assignment

$$(\Pi \curvearrowright M_{\mathbb{TM}}(\Pi)) \mapsto (\Pi/\Delta \curvearrowright M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi))$$

— i.e., for which the definition of the *coric data* coincides with the definition of the coric data of the multiradial environment in the statement of Corollary 1.11].

(v) The reason [cf. the discussion of (iii)] that we wish to consider cyclotomic rigidity algorithms whose input data is **post-anabelian** is that we wish to be able to apply the same algorithms to input data that does not necessarily arise from a group-theoretic construction algorithm — e.g., to **input data** that arises from the [divisor and rational function] **monoid** portion of a Frobenioid, as in Proposition 1.3. In the context of Proposition 1.3, the exterior cyclotome of the mono-theta environment that appears in Corollary 1.10 and the cyclotome arising from "$\mathcal{O}^{\rhd}(-)$" that appears in Corollary 1.11, (a), both correspond to the *same cyclotome* "$\boldsymbol{\mu}_N(S)$" [which arises from the monoid portion of the Frobenioid involved]. On the other hand, at the level of *construction algorithms*, in order to relate the exterior cyclotome "$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_*(\Pi))$" of Corollary 1.10 to the cyclotome "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\mathcal{O}^{\times}(G))$" of Corollary

1.11, (a), it is necessary [cf. Proposition 1.3, (iii)] to pass through the cyclotome "$(l \cdot \Delta_\Theta)(\Pi)$" by applying the cyclotomic rigidity isomorphisms of Corollaries 1.10, 1.11 — which, in the case of Corollary 1.11, results in various *indeterminacies*. Put another way, the Frobenioid-theoretic identification [i.e., via "$\boldsymbol{\mu}_N(S)$"] of Proposition 1.3 between the cyclotomes "$\Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^\Theta(\Pi))$", "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\mathcal{O}^\times(G))$" of Corollaries 1.10; 1.11, (a), may be thought of either as being only **uniradially defined** [cf. (iv)], or as **multiradially defined**, but only up to certain *indeterminacies*.

Remark 1.11.4.

(i) One way to understand the significance of the cyclotomic rigidity isomorphism obtained in Corollary 1.10 — i.e., of the **triviality** of the homomorphism that relates the *cyclotomic portion of "$\mathcal{O}^\rhd(-)$"* to the *coric data "$\mathcal{O}^{\times\boldsymbol{\mu}}(-)$"* [cf. Remark 1.11.3, (iii), (b)] — relative to the cyclotomic rigidity isomorphism of Corollary 1.11, which involves substantial indeterminacies arising from the highly **nontrivial** relationship between the *input data "$\mathcal{O}^\rhd(-)$"* of the cyclotomic rigidity algorithm involved and the *coric data "$\mathcal{O}^{\times\boldsymbol{\mu}}(-)$"* [cf. Remark 1.11.3, (iii), (a)], is as a sort of

> **splitting**, or **decoupling**, that serves to separate the **"purely radial data"** that appears in the cyclotomic rigidity isomorphism of Corollary 1.10 from the **"purely coric data"** constituted by "$\mathcal{O}^{\times\boldsymbol{\mu}}(-)$".

This point of view is discussed further in Remark 1.12.2, (vi), below.

(ii) From the point of view of the discussion of Remark 1.9.2, (iii), the "purely radial data" that appears in the cyclotomic rigidity isomorphism of Corollary 1.10 depends on the tautological collection of *"***labels** *of all possible arithmetic holomorphic structures"*. That is to say, the algorithms of Corollary 1.10 do not give rise to a *"detailed, explicit description"* of these labels in terms of the "purely coric data $\mathcal{O}^{\times\boldsymbol{\mu}}(-)$". On the other hand, one may also consider a *modified version* of Corollary 1.10 in which

(∗) one replaces "$\mathcal{O}^{\times\boldsymbol{\mu}}(-)$" by "$\mathcal{O}^\times(-)$" [i.e., so that the crucial *triviality* discussed in Remark 1.11.3, (iii), (b), no longer holds!] and applies the **tautological approach** discussed in Example 1.9, (iv), (a), to constructing the cyclotomic rigidity isomorphism [without indeterminacies!] under consideration.

If one works with this modified version (∗), then the codomain of the cyclotomic rigidity isomorphism under consideration may be thought of as the submodule "$\mathcal{O}^{\boldsymbol{\mu}}(-)$" of the *"purely coric data $\mathcal{O}^\times(-)$"*, equipped with a *"certain rigidity"* that **depends** on the choice of an element of the collection of *"***labels** *of all possible arithmetic holomorphic structures"*. That is to say, whereas Corollary 1.10 has the *drawback* of being *"not entirely free of label-dependence"*, the significance of Corollary 1.10 [as stated!] relative to the tautological modified version (∗) lies in the fact that the **label-dependence** inherent in Corollary 1.10 is **confined** to the **"purely radial component"** of the *splitting*, or *decoupling*, discussed in (i) — i.e., unlike the case with (∗), the algorithms of Corollary 1.10 yield a **"purely coric component"** that is **free** *of such "unwanted"* **label-dependent data**. Thus, in summary, unlike the case with (∗), the algorithms of Corollary 1.10 yield output data equipped with a **splitting**, or **decoupling**, into **label-dependent** [i.e., "purely radial"] and **label-independent** [i.e. "purely coric"] components.

Remark 1.11.5. Suppose that we are in the situation of Corollary 1.11.

(i) Recall the *natural surjection*

$$H^1(G, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G)) \twoheadrightarrow \widehat{\mathbb{Z}}$$

— which is constructed via a *functorial group-theoretic algorithm* in [AbsTopIII], Corollary 1.10, (b). That is to say, when $G = G_k$, this surjection is the surjection determined by the valuation of k on the image of the natural Kummer map

$$k^\times \hookrightarrow H^1(G_k, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_k))$$

— where we recall that the image of this Kummer map is equal to the inverse image of $\mathbb{Z} \subseteq \widehat{\mathbb{Z}}$ via the surjection under consideration. In particular, the existence of this functorial group-theoretic algorithm implies that the data consisting of this *natural surjection* — hence, in particular, its **kernel**, i.e., "$\mathcal{O}_k^\times$" — may be formulated as a **corically**, hence, in particular, as a **multiradially** [cf. Example 1.7, (iv)], **defined** functor. [We leave the routine details to the reader.]

(ii) On the other hand, if one applies the isomorphisms $(*^{\text{bs-Gal}}_{G,\triangleright})$ [cf. also the poly-isomorphism $\alpha_\triangleright$ of Example 1.8, (ii)] and $(*^{\text{bs-Gal}}_{G,\Pi})$, of Corollary 1.11, then the natural surjection of (i) gives rise to *natural surjections*

$$H^1(G, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(M_{\mathbb{TM}}(\Pi))) \twoheadrightarrow \widehat{\mathbb{Z}}; \quad H^1(G, (l \cdot \Delta_\Theta)(\Pi)) \twoheadrightarrow \widehat{\mathbb{Z}}$$

— which yield data that may be formulated either as a **uniradially defined** functor [cf. Remark 1.11.3, (iv)] or, when considered up to a $\widehat{\mathbb{Z}}^\times$-*indeterminacy*, as a **multiradially defined** functor [cf. Corollary 1.11]. In particular, the *kernels* of these natural surjections yield data that may be formulated as a **multiradially defined** functor. [We leave the routine details to the reader.]

Remark 1.11.6. The importance of *cyclotomic rigidity* in the theory of the present series of papers is interesting in light of the analogy between the ideas of the present series of papers and the *p-adic Teichmüller theory* of [pTeich] [cf. the discussion of [AbsTopIII], §I5]. Indeed, the proof of a fundamental *absolute p-adic anabelian* result concerning the *canonical curves* that arise in the theory of [pTeich] [cf. [CanLift], Theorem 3.6] depends, in an essential way, on a certain *cyclotomic rigidity* result proven in an earlier paper [cf. [AbsAnab], Lemma 2.5, (ii)]. In this context, we observe that one important theme that appears both in the present series of papers and in the theory of [CanLift], §3, is the idea that *cyclotomes should be thought of as the* **"skeleta of arithmetic holomorphic structures"** — cf. the relation of $\mathbb{S}^1$ to $\mathbb{C}^\times$ in the complex archimedean theory.

We are now ready to discuss the *main result* of the present §1.

Corollary 1.12. **(Multiradial Constant Multiple Rigidity)** *Write* $(\mathcal{R}, \mathcal{C}, \Phi : \mathcal{R} \to \mathcal{C})$ *— i.e., in the notation of Example 1.8, (v), (vi),*

$$(\Pi \curvearrowright \Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta_*(\Pi)) \otimes \mathbb{Q}/\mathbb{Z}, G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G), \alpha_{\boldsymbol{\mu},\times\boldsymbol{\mu}}) \mapsto (G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G))$$

— *for the* **multiradial environment** *discussed in Example 1.8, (v), (vi), where we take* $\Gamma^{\times\boldsymbol{\mu}} \stackrel{\mathrm{def}}{=} \mathrm{Ism}$. *Consider the* **functorial algorithm** *that associates to*

$$\Pi$$

the following **commutative diagram** $(\dagger_{\times\theta})(\Pi)$

$$\begin{array}{ccccc} M^{\times}_{\mathbb{TM}}(\Pi) & \bigcup & M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}(\Pi) & \hookrightarrow & \varinjlim_J \ H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\Pi)|_J, (l \cdot \Delta_{\Theta})(\Pi)) \\ & & \downarrow & & \downarrow \\ M^{\times}_{\mathbb{TM}}(\mathbb{M}^{\Theta}_{*}(\Pi)) & \bigcup & M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi)) & \hookrightarrow & \varinjlim_J \ H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_{*}(\Pi))|_J, \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(\Pi))) \end{array}$$

— *where*

(a) *J ranges over the finite index open subgroups of Π; "$|_J$" denotes the fiber product "$\times_{\Pi} J$";*

(b) *the right-hand vertical arrow is the isomorphism of modules induced by the* **cyclotomic rigidity isomorphism** *obtained via the functorial algorithm of Corollary 1.10;*

(c) *we recall that it follows from the definitions [cf. Example 1.8, (ii), (iii); [AbsTopIII], Definition 3.1, (vi); [IUTchI], Remark 3.1.2] that one has a natural inclusion $M^{\times}_{\mathbb{TM}}(\Pi) \hookrightarrow \varinjlim_J \ H^1(J, (l \cdot \Delta_{\Theta})(\Pi))$, hence a natural inclusion of $M^{\times}_{\mathbb{TM}}(\Pi)$ into the inductive limit of the first line;*

(d) *we define $M^{\times}_{\mathbb{TM}}(\mathbb{M}^{\Theta}_{*}(\Pi))$ and the left-hand vertical arrow to be the submodule and bijection induced by the cyclotomic rigidity isomorphism of (b);*

(e) *we define $M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}(\Pi) \stackrel{\mathrm{def}}{=} M^{\times}_{\mathbb{TM}}(\Pi) \cdot {}_{\infty}\underline{\underline{\theta}}(\Pi)$; here, ${}_{\infty}\underline{\underline{\theta}}(\Pi)$ is obtained via the functorial algorithm of Proposition 1.4, applied to Π, and the "$\cdot$" is to be understood as being taken with respect to the module structure [i.e., which is usually denoted* **additively***!] of the ambient cohomology module;*

(f) *we define $M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \stackrel{\mathrm{def}}{=} M^{\times}_{\mathbb{TM}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \cdot {}_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi))$; here, ${}_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi))$ is obtained via the functorial algorithm of Proposition 1.5, (iii), applied to $\mathbb{M}^{\Theta}_{*}(\Pi)$ [cf. Propositions 1.2, (i); 1.5, (i)]; the "$\cdot$" is as in (e);*

(g) *the horizontal arrows "$\hookrightarrow$" are the natural inclusions.*

Also, let us write $M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(-) \stackrel{\mathrm{def}}{=} M^{\times}_{\mathbb{TM}}(-)/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(-)$, where $M^{\boldsymbol{\mu}}_{\mathbb{TM}}(-) \subseteq M^{\times}_{\mathbb{TM}}(-)$ denotes the submodule of torsion elements. Then:

(i) There is a **functorial group-theoretic algorithm**

$$\Pi \quad \mapsto \quad \{(\iota, D)\}(\Pi)$$

that assigns to the topological group Π a collection of pairs (ι, D) — where $\Delta_{\underline{\underline{\ddot{Y}}}}(\Pi) \stackrel{\mathrm{def}}{=} \Pi_{\underline{\underline{\ddot{Y}}}}(\Pi) \bigcap \Delta$, ι is a $\Delta_{\underline{\underline{\ddot{Y}}}}(\Pi)$-outer automorphism of $\Pi_{\underline{\underline{\ddot{Y}}}}(\Pi)$ [cf. Proposition 1.4], and *[by abuse of notation] $D \subseteq \Pi_{\underline{\underline{\ddot{Y}}}}(\Pi)$ is a $\Delta_{\underline{\underline{\ddot{Y}}}}(\Pi)$-conjugacy class of closed subgroups — with the property that when $\Pi = \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_k}$, the resulting collection of pairs coincides with the collection of "pointed inversion automorphisms" of Remark 1.4.1, (ii). Here, each pair (ι, D) will be referred to as a* **pointed inversion automorphism**. *If (ι, D) is a pointed inversion automorphism, and ι* **induces** *an* **"action up to torsion"** *on some subset "$(-)$" of an abelian group [i.e., an action on the image of this subset in the quotient of the abelian group by its torsion subgroup], then we shall denote by a superscript "ι" on "$(-)$" the subset of* **ι-invariants** *with respect to this "action up to torsion", i.e., the subset of "$(-)$" that consists precisely of those elements of "$(-)$" whose images in the quotient of the abelian group by its torsion subgroup are* **fixed** *by the induced action of ι.*

(ii) Let (ι, D) be a pointed inversion automorphism associated to Π [cf. (i)]. Then **restriction** *to the subgroup $D \subseteq \Pi_{\underline{\underline{\ddot{Y}}}}(\Pi)$ determines [the horizontal arrows in] a* **commutative diagram**

$$\begin{array}{ccc} \{M^{\times}_{\mathbb{TM}} \cdot {}_\infty\underline{\underline{\theta}}(\Pi)\}^{\iota} & \longrightarrow & M^{\times}_{\mathbb{TM}}(\Pi) \quad \left(\subseteq \varinjlim_J \ H^1(J, (l \cdot \Delta_{\Theta})(\Pi))\right) \\ \downarrow & & \downarrow \\ \{M^{\times}_{\mathbb{TM}} \cdot {}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_*(\Pi))\}^{\iota} & \longrightarrow & M^{\times}_{\mathbb{TM}}(\mathbb{M}^{\Theta}_*(\Pi)) \quad \left(\subseteq \varinjlim_J \ H^1(J, \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_*(\Pi)))\right) \end{array}$$

— where J ranges over the finite index open subgroups of Π [cf. (a)]; the vertical arrows are the isomorphisms induced by the cyclotomic rigidity isomorphism of Corollary 1.10 [cf. (b)]. Here, the inverse images of the submodules of torsion elements — i.e., [up to various natural isomorphisms] the submodules "$M^{\boldsymbol{\mu}}_{\mathbb{TM}}(-)$" — via the upper and lower horizontal arrows are given, respectively, by ${}_\infty\underline{\underline{\theta}}(\Pi)^{\iota}$ and ${}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_*(\Pi))^{\iota}$. In particular, we obtain a functorial algorithm [in the topological group Π] for constructing **splittings**

$$\begin{gathered} M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi) \ \times \ \{{}_\infty\underline{\underline{\theta}}(\Pi)^{\iota}/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi)\}; \\ M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(\mathbb{M}^{\Theta}_*(\Pi)) \ \times \ \{{}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_*(\Pi))^{\iota}/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\mathbb{M}^{\Theta}_*(\Pi))\} \end{gathered} \qquad (\dagger_{\boldsymbol{\mu}\theta})(\Pi)$$

— i.e., direct product decompositions inside the quotients of the inductive limits on the right-hand side of the diagram $(\dagger_{\times\theta})(\Pi)$ by "$M^{\boldsymbol{\mu}}_{\mathbb{TM}}(-)$" — of the respective images of $\{M^{\times}_{\mathbb{TM}} \cdot {}_\infty\underline{\underline{\theta}}(\Pi)\}^{\iota}$, $\{M^{\times}_{\mathbb{TM}} \cdot {}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_*(\Pi))\}^{\iota}$ in these quotients.

(iii) Consider the assignment that associates to the data

$$(\Pi \curvearrowright \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_*(\Pi)) \otimes \mathbb{Q}/\mathbb{Z}, G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G), \alpha_{\boldsymbol{\mu},\times\boldsymbol{\mu}})$$

the data consisting of

- $\mathbb{M}^{\Theta}_*(\Pi)$ — *i.e., the* **projective systems of mono-theta environments** *of Propositions 1.2, (i); 1.5, (i);*
- $(\dagger_{\times\theta})(\Pi)$ — *i.e.,* **"subsets"***;*
- $(\dagger_{\boldsymbol{\mu}\theta})(\Pi)$ — *i.e.,* **"splittings"***;*

· *the* **diagram**

$$\begin{array}{ccccc} \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(\Pi)) \otimes \mathbb{Q}/\mathbb{Z} & \overset{\sim}{\to} & M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\mathbb{M}^{\Theta}_{*}(\Pi)) & \overset{\sim}{\to} & M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi) \\ & \to & M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi) & \overset{\sim}{\to} & \mathcal{O}^{\times\boldsymbol{\mu}}(G) \end{array} \qquad (\dagger_{\boldsymbol{\mu},\times\boldsymbol{\mu}})$$

— where the first " $\overset{\sim}{\to}$ *" is the isomorphism determined by the injection of Remark 1.5.2; the second "* $\overset{\sim}{\to}$ *" is the isomorphism determined by the vertical arrows of* $(\dagger_{\times\theta})(\Pi)$*; the "*$\to$*" is the* **trivial** *homomorphism; the final "* $\overset{\sim}{\to}$ *" denotes the* **poly-isomorphism** *induced by the poly-isomorphism "*$\alpha_{\times}$*" of Example 1.8, (iii) [cf. also the discussion of "*$\Gamma^{\times\boldsymbol{\mu}}$*" in Example 1.8, (iv)].*

Then this assignment determines a functor $\mathcal{R} \to \mathcal{F}$ *which arises from a functorial algorithm; denote the corresponding* **graph** *[cf. Example 1.9, (i)] by* $\mathcal{R}^{\dagger}$*. In particular, the resulting natural functor* $\Psi_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}^{\dagger}$ *[cf. Example 1.9, (i)] is* **multiradially defined**.

Proof. Assertion (i) follows immediately from the discussion of Remark 1.4.1 and the references quoted in this discussion. Assertion (ii) follows immediately from the structure of the objects under consideration, as described in [EtTh], Proposition 1.5, (ii), (iii) [cf. also the proofs of [EtTh], Theorems 1.6, 1.10]. Finally, the *multiradiality* of assertion (iii) follows immediately from the *characteristic* nature of the various torsion submodules "$M^{\boldsymbol{\mu}}_{\mathbb{TM}}(-)$" that appear [cf. the discussion of Remark 1.10.2; the discussion of Remark 1.12.2 below]. ◯

Remark 1.12.1. One verifies immediately that Corollaries 1.10, 1.11, and 1.12 admit *"log-shell versions"* [cf. Example 1.8, (ix)]. The various interpretations of these corollaries discussed in the remarks following the corollaries also apply to such "log-shell versions".

Remark 1.12.2.

(i) Modulo the multiradiality of the cyclotomic rigidity isomorphism of Corollary 1.10 [cf. Corollary 1.12, (b)], the essential content of the multiradiality of Corollary 1.12 lies in

> the functorial group-theoretic algorithm implicit in the proof of [EtTh], Theorem 1.10, (i), for constructing $\underline{\underline{\theta}}(\Pi)$ up to a $\boldsymbol{\mu}_{2l}$**-indeterminacy** — i.e., as opposed to only up to a *"$\mathcal{O}_k^{\times}$-indeterminacy"*, as is done in the proof of [EtTh], Theorem 1.6, (iii) — together with the [elementary] observation that the submodule of [any isomorph of] $\mathcal{O}_k^{\times}$ constituted by the $2l$-torsion is *characteristic* [cf. the proof of Corollary 1.12, (iii)].

That is to say, it is this "essential content" that implies that the crucial splittings $(\dagger_{\boldsymbol{\mu}\theta})(\Pi)$ are **compatible with gluing** together the various collections of coric data "$(G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G))$" that arise from distinct arithmetic holomorphic structures.

(ii) Here, we recall in passing [cf. also the discussion of Remark 1.4.1] that the functorial group-theoretic algorithm implicit in the proof of [EtTh], Theorem 1.10, (i), for constructing $\underline{\underline{\theta}}(\Pi)$ up to a $\boldsymbol{\mu}_{2l}$-indeterminacy consists of

> *normalizing* the étale theta functions under consideration by requiring that their **values** at points [cf. also the discussion of Remark 1.12.4 below] lying over the 2-torsion point "μ_-" of [IUTchI], Example 4.4, (i), be $\in \boldsymbol{\mu}_{2l}$

— i.e., of considering étale theta functions *"of standard type"* [cf. [EtTh], Definition 1.9, (ii); [EtTh], Theorem 1.10, (i); [EtTh], Definition 2.7]. Also, we recall from the proof of [EtTh], Theorem 1.10, (i), that the decomposition groups $\subseteq \Pi$ corresponding to these points lying over the 2-torsion point "μ_-" are reconstructed by applying, among other tools, the *elliptic cuspidalizations* reviewed in Proposition 1.6, (ii) [cf. also the discussion of Corollary 2.4, (ii), (b), below].

(iii) By contrast, if, in the context of the discussion of (i), the normalization reviewed in (ii) consisted of the requirement that certain values of the étale theta function be equal, for instance, to

$$2 \stackrel{\mathrm{def}}{=} 1 + 1 \in \mathcal{O}_k^\times \subseteq (k^\times)^\wedge$$

[where we recall that the residue characteristic of k is assumed to be *odd* — cf. [IUTchI], Definition 3.1, (b)], i.e., an element of $(k^\times)^\wedge$ whose construction depends, in an essential way, on the *ring structure* relative to some specific $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater — i.e., some specific *arithmetic holomorphic structure* — then the normalization would **fail** *to give rise to a* **multiradially defined** *functor*, although [cf. [AbsTopIII], Corollary 1.10, (h); [IUTchI], Remark 3.1.2] it would nonetheless give rise to a **uniradially defined** functor [cf. the discussion of Example 1.9, (iv), (b); Remark 1.11.5, (ii)].

(iv) From the point of view of the further development of the theory of the present series of papers, the significance of obtaining *"splittings up to a $\boldsymbol{\mu}$-indeterminacy"* may be summarized as follows. Ultimately, we shall be interested, in [IUTchIII], in applying the theory of *log-shells* developed in [AbsTopIII] [cf. Remark 1.12.1]. From the point of view of log-shells, which may be thought of as being contained in $\mathcal{O}^{\times\boldsymbol{\mu}}(G)$, an indeterminacy up to *some larger subgroup* of $\mathcal{O}_k^\times$ — such as, for instance, the subgroup generated by $2 = 1 + 1$, together with its $\mathrm{Aut}(G)$-conjugates [cf. the discussion of (iii)] — would imply that

> one may only work, in an inconsistent fashion, with [for instance, the image of the log-shell in] the **quotient** *of $\mathcal{O}^{\times\boldsymbol{\mu}}(G)$ by such a larger subgroup*

— a situation which is unacceptable from the point of view of the further development of the theory of the present series of papers.

(v) The discussion in (i), (ii), and (iii) above of the **multiradiality** of the crucial splittings $(\dagger_{\boldsymbol{\mu}\theta})(\Pi)$ of Corollary 1.12, (ii), yields another important example [cf. Remark 1.11.3, (iii)] of the phenomenon that sometimes not only the *existence of a single reconstruction algorithm*, but also the **content** *of the reconstruction algorithm* is of crucial importance in the development of the theory. Similar ideas, relative to the point of view of the theory of [EtTh], may also be seen in the discussion of [EtTh], Remarks 1.10.2, 1.10.4.

(vi) In general, multiradiality amounts to a sort of *"surjectivity"* [cf. the definition of a multiradial environment via a *"fullness"* condition in Example 1.7, (ii);

the discussion of Example 1.7, (v)] of the radial data onto the coric data. From this point of view, the content of the multiradiality of the splittings $(\dagger_{\boldsymbol{\mu}\theta})(\Pi)$ of Corollary 1.12, (ii), may be thought of as consisting of a **splitting** *of this "surjection of radial data onto coric data"* into

(a) a **"purely radial component"** constituted by $\{{}_\infty\underline{\underline{\theta}}(\Pi)^\iota/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi)\}$, $\{{}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^\Theta_*(\Pi))^\iota/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\mathbb{M}^\Theta_*(\Pi))\}$ and

(b) a **"purely coric component"** constituted by $M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi)$, $M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(\mathbb{M}^\Theta_*(\Pi))$

[cf. the discussion of Remark 1.11.4].

Remark 1.12.3. From the point of view of the discussion of Remark 1.11.3, it is useful to note that the subsets $M^\times_{\mathbb{TM}} \cdot {}_\infty\underline{\underline{\theta}}(\Pi)$, $M^\times_{\mathbb{TM}} \cdot {}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^\Theta_*(\Pi))$ that appear in Corollary 1.12 may be thought of as ["roots" of] the images, via the **Kummer map**, of a certain generating subset of the monoid of rational functions "$\mathcal{O}^{\rhd}_{\mathcal{C}^\Theta_{\underline{v}}}(-)$" defined in [IUTchI], Example 3.2, (v), which is used to construct the underlying Frobenioid of the *split Frobenioid* "$\mathcal{F}^\Theta_{\underline{v}}$" — cf. also the discussion of Kummer classes in [EtTh], Proposition 5.2, (iii). Here, the *splittings* $(\dagger_{\boldsymbol{\mu}\theta})(\Pi)$ of Corollary 1.12, (ii), correspond to the splitting data of this split Frobenioid $\mathcal{F}^\Theta_{\underline{v}}$. Put another way,

> *this* **monoidal data** *that gives rise to the split Frobenioid*
>
> $$\mathcal{F}^\Theta_{\underline{v}}$$
>
> *may be thought of as the result of* **forgetting the "anabelian structure"** *of* $M^\times_{\mathbb{TM}} \cdot {}_\infty\underline{\underline{\theta}}(\Pi)$, $M^\times_{\mathbb{TM}} \cdot {}_\infty\underline{\underline{\theta}}_{\mathrm{env}}(\mathbb{M}^\Theta_*(\Pi))$, *and* $(\dagger_{\boldsymbol{\mu}\theta})(\Pi)$

— cf. the discussion of Remark 1.11.3, (i), (ii); the theory of §3 below, especially, Proposition 3.4. In particular, the specification of *coric data* "$(G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G))$" in the multiradial environment that appears in Corollary 1.12 arises naturally from the point of view of applying the *"coricity of* $\mathcal{O}^\times$*"* given in [IUTchI], Corollary 3.7, (iii), as in the discussion of Remark 1.11.3, (ii). Finally, we recall from the discussion of Remark 1.11.3, (ii), that this specification of coric data "$(G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G))$" has the effect of **inducing**, in particular, an $(\mathrm{Aut}(G), \mathrm{Im}(\widehat{\mathbb{Z}}^\times)\ (\subseteq \mathrm{Ism}))$*-indeterminacy on "*$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)$*"* [cf. Corollary 1.12, (iii)].

Remark 1.12.4. The fact that the *"theta evaluation"* functorial algorithm of Corollary 1.12, (ii), given by **restriction** *to the decomposition groups associated to the point "*μ_-*"* involves only the topological group "Π" as input data will be of *crucial importance* when we combine the theory developed in the present paper with the theory of log-shells [cf. [AbsTopIII]] in [IUTchIII]. At this point, it is useful to stop and consider to what extent this sort of *"group-theoretic evaluation algorithm"* is an inevitable consequence of various natural conditions. To this end, let us suppose that we are given **some** *"mysterious evaluation algorithm"*

$$\text{(abstract theta function)} \quad \mapsto \quad \text{(theta values)}$$

— i.e., which is not necessarily given by restriction to the decomposition group associated to a closed point. Then [cf. [EtTh], Remark 1.10.4; the theory of the

"$\mathfrak{log}$-wall", as discussed in [AbsTopIII], §I4] it is natural to require [cf., especially, the point of view of the discussion of Remark 1.12.3] that this algorithm be

> **compatible** *with the operation of forming* **Kummer classes** *by extracting N-th roots of the "abstract theta function" and the "theta values".*

In particular, it is natural to require that this algorithm extend to *coverings* [e.g., Galois coverings] on both the input and output data of the algorithm. But then the natural requirement of **functoriality** with respect to the Galois groups on either side leads one [cf. Fig. 1.5 below], in effect, to the *conclusion* — which we shall refer to as the *principle of* **Galois evaluation** — that

> the **"mysterious evaluation algorithm"** under consideration in fact arises from a **section** $G \to \Pi_{\underline{\underline{\ddot{Y}}}}(\Pi)$ of the natural surjection $\Pi_{\underline{\underline{\ddot{Y}}}}(\Pi) \twoheadrightarrow G$.

Moreover, by the *"Section Conjecture"* of anabelian geometry, one expects that such [continuous] sections $G \to \Pi_{\underline{\underline{\ddot{Y}}}}(\Pi)$ necessarily arise from *geometric points*. [Here, we pause to observe that this relation to the "Section Conjecture" is interesting in light of the discussion of [IUTchI], Example 4.5, (i); [IUTchI], Remark 2.5.1.] In this context, it is useful to recall that from the point of view of the theory of [AbsTopIII] [cf., e.g., the discussion of [AbsTopIII], §I5], the *group-theoreticity* of the evaluation algorithm may be thought of as a sort of abstract analogue of the condition, in the p-adic theory, that an operation involving various Frobenius crystals be *compatible with the* **Frobenius crystal** *structures* [i.e., connection and Frobenius action] on the input and output data of the operation.

$\Pi_{\underline{\underline{\ddot{Y}}}}(\Pi)$ $\curvearrowright$	geometric object (+ coverings!) that support(s) the abstract theta function	- - - >	geometric object (+ coverings!) that support(s) the theta values	$\curvearrowleft$ G

Fig. 1.5: Theta evaluation and Galois functoriality

Remark 1.12.5.

(i) Recall that the *scheme-theoretic Hodge-Arakelov theory* reviewed in [HASurI], [HASurII] may be thought of as a sort of *scheme-theoretic version* of the well-known classical computation of the **Gaussian integral**

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx \ = \ \sqrt{\pi}$$

— i.e., by thinking of the square of this integral as an integral over the *cartesian plane* $\mathbb{R}^2$, which may be computed easily by applying a coordinate transformation into *polar coordinates*. That is to say [cf. the left-hand and middle columns of Fig. 1.6 below], the main theorem of scheme-theoretic Hodge-Arakelov theory is a certain *comparison isomorphism* [cf. [HASurI], Theorem A] between a *"de Rham side"* —

which consists of certain sections of an ample line bundle on the universal extension of an elliptic curve — and an *"étale side"* — which consists of arbitrary functions on the set of l-torsion points of the elliptic curve [where l is, say, some odd prime number]. Here, the module on the de Rham side is equipped with a natural *Hodge filtration*, while the module on the étale side is equipped with a natural *Galois action* by $GL_2(\mathbb{F}_l)$. The *ordered, "step-like" structure* of the Hodge filtration is reminiscent of the cartesian structure of the plane $\mathbb{R}^2$, i.e., regarded as an ordered collection [parametrized by one factor of $\mathbb{R}^2$] of lines [corresponding to the other factor of $\mathbb{R}^2$]. On the other hand, the $GL_2(\mathbb{F}_l)$-symmetry of the étale side is reminiscent of the *rotational symmetry* of the representation of the Gaussian integral on the plane via polar coordinates. Moreover, the function "e^{-x^2}" itself appears in the *Gaussian poles* that appear in the de Rham side [cf. [HASurI], §1.1], while the "$\sqrt{\pi}$" may be thought of as corresponding to the [negative] *tensor powers of the sheaf "ω"* of invariant differentials on the elliptic curve that appear in the subquotients of the Hodge filtration, which give rise to a Kodaira-Spencer isomorphism [cf. [HASurII], Theorems 2.8, 2.10] between $\omega^{\otimes 2}$ and the restriction to the base scheme of the sheaf of logarithmic differentials on the moduli stack of elliptic curves — i.e., between ω and the *"square root"* of this sheaf of logarithmic differentials. Finally, we recall that this relationship between the theory of [HASurI], [HASurII] and the classical Gaussian integral may be seen more *explicitly* when this theory is restricted to the archimedean primes of a number field via the *"Hermite model"* [cf. [HASurI], §1.1].

classical Gaussian integral	*scheme-theoretic Hodge-Arakelov theory*	*inter-universal Teichmüller theory*
cartesian coordinates	de Rham side, Hodge filtration	Frobenius-like structures, Frobenius-picture
polar coordinates	étale side, Galois action on torsion points	étale-like structures, étale-picture

Fig. 1.6: Analogy with the classical Gaussian integral

(ii) Just as the theory of [HASurI], [HASurII] may be thought of as a *scheme-theoretic* version of the classical theory of the Gaussian integral,

> *the* **"inter-universal Teichmüller theory"** *developed in the present series of papers may be thought of as a sort of* **global arithmetic/Galois-theoretic** *version of the* **classical Gaussian integral**

— cf. the right-hand column of Fig. 1.6. Indeed, the **ordered, "step-like"** nature of the cartesian representation of the Gaussian integral on the plane is reminiscent of the structure of the **Frobenius-picture** discussed in [IUTchI], Corollary 3.8; [IUTchI], Remark 3.8.1 — i.e., in particular, of the notion of a *Frobenius-like* mathematical structure that appears in the discussion of [FrdI], Introduction.

On the other hand, the *rotational symmetry* of the representation of the Gaussian integral on the plane via polar coordinates is reminiscent of the *étale-picture* discussed in [IUTchI], Corollary 3.9, and the following remarks — i.e., in particular, of the notion of an *étale-like* mathematical structure that appears in the discussion of [FrdI], Introduction. The étale-picture that arises from the multiradially defined functor of Corollary 1.12 is depicted in Fig. 1.7 below [where we recall the notation of Proposition 1.4; Example 1.8, (iv)]. From the point of view of the classical series representation of a theta function — i.e., roughly speaking, the series "$\sum_{n\in\mathbb{Z}} q^{n^2} \cdot U^n$" [cf. [EtTh], Proposition 1.4] —

> *this* **étale-picture** *of various copies of the Gaussian function "q^{n^2}" defined on* **spokes** *emanating from a single common* **core**

$$\begin{array}{ccccc}
 & & {}_{\infty}\underline{\underline{\theta}}({}^{i}\Pi) & & \\
 & \dots & \downarrow & \dots & \\
{}_{\infty}\underline{\underline{\theta}}({}^{i'}\Pi) & \longrightarrow & \boxed{\begin{array}{c}\textbf{mono-analytic core} \\ G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G) \curvearrowleft \mathrm{Ism}\end{array}} & \longleftarrow & {}_{\infty}\underline{\underline{\theta}}({}^{i''}\Pi) \\
 & \dots & \uparrow & \dots & \\
 & & {}_{\infty}\underline{\underline{\theta}}({}^{i'''}\Pi) & &
\end{array}$$

Fig. 1.7: Multiradial étale theta functions

[cf. also the point of view of Remark 1.12.2, (vi)] is highly reminiscent of the polar coordinate representation of the Gaussian integral on the plane. In this context, it is also of interest to observe that the **coordinate transformation**

$$e^{-r^2} \rightsquigarrow u$$

that appears in the *radial portion* of the integrand of the Gaussian integral that arises from the *transformation* from cartesian to polar coordinates

$$\begin{aligned} 2 \cdot (\textstyle\int e^{-x^2} dx)^2 \;&=\; 2 \cdot \textstyle\int\int e^{-x^2-y^2} \; dx \; dy \;=\; \textstyle\int\int e^{-r^2} \cdot 2rdr \; d\theta \\ &=\; \textstyle\int\int d(e^{-r^2}) \; d\theta \;=\; \textstyle\int\int du \; d\theta \end{aligned}$$

is *formally reminiscent* of the Θ**-link** "${}^{\dagger}\underline{\underline{\Theta}}_{v} \mapsto {}^{\ddagger}\underline{\underline{q}}_{v}$" [cf. [IUTchI], Remark 3.8.1, (i)], various versions of which play a *central role* in the theory of the present series of papers.

(iii) Just as the equivalence between cartesian and polar representations of the classical Gaussian integral is used effectively to compute the value of this Gaussian integral, the relationship between the *Frobenius-* and *étale-pictures* will play a **central role** [cf., especially, the computations of [IUTchIII], §3; [IUTchIV], §1] in the theory of the present series of papers.

Section 2: Galois-theoretic Theta Evaluation

In the present §2, we develop the theory of *group-theoretic algorithms* surrounding the **Hodge-Arakelov-theoretic evaluation** *of the étale theta function on l-torsion points*. At a more technical level, this theory depends on a careful analysis of the issue of **conjugate synchronization** [cf. Remark 2.6.1] — i.e., of synchronizing conjugates of various copies of objects associated to the absolute Galois group of the base field that occur at the evaluation points — as well as on the computation, via the theory of [IUTchI], §2, of various **conjugacy indeterminacies** [cf. Corollaries 2.4, 2.5] that arise from the consideration of certain closed subgroups of various topological groups. In fact, these various technical issues arise, ultimately, as a consequence of the requirement of performing the Hodge-Arakelov-theoretic evaluation in question with respect to a **single basepoint** [cf. the discussions of Remark 1.12.4; [IUTchI], Remark 6.12.6]. This Hodge-Arakelov-theoretic evaluation will play a *central role* in the theory developed in the present series of papers.

In the present §2, we shall work mainly with the *local portion* at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ of the various mathematical objects considered in [IUTchI], §3, §4, §5, §6. In fact, however, many of the constructions carried out in the present §2 will be *valid for strictly local data* [as in §1], i.e., that does not necessarily arise from global data over a number field. Nevertheless, in order to keep the notation simple from the point of view of discussing the *compatibility* of the theory of the present §2 with the theory of [IUTchI], we shall carry out the discussion of the present §2 only for the localized objects that arise from the theory of [IUTchI], §3, §4, §5, §6, leaving the routine details of a corresponding purely local theory to the interested reader.

Proposition 2.1. **(Review of Certain Tempered Coverings)** *Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Write*

$$\begin{array}{ccccc} \Pi^{\mathrm{tp}}_{\ddot{\underline{\underline{Y}}}_{\underline{v}}} & \longrightarrow & \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}_{\underline{v}}} & \longrightarrow & \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}} = \Pi_{\underline{v}} \\ \downarrow & & \downarrow & & \downarrow \\ \Pi^{\mathrm{tp}}_{\ddot{Y}_{\underline{v}}} & \longrightarrow & \Pi^{\mathrm{tp}}_{Y_{\underline{v}}} & \longrightarrow & \Pi^{\mathrm{tp}}_{\underline{X}_{\underline{v}}} \end{array}$$

for the diagram of **open injections of topological groups** *arising from the theory of [EtTh], §2 — where*

(a) *$\Pi^{\mathrm{tp}}_{\underline{X}_{\underline{v}}}$, $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}}$ are the tempered fundamental groups determined by the hyperbolic [orbi]curves $\underline{X}_{\underline{v}}$, $\underline{\underline{X}}_{\underline{v}}$ of [IUTchI], Definition 3.1, (e);*

(b) *$\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}_{\underline{v}}} \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}}$, $\Pi^{\mathrm{tp}}_{Y_{\underline{v}}} \subseteq \Pi^{\mathrm{tp}}_{\underline{X}_{\underline{v}}}$ are the open subgroups corresponding to the tempered coverings $\underline{\underline{Y}}_{\underline{v}} \to \underline{\underline{X}}_{\underline{v}}$, $Y_{\underline{v}} \to \underline{X}_{\underline{v}}$ determined by the objects "$\underline{\underline{Y}}^{\log}$", "$Y^{\log}$" in the discussion preceding [EtTh], Definition 2.7;*

(c) *$\Pi^{\mathrm{tp}}_{\ddot{\underline{\underline{Y}}}_{\underline{v}}} \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}}$ is the open subgroup determined by the tempered covering $\ddot{\underline{\underline{Y}}}_{\underline{v}} \to \underline{\underline{X}}_{\underline{v}}$ of [IUTchI], Example 3.2, (ii); $\Pi^{\mathrm{tp}}_{\ddot{Y}_{\underline{v}}} \subseteq \Pi^{\mathrm{tp}}_{\underline{X}_{\underline{v}}}$ is the open subgroup* *corresponding to the tempered covering* $\ddot{Y}_{\underline{v}} \to \underline{X}_{\underline{v}}$ *determined by the object "*$\ddot{Y}^{\log}$*" in the discussion preceding [EtTh], Lemma 1.2;*

(d) the arrows are the natural inclusions, and both squares are **cartesian**.

Then this diagram may be reconstructed via a **functorial group-theoretic algorithm** *[cf. [EtTh], Proposition 2.4] from the [***temp-slim***! — cf., e.g., [SemiAnbd], Example 3.10] topological group* $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}}$.

Proof. The assertions of Proposition 2.1 follow immediately from the results of [EtTh], [SemiAnbd] that are quoted in the statements of these assertions. ◯

Remark 2.1.1. In the notation of Proposition 2.1:

(i) Recall that the *special fiber* of any model of $\ddot{Y}_{\underline{v}}$ that arises from a stable model of $\underline{X}_{\underline{v}}$ consists of a **chain** *of copies of the* **projective line** *joined together at the points "0", "∞"* [cf. the discussion preceding [EtTh], Proposition 1.1]. The set of irreducible components of this special fiber may be thought of as a **torsor** *over the group* $\mathbb{Z}$. Moreover, the *natural action* of $\mathrm{Gal}(\ddot{Y}_{\underline{v}}/Y_{\underline{v}}) \cong \{\pm 1\}$ on $\ddot{Y}_{\underline{v}}$ fixes each of the irreducible components of the special fiber of $\ddot{Y}_{\underline{v}}$ and fits into an *exact sequence* $1 \to \mathrm{Gal}(\ddot{Y}_{\underline{v}}/Y_{\underline{v}}) \to \mathrm{Gal}(\ddot{Y}_{\underline{v}}/\underline{X}_{\underline{v}}) \to \mathrm{Gal}(Y_{\underline{v}}/\underline{X}_{\underline{v}}) \to 1$, where $\mathrm{Gal}(Y_{\underline{v}}/\underline{X}_{\underline{v}})$ may be identified with the subgroup $l \cdot \mathbb{Z} \subseteq \mathbb{Z}$. Since the degree l covering $\underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}}$ is *totally ramified at the cusps*, it thus follows that each of the maps

$$\Gamma_{\underline{\underline{\ddot{Y}}}} \to \Gamma_{\underline{\underline{Y}}}; \quad \Gamma_{\ddot{Y}} \to \Gamma_{Y}; \quad \Gamma_{\underline{\underline{\ddot{Y}}}} \to \Gamma_{\ddot{Y}}; \quad \Gamma_{\underline{\underline{Y}}} \to \Gamma_{Y}; \quad \Gamma_{\underline{\underline{X}}} \to \Gamma_{\underline{X}}$$

on *dual graphs associated to the special fibers of stable models* [where we *omit* the various subscript "$\underline{v}$'s" in order to simplify the notation] determined by the various coverings discussed in Proposition 2.1 induces a *bijection on vertices*.

(ii) Let $\iota_{\underline{\underline{X}}}$, $\iota_{\underline{X}}$, $\iota_{\underline{\underline{\ddot{Y}}}}$ be as in Remark 1.4.1, where we take "$\underline{\underline{X}}_k$" to be $\underline{\underline{X}}_{\underline{v}}$. Write $\iota_{\ddot{Y}}$ for the automorphism of $\ddot{Y}_{\underline{v}}$ induced by $\iota_{\underline{\underline{\ddot{Y}}}}$;

$$\Gamma^{\blacktriangleright}_{\underline{X}} \quad \subseteq \quad \Gamma_{\underline{X}}$$

for the *unique connected subgraph* of $\Gamma_{\underline{X}}$ which is a *tree* that is *stabilized* by $\iota_{\underline{X}}$ and contains *every vertex* of $\Gamma_{\underline{X}}$;

$$\Gamma^{\bullet}_{\underline{X}} \quad \subseteq \quad \Gamma^{\blacktriangleright}_{\underline{X}}$$

for the *unique connected subgraph* of $\Gamma_{\underline{X}}$ *stabilized* by $\iota_{\underline{X}}$ that contains *precisely one vertex* and *no edges*. Thus, if one thinks of the vertices of $\Gamma_{\underline{X}}$ as being labeled by elements $\in$

$$\{-l^{*}, -l^{*}+1, \ldots, -1, 0, 1, \ldots, l^{*}-1, l^{*}\}$$

— where the vertex labeled 0 is fixed by $\iota_{\underline{X}}$ — then $\Gamma^{\blacktriangleright}_{\underline{X}}$ is obtained from $\Gamma_{\underline{X}}$ by eliminating the *unique edge* joining the vertices with labels $\pm l^{*}$; $\Gamma^{\bullet}_{\underline{X}}$ consists of the *unique vertex* 0 and no edges. In particular, by taking appropriate *connected* *components of inverse images*, one concludes [cf. (i)] that $\Gamma^{\blacktriangleright}_{\underline{X}}$ determines *finite, connected subgraphs*

$$\Gamma^{\bullet}_{\underline{\underline{X}}} \subseteq \Gamma^{\blacktriangleright}_{\underline{\underline{X}}} \subseteq \Gamma_{\underline{\underline{X}}}, \quad \Gamma^{\bullet}_{\ddot{Y}} \subseteq \Gamma^{\blacktriangleright}_{\ddot{Y}} \subseteq \Gamma_{\ddot{Y}}, \quad \Gamma^{\bullet}_{\underline{\underline{\ddot{Y}}}} \subseteq \Gamma^{\blacktriangleright}_{\underline{\underline{\ddot{Y}}}} \subseteq \Gamma_{\underline{\underline{\ddot{Y}}}}$$

of the dual graphs corresponding to $\underline{\underline{X}}_{\underline{v}}$, $\ddot{Y}_{\underline{v}}$, $\underline{\underline{\ddot{Y}}}_{\underline{v}}$ which are *stabilized* by the respective *"inversion automorphisms"* $\iota_{\underline{\underline{X}}}$, $\iota_{\ddot{Y}}$, $\iota_{\underline{\underline{\ddot{Y}}}}$. Here, each subgraph $\Gamma^{\bullet}_{(-)}$ consists of *precisely one vertex* and no edges, while the set of vertices of each subgraph $\Gamma^{\blacktriangleright}_{(-)}$ maps *bijectively* to the set of vertices of $\Gamma^{\blacktriangleright}_{\underline{X}}$. In fact, [although we shall not use this fact in the present series of papers] it is not difficult to verify, by considering the *divisibility at the edges [i.e., nodes] of the divisor of poles of the theta function* [cf. [EtTh], Proposition 1.4, (i)], that

each subgraph $\Gamma^{\blacktriangleright}_{(-)}$ maps *isomorphically* to $\Gamma^{\blacktriangleright}_{\underline{X}}$.

Proposition 2.2. **(Decomposition Groups Associated to Subgraphs)** *In the notation of Proposition 2.1, write*

$$\Pi_{\underline{v}^{\bullet}} \subseteq \Pi_{\underline{v}^{\blacktriangleright}} \subseteq \Pi_{\underline{v}}$$

for the **decomposition groups** *determined, respectively, by the subgraphs* $\Gamma^{\bullet}_{\underline{\underline{X}}}$ *and* $\Gamma^{\blacktriangleright}_{\underline{\underline{X}}}$ *— i.e., more precisely, the group "*$\Pi^{\mathrm{tp}}_{X,\mathbb{H}}$*" of [IUTchI], Corollary 2.3, (iii), where we take "*X*" to be* $\underline{\underline{X}}_{\underline{v}}$*, "*$\mathbb{H}$*" to be* $\Gamma^{\bullet}_{\underline{\underline{X}}}$ *or* $\Gamma^{\blacktriangleright}_{\underline{\underline{X}}}$*, "*$\Sigma$*" to be* $\{l\}$*, and "*$\widehat{\Sigma}$*" to be* $\mathfrak{Primes}$*. Thus,* $\Pi_{\underline{v}^{\blacktriangleright}}$ *is well-defined up to* $\Pi_{\underline{v}}$**-conjugacy***; once one fixes* $\Pi_{\underline{v}^{\blacktriangleright}}$*, then the subgroup* $\Pi_{\underline{v}^{\bullet}} \subseteq \Pi_{\underline{v}^{\blacktriangleright}}$ *is well-defined up to* $\Pi_{\underline{v}^{\blacktriangleright}}$**-conjugacy** *[cf. Remark 2.2.1 below];* $\Pi_{\underline{v}^{\blacktriangleright}} \subseteq \Pi^{\mathrm{tp}}_{Y_{\underline{v}}} \bigcap \Pi_{\underline{v}} = \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}_{\underline{v}}}$*. Note, moreover, that we may assume that* $\Pi_{\underline{v}^{\bullet}}$*,* $\Pi_{\underline{v}^{\blacktriangleright}}$*, and* $\iota \stackrel{\mathrm{def}}{=} \iota_{\underline{\underline{\ddot{Y}}}}$ *[cf. Remarks 1.4.1, (ii); 2.1.1, (ii)] have been chosen so that some representative of* ι **stabilizes** $\Pi_{\underline{v}^{\bullet}}$ *and* $\Pi_{\underline{v}^{\blacktriangleright}}$*. Then:*

(i) The collection of data $(\Pi_{\underline{v}^{\bullet}} \subseteq \Pi_{\underline{v}^{\blacktriangleright}} \subseteq \Pi_{\underline{v}}, \iota)$*, regarded up to* $\Pi_{\underline{v}}$**-conjugacy***, may be reconstructed via a* **functorial group-theoretic algorithm** *from the topological group* $\Pi_{\underline{v}}$*.*

(ii) The **functorial group-theoretic algorithms**

$$\Pi_{\underline{v}} \quad \mapsto \quad \underline{\underline{\theta}}(\Pi_{\underline{v}}) \quad \subseteq \quad {}_{\infty}\underline{\underline{\theta}}(\Pi_{\underline{v}}) \quad \subseteq \quad \varinjlim_{J} \ H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\Pi_{\underline{v}})|_J, (l \cdot \Delta_{\Theta})(\Pi_{\underline{v}}))$$

of Proposition 1.4 [i.e., where we take "Π*" to be* $\Pi_{\underline{v}}$*], together with the condition of* **invariance** *with respect to* ι *[cf. [EtTh], Proposition 1.4, (ii); the proof of [EtTh], Theorem 1.6, (iii)], determines a* **specific** $\boldsymbol{\mu}_{2l}$**-** *(respectively,* $\boldsymbol{\mu}$ $(= M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi_{\underline{v}}))$**-***)* **orbit**

$$\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}}) \quad \subseteq \quad \underline{\underline{\theta}}(\Pi_{\underline{v}}) \qquad (\text{respectively,} \quad {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}}) \quad \subseteq \quad {}_{\infty}\underline{\underline{\theta}}(\Pi_{\underline{v}}))$$

within the unique $\{(l \cdot \underline{\mathbb{Z}}) \times \boldsymbol{\mu}_{2l}\}$*- (respectively, each* $\{(l \cdot \underline{\mathbb{Z}}) \times \boldsymbol{\mu}\}$*-) orbit contained in the set* $\underline{\underline{\theta}}(\Pi_{\underline{v}})$ *(respectively,* ${}_{\infty}\underline{\underline{\theta}}(\Pi_{\underline{v}})$*) [cf. Proposition 1.4; Corollary 1.12, (ii)].*

Proof. Assertion (i) follows immediately from the fact that dual graphs of stable models may be *reconstructed via a functorial group-theoretic algorithm* from the corresponding tempered fundamental group [cf., e.g., [SemiAnbd], Corollary 3.11, or, alternatively, [AbsTopI], Theorem 2.14, (i)]. Assertion (ii) follows immediately from the results of [EtTh] that are quoted in the statements of assertion (ii). ◯

Remark 2.2.1. In the notation of Proposition 2.2, we recall that since the subgroup $\Pi_{\underline{v}^\blacktriangleright} \subseteq \Pi_{\underline{v}}$ is *commensurably terminal* [cf. [IUTchI], Corollary 2.3, (iv)], it follows that even when this subgroup is subject to a $\Pi_{\underline{v}}$-conjugacy indeterminacy, the indeterminacy induced on any specific $\Pi_{\underline{v}}$-conjugate of this subgroup $\Pi_{\underline{v}^\blacktriangleright}$ is an indeterminacy with respect to *inner automorphisms* [i.e., of the specific $\Pi_{\underline{v}}$-conjugate of $\Pi_{\underline{v}^\blacktriangleright}$].

Definition 2.3.

(i) In the notation of Proposition 2.2; [IUTchI], Definition 3.1, (e); [IUTchI], Remark 3.1.1: Write $\Delta_{\underline{v}} \stackrel{\mathrm{def}}{=} \Delta^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}}$, $\Delta^{\pm}_{\underline{v}} \stackrel{\mathrm{def}}{=} \Delta^{\mathrm{tp}}_{\underline{X}_{\underline{v}}}$, $\Pi^{\pm}_{\underline{v}} \stackrel{\mathrm{def}}{=} \Pi^{\mathrm{tp}}_{\underline{X}_{\underline{v}}}$, $\Delta^{\mathrm{cor}}_{\underline{v}} \stackrel{\mathrm{def}}{=} \Delta^{\mathrm{tp}}_{C_{\underline{v}}}$, $\Pi^{\mathrm{cor}}_{\underline{v}} \stackrel{\mathrm{def}}{=} \Pi^{\mathrm{tp}}_{C_{\underline{v}}}$; denote the respective profinite completions by means of a "$\wedge$". Thus, we have *natural diagrams of outer inclusions* of topological groups

$$\begin{array}{ccccc} \Delta_{\underline{v}} & \longrightarrow & \Delta^{\pm}_{\underline{v}} & \longrightarrow & \Delta^{\mathrm{cor}}_{\underline{v}} \\ \downarrow & & \downarrow & & \downarrow \\ \Pi_{\underline{v}} & \longrightarrow & \Pi^{\pm}_{\underline{v}} & \longrightarrow & \Pi^{\mathrm{cor}}_{\underline{v}} \end{array} \qquad \begin{array}{ccccc} \widehat{\Delta}_{\underline{v}} & \longrightarrow & \widehat{\Delta}^{\pm}_{\underline{v}} & \longrightarrow & \widehat{\Delta}^{\mathrm{cor}}_{\underline{v}} \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\Pi}_{\underline{v}} & \longrightarrow & \widehat{\Pi}^{\pm}_{\underline{v}} & \longrightarrow & \widehat{\Pi}^{\mathrm{cor}}_{\underline{v}} \end{array}$$

— where the left-hand diagram admits a *natural outer inclusion* into the right-hand diagram, in the evident fashion. Here, we recall that $\widehat{\Delta}_{\underline{v}}$ includes as a normal open subgroup of $\widehat{\Delta}^{\pm}_{\underline{v}}$ of index l [cf. [EtTh], Proposition 2.2, (ii); [EtTh], Remark 2.6.1], that $\widehat{\Delta}^{\pm}_{\underline{v}}$ includes as a normal open subgroup of $\widehat{\Delta}^{\mathrm{cor}}_{\underline{v}}$ of index $2l$ [cf. the discussion preceding [EtTh], Definition 2.1], and that $\Pi^{\pm}_{\underline{v}}$ and $\Pi^{\mathrm{cor}}_{\underline{v}}$ may be reconstructed *group-theoretically* from $\Pi_{\underline{v}}$ [cf. [EtTh], Proposition 2.4]. We shall use these diagrams to regard the various groups appearing in the diagrams as subgroups, well-defined up to $\widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}$-conjugacy, of $\widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}$. Recall the collection of data $(\Pi_{\underline{v}^\bullet} \subseteq \Pi_{\underline{v}^\blacktriangleright} \subseteq \Pi_{\underline{v}}, \iota)$, well-defined up to $\Pi_{\underline{v}}$-*conjugacy*, of Proposition 2.2, (i). Write

$$\Pi^{\pm}_{\underline{v}^\bullet} \stackrel{\mathrm{def}}{=} N_{\Pi^{\pm}_{\underline{v}}}(\Pi_{\underline{v}^\bullet}) \subseteq \Pi^{\pm}_{\underline{v}^\blacktriangleright} \stackrel{\mathrm{def}}{=} N_{\Pi^{\pm}_{\underline{v}}}(\Pi_{\underline{v}^\blacktriangleright}) \subseteq \Pi^{\pm}_{\underline{v}}$$

[cf. Remark 2.1.1, (ii); [IUTchI], Corollary 2.3, (iv)] — so we have natural isomorphisms

$$\Pi^{\pm}_{\underline{v}^\bullet}/\Pi_{\underline{v}^\bullet} \xrightarrow{\sim} \Pi^{\pm}_{\underline{v}^\blacktriangleright}/\Pi_{\underline{v}^\blacktriangleright} \xrightarrow{\sim} \Pi^{\pm}_{\underline{v}}/\Pi_{\underline{v}} \xrightarrow{\sim} \widehat{\Delta}^{\pm}_{\underline{v}}/\widehat{\Delta}_{\underline{v}} \xrightarrow{\sim} \mathrm{Gal}(\underline{\underline{X}}_{\underline{v}}/\underline{X}_{\underline{v}}) \ (\cong \mathbb{Z}/l\mathbb{Z})$$

and equalities $\Pi^{\pm}_{\underline{v}^\bullet} \bigcap \Pi_{\underline{v}} = \Pi_{\underline{v}^\bullet}$, $\Pi^{\pm}_{\underline{v}^\blacktriangleright} \bigcap \Pi_{\underline{v}} = \Pi_{\underline{v}^\blacktriangleright}$ [cf. [IUTchI], Corollary 2.3, (iv)].

(ii) Let $\Pi_{\supseteq}$, $\Pi_{\subseteq}$ be any of the topological groups $\Pi_{\underline{v}}$, $\Pi^{\pm}_{\underline{v}}$, $\Pi^{\mathrm{cor}}_{\underline{v}}$, $\widehat{\Pi}_{\underline{v}}$, $\widehat{\Pi}^{\pm}_{\underline{v}}$, $\widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}$ of (i); suppose that $\Pi_{\subseteq} \subseteq \Pi_{\supseteq}$ relative to one of the *natural outer inclusions* discussed in (i). Then we recall that the *cuspidal inertia groups of* $\Pi_\supseteq$ may be reconstructed *group-theoretically* from the topological group $\Pi_\supseteq$ via the algorithms of [AbsTopI], Lemma 4.5 [cf. also [IUTchI], Remark 1.2.2, (ii)]; [AbsTopI], Proposition 4.10, (vi), and that the *cuspidal inertia groups of* $\Pi_\subseteq$ may be obtained as the intersections with $\Pi_\subseteq$ of those cuspidal inertia groups of $\Pi_\supseteq$ that contain a finite index subgroup that lies inside $\Pi_\subseteq$ [cf. [IUTchI], Corollary 2.5; [IUTchI], Remark 2.5.2], while the *cuspidal inertia groups of* $\Pi_\supseteq$ may be obtained as the $\Pi_\supseteq$-*conjugates* of the *commensurators* [or, alternatively, the *normalizers*] in $\Pi_\supseteq$ of the cuspidal inertia groups of $\Pi_\subseteq$ [cf. [CombGC], Proposition 1.2, (ii)].

(iii) Let $\Pi_\subseteq$ be any of the topological groups $\Pi_{\underline{v}}$, $\Pi^\pm_{\underline{v}}$, $\widehat{\Pi}_{\underline{v}}$, $\widehat{\Pi}^\pm_{\underline{v}}$ of (i); if $\Pi_\subseteq$ is equal to $\Pi_{\underline{v}}$ or $\Pi^\pm_{\underline{v}}$, then set $\Pi_\supseteq \stackrel{\mathrm{def}}{=} \Pi^\pm_{\underline{v}}$; if $\Pi_\subseteq$ is equal to $\widehat{\Pi}_{\underline{v}}$ or $\widehat{\Pi}^\pm_{\underline{v}}$, then set $\Pi_\supseteq \stackrel{\mathrm{def}}{=} \widehat{\Pi}^\pm_{\underline{v}}$. Thus, $\Pi_\subseteq \subseteq \Pi_\supseteq$. Then [cf. [IUTchI], Definition 6.1, (iii)] we define a $\pm$-*label class of cusps* of $\Pi_\subseteq$ to be the set of $\Pi_\subseteq$-conjugacy classes of cuspidal inertia subgroups of $\Pi_\subseteq$ whose commensurators in $\Pi_\supseteq$ [cf. the discussion of (ii)] determine a single $\Pi_\supseteq$-conjugacy class of subgroups in $\Pi_\supseteq$. [Here, we remark in passing that since the inclusion $\Pi_\subseteq \subseteq \Pi_\supseteq$ corresponds to a *totally ramified* covering of curves, it is not difficult to verify that such a set of $\Pi_\subseteq$-conjugacy classes is, in fact, of cardinality *one*.] Write

$$\mathrm{LabCusp}^\pm(\Pi_\subseteq)$$

for the *set of* $\pm$-*label classes of cusps* of $\Pi_\subseteq$. Thus, when $\Pi_\subseteq = \Pi_{\underline{v}}$, if we set ${}^\dagger\mathcal{D}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{B}^{\mathrm{temp}}(\Pi_\subseteq)^0$, then the set $\mathrm{LabCusp}^\pm(\Pi_\subseteq)$ may be *naturally identified* with the set $\mathrm{LabCusp}^\pm({}^\dagger\mathcal{D}_{\underline{v}})$ of [IUTchI], Definition 6.1, (iii). In particular, $\mathrm{LabCusp}^\pm(\Pi_{\underline{v}}) = \mathrm{LabCusp}^\pm({}^\dagger\mathcal{D}_{\underline{v}})$ admits a *natural action by* $\mathbb{F}_l^\times$, as well as a *zero element*

$${}^\dagger\underline{\eta}^0_{\underline{v}} \in \mathrm{LabCusp}^\pm(\Pi_{\underline{v}}) = \mathrm{LabCusp}^\pm({}^\dagger\mathcal{D}_{\underline{v}})$$

and a $\pm$-*canonical element*

$${}^\dagger\underline{\eta}^\pm_{\underline{v}} \in \mathrm{LabCusp}^\pm(\Pi_{\underline{v}}) = \mathrm{LabCusp}^\pm({}^\dagger\mathcal{D}_{\underline{v}})$$

— well-defined up to multiplication by ± 1, which may be constructed solely from ${}^\dagger\mathcal{D}_{\underline{v}}$ [cf. [IUTchI], Definition 6.1, (iii)].

(iv) Let $t \in \mathrm{LabCusp}^\pm(\Pi_{\underline{v}})$. Then t determines a *unique vertex* of $\Gamma^{\blacktriangleright}_{\underline{\underline{X}}}$ [cf. [CombGC], Proposition 1.5, (i)]. Write $\Gamma^{\bullet t}_{\underline{\underline{X}}} \subseteq \Gamma^{\blacktriangleright}_{\underline{\underline{X}}}$ for the connected subgraph with no edges whose *unique vertex* is the vertex determined by t. Then just as in the case of $\Gamma^{\bullet}_{\underline{\underline{X}}}$ [i.e., the case where t is the *zero element*] discussed in Proposition 2.2, the subgraph $\Gamma^{\bullet t}_{\underline{\underline{X}}}$ determines — via a *functorial group-theoretic algorithm* — a *decomposition group*

$$\Pi_{\underline{v}\bullet t} \subseteq \Pi_{\underline{v}\blacktriangleright} \subseteq \Pi_{\underline{v}}$$

— which is well-defined up to $\Pi_{\underline{v}\blacktriangleright}$-*conjugacy.* Finally, we shall write $\Pi^\pm_{\underline{v}\bullet t} \stackrel{\mathrm{def}}{=} N_{\Pi^\pm_{\underline{v}}}(\Pi_{\underline{v}\bullet t})$ [cf. (i)]; thus, we have a natural isomorphism $\Pi^\pm_{\underline{v}\bullet t}/\Pi_{\underline{v}\bullet t} \xrightarrow{\sim} \mathrm{Gal}(\underline{\underline{X}}_{\underline{v}}/\underline{X}_{\underline{v}})$.

(v) Let $\Pi_{\subseteq}$ be either of the topological groups $\Pi^{\pm}_{\underline{v}}$, $\widehat{\Pi}^{\pm}_{\underline{v}}$ of (i); if $\Pi_{\subseteq} = \Pi^{\pm}_{\underline{v}}$, then set $\Pi_{\supseteq} \stackrel{\mathrm{def}}{=} \Pi^{\mathrm{cor}}_{\underline{v}}$; if $\Pi_{\subseteq} = \widehat{\Pi}^{\pm}_{\underline{v}}$, then set $\Pi_{\supseteq} \stackrel{\mathrm{def}}{=} \widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}$. Then one verifies immediately that the images [via the natural outer injection $\Pi_{\underline{v}} \hookrightarrow \Pi_{\subseteq}$] in $\mathrm{LabCusp}^{\pm}(\Pi_{\subseteq})$ of the various structures on $\mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$ reviewed in (iii) determine [in the notation and terminology of [IUTchI], Definition 6.1, (i)] a *natural* $\mathbb{F}^{\pm}_l$*-torsor structure* on $\mathrm{LabCusp}^{\pm}(\Pi_{\subseteq})$. Moreover, the natural action of $\Pi_{\supseteq}/\Pi_{\subseteq}$ on $\Pi_{\subseteq}$ preserves this $\mathbb{F}^{\pm}_l$-torsor structure, hence determines a *natural outer isomorphism*

$$\Pi_{\supseteq}/\Pi_{\subseteq} \xrightarrow{\sim} \mathbb{F}^{\rtimes\pm}_l$$

[cf. [IUTchI], Definition 6.1, (i)].

Remark 2.3.1. In the situation of (iii), suppose that the inclusion $\Pi_{\subseteq} \subseteq \Pi_{\supseteq}$ is *strict*. Then one verifies immediately that if $I \subseteq \Pi_{\supseteq}$ is a cuspidal inertia group of $\Pi_{\supseteq}$, then the cuspidal inertia group $I \bigcap \Pi_{\subseteq} \subseteq \Pi_{\subseteq}$ of $\Pi_{\subseteq}$ satisfies

$$I \bigcap \Pi_{\subseteq} = I^l$$

— where the superscript l is relative to the group operation on I, written *multiplicatively*. In particular, [even though $\Pi_{\underline{v}}$ (respectively, $\widehat{\Pi}_{\underline{v}}$) *fails to be normal* in $\Pi^{\mathrm{cor}}_{\underline{v}}$ (respectively, $\widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}$)] it follows — since $\Pi^{\pm}_{\underline{v}}$ (respectively, $\widehat{\Pi}^{\pm}_{\underline{v}}$) is *normal* in $\Pi^{\mathrm{cor}}_{\underline{v}}$ (respectively, $\widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}$) — that *the cuspidal inertia groups of* $\Pi_{\underline{v}}$ *(respectively,* $\widehat{\Pi}_{\underline{v}}$*) are permuted by the conjugation action of* $\Pi^{\mathrm{cor}}_{\underline{v}}$ *(respectively,* $\widehat{\Pi}^{\mathrm{cor}}_{\underline{v}}$*).*

The *theta evaluation algorithm* discussed in the following Corollaries 2.4, 2.5, 2.8, and 2.9 is *central* to the theory of the present §2, and, indeed, of the present series of papers.

Corollary 2.4. **($\mathbb{F}^{\rtimes\pm}_l$-Symmetric Two-torsion Translates of Cusps)** *In the notation of Definition 2.3: Let* $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$*;* $\Box \in \{\bullet t, \blacktriangleright\}$*. Write*

$$\Delta_{\underline{v}\Box} \stackrel{\mathrm{def}}{=} \Delta_{\underline{v}} \bigcap \Pi_{\underline{v}\Box}, \quad \Delta^{\pm}_{\underline{v}\Box} \stackrel{\mathrm{def}}{=} \Delta^{\pm}_{\underline{v}} \bigcap \Pi^{\pm}_{\underline{v}\Box}$$

$$\Pi_{\underline{v}\ddot{\Box}} \stackrel{\mathrm{def}}{=} \Pi_{\underline{v}\Box} \bigcap \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}, \quad \Delta_{\underline{v}\ddot{\Box}} \stackrel{\mathrm{def}}{=} \Delta_{\underline{v}} \bigcap \Pi_{\underline{v}\ddot{\Box}}$$

— so we have

$$[\Pi_{\underline{v}\Box} : \Pi_{\underline{v}\ddot{\Box}}] = [\Delta_{\underline{v}\Box} : \Delta_{\underline{v}\ddot{\Box}}] = 2, \quad [\Pi^{\pm}_{\underline{v}\Box} : \Pi_{\underline{v}\Box}] = [\Delta^{\pm}_{\underline{v}\Box} : \Delta_{\underline{v}\Box}] = l$$

$$[\Pi^{\pm}_{\underline{v}\Box} : \Pi_{\underline{v}\ddot{\Box}}] = [\Delta^{\pm}_{\underline{v}\Box} : \Delta_{\underline{v}\ddot{\Box}}] = 2l$$

[cf. Definition 2.3, (i), (iv)].

(i) **(Inclusions and Conjugates)** *Let* $I_t \subseteq \Pi_{\underline{v}}$ *be a cuspidal inertia group that belongs to the class determined by t such that* $I_t \subseteq \Delta_{\underline{v}\Box}$*. Consider the [*$\widehat{\Pi}^{\pm}_{\underline{v}}$*-conjugacy stable] sets of subgroups of* $\widehat{\Pi}^{\pm}_{\underline{v}}$

$$\{I_t^{\gamma_1}\}_{\gamma_1 \in \widehat{\Pi}^{\pm}_{\underline{v}}} = \{I_t^{\gamma_1}\}_{\gamma_1 \in \widehat{\Delta}^{\pm}_{\underline{v}}}$$

$$\{\Pi^{\gamma_2}_{\underline{v}\Box}\}_{\gamma_2 \in \widehat{\Pi}^{\pm}_{\underline{v}}} = \{\Pi^{\gamma_2}_{\underline{v}\Box}\}_{\gamma_2 \in \widehat{\Delta}^{\pm}_{\underline{v}}}; \quad \{(\Pi^{\pm}_{\underline{v}\Box})^{\gamma_3}\}_{\gamma_3 \in \widehat{\Pi}^{\pm}_{\underline{v}}} = \{(\Pi^{\pm}_{\underline{v}\Box})^{\gamma_3}\}_{\gamma_3 \in \widehat{\Delta}^{\pm}_{\underline{v}}}$$

— where the superscript "γ's" denotes conjugation [i.e., "$(-) \mapsto \gamma \cdot (-) \cdot \gamma^{-1}$"] by γ. Then for $\gamma, \gamma' \in \widehat{\Delta}^{\pm}_{\underline{v}}$, the following three conditions are **equivalent***:*

$$(a)\ \gamma' \in \Delta^{\pm}_{\underline{v}\Box}; \quad (b)\ I_t^{\gamma\cdot\gamma'} \subseteq \Pi^{\gamma}_{\underline{v}\Box}; \quad (c)\ I_t^{\gamma\cdot\gamma'} \subseteq (\Pi^{\pm}_{\underline{v}\Box})^{\gamma}.$$

(ii) **(Two-torsion Translates of Cusps)** *In the situation of (i), if we write $\delta \stackrel{\text{def}}{=} \gamma \cdot \gamma' \in \widehat{\Delta}^{\pm}_{\underline{v}}$, then any inclusion*

$$I_t^{\delta} = I_t^{\gamma\cdot\gamma'} \subseteq \Pi^{\gamma}_{\underline{v}\Box} = \Pi^{\delta}_{\underline{v}\Box}$$

as in (i) **completely determines** *the following data:*

(a) a **decomposition group** *$D_t^{\delta} \stackrel{\text{def}}{=} N_{\Pi^{\delta}_{\underline{v}}}(I_t^{\delta}) \subseteq \Pi^{\delta}_{\underline{v}\ddot{\Box}}$ corresponding to the inertia group I_t^{δ};*

(b) a **decomposition group** *$D^{\delta}_{\mu_-} \subseteq \Pi^{\delta}_{\underline{v}\ddot{\blacktriangleright}}$, well-defined up to $(\Pi^{\pm}_{\underline{v}\blacktriangleright})^{\delta}$- [or, equivalently, $(\Delta^{\pm}_{\underline{v}\blacktriangleright})^{\delta}$-] conjugacy, corresponding to the torsion point "μ_-" of Remark 1.4.1, (i), (ii), via the algorithms of [SemiAnbd], Theorem 6.8, (iii) [concerning the group-theoreticity of the decomposition groups of torsion points], and [SemiAnbd], Corollary 3.11 [concerning the dual semi-graphs of the special fibers of stable models], applied to $\Delta^{\delta}_{\underline{v}} \subseteq \Pi^{\delta}_{\underline{v}}$;*

(c) a **decomposition group** *$D^{\delta}_{t,\mu_-} \subseteq \Pi^{\delta}_{\underline{v}\ddot{\Box}}$, well-defined up to $(\Pi^{\pm}_{\underline{v}\Box})^{\delta}$- [or, equivalently, $(\Delta^{\pm}_{\underline{v}\Box})^{\delta}$-] conjugacy — i.e., the image of an* **evaluation section** *[cf. [IUTchI], Example 4.4, (i)] — corresponding to the "μ_--translate of the cusp that gives rise to I_t^{δ}", via the algorithm of [SemiAnbd], Theorem 6.8, (iii) [concerning the group-theoreticity of the decomposition groups of translates by torsion points of the cusps].*

Moreover, the construction of the above data is **compatible** *with* **conjugation** *by arbitrary $\delta \in \widehat{\Delta}^{\pm}_{\underline{v}}$, as well as with the natural* **inclusion** *$\Pi_{\underline{v}\bullet t} \subseteq \Pi_{\underline{v}\blacktriangleright}$ of Definition 2.3, (iv), as one varies $\Box \in \{\bullet t, \blacktriangleright\}$.*

(iii) **($\mathbb{F}_l^{\rtimes\pm}$-Symmetry)** *Suppose that $\Box = \bullet t$. Then the construction of the data of (ii), (a), (c), is* **compatible** *with conjugation by arbitrary $\delta \in \widehat{\Pi}^{\text{cor}}_{\underline{v}}$ [cf. Remark 2.3.1]. Here, we recall from Definition 2.3, (v), that we have natural outer isomorphisms $\widehat{\Delta}^{\text{cor}}_{\underline{v}}/\widehat{\Delta}^{\pm}_{\underline{v}} \xrightarrow{\sim} \widehat{\Pi}^{\text{cor}}_{\underline{v}}/\widehat{\Pi}^{\pm}_{\underline{v}} \xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm}$.*

Proof. First, we consider assertion (i). The implications (a) $\Longrightarrow$ (b) and (b) $\Longrightarrow$ (c) are immediate from the definitions [cf. also Remark 2.3.1]. Thus, it suffices to verify that (c) $\Longrightarrow$ (a), i.e., that the condition $I_t^{\gamma\cdot\gamma'} \subseteq (\Pi^{\pm}_{\underline{v}\Box})^{\gamma}$ implies that $\gamma' \in \Delta^{\pm}_{\underline{v}\Box}$; we may assume without loss of generality that $\gamma = 1$. Then by [IUTchI], Corollary 2.5 [cf. also [IUTchI], Remark 2.5.2], the inclusion $I_t^{\gamma'} \subseteq \Pi^{\pm}_{\underline{v}\Box} \subseteq \Pi^{\pm}_{\underline{v}}$ implies that $\gamma' \in \Delta^{\pm}_{\underline{v}}$. Now, by applying the equivalence of [IUTchI], Corollary 2.3, (vi) [cf.

also [CombGC], Proposition 1.2, (ii)], to the various finite index open subgroups of $\Delta^{\pm}_{\underline{v}}$, it follows that $\gamma' \in \widehat{\Delta}^{\pm}_{\underline{v}\square}$ — where we use the notation "$\wedge$" to denote the *closure* in $\widehat{\Delta}^{\pm}_{\underline{v}}$ [cf. Proposition 2.2; Definition 2.3, (iv); [IUTchI], Corollary 2.3, (ii)] — hence that $\gamma' \in \Delta^{\pm}_{\underline{v}\square} = \widehat{\Delta}^{\pm}_{\underline{v}\square} \bigcap \Delta^{\pm}_{\underline{v}}$ [cf. [IUTchI], Corollary 2.3, (v)]. This completes the proof of assertion (i). Assertions (ii) and (iii) follow immediately from the definitions and the references quoted in the statements of these assertions. ○

Remark 2.4.1. Note that by applying [IUTchI], Proposition 2.4, (i) [cf. the proof of [IUTchI], Corollary 2.5; [IUTchI], Remark 2.5.2], one may replace "I_t" in Corollary 2.4 by its *maximal pro-l' subgroup* for any $l' \in \mathfrak{Primes} \setminus \{p_{\underline{v}}\}$. The use of such maximal pro-l' subgroups sometimes results in a simplification of arguments involving intersections with other closed subgroups, since every closed subgroup of such a maximal pro-l' subgroup is either *open* or *trivial*.

Corollary 2.5. **(Group-theoretic Theta Evaluation)** *In the notation of Corollary 2.4:*

(i) **(Restriction of Subquotients to Subgraphs)** *Write*

$$(l \cdot \Delta_\Theta)(\Pi_{\underline{v}\ddot{\blacktriangleright}})$$

for the subquotient of $\Pi_{\underline{v}\ddot{\blacktriangleright}}$ *determined by the subquotient* $(l \cdot \Delta_\Theta)(\Pi_{\underline{v}})$ *of* $\Pi_{\underline{v}}$*. Then the inclusion* $\Pi_{\underline{v}\ddot{\blacktriangleright}} \hookrightarrow \Pi_{\underline{v}}$ *induces an* **isomorphism** $(l \cdot \Delta_\Theta)(\Pi_{\underline{v}\ddot{\blacktriangleright}}) \xrightarrow{\sim} (l \cdot \Delta_\Theta)(\Pi_{\underline{v}})$*. Write*

$$\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}}), \quad \Pi_{\underline{v}\ddot{\blacktriangleright}} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$$

for the quotients determined by the natural surjection $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$*. Then there exists a* **functorial group-theoretic algorithm** *for constructing these quotients from the topological group* $\Pi_{\underline{v}}$ *[cf., e.g., [AbsAnab], Lemma 1.3.8, as well as Proposition 2.2, (i); Corollary 2.4 of the present paper]].*

(ii) **(Restriction of Étale Theta Functions to Subgraphs and Evaluation Points)** *Let*

$$I^\delta_t = I^{\gamma\cdot\gamma'}_t \subseteq \Pi^\delta_{\underline{v}\ddot{\blacktriangleright}} \subseteq \Pi^\gamma_{\underline{v}\blacktriangleright} = \Pi^\delta_{\underline{v}\blacktriangleright}$$

be an inclusion as in Corollary 2.4, (ii) [where we take $\square \stackrel{\mathrm{def}}{=} \blacktriangleright$*]. Then* **restriction** *of the* ι^γ*-invariant sets* $\underline{\underline{\theta}}^\iota(\Pi^\gamma_{\underline{v}})$, ${}_\infty\underline{\underline{\theta}}^\iota(\Pi^\gamma_{\underline{v}})$ *of Proposition 2.2, (ii), to the* **subgroup** $\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}} \subseteq \Pi_{\underline{\underline{\ddot{Y}}}}(\Pi^\gamma_{\underline{v}})$ $(\subseteq \Pi^\gamma_{\underline{v}})$ *yields* $\boldsymbol{\mu}_{2l}$*-,* $\boldsymbol{\mu}$*-orbits of elements*

$$\underline{\underline{\theta}}^\iota(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}) \quad \subseteq \quad {}_\infty\underline{\underline{\theta}}^\iota(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}) \quad \subseteq \quad \varinjlim_{\widehat{J}} \ H^1(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}|_{\widehat{J}}, (l \cdot \Delta_\Theta)(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}))$$

— where $\widehat{J} \subseteq \widehat{\Pi}_{\underline{v}}$ *ranges over the open subgroups of* $\widehat{\Pi}_{\underline{v}}$ *— which, upon further* **restriction** *to the decomposition groups* D^δ_{t,μ_-} *of Corollary 2.4, (ii), (c), yield* $\boldsymbol{\mu}_{2l}$*-,* $\boldsymbol{\mu}$*-orbits of elements*

$$\underline{\underline{\theta}}^t(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}) \quad \subseteq \quad {}_\infty\underline{\underline{\theta}}^t(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}) \quad \subseteq \quad \varinjlim_{J_G} \ H^1(G_{\underline{v}}(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}})|_{J_G}, (l \cdot \Delta_\Theta)(\Pi^\gamma_{\underline{v}\ddot{\blacktriangleright}}))$$

for each $t \in \mathrm{LabCusp}^{\pm}(\Pi^{\gamma}_{\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$ *— where* $J_G \subseteq G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$ *ranges over the open subgroups of* $G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$*; the "*$\xrightarrow{\sim}$*" is induced by conjugation by* γ*. Moreover, the sets* $\underline{\underline{\theta}}^{t}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$, ${}_{\infty}\underline{\underline{\theta}}^{t}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$ *depend only on the* **label** $|t| \in |\mathbb{F}_l|$ *determined by* t *[cf. Definition 2.3, (iii); [IUTchI], Example 4.4, (i); [IUTchI], Definition 6.1, (iii)]. Thus, we shall write* $\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \stackrel{\mathrm{def}}{=} \underline{\underline{\theta}}^{t}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$, ${}_{\infty}\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \stackrel{\mathrm{def}}{=} {}_{\infty}\underline{\underline{\theta}}^{t}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$.

(iii) **(Functorial Group-theoretic Evaluation Algorithm)** *If one starts with an* **arbitrary** $\widehat{\Delta}^{\pm}_{\underline{v}}$**-conjugate** $\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}$ *of* $\Pi_{\underline{v}\ddot{\blacktriangleright}}$*, and one considers, as* t *ranges over the elements of* $\mathrm{LabCusp}^{\pm}(\Pi^{\gamma}_{\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$ *[where the "*$\xrightarrow{\sim}$*" is induced by conjugation by* γ*], the resulting* $\boldsymbol{\mu}_{2l}$*-,* $\boldsymbol{\mu}$*-orbits* $\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$, ${}_{\infty}\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$ *arising from an* **arbitrary** $\widehat{\Delta}^{\pm}_{\underline{v}}$**-conjugate** I^{δ}_t *of* I_t *that is contained in* $\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}$ *[cf. (ii)], then one obtains a* **group-theoretic algorithm** *for constructing the collections of* $\boldsymbol{\mu}_{2l}$*-,* $\boldsymbol{\mu}$*-orbits*

$$\{\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})\}_{|t| \in |\mathbb{F}_l|}; \quad \{{}_{\infty}\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})\}_{|t| \in |\mathbb{F}_l|}$$

which is **functorial** *in the topological group* $\Pi_{\underline{v}}$ *and, moreover,* **compatible** *with the* **independent conjugacy actions** *of* $\widehat{\Delta}^{\pm}_{\underline{v}}$ *on the sets* $\{I^{\gamma_1}_t\}_{\gamma_1 \in \widehat{\Pi}^{\pm}_{\underline{v}}} = \{I^{\gamma_1}_t\}_{\gamma_1 \in \widehat{\Delta}^{\pm}_{\underline{v}}}$ *and* $\{\Pi^{\gamma_2}_{\underline{v}\ddot{\blacktriangleright}}\}_{\gamma_2 \in \widehat{\Pi}^{\pm}_{\underline{v}}} = \{\Pi^{\gamma_2}_{\underline{v}\ddot{\blacktriangleright}}\}_{\gamma_2 \in \widehat{\Delta}^{\pm}_{\underline{v}}}$ *[cf. the sets of Corollary 2.4, (i); Remark 2.2.1].*

Proof. Assertions (i), (ii), and (iii) follow immediately from the definitions and the references quoted in the statements of these assertions. Here, in assertion (i), we observe that the fact that the inclusion $\Pi_{\underline{v}\ddot{\blacktriangleright}} \hookrightarrow \Pi_{\underline{v}}$ induces an *isomorphism* $(l \cdot \Delta_{\Theta})(\Pi_{\underline{v}\ddot{\blacktriangleright}}) \xrightarrow{\sim} (l \cdot \Delta_{\Theta})(\Pi_{\underline{v}})$ follows immediately by considering the *cuspidal inertia groups* involved. ◯

Remark 2.5.1.

(i) Recall from the discussion of [IUTchI], Example 4.4, (i), that relative to the "standard" *cyclotomic rigidity isomorphism* $(*^{\text{bs-Gal}})$ of Proposition 1.3, (ii), and the resulting *Kummer map*

$$K^{\times}_{\underline{v}} \hookrightarrow H^1(G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}}), (l \cdot \Delta_{\Theta})(\Pi_{\underline{v}\ddot{\blacktriangleright}}))$$

[i.e., we take "δ" in Corollary 2.5, (ii), to be the identity — without loss of generality, in light of Remark 2.2.1], it follows immediately from the *definition of the connected subgraph* "$\Gamma^{\blacktriangleright}_{\underline{\underline{X}}}$" in Remark 2.1.1, (ii) [cf. also [IUTchI], Corollary 2.3, (vi)], that, for $j \in |\mathbb{F}_l|$, the set $\underline{\underline{\theta}}^{j}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$ consists of precisely the $\boldsymbol{\mu}_{2l}$-orbit of the **"theta value"**

$$\underline{\underline{q}}_{\underline{v}}^{\underline{\underline{j}}^2}$$

[cf. [IUTchI], Example 3.2, (iv); [EtTh], Proposition 1.4, (ii)] — where the "$\underline{\underline{j}}$" in the exponent denotes the element $\in \{0, 1, \ldots, l^{*}\}$ determined by the given element $j \in |\mathbb{F}_l|$.

(ii) Note that [the reciprocals of the l-th powers of] the theta values discussed in (i) are *somewhat unusual* among the various values

$$\ddot{\Theta}(c)$$

— where $c \in K_{\underline{v}}$ — attained by the theta series

$$\ddot{\Theta} = \ddot{\Theta}(\ddot{U}) \stackrel{\text{def}}{=} q_X^{-\frac{1}{8}} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot q_X^{\frac{1}{2}(n+\frac{1}{2})^2} \cdot \ddot{U}^{2n+1}$$

discussed in [EtTh], Proposition 1.4 [cf. the notation of *loc. cit.*] in that they satisfy the following *crucial property* [cf. the discussion of Remark 1.12.2]:

> the **ratio** $\ddot{\Theta}(c)/\ddot{\Theta}(c')$ is a **root of unity**, for any $c' \in K_{\underline{v}}$ [corresponding to a point of $\ddot{Y}_{\underline{v}}$] that occurs as the result of applying an automorphism of $\Pi_{\underline{v}}$ to [the point of $\ddot{Y}_{\underline{v}}$ that corresponds to] c such that c'/c is a *unit*.

That is to say, the reciprocals of the l-th powers of the theta values discussed in (i) correspond to the values $\ddot{\Theta}(\pm\sqrt{-1} \cdot q_X^{\underline{\underline{j}}/2})$, where $\underline{\underline{j}} \in \{0, 1, \ldots, l^*\}$, i.e., the values at points separated by *periods* [i.e., the "$q_X^{\underline{\underline{j}}/2}$"] from the point "$\pm\sqrt{-1}$". These values may be computed easily from the *"functional equations"* given in [EtTh], Proposition 1.4, (ii).

(iii) Note that, in the context of the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** discussed in Corollary 2.4, (iii),

> the various $\boldsymbol{\mu}_{2l}$*-multiple indeterminacies* that occur, for various $j \in |\mathbb{F}_l|$, in the $\boldsymbol{\mu}_{2l}$-orbit $\underline{\underline{\theta}}^j(\Pi_{\underline{v}\blacktriangleright\!\!\ddot{}})$ are **independent**.

That is to say, these indeterminacies are **not** *"synchronized"* so as to arise from a *single indeterminacy* that is independent of j. Indeed, each of these $\boldsymbol{\mu}_{2l}$-multiple indeterminacies arises as a consequence of the action of $(\Delta^{\pm}_{\underline{v}\bullet t}/\Delta_{\underline{v}\ddot{\bullet} t})^{\delta}$, where we recall from Corollary 2.4 that $[\Delta^{\pm}_{\underline{v}\bullet t} : \Delta_{\underline{v}\ddot{\bullet} t}] = 2l$, on the *decomposition groups* "$D^{\delta}_{t,\mu_-} \subseteq \Pi^{\delta}_{\underline{v}\ddot{\blacktriangleright}}$" of Corollary 2.4, (ii), (c), hence is *induced* by the $\widehat{\Delta}^{\pm}_{\underline{v}}$**-outer** nature of the action of $\widehat{\Delta}^{\text{cor}}_{\underline{v}}/\widehat{\Delta}^{\pm}_{\underline{v}} \stackrel{\sim}{\to} \mathbb{F}_l^{\rtimes\pm}$ that appears in Corollary 2.4, (iii) — cf. the discussion of Remarks 2.5.2, 2.6.2 below.

Remark 2.5.2.

(i) If one thinks of the

$$\text{\textit{"set} } \{I_t^{\gamma_1}\}_{\gamma_1 \in \widehat{\Pi}^{\pm}_{\underline{v}}} = \{I_t^{\gamma_1}\}_{\gamma_1 \in \widehat{\Delta}^{\pm}_{\underline{v}}} \text{ \textit{regarded up to} } \widehat{\Delta}^{\pm}_{\underline{v}}\text{\textit{-conjugacy"}}$$

$$\text{(respectively, \textit{"set} } \{\Pi^{\gamma_2}_{\underline{v}\ddot{\blacktriangleright}}\}_{\gamma_2 \in \widehat{\Pi}^{\pm}_{\underline{v}}} = \{\Pi^{\gamma_2}_{\underline{v}\ddot{\blacktriangleright}}\}_{\gamma_2 \in \widehat{\Delta}^{\pm}_{\underline{v}}} \text{ \textit{regarded up to} } \widehat{\Delta}^{\pm}_{\underline{v}}\text{\textit{-conjugacy"}})$$

[cf. Corollary 2.5, (iii)] as a sort of **quotient** *by* $\widehat{\Delta}^{\pm}_{\underline{v}}$, then one may think of the various *inclusion morphisms* $I_t^{\gamma_1} \hookrightarrow \Pi^{\gamma_2}_{\underline{v}\ddot{\blacktriangleright}}$ as a sort of *morphism between quotients*

$$\left(\widehat{\Delta}^{\pm}_{\underline{v}} \curvearrowright \{I_t^{\gamma_1}\}_{\gamma_1 \in \widehat{\Delta}^{\pm}_{\underline{v}}}\right)/\widehat{\Delta}^{\pm}_{\underline{v}} \to \left(\widehat{\Delta}^{\pm}_{\underline{v}} \curvearrowright \{\Pi^{\gamma_2}_{\underline{v}\ddot{\blacktriangleright}}\}_{\gamma_2 \in \widehat{\Delta}^{\pm}_{\underline{v}}}\right)/\widehat{\Delta}^{\pm}_{\underline{v}}$$

which *induces a morphism between quotients*

$$\left(\widehat{\Delta}^{\pm}_{\underline{v}} \curvearrowright \{D^{\gamma_1}_{t,\mu_-}\}_{\gamma_1 \in \widehat{\Delta}^{\pm}_{\underline{v}}}\right)/\widehat{\Delta}^{\pm}_{\underline{v}} \to \left(\widehat{\Delta}^{\pm}_{\underline{v}} \curvearrowright \{\Pi^{\gamma_2}_{\underline{v}\ddot{\blacktriangleright}}\}_{\gamma_2 \in \widehat{\Delta}^{\pm}_{\underline{v}}}\right)/\widehat{\Delta}^{\pm}_{\underline{v}}$$

— cf. Corollary 2.4, (ii); the discussion of [IUTchI], Remark 4.5.1, (i), (iii). Since all of the inclusions involved occur within a single **"ambient container"** — namely, $\widehat{\Pi}^{\pm}_{\underline{v}}$, regarded up to $\widehat{\Pi}^{\pm}_{\underline{v}}$-conjugacy — the *evaluation algorithm* discussed in Corollary 2.5, (iii), may be thought of as a sort of **"nested"** *version* of the principle of **"Galois evaluation"** discussed in Remark 1.12.4. Here, we note that unlike the situation discussed in Remark 1.12.4, in which the subgroup $\Pi_{\ddot{\underline{\underline{Y}}}}(\Pi) \subseteq \Pi$ is *normal*, the subgroups $\Pi_{\underline{v}\blacktriangleright}, \Pi_{\underline{v}\ddot{\blacktriangleright}} \subseteq \widehat{\Pi}^{\pm}_{\underline{v}}$ are *far from being normal*!

(ii) In the notation of [IUTchI], Definition 3.1, (d) [cf. also the notation of [IUTchI], Definition 6.1, (v)], write

$$\Pi^{\circledcirc\pm} \stackrel{\mathrm{def}}{=} \Pi_{\underline{X}_K}; \quad \Delta^{\circledcirc\pm} \stackrel{\mathrm{def}}{=} \Delta_{\underline{X}}$$

— so $\Delta^{\circledcirc\pm}$ may be naturally identified, up to inner automorphism, with $\widehat{\Delta}^{\pm}_{\underline{v}}$. Then note that unlike the *tempered fundamental groups* $\Delta_{\underline{v}}$, $\Delta^{\pm}_{\underline{v}}$, $\Delta_{\underline{v}\blacktriangleright}$, $\Delta_{\underline{v}\ddot{\blacktriangleright}}$ or the *local Galois groups* $\Pi_{\underline{v}}/\Delta_{\underline{v}}$, $\Pi^{\pm}_{\underline{v}}/\Delta^{\pm}_{\underline{v}}$, $\Pi_{\underline{v}\blacktriangleright}/\Delta_{\underline{v}\blacktriangleright}$, $\Pi_{\underline{v}\ddot{\blacktriangleright}}/\Delta_{\underline{v}\ddot{\blacktriangleright}}$ — all of which *depend*, in a quite essential way, on $\underline{v}$ — the topological group $\Delta^{\circledcirc\pm} \cong \widehat{\Delta}^{\pm}_{\underline{v}}$ is **independent** of $\underline{v}$ and, moreover, may be recovered directly from the **global portion** "${}^{\dagger}\mathcal{D}^{\circledcirc\pm}$" of a $\mathcal{D}$-Θ^{ell}-bridge [cf. [IUTchI], Definition 6.4, (ii); [AbsAnab], Lemma 1.1.4, (i)]. On the other hand, $\Delta^{\circledcirc\pm} \cong \widehat{\Delta}^{\pm}_{\underline{v}}$ also serves as an **"ambient container"** for the $\widehat{\Delta}^{\pm}_{\underline{v}}$-conjugates of both I_t and $\Delta_{\underline{v}\ddot{\blacktriangleright}}$. That is to say,

> $\Delta^{\circledcirc\pm}$ ($\cong \widehat{\Delta}^{\pm}_{\underline{v}}$) serves as a sort of **"common bridge"** between **local** data [such as $\Delta_{\underline{v}\ddot{\blacktriangleright}}$] and **global** data such as the **labels**
>
> $$t \in \mathrm{LabCusp}^{\pm}(\Pi^{\circledcirc\pm}) \quad (\stackrel{\sim}{\to} \mathrm{LabCusp}^{\pm}(\Pi^{\gamma}_{\underline{v}}) \stackrel{\sim}{\to} \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}}))$$
>
> [where we write $\mathrm{LabCusp}^{\pm}(\Pi^{\circledcirc\pm}) \stackrel{\mathrm{def}}{=} \mathrm{LabCusp}^{\pm}(\mathcal{B}(\Pi^{\circledcirc\pm})^0)$ — cf. [IUTchI], Definition 6.1, (vi)], in the form of conjugacy classes of I_t.

(iii) On the other hand, if, in the discussion of (ii), one passes — as in the theory of [IUTchI], §6 — between **distinct** $\underline{v}$ via this "global bridge" $\Delta^{\circledcirc\pm}$, then one must take into account the fact that, unlike the labels t [i.e., conjugacy classes of I_t], the groups $\Pi_{\underline{v}\ddot{\blacktriangleright}}$ *do not admit globalizations or extensions to multiple* $\underline{v}$*'s.* This is precisely the reason for

> *the* **independence** *of the* $\widehat{\Delta}^{\pm}_{\underline{v}}$ ($\cong \Delta^{\circledcirc\pm}$)*- [or, equivalently,* $\widehat{\Pi}^{\pm}_{\underline{v}}$*-] conjugacy indeterminacies that act on the conjugates of* I_t *and* $\Pi_{\underline{v}\ddot{\blacktriangleright}}$

[cf. the *"quotient interpretation"* of (i) above; the statement of Corollary 2.5, (iii)]. Here, we observe that since [in the notation of [IUTchI], Definition 3.1] neither of the natural surjections $\widehat{\Pi}^{\pm}_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$, $\Pi^{\circledcirc\pm} \twoheadrightarrow G_K$ admits a section that *simultaneously* *normalizes* the subgroups I_t, as t ranges over the elements of $\mathrm{LabCusp}^{\pm}(\Pi^{\circledcirc\pm}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(\Pi^{\gamma}_{\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$ [cf., e.g., [AbsSect], Theorem 1.3, (ii); [pGC], Theorem C], it follows that any $G_{\underline{v}}$- (respectively, G_K-) conjugacy indeterminacy necessarily results in a $\widehat{\Delta}^{\pm}_{\underline{v}} \cong \Delta^{\circledcirc\pm}$-conjugacy indeterminacy acting on the various I_t, i.e.,

$$G_{\underline{v}}\text{-}\textit{conjugacy indeterminacy} \Longrightarrow \widehat{\Delta}^{\pm}_{\underline{v}}\text{-}\textit{conjugacy indeterminacy,}$$

$$G_K\text{-}\textit{conjugacy indeterminacy} \Longrightarrow \Delta^{\circledcirc\pm}\text{-}\textit{conjugacy indeterminacy.}$$

Since, moreover, the natural surjection $\widehat{\Delta}^{\mathrm{cor}}_{\underline{v}} \twoheadrightarrow \widehat{\Delta}^{\mathrm{cor}}_{\underline{v}}/\widehat{\Delta}^{\pm}_{\underline{v}}$ *does not admit a splitting*, it follows that the $\widehat{\Delta}^{\pm}_{\underline{v}}$**-outer action** of $\widehat{\Delta}^{\mathrm{cor}}_{\underline{v}}/\widehat{\Delta}^{\pm}_{\underline{v}} \xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm}$ of Corollary 2.4, (iii), induces

> **independent** $\widehat{\Delta}^{\pm}_{\underline{v}} \cong \Delta^{\circledcirc\pm}$*-conjugacy indeterminacies on the subgroups* I_t, *for* **distinct** t.

In a similar vein, since $G_{\underline{v}}$ *does not determine a direct summand* of G_K — cf. [NSW], Corollary 12.1.3; the phenomenon of the *non-splitting of "prime decomposition trees"* discussed in [IUTchI], Remark 4.3.1, (ii) — it follows that any G_K-conjugacy indeterminacy [which, as just discussed, gives rise to $\Delta^{\circledcirc\pm}$-conjugacy indeterminacy] induces *independent* $G_{\underline{v}}$-conjugacy indeterminacies on the various G_K-conjugates of $G_{\underline{v}}$ [hence also, as just discussed, independent $\widehat{\Delta}^{\pm}_{\underline{v}}$-conjugacy indeterminacies] — i.e.,

$$G_K\text{-}\textit{conjugacy indeterminacy} \Longrightarrow \textbf{independent } G_{\underline{v}}\text{-}\textit{conjugacy indeterminacies}$$

— cf. the discussion of [IUTchI], Remark 4.5.1, (iii).

(iv) One way to visualize the *independent conjugacy indeterminacies* of the discussion of (iii) above is via the illustration given in Fig. 2.1 below.

$$\begin{array}{ccccccccccc} \dots & \circ & & \circ & & \circ & & \circ & & \circ & \dots \\ \dots & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \dots \end{array}$$

Fig. 2.1: Independent conjugacy indeterminacies

That is to say, one thinks of the upper and lower lines of Fig. 2.1 as being equipped with **independent** *actions by groups of horizontal translations* [i.e., each of which is isomorphic to $\mathbb{Z}$]; one thinks of each of the "$\circ$'s" in the upper line as representing a $\Delta^{\circledcirc\pm} \cong \widehat{\Delta}^{\pm}_{\underline{v}}$-conjugate of I_t and of each of the "$\bullet \longrightarrow \bullet$'s" in the lower line as representing a $\Delta^{\circledcirc\pm} \cong \widehat{\Delta}^{\pm}_{\underline{v}}$-conjugate of $\Pi_{\underline{v}_{\blacktriangleright}^{\ddot{}}}$. Thus, since the translation actions on the upper and lower lines are *not synchronized with one another* [cf. the discussion of (iii)],

> there is *no way to* **separate** — i.e., in a fashion that is *compatible with the indeterminacy arising from both translation actions* — the inclusion of a "$\circ$" into a "$\bullet \longrightarrow \bullet$" as the **left-hand** "$\bullet$" from the inclusion of the same "$\circ$" into some "$\bullet \longrightarrow \bullet$" as the **right-hand** "$\bullet$".

Corollary 2.6. **(Splittings Defined on Subgraphs)** *In the notation of Corollary 2.5, (ii):*

(i) **("$M^{\times}_{\mathbb{TM}}$" Defined on Subgraphs)** *The γ-conjugate of the quotient $\Pi_{\underline{v}\blacktriangleright} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}\blacktriangleright})$ of Corollary 2.5, (i), determines* **subsets**

$$\Big(\varinjlim_{J_G} H^1(J_G, (l \cdot \Delta_\Theta)(\Pi^{\gamma}_{\underline{v}\blacktriangleright})) \supseteq \Big) M^{\times}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}) \subseteq \varinjlim_{\widehat{J}} H^1(\Pi^{\gamma}_{\underline{v}\blacktriangleright}|_{\widehat{J}}, (l \cdot \Delta_\Theta)(\Pi^{\gamma}_{\underline{v}\blacktriangleright})),$$

$$M^{\times}_{\mathbb{TM}} \cdot \underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}) \subseteq M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}) \subseteq \varinjlim_{\widehat{J}} H^1(\Pi^{\gamma}_{\underline{v}\blacktriangleright}|_{\widehat{J}}, (l \cdot \Delta_\Theta)(\Pi^{\gamma}_{\underline{v}\blacktriangleright}))$$

— where $J_G \subseteq G_{\underline{v}}(\Pi_{\underline{v}\blacktriangleright})$, $\widehat{J} \subseteq \widehat{\Pi}_{\underline{v}}$ range over the open subgroups of $G_{\underline{v}}(\Pi_{\underline{v}\blacktriangleright})$, $\widehat{\Pi}_{\underline{v}}$, respectively; $M^{\times}_{\mathbb{TM}} \cdot \underline{\underline{\theta}}^{\iota}(-) \stackrel{\mathrm{def}}{=} M^{\times}_{\mathbb{TM}}(-) \cdot \underline{\underline{\theta}}^{\iota}(-)$, $M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}(-) \stackrel{\mathrm{def}}{=} M^{\times}_{\mathbb{TM}}(-) \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}(-)$ — which are **compatible**, *relative to the* **first restriction** *operation discussed in Corollary 2.5, (ii), with the corresponding* **subsets** *"$M^{\times}_{\mathbb{TM}}(-)$", "$M^{\times}_{\mathbb{TM}} \cdot \underline{\underline{\theta}}^{\iota}(-)$", "$M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}(-)$" of Proposition 1.4 and Corollary 1.12 [cf. Corollary 1.12, (a), (c), (e); Corollary 1.12, (i); Remark 1.11.5, (i), (ii)]. Also, [cf. Corollary 1.12] let us write*

$$M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}) \stackrel{\mathrm{def}}{=} M^{\times}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})$$

— where $M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}) \subseteq M^{\times}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})$ denotes the submodule of torsion elements.

(ii) **(Splittings at Zero-labeled Evaluation Points)** *In the situation of Corollary 2.5, (ii), suppose that t is taken to be the* **zero element**. *Then the set $\underline{\underline{\theta}}^{t}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})$ (respectively, ${}_{\infty}\underline{\underline{\theta}}^{t}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})$) is equal to the $\boldsymbol{\mu}_{2l}$- (respectively, $\boldsymbol{\mu}$-) orbit of the* **identity** *element [i.e., the zero element of cohomology module in question, if one denotes the module structure additively]. In particular, if one considers the* **quotient** *of the diagram of the first display of (i) by $M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})$, then* **restriction** *to the decomposition groups D^{δ}_{t,μ_-} of Corollary 2.4, (ii), (c), determines* **splittings**

$$M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright}) \times \{{}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})\}$$

of $M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\blacktriangleright})$ which are **compatible**, *relative to the* **first restriction** *operation discussed in Corollary 2.5, (ii), with the* **splittings** *of Corollary 1.12, (ii).*

Proof. Assertions (i) and (ii) follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 2.6.1.

(i) One of the most *central* properties, from the point of view of the theory of the present series of papers, of the *evaluation algorithm* of Corollary 2.5, (iii), consists of the observation that this algorithm is performed

> relative to a **single basepoint** — i.e., from a more *geometric* point of view, relative to the "fundamental group" $\Pi^{\gamma}_{\underline{v}\blacktriangleright}$ corresponding to the **connected** subgraph $\Gamma^{\blacktriangleright}_{\underline{\underline{X}}} \subseteq \Gamma_{\underline{\underline{X}}}$ [cf. Remark 2.1.1, (ii)].

In particular, despite the fact that we are ultimately interested in [not a single, but rather] a *plurality* of theta values, associated to the various $|t| \in |\mathbb{F}_l|$, these theta values

$$\underline{\underline{\theta}}^{|t|}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \quad \subseteq \quad H^1(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}), (l \cdot \Delta_{\Theta})(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}))$$

for various $|t| \in |\mathbb{F}_l|$ are all computed relative to the **single copy** [i.e., which is *independent* of $|t|$!] of the Galois group $G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$ and the **single cyclotome** $(l \cdot \Delta_{\Theta})(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$ [i.e., which is *independent* of $|t|$!] arising from $\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}$ — i.e., arising from the "single basepoint" under consideration. We shall refer to this phenomenon by the term **conjugate synchronization**. This conjugate synchronization is necessary in order to perform **Kummer theory** [cf. the discussion of *Galois evaluation* in Remark 1.12.4], as we shall do in §3.

(ii) Put another way, the significance of conjugate synchronization in the context of Kummer theory — especially, in the context of the theory of **Gaussian Frobenioids**, to be developed in §3 below — may be understood as arising from the requirement that the *collection of theta values*, for $|t| \in \mathbb{F}_l^{\circledast}$, be treated as

> a **single unified entity**, whose Kummer theory may be described by considering the action of a **single Galois group** in the context of the *simultaneous extraction of N-th roots* of all theta values, relative to a **single cyclotome** [i.e., copy of the module of N-th roots of unity] that acts *simultaneously* on the N-th roots of all of the theta values, and in a fashion that is *compatible with the Kummer theory of the* **"base field"** [i.e., arising from the quotient $\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}} \twoheadrightarrow G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$].

This point of view may only be realized by means of a **"single basepoint"** of a suitable category of coverings of a geometric object that consists of a **single connected component** [cf. the discussion of *Galois evaluation* in Remark 1.12.4; the discussion of [EtTh], Remark 1.10.4]. Also, we recall [cf. the discussion of *Galois evaluation* in Remark 1.12.4] that this *"Kummer-theoretic representation"* of the ["Frobenioid-theoretic"] monoid generated by the ["Frobenioid-theoretic"] theta function satisfies the crucial property of being compatible [unlike the various ring structures involved!] with the **"$\mathfrak{log}$-wall"** [cf. the theory of [AbsTopIII]]. This crucial property will play a fundamental role in the theory to be developed in [IUTchIII].

Remark 2.6.2.

(i) In the context of the discussion of conjugate synchronization in Remark 2.6.1, it is useful to recall the theory of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*-Hodge theaters*

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} = ({}^{\dagger}\mathfrak{D}_{\succ} \quad \stackrel{{}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}}{\longleftarrow} \quad {}^{\dagger}\mathfrak{D}_{T} \quad \stackrel{{}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}}{\longrightarrow} \quad {}^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

[cf. [IUTchI], Definition 6.4, (iii)] developed in [IUTchI], §6. That is to say, from the point of view of the theory of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters, it is natural to think

(a) of the *topological group* $\Pi_{\underline{v}}$ that appears in Corollaries 2.4, 2.5, and 2.6 as the tempered fundamental group of ${}^{\dagger}\mathcal{D}_{\succ,\underline{v}}$,

(b) of the *topological group* $\widehat{\Pi}^{\pm}_{\underline{v}}$ that appears in Corollaries 2.4, 2.5, and 2.6 as the *commensurator of the closure* of $\Pi_{\underline{v}}$ [i.e., relative to the interpretation of (a)] inside the profinite fundamental group of ${}^{\dagger}\mathcal{D}^{\circledcirc\pm}$ relative to the composite poly-morphism

$$ {}^{\dagger}\mathcal{D}_{\succ,\underline{v}} \quad \xrightarrow{({}^{\dagger}\phi^{\Theta^{\pm}}_{\underline{v}_0})^{-1}} \quad {}^{\dagger}\mathcal{D}_{\underline{v}_0} \quad \xrightarrow{{}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\underline{v}_0}} \quad {}^{\dagger}\mathcal{D}^{\circledcirc\pm} $$

determined by the portions of ${}^{\dagger}\phi^{\Theta^{\pm}}_{\pm}$, ${}^{\dagger}\phi^{\Theta^{\mathrm{ell}}}_{\pm}$ labeled by $0 \in T$, $\underline{v} \in \underline{\mathbb{V}}$ [cf. the discussions of [IUTchI], Examples 6.2, (i); 6.3, (i)], and

(c) of the $\widehat{\Delta}^{\pm}_{\underline{v}}$-*outer action of* $\widehat{\Delta}^{\mathrm{cor}}_{\underline{v}}/\widehat{\Delta}^{\pm}_{\underline{v}} \xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm}$ that appears in Corollary 2.4, (iii), as corresponding to the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* of [IUTchI], Proposition 6.8, (i).

Relative to the interpretation of (a), (b), and (c), one has the following *fundamental observation* concerning the discussion of Remark 2.6.1:

> the **single basepoint** that underlies the *conjugate synchronization* discussed in Remark 2.6.1 is *compatible* with the **single basepoint** that underlies the *label synchronization* discussed in [IUTchI], Remark 6.12.4.

That is to say, both of these basepoints may be thought of as arising from a *single basepoint* that gives rise to the various topological groups $\Pi_{\underline{v}}$, $\widehat{\Pi}^{\pm}_{\underline{v}}$, etc. that appear in Corollaries 2.4, 2.5, and 2.6. In particular,

> the **conjugate synchronization** discussed in Remark 2.6.1 is **compatible** with the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** of [IUTchI], Proposition 6.8, (i) [cf. also Remark 3.8.3 below].

Indeed, this compatibility is essentially the content of Corollary 2.4, (iii) [cf. (c) above].

(ii) Note that the compatibility of basepoints discussed in (i) *contrasts sharply* with the *incompatibility* of the conjugate synchronization basepoint of Remark 2.6.1 with the $\mathbb{F}_l^{\circledast}$-*symmetry* of [IUTchI], Proposition 4.9, (i), in the case of $\mathcal{D}$-ΘNF-*Hodge theaters*. At a more concrete level, this difference between $\mathbb{F}_l^{\rtimes\pm}$- and $\mathbb{F}_l^{\circledast}$-symmetries may be understood as a consequence of the fact that whereas the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is defined relative to a **single copy** of a local geometric object at $\underline{v}$ — i.e., "$\widehat{\Pi}^{\pm}_{\underline{v}}$" [cf. (a), (b), (c) above] — the $\mathbb{F}_l^{\circledast}$-symmetry involves **permuting multiple copies** of local geometric objects in such a way that one may only identify these multiple copies with one another at the expense of allowing the phenomenon of **"label crushing"** [cf. the discussions of [IUTchI], Remark 4.9.2, (i), (ii); 6.12.6, (i), (ii), (iii)].

(iii) Another important property of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry — which is not satisfied by the $\mathbb{F}_l^{\circledast}$-symmetry! — is that the $\mathbb{F}_l^{\rtimes\pm}$-symmetry allows **comparison** *with the label* **zero** [cf. the discussion of [IUTchI], Remark 6.12.5], hence, in particular, comparison with the copies of "$\mathcal{O}^{\times}_{\underline{k}}$" [cf. the discussion of Remark 1.12.2] that occur in the *splittings* of Corollary 1.12, (ii), that give rise to the crucial *constant multiple rigidity of the étale theta function*. This important property is precisely the content of Corollary 2.6.

Remark 2.6.3.

(i) The discussion of *independent conjugacy indeterminacies* in Remark 2.5.2 and of *"single basepoints"* that are compatible with the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* of [IUTchI], §6, in Remarks 2.6.1, 2.6.2 imply rather **severe restrictions** concerning the **subgraph** "$\Gamma^{\blacktriangleright}_{\ddot{Y}} \subseteq \Gamma_{\ddot{Y}}$" of Remark 2.1.1, (ii). That is to say, suppose that one attempts to develop the theory of the present §2 for *another subgraph* Γ' of the graph $\Gamma_{\ddot{Y}}$. Recall from the discussion of Remark 2.1.1, (i), that the graph $\Gamma_{\ddot{Y}}$ may be thought of as a *"copy of the real line* $\mathbb{R}$*"*, in which the integers $\mathbb{Z} \subseteq \mathbb{R}$ are taken to be the *vertices*, and the line segments joining the integers are taken to be the *edges*. Then the discussion of **"single basepoints"** [cf. Remark 2.6.1] implies, first of all, that

(a) this subgraph Γ' must be **connected**.

Since, moreover, one wishes to consider the crucial **splittings** of Corollary 2.6, (ii) [cf. Remark 2.6.2, (iii)], it follows that

(b) this subgraph Γ' must **contain** *the vertex of* $\Gamma_{\ddot{Y}}$ *labeled "0"*.

The conditions (a) and (b) already impose substantial restrictions on Γ' and hence on the *collection of values of the étale theta function* that may arise by restricting to the μ_--translates of the cusps that lie in Γ' [cf. Remark 2.5.1, (ii)] — i.e., on the collection of

$$\underline{\underline{q}}_{\underline{v}}^{\underline{\underline{j}}^2}$$

obtained by allowing $\underline{\underline{j}} \in \mathbb{Z}$ to range [relative to the identification of the vertices of $\Gamma_{\ddot{Y}}$ with the *integral points of the real line*] over the "vertices" of Γ' [cf. Remark 2.5.1, (i)].

(ii) By abuse of notation, let us write "$\underline{\underline{j}} \in \Gamma'$" for "vertices" $\underline{\underline{j}} \in \mathbb{Z}$ that lie in Γ'. Also, for simplicity, let us assume that the subgraph Γ' is *finite* [cf. (iii) below]. Then ultimately, in the theory of [IUTchIV], when we consider various **height inequalities**, we shall be concerned with the issue of **maximizing** the quantity

$$||\Gamma'|| \quad \stackrel{\text{def}}{=} \quad |\Gamma'|^{-1} \cdot \sum_{j \in \mathbb{F}_l^{*}} \ \min_{\underline{\underline{j}} \in j \bigcap \Gamma'} \{ \ \underline{\underline{j}}^2 \ \}$$

— where we write $|\Gamma'|$ for the *cardinality* of the image in $\mathbb{F}_l^{*}$ of the nonzero elements of Γ'; we regard the "min" over an empty set as being equal to *zero*; we think of the various $j \in \mathbb{F}_l^{*}$ as corresponding to the *subsets of* $\mathbb{Z}$ determined by the fibers of the natural projection $\mathbb{Z} \twoheadrightarrow |\mathbb{F}_l|$ ($\supseteq \mathbb{F}_l^{*}$). Here, we observe that

(c) the set of "$\underline{\underline{j}}$'s" that occur in the "min" ranging over "$\underline{\underline{j}}$" [i.e., not over "$j$"!] that appears in the definition of $||\Gamma'||$ is always equal to a **fiber** of the restriction to the set of vertices of Γ' of the **natural projection** $\mathbb{Z} \twoheadrightarrow |\mathbb{F}_l|$.

In fact, this observation (c) is, in essence, a consequence of the phenomenon discussed in Remark 2.5.2 of **independent conjugacy indeterminacies** [cf., especially, Remark 2.5.2, (iv)] — i.e., roughly speaking, that one cannot restrict the étale theta function to **"one** $\underline{\underline{j}} \in \Gamma'$**"** without *also* restricting the étale theta function to the various **"other** $\underline{\underline{j}} \in \Gamma'$**"** that lie in the **same fiber** over $|\mathbb{F}_l|$.

Next, let us make the [easily verified — cf. (a), (b)!] observation that if one thinks of $||\Gamma'||$ *as a function of* $|\Gamma'|$, then as $|\Gamma'|$ ranges over the positive integers, it holds that

(d) the function of $|\Gamma'|$ constituted by $||\Gamma'||$ — which may be thought of as a sort of *average* — is a **monotone increasing** [but not strictly increasing!] function of $|\Gamma'|$ valued in the positive rational numbers which attains its **maximum** when $|\Gamma'| = l^*$ and is **constant** for $|\Gamma'| \geq l^*$.

Now it follows formally from (d) that, as $|\Gamma'|$ ranges over the positive integers, the quantity $||\Gamma'||$ attains its *maximum* when $|\Gamma'| = l^*$ — hence, in particular, when Γ' is taken to be $\Gamma^{\blacktriangleright}_{\ddot{Y}}$. Thus, from the point of view of the issue of maximizing this quantity $||\Gamma'||$, there is *"no loss of generality" in assuming that* $\Gamma' = \Gamma^{\blacktriangleright}_{\ddot{Y}}$ [cf. also the discussion of (iv) below].

(iii) Although in the discussion of (ii) above we assumed that Γ' is *finite*, this assumption does not in fact result in any loss of generality. Indeed, one verifies immediately that $||\Gamma'||$ is *defined*, *finite*, and satisfies the evident analogue of (d) even for *infinite* Γ'. Thus, the case of infinite Γ' may be excluded without loss of generality.

(iv) Ultimately, in §4 of the present paper, we shall be concerned with the issue of **globalizing**, via the construction of various *global realified Frobenioids*, the monoids determined by the *theta values* at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ that appear in the present §2. This globalization will be achieved, in effect, by imposing the condition that the **product formula** be satisfied. On the other hand, the *indeterminacies* discussed in (ii) above [cf., especially, (ii), (c)] that arise when a fiber of Γ' over $|\mathbb{F}_l|$ contains *more than one element* are easily seen to be *fundamentally incompatible* with the product formula. In particular, from the point of view of the issue of maximizing the quantity $||\Gamma'||$, in fact the *only choice* for Γ' that is compatible with the "globalization via the product formula" to be performed in §4 is $\Gamma^{\blacktriangleright}_{\ddot{Y}}$.

(v) One may summarize the discussion of (i), (ii), (iii), and (iv) as follows:

the *collection of values "*$\underline{\underline{q}}_{\underline{v}}^{\underline{\underline{j}}^2}$*" of the étale theta function* determined by the subgraph $\Gamma^{\blacktriangleright}_{\ddot{Y}}$ is of a **highly distinguished** nature

— and, indeed, is *essentially determined* [cf. the discussion at the end of (ii); the discussion of (iv)] by the requirement of **maximizing** the quantity "$||\Gamma'||$" in a fashion compatible with the **global product formula**, together with various *qualitative considerations* that arise from Corollaries 2.4, 2.5, 2.6; the discussion of Remarks 2.5.1, 2.5.2, 2.6.1, 2.6.2.

Definition 2.7. In the notation of Definition 2.3: Let

$$\mathbb{M}^{\Theta}_{*} \quad = \quad \{\ldots \rightarrow \ \mathbb{M}^{\Theta}_{M'} \ \rightarrow \mathbb{M}^{\Theta}_{M} \ \rightarrow \ \ldots\}$$

be a *projective system of mono-theta environments* as in Proposition 1.5, such that $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*) \cong \Pi_{\underline{v}}$.

(i) Write

$$\Pi_{\mathbb{M}^{\Theta}_*}$$

for the *inverse limit* of the induced projective system of topological groups $\{\ldots \to \Pi_{\mathbb{M}^{\Theta}_{M'}} \to \Pi_{\mathbb{M}^{\Theta}_{M}} \to \ldots\}$ [cf. the notation discussed at the beginning of Definition 1.1]. Thus, [in the notation of Proposition 1.5] we have a *natural homomorphism of topological groups*

$$\Pi_{\mathbb{M}^{\Theta}_*} \ \to \ \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*)$$

whose *kernel* may be identified with the *exterior cyclotome* $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_*)$, and whose *image* is the subgroup of $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*) \cong \Pi_{\underline{v}}$ determined by $\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}_{\underline{v}}}$.

(ii) Write

$$\Pi_{\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}} \subseteq \Pi_{\mathbb{M}^{\Theta}_{*\blacktriangleright}} \subseteq \Pi_{\mathbb{M}^{\Theta}_*}$$

for the respective *inverse images* of $\Pi_{\underline{v}\ddot{\blacktriangleright}} \subseteq \Pi_{\underline{v}\blacktriangleright} \subseteq \Pi_{\underline{v}} \cong \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*)$ in $\Pi_{\mathbb{M}^{\Theta}_*}$;

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}), \quad (l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}), \quad \Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}), \quad G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$$

for the *subquotients* of $\Pi_{\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}}$ determined by the subquotient $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_*)$ of $\Pi_{\mathbb{M}^{\Theta}_*}$ and the subquotients $(l \cdot \Delta_{\Theta})(\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*))$ [cf. Proposition 1.4], $\Pi_{\underline{v}\ddot{\blacktriangleright}}$, and $G_{\underline{v}}(\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*))$ [cf. Corollary 2.5, (i)] of $\Pi_{\underline{v}} \cong \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*)$. Thus, we obtain a **cyclotomic rigidity isomorphism**

$$(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \xrightarrow{\sim} \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$$

— i.e., by *restricting* the cyclotomic rigidity isomorphism $(l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_*) \xrightarrow{\sim} \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_*)$ of Proposition 1.5, (iii), to $\Pi_{\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}}$.

Corollary 2.8. (Mono-theta-theoretic Theta Evaluation) *In the notation of Definition 2.7: Suppose that we are in the situation of Proposition 2.2, (ii); Corollary 2.5, (ii); to simplify notation, we assume that $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*) = \Pi_{\underline{v}}$, and we use the notation for objects constructed from "$\Pi_{\underline{v}}$" to denote the corresponding objects constructed from $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*)$. Also, let us write*

$$(\mathbb{M}^{\Theta}_*)^{\gamma}$$

for the projective system of mono-theta environments obtained via transport of structure from the isomorphism $\Pi_{\underline{v}} \xrightarrow{\sim} \Pi^{\gamma}_{\underline{v}}$ determined by conjugation by γ.

(i) **(Restriction of Étale Theta Functions to Subgraphs and Evaluation Points)** *In the situation of Proposition 2.2, (ii); Corollary 2.5, (ii), let us apply the* **cyclotomic rigidity isomorphisms**

$$(l \cdot \Delta_{\Theta})((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \xrightarrow{\sim} \Pi_{\boldsymbol{\mu}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}); \quad (l \cdot \Delta_{\Theta})((\mathbb{M}^{\Theta}_*)^{\gamma}) \xrightarrow{\sim} \Pi_{\boldsymbol{\mu}}((\mathbb{M}^{\Theta}_*)^{\gamma})$$

[cf. Definition 2.7, (ii), applied to $(\mathbb{M}^{\Theta}_{*})^{\gamma}$*] to* **replace** *"*$(l \cdot \Delta_{\Theta})(-)$*" by "*$\Pi_{\boldsymbol{\mu}}(-)$*". Then the* ι^{γ}*-invariant subsets* $\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}}) \subseteq \underline{\underline{\theta}}(\Pi^{\gamma}_{\underline{v}})$, $_{\infty}\underline{\underline{\theta}}^{\iota}(\Pi^{\gamma}_{\underline{v}}) \subseteq {}_{\infty}\underline{\underline{\theta}}(\Pi^{\gamma}_{\underline{v}})$ *[cf. Proposition 2.2, (ii); Corollary 2.5, (ii)] determine* ι^{γ}*-invariant subsets*

$$\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*})^{\gamma}) \quad \subseteq \quad \underline{\underline{\theta}}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*})^{\gamma}); \quad {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*})^{\gamma}) \quad \subseteq \quad {}_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*})^{\gamma})$$

[cf. Proposition 1.5, (iii), applied to $(\mathbb{M}^{\Theta}_{*})^{\gamma}$*];* **restriction** *of these subsets* $\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*})^{\gamma})$, $_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*})^{\gamma})$ *to* $\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$ *yields* $\boldsymbol{\mu}_{2l}$*-,* $\boldsymbol{\mu}$*-orbits of elements*

$$\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \quad \subseteq \quad {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \quad \subseteq \quad \varinjlim_{\widehat{J}} \ H^1(\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})|_{\widehat{J}}, \Pi_{\boldsymbol{\mu}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}))$$

— where $\widehat{J} \subseteq \widehat{\Pi}_{\underline{v}}$ *ranges over the open subgroups of* $\widehat{\Pi}_{\underline{v}}$ *— which, upon further* **restriction** *to the decomposition groups* $D^{\delta}_{t,\mu_{-}}$ *of Corollary 2.4, (ii), (c), yield* $\boldsymbol{\mu}_{2l}$*-,* $\boldsymbol{\mu}$*-orbits of elements*

$$\underline{\underline{\theta}}^{t}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \quad \subseteq \quad {}_{\infty}\underline{\underline{\theta}}^{t}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \quad \subseteq \quad \varinjlim_{J_G} \ H^1(G_{\underline{v}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})|_{J_G}, \Pi_{\boldsymbol{\mu}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}))$$

for each $t \in \mathrm{LabCusp}^{\pm}(\Pi^{\gamma}_{\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$ *— where* $J_G \subseteq G_{\underline{v}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$ *ranges over the open subgroups of* $G_{\underline{v}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$*; the "* $\xrightarrow{\sim}$ *" is induced by conjugation by* γ*. Moreover, the sets* $\underline{\underline{\theta}}^{t}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$, $_{\infty}\underline{\underline{\theta}}^{t}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$ *depend only on the* **label** $|t| \in |\mathbb{F}_l|$ *determined by t [cf. Corollary 2.5, (ii)]. Thus, we shall write* $\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \stackrel{\mathrm{def}}{=} \underline{\underline{\theta}}^{t}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$, $_{\infty}\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}) \stackrel{\mathrm{def}}{=} {}_{\infty}\underline{\underline{\theta}}^{t}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$*.*

(ii) **(Functorial Group-theoretic Evaluation Algorithm)** *If one starts with an* **arbitrary** $\widehat{\Delta}^{\pm}_{\underline{v}}$**-conjugate** $\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$ *of* $\Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$*, and one considers, as t ranges over the elements of* $\mathrm{LabCusp}^{\pm}(\Pi^{\gamma}_{\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$ *[where the "* $\xrightarrow{\sim}$ *" is induced by conjugation by* γ*], the resulting* $\boldsymbol{\mu}_{2l}$*-,* $\boldsymbol{\mu}$*-orbits* $\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$, $_{\infty}\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$ *arising from an* **arbitrary** $\widehat{\Delta}^{\pm}_{\underline{v}}$**-conjugate** I^{δ}_t *of* I_t *that is contained in* $\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})$ *[cf. (i)], then one obtains an* **algorithm** *for constructing the collections of* $\boldsymbol{\mu}_{2l}$*-,* $\boldsymbol{\mu}$*-orbits*

$$\{\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})\}_{|t| \in |\mathbb{F}_l|}; \quad \{{}_{\infty}\underline{\underline{\theta}}^{|t|}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma})\}_{|t| \in |\mathbb{F}_l|}$$

which is **functorial** *in the projective system of mono-theta environments* $\mathbb{M}^{\Theta}_{*}$ *and, moreover,* **compatible** *with the* **independent conjugacy actions** *of* $\widehat{\Delta}^{\pm}_{\underline{v}}$ *on the sets* $\{I^{\gamma_1}_t\}_{\gamma_1 \in \widehat{\Pi}^{\pm}_{\underline{v}}} = \{I^{\gamma_1}_t\}_{\gamma_1 \in \widehat{\Delta}^{\pm}_{\underline{v}}}$ *and* $\{\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma_2})\}_{\gamma_2 \in \widehat{\Pi}^{\pm}_{\underline{v}}} = \{\Pi_{\underline{v}\ddot{\blacktriangleright}}((\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma_2})\}_{\gamma_2 \in \widehat{\Delta}^{\pm}_{\underline{v}}}$ *[cf. the sets of Corollary 2.4, (i); Remark 2.2.1].*

(iii) **(Splittings at Zero-labeled Evaluation Points)** *In the situation of (i), suppose that t is taken to be the* **zero element***. Then, by applying the cyclotomic rigidity isomorphisms of (i) to replace "*$(l \cdot \Delta_{\Theta})(-)$*" by "*$\Pi_{\boldsymbol{\mu}}(-)$*" — an operation that, when applied to "*$M^{??}_{\mathbb{TM}}(-)$*" [where "??"* $\in \{\times, \boldsymbol{\mu}, \times\boldsymbol{\mu}\}$*], we shall denote by replacing the notation "*$\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}$*" by "*$(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})^{\gamma}$*" — in Corollary 2.6, (ii), the*

second restriction operation discussed in (i) determines **splittings** *[cf. Corollary 2.6, (ii)]*

$$M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}((\mathbb{M}^{\Theta}_{*\blacktriangleright\!\!\!\cdot\cdot})^{\gamma}) \times \{{}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\blacktriangleright\!\!\!\cdot\cdot})^{\gamma})/M^{\boldsymbol{\mu}}_{\mathbb{TM}}((\mathbb{M}^{\Theta}_{*\blacktriangleright\!\!\!\cdot\cdot})^{\gamma})\}$$

of $M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{*\blacktriangleright\!\!\!\cdot\cdot})^{\gamma})/M^{\boldsymbol{\mu}}_{\mathbb{TM}}((\mathbb{M}^{\Theta}_{*\blacktriangleright\!\!\!\cdot\cdot})^{\gamma})$ *which are* **compatible***, relative to the first restriction operation discussed in (i), with the splittings of Corollary 1.12, (ii) [i.e., relative to any isomorphism* $\mathbb{M}^{\Theta}_{*} \xrightarrow{\sim} \mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})$ *— cf. Proposition 1.2, (i); Proposition 1.5, (i); Remarks 2.8.1, 2.8.2 below].*

Proof. Assertions (i), (ii), and (iii) follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 2.8.1. One may regard Corollaries 2.5, 2.6 as a special case of Corollary 2.8, i.e., the case where the projective system of mono-theta environments $\mathbb{M}^{\Theta}_{*}$ arises from the topological group $\Pi_{\underline{v}}$ by applying the *functorial group-theoretic algorithm* of Proposition 1.2, (i) [cf. also Proposition 1.5, (i)].

Remark 2.8.2. The significance of the **mono-theta-theoretic** version of Corollaries 2.5, 2.6 constituted by Corollary 2.8 lies in the fact that this mono-theta-theoretic version allows one to relate the **group-theoretic** *theta evaluation theory* of the present §2 to the theory of **Frobenioid-theoretic** *theta functions* associated to *tempered Frobenioids* [cf. [EtTh], §5], i.e., by considering the case where $\mathbb{M}^{\Theta}_{*}$ arises from a tempered Frobenioid [cf. Proposition 1.2, (ii)]. For instance, by considering the case where $\mathbb{M}^{\Theta}_{*}$ arises from a tempered Frobenioid, one may treat the *Frobenioid-theoretic cyclotomes* [i.e., cyclotomes that arise from the *units* of the Frobenioid] of Proposition 1.3, (i), in the context of the theory of the present §2.

Remark 2.8.3.

(i) The use of the **archimedean** *line segment* $\Gamma^{\blacktriangleright}_{\underline{X}} \subseteq \Gamma_{\underline{X}}$ [cf. Remark 2.1.1, (ii)] to single out the elements $\in \{-l^{*}, -l^{*}+1, \dots, -1, 0, 1, \dots, l^{*}-1, l^{*}\}$ — i.e., the elements with *absolute value* $\leq l^{*}$ — within the **nonarchimedean** *congruence classes modulo l* constituted by an element $\in \mathbb{F}^{*}_{l}$ is reminiscent of the computation of the *set of global sections* of an arithmetic line bundle on a number field [cf., e.g., [Szp], pp. 13-14], as well as of the *arithmetic* inherent in the *graph theory* associated to the loop $\Gamma_{\underline{X}}$ [cf. [SemiAnbd], Remark 1.5.1].

(ii) The sort of argument discussed in (i) involving the **connected***, "archimedean" line segment* $\Gamma^{\blacktriangleright}_{\underline{X}} \subseteq \Gamma_{\underline{X}}$ [cf. Remark 2.6.1 for more on the importance of this *connectedness*] depends, in an essential way, on the *discreteness* of $\underline{\mathbb{Z}}$ ($\cong \mathbb{Z}$). Put another way, this sort of argument may be thought of as an application of the **discrete rigidity** that forms one of the central themes of [EtTh]. Note, moreover, that in the context of Corollary 2.8, this application of discrete rigidity is closely related to the application of **cyclotomic rigidity**. This is perhaps not so surprising, since *discrete rigidity* — in the form of the discreteness of *squares of elements of* $\underline{\mathbb{Z}}$, i.e., in effect, the quotient of $\underline{\mathbb{Z}}$ by the action of $\{\pm 1\}$ — may be thought of as a sort of **dual** property to the *cyclotomic rigidity* of "$(l \cdot \Delta_{\Theta})(-)$". Indeed, one may think of this duality as being embodied in the very *structure* and *values* of the étale theta function [cf. [EtTh], Proposition 1.4, (ii), (iii); [EtTh], Proposition 1.5, (ii)].

In a similar vein, one may also consider the theory of group-theoretic theta evaluation developed in the present §2 in the context of the *natural isomorphism* "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_X)$" of [AbsTopIII], Corollary 1.10, (c) [cf. also Proposition 1.3, (ii); Corollary 1.11, (b)].

Corollary 2.9. **(Theta Evaluation via Base-field-theoretic Cyclotomes)** *Suppose that we are in the situation of Proposition 2.2, (ii); Corollary 2.5, (ii). Also, let us write*

$$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi_{\underline{v}})) \xrightarrow{\sim} (l \cdot \Delta_{\Theta})(\Pi_{\underline{v}}); \qquad \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})) \xrightarrow{\sim} (l \cdot \Delta_{\Theta})(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$$

for the **cyclotomic rigidity isomorphisms** *determined by the natural isomorphism "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_X)$" of [AbsTopIII], Corollary 1.10, (c) [cf. also Proposition 1.3, (ii); Corollary 1.11, (b)] and its restriction to $\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}$ [cf. Corollary 2.5, (i)].*

(i) **(Restriction of Étale Theta Functions to Subgraphs and Evaluation Points)** *In the situation of Proposition 2.2, (ii); Corollary 2.5, (ii), let us apply the above cyclotomic rigidity isomorphisms to* **replace** *"$(l \cdot \Delta_{\Theta})(-)$" by "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(-))$". Then the ι^{γ}-invariant subsets $\underline{\underline{\boldsymbol{\theta}}}^{\iota}(\Pi^{\gamma}_{\underline{v}}) \subseteq \underline{\underline{\boldsymbol{\theta}}}(\Pi^{\gamma}_{\underline{v}})$, ${}_{\infty}\underline{\underline{\boldsymbol{\theta}}}^{\iota}(\Pi^{\gamma}_{\underline{v}}) \subseteq {}_{\infty}\underline{\underline{\boldsymbol{\theta}}}(\Pi^{\gamma}_{\underline{v}})$ [cf. Proposition 2.2, (ii); Corollary 2.5, (ii)] determine ι^{γ}-invariant subsets*

$$\underline{\underline{\boldsymbol{\theta}}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}}) \quad \subseteq \quad \underline{\underline{\boldsymbol{\theta}}}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}}); \quad {}_{\infty}\underline{\underline{\boldsymbol{\theta}}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}}) \quad \subseteq \quad {}_{\infty}\underline{\underline{\boldsymbol{\theta}}}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}})$$

*— where one may think of the "*bs*" as an abbreviation of the term* **"base-field-theoretic"***;* **restriction** *of these subsets $\underline{\underline{\boldsymbol{\theta}}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}})$, ${}_{\infty}\underline{\underline{\boldsymbol{\theta}}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}})$ to $\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}$ yields $\boldsymbol{\mu}_{2l}$-, $\boldsymbol{\mu}$-orbits of elements*

$$\underline{\underline{\boldsymbol{\theta}}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \quad \subseteq \quad {}_{\infty}\underline{\underline{\boldsymbol{\theta}}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \quad \subseteq \quad \varinjlim_{\widehat{J}} \ H^1(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}|_{\widehat{J}}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})))$$

— where $\widehat{J} \subseteq \widehat{\Pi}_{\underline{v}}$ ranges over the open subgroups of $\widehat{\Pi}_{\underline{v}}$ — which, upon further **restriction** *to the decomposition groups D^{δ}_{t,μ_-} of Corollary 2.4, (ii), (c), yield $\boldsymbol{\mu}_{2l}$-, $\boldsymbol{\mu}$-orbits of elements*

$$\underline{\underline{\boldsymbol{\theta}}}^{t}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \quad \subseteq \quad {}_{\infty}\underline{\underline{\boldsymbol{\theta}}}^{t}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \quad \subseteq \quad \varinjlim_{J_G} \ H^1(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})|_{J_G}, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})))$$

for each $t \in \mathrm{LabCusp}^{\pm}(\Pi^{\gamma}_{\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$ — where $J_G \subseteq G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$ ranges over the open subgroups of $G_{\underline{v}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$; the "$\xrightarrow{\sim}$" is induced by conjugation by γ. Moreover, the sets $\underline{\underline{\boldsymbol{\theta}}}^{t}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$, ${}_{\infty}\underline{\underline{\boldsymbol{\theta}}}^{t}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$ depend only on the **label** *$|t| \in |\mathbb{F}_l|$ determined by t [cf. Corollary 2.5, (ii)]. Thus, we shall write $\underline{\underline{\boldsymbol{\theta}}}^{|t|}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \stackrel{\mathrm{def}}{=} \underline{\underline{\boldsymbol{\theta}}}^{t}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$, ${}_{\infty}\underline{\underline{\boldsymbol{\theta}}}^{|t|}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}) \stackrel{\mathrm{def}}{=} {}_{\infty}\underline{\underline{\boldsymbol{\theta}}}^{t}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$.*

(ii) **(Functorial Group-theoretic Evaluation Algorithm)** *If one starts with an* **arbitrary** $\widehat{\Delta}^{\pm}_{\underline{v}}$**-conjugate** $\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}$ *of* $\Pi_{\underline{v}\ddot{\blacktriangleright}}$*, and one considers, as* t *ranges over the elements of* $\mathrm{LabCusp}^{\pm}(\Pi^{\gamma}_{\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$ *[where the "*$\xrightarrow{\sim}$*" is induced by conjugation by* γ*], the resulting* $\boldsymbol{\mu}_{2l}$*-,* $\boldsymbol{\mu}$*-orbits* $\underline{\underline{\boldsymbol{\theta}}}^{|t|}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$*,* ${}_{\infty}\underline{\underline{\boldsymbol{\theta}}}^{|t|}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})$ *arising from an* **arbitrary** $\widehat{\Delta}^{\pm}_{\underline{v}}$**-conjugate** I^{δ}_t *of* I_t *that is contained in* $\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}}$ *[cf. (i)], then one obtains an* **algorithm** *for constructing the collections of* $\boldsymbol{\mu}_{2l}$*-,* $\boldsymbol{\mu}$*-orbits*

$$\{\underline{\underline{\boldsymbol{\theta}}}^{|t|}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})\}_{|t|\in|\mathbb{F}_l|}; \quad \{{}_{\infty}\underline{\underline{\boldsymbol{\theta}}}^{|t|}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})\}_{|t|\in|\mathbb{F}_l|}$$

which is **functorial** *in the topological group* $\Pi_{\underline{v}}$ *and, moreover,* **compatible** *with the* **independent conjugacy actions** *of* $\widehat{\Delta}^{\pm}_{\underline{v}}$ *on the sets* $\{I^{\gamma_1}_t\}_{\gamma_1\in\widehat{\Pi}^{\pm}_{\underline{v}}} = \{I^{\gamma_1}_t\}_{\gamma_1\in\widehat{\Delta}^{\pm}_{\underline{v}}}$ *and* $\{\Pi^{\gamma_2}_{\underline{v}\ddot{\blacktriangleright}}\}_{\gamma_2\in\widehat{\Pi}^{\pm}_{\underline{v}}} = \{\Pi^{\gamma_2}_{\underline{v}\ddot{\blacktriangleright}}\}_{\gamma_2\in\widehat{\Delta}^{\pm}_{\underline{v}}}$ *[cf. the sets of Corollary 2.4, (i); Remark 2.2.1].*

(iii) **(Splittings at Zero-labeled Evaluation Points)** *In the situation of (i), suppose that* t *is taken to be the* **zero element***. Then, by applying the cyclotomic rigidity isomorphisms reviewed above to replace "*$(l\cdot\Delta_{\Theta})(-)$*" by "*$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(-))$*" — an operation that, when applied to "*$M^{??}_{\mathbb{TM}}(-)$*" [where "??"* $\in \{\times, \boldsymbol{\mu}, \times\boldsymbol{\mu}\}$*], we shall denote by means of a subscript "*bs*" — in Corollary 2.6, (ii), the second restriction operation discussed in (i) determines* **splittings** *[cf. Corollary 2.6, (ii)]*

$$M^{\times\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})_{\mathrm{bs}} \times \{{}_{\infty}\underline{\underline{\boldsymbol{\theta}}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})_{\mathrm{bs}}\}$$

of $M^{\times}_{\mathbb{TM}} \cdot {}_{\infty}\underline{\underline{\boldsymbol{\theta}}}^{\iota}_{\mathrm{bs}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})/M^{\boldsymbol{\mu}}_{\mathbb{TM}}(\Pi^{\gamma}_{\underline{v}\ddot{\blacktriangleright}})_{\mathrm{bs}}$ *which are* **compatible***, relative to the first restriction operation discussed in (i) and the cyclotomic rigidity isomorphisms reviewed above, with the splittings of Corollary 1.12, (ii).*

Proof. Assertions (i), (ii), and (iii) follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 2.9.1.

(i) Let us recall that [the cyclotomic rigidity isomorphisms involving] the cyclotomes "$\Pi_{\boldsymbol{\mu}}(-)$" that appear in Corollary 2.8 admit a **multiradial** formulation [cf. Corollary 1.10]. By contrast, at least relative to the point of view of Remark 1.11.3, (iv), [the cyclotomic rigidity isomorphisms involving] the cyclotomes "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(-))$" that appear in Corollary 2.9 only admit a **uniradial** formulation — i.e., unless one is willing to *sacrifice the crucial cyclotomic rigidity* under consideration as in the formulation of Corollary 1.11.

(ii) On the other hand, the use of [the cyclotomic rigidity isomorphisms involving] the cyclotomes "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(-))$" has the crucial advantage that it allows one to apply the [not *multiradially* (!), but rather] **uniradially defined** *natural surjection*

$$H^1(G_{\underline{v}}(-), \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(-))) \twoheadrightarrow \widehat{\mathbb{Z}}$$

of Remark 1.11.5, (i), (ii).

(iii) One immediate consequence of the discussion of (i) is the observation that, at least relative to the point of view of Remark 1.11.3, (iv), the *algorithms* of Corollary 2.9, (ii), (iii), only give rise to a **uniradially defined** functor. On the other hand, one important consequence of the theory to be developed in [IUTchIII] is the result that,

> by applying the theory of *log-shells* [cf. [AbsTopIII]], one may modify these algorithms in such a way as to obtain algorithms that [yield functors which] are *manifestly* **multiradially defined**

— albeit at the cost of allowing for certain [relatively mild!] **indeterminacies**.

Section 3: Tempered Gaussian Frobenioids

In the present §3, we relate the theory of group-theoretic algorithms surrounding the **Hodge-Arakelov-theoretic evaluation** *of the étale theta function on l-torsion points* developed in §1, §2 to the local portion at bad primes [i.e., at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] of the various *Frobenioids* considered in [IUTchI], §3, §4, §5, §6. In particular, we shall discuss how the various **multiradial** formulations developed in §1 and the theory of **conjugate synchronization** developed in §2 may be applied in the context of the **"tempered Gaussian Frobenioids"** that arise from the Hodge-Arakelov-theoretic evaluation of the étale theta function on l-torsion points.

In the present §3, we shall continue to use the notation of §2. In particular, our discussion concerns the *local portion* at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ of the various mathematical objects considered in [IUTchI], §3, §4, §5, §6.

Proposition 3.1. **(Mono-theta-theoretic Theta Monoids)** *Let*

$$\mathbb{M}_*^{\Theta} \quad = \quad \{\ldots \ \to \ \mathbb{M}_{M'}^{\Theta} \ \to \mathbb{M}_M^{\Theta} \ \to \ \ldots\}$$

be a **projective system of mono-theta environments** *[cf. Proposition 1.5, Corollary 2.8] such that $\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta}) \cong \Pi_{\underline{v}}$. In the following, to simplify the notation, we shall denote a "$\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta})$" in parenthesis by means of the abbreviated notation "$\mathbb{M}_*^{\Theta}$".*

(i) **(Split Theta Monoids)** *By applying the constructions of Proposition 1.5, (iii); Corollary 2.8, (i) [cf. also Corollary 1.12, (d)], one obtains a* **functorial algorithm**

$$\mathbb{M}_*^{\Theta} \quad \mapsto \quad \Big\{ M_{\mathbb{TM}}^{\times}(\mathbb{M}_*^{\Theta}),\ \underline{\underline{\theta}}_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta}),\ {}_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta}),$$
$$M_{\mathbb{TM}}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta}) \ \subseteq \ \varinjlim_{J} \ H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}_*^{\Theta})|_J, \Pi_{\boldsymbol{\mu}}(\mathbb{M}_*^{\Theta})) \Big\}_{\iota}$$

— where J ranges over the finite index open subgroups of $\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta})$, and ι ranges over the inversion automorphisms of Proposition 2.2, (i) — for constructing various subsets of the direct limit of cohomology modules in the above display; this collection of subsets is equipped with a natural **conjugation action** *by $\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta})$. In particular, one obtains a functorial algorithm for constructing the data*

$$\Psi_{\mathrm{env}}(\mathbb{M}_*^{\Theta}) \ \stackrel{\mathrm{def}}{=} \ \Big\{ \ \Psi_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta}) \ = \ M_{\mathbb{TM}}^{\times}(\mathbb{M}_*^{\Theta}) \cdot \underline{\underline{\theta}}_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta})^{\mathbb{N}} \ \Big\}_{\iota};$$
$${}_{\infty}\Psi_{\mathrm{env}}(\mathbb{M}_*^{\Theta}) \ \stackrel{\mathrm{def}}{=} \ \Big\{ \ {}_{\infty}\Psi_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta}) \ = \ M_{\mathbb{TM}}^{\times}(\mathbb{M}_*^{\Theta}) \cdot {}_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta})^{\mathbb{N}} \ \Big\}_{\iota}$$

consisting of the **submonoids** *$\{\Psi_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta})\}_{\iota}$, $\{{}_{\infty}\Psi_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta})\}_{\iota}$ [of the direct limit of cohomology modules in the first display of the present (i)] generated, respectively, by the subsets "$M_{\mathbb{TM}}^{\times} \cdot \underline{\underline{\theta}}_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta})$", "$M_{\mathbb{TM}}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta})$", as well as a functorial algorithm for constructing the* **splittings** *up to torsion determined by the subsets "$M_{\mathbb{TM}}^{\times}(\mathbb{M}_*^{\Theta})$", "$\underline{\underline{\theta}}_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta})$", "${}_{\infty}\underline{\underline{\theta}}_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta})$" [cf. Corollary 2.8, (iii)]. We shall refer to each $\Psi_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta})$, ${}_{\infty}\Psi_{\mathrm{env}}^{\iota}(\mathbb{M}_*^{\Theta})$ as a* **theta monoid**.

(ii) **(Constant Monoids)** *By applying the* **cyclotomic rigidity isomorphisms** *of Corollaries 2.8, (i); 2.9, and considering the inverse image of* $\mathbb{Z} \subseteq \widehat{\mathbb{Z}}$ *via the surjection of Remark 1.11.5, (i), applied to* $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})$ $(= G_{\underline{v}}(\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})))$ *[cf. the notation of Corollary 2.5, (i)], one obtains a* **functorial algorithm**

$$\mathbb{M}^{\Theta}_{*} \quad \mapsto \quad \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}) \ \stackrel{\mathrm{def}}{=}\ M_{\mathbb{TM}}(\mathbb{M}^{\Theta}_{*}) \ \subseteq\ \varinjlim_{J} \ H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta}_{*})|_{J}, \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}))$$

[where J *is as in (i)] for constructing a "monoid of constants" — i.e., which is naturally isomorphic to* $\mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$ *[cf. Example 1.8, (ii)] — equipped with a natural* **conjugation action** *by* $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$*. We shall refer to* $\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})$ *as a* **constant monoid***.*

Proof. Assertions (i) and (ii) follow immediately from the definitions and the references quoted in the statements of these assertions. ○

Before proceeding, we pause to review the theory of *tempered Frobenioids* discussed in [IUTchI], Example 3.2.

Example 3.2. Theta Monoids Constructed from Tempered Frobenioids. In the situation of [IUTchI], Example 3.2:

(i) Recall the *tempered Frobenioid* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ of [IUTchI], Example 3.2, (i), (ii), (v) [cf. also [IUTchI], Remark 3.2.3, (i), (ii)]. Then, in the notation of *loc. cit.*, the choice of a *Frobenioid-theoretic theta function*

$$\underline{\underline{\Theta}}_{\underline{v}} \in \mathcal{O}^{\times}(\mathbb{T}^{\div}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$$

— i.e., among the $\boldsymbol{\mu}_{2l}(\mathbb{T}^{\div}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$*-multiples* of the $\mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{\underline{v}})$*-conjugates* of $\underline{\underline{\Theta}}_{\underline{v}}$ — determines a *monoid* $\mathcal{O}^{\triangleright}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ *on* $\mathcal{D}^{\Theta}_{\underline{v}}$. Now suppose, for simplicity, that the topological group $\Pi_{\underline{v}}$ arises from a *basepoint*, i.e., more concretely, from a *"universal covering pro-object"* A^{Θ}_{∞} of $\mathcal{D}_{\underline{v}}$ [i.e., a pro-object determined by a cofinal projective system of Galois objects of $\mathcal{D}_{\underline{v}}$]. Then by evaluating $\mathcal{O}^{\triangleright}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ on [the "universal covering pro-object" of $\mathcal{D}^{\Theta}_{\underline{v}}$ determined by] A^{Θ}_{∞}, we obtain *submonoids* [in the usual sense]

$$\begin{array}{ccccccc} \Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\mathrm{id}} & \stackrel{\mathrm{def}}{=} & \mathcal{O}^{\triangleright}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) & = & \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) \cdot \underline{\underline{\Theta}}_{\underline{v}}^{\mathbb{N}}|_{A^{\Theta}_{\infty}} & & \\ & \subseteq & {}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\mathrm{id}} & \stackrel{\mathrm{def}}{=} & \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) \cdot \underline{\underline{\Theta}}_{\underline{v}}^{\mathbb{Q}_{\geq 0}}|_{A^{\Theta}_{\infty}} & \subseteq & \mathcal{O}^{\times}(\mathbb{T}^{\div}_{A^{\Theta}_{\infty}}) \end{array}$$

— where the superscript "$\mathbb{Q}_{\geq 0}$" denotes the set of elements for which some [positive integer] *power* is equal to a [positive integer] power of $\underline{\underline{\Theta}}_{\underline{v}}|_{A^{\Theta}_{\infty}}$. In a similar vein, by considering [cf. [IUTchI], Remark 3.2.3, (i)] the various *conjugates* $\underline{\underline{\Theta}}^{\alpha}_{\underline{v}}$ of $\underline{\underline{\Theta}}_{\underline{v}}$, for $\alpha \in \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{\underline{v}})$, we also obtain submonoids $\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha} \subseteq {}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha} \subseteq \mathcal{O}^{\times}(\mathbb{T}^{\div}_{A^{\Theta}_{\infty}})$.

Moreover, one has a *natural surjection* $\Pi_{\underline{v}} \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{v})$, as well as a *natural conjugation action* of $\Pi_{\underline{v}}$ on the *collections of submonoids*

$$\Psi_{\mathcal{F}^{\Theta}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \left\{\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}\right\}_{\alpha \in \Pi_{\underline{v}}}; \quad {}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \left\{{}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}\right\}_{\alpha \in \Pi_{\underline{v}}}$$

— i.e., where, by abuse of notation, we think of the subscripted "α's" as denoting the image of "α" via the surjection $\Pi_{\underline{v}} \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{v})$. Also, we recall from *loc. cit.* that $\underline{\underline{\Theta}}_{v}^{\mathbb{Q}_{\geq 0}}|_{A^{\Theta}_{\infty}}$ determines *characteristic splittings*, up to torsion, of the monoids $\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}$ [cf. the "$\tau^{\Theta}_{\underline{v}}$" of [IUTchI], Example 3.2, (v)], ${}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}$ which are *compatible* with the action of $\Pi_{\underline{v}}$. Finally, we recall that the **collection of data**

$$\Pi_{\underline{v}} \quad \curvearrowright \quad \Psi_{\mathcal{F}^{\Theta}_{\underline{v}}} = \left\{\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}\right\}_{\alpha \in \Pi_{\underline{v}}}, \quad {}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}}} = \left\{{}_{\infty}\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha}\right\}_{\alpha \in \Pi_{\underline{v}}}$$

— i.e., consisting of two collections of submonoids of the group of units [namely, $\mathcal{O}^{\times}(\mathbb{T}^{\div}_{A^{\Theta}_{\infty}})$] associated to the birationalization of a certain characteristic pro-object of $\underline{\underline{\mathcal{F}}}_{v}$, equipped with the conjugation action by an automorphism group of a certain characteristic pro-object of $\mathcal{D}_{\underline{v}}$ — as well as the **characteristic splittings**, up to torsion, just discussed, may be **reconstructed category-theoretically** from $\underline{\underline{\mathcal{F}}}_{v}$ [cf. [IUTchI], Example 3.2, (vi), (e)], up to an indeterminacy arising from the **inner automorphisms** of $\Pi_{\underline{v}}$.

(ii) In a similar, but somewhat simpler, vein, the Frobenioid structure on the subcategory $\mathcal{C}_{\underline{v}} \subseteq \underline{\underline{\mathcal{F}}}_{v}$ — i.e., the *"base-field-theoretic hull"* of the tempered Frobenioid $\underline{\underline{\mathcal{F}}}_{v}$ [cf. [IUTchI], Example 3.2, (iii)] — determines, via the general theory of Frobenioids [cf. [FrdI], Proposition 2.2], a *monoid* $\mathcal{O}^{\triangleright}_{\mathcal{C}_{\underline{v}}}(-)$ on $\mathcal{D}_{\underline{v}}$. Then by evaluating $\mathcal{O}^{\triangleright}_{\mathcal{C}_{\underline{v}}}(-)$ on A^{Θ}_{∞}, we obtain a *monoid* [in the usual sense]

$$\Psi_{\mathcal{C}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathcal{O}^{\triangleright}_{\mathcal{C}_{\underline{v}}}(A^{\Theta}_{\infty})$$

which is equipped with a *natural action* by $\Pi_{\underline{v}}$. Finally, we recall that the **collection of data**

$$\Pi_{\underline{v}} \quad \curvearrowright \quad \Psi_{\mathcal{C}_{\underline{v}}}$$

— i.e., consisting of a submonoid of the group of units [namely, $\mathcal{O}^{\times}(\mathbb{T}^{\div}_{A^{\Theta}_{\infty}})$] associated to the birationalization of a certain characteristic pro-object of $\underline{\underline{\mathcal{F}}}_{v}$, equipped with the conjugation action by an automorphism group of a certain characteristic pro-object of $\mathcal{D}_{\underline{v}}$ — may be **reconstructed category-theoretically** from $\underline{\underline{\mathcal{F}}}_{v}$ [cf. [IUTchI], Example 3.2, (iii); [IUTchI], Example 3.2, (vi), (d); [FrdI], Theorem 3.4, (iv); [FrdII], Theorem 1.2, (i); [FrdII], Example 1.3, (i)], up to an indeterminacy arising from the **inner automorphisms** of $\Pi_{\underline{v}}$.

Proposition 3.3. (Frobenioid-theoretic Theta Monoids) *Suppose, in the situation of Proposition 3.1, that* $\mathbb{M}^{\Theta}_{*}$ *arises [cf. Proposition 1.2, (ii)] from a* **tempered Frobenioid** ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{v}$ *— i.e.,*

$$\mathbb{M}^{\Theta}_{*} = \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{v})$$

— that appears in a Θ*-Hodge theater* ${}^\dagger\mathcal{HT}^\Theta = (\{{}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{w}}\}_{\underline{w}\in\mathbb{V}},\ {}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$ *[cf. [IUTchI], Definition 3.6] — cf., for instance, the Frobenioid "*$\underline{\underline{\mathcal{F}}}_{\underline{v}}$*" of [IUTchI], Example 3.2, (i). Observe that by applying the category-theoretic constructions of Example 3.2, (i), (ii), to* ${}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}$*, one obtains data*

$$\Pi_{\underline{\underline{X}}}(\mathbb{M}^\Theta_*) \curvearrowright \Psi_{{}^\dagger\mathcal{F}^\Theta_{\underline{v}}} = \left\{\Psi_{{}^\dagger\mathcal{F}^\Theta_{\underline{v}},\alpha}\right\}_{\alpha\in\Pi_{\underline{\underline{X}}}(\mathbb{M}^\Theta_*)},\quad {}_\infty\Psi_{{}^\dagger\mathcal{F}^\Theta_{\underline{v}}} = \left\{{}_\infty\Psi_{{}^\dagger\mathcal{F}^\Theta_{\underline{v}},\alpha}\right\}_{\alpha\in\Pi_{\underline{\underline{X}}}(\mathbb{M}^\Theta_*)};$$
$$\Pi_{\underline{\underline{X}}}(\mathbb{M}^\Theta_*) \curvearrowright \Psi_{{}^\dagger\mathcal{C}_{\underline{v}}}$$

as well as splittings, up to torsion, of each of the monoids $\Psi_{{}^\dagger\mathcal{F}^\Theta_{\underline{v}},\alpha}$*,* ${}_\infty\Psi_{{}^\dagger\mathcal{F}^\Theta_{\underline{v}},\alpha}$*.*

(i) **(Split Theta Monoids)** *By forming* **Kummer classes** *relative to the Frobenioid structure of* ${}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}$ *— i.e., in essence, by considering the Galois cohomology classes that arise when one extracts* N*-th roots of unity for* $N \in \mathbb{N}_{\geq 1}$ *[cf. [FrdII], Definition 2.1, (ii); [IUTchI], Remark 3.2.3, (ii); the discussion of [EtTh], §5] — and applying the description given in Proposition 1.3, (i), of the* **exterior cyclotome** *of a mono-theta environment that arises from a tempered Frobenioid, one obtains, for a suitable* **bijection** *of* $l \cdot \underline{\mathbb{Z}}$**-torsors** *between [*$\mathrm{Gal}(\ddot{\underline{\underline{Y}}}_{\underline{v}}/\underline{\underline{Y}}_{\underline{v}})$*-orbits of] "*$\iota$*" as in Proposition 2.2, (i), and images of "*α*" via the natural surjection* $\Pi_{\underline{v}} \twoheadrightarrow l \cdot \underline{\mathbb{Z}}$*, collections of* **isomorphisms of monoids**

$$\Psi_{{}^\dagger\mathcal{F}^\Theta_{\underline{v}},\alpha} \ \stackrel{\sim}{\to}\ \Psi^\iota_{\mathrm{env}}(\mathbb{M}^\Theta_*);\quad {}_\infty\Psi_{{}^\dagger\mathcal{F}^\Theta_{\underline{v}},\alpha} \ \stackrel{\sim}{\to}\ {}_\infty\Psi^\iota_{\mathrm{env}}(\mathbb{M}^\Theta_*)$$

— each of which is **well-defined** *up to composition with an inner automorphism [cf. the discussion of Example 3.2, (i)] and* **compatible** *with both the respective* **conjugation actions** *by* $\Pi_{\underline{\underline{X}}}(\mathbb{M}^\Theta_*)$ *and the* **splittings** *up to torsion on the monoids under consideration. We shall denote these collections of isomorphisms by means of the notation*

$$\Psi_{{}^\dagger\mathcal{F}^\Theta_{\underline{v}}} \ \stackrel{\sim}{\to}\ \Psi_{\mathrm{env}}(\mathbb{M}^\Theta_*);\quad {}_\infty\Psi_{{}^\dagger\mathcal{F}^\Theta_{\underline{v}}} \ \stackrel{\sim}{\to}\ {}_\infty\Psi_{\mathrm{env}}(\mathbb{M}^\Theta_*)$$

[cf. the notation of Proposition 3.1, (i); Example 3.2, (i)].

(ii) **(Constant Monoids)** *By forming* **Kummer classes** *relative to the Frobenioid structure of* ${}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}$ *— i.e., in essence, by considering the Galois cohomology classes that arise when one extracts* N*-th roots of unity for* $N \in \mathbb{N}_{\geq 1}$ *[cf. [FrdII], Definition 2.1, (ii); [IUTchI], Remark 3.2.3, (ii); [FrdII], Theorem 2.4] — and applying the description given in Proposition 1.3, (i), of the* **exterior cyclotome** *of a mono-theta environment that arises from a tempered Frobenioid, one obtains an* **isomorphism of monoids**

$$\Psi_{{}^\dagger\mathcal{C}_{\underline{v}}} \ \stackrel{\sim}{\to}\ \Psi_{\mathrm{cns}}(\mathbb{M}^\Theta_*)$$

— which is **well-defined** *up to composition with an inner automorphism [cf. the discussion of Example 3.2, (ii)] and* **compatible** *with the respective* **conjugation actions** *by* $\Pi_{\underline{\underline{X}}}(\mathbb{M}^\Theta_*)$*.*

Proof. Assertions (i) and (ii) follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Proposition 3.4. (Group-theoretic Theta Monoids) *Let* $^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}$ *be a* **tempered Frobenioid** *as in Proposition 3.3. Consider the* **full poly-isomorphism**

$$\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \quad \xrightarrow{\sim} \quad \mathbb{M}^{\Theta}_{*}(^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

— where $\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})$ *is the projective system of mono-theta environments arising from the algorithm of Proposition 1.2, (i) [cf. also Proposition 1.5, (i)] — of* **projective systems of mono-theta environments***.*

(i) **(Multiradiality of Split Theta Monoids)** *Each isomorphism of projective systems of mono-theta environments* $\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \xrightarrow{\sim} \mathbb{M}^{\Theta}_{*}(^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})$ *induces* **compatible** *[in the evident sense]* **collections of isomorphisms**

$$\begin{array}{ccccc}
\Pi_{\underline{v}} \xrightarrow{\sim} \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}(^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})) & = & \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}(^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})) \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
{}_\infty\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & {}_\infty\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})) & \xrightarrow{\sim} & {}_\infty\Psi_{^\dagger\mathcal{F}^{\Theta}_{\underline{v}}}; \\
\bigcup & & \bigcup & & \bigcup \\
\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & \Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})) & \xrightarrow{\sim} & \Psi_{^\dagger\mathcal{F}^{\Theta}_{\underline{v}}}
\end{array}$$

and

$$\begin{array}{ccccc}
G_{\underline{v}} \xrightarrow{\sim} G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}(^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})) & = & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}(^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})) \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))^{\times} & \xrightarrow{\sim} & \Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}))^{\times} & \xrightarrow{\sim} & (\Psi_{^\dagger\mathcal{F}^{\Theta}_{\underline{v}}})^{\times}
\end{array}$$

— where the upper horizontal isomorphisms in each diagram are isomorphisms of topological groups; the lower/middle horizontal isomorphisms in each diagram are isomorphisms of [ind-topological] monoids; the lower/middle horizontal isomorphisms in the first diagram are compatible with the respective **splittings** *up to torsion; the left-hand square in each diagram arises from the* **functoriality** *of the algorithms involved, relative to isomorphisms of projective systems of mono-theta environments; the right-hand square in each diagram arises from the* **inverses** *of the isomorphisms of the second display of Proposition 3.3, (i); the superscript "×" denotes the submonoid of units; the notation "*$G_{\underline{v}}(-)$*" is as in Proposition 3.1, (ii). Finally, if we write* $(\Psi_{^\dagger\mathcal{F}^{\Theta}_{\underline{v}}})^{\times\boldsymbol{\mu}}$ *for the ind-topological monoid obtained by forming the quotient of* $(\Psi_{^\dagger\mathcal{F}^{\Theta}_{\underline{v}}})^{\times}$ *by its torsion subgroup, then the* **functorial algorithms**

$$\Pi_{\underline{v}} \;\mapsto\; \Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})); \quad \Pi_{\underline{v}} \;\mapsto\; {}_\infty\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$$

— where we think of $\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$, ${}_\infty\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$ *as being equipped with their natural* $\Pi_{\underline{v}}$*-actions and splittings up to torsion [cf. Proposition 3.1, (i)] — obtained by composing the algorithms of Propositions 1.2, (i); 3.1, (i), are* **compatible***, relative to the above displayed diagrams, with* **arbitrary automorphisms** *of [the underlying pair, consisting of an ind-topological monoid equipped with the action of a topological group, determined by] the pair*

$$G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}(^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})) \quad \curvearrowright \quad (\Psi_{^\dagger\mathcal{F}^{\Theta}_{\underline{v}}})^{\times\boldsymbol{\mu}}$$

which arise as Ism**-multiples** *of automorphisms induced by automorphisms of [the underlying pair, consisting of an ind-topological monoid equipped with the action of a topological group, determined by] the pair* $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \curvearrowright (\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times}$ *[cf. Example 1.8, (iv); Remark 1.8.1; Remark 1.11.1, (i), (b)] — in the sense that the natural functor "*$\Psi_{\mathcal{R}}$*" of Corollary 1.12, (iii), is* **multiradially defined***.*

(ii) **(Uniradiality of Constant Monoids)** *Each isomorphism of projective systems of mono-theta environments* $\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \xrightarrow{\sim} \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$ *induces* **compatible collections of isomorphisms**

$$\begin{array}{ccccc}
\Pi_{\underline{v}} \xrightarrow{\sim} \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) & = & \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) & \xrightarrow{\sim} & \Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}}
\end{array}$$

and

$$\begin{array}{ccccc}
G_{\underline{v}} \xrightarrow{\sim} G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) & = & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \\
\curvearrowright & & \curvearrowright & & \curvearrowright \\
\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))^{\times} & \xrightarrow{\sim} & \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))^{\times} & \xrightarrow{\sim} & (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})^{\times}
\end{array}$$

— where the upper horizontal isomorphisms in each diagram are isomorphisms of topological groups; the lower horizontal isomorphisms in each diagram are isomorphisms of [ind-topological] monoids; the second diagram may be **naturally identified** *with the second displayed commutative diagram of (i); the left-hand square in each diagram arises from the* **functoriality** *of the algorithms involved, relative to isomorphisms of projective systems of mono-theta environments; the right-hand square in each diagram arises from the* **inverse** *of the displayed isomorphism of Proposition 3.3, (ii); the superscript "*$\times$*" denotes the submonoid of units; the notation "*$G_{\underline{v}}(-)$*" is as in Proposition 3.1, (ii). Finally, if we write* $(\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})^{\times\boldsymbol{\mu}}$ *for the ind-topological monoid obtained by forming the quotient of* $(\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})^{\times}$ *by its torsion subgroup, then the* **functorial algorithm**

$$\Pi_{\underline{v}} \mapsto \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$$

— where we think of $\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$ *as being equipped with its natural* $\Pi_{\underline{v}}$*-action [cf. Proposition 3.1, (ii)] — obtained by composing the algorithms of Proposition 1.2, (i); 3.1, (ii), depends on the* **cyclotomic rigidity isomorphism** *of Corollary 1.11, (b) [cf. Remark 1.11.5, (ii); the use of the surjection of Remark 1.11.5, (i), in the algorithm of Proposition 3.1, (ii)], hence* **fails to be compatible***, relative to the above displayed diagrams, with* **automorphisms** *of [the underlying pair, consisting of an ind-topological monoid equipped with the action of a topological group, determined by] the pair*

$$G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \curvearrowright (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})^{\times\boldsymbol{\mu}}$$

which arise from automorphisms of [the underlying pair, consisting of an ind-topological monoid equipped with the action of a topological group, determined by] *the pair* $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \curvearrowright (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})^{\times}$ *[cf. Remarks 1.11.1, (i), (b); 1.8.1] — in the sense that this algorithm, as given, only admits a* **uniradial** *formulation [cf. Remarks 1.11.3, (iv); 1.11.5, (ii)].*

Proof. Assertions (i) and (ii) follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 3.4.1.

(i) Note that the pairs

$$\text{“}G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \curvearrowright (\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times\boldsymbol{\mu}}\text{”} \quad \text{and} \quad \text{“}G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \curvearrowright (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})^{\times\boldsymbol{\mu}}\text{”}$$

that appear in Proposition 3.4, (i), (ii), correspond to the pair “$G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G)$” that appears in the discussion of Remark 1.11.3, (ii) — i.e., the data that arises by replacing the “$\mathcal{O}^{\times}$” that appears in the Θ-*link* of [IUTchI], Corollary 3.7, (iii), by “$\mathcal{O}^{\times\boldsymbol{\mu}}$”. That is to say, from the point of view of the present series of papers, the *significance* of Proposition 3.4 lies in the point of view that

> the **multiradiality** (respectively, **uniradiality**) asserted in Proposition 3.4, (i) (respectively, (ii)), may be thought of as a statement of the **compatibility** (respectively, **incompatibility**) of the algorithm in question with the *“$\mathcal{O}^{\times\boldsymbol{\mu}}$-version” of the* **$\Theta$-link** *of [IUTchI], Corollary 3.7, (iii).*

(ii) One important consequence of the theory to be developed in [IUTchIII] [cf. Remark 2.9.1, (iii)] is the result that,

> by applying the theory of *log-shells* [cf. [AbsTopIII]], one may construct certain algorithms related to the algorithm of Proposition 3.4, (ii), that [yield functors which] are *manifestly* **multiradially defined**

— albeit at the cost of allowing for certain [relatively mild!] **indeterminacies**.

The following two corollaries will play a fundamental role in the present series of papers.

Corollary 3.5. **(Mono-theta-theoretic Gaussian Monoids)** *Let* $\mathbb{M}^{\Theta}_{*}$ *be as in Proposition 3.1 [cf. also Corollary 2.8, in the case where* $\gamma = 1$*; Remark 3.5.1 below]. For* $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}))$*, we shall denote copies labeled by* t *of various objects functorially constructed from* $\mathbb{M}^{\Theta}_{*}$ *by means of a* **subscript** *“t”. Also, we shall write*

$$\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \quad \subseteq \quad \Pi_{\underline{X}}(\mathbb{M}^{\Theta}_{*}) \quad \subseteq \quad \Pi_{C}(\mathbb{M}^{\Theta}_{*})$$

$$\Delta_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \quad \subseteq \quad \Delta_{\underline{X}}(\mathbb{M}^{\Theta}_{*}) \quad \subseteq \quad \Delta_{C}(\mathbb{M}^{\Theta}_{*})$$

for the inclusions — which may be functorially constructed from $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$ *— corresponding to the inclusions* $\Pi_{\underline{v}} \subseteq \Pi^{\pm}_{\underline{v}} \subseteq \Pi^{\mathrm{cor}}_{\underline{v}}$, $\Delta_{\underline{v}} \subseteq \Delta^{\pm}_{\underline{v}} \subseteq \Delta^{\mathrm{cor}}_{\underline{v}}$ *of Definition 2.3, (i).*

(i) **(Labels, $\mathbb{F}_l^{\rtimes\pm}$-Symmetries, and Conjugate Synchronization)** *If we think of the* **cuspidal inertia groups** $\subseteq \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$ *corresponding to* t *as subgroups of cuspidal inertia groups of* $\Pi_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$ *[cf. Remark 2.3.1], then the* $\Delta_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$*-outer action of* $\mathbb{F}_l^{\rtimes\pm} \cong \Delta_C(\mathbb{M}^{\Theta}_{*})/\Delta_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$ *on* $\Pi_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$ *[cf. Corollary 2.4, (iii); Remark 1.1.1, (iv), or, alternatively, when applicable, Proposition 1.3, (ii), (iii)] induces* **isomorphisms** *between the pairs*

$$G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_t \curvearrowright \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_t$$

— consisting of a **labeled** *ind-topological monoid equipped with the action of a* **labeled** *topological group [cf. Proposition 3.1, (ii)] — for* **distinct** $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}))$*. We shall refer to these isomorphisms as* **[$\mathbb{F}_l^{\rtimes\pm}$-]symmetrizing isomorphisms** *[cf. Remark 3.5.2 below]. We shall denote by means of a subscript "$|t| \in |\mathbb{F}_l|$" the result of identifying copies labeled by t, $-t$ via a suitable symmetrizing isomorphism. We shall denote by means of a subscript "$\langle|\mathbb{F}_l|\rangle$" (respectively, "$\langle\mathbb{F}_l^{\circledast}\rangle$") the* **diagonal embedding***, determined by suitable symmetrizing isomorphisms, inside the direct product of copies labeled by $|t| \in |\mathbb{F}_l|$ (respectively, $|t| \in \mathbb{F}_l^{\circledast}$). In particular, by* **restricting** *the monoid* $\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})$ *of Proposition 3.1, (ii), via the restriction operations [i.e., to "$\Pi_{\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}}$" and "$D^{\delta}_{t,\mu_-}$"] described in detail in Corollary 2.8, (i), (ii), one obtains a collection of* **compatible morphisms**

$$\begin{array}{ccccc}
\left(\ \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \hookleftarrow\ \right) & \Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) & \twoheadrightarrow & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_{\langle|\mathbb{F}_l|\rangle} \\
 & \curvearrowright & & \curvearrowright \\
 & \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}) & \stackrel{\sim}{\rightarrow} & \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle|\mathbb{F}_l|\rangle}
\end{array}$$

— where the notation "$\curvearrowright$" denotes the natural actions; the bottom horizontal arrow is an isomorphism of monoids — which are **compatible** *with the various* **symmetrizing isomorphisms** *and* **well-defined up to** *composition with an* **inner automorphism** *of* $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$ *[i.e., up to composition with the conjugation action by* $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$ *on the pair* $\Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \curvearrowright \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})$*]. Put another way, this inner automorphism indeterminacy — which,* **a priori***, depends on the index $|t|$ — is, in fact,* **independent** *of $|t| \in |\mathbb{F}_l|$.*

(ii) **(Gaussian Monoids)** *We shall refer to an element of the set*

$$\underline{\underline{\theta}}^{\mathbb{F}_l^{\circledast}}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \stackrel{\mathrm{def}}{=} \prod_{|t|\in\mathbb{F}_l^{\circledast}} \underline{\underline{\theta}}^{|t|}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \subseteq \prod_{|t|\in\mathbb{F}_l^{\circledast}} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{|t|}$$

[cf. the notation of Corollary 2.8, (i), (ii)] — which is of **cardinality** $(2l)^{l^{\circledast}}$ *— as a* **value-profile***. Then by applying [the various objects constructed from] the* **symmetrizing isomorphisms** *of (i), together with the functorial algorithm [for restricting elements of* $\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$, $_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$*] of Corollary 2.8, (i), (ii), one obtains a* **functorial algorithm** *for constructing two collections of* **submonoids**

$$\mathbb{M}^{\Theta}_{*} \mapsto$$

$$\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}) \stackrel{\mathrm{def}}{=} \left\{ \Psi_{\xi}(\mathbb{M}^{\Theta}_{*}) \stackrel{\mathrm{def}}{=} \Psi^{\times}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle\mathbb{F}_l^{\circledast}\rangle} \cdot \xi^{\mathbb{N}} \subseteq \prod_{|t|\in\mathbb{F}_l^{\circledast}} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{|t|} \right\}_{\xi},$$

$$_{\infty}\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}) \stackrel{\mathrm{def}}{=} \left\{ _{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}) \stackrel{\mathrm{def}}{=} \Psi^{\times}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle\mathbb{F}_l^{\circledast}\rangle} \cdot \xi^{\mathbb{Q}_{\geq 0}} \subseteq \prod_{|t|\in\mathbb{F}_l^{\circledast}} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{|t|} \right\}_{\xi}$$

— *where the superscript "$\times$" denotes the submonoid of units; ξ ranges over the* **value-profiles***; "$\xi^{\mathbb{Q}_{\geq 0}}$" denotes the submonoid generated by the N-th roots [for $N \in \mathbb{N}_{\geq 1}$] of ξ [which are* **uniquely determined***, up to multiplication by an element of the N-torsion subgroup of $\Psi^{\times}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle \mathbb{F}^{\circledast}_{l} \rangle}$!] that arise by restricting elements of ${}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$; each $\Psi_{\xi}(\mathbb{M}^{\Theta}_{*})$ is equipped with a* **natural action** *by $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_{\langle \mathbb{F}^{\circledast}_{l} \rangle}$. We shall refer to each $\Psi_{\xi}(\mathbb{M}^{\Theta}_{*})$ or ${}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*})$ as a* **Gaussian monoid***. Here, the submonoid $\Psi_{2l\cdot\xi}(\mathbb{M}^{\Theta}_{*}) \subseteq \Psi_{\xi}(\mathbb{M}^{\Theta}_{*})$ generated by $\Psi^{\times}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle \mathbb{F}^{\circledast}_{l} \rangle}$ and $\xi^{2l\cdot\mathbb{N}}$ is* **independent** *of the value-profile ξ. Finally, the* **restriction** *operations described in detail in Corollary 2.8, (i), (ii), determine a collection of* **compatible** *[in the evident sense]* **morphisms** *[cf. Remark 3.6.1 below]*

$$\begin{array}{ccc}
\Big(\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \hookleftarrow \Big) \quad \Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) & \dashleftarrow & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_{|t|}\}_{|t| \in \mathbb{F}^{\circledast}_{l}} \\
\curvearrowright & & \curvearrowright \\
{}_{\infty}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) & \stackrel{\sim}{\to} & {}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}) \\
\bigcup & & \bigcup \\
\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) & \stackrel{\sim}{\to} & \Psi_{\xi}(\mathbb{M}^{\Theta}_{*})
\end{array}$$

— *where the "$\dashleftarrow$" in the first line denotes the* **compatibility** *of the action [denoted by the second "$\curvearrowright$" in the second line] of $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_{|t|}$ on the factor labeled "$|t|$" of the direct product containing ${}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*})$ [cf. the definition of ${}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*})$] with the* **inclusions** *$G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}) \hookrightarrow \Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$ determined by the various choices of the "D^{δ}_{t,μ_-}" [cf. Corollary 2.8, (i), (ii)] that gave rise to the value-profile ξ; the first "$\curvearrowright$" in the second line denotes the natural action; the lower/middle horizontal arrows are isomorphisms of monoids — which is* **well-defined up to** *composition with a(n) [single!]* **inner automorphism** *of $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$ and* **compatible** *[in the evident sense] with the equalities of submonoids $\Psi_{2l\cdot\xi_1}(\mathbb{M}^{\Theta}_{*}) = \Psi_{2l\cdot\xi_2}(\mathbb{M}^{\Theta}_{*})$ for distinct value-profiles ξ_1, ξ_2. For simplicity, we shall use the notation*

$$\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) \stackrel{\sim}{\to} \Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}); \quad {}_{\infty}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}) \stackrel{\sim}{\to} {}_{\infty}\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*})$$

to denote these collections of compatible morphisms induced by restriction.

(iii) **(Constant Monoids and Splittings)** *Denote the zero element of $|\mathbb{F}_l|$ by $0 \in |\mathbb{F}_l|$. Then [in the notation of (i)] the diagonal submonoid $\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle |\mathbb{F}_l| \rangle}$ determines — i.e., may be thought of as the* **graph** *of — an* **isomorphism of monoids**

$$\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_0 \stackrel{\sim}{\to} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle \mathbb{F}^{\circledast}_{l} \rangle}$$

that is **compatible** *with the respective labeled $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$-actions. Moreover, the* **restriction** *operations to* **zero-labeled evaluation points** *described in detail in Corollary 2.8, (i), (ii), (iii), determine a* **splitting** *up to torsion of each of the Gaussian monoids*

$$\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}) = \Psi^{\times}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle \mathbb{F}^{\circledast}_{l} \rangle} \cdot \xi^{\mathbb{N}}, \quad {}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}) = \Psi^{\times}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle \mathbb{F}^{\circledast}_{l} \rangle} \cdot \xi^{\mathbb{Q}_{\geq 0}}$$

[cf. the definition of $\Psi_{\xi}(\mathbb{M}^{\Theta}_{*})$, ${}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*})$ in (ii)] which is **compatible***, relative to the* **restriction isomorphisms** *of the third display of (ii), with the splittings up to torsion of Proposition 3.1, (i).*

Proof. The various assertions of Corollary 3.5 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 3.5.1.

(i) Note that in Corollary 3.5, unlike the situation of Corollary 2.8, we took γ to be $= 1$. This was done primarily to *simplify the notation* and does not result in any substantive loss of generality. Indeed, one may always simply take the "$\mathbb{M}_*^{\Theta}$" of Corollary 3.5 to be the "$(\mathbb{M}_*^{\Theta})^{\gamma}$" of Corollary 2.8. Alternatively, one may observe that the "δ" that appears in the "D_{t,μ_-}^{δ}" that occurs in the various restriction operations invoked in Corollary 3.5 [cf. Corollary 2.8, (i), (ii)] is *arbitrary*, i.e., it is subject to the *independent conjugation indeterminacies* discussed in Corollary 2.5, (iii); Remark 2.5.2.

(ii) In the present context, it is useful to recall that from the point of view of the discussion of [IUTchI], Remark 3.2.3, (i), the various $\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta})$**-conjugacy indeterminacies** that appear in Corollary 3.5 are applied, in the context of the theory of the present series of papers, to **identify** *the various* $\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta})$*-conjugates of* $\Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta})$ [or, alternatively, "ι's"] with one another.

Remark 3.5.2. Before proceeding, it is useful to pause to consider the significance of the **symmetrizing isomorphisms** of Corollary 3.5, (i).

(i) We begin by discussing a simple **combinatorial model** of the phenomenon of interest. Consider the *totally ordered set* $E = \{0, 1\}$ whose ordering is completely determined by the *inequality*

$$0 \quad < \quad 1$$

— which we shall denote, in the following discussion, by the notation "$\prec$". Then one may consider *labeled copies*

$$\prec_0, \quad \prec_1$$

of $\prec$. Now suppose that one attempts to *identify* these labeled copies $\prec_0$, $\prec_1$ by simply **forgetting the labels**. This amounts, in effect, to sending the *two distinct subscripted labels*

$$E \ \ni \ 0,\ 1 \ \mapsto \ *$$

to a *single point* "$*$". In particular, this **naive approach** to identifying the labeled copies $\prec_0$, $\prec_1$ **fails** to be **compatible** — in a sense that we shall examine in more detail in the discussion to follow — with operations that require one to **distinguish** *the two labels* $0, 1 \in E$. Now if, to avoid confusion, one writes S for the underlying set of E [i.e., obtained from E by *forgetting the ordering* on E], then one has a *natural* $\mathrm{Aut}(S)$*-orbit of bijections*

$$E \ \stackrel{\sim}{\rightarrow} \ S \ \curvearrowleft \ \mathrm{Aut}(S)$$

— where $\mathrm{Aut}(S) \cong \mathbb{Z}/2\mathbb{Z}$. Next, let us suppose that we are given an object $F(\prec)$ *functorially constructed* from [the "totally ordered set of cardinality two"] $\prec$. Then any **"factorization"** of the functorial construction $F(-)$ [i.e., on "totally ordered sets of cardinality two"] through a functorial construction

$$F^{\mathrm{sym}}(S) \curvearrowleft \mathrm{Aut}(S)$$

on *unordered sets of cardinality two* [i.e., relative to the "forgetful functor" that associates to an ordered set the underlying unordered set] may be thought of as a collection of **"symmetrizing isomorphisms"** [cf. the discussion of (ii) below; Corollary 3.5, (i)], or, alternatively, as **"descent data"** for $F(-)$ from E to the **"orbiset quotient"** of S by $\mathrm{Aut}(S)$. Moreover, this "descent data" satisfies the crucial property that it allows one to perform this "descent to the orbiset quotient" in such a way that one is

> *never required to* **violate** *the* **bijective** *relationship — albeit via an* **indeterminate** *bijection! — between E and S.*

By contrast, the "naive approach" discussed above may be thought of as corresponding to working with the **"coarse set-theoretic quotient"** Q of S by $\mathrm{Aut}(S)$ — which we shall think of as consisting of a single point $* \stackrel{\mathrm{def}}{=} \{0,1\} \in Q = \{*\}$. Now suppose, for instance, in the case $F(\prec) \stackrel{\mathrm{def}}{=} \prec$, that one attempts to regard $F(\prec)_{(-)} \stackrel{\mathrm{def}}{=} \prec_{(-)}$ [where $(-) \in S$] as an object *"pulled back"* from a copy $\prec_Q$ [i.e., "$0_Q < 1_Q$"] of $\prec$ over Q. On the other hand, if one wishes to relate each point $s \in S$ to one or more points $\in E_Q \stackrel{\mathrm{def}}{=} \{0_Q, 1_Q\}$ via an $\mathrm{Aut}(S)$-*equivariant* assignment in such a way that every point of E_Q appears in the image of this assignment, then one has no choice but to assign to each point $s \in S$ the collection of *all points* $\in E_Q$. Put another way, one must contend with an **independent indeterminacy**

$$s \quad \mapsto \quad 0_Q? \quad 1_Q?$$

for each $s \in S$ — i.e., if we write $S = \{0_S, 1_S\}$, then these indeterminacies give rise to a total of 4 *possibilities*

$$\begin{array}{ccc} 0_S & \mapsto & 0_Q? \quad 1_Q? \\ 1_S & \mapsto & 0_Q? \quad 1_Q? \end{array}$$

for the desired *assignment*, certain of which [i.e., $0_S, 1_S \mapsto 0_Q$ and $0_S, 1_S \mapsto 1_Q$] **fail** to be **bijective**. Here, it is useful to note that to *synchronize* these indeterminacies amounts, *tautologically*, to the requirement of an "automorphism of $\prec_Q$ that induces the unique nontrivial automorphism of the set $E_Q = \{0_Q, 1_Q\}$". On the other hand, by the *definition* of an "inequality", it is a *tautology* that such an automorphism of $\prec_Q$ *cannot exist*. Finally, in this context, it is useful to recall that this difference between *"crushing the set E to a single point"* and *"symmetrizing without violating the bijective relationship to E"* is precisely the topic of the discussion of [IUTchI], Remark 4.9.2, (i); [IUTchI], Remark 6.12.4, (i) — cf., especially, [IUTchI], Fig. 4.5.

(ii) The starting point of the theory surrounding the symmetrizing isomorphisms of Corollary 3.5, (i), is the *connectedness* — or **"single basepoint"** — observed in the discussion of Remark 2.6.1, (i), together with the compatibility of this connectedness with a certain $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, as discussed in Remark 2.6.2, (i). These symmetrizing isomorphisms may be applied to labeled copies of various objects constructed from $\mathbb{M}^{\Theta}_{*}$ — e.g., $\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})$, $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})$, $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*})$ — cf. the discussion of *"conjugate synchronization"* in Remark 2.6.1, (i). Note that in the *absence* of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry involved, the "single basepoint" under consideration has a **rigidifying** effect *not only* on the various *conjugates* involved, but *also* on the *labels* under consideration. That is to say, *a priori*, it is quite possible that

> *the desired rigidity of the conjugates involved* **depends** *on the rigidity of the labels under consideration.*

Indeed, this is precisely what happens when the data that one wishes to synchronize — i.e., such as monoids, absolute Galois groups, or cyclotomes — consists, for instance, of an *arrow from one label to another*, as was [essentially] the case in the discussion of the combinatorial model of (i). Put another way,

> *the significance of the* $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** *under consideration lies precisely in the observation that this symmetry serves to* **eliminate** *this* **unwanted "a priori" possibility**.

This is in some sense the *central principle* illustrated by the combinatorial model of (i). Put in other words, this "central principle" discussed in (i) may be summarized, in the situation of Corollary 3.5, as follows: the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** under consideration allows one to construct

(a) **symmetrizing isomorphisms** [cf. Corollary 3.5, (i)]

in a fashion that is *compatible* with maintaining a

(b) **bijective** link with the set of **labels** $\mathrm{LabCusp}^{\pm}(\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}))$

— which is necessary in order to construct the **Gaussian monoids** [i.e., which involve *distinct* values at *distinct labels*!] in Corollary 3.5, (ii) — all relative to

(c) a **single basepoint** [i.e., which gives rise to the *single* topological group $\Pi_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$ — cf. the discussion of Remark 2.6.2, (i)]

— which is necessary in order to establish **conjugate synchronization**.

(iii) In the context of Corollary 3.5, (i), one essential aspect of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry under consideration is that this symmetry arises from a $\Delta_{\underline{X}}(\mathbb{M}^{\Theta}_{*})$-*outer action* of $\Delta_C(\mathbb{M}^{\Theta}_{*})/\Delta_{\underline{X}}(\mathbb{M}^{\Theta}_{*}) \xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm}$ [cf. the discussion of Remark 2.6.2, (i)]. That is to say, the fact that this action may be formulated entirely in terms of conjugation by elements of **geometric** [i.e., "Δ"] fundamental groups — that is to say, as opposed to *arithmetic* [i.e., "Π"] fundamental groups — plays a *crucial role* in establishing the **conjugate synchronization** of the various copies of "$G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})$" [and objects constructed from "$G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})$"] under consideration [cf. the discussion of [IUTchI], Remark 6.12.6, (ii)].

(iv) If one thinks of the $\mathbb{F}_l^{\rtimes\pm}$-*symmetries* that appear in the conjugate synchronization of Corollary 3.5, (i), as *"connecting"* the various copies of objects at distinct evaluation points, then it is perhaps natural to regard the **"conjugate synchronization via symmetry"** of Corollary 3.5, (i), as a sort of **nonarchimedean** version of the **"conjugate synchronization via connectedness"** discussed in Remark 2.6.1, (i), which may be thought of as being based on the **"archimedean"** connectedness of the subgraph $\Gamma^{\blacktriangleright}_{\underline{\underline{X}}} \subseteq \Gamma_{\underline{\underline{X}}}$ [cf. the discussion of Remarks 2.6.1, (i); 2.8.3].

(v) In §4 below, we shall generalize the ideas discussed in the present Remark 3.5.2 concerning conjugate synchronization in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ to the **global** portion, as well as to the portion at **good** $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, of a $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-Hodge theater [cf. the discussions of Remark 2.6.2, (i); Remark 3.8.2 below].

Remark 3.5.3. The delicacy and subtlety of the theory surrounding Corollary 3.5, (i), may be thought of as a consequence of the requirement of *simultaneously satisfying* the conditions (a), (b), (c) discussed in Remark 3.5.2, (ii). On the other hand, if one is willing to eliminate condition (c) from one's arguments, then one may obtain *symmetrizing isomorphisms* by simply applying the *functors* of [IUTchI], Proposition 6.8, (i), (ii), (iii); [IUTchI], Proposition 6.9, (i), (ii) — i.e., by passing to $\mathcal{D}$-Θ^{ell}-*bridges* or [holomorphic or mono-analytic] *capsules* or *processions*. Here, we observe that this **"multi-basepoint"** approach to constructing symmetrizing isomorphisms is **compatible** with the single basepoint $\mathbb{F}_l^{\rtimes\pm}$-symmetric approach of Corollary 3.5, (i), relative to the evident *"forgetful functors"*. We leave the routine details to the reader.

Corollary 3.6. **(Frobenioid-theoretic Gaussian Monoids)** *Suppose that we are in the situation of Proposition 3.3, i.e., that*

$$\mathbb{M}_*^{\Theta} \quad = \quad \mathbb{M}_*^{\Theta}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

— *where* ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ *is a* **tempered Frobenioid**. *We continue to use the conventions introduced in Corollary 3.5 concerning* **subscripted labels**.

(i) **(Labels, $\mathbb{F}_l^{\rtimes\pm}$-Symmetries, and Conjugate Synchronization)** *The isomorphism of Proposition 3.3, (ii) [or, alternatively, Proposition 1.3, (ii), (iii)], determines, for each $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta}))$, a collection of* **compatible morphisms**

$$\begin{array}{ccccc}
\Big(\ \Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta})_t \ \twoheadrightarrow \ \Big) & G_{\underline{v}}(\mathbb{M}_*^{\Theta})_t & \xrightarrow{\sim} & G_{\underline{v}}(\mathbb{M}_{*\ddot{\blacktriangleright}}^{\Theta})_t \\
 & \curvearrowright & & \curvearrowright \\
 & (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})_t & \xrightarrow{\sim} & \Psi_{\mathrm{cns}}(\mathbb{M}_*^{\Theta})_t
\end{array}$$

— *which are* **well-defined** *up to composition with an* **inner automorphism** *of $\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta})$ which is* **independent** *of $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta}))$ — as well as* **[$\mathbb{F}_l^{\rtimes\pm}$-]symmetrizing isomorphisms**, *induced by the $\Delta_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta})$-outer action of $\mathbb{F}_l^{\rtimes\pm} \cong \Delta_C(\mathbb{M}_*^{\Theta})/\Delta_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta})$ on $\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta})$ [cf. Corollary 3.5, (i); Remark 1.1.1, (iv), or, alternatively, Proposition 1.3, (ii), (iii)], between the data indexed by distinct $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{\underline{X}}}(\mathbb{M}_*^{\Theta}))$.*

(ii) **(Gaussian Monoids)** *For each* **value-profile** *ξ [cf. Corollary 3.5, (ii)], write*

$$\Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \ \subseteq \ {}_{\infty}\Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \ \subseteq \ \prod_{|t| \in \mathbb{F}_l^{*}} (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})_{|t|}$$

for the **submonoids** *determined, respectively, via the isomorphisms* $(\Psi_{{}^\dagger\mathcal{C}_{\underline{v}}})_{|t|} \xrightarrow{\sim} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_*)_{|t|}$ *of (i), by the monoids* $\Psi_\xi(\mathbb{M}^{\Theta}_*)$, ${}_\infty\Psi_\xi(\mathbb{M}^{\Theta}_*)$ *of Corollary 3.5, (ii), and*

$$\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \left\{ \Psi_{\mathcal{F}_\xi}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}) \right\}_\xi, \qquad {}_\infty\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \left\{ {}_\infty\Psi_{\mathcal{F}_\xi}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}) \right\}_\xi$$

— where ξ *ranges over the value-profiles. Thus, each monoid* $\Psi_{\mathcal{F}_\xi}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})$ *is equipped with a natural* **action** *by* $G_{\underline{v}}(\mathbb{M}^{\Theta}_*)_{\langle \mathbb{F}^{\circledast}_l \rangle}$*. Then by composing the* **Kummer isomorphisms** *discussed in (i) above and Proposition 3.3, (i), (ii), with the* **restriction isomorphisms** *of Corollary 3.5, (ii), one obtains a diagram of* **compatible morphisms**

$$\begin{array}{ccccccc}
\Pi_{\underline{v}\blacktriangleright}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) & = & \Pi_{\underline{v}\blacktriangleright}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) & \dashleftarrow & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})_{|t|}\}_{|t| \in \mathbb{F}^{\circledast}_l} & \xrightarrow{\sim} & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_*)_{|t|}\}_{|t| \in \mathbb{F}^{\circledast}_l} \\
\curvearrowright & & \curvearrowright & & \curvearrowright & & \curvearrowright \\
{}_\infty\Psi_{{}^\dagger\mathcal{F}^{\Theta}_{\underline{v}},\alpha} & \xrightarrow{\sim} & {}_\infty\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_*) & \xrightarrow{\sim} & {}_\infty\Psi_\xi(\mathbb{M}^{\Theta}_*) & \xrightarrow{\sim} & {}_\infty\Psi_{\mathcal{F}_\xi}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}) \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
\Psi_{{}^\dagger\mathcal{F}^{\Theta}_{\underline{v}},\alpha} & \xrightarrow{\sim} & \Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_*) & \xrightarrow{\sim} & \Psi_\xi(\mathbb{M}^{\Theta}_*) & \xrightarrow{\sim} & \Psi_{\mathcal{F}_\xi}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})
\end{array}$$

— where the "$\dashleftarrow$*" in the first line [cf. also the second and third "*$\curvearrowright$*" in the second line] is as in Corollary 3.5, (ii); we recall the natural inclusion* $\Pi_{\underline{v}\blacktriangleright}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \hookrightarrow \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*)$ *— which is* **well-defined** *up to composition with a(n) [single!]* **inner automorphism** *of* $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_*)$ *and* **compatible** *[in the evident sense] with the equalities of submonoids involving "*$\Psi_{2l\cdot\xi}(-)$*" [cf. Corollary 3.5, (ii)]. For simplicity, we shall use the notation*

$$\Psi_{{}^\dagger\mathcal{F}^{\Theta}_{\underline{v}}} \xrightarrow{\sim} \Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_*) \xrightarrow{\sim} \Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_*) \xrightarrow{\sim} \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}});$$

$${}_\infty\Psi_{{}^\dagger\mathcal{F}^{\Theta}_{\underline{v}}} \xrightarrow{\sim} {}_\infty\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_*) \xrightarrow{\sim} {}_\infty\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_*) \xrightarrow{\sim} {}_\infty\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

to denote these collections of compatible morphisms.

(iii) **(Constant Monoids and Splittings)** *Relative to the notational conventions adopted thus far [cf. also Corollary 3.5, (iii)], the diagonal submonoid* $(\Psi_{{}^\dagger\mathcal{C}_{\underline{v}}})_{\langle |\mathbb{F}_l| \rangle}$ *determines — i.e., may be thought of as the* **graph** *of — an* **isomorphism of monoids**

$$(\Psi_{{}^\dagger\mathcal{C}_{\underline{v}}})_0 \xrightarrow{\sim} (\Psi_{{}^\dagger\mathcal{C}_{\underline{v}}})_{\langle \mathbb{F}^{\circledast}_l \rangle}$$

that is **compatible** *with the respective labeled* $G_{\underline{v}}(\mathbb{M}^{\Theta}_*)$*-actions. Moreover, the splittings of Corollary 3.5, (iii), determine* **splittings** *up to torsion of each of the ["Frobenioid-theoretic"] Gaussian monoids*

$$\Psi_{\mathcal{F}_\xi}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}) = (\Psi^{\times}_{{}^\dagger\mathcal{C}_{\underline{v}}})_{\langle \mathbb{F}^{\circledast}_l \rangle} \cdot \mathrm{Im}(\xi)^{\mathbb{N}}, \quad {}_\infty\Psi_{\mathcal{F}_\xi}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}) = (\Psi^{\times}_{{}^\dagger\mathcal{C}_{\underline{v}}})_{\langle \mathbb{F}^{\circledast}_l \rangle} \cdot \mathrm{Im}(\xi)^{\mathbb{Q}_{\geq 0}}$$

— where "$\mathrm{Im}(\xi)$*" denotes the image of* ξ *via the isomorphisms discussed in (ii) — which are* **compatible**, *relative to the various* **isomorphisms** *of the third display*

of (ii), with the splittings up to torsion of Proposition 3.1, (i); Proposition 3.3, (i); Corollary 3.5, (iii).

Proof. The various assertions of Corollary 3.6 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 3.6.1. The *"Galois compatibility"* denoted by the "$\dashleftarrow$" in the third display of Corollaries 3.5, (ii); 3.6, (ii) — involving the monoids "${}_\infty\Psi$" [i.e., not just the monoids "Ψ"!] — corresponds precisely to the *"Galois functoriality"* [cf. Fig. 1.5] of the discussion of Remark 1.12.4.

Remark 3.6.2. The **diagram** in the third display of Corollary 3.6, (ii) — which may be thought of as a sort of concrete realization of the principle of **Galois evaluation** discussed in Remark 1.12.4 [cf. also Remark 3.6.1] — will play a *central role* in the theory of the present series of papers. Thus, it is of interest to pause here to discuss various aspects of the significance of this diagram.

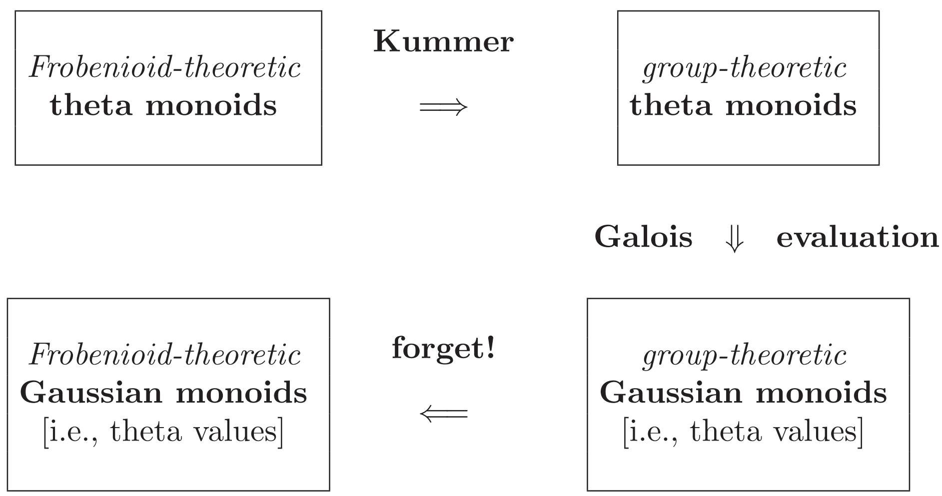

Fig. 3.1: Kummer theory and Galois evaluation

(i) The left-hand, central, and right-hand portions of this diagram are summarized, at a more conceptual level, in Fig. 3.1 above — that is to say, if one thinks of the mono-theta environments "$\mathbb{M}_*^\Theta$" involved as arising *group-theoretically* [i.e., from *étale-like* objects, which is, of course, always the case *up to isomorphism*! — cf. the situation discussed in Corollary 3.7, (i), below], then these portions correspond, respectively, to the arrows "$\Longrightarrow$", "$\Downarrow$", and "$\Longleftarrow$" in Fig. 3.1. Here, we note that the final operation of *"forgetting"* [i.e., "$\Longleftarrow$"] may be thought of as the operation of *forgetting the group-theoretic — i.e., "anabelian" — construction* of the Gaussian monoids, so as to obtain "abstract monoids stripped of any information concerning the group-theoretic algorithms used to construct them" — which we refer to as **"post-anabelian"** [cf. the discussion of Remark 1.11.3, (iii); Corollary 3.7, (i), below; the constructions of Definition 3.8 below]. On the other hand, the composite of the arrows "$\Longrightarrow$" and "$\Downarrow$" may be thought of as a sort of

> **comparison isomorphism** *between* **"Frobenius-like"** *[i.e., "Frobenioid-theoretic"] and* **"étale-like"** *[i.e., "group-theoretic"] structures*

— cf. the discussion of [FrdI], Introduction; [IUTchI], Corollaries 3.8, 3.9. In this context, it is useful to recall that the *comparison isomorphism* of the "classical" scheme-theoretic version of Hodge-Arakelov theory [cf. [HASurI], Theorem A] is obtained precisely by **evaluating theta functions** *and their derivatives at certain* **torsion points** *of an elliptic curve.*

(ii) The existence of both "Frobenius-like" and "étale-like" structures in the theory of the present series of papers, together with the somewhat complicated theory of *comparison isomorphisms* as discussed above in (i), prompts the following question:

> What are the various **merits** and **demerits** of *"Frobenius-like"* and *"étale-like"* structures that require one to avail oneself of *both types of structure* in the theory of the present series of papers [cf. Fig. 3.2 below]?

On the one hand, unlike Frobenius-like structures, étale-like structures — in the form of *étale or tempered fundamental groups* [such as Galois groups] — have the *crucial advantage* of being *functorial* or *invariant* with respect to various **non-ring/scheme-theoretic filters** between distinct ring/scheme theories. In the context of the present series of papers, the main examples of this phenomenon consist of the **Θ-link** [cf., e.g., [IUTchI], Corollary 3.7] and the **$\mathfrak{log}$-wall** [cf. [AbsTopIII], §I1, §I4; this theory will be incorporated into the present series of papers in [IUTchIII]]. Another important characteristic of the étale-like structures constituted by étale or tempered fundamental group is their *"remarkable rigidity"* — a property that is exhibited explicitly [cf., e.g., the theory of [EtTh]; [AbsTopIII]] by various **anabelian algorithms** that may be applied to construct, in a "purely group-theoretic fashion", various structures motivated by **conventional scheme theory**. By contrast, the Frobenius-like structures constituted by various *abstract monoids* — which typically give rise to various *Frobenioids* — satisfy the *crucial property* of **not** being subject to such rigidifying anabelian algorithms that relate various étale-like structures to conventional scheme theory. It is precisely this property of such abstract monoids that allows one to *use these abstract monoids to* **construct such non-scheme-theoretic filters** as the Θ-*link* [cf. [IUTchI], Corollary 3.7] or the $\mathfrak{log}$-*wall* of the theory of [AbsTopIII]. Here, it is interesting to observe that

> these merits/demerits of étale-like and Frobenius-like structures play somewhat **complementary roles** with respect to **binding/not binding** the structures under consideration to **conventional scheme theory**.

Finally, we note that **Kummer theory** serves the crucial role [cf. the discussion of (i)] of relating [via various *comparison isomorphisms* — cf. (i)] — within a given **Hodge theater** — potentially non-scheme-theoretic Frobenius-like structures to étale-like structures which are subject to anabelian rigidifications that bind them to conventional scheme theory.

(iii) If one composes the correspondence "$\underline{\underline{q}}_{\underline{v}} \mapsto \underline{\underline{\Theta}}_{\underline{v}}$" [cf. the discussion of [IUTchI], Remark 3.8.1, (i)] constituted by the Θ-*link* — i.e., which relates the *"$(n+1)$-th generation q-parameter"* to the *"n-th generation Θ-function"* — with the composite of the arrows "$\Longrightarrow$", "$\Downarrow$", and "$\Longleftarrow$" of Fig. 3.1, then one obtains a correspondence

$$\underline{\underline{q}}_{\underline{v}} \quad \mapsto \quad \left\{ \underline{\underline{q}}_{\underline{v}}^{j^2} \right\}_{1 \leq j \leq l^*}$$

[cf. Remark 2.5.1, (i)]. In fact, in the theory of the present series of papers, it is ultimately this "modified version of the Θ-link" — i.e., which takes into account the *Hodge-Arakelov-theoretic evaluation theory* developed so far in §2 and the present §3 — that will be of interest to us. The theory of this "modified version of the Θ-link" will constitute one of the main topics treated in §4 below. Here, we observe that the above correspondence may be thought of as a sort of **"abstract, combinatorial Frobenius lifting"** — i.e., as a sort of **"homotopy"** between

· the **identity** $\underline{\underline{q}}_{\underline{v}} \mapsto \underline{\underline{q}}_{\underline{v}}$ [i.e., which corresponds to *"characteristic zero"*]

and

· the **purely monoid-theoretic/highly non-scheme-theoretic** correspondence $\underline{\underline{q}}_{\underline{v}} \mapsto \underline{\underline{q}}_{\underline{v}}^{(l^*)^2}$ [i.e., which corresponds to the *"positive characteristic Frobenius morphism"*].

Moreover, we recall [cf. the discussion of Remark 2.6.3] that the collection of exponents $\{j^2\}_{1 \leq j \leq l^*}$ that appear in this "abstract, combinatorial Frobenius lifting" is **highly distinguished** — hence, in particular, *far from arbitrary*!

étale-like structures	*Frobenius-like structures*
functoriality/invariance with respect to **$\mathfrak{log}$-wall, Θ-link**	—
rigidified relationship via **Kummer theory** **+ anabelian geom.** to **conventional arith. geom.**	—
—	lack of rigidification allows construction of **non-scheme-theoretic filters**, such as **$\mathfrak{log}$-wall, Θ-link**

Fig. 3.2: Étale-like versus Frobenius-like structures

(iv) In the context of the discussion of (i), it is of interest to recall that various *"Grothendieck Conjecture-type results"* in anabelian geometry [e.g., over p-adic local fields and finite fields] — i.e., which may be thought of as *comparison isomorphisms* between *polynomial-function-theoretic* and *group-theoretic* collections of morphisms — are obtained precisely by combining various considerations particular to the situation of interest with the **"Galois evaluation"** *via* **Kummer theory** *of polynomial functions or differential forms at various rational points* — cf. the theory of [pGC]; [Cusp], §2.

Remark 3.6.3. Before proceeding, we make some observations concerning **basepoints** in the context of the **"non-ring/scheme-theoretic filters"** discussed in Remark 3.6.2.

(i) First, let us recall from the elementary theory of *étale fundamental groups* that the *fiber functor* associated to a **basepoint** is defined by considering the points of a finite étale covering valued in some separably closed field that lie over a fixed point [valued in the same separably closed field] of the base scheme over which the covering is given. Thus, for instance, when this base scheme is the spectrum of a field, the finite set of points associated by the fiber functor to a finite étale covering is obtained by considering the various *ring homomorphisms* from this field into some separably closed field. In particular, it follows that

> the conventional scheme-theoretic definition of a **basepoint** [in the form of a *fiber functor*] depends, in an *essential* fashion, on the **ring/scheme structure** of the rings or schemes under consideration.

One immediate consequence of these elementary considerations — which is of *central importance* in the theory of the present series of papers — is the following *observation* concerning the *"non-ring/scheme-theoretic filters"* discussed in Remark 3.6.2, which relate *one ring to another* in a fashion that is **incompatible** *with the respective ring structures*:

> The **distinct ring structures** on either side of one of the *"non-ring/scheme-theoretic filters"* discussed in Remark 3.6.2 — i.e., the **$\mathfrak{log}$-wall** of [AbsTopIII] and the **Θ-link** of [IUTchI], Corollary 3.7 — give rise to **distinct, unrelated basepoints** [cf. the discussion of [AbsTopIII], Remark 3.7.7, (i)].

In some sense, the above discussion may be thought of as an "expanded, leisurely version" of an observation made at the beginning of the discussion of [AbsTopIII], Remark 3.7.7, (i)].

(ii) The observations of (i) also apply to the *"N-th power morphisms"* [where $N > 1$] — i.e., *"morphisms of Frobenius type"* — that appear in the theory of Frobenioids [cf. [FrdI], [FrdII], [EtTh]]. That is to say, in the context of the tempered Frobenioids that appear in the theory of [EtTh], §5, such "morphisms of Frobenius type" [i.e., "N-th power morphisms" regarded as morphisms contained in the underlying categories associated to these tempered Frobenioids] induce "N-th power morphisms" between various monoids [arising from the Frobenioid structure] isomorphic to $\mathcal{O}^{\triangleright}_{K_{\underline{v}}}$. In particular,

> these N-th power morphisms of monoids **fail** [since $N > 1$] *to preserve the ring structure of $K_{\underline{v}}$*, hence give rise to **distinct, unrelated basepoints** on the domain and codomain objects of the original "morphism of Frobenius type" [cf. the discussion of (i)].

On the other hand, let us *observe* that unlike the situations considered in the discussion of (i), the considerations of the present discussion involving N-th power morphisms take place in a fashion that is **compatible** *with the projection functor to the base category of the Frobenioid.* One important consequence of this last observation is that unlike the situations discussed in (i) involving the $\mathfrak{log}$-wall and the Θ-link in which one must consider **arbitrary isomorphisms of topological groups** between the étale [or tempered] fundamental groups that arise in the domain and the codomain of the operation under consideration,

> in the situation of the present discussion of N-th power morphisms, the "distinct, unrelated basepoints" that arise only give rise to **inner automorphisms** of the topological group determined by [i.e., roughly speaking, the "fundamental group" of] the base category.

This phenomenon may be thought of as a reflection of the fact that the application of an N-th power morphism is somewhat *"milder"* than the $\mathfrak{log}$-wall or Θ-link considered in (i) in that it only involves an operation — i.e., raising to the N-th power — that is **"algebraic"**, in the sense that it is defined with respect to the ring structure of the ring [e.g., $K_{\underline{v}}$] involved. This somewhat "milder nature" of an N-th power morphism allows one to consider N-th power morphisms within a **single category** [namely, the tempered Frobenioid under consideration] which can be defined in terms of [formal] flat $\mathcal{O}_{K_{\underline{v}}}$-schemes [cf. the point of view of [EtTh], §1]. By contrast, the operation inherent in the $\mathfrak{log}$-wall or Θ-link considered in (i) is much more **drastic** and **arithmetic** [i.e., "non-algebraic"] in nature, and it is *difficult to see how to fit such an operation into a single category* that somehow "extends" the tempered Frobenioid under consideration in a fashion that "lies over" the same base category as the tempered Frobenioid — cf., e.g., Remark 1.11.2, (ii), in the case of the Θ-link; the discussion of [AbsTopIII], Remark 3.7.7, in the case of the $\mathfrak{log}$-wall. Put another way,

> the *highly nontrivial* study of the mathematical structures *"***generated** *by the* $\mathfrak{log}$*-wall and* Θ*-link"* is, in some sense, one of the **main themes** of the theory of the present series of papers

— cf., especially, the theory of [IUTchIII]!

Remark 3.6.4. Since the theory of **mono-theta environments** developed in [EtTh] plays a *fundamental role* in the theory of the present paper — cf., e.g., Corollaries 1.12, 2.8, 3.5, 3.6 — it is of interest to pause to review the relationship of the theory of [EtTh] to the theory developed so far in the present paper.

(i) The various remarks following [EtTh], Corollary 5.12, discuss the significance of the various *rigidity properties* of a mono-theta environment that are verified in [EtTh]. The logical starting point of this discussion is the situation considered in [EtTh], Remarks 5.12.1, 5.12.2, consisting of an abstract *category* which is only known up to isomorphism [i.e., up to an indeterminate equivalence of categories], and in which each of the *objects* is only known up to *isomorphism*. The main example of such a category, in the context of the theory of [EtTh], is a *tempered Frobenioid* of the sort considered in Propositions 3.3, 3.4; Corollary 3.6. The situation of [EtTh], Remarks 5.12.1, 5.12.2, in which each of the objects in the category is only known up to isomorphism, *contrasts sharply* with the notion of *a system, or tower, of [specific!] coverings* — e.g., of the sort that appears in *Kummer theory*, in which the coverings are related by *[specific!] N-th power morphisms.* Indeed, the various rigidity properties verified in [EtTh] are of interest precisely because

> they yield effective **reconstruction algorithms** for reconstructing the various structures of interest in a fashion that is **invariant** with respect to the *indeterminacies* that arise from a situation in which *each of the objects in the category is only known up to isomorphism.*

This prompts the following question:

> What is the **fundamental reason**, in the context of the theory of the present series of papers, that one must work under the assumption that each of the objects in the category is **only known up to isomorphism**, thus requiring one to avail oneself of the **rigidity theory** of [EtTh]?

To understand the answer to this question, let us first observe that *Kummer towers involving [specific!] N-th power morphisms* are constructed by using the **multiplicative** structure of the "rational functions" [such as the $p_{\underline{v}}$-adic local field $K_{\underline{v}}$] under consideration. That is to say, the N-th power morphisms are compatible with the multiplicative structure, but **not** the **additive** structure of such rational functions. On the other hand, ultimately,

> when, in [IUTchIII], we consider the theory of the $\mathfrak{log}$**-wall** [cf. [AbsTopIII]], it will be of crucial importance to consider, within each Hodge theater, the **ring** structure [i.e., *both* the multiplicative and additive structures] of the fields $K_{\underline{v}}$.

That is to say, without the ring structure on $K_{\underline{v}}$, one cannot even define the $p_{\underline{v}}$-adic logarithm! Put another way, the N-th power morphisms that appear in a Kummer tower may be thought of as *"Frobenius morphisms of a sort"* that relate *distinct ring structures* — i.e., since the N-th power morphism *fails to be compatible with addition*! In particular, the distinct ring structures that exist in the domain and codomain of such a "Frobenius morphism" necessarily give rise to **distinct, unrelated basepoints** [cf. the discussion of Remark 3.6.3, (ii)] — i.e., at an abstract category-theoretic level, to objects which are only known *up to isomorphism*! This is what requires one to contend with the indeterminacies discussed in [EtTh], Remarks 5.12.1, 5.12.2.

(ii) The theory of [EtTh] may be summarized as asserting that one may reconstruct various structures of interest from a *mono-theta environment* without sacrificing certain fundamental *rigidity properties*, even in a situation subject to certain *indeterminacies* [cf. (i)]. Moreover, mono-theta environments serve as a sort of **bridge** [cf. [EtTh], Remark 5.10.1] between *tempered Frobenioids* — i.e., *"Frobenius-like structures"* [cf. Remark 3.6.2] — as in Propositions 3.3, 3.4; Corollary 3.6, on the one hand, and *tempered fundamental groups* [cf. Proposition 3.4] — i.e., *"étale-like structures* [cf. Remark 3.6.2] — on the other.

(iii) One central feature of the theory of [EtTh] is an explanation of the special role played by the **first power** of the [reciprocal of the l-th root of the] *theta function*, a role which is reflected in the theory of **cyclotomic rigidity** developed in [EtTh] [cf. [EtTh], Introduction]. Note that the operation of **Galois evaluation** is necessarily **linear** [cf. the discussion of Remark 1.12.4]. This linearity may be seen in the linearity of the arrows "$\Longrightarrow$", "$\Downarrow$", and "$\Longleftarrow$" of Fig. 3.1. In particular, these arrows are *compatible with the* **ring** *structure on the constants* [i.e., "$K_{\underline{v}}$"] — a property that will be of crucial importance when, in [IUTchIII], we consider the theory of the **$\mathfrak{log}$-wall** [cf. the discussion of (i) above]. Moreover, this linearity property of the operation of Galois evaluation implies that

> the **first power of the theta values** *of the [reciprocal of the l-th root of the] theta function* "inherits", so to speak, the *special role* played by the first power of the [reciprocal of the l-th root of the] theta function.

This observation is interesting in light of the discussions of Remarks 2.6.3; 3.6.2, (iii).

(iv) In the context of (iii), we note that the various *theta monoids* discussed in Propositions 3.1, 3.3, as well as the various *Gaussian monoids* discussed in Corollaries 3.5, 3.6, involve **arbitrary powers/roots** of the [reciprocal of the l-th root of the] theta function. Nevertheless, it is important to remember that

> in order to apply the **Θ-link** — which requires one to work with **"Frobenius-like structures"** [cf. the discussion of Remark 3.6.2, (ii)] — it is necessary to consider the operation of *Galois evaluation* summarized in Fig. 3.1 applied to the **first power** of the [reciprocal of the l-th root of the] Frobenioid-theoretic theta function in order to avail oneself of the **cyclotomic rigidity** furnished by the *delicate* **bridge** constituted by the *mono-theta environment*

— cf. (ii) above. That is to say, the **"narrow bridge"** afforded by the mono-theta environment between the worlds of "Frobenius-like" and "étale-like" structures may only be crossed by the **first power** of the [reciprocal of the l-th root of the] theta function and its theta values. Put another way,

> from the point of view of the **étale-like portion** [i.e., "group-theoretic" portion] of the operation of *Galois evaluation* summarized in Fig. 3.1, the **N-th power** of the [reciprocal of the l-th root of the] Frobenioid-theoretic theta function, for $N > 1$, is **only defined as the N-th power** "$(-)^N$" of the first power of the [reciprocal of the l-th root of the] Frobenioid-theoretic theta function.

That is to say, from the point of view of the étale-like portion of the operation of Galois evaluation summarized in Fig. 3.1, the **N-th power** of the [reciprocal of the l-th root of the] Frobenioid-theoretic theta function, for $N > 1$ — hence, in particular, the Θ-*link* — **may only be calculated by forming the N-th power** "$(-)^N$" of the *first power* of the [reciprocal of the l-th root of the] Frobenioid-theoretic theta function.

(v) The necessity of working with "Frobenius-like structures" [cf. the discussion of (iv)] may also be thought of as the necessity of working with the various **post-anabelian** monoids arising from the group-theoretic "anabelian" algorithms that appear in the operation of Galois evaluation [cf. the discussion of Remark 3.6.2, (i)]. In the context of this observation, it is useful to recall that from the point of view of the theory of §1,

> the **"narrow bridge"** furnished by [for instance, the *cyclotomic rigidity* of] a mono-theta environment satisfies the crucial property of **multiradiality** [cf. Corollaries 1.10, 1.12] — i.e., of being **"horizontal"** with respect to the **"connection structure"** determined by the formulation of this multiradiality [cf. the point of view discussed in Remarks 1.7.1, 1.9.2].

Put another way, to work with powers *other than the first power* of the [reciprocal of the l-th root of the] theta function or its theta values gives rise to structures which are **"not horizontal"** with respect to this "connection structure". This point of view is consistent with the point of view of Remark 3.6.5, (iii), below. A similar observation concerning *multiradiality* will also apply to the "multiradial versions of the Gaussian monoids" that will be constructed in [IUTchIII] [cf. Remark 3.7.1 below].

Remark 3.6.5. In light of the central role played by **mono-theta-theoretic cyclotomic rigidity** in the discussion of Remark 3.6.4, we pause to make some observations — of a somewhat more philosophical nature — concerning this topic.

(i) First of all, we observe that

> a **cyclotome** may be thought of as a sort of **"skeleton of the arithmetic holomorphic structure"** under consideration

— cf. the discussion of Remark 1.11.6. Indeed, this point of view may be thought of as being motivated by the situation at *archimedean primes*, where the circle "$\mathbb{S}^1$" may be thought of as a sort of "representative skeleton of $\mathbb{C}^\times$". This point of view will play a central role in the remainder of the discussion of the present Remark 3.6.5, as well as in the discussion of Remark 3.8.3 below.

(ii) In the theory of [EtTh],

> (a) the **commutator structure** $[-,-]$ *of the theta group* plays a central role in the theory of **mono-theta-theoretic cyclotomic rigidity**

— cf. [EtTh], Introduction; [EtTh], Remark 2.19.2. On the other hand, in the classical theory of algebraic theta functions

> (b) the **commutator structure** $[-,-]$ *of the theta group* plays a central role in the theory via the observation that this commutator structure implies the **irreducibility** of certain representations of the theta group.

At first glance, these two applications (a), (b) of the commutator structure $[-,-]$ of the theta group may appear to be unrelated. In fact, however, they may both be understood as examples of the following phenomenon:

> (c) the **commutator structure** $[-,-]$ *of the theta group* may be thought of as a sort of concrete embodiment of the **"coherence of holomorphic structures"**.

Indeed, as discussed in [EtTh], Introduction, from the point of view of the scheme-theoretic Hodge-Arakelov theory of [HASurI], [HASurII], the irreducible representations that appear in the classical theory of algebraic theta functions as submodules of the module of all set-theoretic functions on the l-torsion points of an elliptic curve [cf. (b)] may be thought of, for instance, when l is *large*, as discrete analogues of the *submodule of* **"holomorphic functions"** *within the module of all real analytic functions.* On the other hand, if one thinks of cyclotomes as *"skeleta of arithmetic holomorphic structures"* [cf. (i)], then the theory of **conjugate synchronization** [cf. Remark 3.5.2, as well as Remark 3.8.3 below] — applied, for instance, in the case of *cyclotomes* — may be thought of as a sort of "discretely parametrized" [in the sense that it is indexed by torsion points] **coherence of arithmetic holomorphic structures**, which is obtained by working with the *connected* subgraph $\Gamma^{\blacktriangleright}_{\underline{\underline{X}}} \subseteq \Gamma_{\underline{\underline{X}}}$ [cf. Remark 2.6.1, (i)]. In this context, mono-theta-theoretic cyclotomic rigidity [cf. (a)] may be thought of as a sort of "continuously parametrized version" [i.e., supported on $\underline{\underline{\ddot{Y}}}_{\underline{v}}$, as opposed to a finite set of torsion points] of this *coherence of arithmetic holomorphic structures.* Finally, we recall that the interaction — i.e., via *restriction* operations — between these "discrete" and "continuous" versions of the "coherence of arithmetic holomorphic structures" plays a central role in the theory of *Galois evaluation* given in Corollaries 2.8, (i); 3.5, (ii); 3.6, (ii).

(iii) If one thinks of cyclotomes at *localizations* [say, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] of a number field [i.e., K] as **local skeleta** *of the arithmetic holomorphic structure* [cf. (i)], then

> the **mono-theta-theoretic cyclotomic rigidity** may be thought of as a sort of **"local uniformization"** of a number field [cf. the *exterior cyclotome* of a mono-theta environment that arises from a tempered Frobenioid, as in Proposition 1.3, (i)] via a local portion [cf. the *interior cyclotome* in the situation of Proposition 1.3, (i)] of the geometric tempered fundamental group $\Delta_{\underline{v}}$ associated to a certain covering of the once-punctured elliptic curve X_F [cf. Definition 2.3, (i); [IUTchI], Definition 3.1, (e)].

Since the cyclotomic rigidity isomorphism arising from mono-theta-theoretic cyclotomic rigidity may be thought of as the *"cyclotomic portion"* of the *theta function*, mono-theta-theoretic cyclotomic rigidity may be interpreted as the statement that the theta function constructed from a mono-theta environment is free of any $\widehat{\mathbb{Z}}^{\times}$-*power indeterminacies.* Moreover, if one takes this point of view, then

> **constant multiple rigidity** may be thought of as the statement that the above *"local uniformization"* is sufficiently rigid as to be free of any *constant multiple indeterminacies.*

Here, it is useful to recall that the once-punctured elliptic curve X_F on the number field F that occurs in the theory of the present series of papers may be thought of as being analogous to the **nilpotent ordinary indigenous bundles** *on a hyperbolic curve in positive characteristic* in p-adic Teichmüller theory [cf. the discussion of [AbsTopIII], §I5]. That it to say, from this point of view, the "local uniformizations" of the above discussion may be thought of as corresponding to the **local uniformizations via canonical coordinates** of p-adic Teichmüller theory [cf., e.g., [pTeich], §0.9], which are also "sufficiently rigid" as to be free of any $\widehat{\mathbb{Z}}^{\times}$-*power* or *constant multiple indeterminacies.* Here, mono-theta-theoretic cyclotomic rigidity may be thought of as corresponding to the **Kodaira-Spencer isomorphism** [associated to the Hodge section of the canonical indigenous bundle], which, in some sense, may be thought of as the *"skeleton"* of the local uniformizations of p-adic Teichmüller theory. Also, it is useful to recall in this context that the canonical coordinates of p-adic Teichmüller theory are constructed by considering *invariants* with respect to certain *canonical Frobenius liftings*. Put another way, the technique of considering *Frobenius-invariants* allows one to pass, in a *canonical* way, from objects defined *modulo* p to objects defined *modulo higher powers of* p. Since the various Θ-*links* of the Frobenius-picture may be regarded as corresponding to the various *transitions from "mod* p^n *to mod* p^{n+1}*"* [where $n \in \mathbb{N}$] in the theory of Witt vectors [cf. the discussion of [IUTchI], §I4; [IUTchIII], Remark 1.4.1, (iii)], it is natural to regard, in the context of the *canonical splittings* furnished by the *étale-picture* [cf. the discussion of [IUTchI], §I1],

> the **multiradiality** of the formulation of *mono-theta-theoretic cyclotomic rigidity* and *constant multiple rigidity* given in Corollary 1.12 as corresponding to the **Frobenius-invariant** nature of the *canonical coordinates* of p-adic Teichmüller theory.

Finally, in this context, we observe that it is perhaps natural to think of the **discrete rigidity** of the theory of [EtTh] as corresponding to the fact that the *canonical coordinates* of p-adic Teichmüller theory, which *a priori* may only be constructed as **PD-formal power series**, may in fact be constructed as **power series** *in the usual sense*, i.e., elements of the completion $\widehat{\mathcal{O}}$ of the local ring at the point under consideration. Indeed, the discrete rigidity of [EtTh] implies that one may restrict oneself to working with the *usual* theta function, canonical multiplicative coordinates [i.e., "U"], and q-parameters on appropriate tempered coverings of the Tate curve, all of which, like the power series arising from canonical parameters in p-adic Teichmüller theory, give rise to *"functions on suitable formal schemes"* in the sense of classical scheme theory. By contrast, if this discrete rigidity were to *fail*, then one would be obliged to work in an *"a priori profinite"* framework that involves, for instance, $\widehat{\mathbb{Z}}$*-powers of "*U*" and "*q*"* [cf. [EtTh], Remarks 1.6.4, 2.19.4]. Such $\widehat{\mathbb{Z}}$-powers appear naturally in the $\widehat{\mathbb{Z}}$*-modules* that arise [e.g., as cohomology modules] in the Kummer theory of the theta function and may be thought of as corresponding to PD-formal power series in the sense that *arbitrary* $\widehat{\mathcal{O}}$*-powers of canonical parameters* [say, for simplicity, at *non-cuspidal* ordinary points of a canonical curve], which arise naturally when one considers such parameters *additively* [cf. the discussion of *"canonical affine coordinates"* in [pOrd], Chapter III], cannot be defined if one restricts oneself to working with conventional power series — i.e., such $\widehat{\mathcal{O}}$-powers may only be defined if one allows oneself to work with *PD-formal power series*.

Corollary 3.7. (Group-theoretic Gaussian Monoids and Uniradiality) *Suppose that we are in the situation of Proposition 3.4, i.e., in the following, we consider the* **full poly-isomorphism**

$$\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \quad \stackrel{\sim}{\rightarrow} \quad \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

— where $\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})$ *is the projective system of mono-theta environments arising from the algorithm of Proposition 1.2, (i) [cf. also Proposition 1.5, (i)];* ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ *is a* **tempered Frobenioid** *as in Proposition 3.3 — of* **projective systems of mono-theta environments**. *When "*$\mathbb{M}^{\Theta}_{*}$*" is taken to be* $\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$*, we shall denote the*

resulting "$\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}$*" by* $\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$ *[cf. Definition 2.7, (ii)]. When "*$\mathbb{M}^{\Theta}_{*}$*" is taken to be* $\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})$*, we shall identify* $\Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$ *and* $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}})$ *[cf. Definition 2.7, (ii)] with* $\Pi_{\underline{v}\ddot{\blacktriangleright}}$ *and* $G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$ *[cf. Corollary 2.5, (i)], respectively, via the tautological isomorphisms* $\Pi_{\underline{v}\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \xrightarrow{\sim} \Pi_{\underline{v}\ddot{\blacktriangleright}}$*,* $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \xrightarrow{\sim} G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$*. Finally, we shall follow the notational conventions of Corollaries 3.5, 3.6 with regard to the* **subscripts** *"*$|t|$*", for* $|t| \in |\mathbb{F}_l|$*, and "*$\langle \mathbb{F}^{\circledast}_l \rangle$*".*

(i) **(From Group-theoretic to Post-anabelian Gaussian Monoids)** *Each isomorphism of projective systems of mono-theta environments* $\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}) \xrightarrow{\sim} \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$ *induces* **compatible** *[in the evident sense]* **collections of isomorphisms**

$$\begin{array}{ccc}
\Pi_{\underline{v}\ddot{\blacktriangleright}} & \dashleftarrow & \{G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})_{|t|}\}_{|t| \in \mathbb{F}^{\circledast}_l} \\
\curvearrowright & & \curvearrowright \\
{}_{\infty}\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & {}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) \\
\bigcup & & \bigcup \\
\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) & \xrightarrow{\sim} & \Psi_{\xi}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\sim} & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))_{|t|}\}_{|t| \in \mathbb{F}^{\circledast}_l} & \xrightarrow{\sim} & \{G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))_{|t|}\}_{|t| \in \mathbb{F}^{\circledast}_l} \\
 & \curvearrowright & & \curvearrowright \\
\xrightarrow{\sim} & {}_{\infty}\Psi_{\xi}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) & \xrightarrow{\sim} & {}_{\infty}\Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \\
 & \bigcup & & \bigcup \\
\xrightarrow{\sim} & \Psi_{\xi}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) & \xrightarrow{\sim} & \Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})
\end{array}$$

and

$$\begin{array}{ccc}
G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}}) & \xrightarrow{\sim} & G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})_{\langle \mathbb{F}^{\circledast}_l \rangle} \\
\curvearrowright & & \curvearrowright \\
\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))^{\times} & \xrightarrow{\sim} & \Psi_{\xi}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))^{\times}
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\sim} & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))_{\langle \mathbb{F}^{\circledast}_l \rangle} & \xrightarrow{\sim} & G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))_{\langle \mathbb{F}^{\circledast}_l \rangle} \\
 & \curvearrowright & & \curvearrowright \\
\xrightarrow{\sim} & \Psi_{\xi}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))^{\times} & \xrightarrow{\sim} & \Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\times}
\end{array}$$

— where the **upper left-hand portion** *of the first display [involving "*$\dashleftarrow$*"] is obtained by applying the third display [involving "*$\dashleftarrow$*"] of Corollary 3.5, (ii), in the case where "*$\mathbb{M}^{\Theta}_{*}$*" is taken to be* $\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})$*; the isomorphisms that relate the upper left-hand portion of the first display to the lower right-hand portion of the first display arise from the* **functoriality** *of the algorithms involved, relative to isomorphisms of projective systems of mono-theta environments; the* **lower right-hand portion** *of the first display is obtained by applying the right-hand portion*

of the third display of Corollary 3.6, (ii), in the case where "$\mathbb{M}^{\Theta}_{*}$" is taken to be $\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$; the **second display** *is obtained from the first display by considering the* **units** *[denoted by means of a superscript "$\times$"].*

(ii) **(Uniradiality of Gaussian Monoids)** *If we write $\Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\times\boldsymbol{\mu}}$ for the ind-topological monoid obtained by forming the quotient of $\Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\times}$ by its torsion subgroup, then the* **functorial algorithms**

$$\Pi_{\underline{v}} \mapsto \Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})); \quad \Pi_{\underline{v}} \mapsto {}_{\infty}\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$$

— where we think of $\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$, ${}_{\infty}\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$ as being equipped with their natural **splittings** *up to torsion [cf. Corollary 3.5, (iii)] and, in the case of $\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$, the natural* $G_{\underline{v}}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$**-action** *[cf. Corollary 3.5, (ii)] — obtained by composing the algorithms of Proposition 1.2, (i); Corollary 3.5, (ii), (iii), depend on the* **cyclotomic rigidity isomorphism** *of Corollary 1.11, (b) [cf. Remark 1.11.5, (ii); the use of the surjection of Remark 1.11.5, (i), in the algorithms of Proposition 3.1, (ii), and Corollary 3.5, (ii)], hence* **fail to be compatible***, relative to the displayed diagrams of (i), with* **automorphisms** *of [the underlying pair, consisting of an ind-topological monoid equipped with the action of a topological group, determined by] the pair*

$$G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))_{\langle \mathbb{F}^{*}_{l} \rangle} \quad \curvearrowright \quad \Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\times\boldsymbol{\mu}}$$

which arise from automorphisms of [the underlying pair, consisting of an ind-topological monoid equipped with the action of a topological group, determined by] the pair $G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))_{\langle \mathbb{F}^{*}_{l} \rangle} \curvearrowright \Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})^{\times}$ [cf. Remarks 1.11.1, (i), (b); 1.8.1] — in the sense that this algorithm, as given, only admits a **uniradial** *formulation [cf. Remarks 1.11.3, (iv); 1.11.5, (ii)].*

Proof. The various assertions of Corollary 3.7 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 3.7.1. One *central* consequence of the theory to be developed in [IUTchIII] [cf. Remarks 2.9.1, (iii); 3.4.1, (ii)] is the result that,

> by applying the theory of *log-shells* [cf. [AbsTopIII]], one may modify the algorithms of Corollary 3.7, (ii), in such a way as to obtain algorithms for computing the **Gaussian monoids** that [yield functors which] are *manifestly* **multiradially defined**

— albeit at the cost of allowing for certain [relatively mild!] **indeterminacies**.

The following definition in some sense *summarizes* the theory of the present §3.

Definition 3.8. Many of the *"monoids equipped with a Galois action"* that appear in the discussion of the present §3 may be thought of as giving rise to *Frobenioids*, as follows.

(i) Each of the monoids equipped with a $\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*})$-action

$$\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \quad \curvearrowright \quad \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}); \qquad \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \quad \curvearrowright \quad \Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}}$$

of Propositions 3.1, (ii); 3.3, (ii), gives rise to a $p_{\underline{v}}$*-adic Frobenioid* of *monoid type* $\mathbb{Z}$ [cf. [FrdII], Example 1.1, (ii)]

$$\mathcal{F}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}); \quad \mathcal{F}_{{}^{\dagger}\mathcal{C}_{\underline{v}}}$$

whose *divisor monoid* associates to every object of $\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}))^0$ a monoid isomorphic to $\mathbb{Q}_{\geq 0}$. It follows immediately from the construction of the data "$\Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}) \curvearrowright \Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}}$" [cf. Example 3.2, (ii)] that one has a *tautological isomorphism of Frobenioids*

$${}^{\dagger}\mathcal{C}_{\underline{v}} \quad \xrightarrow{\sim} \quad \mathcal{F}_{{}^{\dagger}\mathcal{C}_{\underline{v}}}$$

[cf. the discussion of [IUTchI], Example 3.2, (iii), (iv)], which we shall use to *identify* these two Frobenioids. Thus, the isomorphism of monoids of Proposition 3.3, (ii), may be interpreted as an *isomorphism of Frobenioids*

$${}^{\dagger}\mathcal{C}_{\underline{v}} \quad \xrightarrow{\sim} \quad \mathcal{F}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})$$

— which also admits [indeed, induces] a *"mono-analytic version"* ${}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{F}^{\vdash}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})$ [cf. the category "$\mathcal{C}^{\vdash}_{\underline{v}}$" of [IUTchI], Example 3.2, (iv)]. This mono-analytic version admits a *"labeled version"* [cf. Remark 3.8.1 below]

$$({}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}})_{|t|} \quad \xrightarrow{\sim} \quad (\mathcal{F}^{\vdash}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}))_{|t|}$$

— cf. Corollary 3.6, (i). Finally, one has *Frobenioid-theoretic interpretations*

$$(\mathcal{F}^{\vdash}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}))_{\langle|\mathbb{F}_l|\rangle}; \quad (\mathcal{F}^{\vdash}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}))_0 \quad \xrightarrow{\sim} \quad (\mathcal{F}^{\vdash}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}))_{\langle\mathbb{F}^{*}_l\rangle}$$
$$({}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}})_{\langle|\mathbb{F}_l|\rangle}; \quad ({}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}})_0 \quad \xrightarrow{\sim} \quad ({}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}})_{\langle\mathbb{F}^{*}_l\rangle}$$

of the constructions of Corollary 3.5, (iii); 3.6, (iii).

(ii) Each of the monoids equipped with a topological group action

$$G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\blacktriangleright\!\!\ddot{}}) \curvearrowright \Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}); \quad G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\blacktriangleright\!\!\ddot{}}) \curvearrowright \Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}},\alpha}$$
$$G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\blacktriangleright\!\!\ddot{}})_{\langle\mathbb{F}^{*}_l\rangle} \curvearrowright \Psi_{\xi}(\mathbb{M}^{\Theta}_{*}); \quad G_{\underline{v}}(\mathbb{M}^{\Theta}_{*})_{\langle\mathbb{F}^{*}_l\rangle} \curvearrowright \Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

[cf. Proposition 3.1, (i); Proposition 3.3, (i); Corollary 3.5, (ii); Corollary 3.6, (ii)] gives rise to a $p_{\underline{v}}$*-adic Frobenioid* of *monoid type* $\mathbb{Z}$ [cf. [FrdII], Example 1.1, (ii)]

$$\mathcal{F}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}); \quad \mathcal{F}_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}},\alpha}; \quad \mathcal{F}_{\xi}(\mathbb{M}^{\Theta}_{*}); \quad \mathcal{F}_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

whose *divisor monoid* associates to every object of $\mathcal{B}^{\mathrm{temp}}(G_{\underline{v}}(-))^0$ [where "$(-)$" is $\mathbb{M}^{\Theta}_{*\blacktriangleright\!\!\ddot{}}$ or $\mathbb{M}^{\Theta}_{*}$] a monoid isomorphic to $\mathbb{N}$. Moreover, each of these Frobenioids is equipped with a collection of *splittings* [cf. Proposition 3.1, (i); Proposition 3.3, (i); Corollary 3.5, (iii); Corollary 3.6, (iii)]. Also, we shall write

$$\mathcal{F}_{\mathrm{env}}(\mathbb{M}_*^{\Theta}) \stackrel{\mathrm{def}}{=} \left\{ \mathcal{F}^{\iota}_{\mathrm{env}}(\mathbb{M}_*^{\Theta}) \right\}_{\iota}; \quad \mathcal{F}_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \left\{ \mathcal{F}_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}},\alpha} \right\}_{\alpha}$$
$$\mathcal{F}_{\mathrm{gau}}(\mathbb{M}_*^{\Theta}) \stackrel{\mathrm{def}}{=} \left\{ \mathcal{F}_{\xi}(\mathbb{M}_*^{\Theta}) \right\}_{\xi}; \quad \mathcal{F}_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \left\{ \mathcal{F}_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \right\}_{\xi}$$

[cf. the notation of Proposition 3.1, (i); Proposition 3.3, (i); Corollary 3.5, (ii); Corollary 3.6, (ii)]. It follows immediately from the construction of the data "$G_{\underline{v}}(\mathbb{M}^{\Theta}_{*\ddot{\blacktriangleright}}) \curvearrowright \Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}},\alpha}$" [cf. Example 3.2, (i)] that one has a *tautological isomorphism of Frobenioids*

$${}^{\dagger}\mathcal{C}^{\Theta}_{\underline{v}} \quad \stackrel{\sim}{\rightarrow} \quad \mathcal{F}_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}},\alpha}$$

which is *compatible* with the associated *splittings* [cf. the discussion of [IUTchI], Example 3.2, (v)], and which we shall use to *identify* these two split Frobenioids. Thus, the isomorphisms of monoids in the bottom line of the third display of Corollary 3.6, (ii), may be interpreted as *isomorphisms of split Frobenioids*

$$\mathcal{F}_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}},\alpha} \quad \stackrel{\sim}{\rightarrow} \quad \mathcal{F}^{\iota}_{\mathrm{env}}(\mathbb{M}_*^{\Theta}) \quad \stackrel{\sim}{\rightarrow} \quad \mathcal{F}_{\xi}(\mathbb{M}_*^{\Theta}) \quad \stackrel{\sim}{\rightarrow} \quad \mathcal{F}_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

[cf. Proposition 3.3, (i); Corollary 3.5, (iii); Corollary 3.6, (iii)] which are *compatible* with the subcategories

$$\mathcal{F}_{2l\cdot\xi}(\mathbb{M}_*^{\Theta}) \quad \subseteq \quad \mathcal{F}_{\xi}(\mathbb{M}_*^{\Theta}); \qquad \mathcal{F}_{\mathcal{F}_{2l\cdot\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \quad \subseteq \quad \mathcal{F}_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

determined by the submonoids "$\Psi_{2l\cdot\xi}(-)$" [cf. Corollaries 3.5, (ii); 3.6, (ii)] and which yield *isomorphisms of collections of split Frobenioids*

$$\mathcal{F}_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}} \quad \stackrel{\sim}{\rightarrow} \quad \mathcal{F}_{\mathrm{env}}(\mathbb{M}_*^{\Theta}) \quad \stackrel{\sim}{\rightarrow} \quad \mathcal{F}_{\mathrm{gau}}(\mathbb{M}_*^{\Theta}) \quad \stackrel{\sim}{\rightarrow} \quad \mathcal{F}_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

[cf. the fourth display of Corollary 3.6, (ii)].

(iii) The *direct products* in which the submonoids $\Psi_{\xi}(\mathbb{M}_*^{\Theta})$ and $\Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$ are constructed [cf. the second display of Corollary 3.5, (ii); the first display of Corollary 3.6, (ii)] determine *natural embeddings of categories* [cf. Remark 3.8.1 below]

$$\mathcal{F}_{\xi}(\mathbb{M}_*^{\Theta}) \quad \hookrightarrow \quad \prod_{|t| \in \mathbb{F}_l^{\circledast}} \mathcal{F}^{\vdash}_{\mathrm{cns}}(\mathbb{M}_*^{\Theta})_{|t|}; \qquad \mathcal{F}_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \quad \hookrightarrow \quad \prod_{|t| \in \mathbb{F}_l^{\circledast}} ({}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}})_{|t|}$$

which *coincide* on the subcategories $\mathcal{F}_{2l\cdot\xi}(\mathbb{M}_*^{\Theta}) \subseteq \mathcal{F}_{\xi}(\mathbb{M}_*^{\Theta})$, $\mathcal{F}_{\mathcal{F}_{2l\cdot\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \subseteq \mathcal{F}_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$. We shall write [cf. Remark 3.8.1 below]

$$\mathcal{F}_{\mathrm{gau}}(\mathbb{M}_*^{\Theta}) \quad \hookrightarrow \quad \mathcal{F}^{\vdash}_{\mathrm{cns}}(\mathbb{M}_*^{\Theta})_{\mathbb{F}_l^{\circledast}} \stackrel{\mathrm{def}}{=} \prod_{|t| \in \mathbb{F}_l^{\circledast}} \mathcal{F}^{\vdash}_{\mathrm{cns}}(\mathbb{M}_*^{\Theta})_{|t|}$$

$$\mathcal{F}_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \quad \hookrightarrow \quad ({}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}})_{\mathbb{F}_l^{\circledast}} \stackrel{\mathrm{def}}{=} \prod_{|t| \in \mathbb{F}_l^{\circledast}} ({}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}})_{|t|}$$

for the *collections of embeddings of categories* obtained by allowing ξ to vary. These embeddings may be thought of as **"Gaussian distributions"** and are depicted in Fig. 3.3 below. In this context, it is useful to observe that we also have *natural diagonal embeddings of categories*, i.e., **"constant distributions"** [cf. Remark 3.8.1 below]

$$\mathcal{F}^{\vdash}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}) \ \stackrel{\sim}{\to}\ \mathcal{F}^{\vdash}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle \mathbb{F}^{\circledast}_{l} \rangle} \quad \hookrightarrow \quad \mathcal{F}^{\vdash}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\mathbb{F}^{\circledast}_{l}} \ = \ \prod_{|t| \in \mathbb{F}^{\circledast}_{l}} \mathcal{F}^{\vdash}_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{|t|}$$

$$^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}} \ \stackrel{\sim}{\to}\ (^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}})_{\langle \mathbb{F}^{\circledast}_{l} \rangle} \quad \hookrightarrow \quad (^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}})_{\mathbb{F}^{\circledast}_{l}} \ = \ \prod_{|t| \in \mathbb{F}^{\circledast}_{l}} (^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}})_{|t|}$$

— where the " $\stackrel{\sim}{\to}$'s" denote the tautological isomorphisms — cf. the discussion [and notational conventions!] of [IUTchI], Example 5.4, (i); [IUTchI], Fig. 5.1.

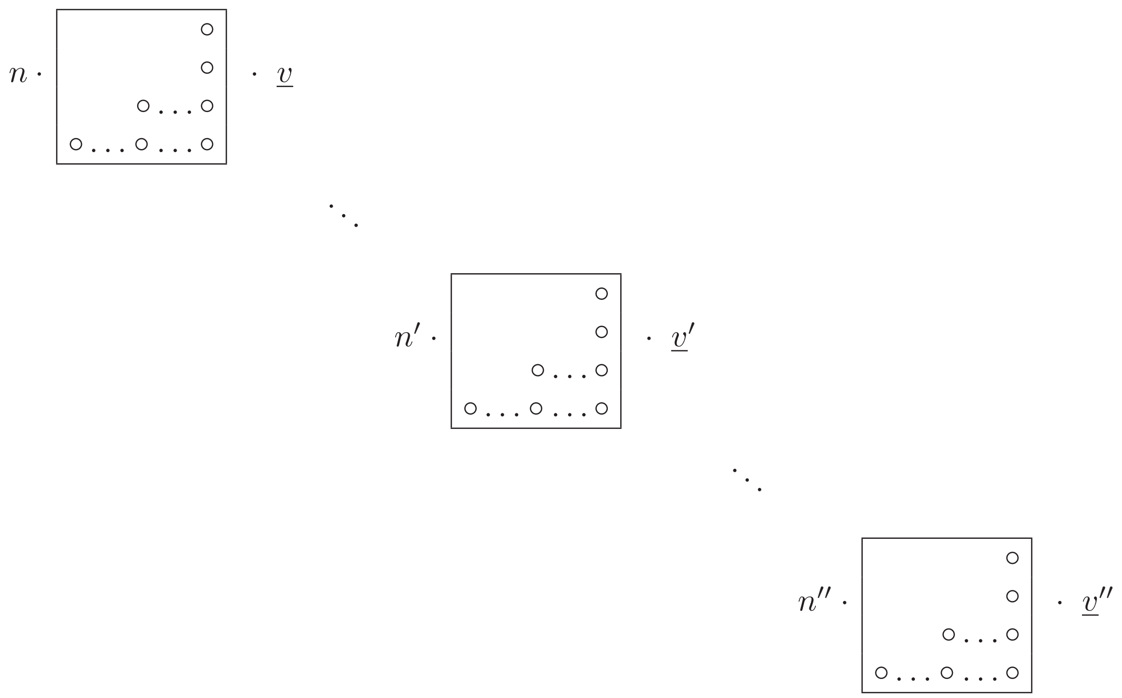

Fig. 3.3: Gaussian distribution

Remark 3.8.1. In the present series of papers, we follow the convention [cf. [IUTchI], §0] that an "isomorphism of categories" is to be understood as an isomorphism class of equivalences of categories. On the other hand, in the context of the discussion of Frobenioids in Definition 3.8, in order to obtain a *precise "Frobenioid-theoretic translation"* of the results obtained so far [in the language of monoids] that involve the phenomenon of **conjugate synchronization** [cf. Remark 3.5.2; the discussion of Remark 3.8.3 below], one is obliged to consider the various Frobenioids indexed by a subscript "$|t| \in |\mathbb{F}_l|$" as being determined up to an *isomorphism of the identity functor* — i.e., corresponding to an *"inner automorphism"* in the context of Corollaries 3.5, (i); 3.6, (i) — which is **independent** of $|t| \in |\mathbb{F}_l|$. In particular, when there is a danger of confusion, perhaps the simplest approach is to resort to the original *"monoid-theoretic formulations"* of Corollaries 3.5, 3.6.

Remark 3.8.2. At this point, it is of interest to pause to discuss the relationship between the theory of the present §3 and the theories of $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** [cf. [IUTchI], §6] and $\mathbb{F}_l^{\circledast}$**-symmetry** [cf. [IUTchI], §4, §5] developed in [IUTchI].

(i) First of all, the construction algorithms for the **Gaussian monoids** discussed in Corollaries 3.5, (ii); 3.6, (ii), as well as for the closely relating **splittings** discussed in Corollaries 3.5, (iii); 3.6, (iii), involve **restriction** to the decomposition groups of torsion points indexed [via a functorial algorithm] by *profinite conjugacy classes of cusps* [cf. Corollary 2.4, (ii)] which are subject to a certain $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** [cf. Corollary 2.4, (iii)]. This $\mathbb{F}_l^{\rtimes\pm}$-symmetry may be thought of as the restriction, to the portion labeled by the valuation $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ under consideration, of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry [cf. [IUTchI], Proposition 6.8, (i)] associated to a $\mathcal{D}$-$\Theta^{\pm\text{ell}}$-*Hodge theater* [cf. Remark 2.6.2, (i)]. From the point of view of the issue of "*which portion of the original once-punctured elliptic curve over a number field* X_F [cf. [IUTchI], Definition 3.1] is involved", this theory of split Gaussian monoids revolves around various *labeled* [i.e., by elements of copies of $\mathbb{F}_l$ or $|\mathbb{F}_l|$] *copies of the* **local Frobenioids** *at* $\underline{v}$ *of the mono-analyticizations of the* $\mathcal{F}$*-prime-strips* that appear in a $\mathcal{D}$-$\Theta^{\pm\text{ell}}$-Hodge theater — cf. the various **natural embeddings** discussed in Definition 3.8, (iii) — i.e., more concretely, copies of the portion of the pair "$G_{\underline{v}}(\Pi_{\underline{v}}) \curvearrowright \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$" determined by a certain submonoid of $\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$. Finally, we recall that after one executes these construction algorithms for split Gaussian monoids and observes the $\mathbb{F}_l^{\rtimes\pm}$-symmetry discussed above, one may then form *holomorphic or mono-analytic* **processions**, indexed by subsets of $|\mathbb{F}_l|$, as discussed in [IUTchI], Proposition 6.9, (i), (ii).

(ii) On the other hand, by applying the algorithm of [IUTchI], Proposition 6.7, one may pass to the local portion at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ of a $\mathcal{D}$-ΘNF-*Hodge theater*. At the level of *labels*, this amounts to *removing the label* $0 \in |\mathbb{F}_l|$ and *identifying this label with the complement of* 0 *in* $|\mathbb{F}_l|$, i.e., with $\mathbb{F}_l^{\circledast}$ — cf. the *assignment*

$$\text{“ } 0,\ \succ \quad \mapsto \quad > \text{ ”}$$

of $\mathcal{D}$-prime-strips discussed in [IUTchI], Proposition 6.7. At the level of *local Frobenioids* at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ [i.e., copies of the pair "$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$"] corresponding to these labels, this assignment may be thought of as corresponding to the *isomorphisms of monoids* "$\Psi_{\text{cns}}(\mathbb{M}^{\Theta}_*)_0 \ \overset{\sim}{\to}\ \Psi_{\text{cns}}(\mathbb{M}^{\Theta}_*)_{\langle \mathbb{F}_l^{\circledast} \rangle}$" and "$(\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})_0 \ \overset{\sim}{\to}\ (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})_{\langle \mathbb{F}_l^{\circledast} \rangle}$" discussed in the first displays of Corollaries 3.5, (iii); 3.6, (iii). This newly obtained situation involving the local portion at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ of a $\mathcal{D}$-ΘNF-*Hodge theater* admits an $\mathbb{F}_l^{\circledast}$**-symmetry** [cf. [IUTchI], Proposition 4.9, (i)] — cf. the discussion of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry in the situation of (i). As we shall see in §4 below, at least at the level of **value groups**, this newly obtained situation involving $\mathbb{F}_l^{\circledast}$-symmetries is well-suited to relating the theory of the present §3 at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ to the **valuations** $\in \underline{\mathbb{V}}^{\text{good}}$, as well as to the **global theory** of [IUTchI], §5. This global theory satisfies the crucial property that it allows one to *relate the* **multiplicative** *and* **additive** *structures of a global number field* [cf. the discussion of [IUTchI], Remark 4.3.2; [IUTchI], Remark 6.12.5, (ii)]. Finally, starting from this newly obtained situation, one may proceed to form *holomorphic or mono-analytic* **processions**, indexed by subsets of $\mathbb{F}_l^{\circledast}$, as discussed in [IUTchI], Proposition 4.11, (i), (ii), which are **compatible** [cf. [IUTchI], Proposition 6.9, (iii)] *with the "*$|\mathbb{F}_l|$*-processions"* discussed in (i).

Remark 3.8.3. One central theme of the theory of the present §3 is the **application** of the phenomenon of **conjugate synchronization** [cf. Remark 3.5.2], which plays a fundamental role in the theory of the group-theoretic version of Hodge-Arakelov-theoretic evaluation of the theta function developed in §2. Thus, it is of interest to pause to discuss *precisely what was gained* in the present §3 by applying the conjugate synchronization obtained in §2.

(i) We begin our discussion by reviewing the following *direct technical consequences* of the conjugate synchronization discussed in Remark 3.5.2:

(a) the isomorphisms of monoids

$$\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{|t_1|} \stackrel{\sim}{\rightarrow} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{|t_2|}; \quad (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})_{|t_1|} \stackrel{\sim}{\rightarrow} (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})_{|t_2|}; \quad (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})_{|t|} \stackrel{\sim}{\rightarrow} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{|t|}$$

— where $|t|, |t_1|, |t_2| \in |\mathbb{F}_l|$; the third isomorphism is well-defined up to an inner automorphism indeterminacy that is **independent of** $|t|$ — discussed in Corollaries 3.5, (i); 3.6, (i);

(b) the construction of well-defined **diagonal submonoids**

$$\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle|\mathbb{F}_l|\rangle} \subseteq \prod_{|t|\in|\mathbb{F}_l|} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{|t|}; \quad \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle\mathbb{F}^{*}_l\rangle} \subseteq \prod_{|t|\in\mathbb{F}^{*}_l} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{|t|}$$

in Corollary 3.5, (i), and the corresponding *diagonal embeddings of categories* — i.e., *"constant distributions"* — discussed in Definition 3.8, (iii);

(c) the well-defined **isomorphisms of monoids**

$$\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{0} \stackrel{\sim}{\rightarrow} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*})_{\langle\mathbb{F}^{*}_l\rangle}; \quad (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})_{0} \stackrel{\sim}{\rightarrow} (\Psi_{{}^{\dagger}\mathcal{C}_{\underline{v}}})_{\langle\mathbb{F}^{*}_l\rangle}$$

of Corollaries 3.5, (iii); 3.6, (iii);

(d) the restriction to the **units** *of the [composite] isomorphism of monoids*

$$\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}},\alpha} \stackrel{\sim}{\rightarrow} \Psi_{\mathcal{F}_{\xi}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

that appears in the third display of Corollary 3.6, (ii) [cf. also Fig. 3.1; the discussion of Remark 3.6.2, (i)].

Here, we observe that (b) and (c) may be thought of as *formal consequences* of (a), while (d) may be thought of as an *alternate formulation* of the portion of (a) concerning the units in the case of $|t| \in \mathbb{F}^{*}_l$. Moreover, as discussed in Remark 3.6.2, (iii), ultimately, in the present series of papers, we shall be interested in composing the **Θ-link** with the composite of the arrows "$\Longrightarrow$", "$\Downarrow$", and "$\Longleftarrow$" of Fig. 3.1 — i.e., with the isomorphism of monoids that appears in the display of (d). Indeed, from the point of view of the theory of the present series of papers,

> our *main application* [cf. §4 below] of the **conjugate synchronization** discussed in Remark 3.5.2 will consist precisely of the **isomorphism of units** of (d), in the context of composition with the **Θ-link** — cf. the **"coricity of $\mathcal{O}^{\times}$"** given in [IUTchI], Corollary 3.7, (iii).

Finally, in this context, we recall that the *isomorphisms of monoids* that appear in the Θ-link or in the third display of Corollary 3.6, (ii), only make sense if one works with **post-anabelian** *abstract monoids/Frobenioids* — i.e., with *"Frobenius-like" structures* [cf. the discussion of Remark 3.6.2, (i), (ii)].

(ii) In [IUTchIII], it will be of central importance to consider the theory of the present paper in the context of the **$\mathfrak{log}$-wall** [i.e., the situation considered in [AbsTopIII]]. In the context of the $\mathfrak{log}$-wall, it will be of fundamental importance to construct versions of the various *Frobenioid-theoretic theta and Gaussian monoids* that appeared in the discussion at the end of (i) that are capable of *"penetrating the* $\mathfrak{log}$*-wall"* [cf. the discussion of [AbsTopIII], §I4] — i.e., to construct *étale-like* versions of these *Frobenioid-theoretic theta and Gaussian monoids*, by availing oneself of the *right-hand portion* of Fig. 3.1. Now to pass from these Frobenioid-theoretic monoids to their étale-like counterparts, one must apply **Kummer theory** — cf. the arrow "$\Longrightarrow$" of Fig. 3.1. Moreover, in order to apply Kummer theory, one must avail oneself of the **cyclotomes** contained in [i.e., the torsion subgroups of] the various groups of units of the relevant monoids. It is at this point that it is necessary to apply, in the fashion discussed in (i), the **conjugate synchronization** discussed in Remark 3.5.2 in an *essential way.* That is to say, if one is in a situation in which one cannot avail oneself of this conjugate synchronization, then it follows from the **distinct, unrelated** nature of the **basepoints** on either side of the $\mathfrak{log}$-wall [cf. the discussion of Remark 3.6.3, (i)] that

> one may only construct *diagonal embeddings* of either submonoids of *Galois-invariants* or sets of *Galois-orbits* of the various *constant monoids* [i.e., "Ψ_{cns}"] involved.

On the other hand, such Galois-invariants or Galois-orbits are clearly *insufficient* for conducting *Kummer theory* [cf. [IUTchIII], Remark 1.5.1, (ii), for a discussion of a related topic]. Moreover, the operation of passing to sets of Galois-orbits *fails to be compatible with the* **ring** *structure* — e.g., the **additive** *structure* — on [the formal union with "$\{0\}$" of] the various constant monoids. Such an incompatibility is unacceptable in the context of the theory of the present series of papers since it is impossible to develop the theory of the **$\mathfrak{log}$-wall** [cf. [AbsTopIII]; [IUTchIII]] without applying the ring structure within each Hodge theater [cf. the discussion of Remark 3.6.4, (i)].

(iii) As discussed at the beginning of §1, the problem of giving an **explicit description** of what **one** *arithmetic holomorphic structure* looks like from the point of view of a **distinct** *arithmetic holomorphic structure* that is only related to the original arithmetic holomorphic structure via some mono-analytic core is one of the central themes of the theory of the present series of papers. The phenomenon of *conjugate synchronization* as discussed in (i) and (ii) above, as well as the closely related phenomenon of *mono-theta-theoretic cyclotomic rigidity* [cf. the discussion of Remark 3.6.5, (ii)], may be thought of as particular instances of this general theme. Indeed, from the point of view of classical discussions of scheme-theoretic arithmetic geometry,

> *the* **"natural isomorphisms"** *that exist between various* **cyclotomes** *that appear in a discussion are* **typically taken for granted**

— i.e., typically no attention is given to the issue of devising *explicit, intrinsic reconstruction algorithms* for these "natural isomorphisms" between cyclotomes.

Section 4: Global Gaussian Frobenioids

In the present §4, we generalize the theory of **Gaussian monoids**, developed in §3 in the case of *bad* $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, first to the case of **nonarchimedean** and **archimedean good** $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$ and then to the **global** case. One important feature of these generalizations, especially in the global case, is the theme of *compatibility* with the theory of ΘNF**-** (respectively, $\Theta^{\pm\text{ell}}$**-**) **Hodge theaters** — and, in particular, the $\mathbb{F}_l^{\circledast}$**-** (respectively, $\mathbb{F}_l^{\rtimes\pm}$**-**) **symmetries** of such Hodge theaters — developed in [IUTchI], §4, §5 (respectively, [IUTchI], §6).

In the following discussion, we assume that we have been given *initial* Θ-*data* as in [IUTchI], Definition 3.1. We begin our discussion by considering *good nonarchimedean* $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \bigcap \underline{\mathbb{V}}^{\text{non}}$.

Proposition 4.1. **(Group-theoretic Gaussian Monoids at Good Nonarchimedean Primes)** *Let* $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \bigcap \underline{\mathbb{V}}^{\text{non}}$. *In the notation of [IUTchI], Definition 3.1, (e), (f), write*

$$\Pi_{\underline{v}} \stackrel{\text{def}}{=} \Pi_{\underline{\underrightarrow{X}}_{\underline{v}}} \quad \subseteq \quad \Pi_{\underline{v}}^{\pm} \stackrel{\text{def}}{=} \Pi_{\underline{X}_{\underline{v}}} \quad \subseteq \quad \Pi_{\underline{v}}^{\text{cor}} \stackrel{\text{def}}{=} \Pi_{C_{\underline{v}}}$$

[cf. Definition 2.3, (i), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$*] — so* $\Pi_{\underline{v}}^{\pm}/\Pi_{\underline{v}} \cong \mathbb{Z}/l\mathbb{Z}$ *[cf. the discussion preceding [IUTchI], Definition 1.1],* $\Pi_{\underline{v}}^{\text{cor}}/\Pi_{\underline{v}}^{\pm} \cong \mathbb{F}_l^{\rtimes\pm}$*;*

$$\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}}), \quad \Pi_{\underline{v}}^{\pm} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}}^{\pm}), \quad \Pi_{\underline{v}}^{\text{cor}} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}}^{\text{cor}})$$

for the quotients — which admit natural isomorphisms $G_{\underline{v}}(\Pi_{\underline{v}}) \stackrel{\sim}{\to} G_{\underline{v}}(\Pi_{\underline{v}}^{\pm}) \stackrel{\sim}{\to} G_{\underline{v}}(\Pi_{\underline{v}}^{\text{cor}}) \stackrel{\sim}{\to} G_{\underline{v}}$ *— determined by the natural surjections to* $G_{\underline{v}}$*;* $\Delta_{\underline{v}}$, $\Delta_{\underline{v}}^{\pm}$, $\Delta_{\underline{v}}^{\text{cor}}$ *for the respective kernels of these quotients. Also, we recall that* $\Pi_{\underline{v}}^{\pm}$, $\Pi_{\underline{v}}^{\text{cor}}$, $G_{\underline{v}}(\Pi_{\underline{v}})$, $G_{\underline{v}}(\Pi_{\underline{v}}^{\pm})$, *and* $G_{\underline{v}}(\Pi_{\underline{v}}^{\text{cor}})$ *may be* **reconstructed algorithmically** *[cf. [IUTchI], Corollary 1.2, and its proof; [AbsAnab], Lemma 1.3.8] from the topological group* $\Pi_{\underline{v}}$.

(i) **(Constant Monoids)** *The functorial group-theoretic algorithm of [AbsTopIII], Corollary 1.10, (b) [cf. also the discussion of Remark 1.11.5, (i), in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$*; the discussion of "*$\mathbb{M}_v(-)$*" in [IUTchI], Definition 5.2, (v)] yields a* **functorial group-theoretic algorithm** *in the topological group* $G_{\underline{v}}$ *for constructing the* **ind-topological submonoid** *[which is naturally isomorphic to* $\mathcal{O}^{\triangleright}_{F_{\underline{v}}}$*]*

$$\Psi_{\text{cns}}(G_{\underline{v}}) \ \subseteq \ \varinjlim_{J} \ H^1(J, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}))$$

— where J *ranges over the open subgroups of* $G_{\underline{v}}$*;* $\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}})$ *is as in [AbsTopIII], Corollary 1.10, (b) — equipped with its natural* $G_{\underline{v}}$*-action. In particular, we obtain a functorial group-theoretic algorithm in the topological group* $\Pi_{\underline{v}}$ *for constructing the ind-topological submonoid*

$$\begin{aligned}\Psi_{\text{cns}}(\Pi_{\underline{v}}) \ \stackrel{\text{def}}{=} \Psi_{\text{cns}}(G_{\underline{v}}(\Pi_{\underline{v}})) \ &\subseteq \ \varinjlim_{J} \ H^1(G_{\underline{v}}(\Pi_{\underline{v}})|_J, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi_{\underline{v}}))) \\ &\subseteq \ \varinjlim_{J} \ H^1(\Pi_{\underline{v}}^{\pm}|_J, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi_{\underline{v}}))) \ \subseteq \ \varinjlim_{J} \ H^1(\Pi_{\underline{v}}|_J, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(G_{\underline{v}}(\Pi_{\underline{v}})))\end{aligned}$$

— *where* J *ranges over the open subgroups of* $G_{\underline{v}}(\Pi_{\underline{v}})$ — *equipped with its natural* $G_{\underline{v}}(\Pi_{\underline{v}})$*-action [cf. Proposition 3.1, (ii), in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*].*

(ii) **(Mono-analytic Semi-simplifications)** *The functorial algorithm discussed in [IUTchI], Example 3.5, (iii), for constructing "*$(\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$*" [cf. also [AbsTopIII], Proposition 5.8, (iii)] yields a* **functorial group-theoretic algorithm** *in the topological group* $G_{\underline{v}}$ *for constructing a* **topological monoid** $\mathbb{R}_{\geq 0}(G_{\underline{v}})$ *equipped with a* **natural isomorphism**

$$\Psi^{\mathbb{R}}_{\mathrm{cns}}(G_{\underline{v}}) \stackrel{\mathrm{def}}{=} (\Psi_{\mathrm{cns}}(G_{\underline{v}})/\Psi_{\mathrm{cns}}(G_{\underline{v}})^{\times})^{\mathrm{rlf}} \quad \stackrel{\sim}{\to} \quad \mathbb{R}_{\geq 0}(G_{\underline{v}})$$

— *where the superscript "*$\times$*" denotes the submonoid of units; the superscript "*rlf*" denotes the realification [which is isomorphic to* $\mathbb{R}_{\geq 0}$*] of the monoid in parentheses [which is isomorphic to* $\mathbb{Q}_{\geq 0}$*] — and a* **distinguished element**

$$\log^{G_{\underline{v}}}(p_{\underline{v}}) \in \mathbb{R}_{\geq 0}(G_{\underline{v}})$$

— *i.e., the element "*$\log^{\mathcal{D}}_{\Phi}(p_{\underline{v}})$*" of [IUTchI], Example 3.5, (iii). Write*

$$\Psi^{\mathrm{ss}}_{\mathrm{cns}}(G_{\underline{v}}) \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{cns}}(G_{\underline{v}})^{\times} \times \mathbb{R}_{\geq 0}(G_{\underline{v}})$$

— *which we shall think of as a sort of* **"semi-simplified version"** *of* $\Psi_{\mathrm{cns}}(G_{\underline{v}})$*. Also, just as in (i), we shall abbreviate notation that denotes a dependence on "*$G_{\underline{v}}(\Pi_{\underline{v}})$*" [e.g., a "*$G_{\underline{v}}(\Pi_{\underline{v}})$*" in parentheses] by means of notation that denotes a dependence on "*$\Pi_{\underline{v}}$*".*

(iii) **(Labels, $\mathbb{F}_l^{\rtimes\pm}$-Symmetries, and Conjugate Synchronization)** *Let* $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}}) \stackrel{\mathrm{def}}{=} \mathrm{LabCusp}^{\pm}(\mathcal{B}(\Pi_{\underline{v}})^0)$ *[cf. [IUTchI], Definition 6.1, (iii)]. In the following, we shall use analogous conventions to the conventions introduced in Corollary 3.5 concerning* **subscripted labels***. Then if we think of the* **cuspidal inertia groups** $\subseteq \Pi_{\underline{v}}$ *corresponding to* t *as subgroups of cuspidal inertia groups of* $\Pi^{\pm}_{\underline{v}}$ *[cf. Remark 2.3.1, in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*], then the* $\Delta^{\pm}_{\underline{v}}$*-outer action of* $\mathbb{F}_l^{\rtimes\pm} \cong \Delta^{\mathrm{cor}}_{\underline{v}}/\Delta^{\pm}_{\underline{v}}$ *on* $\Pi^{\pm}_{\underline{v}}$ *[cf. Corollary 2.4, (iii), in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*] induces* **isomorphisms** *between the pairs*

$$G_{\underline{v}}(\Pi_{\underline{v}})_t \curvearrowright \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_t$$

— *consisting of a* **labeled** *ind-topological monoid equipped with the action of a* **labeled** *topological group — for* **distinct** $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$*. We shall refer to these isomorphisms as* **[$\mathbb{F}_l^{\rtimes\pm}$-]symmetrizing isomorphisms** *[cf. Remark 3.5.2, in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*]. These symmetrizing isomorphisms determine* **diagonal submonoids**

$$\Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_{\langle|\mathbb{F}_l|\rangle} \subseteq \prod_{|t|\in|\mathbb{F}_l|} \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_{|t|}; \quad \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_{\langle\mathbb{F}_l^{*}\rangle} \subseteq \prod_{|t|\in\mathbb{F}_l^{*}} \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_{|t|}$$

of the respective product monoids compatible with the respective actions by subscripted versions of $G_{\underline{v}}(\Pi_{\underline{v}})$ *[cf. the discussion of Corollary 3.5, (i), in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*], as well as an* **isomorphism of ind-topological monoids**

$$\Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_0 \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})_{\langle\mathbb{F}_l^{*}\rangle}$$

compatible with the respective actions by subscripted versions of $G_{\underline{v}}(\Pi_{\underline{v}})$ *[cf. Corollary 3.5, (iii), in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*].*

(iv) **(Theta and Gaussian Monoids)** *Relative to the notational conventions discussed at the end of (ii), let us write*

$$\Psi_{\mathrm{env}}(\Pi_{\underline{v}}) \quad \stackrel{\mathrm{def}}{=} \quad \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})^{\times} \ \times \ \left\{\mathbb{R}_{\geq 0} \cdot \log^{\Pi_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\Pi_{\underline{v}}}(\underline{\underline{\Theta}})\right\}$$

— where the notation "$\log^{\Pi_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\Pi_{\underline{v}}}(\underline{\underline{\Theta}})$*" is to be understood as a* **formal symbol** *[cf. the discussion of [IUTchI], Example 3.3, (ii)] — and*

$$\begin{aligned} \Psi_{\mathrm{gau}}(\Pi_{\underline{v}}) \quad &\stackrel{\mathrm{def}}{=} \quad \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})^{\times}_{\langle \mathbb{F}_l^{\circledast} \rangle} \ \times \ \left\{\mathbb{R}_{\geq 0} \cdot \left(\ldots, j^2 \cdot \log^{\Pi_{\underline{v}}}(p_{\underline{v}}), \ldots\right)\right\} \\ &\subseteq \quad \prod_{j \in \mathbb{F}_l^{\circledast}} \Psi^{\mathrm{ss}}_{\mathrm{cns}}(\Pi_{\underline{v}})_j \quad = \quad \prod_{j \in \mathbb{F}_l^{\circledast}} \Psi_{\mathrm{cns}}(\Pi_{\underline{v}})^{\times}_j \ \times \ \mathbb{R}_{\geq 0}(\Pi_{\underline{v}})_j \end{aligned}$$

— where, by abuse of notation, we also write "j" for the natural number $\in \{1, \ldots, l^{\circledast}\}$ *determined by an element* $j \in \mathbb{F}_l^{\circledast}$*. In particular, [cf. (i), (ii), (iii)] we obtain a* **functorial group-theoretic algorithm** *in the topological group* $\Pi_{\underline{v}}$ *for constructing the* **theta monoid** $\Psi_{\mathrm{env}}(\Pi_{\underline{v}})$ *and the* **Gaussian monoid** $\Psi_{\mathrm{gau}}(\Pi_{\underline{v}})$*, equipped with their [evident] natural* $G_{\underline{v}}(\Pi_{\underline{v}})$**-actions** *and* **splittings***, as well as the* **formal evaluation isomorphism** *[cf. Corollary 3.5, (ii), in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*]*

$$\begin{aligned} \Psi_{\mathrm{env}}(\Pi_{\underline{v}}) \quad &\stackrel{\sim}{\to} \quad \Psi_{\mathrm{gau}}(\Pi_{\underline{v}}) \\ \log^{\Pi_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\Pi_{\underline{v}}}(\underline{\underline{\Theta}}) \ &\mapsto \ \left(\ldots, j^2 \cdot \log^{\Pi_{\underline{v}}}(p_{\underline{v}}), \ldots\right) \end{aligned}$$

— which restricts to the identity on the respective copies of "$\Psi_{\mathrm{cns}}(\Pi_{\underline{v}})^{\times}$*" and is* **compatible** *with the respective natural actions of* $G_{\underline{v}}(\Pi_{\underline{v}})$ *as well as with the natural splittings on the domain and codomain.*

Proof. The various assertions of Proposition 4.1 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 4.1.1.

(i) Proposition 4.1 may be thought of as a sort of **"easy" formal generalization** of much of the theory of §2, §3 — more precisely, the portion constituted by Proposition 3.1 and Corollaries 2.4, 3.5 — to the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$. By comparison to the corresponding portion of the theory of §2, §3, this generalization is somewhat **tautological** and, for the most part, "vacuous". As we shall see later, the reason for considering this formal generalization to $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ is that it allows one to **"globalize"** the theory of §2, §3, i.e., by gluing together the theories at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ and $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$.

(ii) The symmetrizing isomorphisms of Proposition 4.1, (iii), constitute the analogue at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ of the **conjugate synchronization** at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ discussed in Corollary 3.5, (i); Remark 3.5.2. In this context, it is perhaps most natural to think of the *"copies of $G_{\underline{v}}(\Pi_{\underline{v}})$ labeled by $t \in \mathrm{LabCusp}^{\pm}(\Pi_{\underline{v}})$"* as the quotients

$$D_t/I_t$$

— where I_t is a **cuspidal inertia group** $\subseteq \Pi_{\underline{v}}$ corresponding to t; D_t is the corresponding **decomposition group** $\subseteq \Pi_{\underline{v}}$ [i.e., the *normalizer*, or, equivalently, the commensurator, of I_t in $\Pi_{\underline{v}}$ — cf., e.g., [AbsSect], Theorem 1.3, (ii)]; we think of D_t/I_t as being equipped with the isomorphism $D_t/I_t \xrightarrow{\sim} G_{\underline{v}}(\Pi_{\underline{v}})$ induced by the natural surjection $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}(\Pi_{\underline{v}})$.

(iii) One may also formulate an easy *tautological* formal analogue at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ of the *multiradiality* and *uniradiality* assertions of Proposition 3.4, Corollary 3.7 at $\underline{v} \in \underline{\mathbb{V}}$. For instance,

(a) the construction of the monoids $\Psi_{\mathrm{cns}}(\Pi_{\underline{v}})$ [cf. Proposition 4.1, (i)] is **uniradial** [cf. Proposition 3.4, (ii)], while

(b) the construction of the monoids $\Psi^{\mathrm{ss}}_{\mathrm{cns}}(\Pi_{\underline{v}})$, $\Psi_{\mathrm{env}}(\Pi_{\underline{v}})$, and $\Psi_{\mathrm{gau}}(\Pi_{\underline{v}})$ [cf. Proposition 4.1, (ii), (iv)], as well as of the isomorphism $\Psi_{\mathrm{env}}(\Pi_{\underline{v}}) \xrightarrow{\sim} \Psi_{\mathrm{gau}}(\Pi_{\underline{v}})$ [cf. Proposition 4.1, (iv)], is **multiradial**.

We leave the routine details to the reader. Ultimately, in the present series of papers [cf., especially, the theory of [IUTchIII]], we shall be interested in a **global analogue** of the theory of multiradiality and uniradiality developed in §1, §3 at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. This global analogue will "specialize" to the theory of §1, §3 at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ and to the formal analogue just discussed [i.e., (a), (b)] at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$.

Proposition 4.2. (Frobenioid-theoretic Gaussian Monoids at Good Nonarchimedean Primes) *We continue to use the notation of Proposition 4.1. Let ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ be a* $\boldsymbol{p}_{\underline{v}}$**-adic Frobenioid** *that appears in a Θ-Hodge theater ${}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}},\ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$ [cf. [IUTchI], Definition 3.6] — cf., for instance, the Frobenioid "$\underline{\underline{\mathcal{F}}}_{\underline{v}} = \mathcal{C}_{\underline{v}}$" of [IUTchI], Example 3.3, (i); here, we assume [for simplicity] that the base category of ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is equal to $\mathcal{B}^{\mathrm{temp}}({}^{\dagger}\Pi_{\underline{v}})^0$, and we denote by means of a "$\dagger$" the various topological groups associated to ${}^{\dagger}\Pi_{\underline{v}}$ that correspond to the topological groups associated to $\Pi_{\underline{v}}$ in Proposition 4.1. Write*

$$G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}}) \quad \curvearrowright \quad \Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}}$$

for the ind-topological monoid $\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}}$ equipped with a continuous $G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})$-action determined, up to **inner automorphism** *[i.e., up to an automorphism arising from an element of ${}^{\dagger}\Pi_{\underline{v}}$], by ${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [cf. the construction of "$\Psi_{\mathcal{C}_{\underline{v}}}$" in Example 3.2, (ii), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$; the discussion of "${}^{\ddagger}\mathbb{M}_v$" in [IUTchI], Definition 5.2, (vi); the discussion of [AbsTopIII], Remark 3.1.1] and*

$${}^{\dagger}G_{\underline{v}} \quad \curvearrowright \quad \Psi_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$$

for the ind-topological monoid $\Psi_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$ equipped with a continuous ${}^{\dagger}G_{\underline{v}}$-action determined, up to **inner automorphism** *[i.e., up to an automorphism arising from an* *element of* ${}^{\dagger}G_{\underline{v}}$*], by the portion indexed by* $\underline{v}$ *of the* $\mathcal{F}^{\vdash}$**-prime-strip** $\{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$ *determined by the* Θ*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta}$ *[cf. [IUTchI], Definition 3.6; [IUTchI], Definition 5.2, (ii)].*

(i) **(Constant Monoids)** *There exists a* **unique** $G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})$*-equivariant* **isomorphism of monoids** *[cf. Proposition 3.3, (ii), in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*; the discussion of "*${}^{\ddagger}\mathbb{M}_v$*" in [IUTchI], Definition 5.2, (vi)]*

$$\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{cns}}({}^{\dagger}\Pi_{\underline{v}})$$

— cf. Remark 1.11.1, (i), (a); [AbsTopIII], Proposition 3.2, (iv).

(ii) **(Mono-analytic Semi-simplifications)** *There exists a* **unique** ${}^{\dagger}G_{\underline{v}}$*-equivariant* $\widehat{\mathbb{Z}}^{\times}$**-orbit of isomorphisms of topological groups**

$$\Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{cns}}({}^{\dagger}G_{\underline{v}})^{\times}$$

— cf. Remark 1.11.1, (i), (b); [AbsTopIII], Proposition 3.3, (ii) — as well as a **unique isomorphism of monoids**

$$\Psi^{\mathbb{R}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \stackrel{\mathrm{def}}{=} (\Psi_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}/\Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}})^{\mathrm{rlf}} \quad \stackrel{\sim}{\to} \quad \Psi^{\mathbb{R}}_{\mathrm{cns}}({}^{\dagger}G_{\underline{v}})$$

that maps the **distinguished element** *of* $\Psi^{\mathbb{R}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$ *determined by the unique generator of* $\Psi_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}/\Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$ *to the distinguished element of* $\Psi^{\mathbb{R}}_{\mathrm{cns}}({}^{\dagger}G_{\underline{v}})$ *determined by* $\log^{{}^{\dagger}G_{\underline{v}}}(p_{\underline{v}}) \in \mathbb{R}_{\geq 0}({}^{\dagger}G_{\underline{v}})$ *[cf. Proposition 4.1, (ii)]. In particular, one may define a* **"semi-simplified version"** $\Psi^{\mathrm{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \times \Psi^{\mathbb{R}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$ *of* $\Psi_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$*; the isomorphisms discussed above determine a natural poly-isomorphism of ind-topological monoids*

$$\Psi^{\mathrm{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \quad \stackrel{\sim}{\to} \quad \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}G_{\underline{v}})$$

[cf. Proposition 4.1, (ii)] that is compatible with the natural splittings on the domain and codomain. Write $\Psi^{\mathrm{ss}}_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \Psi^{\mathrm{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$*; thus, it follows from the definitions [cf. also the unique isomorphism of (i)] that we have a natural isomorphism [i.e., as opposed to a poly-isomorphism!]* $\Psi^{\mathrm{ss}}_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} \stackrel{\sim}{\to} \Psi^{\mathrm{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$.

(iii) **(Labels,** $\mathbb{F}_l^{\rtimes\pm}$**-Symmetries, and Conjugate Synchronization)** *The isomorphism of (i) determines, for each* $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\Pi_{\underline{v}})$*, a collection of* **compatible isomorphisms**

$$(\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_t \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{cns}}({}^{\dagger}\Pi_{\underline{v}})_t$$

— which are **well-defined** *up to composition with an* **inner automorphism** *of* ${}^{\dagger}\Pi_{\underline{v}}$ *which is* **independent** *of* $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\Pi_{\underline{v}})$ *[cf. Corollary 3.6, (i), in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*] — as well as [*$\mathbb{F}_l^{\rtimes\pm}$*-]***symmetrizing isomorphisms***, induced by*

the ${}^{\dagger}\Delta^{\pm}_{\underline{v}}$*-outer action of* $\mathbb{F}_l^{\rtimes\pm} \cong {}^{\dagger}\Delta^{\mathrm{cor}}_{\underline{v}}/{}^{\dagger}\Delta^{\pm}_{\underline{v}}$ *on* ${}^{\dagger}\Pi^{\pm}_{\underline{v}}$ *[cf. Corollary 2.4, (iii), in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*], between the data indexed by distinct* $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\Pi_{\underline{v}})$*. Moreover, these symmetrizing isomorphisms determine [various* **diagonal submonoids***, as well as] an* **isomorphism of ind-topological monoids**

$$(\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_0 \quad \stackrel{\sim}{\to} \quad (\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_{\langle \mathbb{F}_l^* \rangle}$$

compatible with the respective actions by subscripted versions of $G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})$ *[cf. Corollary 3.6, (iii), in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*].*

(iv) **(Theta and Gaussian Monoids)** *Write*

$$\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}}, \quad \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

for the **monoids** *equipped with* $G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})$*-actions and natural* **splittings** *determined, respectively — via the isomorphisms of (i), (ii), and (iii) — by the monoids* $\Psi_{\mathrm{env}}({}^{\dagger}\Pi_{\underline{v}})$*,* $\Psi_{\mathrm{gau}}({}^{\dagger}\Pi_{\underline{v}})$*, Galois actions, and splittings of Proposition 4.1, (iv). Then the definition of the various monoids involved, together with the formal evaluation isomorphism of Proposition 4.1, (iv), gives rise to a collection of* **natural isomorphisms** *[cf. Corollary 3.6, (ii), in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*]*

$$\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}} \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{env}}({}^{\dagger}\Pi_{\underline{v}}) \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{gau}}({}^{\dagger}\Pi_{\underline{v}}) \quad \stackrel{\sim}{\to} \quad \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

— which restrict to the identity or to the [restriction to "$(-)^{\times}$*" of the] isomorphism of (i) [or its inverse] on the various copies of* $\Psi^{\times}_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}}$*, "*$\Psi_{\mathrm{cns}}({}^{\dagger}\Pi_{\underline{v}})^{\times}$*" and are* **compatible** *with the various natural actions of* $G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})$ *and natural splittings.*

Proof. The various assertions of Proposition 4.2 follow immediately from the definitions and the references quoted in the statements of these assertions. ○

Remark 4.2.1.

(i) In the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ treated in §3, we did not discuss an analogue of the **"mono-analytic semi-simplification"** $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}G_{\underline{v}})$ of Proposition 4.1, (ii). On the other hand, one verifies immediately that one may define, in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ — via the same *group-theoretic algorithms* as those applied in Proposition 4.1, (i), (ii) — *ind-topological monoids* $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}G_{\underline{v}})$, $\mathbb{R}_{\geq 0}({}^{\dagger}G_{\underline{v}})$ equipped with *natural* ${}^{\dagger}G_{\underline{v}}$*-actions*, a *natural isomorphism* [i.e., as in the first display of Proposition 4.1, (ii)], a *distinguished element* $\log^{{}^{\dagger}G_{\underline{v}}}(p_{\underline{v}}) \in \mathbb{R}_{\geq 0}({}^{\dagger}G_{\underline{v}})$, and a *tautological splitting*

$$\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}G_{\underline{v}}) \quad = \quad \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}G_{\underline{v}})^{\times} \ \times \ \mathbb{R}_{\geq 0}({}^{\dagger}G_{\underline{v}})$$

[cf. Proposition 4.1, (ii)]. Moreover, if we write

$$\Psi_{\mathrm{cns}}(\Pi_{\underline{v}}) \quad \stackrel{\mathrm{def}}{=} \quad \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}}))$$

— where the latter "$\Psi_{\mathrm{cns}}(-)$" is as in Proposition 3.1, (ii) — then, by applying the *cyclotomic rigidity isomorphisms* of Definition 1.1, (ii), and the discussion at the beginning of Corollary 2.9, one obtains a *functorial group-theoretic* [i.e., in the topological group $\Pi_{\underline{v}}$] $\Pi_{\underline{v}}$*-equivariant isomorphism*

$$\Psi_{\mathrm{cns}}(\Pi_{\underline{v}})^{\times} \quad \stackrel{\sim}{\to} \quad \Psi^{\mathrm{ss}}_{\mathrm{cns}}(G_{\underline{v}}(\Pi_{\underline{v}}))^{\times}$$

— cf. the discussion of "$\Psi^{\mathrm{ss}}_{\mathrm{cns}}(-)$" in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ in Proposition 4.1, (ii). Finally, we observe that, relative to the above notation, one has analogues of "$\Psi^{\mathrm{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$" and of Proposition 4.2, (i), (ii), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. We leave the routine details to the reader.

(ii) Note that in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$, the monoids $\Psi_{\mathrm{env}}(\Pi_{\underline{v}})$, $\Psi_{\mathrm{gau}}(\Pi_{\underline{v}})$ of Proposition 4.1, (iv), are already *divisible*. Thus, it is natural, in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$, to simply set

$$\begin{array}{cccccc} {}_{\infty}\Psi_{\mathrm{env}}(\Pi_{\underline{v}}) & \stackrel{\mathrm{def}}{=} & \Psi_{\mathrm{env}}(\Pi_{\underline{v}}); & {}_{\infty}\Psi_{\mathrm{gau}}(\Pi_{\underline{v}}) & \stackrel{\mathrm{def}}{=} & \Psi_{\mathrm{gau}}(\Pi_{\underline{v}}) \\ {}_{\infty}\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}} & \stackrel{\mathrm{def}}{=} & \Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}}; & {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) & \stackrel{\mathrm{def}}{=} & \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}) \end{array}$$

— cf. the various monoids "${}_{\infty}\Psi(-)$" that appeared in the discussion of §3.

(iii) In the situation of (ii), if one regards the pairs $G_{\underline{v}}(\Pi_{\underline{v}}) \curvearrowright \Psi_{\mathrm{env}}(\Pi_{\underline{v}})$, $G_{\underline{v}}(\Pi_{\underline{v}}) \curvearrowright \Psi_{\mathrm{gau}}(\Pi_{\underline{v}})$, $G_{\underline{v}}(\Pi_{\underline{v}}) \curvearrowright {}_{\infty}\Psi_{\mathrm{env}}(\Pi_{\underline{v}})$, $G_{\underline{v}}(\Pi_{\underline{v}}) \curvearrowright {}_{\infty}\Psi_{\mathrm{gau}}(\Pi_{\underline{v}})$ up to an *indeterminacy* with respect to $\Pi_{\underline{v}}$*-inner automorphisms*, then one obtains data which we shall denote by means of the notation

$$\Psi_{\mathrm{env}}(\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0), \;\; \Psi_{\mathrm{gau}}(\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0), \;\; {}_{\infty}\Psi_{\mathrm{env}}(\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0), \;\; {}_{\infty}\Psi_{\mathrm{gau}}(\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0)$$

— i.e., since $\Pi_{\underline{v}}$ may only be reconstructed from $\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0$ *up to an inner automorphism indeterminacy* [cf. the discussion of [IUTchI], §0].

(iv) Suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Then the above discussion motivates the following notational conventions. First, let us write

$$\begin{array}{cc} \Psi_{\mathrm{env}}(\Pi_{\underline{v}}) \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})), & \Psi_{\mathrm{gau}}(\Pi_{\underline{v}}) \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) \\ {}_{\infty}\Psi_{\mathrm{env}}(\Pi_{\underline{v}}) \stackrel{\mathrm{def}}{=} {}_{\infty}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})), & {}_{\infty}\Psi_{\mathrm{gau}}(\Pi_{\underline{v}}) \stackrel{\mathrm{def}}{=} {}_{\infty}\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta}_{*}(\Pi_{\underline{v}})) \end{array}$$

— cf. (ii) above; the notation of Corollary 3.5, (ii). When these monoids equipped with various topological group actions are considered only up to a $\Pi_{\underline{v}}$*-inner automorphism indeterminacy*, we shall denote the resulting data by means of the notation

$$\Psi_{\mathrm{env}}(\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0), \;\; \Psi_{\mathrm{gau}}(\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0), \;\; {}_{\infty}\Psi_{\mathrm{env}}(\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0), \;\; {}_{\infty}\Psi_{\mathrm{gau}}(\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0)$$

— cf. (iii) above.

Next, we consider *[good] archimedean* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ $(\subseteq \underline{\mathbb{V}}^{\mathrm{good}})$.

Proposition 4.3. (Aut-**holomorphic-space-theoretic Gaussian Monoids at Archimedean Primes)** *Let* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ $(\subseteq \underline{\mathbb{V}}^{\mathrm{good}})$. *Recall the* Aut-**holomorphic orbispaces** *of [IUTchI], Example 3.4, (i),*

$$\mathbb{U}_{\underline{v}} \stackrel{\mathrm{def}}{=} \underline{\mathbb{X}}_{\to_{\underline{v}}} \quad \to \quad \mathbb{U}^{\pm}_{\underline{v}} \stackrel{\mathrm{def}}{=} \underline{\mathbb{X}}_{\underline{v}} \quad \to \quad \mathbb{U}^{\mathrm{cor}}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathbb{C}_{\underline{v}}$$

— so $\mathrm{Gal}(\mathbb{U}_{\underline{v}}/\mathbb{U}^{\pm}_{\underline{v}}) \cong \mathbb{Z}/l\mathbb{Z}$ *[cf. the discussion preceding [IUTchI], Definition 1.1],* $\mathrm{Gal}(\mathbb{U}^{\pm}_{\underline{v}}/\mathbb{U}^{\mathrm{cor}}_{\underline{v}}) \cong \mathbb{F}_l^{\rtimes\pm}$*; we shall apply the notation "*$\overline{\mathcal{A}}_\square$*", "*$\mathcal{A}_\square$*" of [IUTchI], Example 3.4, (i), to these* Aut-*holomorphic orbispaces. Also, we shall write* $\mathcal{A}^{\triangleright}_\square \subseteq \mathcal{A}_\square \subseteq \overline{\mathcal{A}}_\square$ *for the topological monoid of nonzero elements of absolute value* ≤ 1 *of the complex archimedean field* $\overline{\mathcal{A}}_\square$ *[cf. the slightly different notation of [AbsTopIII], Corollary 4.5, (ii)]. Finally, we recall the object* $\mathcal{D}^{\vdash}_{\underline{v}}$ *of the category "*$\mathbb{TM}^{\vdash}$*" of split topological monoids discussed in [IUTchI], Example 3.4, (ii); we shall write* $\mathcal{D}^{\vdash}_{\underline{v}}(\mathbb{U}_{\underline{v}})$ *when we wish to regard* $\mathcal{D}^{\vdash}_{\underline{v}}$ *as an object algorithmically constructed from* $\mathbb{U}_{\underline{v}}$.

(i) **(Constant Monoids)** *There is a* **functorial algorithm** *in the* Aut-*holomorphic space* $\mathbb{U}_{\underline{v}}$ *for constructing the* **topological monoid**

$$\Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}}) \quad \stackrel{\mathrm{def}}{=} \quad \mathcal{A}^{\triangleright}_{\mathbb{U}_{\underline{v}}}$$

— cf. [IUTchI], Example 3.4, (i); the discussion of "$\mathbb{M}_v(-)$*" in [IUTchI], Definition 5.2, (vii); [AbsTopIII], Definition 4.1, (i); [AbsTopIII], Corollary 2.7, (e). Moreover, if we write* $\Psi_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}})$ *for the underlying topological monoid of* $\mathcal{D}^{\vdash}_{\underline{v}}$*, then we have a tautological isomorphism of topological monoids*

$$\Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}}) \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}}(\mathbb{U}_{\underline{v}}))$$

[cf. [IUTchI], Example 3.4, (ii)] — which we shall use to **identify** *these two topological monoids.*

(ii) **(Mono-analytic Semi-simplifications)** *The functorial algorithm discussed in [IUTchI], Example 3.5, (iii), for constructing "*$(\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$*" [cf. also [AbsTopIII], Proposition 5.8, (vi)] yields a* **functorial algorithm** *in the object* $\mathcal{D}^{\vdash}_{\underline{v}}$ *of* $\mathbb{TM}^{\vdash}$ *for constructing a* **topological monoid** $\mathbb{R}_{\geq 0}(\mathcal{D}^{\vdash}_{\underline{v}})$ *equipped with a* **distinguished element**

$$\log^{\mathcal{D}^{\vdash}_{\underline{v}}}(p_{\underline{v}}) \in \mathbb{R}_{\geq 0}(\mathcal{D}^{\vdash}_{\underline{v}})$$

— i.e., the element "$\log^{\mathcal{D}}_{\Phi}(p_{\underline{v}})$*" of [IUTchI], Example 3.5, (iii). Write*

$$\Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}})^{\times} \times \mathbb{R}_{\geq 0}(\mathcal{D}^{\vdash}_{\underline{v}})$$

— where the superscript "$\times$*" denotes the submonoid of units — which we shall think of as a sort of* **"semi-simplified version"** *of* $\Psi_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}})$. *We shall abbreviate notation that denotes a dependence on "*$\mathcal{D}^{\vdash}_{\underline{v}}(\mathbb{U}_{\underline{v}})$*" [e.g., a "*$\mathcal{D}^{\vdash}_{\underline{v}}(\mathbb{U}_{\underline{v}})$*" in parentheses] by means of notation that denotes a dependence on "*$\mathbb{U}_{\underline{v}}$*". Finally, there is a functorial algorithm in the* Aut-*holomorphic space* $\mathbb{U}_{\underline{v}}$ *for constructing the* **natural isomorphism** *[which arises immediately from the definitions]*

$$\Psi^{\mathbb{R}}_{\mathrm{cns}}(\mathbb{U}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})/\Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})^{\times} \quad \stackrel{\sim}{\to} \quad \mathbb{R}_{\geq 0}(\mathbb{U}_{\underline{v}})$$

— *cf. [IUTchI], Example 3.4, (i).*

(iii) **(Labels, $\mathbb{F}_l^{\rtimes\pm}$-Symmetries, and Conjugate Synchronization)** *Let $t \in \mathrm{LabCusp}^{\pm}(\mathbb{U}_{\underline{v}})$ [cf. [IUTchI], Definition 6.1, (iii)]. In the following, we shall use analogous conventions to the conventions introduced in Corollary 3.5 concerning* **subscripted labels***. Then the action of $\mathbb{F}_l^{\rtimes\pm} \cong \mathrm{Gal}(\mathbb{U}_{\underline{v}}^{\pm}/\mathbb{U}_{\underline{v}}^{\mathrm{cor}})$ on the various $\mathrm{Gal}(\mathbb{U}_{\underline{v}}/\mathbb{U}_{\underline{v}}^{\pm})$-orbits of cusps of $\mathbb{U}_{\underline{v}}$ [cf. the definition of "$\mathrm{LabCusp}^{\pm}(-)$" in [IUTchI], Definition 6.1, (iii)] induces* **isomorphisms** *between the* **labeled** *topological monoids*

$$\Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_t$$

for **distinct** *$t \in \mathrm{LabCusp}^{\pm}(\mathbb{U}_{\underline{v}})$. We shall refer to these isomorphisms as* **[$\mathbb{F}_l^{\rtimes\pm}$-]symmetrizing isomorphisms** *[cf. Remark 3.5.2, in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$]. These symmetrizing isomorphisms determine* **diagonal submonoids**

$$\Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_{\langle|\mathbb{F}_l|\rangle} \subseteq \prod_{|t|\in|\mathbb{F}_l|} \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_{|t|}; \quad \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_{\langle\mathbb{F}_l^{\circledast}\rangle} \subseteq \prod_{|t|\in\mathbb{F}_l^{\circledast}} \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_{|t|}$$

of the respective product monoids [cf. the discussion of Corollary 3.5, (i), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], as well as an **isomorphism of topological monoids**

$$\Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_0 \quad \xrightarrow{\sim} \quad \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})_{\langle\mathbb{F}_l^{\circledast}\rangle}$$

[cf. Corollary 3.5, (iii), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$].

(iv) **(Theta and Gaussian Monoids)** *Relative to the notational conventions discussed in (ii), let us write*

$$\Psi_{\mathrm{env}}(\mathbb{U}_{\underline{v}}) \quad \stackrel{\mathrm{def}}{=} \quad \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})^{\times} \times \left\{\mathbb{R}_{\geq 0} \cdot \log^{\mathbb{U}_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\mathbb{U}_{\underline{v}}}(\underline{\underline{\Theta}})\right\}$$

— *where the notation "$\log^{\mathbb{U}_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\mathbb{U}_{\underline{v}}}(\underline{\underline{\Theta}})$" is to be understood as a* **formal symbol** *[cf. the discussion of [IUTchI], Example 3.4, (iii)] — and*

$$\begin{aligned}\Psi_{\mathrm{gau}}(\mathbb{U}_{\underline{v}}) \quad &\stackrel{\mathrm{def}}{=} \quad \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})^{\times}_{\langle\mathbb{F}_l^{\circledast}\rangle} \times \left\{\mathbb{R}_{\geq 0} \cdot \Big(\ldots, j^2 \cdot \log^{\mathbb{U}_{\underline{v}}}(p_{\underline{v}}), \ldots\Big)\right\} \\ &\subseteq \quad \prod_{j\in\mathbb{F}_l^{\circledast}} \Psi_{\mathrm{cns}}^{\mathrm{ss}}(\mathbb{U}_{\underline{v}})_j \quad = \quad \prod_{j\in\mathbb{F}_l^{\circledast}} \Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})^{\times}_j \times \mathbb{R}_{\geq 0}(\mathbb{U}_{\underline{v}})_j\end{aligned}$$

— *where, by abuse of notation, we also write "j" for the natural number $\in \{1, \ldots, l^{\circledast}\}$ determined by an element $j \in \mathbb{F}_l^{\circledast}$. In particular, [cf. (i), (ii), (iii)] we obtain a* **functorial algorithm** *in the* Aut*-holomorphic space $\mathbb{U}_{\underline{v}}$ for constructing the* **theta monoid** *$\Psi_{\mathrm{env}}(\mathbb{U}_{\underline{v}})$ and the* **Gaussian monoid** *$\Psi_{\mathrm{gau}}(\mathbb{U}_{\underline{v}})$, equipped with their [evident] natural* **splittings***, as well as the* **formal evaluation isomorphism** *[cf. Corollary 3.5, (ii), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$]*

$$\begin{aligned}\Psi_{\mathrm{env}}(\mathbb{U}_{\underline{v}}) \quad &\xrightarrow{\sim} \quad \Psi_{\mathrm{gau}}(\mathbb{U}_{\underline{v}}) \\ \log^{\mathbb{U}_{\underline{v}}}(p_{\underline{v}}) \cdot \log^{\mathbb{U}_{\underline{v}}}(\underline{\underline{\Theta}}) \quad &\mapsto \quad \Big(\ldots, j^2 \cdot \log^{\mathbb{U}_{\underline{v}}}(p_{\underline{v}}), \ldots\Big)\end{aligned}$$

— *which restricts to the identity on the respective copies of "*$\Psi_{\mathrm{cns}}(\mathbb{U}_{\underline{v}})^{\times}$*" and is* **compatible** *with the natural splittings on the domain and codomain.*

Proof. The various assertions of Proposition 4.3 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 4.3.1. Analogous observations to the observations made in Remark 4.1.1, (i), (ii), (iii), may be made in the present case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. We leave the routine details to the reader. In this context, we note that the *cuspidal decomposition groups* that appear in the discussion of Remark 4.1.1, (ii), may be thought of as corresponding to the "$\mathcal{A}_p$" that appear in [AbsTopIII], Corollary 2.7, (e) — i.e., in the construction of $\overline{\mathcal{A}}_{\mathbb{U}_{\underline{v}}}$ — in the case of points p that belong to *"sufficiently small" neighborhoods of the cusps* that correspond to an element $t \in \mathrm{LabCusp}^{\pm}(\mathbb{U}_{\underline{v}})$.

Proposition 4.4. **(Frobenioid-theoretic Gaussian Monoids at Archimedean Primes)** *We continue to use the notation of Proposition 4.3. Let* $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}} = (^{\dagger}\mathcal{C}_{\underline{v}}, {}^{\dagger}\mathcal{D}_{\underline{v}}, {}^{\dagger}\kappa_{\underline{v}})$ *be the collection of data indexed by* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ *of a* Θ*-Hodge theater* $^{\dagger}\mathcal{HT}^{\Theta} = (\{^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}, {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$ *[cf. [IUTchI], Definition 3.6; [IUTchI], Example 3.4, (i)]. Write* $^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}} = (^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}, {}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}, {}^{\dagger}\tau^{\vdash}_{\underline{v}})$ *for the data indexed by* $\underline{v}$ *[cf. the discussion of [IUTchI], Example 3.4, (ii)] of the* $\mathcal{F}^{\vdash}$**-prime-strip** *determined by the* Θ*-Hodge theater* $^{\dagger}\mathcal{HT}^{\Theta}$ *[cf. [IUTchI], Definition 3.6; [IUTchI], Definition 5.2, (ii)]. Also, let us write* $^{\dagger}\mathbb{U}_{\underline{v}} \stackrel{\mathrm{def}}{=} {}^{\dagger}\mathcal{D}_{\underline{v}}$ *and* $^{\dagger}\mathbb{U}^{\pm}_{\underline{v}}$, $^{\dagger}\mathbb{U}^{\mathrm{cor}}_{\underline{v}}$ *for the* Aut*-holomorphic orbispaces associated to* $^{\dagger}\mathbb{U}_{\underline{v}}$ *that correspond to "*$\mathbb{U}^{\pm}_{\underline{v}}$*", "*$\mathbb{U}^{\mathrm{cor}}_{\underline{v}}$*", respectively [cf. the discussion of [IUTchI], Definition 6.1, (ii)].*

(i) **(Constant Monoids)** *In the notation of [IUTchI], Definition 3.6; [IUTchI], Example 3.4, (i) [cf. also the discussion of "*$^{\ddagger}\mathbb{M}_{v}$*" in [IUTchI], Definition 5.2, (viii)], the* **Kummer structure**

$$^{\dagger}\kappa_{\underline{v}} : \Psi_{^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathcal{O}^{\triangleright}(^{\dagger}\mathcal{C}_{\underline{v}}) \hookrightarrow \mathcal{A}_{^{\dagger}\mathcal{D}_{\underline{v}}}$$

on the category $^{\dagger}\mathcal{C}_{\underline{v}}$*, together with the tautological equality* $^{\dagger}\mathcal{D}_{\underline{v}} = {}^{\dagger}\mathbb{U}_{\underline{v}}$ *of* Aut*-holomorphic spaces, determine a* **unique isomorphism**

$$\Psi_{^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{cns}}(^{\dagger}\mathbb{U}_{\underline{v}})$$

of topological monoids.

(ii) **(Mono-analytic Semi-simplifications)** *Write* $\Psi_{^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathcal{O}^{\triangleright}(^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}})$ *[cf. [IUTchI], Example 3.4, (ii)]. Then there exists a* **unique** $\{\pm 1\}$**-orbit of isomorphisms of topological groups**

$$\Psi^{\times}_{^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{cns}}(^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})^{\times}$$

as well as a **unique isomorphism of monoids**

$$\Psi^{\mathbb{R}}_{^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \Psi_{^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}/\Psi^{\times}_{^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \quad \stackrel{\sim}{\to} \quad \Psi^{\mathbb{R}}_{\mathrm{cns}}(^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \mathbb{R}_{\geq 0}(^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$$

that maps the **distinguished element** *of* $\Psi^{\mathbb{R}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$ *determined by* $p_{\underline{v}} = e = 2.71828\ldots$ *[i.e., the element of the complex archimedean field that gives rise to* $\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}}$ *whose natural logarithm is equal to* 1*] to the distinguished element of* $\Psi^{\mathbb{R}}_{\mathrm{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$ *determined by* $\log^{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}}(p_{\underline{v}}) \in \mathbb{R}_{\geq 0}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$ *[cf. the first display of Proposition 4.3, (ii)]. In particular, if we write* $\Psi^{\mathrm{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \Psi^{\times}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \times \Psi^{\mathbb{R}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$ *for the* **"semi-simplified version"** *of* $\Psi_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$*, then the former distinguished element, together with the poly-isomorphism of the first display of the present (ii), determine a natural poly-isomorphism of topological monoids*

$$\Psi^{\mathrm{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \quad \stackrel{\sim}{\to} \quad \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$$

[cf. Proposition 4.3, (ii)] that is compatible with the natural splittings on the domain and codomain. Write $\Psi^{\mathrm{ss}}_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \Psi^{\mathrm{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$*; thus, it follows from the definitions that we have a natural isomorphism* $\Psi^{\mathrm{ss}}_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}} \stackrel{\sim}{\to} \Psi^{\mathrm{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}}$.

(iii) **(Labels,** $\mathbb{F}^{\rtimes\pm}_{l}$**-Symmetries, and Conjugate Synchronization)** *The isomorphism of (i) determines, for each* $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathbb{U}_{\underline{v}})$*, a collection of* **compatible isomorphisms**

$$(\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_t \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{cns}}({}^{\dagger}\mathbb{U}_{\underline{v}})_t$$

[cf. Corollary 3.6, (i), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*], as well as* **[**$\mathbb{F}^{\rtimes\pm}_{l}$**-]symmetrizing isomorphisms***, induced by the action of* $\mathbb{F}^{\rtimes\pm}_{l} \cong \mathrm{Gal}({}^{\dagger}\mathbb{U}^{\pm}_{\underline{v}}/{}^{\dagger}\mathbb{U}^{\mathrm{cor}}_{\underline{v}})$ *on the various* $\mathrm{Gal}({}^{\dagger}\mathbb{U}_{\underline{v}}/{}^{\dagger}\mathbb{U}^{\pm}_{\underline{v}})$*-orbits of cusps of* ${}^{\dagger}\mathbb{U}_{\underline{v}}$ *[cf. the definition of "*$\mathrm{LabCusp}^{\pm}(-)$*" in [IUTchI], Definition 6.1, (iii)], between the data indexed by distinct* $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathbb{U}_{\underline{v}})$*. Moreover, these symmetrizing isomorphisms determine [various* **diagonal submonoids***, as well as] an* **isomorphism of topological monoids**

$$(\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_0 \quad \stackrel{\sim}{\to} \quad (\Psi_{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}})_{\langle \mathbb{F}^{*}_{l} \rangle}$$

[cf. Corollary 3.6, (iii), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*].*

(iv) **(Theta and Gaussian Monoids)** *Write*

$$\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}}, \quad \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

for the **topological monoids** *equipped with natural* **splittings** *determined, respectively — via the isomorphisms of (i), (ii), and (iii) — by the monoids* $\Psi_{\mathrm{env}}({}^{\dagger}\mathbb{U}_{\underline{v}})$*,* $\Psi_{\mathrm{gau}}({}^{\dagger}\mathbb{U}_{\underline{v}})$ *and splittings of Proposition 4.3, (iv). Then the definition of the various monoids involved, together with the formal evaluation isomorphism of Proposition 4.3, (iv), gives rise to a collection of* **natural isomorphisms** *[cf. Corollary 3.6, (ii), in the case of* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*]*

$$\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}} \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{env}}({}^{\dagger}\mathbb{U}_{\underline{v}}) \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{gau}}({}^{\dagger}\mathbb{U}_{\underline{v}}) \quad \stackrel{\sim}{\to} \quad \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

— *which restrict to the identity or to the [restriction to "$(-)^\times$" of the] isomorphism of (i) [or its inverse] on the various copies of* $\Psi^\times_{{}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}}$, "$\Psi_{\mathrm{cns}}({}^\dagger\mathbb{U}_{\underline{v}})^\times$" *and are* **compatible** *with the various natural splittings.*

Proof. The various assertions of Proposition 4.4 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 4.4.1. In the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, one verifies immediately that one can make a remark analogous to Remark 4.2.1, (ii).

Corollary 4.5. (Group-theoretic Monoids Associated to Base-$\Theta^{\pm\mathrm{ell}}$-Hodge Theaters) *Let*

$$ {}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} = ({}^\dagger\mathfrak{D}_{\succ} \quad \xleftarrow{{}^\dagger\phi^{\Theta^\pm}_{\pm}} \quad {}^\dagger\mathfrak{D}_T \quad \xrightarrow{{}^\dagger\phi^{\Theta^{\mathrm{ell}}}_{\pm}} \quad {}^\dagger\mathcal{D}^{\circledcirc\pm}) $$

be a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-**Hodge theater** *[relative to the given initial* Θ*-data — cf. [IUTchI], Definition 6.4, (iii)] and*

$$ {}^\ddagger\mathfrak{D} = \{{}^\ddagger\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} $$

a $\mathcal{D}$*-prime-strip; here, we assume [for simplicity] that* ${}^\ddagger\mathcal{D}_{\underline{v}} = \mathcal{B}^{\mathrm{temp}}({}^\ddagger\Pi_{\underline{v}})^0$ *for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*. Also, we shall denote the* $\mathcal{D}^\vdash$*-prime-strip associated to — i.e., the mono-analyticization of — a* $\mathcal{D}$*-prime-strip [cf. [IUTchI], Definition 4.1, (iv)] by means of a superscript "*$\vdash$*" and assume [for simplicity] that* ${}^\ddagger\mathcal{D}^\vdash_{\underline{v}} = \mathcal{B}^{\mathrm{temp}}({}^\ddagger G_{\underline{v}})^0$ *for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$.

(i) **(Constant Monoids)** *There is a* **functorial algorithm** *in the* $\mathcal{D}$*-prime-strip* ${}^\ddagger\mathfrak{D}$ *for constructing the assignment* $\Psi_{\mathrm{cns}}({}^\ddagger\mathfrak{D})$ *given by*

$$ \underline{\mathbb{V}}^{\mathrm{non}} \ni \underline{v} \;\mapsto\; \Psi_{\mathrm{cns}}({}^\ddagger\mathfrak{D})_{\underline{v}} \;\stackrel{\mathrm{def}}{=}\; \Big\{ G_{\underline{v}}({}^\ddagger\Pi_{\underline{v}}) \;\curvearrowright\; \Psi_{\mathrm{cns}}({}^\ddagger\Pi_{\underline{v}}) \Big\} $$
$$ \underline{\mathbb{V}}^{\mathrm{arc}} \ni \underline{v} \;\mapsto\; \Psi_{\mathrm{cns}}({}^\ddagger\mathfrak{D})_{\underline{v}} \;\stackrel{\mathrm{def}}{=}\; \Psi_{\mathrm{cns}}({}^\ddagger\mathcal{D}_{\underline{v}}) $$

— *where the data in brackets "*$\{-\}$*" is to be regarded as being well-defined only up to a* ${}^\ddagger\Pi_{\underline{v}}$-**conjugacy indeterminacy** *— cf. Remark 4.2.1, (i), and Propositions 3.1, (ii); 4.1, (i); 4.3, (i).*

(ii) **(Mono-analytic Semi-simplifications)** *There is a* **functorial algorithm** *in the* $\mathcal{D}^\vdash$*-prime-strip* ${}^\ddagger\mathfrak{D}^\vdash$ *for constructing the assignment* $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\ddagger\mathfrak{D}^\vdash)$ *given by*

$$ \underline{\mathbb{V}}^{\mathrm{non}} \ni \underline{v} \;\mapsto\; \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\ddagger\mathfrak{D}^\vdash)_{\underline{v}} \;\stackrel{\mathrm{def}}{=}\; \Big\{ {}^\ddagger G_{\underline{v}} \;\curvearrowright\; \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\ddagger G_{\underline{v}}) \Big\} $$
$$ \underline{\mathbb{V}}^{\mathrm{arc}} \ni \underline{v} \;\mapsto\; \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\ddagger\mathfrak{D}^\vdash)_{\underline{v}} \;\stackrel{\mathrm{def}}{=}\; \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\ddagger\mathcal{D}^\vdash_{\underline{v}}) $$

— *where the data in brackets "*$\{-\}$*" is to be regarded as being well-defined only up to a* ${}^\ddagger G_{\underline{v}}$-**conjugacy indeterminacy***; each "*$\Psi^{\mathrm{ss}}_{\mathrm{cns}}(-)$*" is equipped with a* **splitting***, i.e., a direct product decomposition*

$$ \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\ddagger\mathfrak{D}^\vdash)_{\underline{v}} \;=\; \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\ddagger\mathfrak{D}^\vdash)^\times_{\underline{v}} \;\times\; \mathbb{R}_{\geq 0}({}^\ddagger\mathfrak{D}^\vdash)_{\underline{v}} $$

as the product of the submonoid of units and a submonoid with no nontrivial units [each of which is equipped with the action of a topological group when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$]; each submonoid $\mathbb{R}_{\geq 0}({}^{\ddagger}\mathfrak{D}^{\vdash})_{\underline{v}}$ is equipped with a **distinguished element**

$$\log^{{}^{\ddagger}\mathfrak{D}^{\vdash}}(p_{\underline{v}}) \in \mathbb{R}_{\geq 0}({}^{\ddagger}\mathfrak{D}^{\vdash})_{\underline{v}}$$

— cf. Remark 4.2.1, (i); Propositions 4.1, (ii), and 4.3, (ii). Here, if we regard ${}^{\ddagger}\mathfrak{D}^{\vdash}$ as an object functorially constructed from ${}^{\ddagger}\mathfrak{D}$, then there is a **functorial algorithm** *in the $\mathcal{D}$-prime-strip ${}^{\ddagger}\mathfrak{D}$ for constructing isomorphisms [of ind-topological abelian groups, equipped with the action of a topological group when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$]*

$$\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})^{\times}_{\underline{v}} \quad \overset{\sim}{\to} \quad \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})^{\times}_{\underline{v}}$$

for each $\underline{v} \in \underline{\mathbb{V}}$ — cf. Remark 4.2.1, (i); Propositions 4.1, (i), (ii), and 4.3, (i), (ii). Finally, there is a **functorial algorithm** *in the $\mathcal{D}^{\vdash}$-prime-strip ${}^{\ddagger}\mathfrak{D}^{\vdash}$ for constructing a Frobenioid*

$$\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$$

[cf. the Frobenioid "$\mathcal{D}^{\Vdash}_{\mathrm{mod}}$" of [IUTchI], Example 3.5, (iii)] isomorphic to the Frobenioid "$\mathcal{C}^{\Vdash}_{\mathrm{mod}}$" of [IUTchI], Example 3.5, (i), equipped with a **bijection**

$$\mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})) \overset{\sim}{\to} \underline{\mathbb{V}}$$

— where we write "$\mathrm{Prime}(-)$" for the set of primes associated to the divisor monoid of the Frobenioid in parentheses [cf. the discussion of [IUTchI], Example 3.5, (i)] — and, for each $\underline{v} \in \underline{\mathbb{V}}$, an **isomorphism of topological monoids** *${}^{\ddagger}\rho_{\mathcal{D}^{\Vdash},\underline{v}} : \Phi_{\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash}),\underline{v}} \overset{\sim}{\to} \mathbb{R}_{\geq 0}({}^{\ddagger}\mathfrak{D}^{\vdash})_{\underline{v}}$, where we write "$\Phi_{\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash}),\underline{v}}$" for the submonoid [isomorphic to $\mathbb{R}_{\geq 0}$] of the divisor monoid of $\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$ associated to $\underline{v}$ [cf. the isomorphism "$\rho^{\mathcal{D}}_{\underline{v}}$" of [IUTchI], Example 3.5, (iii)].*

(iii) **(Labels, $\mathbb{F}_l^{\rtimes\pm}$-Symmetries, and Conjugate Synchronization)** *Write*

$${}^{\dagger}\zeta_{\succ} : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ}) \overset{\sim}{\to} T$$

for the **bijection** *${}^{\dagger}\zeta_{\pm} \circ {}^{\dagger}\zeta_0^{\Theta^{\mathrm{ell}}} \circ ({}^{\dagger}\zeta_0^{\Theta^{\pm}})^{-1}$ arising from the bijections discussed in [IUTchI], Proposition 6.5, (i), (ii), (iii). Let $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ})$. In the following, we shall use analogous conventions to the conventions introduced in Corollary 3.5 concerning* **subscripted labels**. *Then the various* **local $\mathbb{F}_l^{\rtimes\pm}$-actions** *discussed in Corollary 3.5, (i), and Propositions 4.1, (iii); 4.3, (iii), induce* **isomorphisms** *between the* **labeled** *data*

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_t$$

[cf. (i)] for **distinct** *$t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ})$. We shall refer to these isomorphisms as* **[$\mathbb{F}_l^{\rtimes\pm}$-]symmetrizing isomorphisms**. *These symmetrizing isomorphisms are* **compatible**, *relative to ${}^{\dagger}\zeta_{\succ}$, with the* **$\mathbb{F}_l^{\rtimes\pm}$-symmetry** *of the associated $\mathcal{D}$-Θ^{ell}-bridge [cf. [IUTchI], Proposition 6.8, (i)] and determine* **diagonal submonoids**

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{\langle|\mathbb{F}_l|\rangle} \subseteq \prod_{|t|\in|\mathbb{F}_l|} \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{|t|}; \quad \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{\langle\mathbb{F}_l^{*}\rangle} \subseteq \prod_{|t|\in\mathbb{F}_l^{*}} \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{|t|}$$

— *where the "$\subseteq$'s" denote the various* **local** *inclusions of diagonal submonoids of Corollary 3.5, (i), and Propositions 4.1, (iii); 4.3, (iii) — as well as an* **isomorphism**

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_0 \quad \xrightarrow{\sim} \quad \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_{\langle \mathbb{F}_l^{*} \rangle}$$

constituted by the various corresponding **local** *isomorphisms of Corollary 3.5, (iii), and Propositions 4.1, (iii); 4.3, (iii).*

(iv) **(Local Theta and Gaussian Monoids)** *There is a* **functorial algorithm** *in the $\mathcal{D}$-prime-strip ${}^{\dagger}\mathfrak{D}_{\succ}$ for constructing assignments $\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{\succ})$, $\Psi_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}_{\succ})$, ${}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{\succ})$, ${}_{\infty}\Psi_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}_{\succ})$*

$$\underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ \Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{\succ})_{\underline{v}} \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{env}}({}^{\dagger}\mathcal{D}_{\succ,\underline{v}}); \quad \underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ \Psi_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}_{\succ})_{\underline{v}} \stackrel{\mathrm{def}}{=} \Psi_{\mathrm{gau}}({}^{\dagger}\mathcal{D}_{\succ,\underline{v}})$$

$$\underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{\succ})_{\underline{v}} \stackrel{\mathrm{def}}{=} {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathcal{D}_{\succ,\underline{v}})$$

$$\underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ {}_{\infty}\Psi_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}_{\succ})_{\underline{v}} \stackrel{\mathrm{def}}{=} {}_{\infty}\Psi_{\mathrm{gau}}({}^{\dagger}\mathcal{D}_{\succ,\underline{v}})$$

— *where the various* **local** *data are equipped with actions by* **topological groups** *when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ and* **splittings** *[for all $\underline{v} \in \underline{\mathbb{V}}$], as described in detail in Corollary 3.5, (ii), (iii), and Propositions 4.1, (iv); 4.3, (iv) [cf. also Remarks 4.2.1, (ii), (iii), (iv); 4.4.1] — as well as compatible* **evaluation isomorphisms**

$$\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{\succ}) \quad \xrightarrow{\sim} \quad \Psi_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}_{\succ}); \quad {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{\succ}) \quad \xrightarrow{\sim} \quad {}_{\infty}\Psi_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}_{\succ})$$

as described in detail in Corollary 3.5, (ii), and Propositions 4.1, (iv); 4.3, (iv).

(v) **(Global Realified Theta and Gaussian Frobenioids)** *There is a* **functorial algorithm** *in the $\mathcal{D}^{\vdash}$-prime-strip ${}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}$ for constructing a Frobenioid*

$$\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$$

— *namely, as a copy of the Frobenioid "$\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$" of (ii) above, multiplied by a* **formal symbol** *"$\log^{{}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}}(\underline{\underline{\Theta}})$" [cf. the constructions of Propositions 4.1, (iv), and 4.3, (iv), as well as of [IUTchI], Example 3.5, (ii)] — isomorphic to the Frobenioid "$\mathcal{C}^{\Vdash}_{\mathrm{mod}}$" of [IUTchI], Example 3.5, (i), equipped with a* **bijection** $\mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})) \xrightarrow{\sim} \underline{\mathbb{V}}$ *[cf. (ii) above] and, for each $\underline{v} \in \underline{\mathbb{V}}$, an* **isomorphism of topological monoids**

$$\Phi_{\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}),\underline{v}} \ \xrightarrow{\sim} \ \Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})^{\mathbb{R}}_{\underline{v}}$$

— *where we write "$\Phi_{\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}),\underline{v}}$" for the submonoid [isomorphic to $\mathbb{R}_{\geq 0}$] of the divisor monoid of $\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$ associated to $\underline{v}$; we write $\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})^{\mathbb{R}}_{\underline{v}}$ for the data [which, as is easily verified, is* **completely determined** *by ${}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}$ — cf. Propositions 4.1, (ii), (iv), and 4.3, (ii), (iv), as well as the evident analogues of these results at bad primes, i.e., in the spirit of Remark 4.2.1, (i)] obtained from $\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{\succ})_{\underline{v}}$ [cf. (iv) above] by replacing the ind-topological monoid portion of $\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{\succ})_{\underline{v}}$ by the realification of the quotient of this ind-topological monoid by its submonoid of units. There is a* **functorial algorithm** *in the $\mathcal{D}^{\vdash}$-prime-strip ${}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}$ for constructing a subcategory, equipped with a Frobenioid structure,*

$$\mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ}) \subseteq \prod_{j \in \mathbb{F}_l^{*}} \mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})_j$$

— [cf. Remark 4.5.2, (i), below concerning the subscript "j's"] whose divisor and rational function monoids are determined [relative to the divisor and rational function monoids of each factor in the product category of the display] by the "vector of ratios"

$$\left(\ldots, j^2 \cdot, \ldots \right)$$

whose components are indexed by $j \in \mathbb{F}_l^*$ [cf. Remark 4.5.4 below; the notational conventions of Propositions 4.1, (iv); 4.3, (iv)] — equipped with a **bijection** $\mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^\dagger\mathfrak{D}^{\vdash}_{\succ})) \xrightarrow{\sim} \underline{\mathbb{V}}$ *[cf. (ii) above] and, for each $\underline{v} \in \underline{\mathbb{V}}$, an* **isomorphism of topological monoids**

$$\Phi_{\mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^\dagger\mathfrak{D}^{\vdash}_{\succ}), \underline{v}} \xrightarrow{\sim} \Psi_{\mathrm{gau}}({}^\dagger\mathfrak{D}^{\vdash}_{\succ})^{\mathbb{R}}_{\underline{v}}$$

— where we write "$\Phi_{\mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^\dagger\mathfrak{D}^{\vdash}_{\succ}), \underline{v}}$" for the submonoid [isomorphic to $\mathbb{R}_{\geq 0}$] of the divisor monoid of $\mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^\dagger\mathfrak{D}^{\vdash}_{\succ})$ associated to $\underline{v}$; we write $\Psi_{\mathrm{gau}}({}^\dagger\mathfrak{D}^{\vdash}_{\succ})^{\mathbb{R}}_{\underline{v}}$ for the data [which, as is easily verified, is **completely determined** *by ${}^\dagger\mathfrak{D}^{\vdash}_{\succ}$ — cf. Propositions 4.1, (ii), (iv), and 4.3, (ii), (iv), as well as the evident analogues of these results at bad primes, i.e., in the spirit of Remark 4.2.1, (i)] obtained from $\Psi_{\mathrm{gau}}({}^\dagger\mathfrak{D}_{\succ})_{\underline{v}}$ [cf. (iv) above] by replacing the ind-topological monoid portion of $\Psi_{\mathrm{gau}}({}^\dagger\mathfrak{D}_{\succ})_{\underline{v}}$ by the realification of the quotient of this ind-topological monoid by its submonoid of units. Finally, there is a* **functorial algorithm** *in the $\mathcal{D}^{\vdash}$-prime-strip ${}^\dagger\mathfrak{D}^{\vdash}_{\succ}$ for constructing a* **global formal evaluation isomorphism** *of Frobenioids*

$$\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^\dagger\mathfrak{D}^{\vdash}_{\succ}) \xrightarrow{\sim} \mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^\dagger\mathfrak{D}^{\vdash}_{\succ})$$

which is **compatible***, relative to the* **bijections** *and* **local isomorphisms of topological monoids** *associated to these Frobenioids, with the* **local evaluation isomorphisms** *of (iv) above.*

Proof. The various assertions of Corollary 4.5 follow immediately from the definitions and the references quoted in the statements of these assertions. ○

Remark 4.5.1.

(i) Just as was done in Definition 3.8, one may interpret the various collections of monoids constructed in Corollary 4.5, (i), (iv) as collections of Frobenioids. That is to say, the collection of monoids discussed in Corollary 4.5, (i), gives rise to an *$\mathcal{F}$-prime-strip*, hence also to an associated *$\mathcal{F}^{\vdash}$-prime-strip*. In a similar vein, the theta and Gaussian monoids of Corollary 4.5, (iv), give rise to a well-defined $\mathcal{F}^{\vdash}$-*prime-strip* — up to an *indeterminacy*, at the $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [corresponding to the various $2l$*-th roots of the square of the theta function* and *"value-profiles"*], relative to *automorphisms of the split Frobenioid* at such $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ that induce the identity automorphism on the subcategory of *isometries* [cf. [FrdI], Theorem 5.1, (iii)] of the underlying category of the split Frobenioid — cf. Remark 4.10.1 below. On the other hand, as discussed in Remark 3.8.1, this Frobenioid-theoretic formulation is — by comparison to the original monoid-theoretic formulation — *technically ill-suited to discussions of* **conjugate synchronization**.

(ii) On the other hand, such technical complications do not occur if one restricts oneself to discussions of **realifications** — cf., e.g., the objects "$\mathbb{R}_{\geq 0}({}^{\ddagger}\mathfrak{D}^{\vdash})_{\underline{v}}$", "$\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$" discussed in Corollary 4.5, (ii). In general, Frobenioid-theoretic formulations are typically *technically easier* to work with than monoid-theoretic formulations when the associated *"Picard groups $Pic_{\Phi}(-)$"* [cf. [FrdI], Theorem 5.1; [FrdI], Theorem 6.4, (i); [IUTchI], Remark 3.1.5] contain *nontorsion* elements — i.e., at a more intuitive level, when there is a *nontrivial notion of the* **"degree"** *of a line bundle*. Examples of such Frobenioids include global arithmetic Frobenioids such as the Frobenioid "$\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$" of Corollary 4.5, (ii), as well as the *tempered Frobenioids* that appeared in Propositions 3.3 and 3.4; Corollary 3.6.

Remark 4.5.2.

(i) One may also construct **symmetrizing isomorphisms** as in Corollary 4.5, (iii), for versions *labeled* by $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ})$ of the *semi-simplifications* $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$, equipped with *splittings* and *distinguished elements*, and the *global realified Frobenioids* $\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$, equipped with *bijections* and *local isomorphisms of topological monoids*, as discussed in Corollary 4.5, (iii). We leave the routine details to the reader.

(ii) Just as was discussed in Remark 3.5.3, one may also consider **"multi-basepoint"** versions of the *symmetrizing isomorphisms* of Corollary 4.5, (iii) [cf. also the discussion of (i) above] — i.e., by passing to $\mathcal{D}$-Θ^{ell}-*bridges* or [holomorphic or mono-analytic] *capsules* or *processions* [cf. [IUTchI], Proposition 6.8, (i), (ii), (iii); [IUTchI], Proposition 6.9, (i), (ii)]. We leave the routine details to the reader.

Remark 4.5.3. Before proceeding, we pause to review the *significance* of the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** that gives rise to the *symmetrizing isomorphisms* of Corollary 4.5, (iii) [cf. Remark 3.5.2].

(i) First, we recall that the crucial **conjugate synchronization** established in Corollaries 3.5, (i); 4.5, (iii) [cf. also Propositions 4.1, (iii); 4.3, (iii)], is possible in the case of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry — but *not* in the case of the $\mathbb{F}_l^{\circledast}$-symmetry! — precisely because of the **connectedness**, at each $\underline{v} \in \underline{\mathbb{V}}$, of the local components involved — cf. the discussion of Remarks 2.6.1, (i); 2.6.2, (i); 3.5.2, (ii), as well as [IUTchI], Remark 6.12.4, (i), (ii). Here, we note in passing that although these remarks essentially only concern $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, similar [but, in some sense, easier!] remarks hold at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$. A related property of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry — which, again, is not satisfied by the $\mathbb{F}_l^{\circledast}$-symmetry! — is the **"geometric"** nature of the automorphisms that give rise to this symmetry [cf. Remark 3.5.2, (iii)].

(ii) One way to understand the significance of the *"single basepoint"* symmetrizing isomorphisms arising from the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is to *compare* these symmetrizing isomorphisms with the symmetrizing isomorphisms that arise from the various *"multi-basepoint"* [i.e., "multi-connected component"] symmetries discussed in Remarks 3.5.3; 4.5.2, (ii). That is to say:

(a) By comparison to the symmetries that arise from **mono-analytic capsules/processions**: the ring structure — i.e., *"arithmetic holomorphic *structure"* — that remains intact in the case of the symmetrizing isomorphisms of Corollary 4.5, (iii), will play an essential role in the theory of the $\mathfrak{log}$-wall [cf. the discussion of Remark 3.6.4, (i)], which we shall apply in [IUTchIII].

(b) By comparison to the symmetries that arise from **holomorphic capsules/processions**: the *"single basepoint"* that remains intact in the case of the symmetrizing isomorphisms of Corollary 4.5, (iii), is used not only to establish *conjugate synchronization*, but also to maintain a *bijective link* with the set of labels in "LabCusp$^{\pm}(-)$" [cf. the discussion of Remark 3.5.2]. Both conjugate synchronization and the bijective link with the set of labels play *crucial roles* in the theory of *Galois-theoretic theta evaluation* developed in §3 [cf. the various remarks following Corollaries 3.5, 3.6; Remark 3.8.3].

(c) By comparison to the symmetries that arise from the $\mathbb{F}_l^{\rtimes\pm}$-symmetries of **$\mathcal{D}$-$\Theta^{\mathrm{ell}}$-bridges**: Although the structure of a $\mathcal{D}$-Θ^{ell}-bridge allows one to maintain a *bijective link* with the set of labels in "LabCusp$^{\pm}(-)$" [cf. the discussion of [IUTchI], Remark 4.9.2, (i); [IUTchI], Remark 6.12.4, (i)], the *multi-basepoint* nature of the $\mathbb{F}_l^{\rtimes\pm}$-symmetries of $\mathcal{D}$-Θ^{ell}-bridges does *not* allow one to establish *conjugate synchronization* [cf. (b)].

(iii) Note that in order to *glue together* the various *local $\mathbb{F}_l^{\rtimes\pm}$-symmetries* of Corollary 3.5, (i), and Propositions 4.1, (iii); 4.3, (iii), so as to obtain the **global $\mathbb{F}_l^{\rtimes\pm}$-symmetry** of Corollary 4.5, (iii), it is necessary to make use of the *global portion* "${}^{\dagger}\mathcal{D}^{\circledcirc\pm}$" of the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater under consideration — i.e., by applying the theory of [IUTchI], Proposition 6.5 [cf. also [IUTchI], Remark 6.12.4, (iii)]. That is to say, the **global portion** *of the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater* under consideration plays, in particular, the role of

synchronizing *the $\pm$-indeterminacies* at each $\underline{v} \in \underline{\mathbb{V}}$.

Indeed, in some sense, this is precisely the content of [IUTchI], Proposition 6.5. In particular, the essential role played in this context by "${}^{\dagger}\mathcal{D}^{\circledcirc\pm}$" in synchronizing, or coordinating, the various local $\pm$-indeterminacies is one important underlying cause for the **profinite conjugacy indeterminacies** — i.e., *"$\widehat{\Delta}$"-conjugacy indeterminacies* — that occur in Corollaries 2.4, 2.5 — cf. the discussion of Remark 2.5.2. Thus, in summary, these local $\pm$-indeterminacies constitute one important reason for the need to apply the "complements on tempered coverings" developed in [IUTchI], §2, in the proof of Corollary 2.4 of the present paper.

Remark 4.5.4. In the situation of Corollary 4.5, (v), we remark that the Frobenioid $\mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$ may be thought of as a sort of *"weighted diagonal"*, relative to the weights determined by the vector "$(\dots, j^2\cdot, \dots)$". That is to say, at a more concrete level, the *divisor monoid* (respectively, *rational function monoid*) of this Frobenioid consists of elements of the form

$$(1^2\cdot\phi,\ 2^2\cdot\phi, \dots,\ j^2\cdot\phi, \dots) \quad \text{(respectively, } (1^2\cdot\beta,\ 2^2\cdot\beta, \dots,\ j^2\cdot\beta, \dots))$$

— where ϕ (respectively, β) is an element of the divisor monoid (respectively, rational function monoid) associated to the Frobenioid $\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$.

Corollary 4.6. (Frobenioid-theoretic Monoids Associated to $\Theta^{\pm\text{ell}}$-Hodge Theaters) *Let*

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}} = (^{\dagger}\mathfrak{F}_{\succ} \quad \overset{^{\dagger}\psi^{\Theta^{\pm}}_{\pm}}{\longleftarrow} \quad ^{\dagger}\mathfrak{F}_{T} \quad \overset{^{\dagger}\psi^{\Theta^{\text{ell}}}_{\pm}}{\longrightarrow} \quad ^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

be a $\Theta^{\pm\text{ell}}$-**Hodge theater** *[relative to the given initial* Θ*-data — cf. [IUTchI], Definition 6.11, (iii)] and*

$$^{\ddagger}\mathfrak{F} = \{^{\ddagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

an $\mathcal{F}$*-prime-strip; here, we assume [for simplicity] that the* $\mathcal{D}$-$\Theta^{\pm\text{ell}}$*-Hodge theater associated to* $^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}}$ *[cf. [IUTchI], Definition 6.11, (iii)] is the* $\mathcal{D}$-$\Theta^{\pm\text{ell}}$*-Hodge theater* $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}}$ *of Corollary 4.5, and that the* $\mathcal{D}$*-prime-strip associated to* $^{\ddagger}\mathfrak{F}$ *[cf. [IUTchI], Remark 5.2.1, (i)] is the* $\mathcal{D}$*-prime-strip* $^{\ddagger}\mathfrak{D}$ *of Corollary 4.5. Also, we shall denote the* $\mathcal{F}^{\vdash}$*-prime-strip associated to — i.e., the mono-analyticization of — an* $\mathcal{F}$*-prime-strip [cf. [IUTchI], Definition 5.2.1, (ii)] by means of a superscript "*$\vdash$*".*

(i) **(Constant Monoids)** *There is a* **functorial algorithm** *in the* $\mathcal{F}$*-prime-strip* $^{\ddagger}\mathfrak{F}$ *for constructing the assignment* $\Psi_{\text{cns}}(^{\ddagger}\mathfrak{F})$ *given by*

$$\underline{\mathbb{V}}^{\text{non}} \ni \underline{v} \;\mapsto\; \Psi_{\text{cns}}(^{\ddagger}\mathfrak{F})_{\underline{v}} \;\overset{\text{def}}{=}\; \left\{G_{\underline{v}}(^{\ddagger}\Pi_{\underline{v}}) \;\curvearrowright\; \Psi_{^{\ddagger}\mathcal{F}_{\underline{v}}}\right\}$$

$$\underline{\mathbb{V}}^{\text{arc}} \ni \underline{v} \;\mapsto\; \Psi_{\text{cns}}(^{\ddagger}\mathfrak{F})_{\underline{v}} \;\overset{\text{def}}{=}\; \Psi_{^{\ddagger}\mathcal{F}_{\underline{v}}}$$

— where the data in brackets "$\{-\}$*" is to be regarded as being well-defined only up to a* $^{\ddagger}\Pi_{\underline{v}}$**-conjugacy indeterminacy** *— cf. [IUTchI], Definition 5.2, (i); Propositions 3.3, (ii) [i.e., where we take "*$^{\dagger}\mathcal{C}_{\underline{v}}$*" to be* $^{\ddagger}\mathcal{F}_{\underline{v}}$*]; 4.2, (i); 4.4, (i). We shall write*

$$\Psi_{\text{cns}}(^{\ddagger}\mathfrak{F}) \quad \overset{\sim}{\to} \quad \Psi_{\text{cns}}(^{\ddagger}\mathfrak{D})$$

for the collection of **isomorphisms** *of data indexed by* $\underline{v} \in \underline{\mathbb{V}}$ *determined by the* **"Kummer-theoretic"** *isomorphisms of Propositions 3.3, (ii) [i.e., where we take "*$^{\dagger}\mathcal{C}_{\underline{v}}$*" to be* $^{\ddagger}\mathcal{F}_{\underline{v}}$ *and apply the conventions discussed in Remark 4.2.1., (i); cf. also Proposition 1.3, (ii), (iii)]; 4.2, (i); 4.4, (i).*

(ii) **(Mono-analytic Semi-simplifications)** *There is a* **functorial algorithm** *in the* $\mathcal{F}^{\vdash}$*-prime-strip* $^{\ddagger}\mathfrak{F}^{\vdash}$ *for constructing the assignment* $\Psi^{\text{ss}}_{\text{cns}}(^{\ddagger}\mathfrak{F}^{\vdash})$ *given by*

$$\underline{\mathbb{V}} \ni \underline{v} \quad \mapsto \quad \Psi^{\text{ss}}_{\text{cns}}(^{\ddagger}\mathfrak{F}^{\vdash})_{\underline{v}} \;\overset{\text{def}}{=}\; \Psi^{\text{ss}}_{^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}}$$

— where we regard each "$\Psi^{\text{ss}}_{^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}}$*" as being equipped with its natural* **splitting** *and, when* $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$*, its associated* **distinguished element***; for* $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$*, "*$\Psi^{\text{ss}}_{^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}}$*" is to be regarded as being well-defined only up to a* $^{\dagger}G_{\underline{v}}$**-conjugacy indeterminacy** *— cf. Remark 4.2.1, (i), and Propositions 4.2, (ii); 4.4, (ii). We shall write*

$$\Psi^{\text{ss}}_{\text{cns}}(^{\ddagger}\mathfrak{F}^{\vdash}) \quad \overset{\sim}{\to} \quad \Psi^{\text{ss}}_{\text{cns}}(^{\ddagger}\mathfrak{D}^{\vdash})$$

for the collection of **isomorphisms** *of data indexed by* $\underline{v} \in \underline{\mathbb{V}}$ *determined by the* **"Kummer-theoretic"** *isomorphisms of Propositions 4.2, (ii); 4.4, (ii) — cf. also Remark 4.2.1, (i); Corollary 4.5, (ii). Now recall the* $\mathcal{F}^{\Vdash}$*-prime-strip*

$$^{\ddagger}\mathfrak{F}^{\Vdash} \;=\; (^{\ddagger}\mathcal{C}^{\Vdash},\; \text{Prime}(^{\ddagger}\mathcal{C}^{\Vdash}) \overset{\sim}{\to} \underline{\mathbb{V}},\; ^{\ddagger}\mathfrak{F}^{\vdash},\; \{^{\ddagger}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

associated to ${}^{\ddagger}\mathfrak{F}$ *in [IUTchI], Remark 5.2.1, (ii). Then, in the notation of Corollary 4.5, (ii); [IUTchI], Remark 5.2.1, (ii), there is an* **isomorphism of Frobenioids**

$${}^{\ddagger}\mathcal{C}^{\Vdash} \quad \xrightarrow{\sim} \quad \mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$$

that is **uniquely** *determined by the condition that it be* **compatible** *with the respective* **bijections** $\mathrm{Prime}(-) \xrightarrow{\sim} \underline{\mathbb{V}}$ *and* **local isomorphisms of topological monoids** *for each* $\underline{v} \in \underline{\mathbb{V}}$*, relative to the above collection of isomorphisms* $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F}^{\vdash}) \xrightarrow{\sim} \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})$*. Finally, there is a* **functorial algorithm** *for constructing from the* $\mathcal{F}^{\Vdash}$*-prime-strip* ${}^{\ddagger}\mathfrak{F}^{\Vdash}$ *[recalled above] the isomorphism* ${}^{\ddagger}\mathcal{C}^{\Vdash} \xrightarrow{\sim} \mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$ *[of the preceding display] and the [necessarily compatible] collection of isomorphisms* $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F}^{\vdash}) \xrightarrow{\sim} \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})$ *[cf. Remark 4.6.1 below].*

(iii) **(Labels,** $\mathbb{F}_l^{\rtimes\pm}$**-Symmetries, and Conjugate Synchronization)** *In the notation of Corollary 4.5, (iii), the collection of isomorphisms of (i) determines, for each* $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ})$*, a collection of* **compatible isomorphisms**

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_t \quad \xrightarrow{\sim} \quad \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}_{\succ})_t$$

— where the ${}^{\dagger}\Pi_{\underline{v}}$**-conjugacy indeterminacy** *at each* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ *[cf. (i)] is* **independent** *of* $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ})$ *— as well as* **[**$\mathbb{F}_l^{\rtimes\pm}$**-]symmetrizing isomorphisms***, induced by the various* **local** $\mathbb{F}_l^{\rtimes\pm}$**-actions** *discussed in Corollary 3.6, (i), and Propositions 4.2, (iii); 4.4, (iii), between the data indexed by distinct* $t \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ})$*. Moreover, these symmetrizing isomorphisms are* **compatible***, relative to* ${}^{\dagger}\zeta_{\succ}$ *[cf. Corollary 4.5, (iii)], with the* $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** *of the associated* $\mathcal{D}$*-*Θ^{ell}*-bridge [cf. [IUTchI], Proposition 6.8, (i)] and determine [various* **diagonal submonoids***, as well as] an* **isomorphism**

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_0 \quad \xrightarrow{\sim} \quad \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_{\langle \mathbb{F}_l^{*} \rangle}$$

constituted by the various corresponding **local** *isomorphisms of Corollary 3.6, (iii), and Propositions 4.2, (iii); 4.4, (iii).*

(iv) **(Local Theta and Gaussian Monoids)** *Let*

$$({}^{\dagger}\mathfrak{F}_J \quad \xrightarrow{{}^{\dagger}\psi^{\Theta}_{*}} \quad {}^{\dagger}\mathfrak{F}_{>} \quad \dashrightarrow \quad {}^{\dagger}\mathcal{HT}^{\Theta})$$

be a Θ**-bridge** *[relative to the given initial* Θ*-data — cf. [IUTchI], Definition 5.5, (ii)] which is* **glued** *to the* $\Theta^{\pm}$**-bridge** *associated to the* $\Theta^{\pm\mathrm{ell}}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$ *via the functorial algorithm of [IUTchI], Proposition 6.7 [so* $J = T^{*}$*] — cf. the discussion of [IUTchI], Remark 6.12.2, (i). Then there is a* **functorial algorithm** *in the* Θ**-bridge** *of the above display, equipped with its* **gluing** *to the* $\Theta^{\pm}$*-bridge associated to* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$*, for constructing assignments* $\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})$*,* $\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})$*,* ${}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})$*,* ${}_{\infty}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})$ *[where we make a slight abuse of the notation "*${}^{\dagger}\mathcal{HT}^{\Theta}$*"]*

$$\underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ \Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} \stackrel{\mathrm{def}}{=} \Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}}; \quad \underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} \stackrel{\mathrm{def}}{=} \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$
$$\underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} \stackrel{\mathrm{def}}{=} {}_{\infty}\Psi_{{}^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}}$$
$$\underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} \stackrel{\mathrm{def}}{=} {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

— *where the various* **local** *data are equipped with actions by* **topological groups** *when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ *and* **splittings** *[for all* $\underline{v} \in \underline{\mathbb{V}}$*], as described in detail in Corollary 3.6, (ii), (iii), and Propositions 4.2, (iv); 4.4, (iv) [cf. also Remarks 4.2.1, (ii); 4.4.1] — as well as compatible* **evaluation isomorphisms**

$$\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta}) \quad \overset{\sim}{\to} \quad \Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \quad \overset{\sim}{\to} \quad \Psi_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}_{>}) \quad \overset{\sim}{\to} \quad \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta});$$

$$_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta}) \quad \overset{\sim}{\to} \quad {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \quad \overset{\sim}{\to} \quad {}_{\infty}\Psi_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}_{>}) \quad \overset{\sim}{\to} \quad {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})$$

as described in detail in Corollary 3.6, (ii) [cf. also Remark 4.2.1, (iv); the left-hand portion of the first display of Proposition 3.4, (i); the first display of Proposition 3.7, (i)], and Propositions 4.2, (iv); 4.4, (iv) [cf. also Corollary 4.5, (iv)].

(v) **(Global Realified Theta and Gaussian Frobenioids)** *By applying — i.e., in the fashion of the constructions of Propositions 4.2, (iv); 4.4, (iv) — both* **labeled** *[as in (iii) — cf. Remark 4.6.2, (ii), below] and non-labeled versions of the isomorphism "*${}^{\ddagger}\mathcal{C}^{\Vdash} \overset{\sim}{\to} \mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$*" of (ii) to the global Frobenioids "*$\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$*", "*$\mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\succ})$*" constructed in Corollary 4.5, (v), one obtains a* **functorial algorithm** *in the* **Θ-bridge** *of the first display of (iv), equipped with its* **gluing** *to the $\Theta^{\pm}$-bridge associated to* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$*, for constructing* **Frobenioids**

$$\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}), \quad \mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})$$

— *where again we make a slight abuse of the notation "*${}^{\dagger}\mathcal{HT}^{\Theta}$*"; we note in passing that the construction of "*$\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta})$*" is essentially similar to the construction of "*$\mathcal{C}^{\Vdash}_{\mathrm{tht}}$*" in [IUTchI], Example 3.5, (ii) — together with* **bijections** $\mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta})) \overset{\sim}{\to} \underline{\mathbb{V}}$, $\mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})) \overset{\sim}{\to} \underline{\mathbb{V}}$ *and* **isomorphisms of topological monoids**

$$\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}),\underline{v}} \overset{\sim}{\to} \Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathbb{R}}_{\underline{v}}; \quad \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}),\underline{v}} \overset{\sim}{\to} \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathbb{R}}_{\underline{v}}$$

[cf. the notational conventions of Corollary 4.5, (v)] for each $\underline{v} \in \underline{\mathbb{V}}$*, as well as* **evaluation isomorphisms**

$$\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}) \quad \overset{\sim}{\to} \quad \mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>}) \quad \overset{\sim}{\to} \quad \mathcal{D}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>}) \quad \overset{\sim}{\to} \quad \mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})$$

— *i.e., in the fashion of the constructions of Propositions 4.2, (iv); 4.4, (iv), by "conjugating" the evaluation isomorphism of Corollary 4.5, (v), by the isomorphism "*${}^{\ddagger}\mathcal{C}^{\Vdash} \overset{\sim}{\to} \mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$*" of (ii) — which are* **compatible***, relative to the* **local isomorphisms of topological monoids** *for each* $\underline{v} \in \underline{\mathbb{V}}$ *discussed above, with the* **local evaluation isomorphisms** *of (iv).*

Proof. The various assertions of Corollary 4.6 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 4.6.1. One verifies easily that, in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the poly-isomorphism $\Psi^{\mathrm{ss}}_{{}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}} \overset{\sim}{\to} \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}G_{\underline{v}})$ of Proposition 4.2, (ii) [cf. also Remark 4.2.1, (i)], may be *reconstructed algorithmically* from ${}^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$. By contrast, in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, it is *not possible* to reconstruct algorithmically [the non-unit portion of] the corresponding poly-isomorphism $\Psi^{\mathrm{ss}}_{{}^\dagger\mathcal{F}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} \Psi^{\mathrm{ss}}_{\mathrm{cns}}(\mathcal{D}^{\vdash}_{\underline{v}})$ of Proposition 4.4, (ii), from ${}^\dagger\mathcal{F}^{\vdash}_{\underline{v}}$. That is to say, in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, the *distinguished element* of $\Psi^{\mathrm{ss}}_{{}^\dagger\mathcal{F}^{\vdash}_{\underline{v}}}$ [i.e., of $\Psi^{\mathbb{R}}_{{}^\dagger\mathcal{F}^{\vdash}_{\underline{v}}}$] is *not preserved* by arbitrary automorphisms of ${}^\dagger\mathcal{F}^{\vdash}_{\underline{v}}$. On the other hand, in the context of Corollary 4.6, (ii), if one reconstructs *both* $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\ddagger\mathfrak{F}^{\vdash}) \xrightarrow{\sim} \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\ddagger\mathfrak{D}^{\vdash})$ and ${}^\ddagger\mathcal{C}^{\Vdash} \xrightarrow{\sim} \mathcal{D}^{\Vdash}({}^\ddagger\mathfrak{D}^{\vdash})$ in a *compatible* fashion, then the distinguished elements at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ may be computed [in the evident fashion] from the distinguished elements at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, together with the *structure of the global Frobenioids* ${}^\ddagger\mathcal{C}^{\Vdash}$, $\mathcal{D}^{\Vdash}({}^\ddagger\mathfrak{D}^{\vdash})$, i.e., by thinking of these global Frobenioids as *"devices for currency exchange"* between the various *"local currencies"* constituted by the divisor monoids at the various $\underline{v} \in \underline{\mathbb{V}}$ [cf. [IUTchI], Remark 3.5.1, (ii)].

Remark 4.6.2.

(i) Similar observations to the observations made in Remark 4.5.1, (i), concerning the content of Corollary 4.5, (i), (iv), may be made in the case of Corollary 4.6, (i), (iv).

(ii) Similar observations to the observations made in Remark 4.5.2, (i), (ii), concerning the content of Corollary 4.5, (iii), may be made in the case of Corollary 4.6, (iii).

Corollary 4.7. (Group-theoretic Monoids Associated to Base-ΘNF-Hodge Theaters) *Let*

$$ {}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} = ({}^\dagger\mathcal{D}^{\circledcirc} \quad \xleftarrow{{}^\dagger\phi^{\mathrm{NF}}_{\circledast}} \quad {}^\dagger\mathfrak{D}_J \quad \xrightarrow{{}^\dagger\phi^{\Theta}_{\circledast}} \quad {}^\dagger\mathfrak{D}_{>}) $$

be a $\mathcal{D}$**-ΘNF-Hodge theater** *[cf. [IUTchI], Definition 4.6, (iii)] which is* **glued** *to the* $\mathcal{D}$**-$\Theta^{\pm\mathrm{ell}}$-Hodge theater** ${}^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ *of Corollary 4.5 via the functorial algorithm of [IUTchI], Proposition 6.7 [so $J = T^{\circledast}$] — cf. the discussion of [IUTchI], Remark 6.12.2, (i), (ii).*

(i) **(Non-realified Global Structures)** *There is a* **functorial algorithm** *in the category ${}^\dagger\mathcal{D}^{\circledcirc}$ for constructing the* **morphism**

$$ {}^\dagger\mathcal{D}^{\circledcirc} \to {}^\dagger\mathcal{D}^{\circledast} $$

[i.e., a "category-theoretic version" of the natural morphism of hyperbolic orbicurves $\underline{C}_K \to C_{F_{\mathrm{mod}}}$] of [IUTchI], Example 5.1, (i), the **monoid/field/pseudo-monoid** *equipped with natural $\pi_1({}^\dagger\mathcal{D}^{\circledast})$-/$(\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^{\circledast}) \twoheadrightarrow)\pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^{\circledast})$-actions*

$$ \pi_1({}^\dagger\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc}), \quad \pi_1({}^\dagger\mathcal{D}^{\circledast}) \curvearrowright \overline{\mathbb{M}}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc}), \quad \pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^\dagger\mathcal{D}^{\circledcirc}) $$

— which are well-defined up to $\pi_1({}^\dagger\mathcal{D}^{\circledast})$**-/**$\pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^{\circledast})$**-conjugacy indeterminacies** *— of [IUTchI], Example 5.1, (i), the* **submonoids/subfield/subset** *of $\pi_1({}^\dagger\mathcal{D}^{\circledast})$-/$\pi_1^{\mathrm{rat}/\kappa\text{-sol}}({}^\dagger\mathcal{D}^{\circledast})$-/$\pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^{\circledast})$-invariants*

$$ \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^\dagger\mathcal{D}^{\circledcirc}) \quad \subseteq \quad (\pi_1^{\kappa\text{-sol}}({}^\dagger\mathcal{D}^{\circledast}) \curvearrowright) \quad \mathbb{M}^{\circledast}_{\mathrm{sol}}({}^\dagger\mathcal{D}^{\circledcirc}) \quad \subseteq \quad \mathbb{M}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc}), $$

$$\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \quad \subseteq \quad \overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}), \qquad \mathbb{M}^{\circledast}_{\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}) \quad \subseteq \quad \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

[cf. [IUTchI], Example 5.1, (i)], the ["corresponding"] **Frobenioids**

$$\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \quad \subseteq \quad \mathcal{F}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \quad \leftarrow \quad \mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

— where we write $\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})$, $\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$ for the categories "${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}$", "${}^{\dagger}\mathcal{F}^{\circledcirc}$" obtained in [IUTchI], Example 5.1, (iii), by taking the "${}^{\dagger}\mathcal{F}^{\circledast}$" of loc. cit. to be $\mathcal{F}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$, and, by abuse of notation, we regard the Frobenioid $\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})$ as being equipped with a natural **bijection**

$$\mathrm{Prime}(\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})) \quad \xrightarrow{\sim} \quad \underline{\mathbb{V}}$$

[cf. the final portion of [IUTchI], Example 5.1, (v)] — of [IUTchI], Example 5.1, (ii), (iii), and the natural **realification functor**

$$\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \quad \to \quad \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

[cf. [IUTchI], Example 5.1, (vii); [FrdI], Proposition 5.3].

(ii) **(Labels and $\mathbb{F}^{\rtimes}_{l}$-Symmetry)** *Recall the* **bijection**

$${}^{\dagger}\zeta_{\rtimes} : \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} J \quad (\xrightarrow{\sim} \mathbb{F}^{\rtimes}_{l})$$

of [IUTchI], Proposition 4.7, (iii). In the following, we shall use analogous conventions to the conventions applied in Corollary 4.5 concerning **subscripted labels**. *Let $j \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$. Then there is a* **functorial algorithm** *in the category ${}^{\dagger}\mathcal{D}^{\circledcirc}$ for constructing an* $\mathcal{F}$**-prime-strip**

$$\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_{j}$$

— which is well-defined up to **isomorphism** *— from $\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$ [cf. [IUTchI], Example 5.4, (iv), where we take the "δ" of loc. cit. to be j]. Moreover, the natural poly-action of $\mathbb{F}^{\rtimes}_{l}$ on ${}^{\dagger}\mathcal{D}^{\circledcirc}$ [cf. [IUTchI], Example 4.3, (iv)] induces* **isomorphisms** *between the* **labeled** *data*

$$\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_{j}, \quad \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{j}, \quad \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{j},$$

$$\{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\mathrm{sol}}({}^{\dagger}\mathcal{D}^{\circledcirc})\}_{j}, \quad \{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})\}_{j},$$

$$\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{j} \to \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{j}$$

[cf. (i)] for **distinct** *$j \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$ [cf. Remark 4.7.2 below]. We shall refer to these isomorphisms as* $[\mathbb{F}^{\rtimes}_{l}$**-]symmetrizing isomorphisms**. *Here, the objects equipped with $\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})(\twoheadrightarrow \pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}))$-actions are to be regarded as being subject to* **independent** $\pi_1^{\mathrm{rat}/\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledcirc})$**-conjugacy indeterminacies** *for* **distinct** *j, together with a* **single** $(\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \twoheadrightarrow)\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast})$**-conjugacy indeterminacy** *that is* **independent** *of j [cf. the discussion of the final portion of [IUTchI], Example 5.1, (i)]. These symmetrizing isomorphisms are* **compatible**, *relative to ${}^{\dagger}\zeta_{\rtimes}$, with the* $\mathbb{F}^{\rtimes}_{l}$**-symmetry** *of the associated $\mathcal{D}$-NF-bridge [cf. [IUTchI], Proposition*

4.9, (i)] and determine **diagonal $\mathcal{F}$-prime-strips/submonoids/subrings/sub-pseudo-monoids** *[equipped with a group action subject to conjugacy indeterminacies as described above]/***subcategories** *[cf. Remark 4.7.2 below]*

$$(-)_{\langle \mathbb{F}_l^{\circledast} \rangle} \quad \subseteq \quad \prod_{j \in \mathbb{F}_l^{\circledast}} (-)_j$$

— where "$(-)_{\ldots}$" may be taken to be $\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_{\ldots}$ *[cf. the discussion of [IUTchI], Example 5.4, (i)],* $\mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{\ldots}$, $\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{\ldots}$, $\{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\mathrm{sol}}({}^{\dagger}\mathcal{D}^{\circledcirc})\}_{\ldots}$, $\{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})\}_{\ldots}$, $\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{\ldots}$, *or* $\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{\ldots}$ *[cf. the discussion of [IUTchI], Example 5.1, (vii)]. [Here, the notion of a "diagonal $\mathcal{F}$-prime-strip", of a "diagonal sub-pseudo-monoid equipped with a group action subject to conjugacy indeterminacies as described above", or of a "diagonal subcategory" is to be understood in a* **purely formal** *sense, i.e., as a purely formal notational shorthand for the $\mathbb{F}_l^{\circledast}$-symmetrizing isomorphisms discussed above.]*

(iii) **(Localization Functors and Realified Global Structures)** *Let $j \in$* $\mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$*. For simplicity, write* ${}^{\dagger}\mathfrak{D}_j = \{{}^{\dagger}\mathcal{D}_{\underline{v}_j}\}_{\underline{v} \in \underline{\mathbb{V}}}$, ${}^{\dagger}\mathfrak{D}^{\vdash}_j = \{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}_j}\}_{\underline{v} \in \underline{\mathbb{V}}}$ *for the $\mathcal{D}$-, $\mathcal{D}^{\vdash}$-prime-strips associated [cf. [IUTchI], Definition 4.1, (iv); [IUTchI], Remark 5.2.1, (i)] to the $\mathcal{F}$-prime-strip* $\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j$*. Then there is a* **functorial algorithm** *in the category* ${}^{\dagger}\mathcal{D}^{\circledcirc}$ *for constructing* [1-]**compatible** *collections of* **"localization" functors/poly-morphisms** *[up to isomorphism]*

$$\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j \quad \to \quad \mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j, \qquad \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j \quad \to \quad (\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j)^{\mathbb{R}}$$

$$\Big\{\{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})\}_j \quad \to \quad \mathbb{M}_{\infty\kappa v}({}^{\dagger}\mathcal{D}_{\underline{v}_j}) \subseteq \mathbb{M}_{\infty\kappa\times v}({}^{\dagger}\mathcal{D}_{\underline{v}_j})\Big\}_{\underline{v} \in \underline{\mathbb{V}}}$$

— where the superscript "$\mathbb{R}$" denotes the **realification** *— as in the discussion of [IUTchI], Example 5.4, (iv), (vi) [cf. also [IUTchI], Definition 5.2, (v), (vii)], together with a natural* **isomorphism of Frobenioids**

$$\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_j) \quad \xrightarrow{\sim} \quad \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$$

[cf. the notation of Corollary 4.5, (ii)] and, for each $\underline{v} \in \underline{\mathbb{V}}$, a natural **isomorphism of topological monoids**

$$\mathbb{R}_{\geq 0}({}^{\dagger}\mathfrak{D}^{\vdash}_j)_{\underline{v}} \quad \xrightarrow{\sim} \quad \Psi_{(\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j)^{\mathbb{R}}, \underline{v}}$$

— where "$\Psi_{(\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j)^{\mathbb{R}}, \underline{v}}$" denotes the **divisor monoid** *associated to the Frobenioid that constitutes* $(\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j)^{\mathbb{R}}$ *at $\underline{v}$ — which are* **compatible** *[cf. Remark 4.7.1 below] with the respective* **bijections** *involving* "Prime$(-)$" *and the respective* **local isomorphisms of topological monoids** *[cf. the arrow* $\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j \to (\mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j)^{\mathbb{R}}$ *discussed above; Corollary 4.5, (ii)]. Finally, all of these structures are* **compatible** *with the respective* **$\mathbb{F}_l^{\circledast}$-symmetrizing isomorphisms** *[cf. (ii)].*

Proof. The various assertions of Corollary 4.7 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 4.7.1. Similar observations to the observations made in Remark 4.5.2, (i), (ii), concerning the $\mathbb{F}_l^{\rtimes\pm}$-symmetrizing isomorphisms of Corollary 4.5, (iii), may be made in the case of the $\mathbb{F}_l^{\circledast}$-*symmetrizing isomorphisms* of Corollary 4.7, (ii).

Remark 4.7.2. In the context of Corollary 4.7, (ii), we recall from Remarks 3.5.2, (iii); 4.5.3, (i), that unlike the case with $\mathbb{F}_l^{\rtimes\pm}$-symmetry, in the case of $\mathbb{F}_l^{\circledast}$**-symmetry**, it is **not** *possible* to establish the sort of **conjugate synchronization** given in Corollary 4.5, (iii), since the $\mathbb{F}_l^{\circledast}$-symmetry involves — i.e., more precisely, arises from *conjugation by elements with nontrivial image* in — the **arithmetic** portion [i.e., the absolute Galois group of the base field] of the global arithmetic fundamental groups involved [cf. the discussion of how *"G_K-conjugacy indeterminacies give rise to $G_{\underline{v}}$-conjugacy indeterminacies"* in Remark 2.5.2, (iii)]. It is precisely this state of affairs that obliges us, in Corollary 4.7, (ii), to work with

(a) $\mathcal{F}$**-prime-strips**, as opposed to the corresponding *ind-topological monoids with Galois actions* as in Corollary 4.5, (iii), and with

(b) the various objects introduced in Corollary 4.7, (i), that are equipped with **sub-/super-scripts**

"mod", "sol", "κ-sol", or "$_\infty\kappa$"

— corresponding to "F_{mod}", "F_{sol}", "$\pi_1^{\kappa\text{-sol}}(-)$", or "$_\infty\kappa$-coric rational functions" — **or** [as in the case of "$\pi_1^{\mathrm{rat}/\kappa\text{-sol}}(-)$"] are only defined up to **certain conjugacy indeterminacies**, as opposed to the objects not equipped with such subscripts or not subject to such conjugacy indeterminacies.

That is to say, both (a) and (b) allow one to *ignore* the various *independent* — i.e., *non-synchronizable* — *conjugacy indeterminacies* that occur at the various distinct labels as a consequence of the **single basepoint** with respect to which one considers both the *labels* and the *labeled objects* [cf. the discussion of Remark 3.5.2, (ii)]. Here, it is also useful to observe that by working with the various objects introduced in Corollary 4.7, (i), that are equipped with a *sub-/super-script "mod", "sol", or "κ-sol"* — i.e., on which the various conjugacy indeterminacies involved act in a *synchronized* fashion — one may construct the various *diagonal subcategories* associated to the corresponding *Frobenioids* in a fashion in which one is not obliged to contend with the technical subtleties that arise from independent conjugacy indeterminacies at distinct labels [cf. the discussion of "Galois-invariants/Galois-orbits" in Remark 3.8.3, (ii)]. In [IUTchIII], the **ring structure** on these objects equipped with a subscript "mod" will be applied as a sort of **translation apparatus** between **"⊞-line bundles"** [i.e., arithmetic line bundles thought of as *additive modules* with additional structure] and **"⊠-line bundles"** [i.e., arithmetic line bundles thought of "multiplicatively" or "idèlically", as in the theory of Frobenioids] — cf. [AbsTopIII], Definition 5.3, (i), (ii).

Remark 4.7.3. At this point, it is of interest to review the significance of the $\mathbb{F}_l^{\rtimes\pm}$- and $\mathbb{F}_l^{\circledast}$**-symmetries** in the context of the theory of the present §4.

(i) First, we recall that, in the context of the present series of papers, the "$\mathbb{F}_l$" that appears in the notation "$\mathbb{F}_l^{\rtimes\pm}$" and "$\mathbb{F}_l^{\circledast}$" is to be thought of — since l is *"large"* — as a sort of **finite approximation** *of the ring of rational integers* $\mathbb{Z}$ [cf. the discussion of [IUTchI], Remark 6.12.3, (i)]. That is to say, the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* corresponds to the **additive** structure of $\mathbb{Z}$, while the $\mathbb{F}_l^{\circledast}$-*symmetry* corresponds to the **multiplicative** structure of $\mathbb{Z}$. Since the "$\mathbb{F}_l$" under consideration arises from the torsion points of an elliptic curve, it is natural — especially in light of the central role played in the present series of papers by $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ — to think of the "$\mathbb{Z}$" under consideration as the Galois group "$\underline{\mathbb{Z}}$" of the *universal combinatorial covering of the Tate curves* that appear at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. the discussion at the beginning of [EtTh], §1]. In particular, in light of the theory of Tate curves, it is natural to think of this "$\mathbb{Z}$" as representing a sort of universal version of the **value group** associated to a local field that occurs at a $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, and to think of the element $0 \in \mathbb{Z}$ — hence, the **label**

$$0 \in |\mathbb{F}_l|$$

— as representing the **units**.

(ii) Perhaps the most fundamental difference between the $\mathbb{F}_l^{\rtimes\pm}$- and $\mathbb{F}_l^{\circledast}$-symmetries lies in the fact that the $\mathbb{F}_l^{\rtimes\pm}$-symmetry involves the **zero label** $0 \in |\mathbb{F}_l|$ [cf. the discussion of [IUTchI], Remark 6.12.5]. In particular, the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is suited to application to the *"units"* — i.e., to the various local "$\mathcal{O}^\times$" and "$\mathcal{O}^{\times\boldsymbol{\mu}}$" that appear in the theory. At a more technical level, this relationship between the $\mathbb{F}_l^{\rtimes\pm}$-symmetry and "$\mathcal{O}^\times$" may be seen in the theory of §3 [cf. also Corollaries 4.5, (iii); 4.6, (iii)]. That is to say, in §3 [cf. the discussion of Remark 3.8.3], the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is applied precisely to establish **conjugate synchronization**, which, in turn, will be applied eventually to establish the crucial **coricity of "$\mathcal{O}^{\times\boldsymbol{\mu}}$"** in the context of the $\Theta_{\mathrm{gau}}^{\times\boldsymbol{\mu}}$-link [cf. Corollary 4.10, (iv), below]. Here, let us observe that the conjugate synchronization, established by means of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry, of copies of the absolute Galois group of the local base field at various $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ is a very delicate property that *depends quite essentially on the "arithmetic holomorphic structure" of the Hodge theaters under consideration.* That is to say, from the point of view of the theory of §1, conjugate synchronization in one Hodge theater *fails to be compatible* with conjugate synchronization in another Hodge theater with a *distinct* arithmetic holomorphic structure. Put another way, from the point of view of the theory of §1, conjugate synchronization can only be naturally formulated in a **uniradial** fashion. This uniradiality may also be seen at a *purely combinatorial level*, as we shall discuss in Remark 4.7.4 below. On the other hand, if one passes to *mono-analyticizations* — e.g., to **mono-analytic processions** — then the *mono-analytic* "$\mathcal{O}^{\times\boldsymbol{\mu}}$" that appears in the $\Theta_{\mathrm{gau}}^{\times\boldsymbol{\mu}}$-link [cf. Corollary 4.10, (iv), below] is, by contrast, **coric**. That is to say, by relating the *zero label*, which is common to *distinct arithmetic holomorphic structures*, to the various *nonzero labels*, which belong to a *single fixed arithmetic holomorphic structure*, the condition of invariance with respect to the $\mathbb{F}_l^{\rtimes\pm}$-symmetry may — e.g., in the case of the *mono-analytic* "$\mathcal{O}^{\times\boldsymbol{\mu}}$" — amount to a condition of **coricity**. In particular, in the case of the *mono-analytic* "$\mathcal{O}^{\times\boldsymbol{\mu}}$",

> the $\mathbb{F}_l^{\rtimes\pm}$-symmetry plays the role of establishing the **coric pieces** — i.e., components which are **"uniform"** with respect to *all of the distinct arithmetic holomorphic structures* involved — of the **apparatus** to be established in the present series of papers.

This **dual role** — i.e., consisting of both **uniradial** and **coric** aspects — played by the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is to be considered in contrast to the **strictly multiradial** role [cf. (iii) below] played by the $\mathbb{F}_l^{\circledast}$-symmetry. Also, in this context, we observe that the *symmetrization*, effected by the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, between *zero* and *nonzero labels* may be thought of, from the point of view of (i), as a symmetrization between [local] **units** and **value groups** and, hence, in particular, is *reminiscent* of the **intertwining** of units and value groups effected by the $\mathfrak{log}$**-link** [cf. [IUTchIII], Remark 3.12.2, (i), (ii)], as well as of the *crucial compatibility* between the $\mathbb{F}_l^{\rtimes\pm}$-*symmetrizing isomorphisms* [i.e., that give rise to the *conjugate synchronization*] and the $\mathfrak{log}$-*link* [cf. [IUTchIII], Remark 1.3.2].

(iii) The significance of the $\mathbb{F}_l^{\circledast}$-symmetry lies, in a word, in the fact that it allows one to **separate the zero label** from the nonzero labels. From the point of view of the theory of the present series of papers, this property makes the $\mathbb{F}_l^{\circledast}$-symmetry well-suited for the *construction/description of the internal structure of the* **Gaussian monoids**, which are, in effect, "distributions" or "functions" of a parameter $j \in \mathbb{F}_l^{\circledast}$ [cf. Corollaries 4.5, (iv), (v); 4.6, (iv), (v)]. Here, we note that this separation of the zero label — which parametrizes *coric data* that is *common to distinct arithmetic holomorphic structures* — from the nonzero labels — which parametrize the components of the Gaussian monoid associated to a *particular arithmetic holomorphic structure* — is crucial from the point of view of *describing the Gaussian monoid* associated to a particular arithmetic holomorphic structure in terms that may be understood from the point of view of some *"alien" arithmetic holomorphic structure*. Put another way, from the point of view of the theory of §1, the $\mathbb{F}_l^{\circledast}$-symmetry admits a natural **multiradial** formulation. This multiradiality may also be seen at a *purely combinatorial level*, as we shall discuss in Remark 4.7.4 below. In this context, it is important to note that if one thinks of the **coric** *constant distribution*, labeled by zero, as embedded via the *diagonal embedding* into the various products parametrized by $j \in \mathbb{F}_l^{\circledast}$ that appear in the construction of the Gaussian monoids [cf. the isomorphisms that appear in the final displays of Corollaries 4.5, (iii); 4.6, (iii)], then it is natural to think of the *volumes* computed at each $j \in \mathbb{F}_l^{\circledast}$ as being assigned a **weight** $1/l^{\circledast}$ — i.e., so that the diagonal embedding of the constant distribution is compatible with taking the *constant distribution* to be of **weight** 1 [cf. the discussion of [IUTchI], Remark 5.4.2]. Put another way, from the point of view of *"computation of weighted volumes"*, the various **nonzero** $j \in \mathbb{F}_l^{\circledast}$ are **"subordinate"** to $0 \in |\mathbb{F}_l|$ — i.e., $\mathbb{F}_l^{\circledast} \ni j \lll 0$. In particular, to *symmetrize*, in the context of the internal structure of the Gaussian monoids, the zero and nonzero labels [i.e., as in the case of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry!] amounts to allowing a relation

$$\text{“}0 \lll 0\text{”}$$

— which is *absurd* [i.e., in the sense that it *fails to be compatible* with weighted volume computations]!

Remark 4.7.4.

(i) One way to understand the underlying **combinatorial structure** of the **uniradiality** of the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* and the **multiradiality** of the $\mathbb{F}_l^{\circledast}$-*symmetry* [cf. the discussion of Remark 4.7.3, (ii), (iii)] is to consider these symmetries — which are defined relative to some given *arithmetic holomorphic structure* [or, at a more technical level, some given $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* — cf. [IUTchI], Definition 6.13, (i)] — in the context of the **étale-pictures** that arise from each of these symmetries [cf. [IUTchI], Corollaries 4.12, 6.10]. In the case of the $\mathbb{F}_l^{\rtimes\pm}$- (respectively, $\mathbb{F}_l^{\circledast}$-) symmetry, this étale-picture consists of a collection of copies of $\mathbb{F}_l$ (respectively, $|\mathbb{F}_l| = \mathbb{F}_l^{\circledast} \bigcup \{0\}$), each copy corresponding to a *single arithmetic holomorphic structure*, which are *glued together* at the *coric label* $0 \in \mathbb{F}_l$ (respectively, $0 \in |\mathbb{F}_l|$). In Fig. 4.1 (respectively, 4.2) below, an illustration is given of such an étale-picture, in which the notation "$\pm$" (respectively, "$\circledast$") is used to denote the various elements of $\mathbb{F}_l \setminus \{0\}$ (respectively, $\mathbb{F}_l^{\circledast}$) in each copy of $\mathbb{F}_l$ (respectively, $|\mathbb{F}_l|$). Moreover, on each copy of $\mathbb{F}_l$ (respectively, $|\mathbb{F}_l|$) — labeled, say, by some *spoke* α [corresponding to a single arithmetic holomorphic structure] — one has a natural action of a "corresponding copy" of $\mathbb{F}_l^{\rtimes\pm}$ (respectively, $\mathbb{F}_l^{\circledast}$).

(ii) The fundamental difference between the simple combinatorial models of the étale-pictures considered in (i) lies in the fact that whereas

(a) in the case of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry, the $\mathbb{F}_l^{\rtimes\pm}$-actions on distinct spokes **fail to commute** with one another,

(b) in the case of the $\mathbb{F}_l^{\circledast}$-symmetry, the $\mathbb{F}_l^{\circledast}$-actions on distinct spokes **commute** with one another and, moreover, are **compatible** with the **permutations of spokes** discussed in [IUTchI], Corollary 4.12, (iii).

Indeed, the noncommutativity, or *"incompatibility with simultaneous execution at distinct spokes"* [cf. Remark 1.9.1], of (a) is a direct consequence of the *inclusion of the zero label* in the $\mathbb{F}_l^{\rtimes\pm}$-symmetry and may be thought of as a sort of **prototypical combinatorial representation** of the phenomenon of **uniradiality**. By contrast, the commutativity, or *"compatibility with simultaneous execution at distinct spokes"* [cf. Remark 1.9.1], of (b) is a direct consequence of the *exclusion of the zero label* from the $\mathbb{F}_l^{\circledast}$-symmetry and may be thought of as a sort of prototypical combinatorial representation of the phenomenon of **multiradiality**. Note that in the case of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry, it is also a direct consequence of the inclusion of the zero label that the condition of *invariance* with respect to the $\mathbb{F}_l^{\rtimes\pm}$-actions on all of the spokes may be thought of as a condition of *"uniformity"* among the elements of the copies of $\mathbb{F}_l$ at the various spokes, hence as a sort of *coricity* [cf. the discussion of Remark 4.7.3, (ii)].

$$\begin{array}{ccccc}
\dots & & \boxed{\begin{array}{cc}\pm & \pm \\ \multicolumn{2}{c}{\curvearrowright} \\ \pm & \pm\end{array}} & & \dots \\
 & & \downarrow\uparrow & & \\
\boxed{\begin{array}{cc}\pm & \pm \\ \multicolumn{2}{c}{\curvearrowright} \\ \pm & \pm\end{array}} & \rightleftarrows & 0 & \leftrightarrows & \boxed{\begin{array}{cc}\pm & \pm \\ \multicolumn{2}{c}{\curvearrowright} \\ \pm & \pm\end{array}}
\end{array}$$

Fig. 4.1: Étale-picture of $\mathbb{F}_l^{\rtimes\pm}$-symmetries

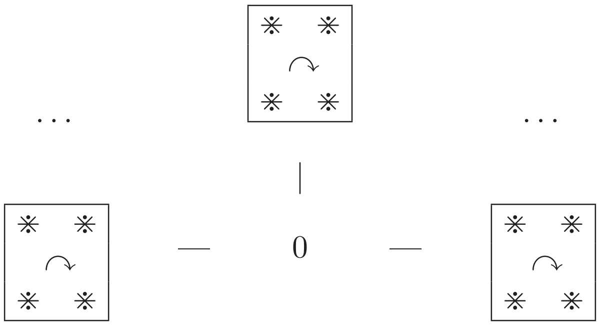

Fig. 4.2: Étale-picture of $\mathbb{F}_l^{\circledast}$-symmetries

(iii) Although the combinatorial versions of uniradiality and multiradiality discussed in (ii) above are not formulated in terms of the formalism of *uniradial* and *multiradial environments* developed in §1 [cf. Example 1.7, (ii)], it is not difficult to produce such a formulation. For instance, one may take the *coric data* to consist of objects of the form "0_α" — i.e., the zero label, subscripted by the label α associated to some spoke. For any two spokes α, β, we define the set of arrows

$$0_\alpha \ \to \ 0_\beta$$

to consist of precisely one element (α, β). We then take, in the case of the $\mathbb{F}_l^{\rtimes\pm}$- (respectively, $\mathbb{F}_l^{\circledast}$-) symmetry, the *radial data* to consist of a copy $(\mathbb{F}_l)_\alpha$ (respectively, $|\mathbb{F}_l|_\alpha$) of $\mathbb{F}_l$ (respectively, $|\mathbb{F}_l|$) subscripted by the label α associated to some spoke. For any two spokes α, β, we define the set of arrows

$$(\mathbb{F}_l)_\alpha \ \to \ (\mathbb{F}_l)_\beta \quad (\text{respectively, } |\mathbb{F}_l|_\alpha \ \to \ |\mathbb{F}_l|_\beta)$$

to consist of *precisely one element* if the actions $(\mathbb{F}_l^{\rtimes\pm})_\gamma \ \curvearrowright \ (\mathbb{F}_l)_\gamma$ (respectively, $(\mathbb{F}_l^{\circledast})_\gamma \ \curvearrowright \ (|\mathbb{F}_l|)_\gamma$), for $\gamma = \alpha, \beta$, determine an action of

$$(\mathbb{F}_l^{\rtimes\pm})_\alpha \ \times \ (\mathbb{F}_l^{\rtimes\pm})_\beta \quad (\text{respectively, } (\mathbb{F}_l^{\circledast})_\alpha \ \times \ (\mathbb{F}_l^{\circledast})_\beta)$$

on the co-product $(\mathbb{F}_l)_\alpha \coprod_0 (\mathbb{F}_l)_\beta$ (respectively, $(|\mathbb{F}_l|)_\alpha \coprod_0 (|\mathbb{F}_l|)_\beta$) obtained by *identifying* the respective zero labels 0_α, 0_β, and to equal the *empty set* if such an action does not exist. Then one has a natural *radial functor* $(\mathbb{F}_l)_\alpha \ \mapsto \ 0_\alpha$ (respectively, $|\mathbb{F}_l|_\alpha \ \mapsto \ 0_\alpha$) that associates coric data to radial data. Moreover, the resulting *radial environment* is easily seen to be *uniradial* (respectively, *multiradial*). We leave the routine details to the reader. Finally, we note in passing that the formulation involving products given above is reminiscent both of the discussion of the *switching functor* in Example 1.7, (iii), and of the discussion of *parallel transport via connections* in Remark 1.7.1.

Remark 4.7.5. In the context of the discussion of the combinatorial models of the $\mathbb{F}_l^{\rtimes\pm}$- and $\mathbb{F}_l^{\circledast}$-symmetries in Remark 4.7.4, it is useful to recall that the $\mathbb{F}_l^{\rtimes\pm}$- and $\mathbb{F}_l^{\circledast}$-symmetries correspond, respectively, to the *additive* and *multiplicative* structures of the field $\mathbb{F}_l$ — which [cf. Remark 4.7.3, (i)] we wish to think of as a sort of *finite approximation of the ring* $\mathbb{Z}$. That is to say, from the point of view of the theory of the present series of papers,

(a) the $\mathbb{F}_l^{\rtimes\pm}$- and $\mathbb{F}_l^{\circledast}$-symmetries correspond, respectively, to the *two combinatorial dimensions* — i.e., *addition* and *multiplication* — of a *ring* [cf. the discussion of [AbsTopIII], §I3].

Moreover, in the context of the discussion of Remark 4.7.3, (i), concerning *units* and *value groups*, it is useful to recall that these two combinatorial dimensions may be thought of as corresponding to

(b) the *units* and *value group* of a mixed-characteristic nonarchimedean or complex archimedean local field [cf. the discussion of [AbsTopIII], §I3]

or, alternatively, to

(c) the *two cohomological dimensions* of the absolute Galois group of a mixed-characteristic nonarchimedean local field or the *two underlying real dimensions* of a complex archimedean local field [cf. the discussion of [AbsTopIII], §I3].

Finally, the *hierarchical structure* of these two dimensions — i.e., the way in which *"one dimension [i.e., multiplication] is piled on top of the other [i.e., addition]"* — is reflected in the

(d) *subordination structure "$\lll$"*, relative to the computation of weighted volumes, of *nonzero labels* with respect to the *zero label* [cf. the discussion of Remark 4.7.3, (iii)].

as well as in the fact that

(e) the $\mathbb{F}_l^{\rtimes\pm}$-symmetry arises from the conjugation action of the *geometric fundamental group* [cf. Remarks 3.5.2, (iii); 4.5.3, (i)], whereas the $\mathbb{F}_l^{\circledast}$-symmetry arises from the conjugation action of the *absolute Galois group of the global base field* [cf. Remark 4.7.2]

— i.e., where we recall that the *arithmetic fundamental groups* involved may be thought of as having a natural *hierarchical structure* constituted by their *extension structure* [corresponding to the natural outer action of the absolute Galois group of the base field on the geometric fundamental group].

Remark 4.7.6. One important observation in the context of Corollary 4.7, (i), is that it makes sense to consider **non-realified global Frobenioids** [corresponding, e.g., to "F_{mod}"] *only* in the case of the **$\mathbb{F}_l^{\circledast}$-symmetry**. Indeed, in order to consider the field "F_{mod}" from an *anabelian*, or *Galois-theoretic*, point of view, it is necessary to consider the *full profinite group* Π_{C_F} — i.e., not just the open subgroups $\Pi_{\underline{C}_K}$, $\Pi_{\underline{X}_K}$ of Π_{C_F} which give rise, respectively, to the global portions of the $\mathbb{F}_l^{\circledast}$- and $\mathbb{F}_l^{\rtimes\pm}$-symmetries [cf. [IUTchI], Definition 4.1, (v); [IUTchI], Definition 6.1, (v)]. On the other hand, to work with the *abstract topological group* Π_{C_F} means that the subgroups $\Pi_{\underline{C}_K}$, $\Pi_{\underline{X}_K}$ of Π_{C_F} are *only well-defined up to* Π_{C_F}*-conjugacy.* That is to say, in this context, the subgroups $\Pi_{\underline{C}_K}$, $\Pi_{\underline{X}_K}$ are only well-defined up to automorphisms arising from their *normalizers* in Π_{C_F} [cf. the discussion of [IUTchI], Remark 6.12.6, (iii), (iv)]. In particular, in the present context, one is obliged to regard these groups $\Pi_{\underline{C}_K}$, $\Pi_{\underline{X}_K}$ as being subject to indeterminacies arising from the natural $\mathbb{F}_l^{\circledast}$*-poly-actions* [i.e., actions by a group that surjects naturally onto $\mathbb{F}_l^{\circledast}$ — cf. [IUTchI], Example 4.3, (iv)] on these groups — that is to say, subject to indeterminacies arising from the natural **$\mathbb{F}_l^{\circledast}$-symmetries** of these groups. Here, it is important to note that one cannot simply "form the quotient by the indeterminacy constituted by these $\mathbb{F}_l^{\circledast}$-symmetries" since this would give rise to **"label-crushing"**, i.e., to identifying to a single point the *distinct labels* $j \in \mathbb{F}_l^{\circledast}$, which play a crucial role in the construction of the *Gaussian monoids* [cf. the discussion of [IUTchI], Remark 4.7.1]. But then the $\mathbb{F}_l^{\circledast}$-symmetries of $\Pi_{\underline{C}_K}$, $\Pi_{\underline{X}_K}$ that one must contend with necessarily involve *conjugation by elements of the absolute Galois groups of the global base fields* involved, hence are *fundamentally incompatible* with the establishment of *conjugate synchronization* [cf. the discussion of Remark 4.7.2]. That is to say, just as it is necessary to

(a) **isolate** the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** from the $\mathbb{F}_l^{\circledast}$-symmetry in order to establish **conjugate synchronization** [cf. the discussion of Remark 4.7.2],

it is also necessary to

(b) **isolate** the $\mathbb{F}_l^{\circledast}$**-symmetry** from the $\mathbb{F}_l^{\rtimes\pm}$-symmetry in order to work with **Galois-theoretic** representations of the **global base field** F_{mod}.

Indeed, in this context, it is useful to recall that one of the fundamental themes of the theory of the present series of papers consists precisely of the **dismantling** *of the two [a priori intertwined!] combinatorial dimensions of a ring* [cf. Remarks 4.7.3, 4.7.5; [AbsTopIII], §I3].

Remark 4.7.7. The theory of *"tempered versus profinite conjugates"* developed in [IUTchI], §2, is applied in the proof of Corollary 2.4, (i), in a setting which ultimately [cf. Remark 2.6.2, (i); Corollary 4.5, (iii)] is seen to amount to a certain local portion [at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] of a **[$\mathcal{D}$-]$\Theta^{\pm\mathrm{ell}}$-Hodge theater** — i.e., a setting in which one considers the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**. On the other hand, in [IUTchI], Remark 4.5.1, (iii), a discussion is given in which this theory of *"tempered versus profinite conjugates"* developed in [IUTchI], §2, is applied in a setting which constitutes a certain local portion [at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] of a **[$\mathcal{D}$-]$\Theta$NF-Hodge theater**. In this context, it is useful to note that the point of view of this discussion given in [IUTchI], Remark 4.5.1, (iii), may be regarded as *"implicit"* in the point of view of the theory of the present §4 in the following sense: The *profinite conjugacy indeterminacies* that occur in an [$\mathcal{D}$-]ΘNF-Hodge theater [cf. [IUTchI], Remark 4.5.1, (iii)] are linked via the *gluing operation* discussed in [IUTchI], Remark 6.12.2, (i), (ii) — cf. Corollaries 4.6, (iv); 4.7 — to the *profinite conjugacy indeterminacies* that occur in an [$\mathcal{D}$-]$\Theta^{\pm\mathrm{ell}}$-Hodge theater [cf. Remarks 2.5.2, (ii), (iii); 2.6.2, (i); 4.5.3, (iii)], i.e., to the profinite conjugacy indeterminacies that are *"resolved"* in the proof of Corollary 2.4, (i), by applying the theory of [IUTchI], §2.

Corollary 4.8. (Frobenioid-theoretic Monoids Associated to ΘNF-Hodge Theaters) *Let*

$$ {}^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}} = ({}^{\dagger}\mathcal{F}^{\circledast} \dashleftarrow {}^{\dagger}\mathcal{F}^{\circledcirc} \xleftarrow{{}^{\dagger}\psi^{\mathrm{NF}}_{\circledast}} {}^{\dagger}\mathfrak{F}_J \xrightarrow{{}^{\dagger}\psi^{\Theta}_{\circledast}} {}^{\dagger}\mathfrak{F}_{>} \dashrightarrow {}^{\dagger}\mathcal{HT}^{\Theta}) $$

be a ΘNF**-Hodge theater** *[cf. [IUTchI], Definition 5.5, (iii)] which lifts the $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ of Corollary 4.7 and is* **glued** *to the* $\Theta^{\pm\mathrm{ell}}$**-Hodge theater** ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$ *of Corollary 4.6 via the functorial algorithm of [IUTchI], Proposition 6.7 [so $J = T^{\circledast}$] — cf. the discussion of [IUTchI], Remark 6.12.2, (i), (ii).*

(i) **(Non-realified Global Structures)** *There is a* **functorial algorithm** *in the category* ${}^{\dagger}\mathcal{F}^{\circledcirc}$ *[or in the category* ${}^{\dagger}\mathcal{F}^{\circledast}$*] — cf. the discussion of [IUTchI], Example 5.1, (v), (vi), concerning* **isomorphisms of cyclotomes** *and related* **Kummer maps** *— for constructing* **Kummer isomorphisms of pseudo-monoids** *[the first two of which are equipped with group actions and well-defined up to a single conjugacy indeterminacy]*

$$\left\{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}\right\} \xrightarrow{\sim} \left\{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})\right\}, \quad {}^{\dagger}\mathbb{M}^{\circledast}_{\kappa} \xrightarrow{\sim} \mathbb{M}^{\circledast}_{\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

and, hence, by **restricting Kummer classes** *as in the discussion of [IUTchI], Example 5.1, (v), natural* **"Kummer-theoretic" isomorphisms**

$$\left\{\pi_1({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}\right\} \xrightarrow{\sim} \left\{\pi_1({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})\right\}$$

$$\left\{\pi_1({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\overline{\mathbb{M}}^{\circledast}\right\} \xrightarrow{\sim} \left\{\pi_1({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})\right\}$$

$$\left\{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{sol}}\right\} \xrightarrow{\sim} \left\{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\mathrm{sol}}({}^{\dagger}\mathcal{D}^{\circledcirc})\right\}$$

$${}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}), \quad {}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

— which may be interpreted as a **compatible** *collection of* **isomorphisms of Frobenioids**

$${}^{\dagger}\mathcal{F}^{\circledcirc} \xrightarrow{\sim} \mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc}), \quad {}^{\dagger}\mathcal{F}^{\circledast} \xrightarrow{\sim} \mathcal{F}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

$${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}), \quad {}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

[cf. the discussion of [IUTchI], Example 5.1, (ii), (iii)].

(ii) **(Labels and $\mathbb{F}_l^{\circledast}$-Symmetry)** *In the notation of Corollary 4.7, (ii), the collection of isomorphisms of Corollary 4.6, (i) [applied to the $\mathcal{F}$-prime-strips of the capsule* ${}^{\dagger}\mathfrak{F}_J$*; cf. also the discussion of [IUTchI], Example 5.4, (iv)], together with the isomorphisms of (i) above, determine, for each* $j \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$ $(\xrightarrow{\sim} J)$ *[cf. the bijection* ${}^{\dagger}\zeta_{\circledast}$ *of Corollary 4.7, (ii)], a collection of* **isomorphisms**

$${}^{\dagger}\mathfrak{F}_j \xrightarrow{\sim} {}^{\dagger}\mathcal{F}^{\circledcirc}|_j \xrightarrow{\sim} \mathcal{F}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})|_j$$

$$({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{mod}})_j \xrightarrow{\sim} \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j, \quad ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_j \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$$

$$\{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{sol}}\}_j \xrightarrow{\sim} \{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\mathrm{sol}}({}^{\dagger}\mathcal{D}^{\circledcirc})\}_j$$

$$\{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}\}_j \xrightarrow{\sim} \{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc})\}_j$$

$$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_j \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j, \quad ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j \xrightarrow{\sim} \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$$

as well as **[$\mathbb{F}_l^{\circledast}$-]symmetrizing isomorphisms***, induced by the natural poly-action of* $\mathbb{F}_l^{\circledast}$ *on* ${}^{\dagger}\mathcal{F}^{\circledcirc}$ *[cf. [IUTchI], Example 4.3, (iv); [IUTchI], Corollary 5.3, (i)], between the data indexed by distinct* $j \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$*. Here, [just as in Corollary 4.7, (ii)] the objects equipped with* $\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast})(\twoheadrightarrow \pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}))$*-actions are to be regarded as being subject to* **independent** $\pi_1^{\mathrm{rat}/\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledcirc})$**-conjugacy indeterminacies** *for* **distinct** *j, together with a* **single** $(\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \twoheadrightarrow)\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast})$**-conjugacy**

indeterminacy *that is* **independent** *of j [cf. the discussion of the final portion of [IUTchI], Example 5.1, (i)]. Moreover, these symmetrizing isomorphisms are* **compatible**, *relative to* $^{\dagger}\zeta_{\circledast}$ *[cf. Corollary 4.7, (ii)], with the* $\mathbb{F}_l^{\circledast}$**-symmetry** *of the associated NF-bridge [cf. [IUTchI], Proposition 4.9, (i); [IUTchI], Corollary 5.6, (ii)] and determine various* **diagonal** $\mathcal{F}$**-prime-strips/submonoids/subrings/sub-pseudo-monoids** *[equipped with a group action subject to conjugacy indeterminacies as described above]*/**subcategories**

$$(-)_{\langle \mathbb{F}_l^{\circledast} \rangle} \quad \subseteq \quad \prod_{j \in \mathbb{F}_l^{\circledast}} (-)_j$$

[i.e., relative to the conventions discussed in Corollary 4.7, (ii); cf. also Remark 4.7.2].

(iii) **(Localization Functors and Realified Global Structures)** *Let* $j \in \mathrm{LabCusp}(^{\dagger}\mathcal{D}^{\circledcirc})$. *In the following, objects associated to an $\mathcal{F}$-prime-strip labeled by j at an element $v \in \mathbb{V}_{\mathrm{mod}}$ will be denoted by means of a label "v_j". Then there is a* **functorial algorithm** *in the NF-bridge* $(^{\dagger}\mathfrak{F}_J \to {^{\dagger}\mathcal{F}^{\circledcirc}} \dashrightarrow {^{\dagger}\mathcal{F}^{\circledast}})$ *for constructing* **mutually** [1-]**compatible** *collections of* **"localization" functors/poly-morphisms** *[up to isomorphism]*

$$(^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_j \quad \to \quad {^{\dagger}\mathfrak{F}_j}, \qquad (^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j \quad \to \quad {^{\dagger}\mathfrak{F}^{\mathbb{R}}_j}$$

$$\left\{ \{\pi_1^{\kappa\text{-sol}}(^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}}\}_j \quad \to \quad {^{\dagger}\mathbb{M}_{\infty\kappa v_j}} \subseteq {^{\dagger}\mathbb{M}_{\infty\kappa\times v_j}} \right\}_{\underline{v}\in\underline{\mathbb{V}}}$$

as in the discussion of [IUTchI], Example 5.4, (iv), (vi) [cf. also [IUTchI], Definition 5.2, (vi), (viii)] — which are **compatible**, *relative to the various [Kummer/ "Kummer-theoretic"] isomorphisms of (i), (ii) [cf. also [IUTchI], Definition 5.2, (vi), (viii)], with the collections of functors/poly-morphisms of Corollary 4.7, (iii) — together with a natural* **isomorphism of Frobenioids**

$$^{\dagger}\mathcal{C}^{\Vdash}_j \quad \xrightarrow{\sim} \quad (^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j$$

[cf. the notation of Corollary 4.6, (ii); [IUTchI], Remark 5.2.1, (ii), applied to the $\mathcal{F}$-prime-strip $^{\dagger}\mathfrak{F}_j$] which is **compatible** *[cf. Remark 4.8.3 below] with the respective* **bijections** *involving "*$\mathrm{Prime}(-)$*", the respective* **local isomorphisms of topological monoids** *[cf. the arrow* $(^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j \to {^{\dagger}\mathfrak{F}^{\mathbb{R}}_j}$ *discussed above; [IUTchI], Remark 5.2.1, (ii)], the isomorphisms of Corollary 4.7, (iii), and the various ["Kummer-theoretic"] isomorphisms of (i), (ii) [cf. also Corollary 4.6, (ii)]. Finally, all of these structures are* **compatible** *with the respective* $\mathbb{F}_l^{\circledast}$**-symmetrizing isomorphisms** *[cf. (ii)].*

Proof. The various assertions of Corollary 4.8 follow immediately from the definitions and the references quoted in the statements of these assertions. ○

Remark 4.8.1.

(i) The Frobenioid $\mathcal{C}^{\Vdash}_{\mathrm{gau}}(^{\dagger}\mathcal{HT}^{\Theta})$ of Corollary 4.6, (v), is constructed as a subcategory of a product over $j \in \mathbb{F}_l^{\circledast}$ of copies $^{\dagger}\mathcal{C}^{\Vdash}_j$ of the category $^{\dagger}\mathcal{C}^{\Vdash}$. In particular, one may apply the *isomorphism* ${}^{\dagger}\mathcal{C}^{\Vdash}_{j} \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j$ of Corollary 4.8, (iii), to regard this Frobenioid $\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})$ as a subcategory

$$\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}) \quad \hookrightarrow \quad \prod_{j \in \mathbb{F}^{*}_{l}} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j$$

of the product over $j \in \mathbb{F}^{*}_{l}$ of the $({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j$.

(ii) In a similar vein, the local data at $\underline{v} \in \underline{\mathbb{V}}$ of the objects $\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})$ constructed in Corollary 4.6, (iv), gives rise to [the local data at $\underline{v}$ of an $\mathcal{F}^{\vdash}$-prime-strip, i.e., in particular, to] *split Frobenioids* $\mathcal{F}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$ [cf. Definition 3.8, (ii), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$]. Write $\mathcal{F}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})$ for the $\mathcal{F}^{\vdash}$*-prime-strip* determined by this local data $\mathcal{F}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$ at $\underline{v}$, for $\underline{v} \in \underline{\mathbb{V}}$, and

$$\mathcal{F}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathbb{R}}$$

for the object obtained by forming, at each $\underline{v} \in \underline{\mathbb{V}}$, the *realification* of the underlying Frobenioid of $\mathcal{F}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})$ at $\underline{v}$. Then it follows from the construction discussed in Corollary 4.6, (iv), that one may think of the realified Frobenioid, at each $\underline{v} \in \underline{\mathbb{V}}$, of $\mathcal{F}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathbb{R}}$ as being naturally ["poly-"]embedded

$$\mathcal{F}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathbb{R}} \quad \hookrightarrow \quad \prod_{j \in \mathbb{F}^{*}_{l}} ({}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{>})_j$$

[where we use this notation to denote the collection of ["poly-"]embeddings indexed by $\underline{v} \in \underline{\mathbb{V}}$] in the product of copies of realifications of [the underlying Frobenioids of] the $\mathcal{F}$-prime-strip ${}^{\dagger}\mathfrak{F}_{>}$ labeled by $j \in \mathbb{F}^{*}_{l}$. Moreover, by applying the *full poly-isomorphisms* $({}^{\dagger}\mathfrak{F}_{>})_j \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}_j$ — which are *tautologically* compatible with the labels $j \in \mathbb{F}^{*}_{l}$! — we may think of $\mathcal{F}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathbb{R}}$ as being naturally ["poly-"]embedded

$$\mathcal{F}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathbb{R}} \quad \hookrightarrow \quad \prod_{j \in \mathbb{F}^{*}_{l}} {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{j}$$

[where we use this notation to denote the collection of ["poly-"]embeddings indexed by $\underline{v} \in \underline{\mathbb{V}}$] in the product associated to the realifications of [the underlying Frobenioids of] the $\mathcal{F}$-prime-strips ${}^{\dagger}\mathfrak{F}_j$.

(iii) Thus, by applying the various ["poly-"]embeddings considered in (i), (ii), one may think of the *"realified localization" functors*

$$({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_j \quad \to \quad {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{j}$$

of Corollary 4.8, (iii), as inducing a *"realified localization" functor* [up to isomorphism]

$$\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}) \quad \to \quad \mathcal{F}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathbb{R}}$$

— which [as one verifies immediately] is *compatible* [cf. the various compatibilities discussed in Corollary 4.8, (iii)] with the realified localization isomorphisms $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}),\underline{v}} \xrightarrow{\sim} \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathbb{R}}_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}$, considered in Corollary 4.6, (v).

Remark 4.8.2.

(i) The realified localization functor discussed in Remark 4.8.1, (iii), only concerns the **realification** of the Frobenioid-theoretic version $\mathcal{F}_{\text{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})$ of the **Gaussian monoids**. The **unit** portion of the Gaussian monoids will be used, in the context of the theory involving the $\mathfrak{log}$-wall that will be developed in [IUTchIII], not in its capacity as a *"multiplicative object"*, but rather — i.e., by applying the operation "$\mathfrak{log}$" to the units at the various $\underline{v} \in \underline{\mathbb{V}}$, as in the theory of [AbsTopIII] — as an **"additive object"**. In this theory, the *non-realified* global Frobenioids of Corollary 4.8, (i), will appear in the context of localization functors/morphisms — i.e., as a sort of **translation apparatus** between $\boxtimes$- and $\boxplus$**-line bundles** [cf. the discussion of Remark 4.7.2] — that relate these *[multiplicative!] non-realified global Frobenioids* to the *[additive!] images via "$\mathfrak{log}$" of the units*. Note that this sort of construction — i.e., in which the localization operations involving *units* and *value groups* differ by a *shift* via the operation "$\mathfrak{log}$" — *depends, in an essential way* [cf. the discussion of Remark 1.12.2, (iv)], on the **natural splittings** with which the Gaussian monoids are equipped [cf. Corollary 4.6, (iv)].

(ii) In the context of (i), it is useful to observe that, although the *non-realified global Frobenioids* of Corollary 4.8, (i), may only be considered in the context of the $\mathbb{F}_l^{\circledast}$*-symmetry* [cf. the discussion of Remark 4.7.6], this does not yield any obstacles, relative to the discussion in (i) of Gaussian monoids, since *Gaussian monoids* are most naturally considered as *"functions" of a parameter* $j \in \mathbb{F}_l^{\circledast}$ [cf. the discussion of Remark 4.7.3, (iii)].

(iii) From the point of view of the analogy of the theory of the present series of papers with *p-adic Teichmüller theory* [cf. the discussion of [AbsTopIII], §I5], it is of interest to note that the construction discussed in (i) involving the use of the natural splittings of Gaussian monoids to consider *"$\mathfrak{log}$-shifted units"* together with *"non-$\mathfrak{log}$-shifted value groups"* may be thought of as corresponding to the situation that frequently occurs in p-adic Teichmüller theory in which an *indigenous bundle* $(\mathcal{E}, \nabla_{\mathcal{E}})$ equipped with a *Hodge filtration* $0 \to \omega \to \mathcal{E} \to \tau \to 0$ on a hyperbolic curve in positive characteristic is represented, in the context of local Frobenius liftings modulo higher powers of p, as a *direct sum*

$$\Phi^* \tau \ \oplus \ \omega$$

— where Φ denotes the *Frobenius morphism* on the curve, which, as may be recalled from the discussion of [AbsTopIII], §I5, corresponds, relative to the analogy under consideration, to the operation "$\mathfrak{log}$" studied in [AbsTopIII].

Remark 4.8.3. Similar observations to the observations made in Remark 4.5.2, (i), (ii), concerning the $\mathbb{F}_l^{\rtimes\pm}$-symmetrizing isomorphisms of Corollary 4.5, (iii), may be made in the case of the $\mathbb{F}_l^{\circledast}$*-symmetrizing isomorphisms* of Corollary 4.8, (ii).

Definition 4.9.

(i) Let $\mathcal{C}$ be an *arbitrary Frobenioid.* Write $\mathcal{D}$ for the *base category* of $\mathcal{C}$. Suppose that $\mathcal{D}$ is isomorphic to the category of *connected finite étale coverings* of the spectrum of an *MLF* or a *CAF*. Let A be a *"universal covering pro-object"* of $\mathcal{D}$ [cf. the discussion of Example 3.2, (i), (ii)]. Write $G \stackrel{\mathrm{def}}{=} \mathrm{Aut}(A)$ [so G is isomorphic to the absolute Galois group of an MLF or a CAF]. Now by evaluating the monoid "$\mathcal{O}^{\rhd}(-)$" on $\mathcal{D}$ that arises from the general theory of Frobenioids [cf. [FrdI], Proposition 2.2] at A, we thus obtain a *monoid* [in the usual sense] equipped with a *natural action* by G

$$G \curvearrowright \mathcal{O}^{\rhd}(A)$$

[cf. the discussion of Example 3.2, (ii)]. If N is a positive integer, then we shall write

$$\boldsymbol{\mu}_N(A) \subseteq \mathcal{O}^{\boldsymbol{\mu}}(A) \subseteq \mathcal{O}^{\times}(A)$$

for the subgroups of N*-torsion elements* [cf. [FrdII], Definition 2.1, (i)] and *torsion elements of arbitrary order*;

$$\mathcal{O}^{\times}(A) \quad \twoheadrightarrow \quad \mathcal{O}^{\times \boldsymbol{\mu}_N}(A) \quad \twoheadrightarrow \quad \mathcal{O}^{\times \boldsymbol{\mu}}(A)$$

for the respective quotients of the submonoid of units $\mathcal{O}^{\times}(A) \subseteq \mathcal{O}^{\rhd}(A)$ by $\boldsymbol{\mu}_N(A)$, $\mathcal{O}^{\boldsymbol{\mu}}(A)$. Thus, $\mathcal{O}^{\rhd}(A)$, $\mathcal{O}^{\times}(A)$, $\mathcal{O}^{\times \boldsymbol{\mu}_N}(A)$, $\mathcal{O}^{\times \boldsymbol{\mu}}(A)$, $\boldsymbol{\mu}_N(A)$, and $\mathcal{O}^{\boldsymbol{\mu}}(A)$ are all equipped with *natural* G*-actions*. Next, let us suppose that G is *nontrivial* [i.e., arises from an MLF]. Recall the *group-theoretic algorithms* "$G \mapsto (G \curvearrowright \mathcal{O}^{\times}(G))$" and "$G \mapsto (G \curvearrowright \mathcal{O}^{\times \boldsymbol{\mu}}(G))$" discussed in Example 1.8, (iii), (iv). We define a $\times$*-Kummer structure* (respectively, $\times\boldsymbol{\mu}$*-Kummer structure*) on $\mathcal{C}$ to be a $\widehat{\mathbb{Z}}^{\times}$- (respectively, Ism- [cf. Example 1.8, (iv)]) **orbit** *of isomorphisms*

$$\kappa^{\times} : \mathcal{O}^{\times}(G) \quad \stackrel{\sim}{\rightarrow} \quad \mathcal{O}^{\times}(A) \qquad \text{(respectively,} \quad \kappa^{\times \boldsymbol{\mu}} : \mathcal{O}^{\times \boldsymbol{\mu}}(G) \quad \stackrel{\sim}{\rightarrow} \quad \mathcal{O}^{\times \boldsymbol{\mu}}(A))$$

of ind-topological G-modules. Note that since any two "universal covering pro-objects" of $\mathcal{D}$ are *isomorphic*, it follows immediately that the definition of a $\times$- (respectively, $\times\boldsymbol{\mu}$-) Kummer structure is *independent* of the choice of A. Next, let us recall from Remark 1.11.1, (b), that

any $\times$**-Kummer structure** on $\mathcal{C}$ is **unique**.

In the case of $\times\boldsymbol{\mu}$-Kummer structures, let us observe that a $\times\boldsymbol{\mu}$-Kummer structure $\kappa^{\times \boldsymbol{\mu}}$ on $\mathcal{C}$ determines, for each open subgroup $H \subseteq G$, a submodule

$$\mathcal{I}^{\kappa}_H(A) \quad \stackrel{\mathrm{def}}{=} \quad \mathrm{Im}(\mathcal{O}^{\times}(G)^H) \quad \subseteq \quad \mathcal{O}^{\times \boldsymbol{\mu}}(A)$$

— namely, the image via $\kappa^{\times \boldsymbol{\mu}}$ of the image of $\mathcal{O}^{\times}(G)^H$ in $\mathcal{O}^{\times \boldsymbol{\mu}}(G)^H$ [where the superscript "H's" denote the submodules of H-invariants]. Conversely, it is essentially a *tautology* [cf. the definition of "Ism" given in Example 1.8, (iv)!] that the $\times\boldsymbol{\mu}$-Kummer structure $\kappa^{\times \boldsymbol{\mu}}$ on $\mathcal{C}$ is *completely determined* by the submodules $\{\mathcal{I}^{\kappa}_H(A) \subseteq \mathcal{O}^{\times \boldsymbol{\mu}}(A)\}_H$ [where H ranges over the open subgroups of G], namely, as the *unique* Ism*-orbit of* G*-equivariant isomorphisms* $\mathcal{O}^{\times \boldsymbol{\mu}}(G) \stackrel{\sim}{\rightarrow} \mathcal{O}^{\times \boldsymbol{\mu}}(A)$ that maps $\mathcal{O}^{\times}(G)^H$ onto $\mathcal{I}^{\kappa}_H(A)$ for each open subgroup $H \subseteq G$. That is to say,

> a $\times\boldsymbol{\mu}$**-Kummer structure** $\kappa^{\times \boldsymbol{\mu}}$ on $\mathcal{C}$ may be thought of as — i.e., in the sense that it determines and is uniquely determined by — the **collection of submodules** $\{\mathcal{I}^{\kappa}_H(A) \subseteq \mathcal{O}^{\times \boldsymbol{\mu}}(A)\}_H$ [where H ranges over the open subgroups of G].

Finally, we shall refer to as a *[×-, ×$\boldsymbol{\mu}$-]Kummer Frobenioid* any Frobenioid equipped with a [×-, ×$\boldsymbol{\mu}$-]Kummer structure. We shall refer to as a *split-[×-, ×$\boldsymbol{\mu}$-]Kummer Frobenioid* any split Frobenioid equipped with a [×-, ×$\boldsymbol{\mu}$-]Kummer structure.

(ii) Let

$$^{\ddagger}\mathfrak{F}^{\vdash} \;=\; \{^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

be an $\mathcal{F}^{\vdash}$*-prime-strip*; $\underline{w} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Write $^{\ddagger}\mathfrak{D}^{\vdash} = \{^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ for the $\mathcal{D}^{\vdash}$-prime-strip associated to $^{\ddagger}\mathfrak{F}^{\vdash}$ [cf. [IUTchI], Remark 5.2.1, (i)]. Thus, $^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$ is a *split Frobenioid* [cf. [IUTchI], Definition 5.2, (ii), (a); [IUTchI], Example 3.2, (v)], with base category $^{\ddagger}\mathcal{D}^{\vdash}_{\underline{w}}$. Let $^{\ddagger}A$ be a *"universal covering pro-object"* of $^{\ddagger}\mathcal{D}^{\vdash}_{\underline{w}}$ [cf. the discussion of (i)]. Write $^{\ddagger}G \overset{\mathrm{def}}{=} \mathrm{Aut}(^{\ddagger}A)$ [so $^{\ddagger}G$ is a profinite group isomorphic to $G_{\underline{w}}$]. Then the $2l$*-torsion subgroup* $\boldsymbol{\mu}_{2l}(^{\ddagger}A) \subseteq \mathcal{O}^{\times}(^{\ddagger}A)$ of the submonoid of units $\mathcal{O}^{\times}(^{\ddagger}A) \subseteq \mathcal{O}^{\rhd}(^{\ddagger}A)$ of $\mathcal{O}^{\rhd}(^{\ddagger}A)$, together with the images of the *splittings* with which $^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$ is equipped, generate a *submonoid* $\mathcal{O}^{\perp}(^{\ddagger}A) \subseteq \mathcal{O}^{\rhd}(^{\ddagger}A)$, whose quotient by $\boldsymbol{\mu}_{2l}(^{\ddagger}A)$ we denote by

$$\mathcal{O}^{\rhd}(^{\ddagger}A) \;\supseteq\; \mathcal{O}^{\perp}(^{\ddagger}A) \;\twoheadrightarrow\; \mathcal{O}^{\blacktriangleright}(^{\ddagger}A) \;\overset{\mathrm{def}}{=}\; \mathcal{O}^{\perp}(^{\ddagger}A)/\boldsymbol{\mu}_{2l}(^{\ddagger}A)$$

[so we have a natural isomorphism $\mathcal{O}^{\rhd}(^{\ddagger}A)/\mathcal{O}^{\times}(^{\ddagger}A) \overset{\sim}{\to} \mathcal{O}^{\blacktriangleright}(^{\ddagger}A)$]. Write

$$\mathcal{O}^{\blacktriangleright\times\boldsymbol{\mu}}(^{\ddagger}A) \;\overset{\mathrm{def}}{=}\; \mathcal{O}^{\blacktriangleright}(^{\ddagger}A) \times \mathcal{O}^{\times\boldsymbol{\mu}}(^{\ddagger}A)$$

for the direct product monoid. Thus, the monoids $\mathcal{O}^{\rhd}(^{\ddagger}A)$, $\mathcal{O}^{\perp}(^{\ddagger}A)$, $\mathcal{O}^{\blacktriangleright}(^{\ddagger}A)$, $\mathcal{O}^{\times}(^{\ddagger}A)$, $\mathcal{O}^{\times\boldsymbol{\mu}}(^{\ddagger}A)$, $\mathcal{O}^{\boldsymbol{\mu}}(^{\ddagger}A)$, and $\mathcal{O}^{\blacktriangleright\times\boldsymbol{\mu}}(^{\ddagger}A)$ are all equipped with *natural* $^{\ddagger}G$*-actions*. Next, we consider the *group-theoretic algorithms* "$G \mapsto (G \curvearrowright \mathcal{O}^{\times}(G))$" and "$G \mapsto (G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G))$" discussed in Example 1.8, (iii), (iv). If we apply the first of these algorithms to $^{\ddagger}G$, then it follows from Remark 1.11.1, (b), that there exists a **unique** $\widehat{\mathbb{Z}}^{\times}$**-orbit** *of isomorphisms*

$$^{\ddagger}\kappa^{\vdash\times}_{\underline{w}}: \quad \mathcal{O}^{\times}(^{\ddagger}G) \;\overset{\sim}{\to}\; \mathcal{O}^{\times}(^{\ddagger}A)$$

of ind-topological modules equipped with $^{\ddagger}G$-actions. Moreover, $^{\ddagger}\kappa^{\vdash\times}_{\underline{w}}$ induces an Ism-**orbit** [cf. Example 1.8, (iv)] of isomorphisms

$$^{\ddagger}\kappa^{\vdash\times\boldsymbol{\mu}}_{\underline{w}}: \quad \mathcal{O}^{\times\boldsymbol{\mu}}(^{\ddagger}G) \;\overset{\sim}{\to}\; \mathcal{O}^{\times\boldsymbol{\mu}}(^{\ddagger}A)$$

— i.e., by forming the quotient by "$\mathcal{O}^{\boldsymbol{\mu}}(-)$".

(iii) In the notation of (ii), the [rational function monoid determined by the groupification of the] monoid with $^{\ddagger}G$-action $\mathcal{O}^{\blacktriangleright\times\boldsymbol{\mu}}(^{\ddagger}A)$, together with the divisor monoid of [the underlying Frobenioid of] $^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$, determines a *"model Frobenioid"* [cf. [FrdI], Theorem 5.2, (ii)] equipped with a *splitting*, i.e., the splitting arising from the definition of $\mathcal{O}^{\blacktriangleright\times\boldsymbol{\mu}}(^{\ddagger}A)$ as a direct product. Thus, the $^{\ddagger}G$-module obtained by evaluating at $^{\ddagger}A$ the group of units "$\mathcal{O}^{\times}(-)$" (respectively, the monoid "$\mathcal{O}^{\rhd}(-)$") associated to this Frobenioid may be naturally identified with $\mathcal{O}^{\times\boldsymbol{\mu}}(^{\ddagger}A)$ (respectively, $\mathcal{O}^{\blacktriangleright\times\boldsymbol{\mu}}(^{\ddagger}A)$). In particular, the Ism-orbit of isomorphisms $^{\ddagger}\kappa^{\vdash\times\boldsymbol{\mu}}_{\underline{w}}$ determines a $\times\boldsymbol{\mu}$*-Kummer structure* on this Frobenioid. We shall write

$$^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\underline{w}}$$

for the resulting *split-Kummer Frobenioid* and — by *abuse of notation*! —

$$^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$$

for the *split-Kummer Frobenioid* determined by the split Frobenioid $^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$ equipped with the $\times$-*Kummer structure* determined by $^{\ddagger}\kappa^{\vdash\times}_{\underline{w}}$. Here, we remark that the primary justification for this abuse of notation lies in the *uniqueness* of $\times$-Kummer structures discussed in (i) above.

(iv) Let $^{\ddagger}\mathfrak{F}^{\vdash}$ be as in (ii); $\underline{w} \in \underline{\mathbb{V}}^{\text{good}} \bigcap \underline{\mathbb{V}}^{\text{non}}$. Thus, $^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$ is a *split Frobenioid* [cf. [IUTchI], Definition 5.2, (ii), (a); [IUTchI], Example 3.3, (i)], with base category $^{\ddagger}\mathcal{D}^{\vdash}_{\underline{w}}$. Let $^{\ddagger}A$ be a *"universal covering pro-object"* of $^{\ddagger}\mathcal{D}^{\vdash}_{\underline{w}}$ [cf. the discussion of (i)]. Write $^{\ddagger}G \stackrel{\text{def}}{=} \text{Aut}(^{\ddagger}A)$ [so $^{\ddagger}G$ is a profinite group isomorphic to $G_{\underline{w}}$]. Then the image of the *splitting* with which $^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$ is equipped determines a *submonoid* $\mathcal{O}^{\perp}(^{\ddagger}A) \subseteq \mathcal{O}^{\rhd}(^{\ddagger}A)$. Write $\mathcal{O}^{\blacktriangleright}(^{\ddagger}A) \stackrel{\text{def}}{=} \mathcal{O}^{\perp}(^{\ddagger}A)$,

$$\mathcal{O}^{\blacktriangleright\times\boldsymbol{\mu}}(^{\ddagger}A) \quad \stackrel{\text{def}}{=} \quad \mathcal{O}^{\blacktriangleright}(^{\ddagger}A) \times \mathcal{O}^{\times\boldsymbol{\mu}}(^{\ddagger}A)$$

for the direct product monoid. Thus, the monoids $\mathcal{O}^{\rhd}(^{\ddagger}A)$, $\mathcal{O}^{\perp}(^{\ddagger}A)$, $\mathcal{O}^{\blacktriangleright}(^{\ddagger}A)$, $\mathcal{O}^{\times}(^{\ddagger}A)$, $\mathcal{O}^{\times\boldsymbol{\mu}}(^{\ddagger}A)$, $\mathcal{O}^{\boldsymbol{\mu}}(^{\ddagger}A)$, and $\mathcal{O}^{\blacktriangleright\times\boldsymbol{\mu}}(^{\ddagger}A)$ are all equipped with *natural* $^{\ddagger}G$-*actions*. Next, we consider the *group-theoretic algorithms* "$G \mapsto (G \curvearrowright \mathcal{O}^{\times}(G))$" and "$G \mapsto (G \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}(G))$" discussed in Example 1.8, (iii), (iv). If we apply the first of these algorithms to $^{\ddagger}G$, then it follows from Remark 1.11.1, (b), that there exists a **unique** $\widehat{\mathbb{Z}}^{\times}$**-orbit** *of isomorphisms*

$$^{\ddagger}\kappa^{\vdash\times}_{\underline{w}}: \quad \mathcal{O}^{\times}(^{\ddagger}G) \quad \stackrel{\sim}{\to} \quad \mathcal{O}^{\times}(^{\ddagger}A)$$

of ind-topological modules equipped with $^{\ddagger}G$-actions. Moreover, $^{\ddagger}\kappa^{\vdash\times}_{\underline{w}}$ induces an Ism-**orbit** [cf. Example 1.8, (iv)] of isomorphisms

$$^{\ddagger}\kappa^{\vdash\times\boldsymbol{\mu}}_{\underline{w}}: \quad \mathcal{O}^{\times\boldsymbol{\mu}}(^{\ddagger}G) \quad \stackrel{\sim}{\to} \quad \mathcal{O}^{\times\boldsymbol{\mu}}(^{\ddagger}A)$$

— i.e., by forming the quotient by "$\mathcal{O}^{\boldsymbol{\mu}}(-)$". The [rational function monoid determined by the groupification of the] monoid with $^{\ddagger}G$-action $\mathcal{O}^{\blacktriangleright\times\boldsymbol{\mu}}(^{\ddagger}A)$, together with the divisor monoid of [the underlying Frobenioid of] $^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$, determines a *"model Frobenioid"* [cf. [FrdI], Theorem 5.2, (ii)] equipped with a *splitting*, i.e., the splitting arising from the definition of $\mathcal{O}^{\blacktriangleright\times\boldsymbol{\mu}}(^{\ddagger}A)$ as a direct product. Thus, the $^{\ddagger}G$-module obtained by evaluating at $^{\ddagger}A$ the group of units "$\mathcal{O}^{\times}(-)$" (respectively, the monoid "$\mathcal{O}^{\rhd}(-)$") associated to this Frobenioid may be naturally identified with $\mathcal{O}^{\times\boldsymbol{\mu}}(^{\ddagger}A)$ (respectively, $\mathcal{O}^{\blacktriangleright\times\boldsymbol{\mu}}(^{\ddagger}A)$). In particular, the Ism-orbit of isomorphisms $^{\ddagger}\kappa^{\vdash\times\boldsymbol{\mu}}_{\underline{w}}$ determines a $\times\boldsymbol{\mu}$-*Kummer structure* on this Frobenioid. We shall write

$$^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\underline{w}}$$

for the resulting *split-Kummer Frobenioid* and — by *abuse of notation*! [cf. the discussion of (iii) above] —

$$^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$$

for the *split-Kummer Frobenioid* determined by the split Frobenioid ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$ equipped with the $\times$-*Kummer structure* determined by ${}^{\ddagger}\kappa^{\vdash\times}_{\underline{w}}$.

(v) Let ${}^{\ddagger}\mathfrak{F}^{\vdash}$ be as in (ii); $\underline{w} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Then we shall write

$${}^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\underline{w}}$$

for the collection of data obtained by replacing the *split Frobenioid* that appears in the collection of data ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$ [cf. [IUTchI], Definition 5.2, (ii), (b); [IUTchI], Example 3.4, (ii)] by the *inductive system*, indexed by any ["multiplicatively"] *cofinal* subset of the multiplicative monoid $\mathbb{N}_{\geq 1}$, of *split Frobenioids* obtained [in the evident fashion] from ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$ by forming the *quotients by the* N*-torsion*, for $N \in \mathbb{N}_{\geq 1}$. Here, we identify [in the evident fashion] the inductive systems arising from distinct cofinal subsets of $\mathbb{N}_{\geq 1}$. Thus, [cf. the notation of (i)] the *units* of the split Frobenioids of this inductive system give rise to an inductive system

$$\ldots \quad \twoheadrightarrow \quad \mathcal{O}^{\times\boldsymbol{\mu}_N}(A) \quad \twoheadrightarrow \quad \ldots \quad \twoheadrightarrow \quad \mathcal{O}^{\times\boldsymbol{\mu}_{N\cdot N'}}(A) \quad \twoheadrightarrow \quad \ldots$$

[where $N, N' \in \mathbb{N}_{\geq 1}$]. Now recall that ${}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{w}}$ is an *object of the category* $\mathbb{TM}^{\vdash}$ [cf. [IUTchI], Definition 4.1, (iii), (b)]. In particular, the *units* $({}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{w}})^{\times}$ of this object of $\mathbb{TM}^{\vdash}$ form a *topological group* [noncanonically isomorphic to $\mathbb{S}^1$], which we think of as being related to the above inductive system of units via a *system of compatible surjections*

$$({}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{w}})^{\times} \ \twoheadrightarrow \ \mathcal{O}^{\times\boldsymbol{\mu}_N}(A)$$

[i.e., where the kernel of the displayed surjection is the subgroup of N*-torsion*]. This system of compatible surjections is well-defined up to an *indeterminacy* given by composition with the unique nontrivial automorphism of $({}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{w}})^{\times}$. When considered up to this indeterminacy, this system of compatible surjections may be thought of as a sort of *Kummer structure* on ${}^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\underline{w}}$ [which may be algorithmically reconstructed from the collection of data ${}^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\underline{w}}$].

(vi) Write

$${}^{\ddagger}\mathfrak{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}} = \{{}^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

for the collection of data indexed by $\underline{\mathbb{V}}$ obtained as follows: (a) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, then we take ${}^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\underline{v}}$ to be the *split-Kummer Frobenioid* constructed in (iii); (b) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$, then we take ${}^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\underline{v}}$ to be the *split-Kummer Frobenioid* constructed in (iv); (c) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then we take ${}^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\underline{v}}$ to be the *collection of data* constructed in (v). Moreover, by replacing the various split Frobenioids of ${}^{\ddagger}\mathfrak{F}^{\vdash}$ (respectively, ${}^{\ddagger}\mathfrak{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}$) with the split Frobenioids — i.e., equipped with *trivial splittings*! — obtained by considering the subcategories [of the underlying categories associated to these Frobenioids] determined by the *isometries* [i.e., roughly speaking, the *"units"* — cf. [FrdI], Theorem 5.1, (iii), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$; [FrdII], Example 3.3, (iii), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$], one obtains a collection of data

$${}^{\ddagger}\mathfrak{F}^{\vdash\times} \ = \ \{{}^{\ddagger}\mathcal{F}^{\vdash\times}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \quad \text{(respectively, } {}^{\ddagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}} \ = \ \{{}^{\ddagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

indexed by $\underline{\mathbb{V}}$. Thus, for each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, ${}^{\ddagger}\mathcal{F}^{\vdash\times}_{\underline{v}}$ (respectively, ${}^{\ddagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}$) is a *split-*$\times$- (respectively, *split-*$\times\boldsymbol{\mu}$-) *Kummer Frobenioid*.

(vii) Let $\vdash\Box \ \in \ \{ \vdash\times, \ \vdash\times\boldsymbol{\mu}, \ \vdash \blacktriangleright \times\boldsymbol{\mu} \ \}$. Then we define an $\mathcal{F}^{\vdash\Box}$-*prime-strip* to be a collection of data

$$^*\mathfrak{F}^{\vdash\Box} = \{^*\mathcal{F}^{\vdash\Box}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

such that for each $\underline{v} \in \underline{\mathbb{V}}$, $^*\mathcal{F}^{\vdash\Box}_{\underline{v}}$ is a collection of data that is isomorphic to $^\ddagger\mathcal{F}^{\vdash\Box}_{\underline{v}}$ [cf. (vi)]. A *morphism of* $\mathcal{F}^{\vdash\Box}$-*prime-strips* is defined to be a collection of isomorphisms, indexed by $\underline{\mathbb{V}}$, between the various constituent objects of the prime-strips [cf. [IUTchI], Definition 5.2, (iii)].

(viii) We define an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* to be a collection of data

$$^*\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}} \ = \ (^*\mathcal{C}^{\Vdash}, \ \mathrm{Prime}(^*\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}}, \ ^*\mathfrak{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}, \ \{^*\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

satisfying the conditions (a), (b), (c), (d), (e), (f) of [IUTchI], Definition 5.2, (iv), for an $\mathcal{F}^{\Vdash}$-prime-strip, where the portion of the collection of data constituted by an $\mathcal{F}^{\vdash}$-prime-strip is replaced by an $\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip. Thus, relative to the notation of the above display [cf. also (ii), (iii)], the *generators* of the monoids "$\mathcal{O}^{\blacktriangleright}(-)$" [each of which is abstractly isomorphic to $\mathbb{N}$] of the data at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ ($\neq \emptyset$) [cf. [IUTchI], Definition 3.1, (b)] of $^*\mathfrak{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}} = \{^*\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$, together with the $\{^*\rho_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$, determine a well-defined object, up to isomorphism, of the global realified Frobenioid $^*\mathcal{C}^{\Vdash}$ of *negative "arithmetic degree"* [cf. [FrdI], Example 6.3; [FrdI], Theorem 6.4, (i), (ii)], which we refer to as the *pilot object* associated to the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* $^*\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$. A *morphism of* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strips* is defined to be an isomorphism between collections of data as discussed above.

We conclude the present paper with the following two results, which may be thought of as *enhanced versions* of [IUTchI], Corollaries 3.7, 3.8, 3.9 — i.e., versions that reflect the various enhancements made to the theory in [IUTchI], §4, §5, §6, as well as in the present paper.

Corollary 4.10. **(Frobenius-pictures of $\Theta^{\pm\mathrm{ell}}$NF-Hodge Theaters)** *Fix a collection of* **initial Θ-data** *($\overline{F}/F, \ X_F, \ l, \ \underline{C}_K, \ \underline{\mathbb{V}}, \ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}, \ \underline{\epsilon}$) as in [IUTchI], Definition 3.1. Let*

$$^\dagger\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}; \quad ^\ddagger\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

be $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *[relative to the given initial Θ-data] — cf. [IUTchI], Definition 6.13, (i). Write $^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, $^\ddagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ for the associated* **$\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** *— cf. [IUTchI], Definition 6.13, (ii). Then:*

(i) **(Constant Prime-Strips)** *Let us apply the constructions of Corollary 4.6, (i), (iii), to the underlying $\Theta^{\pm\mathrm{ell}}$-Hodge theater of $^\dagger\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$. Then, for each $t \in \mathrm{LabCusp}^{\pm}(^\dagger\mathfrak{D}_{\succ})$, the collection of data $\Psi_{\mathrm{cns}}(^\dagger\mathfrak{F}_{\succ})_t$ determines, in a natural way, an* **$\mathcal{F}$-prime-strip** *[cf. Remark 4.6.2, (i)]. Let us* **identify** *the collections of data*

$$\Psi_{\mathrm{cns}}(^\dagger\mathfrak{F}_{\succ})_0 \ \textit{and} \ \Psi_{\mathrm{cns}}(^\dagger\mathfrak{F}_{\succ})_{\langle\mathbb{F}^{*}_{l}\rangle}$$

via the **isomorphism** *of the second display of Corollary 4.6, (iii), and denote by*

$$^\dagger\mathfrak{F}^{\Vdash}_{\triangle} \ = \ (^\dagger\mathcal{C}^{\Vdash}_{\triangle}, \ \mathrm{Prime}(^\dagger\mathcal{C}^{\Vdash}_{\triangle}) \xrightarrow{\sim} \underline{\mathbb{V}}, \ ^\dagger\mathfrak{F}^{\vdash}_{\triangle}, \ \{^\dagger\rho_{\triangle,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

the resulting $\mathcal{F}^{\Vdash}$**-prime-strip** *determined by the constructions discussed in [IUTchI], Remark 5.2.1, (ii) [which, as is easily verified, are* **compatible** *with the* $\mathbb{F}_l^{\rtimes\pm}$**-symmetrizing isomorphisms** *of Corollary 4.6, (iii)]. Thus, [it follows immediately from the constructions involved that] one has a* **natural identification isomorphism** *of* $\mathcal{F}^{\Vdash}$*-prime-strips* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\triangle} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ *between* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\triangle}$ *and the collection of data* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ *associated to the underlying* Θ*-Hodge theater of* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *[cf. [IUTchI], Definition 3.6, (c)] — cf. the discussion of the assignment*

$$\text{“}\, 0, \succ \;\; \mapsto \;\; > \,\text{”}$$

in Remark 3.8.2, (ii).

(ii) **(Theta and Gaussian Prime-Strips)** *Let us apply the constructions of Corollary 4.6, (iv), (v), to the underlying* Θ*-bridge and* $\Theta^{\pm\mathrm{ell}}$*-Hodge theater of* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*. Then the collection of data* $\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})$ *[cf. Corollary 4.6, (iv)], the global realified Frobenioid* ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{env}} \stackrel{\mathrm{def}}{=} \mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta})$ *[cf. Corollary 4.6, (v)], and the local isomorphisms* $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}),\underline{v}} \xrightarrow{\sim} \Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathbb{R}}_{\underline{v}}$ *for* $\underline{v} \in \underline{\mathbb{V}}$ *[cf. Corollary 4.6, (v)] give rise, in a natural fashion, to an* $\mathcal{F}^{\Vdash}$*-prime-strip*

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \;=\; ({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{env}},\; \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{env}}) \xrightarrow{\sim} \underline{\mathbb{V}},\; {}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}},\; \{{}^{\dagger}\rho_{\mathrm{env},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

[so, in particular, ${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}}$ *is the* $\mathcal{F}^{\vdash}$*-prime-strip determined by* $\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})$ *— cf. Remark 4.6.2, (i); Remark 4.10.1 below]. Thus, [it follows immediately from the constructions involved that] there is a* **natural identification isomorphism** *of* $\mathcal{F}^{\Vdash}$*-prime-strips* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$ *between* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}$ *and the collection of data* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$ *associated to the underlying* Θ*-Hodge theater of* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *[cf. [IUTchI], Definition 3.6, (c)]. In a similar vein, the collection of data* $\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})$ *[cf. Corollary 4.6, (iv)], the global realified Frobenioid* ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{gau}} \stackrel{\mathrm{def}}{=} \mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})$ *[cf. Corollary 4.6, (v)], and the local isomorphisms* $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}),\underline{v}} \xrightarrow{\sim} \Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathbb{R}}_{\underline{v}}$ *for* $\underline{v} \in \underline{\mathbb{V}}$ *[cf. Corollary 4.6, (v)] give rise, in a natural fashion, to an* $\mathcal{F}^{\Vdash}$*-prime-strip*

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}} \;=\; ({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{gau}},\; \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{gau}}) \xrightarrow{\sim} \underline{\mathbb{V}},\; {}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{gau}},\; \{{}^{\dagger}\rho_{\mathrm{gau},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

[so, in particular, ${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{gau}}$ *is the* $\mathcal{F}^{\vdash}$*-prime-strip determined by* $\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{\dagger}\mathcal{HT}^{\Theta})$ *— cf. Remark 4.6.2, (i); Remark 4.10.1 below]. Finally, the evaluation isomorphisms of Corollary 4.6, (iv), (v), determine an* **evaluation isomorphism**

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \;\;\xrightarrow{\sim}\;\; {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$$

of $\mathcal{F}^{\Vdash}$*-prime-strips.*

(iii) **($\Theta^{\times\boldsymbol{\mu}}$- and $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-Links)** *Write* ${}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle}$ *(respectively,* ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}}$*;* ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{gau}}$*) for the* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip associated to the* $\mathcal{F}^{\Vdash}$*-prime-strip* ${}^{\ddagger}\mathfrak{F}^{\Vdash}_{\triangle}$ *(respectively,* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}$*;* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$*) [cf. Definition 4.9, (viii); the functorial algorithm described in Definition 4.9, (vi)]. Then the functoriality of this algorithm induces maps*

$$\mathrm{Isom}_{\mathcal{F}^{\Vdash}}({}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}, {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\triangle}) \;\;\to\;\; \mathrm{Isom}_{\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}}({}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}}, {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle})$$

$$\mathrm{Isom}_{\mathcal{F}^{\Vdash}}({}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}, {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\triangle}) \;\;\to\;\; \mathrm{Isom}_{\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}}({}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{gau}}, {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle})$$

*from [**nonempty!**] sets of isomorphisms of $\mathcal{F}^{\Vdash}$-prime-strips to [**nonempty!**] sets of isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips. Here, the second map may be regarded as being obtained from the first map via* **composition** *[in the case of the domain "*$\mathrm{Isom}_{\mathcal{F}^{\Vdash}}(-,-)$*"] with the* **evaluation isomorphism** $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$ *of (ii) and composition [in the case of the codomain "*$\mathrm{Isom}_{\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}}(-,-)$*"] with the isomorphism* $^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{gau}}$ *functorially obtained from this isomorphism of (ii). We shall refer to the full poly-isomorphism* $^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle}$ *as the* $\Theta^{\times\boldsymbol{\mu}}$**-link**

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \stackrel{\Theta^{\times\boldsymbol{\mu}}}{\longrightarrow} \quad {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

[cf. the "Θ-link" of [IUTchI], Corollary 3.7, (i)] from $^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *to* $^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*, and to the full poly-isomorphism* $^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{gau}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle}$ *as the* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$**-link**

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \stackrel{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}}{\longrightarrow} \quad {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

from $^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *to* $^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.

(iv) **(Coric** $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$**-Prime-Strips)** *The definition of the* **unit portion** *of the* **theta** *and* **Gaussian monoids** *involved [cf. Corollary 3.5, (ii); Corollary 3.6, (ii); Proposition 4.1, (iv); Proposition 4.2, (iv); Proposition 4.3, (iv); Proposition 4.4, (iv)] gives rise to* **natural isomorphisms**

$$^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \quad \xrightarrow{\sim} \quad {}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}} \quad \xrightarrow{\sim} \quad {}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{gau}}$$

— where we write $^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}$, $^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}$, $^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{gau}}$ *for the* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips associated to the* $\mathcal{F}^{\vdash}$*-prime-strips* $^{\dagger}\mathfrak{F}^{\vdash}_{\triangle}$, $^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}}$, $^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{gau}}$*, respectively [cf. the functorial algorithm described in Definition 4.9, (vi)]. Moreover, by composing these natural isomorphisms with the poly-isomorphisms induced on the respective* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips by the* $\Theta^{\times\boldsymbol{\mu}}$*- and* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$*-links of (iii), one obtains a poly-isomorphism*

$$^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \quad \xrightarrow{\sim} \quad {}^{\ddagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}$$

which coincides with the **full** *poly-isomorphism between these two* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips — that is to say, "*$^{(-)}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}$*" is an* **invariant** *of both the* $\Theta^{\times\boldsymbol{\mu}}$*- and* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$*-links. Finally, this full poly-isomorphism induces [cf. Definition 4.9, (vii); [IUTchI], Remark 5.2.1, (i)] the* **full** *poly-isomorphism*

$$^{\dagger}\mathfrak{D}^{\vdash}_{\triangle} \quad \xrightarrow{\sim} \quad {}^{\ddagger}\mathfrak{D}^{\vdash}_{\triangle}$$

between the associated $\mathcal{D}^{\vdash}$*-prime-strips; we shall refer to this poly-isomorphism as the* $\mathcal{D}$**-**$\Theta^{\pm\mathrm{ell}}$**NF-link** *from* $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *to* $^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.

(v) **(Coric Global Realified Frobenioids)** *The full poly-isomorphism* $^{\dagger}\mathfrak{D}^{\vdash}_{\triangle} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}^{\vdash}_{\triangle}$ *of (iv) induces [cf. Corollary 4.5, (ii)] an isomorphism of collections of data*

$$(\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}),\ \mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{{}^{\dagger}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$
$$\xrightarrow{\sim} \quad (\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash}_{\triangle}),\ \mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash}_{\triangle})) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{{}^{\ddagger}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

— i.e., consisting of a Frobenioid, a bijection, and a collection of isomorphisms of topological monoids indexed by $\underline{\mathbb{V}}$. *Moreover, this isomorphism of collections of data is* **compatible***, relative to the* $\Theta^{\times\boldsymbol{\mu}}$**-** *and* $\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}$**-links** *of (iii), with the* $\mathbb{R}_{>0}$**-orbits** *of the isomorphisms of collections of data*

$$\begin{aligned}&({}^{\dagger}\mathcal{C}^{\Vdash}_{\triangle},\ \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\triangle}) \overset{\sim}{\to} \underline{\mathbb{V}},\ \{{}^{\dagger}\rho_{\triangle,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})\\ &\qquad \overset{\sim}{\to} \quad (\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}),\ \mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})) \overset{\sim}{\to} \underline{\mathbb{V}},\ \{{}^{\dagger}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})\end{aligned}$$

$$\begin{aligned}&({}^{\ddagger}\mathcal{C}^{\Vdash}_{\triangle},\ \mathrm{Prime}({}^{\ddagger}\mathcal{C}^{\Vdash}_{\triangle}) \overset{\sim}{\to} \underline{\mathbb{V}},\ \{{}^{\ddagger}\rho_{\triangle,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})\\ &\qquad \overset{\sim}{\to} \quad (\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash}_{\triangle}),\ \mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash}_{\triangle})) \overset{\sim}{\to} \underline{\mathbb{V}},\ \{{}^{\ddagger}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})\end{aligned}$$

obtained by applying the functorial algorithm discussed in the final portion of Corollary 4.6, (ii). Here, the "$\mathbb{R}_{>0}$*-orbits" are defined relative to the* **natural** $\mathbb{R}_{>0}$**-actions** *on the Frobenioids involved obtained by multiplying the "arithmetic degrees" by a given element* $\in \mathbb{R}_{>0}$ *[cf. [FrdI], Example 6.3; [FrdI], Theorem 6.4, (ii); [IUTchI], Remark 3.1.5].*

(vi) **(Frobenius-pictures)** *Let* $\{{}^{n}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}\}_{n\in\mathbb{Z}}$ *be a* **collection of distinct** $\Theta^{\pm\text{ell}}$**NF-Hodge theaters** *indexed by the integers. Then by applying the* $\Theta^{\times\boldsymbol{\mu}}$**-** *and* $\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}$**-links** *of (iii), we obtain* **infinite chains**

$$\ldots \quad \overset{\Theta^{\times\boldsymbol{\mu}}}{\longrightarrow} \quad {}^{(n-1)}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \quad \overset{\Theta^{\times\boldsymbol{\mu}}}{\longrightarrow} \quad {}^{n}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \quad \overset{\Theta^{\times\boldsymbol{\mu}}}{\longrightarrow} \quad {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \quad \overset{\Theta^{\times\boldsymbol{\mu}}}{\longrightarrow} \quad \ldots$$

$$\ldots \quad \overset{\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}}{\longrightarrow} \quad {}^{(n-1)}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \quad \overset{\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}}{\longrightarrow} \quad {}^{n}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \quad \overset{\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}}{\longrightarrow} \quad {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \quad \overset{\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}}{\longrightarrow} \quad \ldots$$

of $\Theta^{\times\boldsymbol{\mu}}$**-/**$\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}$**-linked** $\Theta^{\pm\text{ell}}$**NF-Hodge theaters**. *Either of these infinite chains may be represented symbolically as an* **oriented graph** $\vec{\Gamma}$ *[cf. [AbsTopIII], §0]*

$$\ldots \quad \to \quad \bullet \quad \to \quad \bullet \quad \to \quad \bullet \quad \to \quad \ldots$$

— i.e., where the arrows correspond to either the " $\overset{\Theta^{\times\boldsymbol{\mu}}}{\longrightarrow}$ *'s" or the "* $\overset{\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}}{\longrightarrow}$ *'s", and the "•'s" correspond to the "*${}^{n}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$*". This oriented graph* $\vec{\Gamma}$ *admits a natural action by* $\mathbb{Z}$ *— i.e., a* **translation symmetry** *— but it does* **not admit arbitrary permutation symmetries***. For instance,* $\vec{\Gamma}$ *does not admit an automorphism that switches two adjacent vertices, but leaves the remaining vertices fixed — cf. the discussion of [IUTchI], Corollary 3.8; [IUTchI], Remark 3.8.1.*

Proof. The various assertions of Corollary 4.10 follow immediately from the definitions and the references quoted in the statements of these assertions. ○

Remark 4.10.1. Strictly speaking [cf. Remark 4.6.2, (i)], the $\mathcal{F}^{\vdash}$*-prime-strips* constructed, in Corollary 4.10, (ii), from the theta and Gaussian monoids of Corollary 4.6, (iv), are only well-defined up to an *indeterminacy*, at the $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, relative to *automorphisms of the split Frobenioid* at such $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ that induce the identity automorphism on the associated $\mathcal{F}^{\vdash\times}$*-prime-strip*. On the other hand, such indeterminacies may, in essence, be ignored, since they are *"absorbed"* in the *full poly-isomorphisms* that appear in the $\Theta^{\times\boldsymbol{\mu}}$- and $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-*links* of Corollary 4.10, (iii).

Remark 4.10.2.

(i) Although both the $\Theta^{\times\boldsymbol{\mu}}$- and $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-*links* are treated, in essence, on an equal footing in Corollary 4.10, in the remainder of the present series of papers, we shall ultimately mainly be interested in [a further enhanced version of] the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-*link*. On the other hand, the *significance* of the $\Theta^{\times\boldsymbol{\mu}}$-link lies in the fact that it is precisely by thinking of [a further enhanced version of] the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link as an object that is constructed as the *composite* of the $\Theta^{\times\boldsymbol{\mu}}$-link with the operation of *Galois evaluation* that one may establish the crucial **multiradiality** properties discussed in [IUTchIII], Theorem 3.11.

(ii) At $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, the $\Theta^{\times\boldsymbol{\mu}}$-link may be thought of as a sort of *equivalence* between the *split theta monoids* of Proposition 3.1, (i) [cf. also Corollary 1.12, (ii)] and certain submonoids of the *constant monoids* of Proposition 3.1, (ii), equipped with the *splittings* that arise from the q-parameter "$\underline{\underline{q}}_{\underline{v}}$" [cf. the discussion of "$\tau^{\vdash}_{\underline{v}}$" in [IUTchI], Example 3.2, (iv)]. On the other hand, it is important to note in this context that unlike the case with the *splittings* that occur in the case of the *theta monoids*, the splittings that occur in the case of the *constant monoids* do **not** arise from the operation of **Galois evaluation** — i.e., from a *splitting "$H \hookrightarrow G_{\underline{v}}$" at the level of Galois groups* of some surjection $G_{\underline{v}} \twoheadrightarrow H$. In particular, the splittings in the case of the constant monoids do **not** *admit a natural* **multiradial** *formulation* [cf. Remark 1.11.5; Proposition 3.4, (ii)], as in the case of the theta monoids [cf. Corollary 1.12, (iii)], that allows one to *decouple* the monoids into *"purely radial"* and *"purely coric"* components [cf. discussion of Remarks 1.11.4, (i); 1.12.2, (vi)].

Remark 4.10.3.

(i) The *"coricity of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips"*

$$^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \quad \stackrel{\sim}{\rightarrow} \quad ^{\ddagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}$$

discussed in Corollary 4.10, (iv), amounts, in essence, to the *"coricity of $\mathcal{D}^{\vdash}$-prime-strips"* $^{\dagger}\mathfrak{D}^{\vdash}_{\triangle} \stackrel{\sim}{\rightarrow} {}^{\ddagger}\mathfrak{D}^{\vdash}_{\triangle}$ [cf. Corollary 4.10, (iv)], together with the *"coricity of [various quotients by torsion of] the units $\mathcal{O}^{\times}(-)$"* of the Frobenioids involved — cf. [IUTchI], Corollary 3.7, (ii), (iii). In [IUTchIII], this **coricity of the units** will play a *central role* when we apply the theory of the $\mathfrak{log}$-*wall* [cf. [AbsTopIII]]. In particular, this coricity of the units will allow us to *compare* **volumes** on either side of the $\Theta^{\times\boldsymbol{\mu}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links.

(ii) Unlike the units [cf. the discussion of (i)!], the *"divisor monoid"*, or *"value group"*, portion of the Frobenioids involved is *by no means preserved by the $\Theta^{\times\boldsymbol{\mu}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links*! Indeed, this "value group" portion of the $\Theta^{\times\boldsymbol{\mu}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links may be thought of as a sort of *"Frobenius morphism"* — cf. the discussion of Remark 3.6.2, (iii), as well as Remark 4.11.1 below. Alternatively, from the point of view of the analogy between [complex or p-adic] Teichmüller theory and the theory of the present series of papers, this portion of the $\Theta^{\times\boldsymbol{\mu}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links may be thought of as a sort of *Teichmüller deformation* [cf. the discussion of [IUTchI], Remark 3.9.3, (ii)]. Indeed, the computation of the **"volume distortion"** arising from this **"arithmetic Teichmüller deformation"** may, in some sense, be regarded as the *ultimate goal* of the present series of papers.

(iii) In the context of the discussion of (ii), it is interesting to note that if one restricts the value group portion of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link — i.e.,

$$\underline{\underline{q}}_{v} \quad \mapsto \quad \left\{ \underline{\underline{q}}_{v}^{j^2} \right\}_{1 \leq j \leq l^{*}}$$

[cf. Remark 3.6.2, (iii)] — to the label $j = 1$, then the resulting correspondence

$$\underline{\underline{q}}_{v} \quad \mapsto \quad \underline{\underline{q}}_{v}$$

may be naturally identified with the *"identity"* — cf. the discussion of Remark 3.6.2, (iii). Put another way, the *restriction to the label* $j = 1$ *of the Gaussian distribution* may be identified, for instance at the level of *realifications*, with the *pivotal distribution* discussed in [IUTchI], Example 5.4, (vii). On the other hand, in this context, it is important to observe that the operation of restriction to various *proper subsets* of the set of all labels $|\mathbb{F}_l|$ *fails*, in general, *to be compatible* with the **crucial** $\mathbb{F}_l^{\rtimes\pm}$**-** *and* $\mathbb{F}_l^{*}$**-symmetries** of Corollaries 4.5, (iii); 4.6, (iii); 4.7, (ii); 4.8, (ii) [cf. also the discussion of Remark 2.6.3].

$$\begin{array}{ccccc}
 & & \boxed{{}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}} & & \\
\dots & & | & & \dots \\
\boxed{{}^{n'}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}} & — & \boxed{{}^{(-)}\mathfrak{D}^{\vdash}_{\triangle}} & — & \boxed{{}^{n''}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}} \\
\dots & & | & & \dots \\
 & & \boxed{{}^{n'''}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}} & &
\end{array}$$

Fig. 4.3: Étale-picture of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge Theaters

Corollary 4.11. (Étale-pictures of Base-$\Theta^{\pm\mathrm{ell}}$NF-Hodge Theaters) *Suppose that we are in the situation of Corollary 4.10, (vi).*

(i) Write

$$\dots \xrightarrow{\mathcal{D}} {}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathcal{D}} {}^{(n+1)}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathcal{D}} \dots$$

— *where* $n \in \mathbb{Z}$ — *for the* **infinite chain** *of* $\mathcal{D}$-$\Theta^{\pm \text{ell}}$**NF-linked** $\mathcal{D}$-$\Theta^{\pm \text{ell}}$**NF-Hodge theaters** *[cf. Corollary 4.10, (iv), (vi)] induced by either of the infinite chains of Corollary 4.10, (vi). Then this infinite chain induces a* **chain of full poly-isomorphisms**

$$\dots \ \stackrel{\sim}{\rightarrow} \ {}^{n}\mathfrak{D}^{\vdash}_{\triangle} \ \stackrel{\sim}{\rightarrow} \ {}^{(n+1)}\mathfrak{D}^{\vdash}_{\triangle} \ \stackrel{\sim}{\rightarrow} \ \dots$$

[cf. Corollary 4.10, (iv)]. That is to say, "${}^{(-)}\mathfrak{D}^{\vdash}_{\triangle}$*" forms a* **constant invariant** *[cf. the discussion of [IUTchI], Remark 3.8.1, (ii)] — i.e., a* **mono-analytic core** *[cf. the situation discussed in [IUTchI], Remark 3.9.1] — of the above infinite chain.*

(ii) If we regard each of the $\mathcal{D}$-$\Theta^{\pm \text{ell}}$*NF-Hodge theaters of the chain of (i) as a* **spoke** *emanating from the mono-analytic core "*${}^{(-)}\mathfrak{D}^{\vdash}_{\triangle}$*" discussed in (i), then we obtain a* **diagram** *— i.e., an* **étale-picture of** $\mathcal{D}$-$\Theta^{\pm \text{ell}}$**NF-Hodge theaters** *— as in Fig. 4.3 above [cf. the situation discussed in [IUTchI], Corollaries 4.12, 6.10]. Thus, each spoke may be thought of as a* **distinct "arithmetic holomorphic structure"** *on the mono-analytic core. Finally, [cf. the situation discussed in [IUTchI], Corollaries 4.12, 6.10] this diagram satisfies the important property of admitting* **arbitrary permutation symmetries** *among the spokes [i.e., the labels* $n \in \mathbb{Z}$ *of the* $\mathcal{D}$-$\Theta^{\pm \text{ell}}$*NF-Hodge theaters].*

(iii) The constructions of (i) and (ii) are **compatible***, in the evident sense, with the constructions of [IUTchI], Corollaries 4.12, 6.10, relative to the* **natural identification isomorphisms** ${}^{(-)}\mathfrak{D}^{\vdash}_{\triangle} \ \stackrel{\sim}{\rightarrow} \ {}^{(-)}\mathfrak{D}^{\vdash}_{>}$ *[cf. Corollary 4.10, (i); the discussion preceding [IUTchI], Example 5.4] and the operation of passing to the underlying* $\mathcal{D}$-ΘNF- *[in the case of [IUTchI], Corollary 4.12] and* $\mathcal{D}$-$\Theta^{\pm \text{ell}}$-**Hodge theaters** *[in the case of [IUTchI], Corollary 6.10].*

Proof. The various assertions of Corollary 4.11 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 4.11.1. The $\Theta^{\times \boldsymbol{\mu}}_{\text{gau}}$**-link** of Corollary 4.10, (iii), may be thought of, roughly, as a sort of *transformation*

$$\underline{\underline{q}} \quad \mapsto \quad \underline{\underline{q}}^{\begin{pmatrix} 1^2 \\ \vdots \\ (l^*)^2 \end{pmatrix}}$$

— cf. the discussion of Remark 3.6.2, (iii). From this point of view, the infinite chain of the **Frobenius-picture** discussed in Corollary 4.10, (vi), may be represented as an *infinite iteration*

$$\underline{\underline{q}} \quad \mapsto \quad \left(\underline{\underline{q}}^{\begin{pmatrix} 1^2 \\ \vdots \\ (l^*)^2 \end{pmatrix}} \right)^{\begin{pmatrix} 1^2 \\ \vdots \\ (l^*)^2 \end{pmatrix}^{\cdots}}$$

of this transformation. By contrast, the associated **étale-picture** discussed in Corollary 4.11 corresponds to a sort of *commutativity* involving these "theta exponents"

$$\underline{\underline{q}} \quad \mapsto \quad \underline{\underline{q}}^{\begin{pmatrix} 1^2 \\ \vdots \\ (l^*)^2 \end{pmatrix} \cdot \begin{pmatrix} 1^2 \\ \vdots \\ (l^*)^2 \end{pmatrix} \cdot \begin{pmatrix} 1^2 \\ \vdots \\ (l^*)^2 \end{pmatrix} \quad \cdots}$$

— cf. the *"arbitrary permutation symmetries"* discussed in Corollary 4.11, (ii). In this context, it is useful to recall the analogy between the **classical Gaussian integral** and the theory of the present series of papers [cf. Remark 1.12.5] — an analogy in which the *"order-conscious"* **Frobenius-picture** corresponds to the **cartesian coordinate** representation of the Gaussian integral, while the *"permutation-symmetric"* **étale-picture** corresponds to the **polar coordinate** representation of the Gaussian integral. Finally, from the point of view of the discussion of Remark 4.7.4, the *l-torsion* that occurs as the index set of the various "$\underline{\underline{q}}^{j^2}$'s" that appear in the Gaussian monoid of each $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater may be thought of as a sort of **multiradial combinatorial representation** of the distinct **"arithmetic holomorphic structures"** corresponding to the various $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters.

Remark 4.11.2. At this point, we pause to review the theory developed so far in the present series of papers.

(i) The notion of a $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater* [cf. [IUTchI], Definition 6.13, (i)] is intended as a *model of* **conventional scheme-theoretic arithmetic geometry** — i.e., more precisely, of the conventional scheme-theoretic arithmetic geometry surrounding the *theta function* at primes of bad reduction $\in \underline{\mathbb{V}}^{\mathrm{bad}}$ of the elliptic curve over a number field under consideration. At a more technical level, a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater may be thought of as an apparatus that allows one to construct a sort of **bridge** between the **number field** and **theta functions** [at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] under consideration. From a more concrete point of view, this bridge is realized by the **Gaussian distribution** — i.e., a globalized version of the *theta values*

$$\left\{ \underline{\underline{q}}_{\underline{v}}^{j^2} \right\}_{1 \leq j \leq l^*}$$

at l-torsion points [cf. Remark 3.6.2, (iii)]. Here, we remark that the term "Gaussian distribution" is intended as an intuitive expression that includes the more technical notions of "Gaussian monoids" and "Gaussian Frobenioids". The Gaussian distribution also plays the *crucial role* of allowing the construction of the *[non-scheme/ring-theoretic!]* $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$**-link** between distinct $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters [cf. Corollary 4.10, (iii)] — i.e., between **distinct** *models of conventional scheme-theoretic arithmetic geometry.*

(ii) Within a single $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater, the theory of étale and Frobenioid-theoretic **theta functions** developed in [EtTh] is applied to construct a *single* **connected** *geometric "Kummer theory-compatible theater for evaluation of the theta function"*, whose *étale-theoretic realization* admits a **multiradial formulation** [cf. the theory of §1, especially Corollary 1.12], and whose connectedness allows one to establish **conjugate synchronization** [cf. the discussion of Remark 2.6.1] between the various copies of the absolute Galois group of the base field at the various l-torsion points at which the theta function is evaluated. Moreover, this conjugate synchronization satisfies the crucial property of *compatibility* with the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** [cf. the discussion of Remark 3.5.2, as well as Corollaries 4.5, (iii); 4.6, (iii)] of the underlying $\mathcal{D}$-Θ^{ell}-bridge [cf. [IUTchI], Proposition 6.8, (i)] of the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater under consideration. Conjugate synchronization plays an essential role in establishing the **coricity of the units** [cf. Corollary 4.10, (iv)] in a fashion which is *compatible* with both the *étale-theoretic* — i.e., **"anabelian"** — and *abstract monoid/Frobenioid-theoretic* — i.e., **"post-anabelian"** — representations of the *Gaussian monoids* [cf. the discussion of Remark 3.8.3]. Here, we recall that the "post-anabelian" representation of the Gaussian monoids is necessary to construct the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link of Corollary 4.10, (iii) [cf. Remarks 3.6.2, (ii); 3.8.3, (i)]. On the other hand, the "anabelian" representation of the Gaussian monoids will play an essential role when we apply the theory of the $\mathfrak{log}$**-wall** [cf. [AbsTopIII]] in [IUTchIII] [cf. Remark 3.8.3, (ii)]. Another important aspect of the theory of Gaussian distibutions, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, is the **canonical splittings** of the monoids involved into **"unit"** and **"value group"** components. These splittings may be thought of, in the context of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link, as corresponding to the **"non-deformed"** [cf. the "coricity of the units"] and **"Teichmüller-dilated"** [cf. the "value group" portion of the Gaussian distribution] real dimensions that appear in classical complex Teichmüller theory [cf. the discussion of Remark 4.10.3, (i), (ii)]. Finally, these splittings will play a crucial role in the theory of **log-shells** [cf. [AbsTopIII]], which we shall apply in [IUTchIII].

(iii) By contrast, the **number fields** that appear in the underlying ΘNF-Hodge theater of the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater under consideration [cf. the theory of [IUTchI], §5] will ultimately, in [IUTchIII], in the context of **log-shells**, play the role of *relating* — via the **ring structure** of these number fields — **⊠-line bundles** [i.e., "idèlic" or "Frobenioid-theoretic" line bundles] to **"⊞-line bundles"** [i.e., line bundles thought of as *modules*] — cf. the discussion of Remark 4.7.2. Such relationships are only possible if one considers **all** of the **primes** of the number fields involved [cf. [AbsTopIII], Remark 5.10.2, (iv)]. Constructions associated to these number fields satisfy the property of being compatible with the $\mathbb{F}_l^{*}$**-symmetry** [cf. [IUTchI], Proposition 4.9, (i)] of the underlying NF-bridge of the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater under consideration. Unlike the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* discussed in (ii), which is *combinatorially* **uniradial** in nature and may be thought of, in the context of the *splittings* discussed in (ii), as being associated with the *"units"*, the $\mathbb{F}_l^{*}$-*symmetry* is *combinatorially* **multiradial** in nature and may be thought of, in the context of the *splittings* discussed in (ii), as being associated with the *"value groups"* [cf. the discussion of Remarks 4.7.3, 4.7.4, 4.7.5]. On the other hand, [cf. the discussion of (ii)] the $\mathbb{F}_l^{\rtimes\pm}$-symmetry satisfies the crucial property of being *compatible with* **conjugate synchronization** — a property which may only be established after one isolates the prime-strips under consideration from the **conjugacy indeterminacies** *inherent in the global structure of the absolute Galois group of a number field* [cf. Remark 4.7.2]. Put another way, conjugate synchronization may only be established once the prime-strips under consideration are treated as objects which are **free** of any **combinatorial constraints** arising from the **"prime-trees"** associated to a number field [cf. the discussion of [IUTchI], Remark 4.3.1]. On the other hand, one important property shared by both the $\mathbb{F}_l^{\rtimes\pm}$- and $\mathbb{F}_l^{*}$-symmetries is the

connectedness of the global objects that appear in the [Θ^{ell}-/NF-bridges of] these symmetries. This connectedness plays an essential role in the bookkeeping operations involving the **labels** *of the evaluation points* [cf. the discussion of Remarks 3.5.2 and 4.5.3, (iii), as well as [IUTchI], Remark 4.9.2, (i)]. In particular, such bookkeeping operations cannot be implemented if, for instance, instead of working with a *global number field*, one attempts to take as one's "global objects" *formal products* of the local objects at the various primes of the number field under consideration [cf. the discussion of [AbsTopIII], Remark 3.7.6, (v)]. Finally, we recall that the essential role played by these "global bookkeeping operations" gives rise, in light of the **profinite** nature of the global geometric étale fundamental groups involved, to a situation in which one must apply the "complements on tempered coverings" developed in [IUTchI], §2 [cf. Remark 4.5.3, (iii)].

(iv) One way to summarize the above discussion is as follows. The *bridge* constituted by the *Gaussian distribution* of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater between *theta functions* and *number fields* may be thought of as being constructed by **dismantling** those aspects of the **"characteristic topography"** of the theta functions and number fields involved that constitute an *obstruction* to relating theta functions to number fields. In the case of theta functions, the main obstruction to constructing such a link to the number field under consideration is constituted by the **geometric dimension** of the tempered coverings of elliptic curves [at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] on which the theta functions are defined. This obstruction is resolved by means of the operation of **evaluation** *at the l-torsion points*. Thus, from the point of view of the scheme-theoretic Hodge-Arakelov theory of [HASurI], [HASurII], one may think of these l-torsion points as a sort of *"rough finite approximation"* of the tempered coverings of elliptic curves under consideration [cf. the discussion of [HASurI], §1.3.4]. By contrast, in the case of number fields, the main obstruction to constructing such a link to the theta functions under consideration is the **"prime-trees"** arising from the global structure of the number field, which give rise to the *conjugacy indeterminacies* that obstruct the establishment of **conjugate synchronization** [cf. the discussion of (iii) above]. This obstruction is resolved by dismantling the global prime-tree structure of the number fields involved by working with various **prime-strips** *labeled by elements* $\in \mathbb{F}_l^{*}$ [cf. the discussion of [IUTchI], Remark 4.3.1]. Thus, one may think of these collections of prime-strips labeled by elements $\in \mathbb{F}_l^{*}$ as *"rough finite approximations"* of the infinite prime-trees associated to the number fields involved. At a more **combinatorial level** [cf. the discussion of Remark 4.7.5], this dismantling process may be thought of as the process of dismantling the **ring structure** of $\mathbb{F}_l$ — which we think of as a *"rough finite approximation" of* $\mathbb{Z}$ [cf. [IUTchI], Remark 6.12.3, (i)] — into its **additive** and **multiplicative** components, which correspond, respectively, to the $\mathbb{F}_l^{\rtimes\pm}$**-** *and* $\mathbb{F}_l^{*}$**-symmetries**.

Remark 4.11.3. In the context of the discussion of Remark 4.11.2, it is interesting to observe that, whereas, from the point of view of the *combinatorics of the* $\mathbb{F}_l^{\rtimes\pm}$*- and* $\mathbb{F}_l^{*}$*-symmetries*, one has correspondences

$$\Theta^{\mathrm{ell}} \quad \longleftrightarrow \quad \boxplus, \qquad\qquad \mathrm{NF} \quad \longleftrightarrow \quad \boxtimes$$

— i.e., the Θ^{ell}-bridge corresponds to the *additive* $\mathbb{F}_l^{\rtimes\pm}$-symmetry, while the NF-bridge corresponds to the *multiplicative* $\mathbb{F}_l^{*}$-symmetry — at the level of *line bundles*, one has correspondences

$$\Theta^{\mathrm{ell}} \quad \longleftrightarrow \quad \boxtimes, \qquad \mathrm{NF} \quad \longleftrightarrow \quad \boxplus$$

— i.e., the arithmetic line bundles under consideration are treated *multiplicatively*, via monoids or Frobenioids, in the context of the Θ^{ell}-bridge, while the equivalence of such $\boxtimes$-line bundles with $\boxplus$-line bundles may only be realized in the context of the *global ring structure* of the number fields associated, via the theory of [IUTchI], §5, to the NF-bridge. This **"juggling of $\boxplus$ and $\boxtimes$"** is reminiscent of the theory of the **$\mathfrak{log}$-wall** developed in [AbsTopIII] [cf., e.g., the discussion of [AbsTopIII], §I3] and, indeed, may be thought of as a sort of **combinatorial counterpart** to the "juggling of $\boxplus$ and $\boxtimes$" that occurs in the theory of the $\mathfrak{log}$-wall.

Remark 4.11.4.

(i) From the point of view of *scheme-theoretic Hodge-Arakelov theory*, the l-torsion points of an elliptic curve may be thought of as a *"rough finite approximation"* of the two real dimensions of the underlying real analytic manifold of a one-dimensional complex torus [cf. the discussion of [HASurI], §1.3.4]. In scheme-theoretic Hodge-Arakelov theory, one considers spaces of functions on these l-torsion points. The two dimensions mentioned above then correspond to a "holomorphic dimension" and a "one-dimensional deformation of this holomorphic dimension" [cf. the discussion of [HASurI], §1.4.2]. In the context of the theory of the present series of papers, we work, in effect, with an elliptic curve which is isogenous to the given elliptic curve via an isogeny of degree l — i.e., with "$\underline{X}$" as opposed to "X" — so that we may neglect the "holomorphic dimension" mentioned above and concentrate instead on the *deformations* of this "holomorphic dimension" [cf. the discussion of the Introduction to [EtTh]]. In particular, the various *possible values* at the various l-torsion points at which the theta function is evaluated in the theory of the present series of papers may be thought of as various *possible deformations* of the holomorphic structure, while the *specific values of the theta function* may be thought of as a *specific deformation* of the holomorphic structure. Here, we note that the parameter "$0 \neq t \in \mathrm{LabCusp}^{\pm}(-)$" that indexes these values — which, like the *tangent space* to the original elliptic curve, is *linear* which respect to the group structure of the elliptic curve — descends naturally [especially in the context of ΘNF-*Hodge theater*!] to the parameter "$j \in \mathbb{F}_l^{\circledast}$" — which may be thought of as the "square $(\mathbb{F}_l^{\times})^2$" of $\mathbb{F}_l^{\times}$, hence, like the *square of the tangent space* of the elliptic curve, which is naturally isomorphic to the *tangent space to the moduli space of elliptic curves* at the point determined by the elliptic curve in question, is *quadratic* in its dependence on the linear group structure of the elliptic curve. Finally, this point of view concerning the values of the theta function is reminiscent of the point of view of Remark 3.6.2, (iii), in which we observe that, in the context of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link, these values of the theta function may be thought of as a sort of *"deformation between the identity and a Frobenius morphism"*. The theta function involved may then be thought of as a sort of *continuous version* [i.e., as opposed to a "rough finite approximation"] of such a deformation.

(ii) From the point of view of the analogy between the theory of the present series of papers and *p-adic Teichmüller theory* [cf. [AbsTopIII], §I5], the portion of the infinite chain of $\Theta^{\times \boldsymbol{\mu}}$-*links* of Corollary 4.10, (vi), parametrized by $n \leq 0$

$$\dots \xrightarrow{\Theta^{\times \boldsymbol{\mu}}} {}^{(n-1)}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta^{\times \boldsymbol{\mu}}} {}^{n}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta^{\times \boldsymbol{\mu}}} \dots \xrightarrow{\Theta^{\times \boldsymbol{\mu}}} {}^{0}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

may be thought of as corresponding to the **canonical liftings** of p-adic Teichmüller theory. That is to say, each $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater — which one may think of as representing the *conventional scheme theory* surrounding the *given* **number field** *equipped with an* **elliptic curve** — corresponds to a **hyperbolic curve** *in positive characteristic equipped with a nilpotent ordinary* **indigenous bundle** [cf. the discussion of [AbsTopIII], §I5]. The theta functions that give rise to the $\Theta^{\times \boldsymbol{\mu}}$-links may be thought of as specifying the *specific canonical deformation* [cf. the discussion of (i)] that gives rise to this "canonical lifting". The **canonical Frobenius lifting** on this canonical lifting may be thought of as corresponding to the theory to be developed in [IUTchIII]. From this point of view, the passage

theta functions, number fields $\rightsquigarrow$ **Gaussian distributions**

[cf. the discussion of Remark 4.11.2] effected in the theory of the present series of papers presented thus far — i.e., at a more concrete level, the passage, via **Hodge-Arakelov-theoretic evaluation**, from the above semi-infinite chain to the corresponding semi-infinite chain

$$\dots \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}} {}^{(n-1)}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}} {}^{n}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}} \dots \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}} {}^{0}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

of $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-*links* — may be thought of as corresponding to the passage

$\mathcal{MF}^{\nabla}$**-objects** $\rightsquigarrow$ **Galois representations**

in the case of the **canonical indigenous bundles** that occur in p-adic Teichmüller theory — cf. the discussion of [pTeich], Introduction, §1.3, §1.7; the discussion in [HASurI], §1.3, §1.4, of the relationship between such canonical indigenous bundles in the case of the moduli stack of elliptic curves and the *scheme-theoretic Hodge-Arakelov theory* of [HASurI], [HASurII]. Put another way, it corresponds to the passage from thinking of the "canonical lifting" as a curve equipped with the $\mathcal{MF}^{\nabla}$-object constituted by a *canonical Frobenius-invariant indigenous bundle* to thinking of the "canonical lifting" as a curve equipped with a *canonical Galois representation*, i.e., a canonical crystalline representation [that is to say, a representation that happens to arise from an $\mathcal{MF}^{\nabla}$-object] of the arithmetic fundamental group of the generic fiber of the curve into $PGL_2(\mathbb{Z}_p)$.

(iii) The analogy between the theory of the present series of papers and p-adic Teichmüller theory may also be seen, at a more *technical* level, in the following correspondences between various aspects of the theory presented thus far in the present series of papers and various aspects of the theory of [CanLift], §3 [cf. also Remark 4.11.5 below]:

(a) The discussion of (ii) above is reminiscent of the important role played by the **canonical Galois representation** in the *absolute p-adic anabelian* theory of [CanLift], §3 [cf. the proof of [CanLift], Lemma 3.5].

(b) In light of the important role played, in the present series of papers, by **mono-theta-theoretic cyclotomic rigidity** [which was reviewed in Definition 1.1, (ii); Remark 1.1.1], it is perhaps of interest to recall [cf. Remark 1.11.6] the important role played by *cyclotomic rigidity isomorphisms* in the theory of [CanLift], §3, via the theory of [AbsAnab], §2 [cf., especially, [AbsAnab], Lemmas 2.5, 2.6]. On the other hand, at the level of direct correspondences between the theory of the present series of papers and p-adic Teichmüller theory, it is perhaps better to think of mono-theta-theoretic cyclotomic rigidity as corresponding to the **local uniformizations** arising from the canonical indigenous bundle [cf. the discussion of Remark 3.6.5, (iii)].

(c) The important role played, in the present series of papers, by the "two-dimensional symmetry" constituted by the $\mathbb{F}_l^{\rtimes\pm}$**-** *and* $\mathbb{F}_l^{\circledast}$**-symmetries** — whose two-dimensionality may be thought of as corresponding to the *two real dimensions of the complex upper half-plane* [cf. the discussion of [IUTchI], Remark 6.12.3, (iii)] — is reminiscent of the important role played in the theory of [CanLift], §3, in effect, by the *vanishing* of the zero-th group cohomology module

$$H^0(\mathrm{Ad}(-))$$

of the canonical Galois representation associated to the canonical indigenous bundle — cf. the various *geometric conditions* over the ordinary locus and at the supersingular points of the mod p representations considered in [CanLift], Lemma 3.2. That is to say, the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* may be regarded as corresponding to the **unipotent monodromy** over the **ordinary locus**

$$\left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \quad \stackrel{\sim}{\to} \quad \mathbb{F}_p$$

— which is isomorphic to the *additive* group underlying $\mathbb{F}_p$ — while the $\mathbb{F}_l^{\circledast}$-*symmetry* may be regarded as corresponding to the **toral monodromy** at the **supersingular points**

$$\left\{ \begin{pmatrix} * & 0 \\ 0 & *^{-1} \end{pmatrix} \right\} \quad \stackrel{\sim}{\to} \quad \mathbb{F}_p^{\times}$$

— which is isomorphic to the *multiplicative* group $\mathbb{F}_p^{\times}$ and arises from extracting a $(p-1)$*-th root of the Hasse invariant.* Moreover, the *"intuitive, conventional" nature* of the theory over any *single* connected component of the ordinary locus — a theory which allows one, for instance, to construct q*-parameters* — is reminiscent of the *uniradial* nature of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry, while the fact that supersingular points lie *simultaneously* on irreducible components obtained as closures of distinct connected components of the ordinary locus is reminiscent of the *multiradiality* — i.e., compatibility with *simultaneous execution* in distinct Hodge theaters —

of the $\mathbb{F}_l^{\circledast}$-symmetry [cf. the discussion of Remark 4.7.4]. The above discussion is summarized, at the level of keywords, in Fig. 4.4 below.

$\mathbb{F}_l^{\rtimes\pm}$-symmetry	**$\mathbb{F}_l^{\circledast}$-symmetry**
additive	**multiplicative**
uniradial	**multiradial**
monodromy over the **ordinary locus**	*monodromy at the* **supersingular points**

Fig. 4.4: Correspondence of symmetries with p-adic Teichmüller theory

(d) The important role played, in the present series of papers, by **conjugate synchronization** at the various evaluation points of the theta function — which gives rise, in the form of the **Gaussian distribution**, to the *links* between the various $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters in the second semi-infinite chain that appeared in the discussion of (ii) — is reminiscent of the important role played in the theory of [CanLift], §3, by the description given in [CanLift], Lemma 3.4, of the first group cohomology module

$$H^1(\mathrm{Ad}(-))$$

of the canonical Galois representation associated to the canonical indigenous bundle — whose "slope -1 portion" may be thought of as governing the *"links"* between the "mod p^n" and "mod p^{n+1}" portions of the canonical Galois representation, as it is considered in the proof of [CanLift], Lemma 3.5. Here, we note that this description may be summarized, in effect, as asserting that the slope -1 portion in question is, up to tensor product with an unramified Galois representation, isomorphic to a *direct product of* $3g - 3 + r$ *copies of* $\mathbb{F}_p(-1)$ [where the "(-1)" denotes a Tate twist] — a situation that is reminiscent of the $l^{\circledast}$ *synchronized copies of cyclotomes* that occur in the context of the evaluation of the theta function considered in the present series of papers. Moreover, the **deformations of the canonical Galois representation** parametrized by this module "$H^1(\mathrm{Ad}(-))$" may be thought of as corresponding, in the theory of the present series of papers, to the *"independent* $\mathrm{Aut}(G_{\underline{v}})$*-indeterminacies"* [i.e., for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$] that occur at each label $\in \mathbb{F}_l^{\circledast}$ when one consider *multiradial representations of Gaussian monoids* — cf. the theory of [IUTchIII], §3; [IUTchIII], Remark 3.12.4, (iii).

[Here, we note that, in fact, the various "-1's" in (d) should be replaced by "1's" — cf. Remark 4.11.5 below.] Finally, we observe, with regard to (d), that the description in question that appears in [CanLift], Lemma 3.4, may be thought of as a reflection of the **ordinarity** [i.e., as opposed to just *admissibility*] of the positive characteristic nilpotent indigenous bundle under consideration, hence is reminiscent of the discussion of [AbsTopIII], Remark 5.10.3, (ii), of the correspondence between ordinarity in p-adic Teichmüller theory and the theory of the **étale theta function** developed in [EtTh].

Remark 4.11.5. We take this opportunity to correct a few notational errors in the statement of the condition $(\dagger_M)$ of [CanLift], Lemma 3.4, which, however, *do not affect the proof* of this lemma in any substantive way. The subquotient "$\mathbb{G}^2(M)$" (respectively, "$\mathbb{G}^{-1}$") should have been denoted "$\mathbb{G}^{-2}(M)$" (respectively, "$\mathbb{G}^1$"). The subquotient $\mathbb{G}^{-2}(M)$ (respectively, $\mathbb{G}^1$) is isomorphic to the tensor product of an unramified module with a Tate twist $\mathbb{F}_p(-2)$ (respectively, $\mathbb{F}_p(1)$). That is to say, there is a *sign error* in the Tate twists stated in $(\dagger_M)$. Finally, in order to obtain the desired *dimensions* over $\mathbb{F}_p$, one must replace the cohomology module

$$\text{“}M \stackrel{\mathrm{def}}{=} H^1(\Delta_{X^{\log}}, \mathrm{Ad}(V_{\mathbb{F}_p}))\text{”}$$

by the submodule of this module consisting of elements whose restriction to each of the cuspidal inertia groups of $\Delta_{X^{\log}}$ is *upper triangular* with respect to the filtration determined by the *nilpotent monodromy* action on $V_{\mathbb{F}_p}$ [i.e., by the cuspidal inertia group in question]. That is to say, an elementary computation shows that the operation of restriction to this submodule has the effect of *lowering* the dimension of $\mathbb{G}^{-2}(M)$ from $3g-3+2r$ to $3g-3+r$, as desired.

Bibliography

[Lehto] O. Lehto, *Univalent Functions and Teichmüller Spaces*, *Graduate Texts in Mathematics* **109**, Springer-Verlag (1987).

[pOrd] S. Mochizuki, A Theory of Ordinary p-adic Curves, *Publ. Res. Inst. Math. Sci.* **32** (1996), pp. 957-1151.

[pTeich] S. Mochizuki, *Foundations of p-adic Teichmüller Theory*, AMS/IP Studies in Advanced Mathematics **11**, American Mathematical Society/International Press (1999).

[pGC] S. Mochizuki, The Local Pro-p Anabelian Geometry of Curves, *Invent. Math.* **138** (1999), pp. 319-423.

[HASurI] S. Mochizuki, A Survey of the Hodge-Arakelov Theory of Elliptic Curves I, *Arithmetic Fundamental Groups and Noncommutative Algebra*, *Proceedings of Symposia in Pure Mathematics* **70**, American Mathematical Society (2002), pp. 533-569.

[HASurII] S. Mochizuki, A Survey of the Hodge-Arakelov Theory of Elliptic Curves II, *Algebraic Geometry 2000, Azumino*, *Adv. Stud. Pure Math.* **36**, Math. Soc. Japan (2002), pp. 81-114.

[AbsAnab] S. Mochizuki, The Absolute Anabelian Geometry of Hyperbolic Curves, *Galois Theory and Modular Forms*, Kluwer Academic Publishers (2004), pp. 77-122.

[CanLift] S. Mochizuki, The Absolute Anabelian Geometry of Canonical Curves, *Kazuya Kato's fiftieth birthday*, *Doc. Math. 2003, Extra Vol.*, pp. 609-640.

[AbsSect] S. Mochizuki, Galois Sections in Absolute Anabelian Geometry, *Nagoya Math. J.* **179** (2005), pp. 17-45.

[SemiAnbd] S. Mochizuki, Semi-graphs of Anabelioids, *Publ. Res. Inst. Math. Sci.* **42** (2006), pp. 221-322.

[CombGC] S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math. J.* **59** (2007), pp. 455-479.

[Cusp] S. Mochizuki, Absolute anabelian cuspidalizations of proper hyperbolic curves, *J. Math. Kyoto Univ.* **47** (2007), pp. 451-539.

[FrdI] S. Mochizuki, The Geometry of Frobenioids I: The General Theory, *Kyushu J. Math.* **62** (2008), pp. 293-400.

[FrdII] S. Mochizuki, The Geometry of Frobenioids II: Poly-Frobenioids, *Kyushu J. Math.* **62** (2008), pp. 401-460.

[EtTh] S. Mochizuki, The Étale Theta Function and its Frobenioid-theoretic Manifestations, *Publ. Res. Inst. Math. Sci.* **45** (2009), pp. 227-349.

[AbsTopI] S. Mochizuki, Topics in Absolute Anabelian Geometry I: Generalities, *J. Math. Sci. Univ. Tokyo* **19** (2012), pp. 139-242.

[AbsTopII] S. Mochizuki, Topics in Absolute Anabelian Geometry II: Decomposition Groups and Endomorphisms, *J. Math. Sci. Univ. Tokyo* **20** (2013), pp. 171-269.

[AbsTopIII] S. Mochizuki, Topics in Absolute Anabelian Geometry III: Global Reconstruction Algorithms, *J. Math. Sci. Univ. Tokyo* **22** (2015), pp. 939-1156.

[IUTchI] S. Mochizuki, *Inter-universal Teichmüller Theory I: Construction of Hodge Theaters*, RIMS Preprint **1756** (August 2012), to appear in *Publ. Res. Inst. Math. Sci.*

[IUTchIII] S. Mochizuki, *Inter-universal Teichmüller Theory III: Canonical Splittings of the Log-theta-lattice*, RIMS Preprint **1758** (August 2012), to appear in *Publ. Res. Inst. Math. Sci.*

[IUTchIV] S. Mochizuki, *Inter-universal Teichmüller Theory IV: Log-volume Computations and Set-theoretic Foundations*, RIMS Preprint **1759** (August 2012), to appear in *Publ. Res. Inst. Math. Sci.*

[NSW] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*, Grundlehren der Mathematischen Wissenschaften **323**, Springer-Verlag (2000).

[Szp] L. Szpiro, *Degrés, intersections, hauteurs* in *Astérisque* **127** (1985), pp. 11-28.

Updated versions of preprints are available at the following webpage:

`http://www.kurims.kyoto-u.ac.jp/~motizuki/papers-english.html`

INTER-UNIVERSAL TEICHMÜLLER THEORY III: CANONICAL SPLITTINGS OF THE LOG-THETA-LATTICE

Shinichi Mochizuki

April 2020

Abstract. The present paper constitutes the third paper in a series of four papers and may be regarded as the *culmination* of the *abstract conceptual* portion of the theory developed in the series. In the present paper, we study the theory surrounding the **log-theta-lattice**, a *highly non-commutative* two-dimensional diagram of *"miniature models of conventional scheme theory"*, called $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters*. Here, we recall that $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters were associated, in the first paper of the series, to certain data, called *initial* Θ*-data*, that includes an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$. Each *arrow* of the log-theta-lattice corresponds to a certain *gluing operation* between the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters in the domain and codomain of the arrow. The **horizontal arrows** of the log-theta-lattice are defined as certain versions of the *"*Θ*-link"* that was constructed, in the second paper of the series, by applying the theory of *Hodge-Arakelov-theoretic evaluation* — i.e., evaluation in the style of the **scheme-theoretic Hodge-Arakelov theory** established by the author in previous papers — of the [reciprocal of the l-th root of the] **theta function** at $\boldsymbol{l}$**-torsion points**. In the present paper, we focus on the theory surrounding the $\mathfrak{log}$**-link** between $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters. The $\mathfrak{log}$-link is obtained, roughly speaking, by applying, at each [say, for simplicity, nonarchimedean] valuation of the number field under consideration, the *local* p*-adic logarithm*. The significance of the $\mathfrak{log}$-link lies in the fact that it allows one to construct **log-shells**, i.e., roughly speaking, slightly adjusted forms of the image of the local units at the valuation under consideration via the local p-adic logarithm. The theory of log-shells was studied extensively in a previous paper by the author. The **vertical arrows** of the log-theta-lattice are given by the $\mathfrak{log}$-link. Consideration of various properties of the log-theta-lattice leads naturally to the establishment of **multiradial algorithms** for constructing **"splitting monoids of logarithmic Gaussian procession monoids"**. Here, we recall that "multiradial algorithms" are algorithms that make sense from the point of view of an **"alien arithmetic holomorphic structure"**, i.e., the ring/scheme structure of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater related to a given $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater by means of a *non-ring/scheme-theoretic* horizontal arrow of the log-theta-lattice. These logarithmic Gaussian procession monoids, or **LGP-monoids**, for short, may be thought of as the log-shell-theoretic versions of the *Gaussian monoids* that were studied in the second paper of the series. Finally, by applying these multiradial algorithms for splitting monoids of LGP-monoids, we obtain **estimates** for the **log-volume** of these LGP-monoids. Explicit computations of these estimates will be applied, in the fourth paper of the series, to derive various *diophantine results*.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Contents:

Introduction

In the following discussion, we shall continue to use the notation of the Introduction to the first paper of the present series of papers [cf. [IUTchI], §I1]. In particular, we assume that are given an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$. In the first paper of the series, we introduced and studied the basic properties of $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters*, which may be thought of as miniature models of the conventional scheme theory surrounding the given elliptic curve E_F over the number field F. In the present paper, which forms the third paper of the series, we study the theory surrounding the **$\mathfrak{log}$-link** between $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters. The $\mathfrak{log}$-link induces an *isomorphism between the underlying* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters* and, roughly speaking, is obtained by applying, at each [say, for simplicity, nonarchimedean] valuation $\underline{v} \in \underline{\mathbb{V}}$, the *local* $p_{\underline{v}}$*-adic logarithm* to the local units [cf. Proposition 1.3, (i)]. The significance of the $\mathfrak{log}$-link lies in the fact that it allows one to construct **log-shells**, i.e., roughly speaking, slightly adjusted forms of the image of the local units at $\underline{v} \in \underline{\mathbb{V}}$ via the local $p_{\underline{v}}$-adic logarithm. The theory of log-shells was studied extensively in [AbsTopIII]. The introduction of log-shells leads naturally to the construction of *new versions* — namely, the **$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-links** [cf. Definition 3.8, (ii)] — of the Θ-/$\Theta^{\times\boldsymbol{\mu}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$*-links* studied in [IUTchI], [IUTchII]. The resulting [*highly non-commutative*!] diagram of iterates of the $\mathfrak{log}$- [i.e., the *vertical arrows*] and $\Theta^{\times\boldsymbol{\mu}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-links [i.e., the *horizontal arrows*] — which we refer to as the **log-theta-lattice** [cf. Definitions 1.4; 3.8, (iii), as well as Fig. I.1 below, in the case of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link] — plays a *central role* in the theory of the present series of papers.

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n+1,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n+1,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

Fig. I.1: The [LGP-Gaussian] log-theta-lattice

Consideration of various properties of the log-theta-lattice leads naturally to the establishment of **multiradial algorithms** for constructing **"splitting monoids of logarithmic Gaussian procession monoids"** [cf. Theorem A below]. Here, we recall that "multiradial algorithms" [cf. the discussion of [IUTchII], Introduction] are algorithms that make sense from the point of view of an **"alien arithmetic holomorphic structure"**, i.e., the ring/scheme structure of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater related to a given $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater by means of a *non-ring/scheme-theoretic* Θ-/$\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-link. These logarithmic Gaussian procession monoids, or **LGP-monoids**, for short, may be thought of as the log-shell-theoretic versions of the *Gaussian monoids* that were studied in [IUTchII]. Finally, by applying these multiradial algorithms for splitting monoids of LGP-monoids, we obtain **estimates** for the **log-volume** of these LGP-monoids [cf. Theorem B below]. These estimates will be applied to verify various *diophantine results* in [IUTchIV].

Recall [cf. [IUTchI], §I1] the notion of an *$\mathcal{F}$-prime-strip*. An $\mathcal{F}$-prime-strip consists of data indexed by the valuations $\underline{v} \in \underline{\mathbb{V}}$; roughly speaking, the data at each $\underline{v}$ consists of a *Frobenioid*, i.e., in essence, a system of *monoids* over a *base category*. For instance, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, this data may be thought of as an isomorphic copy of the *monoid with Galois action*

$$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$$

— where we recall that $\mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$ denotes the multiplicative monoid of nonzero integral elements of the completion of an algebraic closure $\overline{F}$ of F at a valuation lying over $\underline{v}$ [cf. [IUTchI], §I1, for more details]. The *$p_{\underline{v}}$-adic logarithm* $\log_{\underline{v}} : \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \to \overline{F}_{\underline{v}}$ at $\underline{v}$ then defines a natural *$\Pi_{\underline{v}}$-equivariant* isomorphism of ind-topological modules

$$(\mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \otimes \mathbb{Q} \ \stackrel{\sim}{\to}\ ) \quad \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \otimes \mathbb{Q} \ \stackrel{\sim}{\to}\ \overline{F}_{\underline{v}}$$

— where we recall the notation "$\mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} = \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} / \mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" from the discussion of [IUTchI], §1 — which allows one to equip $\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \otimes \mathbb{Q}$ with the *field structure* arising from the field structure of $\overline{F}_{\underline{v}}$. The portion at $\underline{v}$ of the **$\mathfrak{log}$-link** associated to an $\mathcal{F}$-prime-strip [cf. Definition 1.1, (iii); Proposition 1.2] may be thought of as the correspondence

$$\left\{ \Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}} \right\} \quad \stackrel{\mathfrak{log}}{\longrightarrow} \quad \left\{ \Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}} \right\}$$

in which one thinks of the copy of "$\mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$" on the *right* as obtained from the field structure induced by the $p_{\underline{v}}$-adic logarithm on the tensor product with $\mathbb{Q}$ of the copy of the units "$\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \subseteq \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$" on the *left*. Since this correspondence induces an *isomorphism of topological groups* between the copies of $\Pi_{\underline{v}}$ on either side, one may think of $\Pi_{\underline{v}}$ as *"immune to"/"neutral with respect to"* — or, in the terminology of the present series of papers, **"coric"** with respect to — the transformation constituted by the $\mathfrak{log}$-link. This situation is studied in detail in [AbsTopIII], §3, and reviewed in Proposition 1.2 of the present paper.

By applying various results from **absolute anabelian geometry**, one may algorithmically reconstruct a copy of the data "$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$" from $\Pi_{\underline{v}}$. Moreover, by applying *Kummer theory*, one obtains natural isomorphisms between this *"coric version"* of the data "$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$" and the copies of this data that appear on either side of the $\mathfrak{log}$-*link*. On the other hand, one verifies immediately that these Kummer isomorphisms are **not compatible** with the **coricity** of the copy of the data "$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$" algorithmically constructed from $\Pi_{\underline{v}}$. This phenomenon is, in some sense, the *central theme* of the theory of [AbsTopIII], §3, and is reviewed in Proposition 1.2, (iv), of the present paper.

The introduction of the $\mathfrak{log}$-link leads naturally to the construction of **log-shells** at each $\underline{v} \in \underline{\mathbb{V}}$. If, for simplicity, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, then the log-shell at $\underline{v}$ is given, roughly speaking, by the *compact additive module*

$$\mathcal{I}_{\underline{v}} \stackrel{\mathrm{def}}{=} p_{\underline{v}}^{-1} \cdot \log_{\underline{v}}(\mathcal{O}^{\times}_{K_{\underline{v}}}) \subseteq K_{\underline{v}} \subseteq \overline{F}_{\underline{v}}$$

[cf. Definition 1.1, (i), (ii); Remark 1.2.2, (i), (ii)]. One has natural *functorial algorithms* for constructing various versions of the notion of a log-shell — i.e., **mono-analytic/holomorphic** and **étale-like/Frobenius-like** — from $\mathcal{D}^{\vdash}$-/$\mathcal{D}$-/$\mathcal{F}^{\vdash}$-/$\mathcal{F}$-prime-strips [cf. Proposition 1.2, (v), (vi), (vii), (viii), (ix)]. Although, as discussed above, the relevant Kummer isomorphisms are *not compatible* with the $\mathfrak{log}$-link *"at the level of elements"*, the log-shell $\mathcal{I}_{\underline{v}}$ at $\underline{v}$ satisfies the important property

$$\mathcal{O}^{\triangleright}_{K_{\underline{v}}} \subseteq \mathcal{I}_{\underline{v}}; \quad \log_{\underline{v}}(\mathcal{O}^{\times}_{K_{\underline{v}}}) \subseteq \mathcal{I}_{\underline{v}}$$

— i.e., it **contains** *the* **images** *of the Kummer isomorphisms* associated to both the domain and the codomain of the $\mathfrak{log}$-link [cf. Proposition 1.2, (v); Remark 1.2.2, (i), (ii)]. In light of the *compatibility* of the $\mathfrak{log}$-link with *log-volumes* [cf. Propositions 1.2, (iii); 3.9, (iv)], this property will ultimately lead to **upper bounds** — i.e., as opposed to *"precise equalities"* — in the computation of *log-volumes* in Corollary 3.12 [cf. Theorem B below]. Put another way, although iterates [cf. Remark 1.1.1] of the $\mathfrak{log}$-link *fail to be compatible* with the various Kummer isomorphisms that arise, one may nevertheless consider the *entire diagram* that results from considering such iterates of the $\mathfrak{log}$-link and related Kummer isomorphisms [cf. Proposition 1.2, (x)]. We shall refer to such diagrams

$$\begin{array}{ccccccccc} \dots & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \dots \\ & \dots & & \searrow & \downarrow & \swarrow & & \dots & \\ & & & & \circ & & & & \end{array}$$

— i.e., where the *horizontal arrows* correspond to the $\mathfrak{log}$-links [that is to say, to the *vertical arrows* of the log-theta-lattice!]; the "$\bullet$'s" correspond to the Frobenioid-theoretic data within a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater; the "$\circ$" corresponds to the *coric version* of this data [that is to say, in the terminology discussed below, *vertically coric* data of the log-theta-lattice]; the vertical/diagonal arrows correspond to the various *Kummer isomorphisms* — as **$\mathfrak{log}$-Kummer correspondences** [cf. Theorem 3.11, (ii); Theorem A, (ii), below]. Then the inclusions of the above display may be interpreted as a sort of **"upper semi-commutativity"** of such diagrams [cf. Remark 1.2.2, (iii)], which we shall also refer to as the **"upper semi-compatibility"** *of the* $\mathfrak{log}$-*link with the relevant Kummer isomorphisms* — cf. the discussion of the **"indeterminacy"** (Ind3) in Theorem 3.11, (ii).

By considering the $\mathfrak{log}$-links associated to the various *$\mathcal{F}$-prime-strips* that occur in a $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater*, one obtains the notion of a **$\mathfrak{log}$-link** *between* $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters*

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\mathfrak{log}} \quad ^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

[cf. Proposition 1.3, (i)]. As discussed above, by considering the iterates of the $\mathfrak{log}$- [i.e., the *vertical arrows*] and Θ-/$\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-links [i.e., the *horizontal arrows*], one obtains a diagram which we refer to as the **log-theta-lattice** [cf. Definitions 1.4; 3.8, (iii), as well as Fig. I.1, in the case of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link]. As discussed above, this diagram is **highly noncommutative**, since the definition of the $\mathfrak{log}$-link depends, in an essential way, on both the *additive* and the *multiplicative* structures — i.e., on the *ring structure* — of the various local rings at $\underline{v} \in \underline{\mathbb{V}}$, structures which are *not preserved* by the Θ-/$\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-links [cf. Remark 1.4.1, (i)]. So far, in the Introductions to [IUTchI], [IUTchII], as well as in the present Introduction, we have discussed various *"coricity"* properties — i.e., properties of *invariance* with respect to various types of "transformations" — in the context of Θ-/$\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-links, as well as in the context of $\mathfrak{log}$-links. In the context of the log-theta-lattice, it becomes necessary to distinguish between various types of coricity. That is to say, coricity with respect to $\mathfrak{log}$-links [i.e., the vertical arrows of the log-theta-lattice] will be referred to as **vertical coricity**, while coricity with respect to Θ-/$\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-links [i.e., the horizontal arrows of the log-theta-lattice] will be referred to as **horizontal coricity**. On the other hand, coricity properties that hold with respect to *all* of the arrows of the log-theta-lattice will be referred to as **bi-coricity** properties.

Relative to the analogy between the theory of the present series of papers and *p-adic Teichmüller theory* [cf. [IUTchI], §I4], we recall that a $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater*, which may be thought of as a miniature model of the *conventional scheme theory* surrounding the given elliptic curve E_F over the number field F, corresponds to the *positive characteristic scheme theory* surrounding a hyperbolic curve over a positive characteristic perfect field that is equipped with a nilpotent ordinary indigenous bundle [cf. Fig. I.2 below]. Then the **rotation**, or **"juggling"**, effected by the **$\mathfrak{log}$-link** of the **additive** and **multiplicative** structures of the conventional scheme theory represented by a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater may be thought of as corresponding to the **Frobenius morphism** *in positive characteristic* [cf. the discussion of [AbsTopIII], §I1, §I3, §I5]. Thus, just as the Frobenius morphism is completely well-defined in *positive characteristic*, the $\mathfrak{log}$-link may be thought of as a phenomenon that occurs within a *single* **arithmetic holomorphic structure**, i.e., a *vertical* line of the log-theta-lattice. By contrast, the essentially *non-ring/scheme-theoretic* relationship between $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters constituted by the Θ-/$\Theta^{\times \boldsymbol{\mu}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-links corresponds to the relationship between the *"mod p^n" and "mod p^{n+1}"* portions of the ring of Witt vectors, in the context of a *canonical lifting* of the original positive characteristic data [cf. the discussion of Remark 1.4.1, (iii); Fig. I.2 below]. Thus, the **log-theta-lattice**, taken as a whole, may be thought of as corresponding to the **canonical lifting** of the original positive characteristic data, equipped with a corresponding **canonical Frobenius action/lifting** [cf. Fig. I.2 below]. Finally, the **non-commutativity** of the log-theta-lattice may be thought of as corresponding to the complicated **"intertwining"** that occurs in the theory of Witt vectors and canonical liftings between the Frobenius morphism in positive characteristic and the mixed characteristic nature of the ring of Witt vectors [cf. the discussion of Remark 1.4.1, (ii), (iii)].

One important consequence of this *"noncommutative intertwining"* of the two dimensions of the log-theta-lattice is the following. Since each *horizontal arrow* of the log-theta-lattice [i.e., the Θ-/$\Theta^{\times\boldsymbol{\mu}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link] may only be used to relate — i.e., via various *Frobenioids* — the *multiplicative* portions of the ring structures in the domain and codomain of the arrow, one natural approach to relating the *additive* portions of these ring structures is to apply the theory of **log-shells**. That is to say, since each horizontal arrow is compatible with the **canonical splittings** [up to roots of unity] discussed in [IUTchII], Introduction, of the *theta/Gaussian monoids* in the domain of the horizontal arrow into *unit group* and *value group* portions, it is natural to attempt to relate the ring structures on either side of the horizontal arrow by applying the canonical splittings to

· relate the **multiplicative** structures on either side of the horizontal arrow by means of the **value group** portions of the theta/Gaussian monoids;

· relate the **additive** structures on either side of the horizontal arrow by means of the **unit group** portions of the theta/Gaussian monoids, **shifted once** via a *vertical arrow*, i.e., the **$\mathfrak{log}$-link**, so as to *"render additive"* the [*a priori*] multiplicative structure of these unit group portions.

Indeed, this is the approach that will ultimately be taken in Theorem 3.11 [cf. Theorem A below] to relating the ring structures on either side of a horizontal arrow. On the other hand, in order to actually implement this approach, it will be necessary to overcome numerous *technical obstacles*. Perhaps the most immediately obvious such obstacle lies in the observation [cf. the discussion of Remark 1.4.1, (ii)] that, precisely because of the *"noncommutative intertwining"* nature of the log-theta-lattice,

> any sort of algorithmic construction concerning objects lying in the *domain* of a horizontal arrow that involves **vertical shifts** [e.g., such as the approach to relating additive structures in the fashion described above] **cannot be "translated"** in any immediate sense into an algorithm that makes sense from the point of view of the *codomain* of the horizontal arrow.

In a word, our approach to overcoming this technical obstacle consists of working with objects in the *vertical line* of the log-theta-lattice that contains the *domain* of the horizontal arrow under consideration that satisfy the crucial property of being

invariant with respect to **vertical shifts**

— i.e., **shifts** via iterates of the **$\mathfrak{log}$-link** [cf. the discussion of Remarks 1.2.2, (iii); 1.4.1, (ii)]. For instance, *étale-like* objects that are **vertically coric** satisfy this invariance property. On the other hand, as discussed in the beginning of [IUTchII], Introduction, in the theory of the present series of papers, it is of crucial importance to be able to relate *corresponding Frobenius-like* and *étale-like structures* to one another via *Kummer theory*. In particular, in order to obtain structures that are *invariant* with respect to *vertical shifts*, it is necessary to consider **$\mathfrak{log}$-Kummer correspondences**, as discussed above. Moreover, in the context of such $\mathfrak{log}$-Kummer correspondences, typically, one may only obtain structures that are invariant with respect to vertical shifts if one is willing to admit some sort of **indeterminacy**, e.g., such as the **"upper semi-compatibility"** [cf. the discussion of the *"indeterminacy"* (Ind3) in Theorem 3.11, (ii)] discussed above.

Inter-universal Teichmüller theory	*p-adic Teichmüller theory*
number field F	**hyperbolic curve** C over a *positive characteristic perfect field*
[**once-punctured**] **elliptic curve** X over F	*nilpotent ordinary* **indigenous bundle** P over C
Θ-link arrows of the *log-theta-lattice*	**mixed characteristic extension** structure of a ring of *Witt vectors*
$\mathfrak{log}$-link arrows of the *log-theta-lattice*	the **Frobenius morphism** in *positive characteristic*
the entire **log-theta-lattice**	the resulting **canonical lifting** + **canonical Frobenius action**; **canonical Frobenius lifting** over the ordinary locus
relatively straightforward *original construction of* **$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link**	relatively straightforward *original construction of* **canonical liftings**
highly nontrivial *description of* **alien arithmetic holomorphic structure** via *absolute anabelian geometry*	highly nontrivial *absolute anabelian reconstruction of* **canonical liftings**

Fig. I.2: Correspondence between inter-universal Teichmüller theory and p-adic Teichmüller theory

One important property of the $\mathfrak{log}$**-link**, and hence, in particular, of the construction of **log-shells**, is its **compatibility** with the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** discussed in the Introductions to [IUTchI], [IUTchII] — cf. Remark 1.3.2. Here, we recall from the discussion of [IUTchII], Introduction, that the $\mathbb{F}_l^{\rtimes\pm}$-symmetry allows one to relate the various *$\mathcal{F}$-prime-strips* — i.e., more concretely, the various copies of the data "$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [and their analogues for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$] — associated to the various **labels** $\in \mathbb{F}_l$ that appear in the *Hodge-Arakelov-theoretic evaluation* of [IUTchII] in a fashion that is **compatible** with

· the **distinct nature** of distinct labels $\in \mathbb{F}_l$;

· the **Kummer isomorphisms** used to relate *Frobenius-like* and *étale-like* versions of the $\mathcal{F}$-prime-strips that appear, i.e., more concretely, the various copies of the data "$\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [and their analogues for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$];

· the structure of the **underlying $\mathcal{D}$-prime-strips** that appear, i.e., more concretely, the various copies of the *[arithmetic] tempered fundamental group* "$\Pi_{\underline{v}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [and their analogues for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$]

— cf. the discussion of [IUTchII], Introduction; Remark 1.5.1; Step (vii) of the proof of Corollary 3.12 of the present paper. This compatibility with the $\mathbb{F}_l^{\rtimes\pm}$-symmetry gives rise to the construction of

· **vertically coric** *$\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips, log-shells* by means of the **arithmetic holomorphic** structures under consideration;

· **mono-analytic** *$\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips, log-shells* which are **bi-coric**

— cf. Theorem 1.5. These *bi-coric mono-analytic log-shells* play a central role in the theory of the present paper.

One notable aspect of the *compatibility* of the $\mathfrak{log}$-*link* with the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* in the context of the theory of *Hodge-Arakelov-theoretic evaluation* developed in [IUTchII] is the following. One important property of *mono-theta environments* is the property of **"isomorphism class compatibility"**, i.e., in the terminology of [EtTh], *"compatibility with the* **topology** *of the* **tempered fundamental group***"* [cf. the discussion of Remark 2.1.1]. This "isomorphism class compatibility" allows one to apply the Kummer theory of mono-theta environments [i.e., the theory of [EtTh]] relative to the **ring-theoretic basepoints** that occur on either side of the $\mathfrak{log}$**-link** [cf. Remark 2.1.1, (ii); [IUTchII], Remark 3.6.4, (i)], for instance, in the context of the $\mathfrak{log}$-*Kummer correspondences* discussed above. Here, we recall that the significance of working with such "ring-theoretic basepoints" lies in the fact that the full *ring structure* of the local rings involved [i.e., as opposed to, say, just the multiplicative portion of this ring structure] is necessary in order to construct the $\mathfrak{log}$-link. That is to say, it is precisely by establishing the *conjugate synchronization* arising from the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* relative to *these basepoints* that occur on either side of the $\mathfrak{log}$-link that one is able to conclude the crucial **compatibility** *of this conjugate synchronization with the* $\mathfrak{log}$-*link* discussed in Remark 1.3.2. Thus, in summary, one important consequence of the "isomorphism class compatibility" of mono-theta environments is the **simultaneous compatibility** of

- the **Kummer theory** of **mono-theta environments**;
- the **conjugate synchronization** arising from the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**;
- the construction of the $\mathfrak{log}$**-link**.

This simultaneous compatibility is necessary in order to perform the construction of the [crucial!] *splitting monoids of LGP-monoids* referred to above — cf. the discussion of Step (vi) of the proof of Corollary 3.12.

In §2 of the present paper, we continue our preparation for the *multiradial construction of splitting monoids of LGP-monoids* given in §3 [of the present paper] by presenting a **global formulation** of the essentially *local theory* at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. [IUTchII], §1, §2, §3] concerning the interpretation, via the notion of **multiradiality**, of various **rigidity** properties of **mono-theta environments**. That is to say, although much of the [essentially routine!] task of formulating the local theory of [IUTchII], §1, §2, §3, in global terms was accomplished in [IUTchII], §4, the [again essentially routine!] task of formulating the portion of this local theory that concerns *multiradiality* was not explicitly addressed in [IUTchII], §4. One reason for this lies in the fact that, from the point of view of the theory to be developed in §3 of the present paper, this global formulation of multiradiality properties of the mono-theta environment may be presented most naturally in the framework developed in §1 of the present paper, involving the **log-theta-lattice** [cf. Theorem 2.2; Corollary 2.3]. Indeed, the **étale-like** versions of the mono-theta environment, as well as the various objects constructed from the mono-theta environment, may be interpreted, from the point of view of the log-theta-lattice, as **vertically coric** structures, and are **Kummer-theoretically** related to their **Frobenius-like** [i.e., Frobenioid-theoretic] counterparts, which arise from the [Frobenioid-theoretic portions of the] various $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters in a vertical line of the log-theta-lattice [cf. Theorem 2.2, (ii); Corollary 2.3, (ii), (iii), (iv)]. Moreover, it is precisely the **horizontal arrows** of the log-theta-lattice that give rise to the $\widehat{\mathbb{Z}}^\times$**-indeterminacies** acting on copies of "$\mathcal{O}^{\times\boldsymbol{\mu}}$" that play a prominent role in the local multiradiality theory developed in [IUTchII] [cf. the discussion of [IUTchII], Introduction]. In this context, it is useful to recall from the discussion of [IUTchII], Introduction [cf. also Remark 2.2.1 of the present paper], that the essential content of this local multiradiality theory consists of the *observation* [cf. Fig. I.3 below] that, since *mono-theta-theoretic cyclotomic* and *constant multiple rigidity* only require the use of the portion of $\mathcal{O}^\times_{\overline{F}_{\underline{v}}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, given by the *torsion subgroup* $\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \subseteq \mathcal{O}^\times_{\overline{F}_{\underline{v}}}$ [i.e., the roots of unity], the *triviality* of the composite of natural morphisms

$$\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \hookrightarrow \mathcal{O}^\times_{\overline{F}_{\underline{v}}} \twoheadrightarrow \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$$

has the effect of **insulating** the **Kummer theory** of the **étale theta function** — i.e., via the theory of the mono-theta environments developed in [EtTh] — from the $\widehat{\mathbb{Z}}^\times$**-indeterminacies** that act on the copies of "$\mathcal{O}^{\times\boldsymbol{\mu}}$" that arise in the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-*prime-strips* that appear in the Θ-/$\Theta^{\times\boldsymbol{\mu}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link.

$$\begin{array}{ccc} \mathrm{id} \curvearrowright & & \widehat{\mathbb{Z}}^{\times} \ \curvearrowright \\ \boxed{\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}} & \rightarrow & \boxed{\mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}} \end{array}$$

Fig. I.3: Insulation from $\widehat{\mathbb{Z}}^{\times}$-indeterminacies in the context of mono-theta-theoretic cyclotomic, constant multiple rigidity

In §3 of the present paper, which, in some sense, constitutes the *conclusion* of the theory developed thus far in the present series of papers, we present the construction of the [splitting monoids of] **LGP-monoids**, which may be thought of as a **multiradial** version of the [splitting monoids of] **Gaussian monoids** that were constructed via the theory of *Hodge-Arakelov-theoretic evaluation* developed in [IUTchII]. In order to achieve this multiradiality, it is necessary to "multiradialize" the various components of the construction of the Gaussian monoids given in [IUTchII]. The first step in this process of "multiradialization" concerns the **labels** $j \in \mathbb{F}_l^{\circledast}$ that occur in the Hodge-Arakelov-theoretic evaluation performed in [IUTchII]. That is to say, the construction of these labels, together with the closely related theory of $\mathbb{F}_l^{\circledast}$**-symmetry**, depend, in an essential way, on the *full arithmetic tempered fundamental groups* "$\Pi_{\underline{v}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, i.e., on the portion of the *arithmetic holomorphic structure* within a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater which is *not shared* by an **alien** *arithmetic holomorphic structure* [i.e., an arithmetic holomorphic structure related to the original arithmetic holomorphic structure via a horizontal arrow of the log-theta-lattice]. One naive approach to remedying this state of affairs is to simply consider the *underlying set*, of cardinality $l^{\circledast}$, associated to $\mathbb{F}_l^{\circledast}$, which we regard as being equipped with the full set of *symmetries* given by arbitrary permutation automorphisms of this underlying set. The problem with this approach is that it yields a situation in which, for each label $j \in \mathbb{F}_l^{\circledast}$, one must contend with an *indeterminacy of* $l^{\circledast}$ *possibilities* for the element of this underlying set that corresponds to j [cf. [IUTchI], Propositions 4.11, (i); 6.9, (i)]. From the point of view of the *log-volume computations* to be performed in [IUTchIV], this degree of indeterminacy gives rise to log-volumes which are *"too large"*, i.e., to estimates that are not sufficient for deriving the various *diophantine results* obtained in [IUTchIV]. Thus, we consider the following alternative approach, via **processions** [cf. [IUTchI], Propositions, 4.11, 6.9]. Instead of working just with the underlying set associated to $\mathbb{F}_l^{\circledast}$, we consider the *diagram of inclusions* of finite sets

$$\mathbb{S}_1^{\pm} \quad \hookrightarrow \quad \mathbb{S}_{1+1=2}^{\pm} \quad \hookrightarrow \quad \ldots \quad \hookrightarrow \quad \mathbb{S}_{j+1}^{\pm} \quad \hookrightarrow \quad \ldots \quad \hookrightarrow \quad \mathbb{S}_{1+l^{\circledast}=l^{\pm}}^{\pm}$$

— where we write $\mathbb{S}_{j+1}^{\pm} \stackrel{\mathrm{def}}{=} \{0, 1, \ldots, j\}$, for $j = 0, \ldots, l^{\circledast}$, and we think of each of these finite sets as being subject to arbitrary permutation automorphisms. That is to say, we think of the set $\mathbb{S}_{j+1}^{\pm}$ as a **container** for the labels $0, 1, \ldots, j$. Thus, for each j, one need only contend with an *indeterminacy of* $j + 1$ *possibilities* for the element of this container that corresponds to j. In particular, if one allows $j = 0, \ldots, l^{\circledast}$ to vary, then this approach allows one to *reduce* the resulting label indeterminacy from a total of $(l^{\pm})^{l^{\pm}}$ possibilities [where we write $l^{\pm} = 1 + l^{\circledast} =$

$(l+1)/2]$ to a total of $l^{\pm}!$ possibilities. It turns out that this reduction will yield just the right estimates in the log-volume computations to be performed in [IUTchIV]. Moreover, this approach satisfies the important property of *insulating the* **"core label 0"** *from the various label indeterminacies* that occur.

Each element of each of the containers $\mathbb{S}^{\pm}_{j+1}$ may be thought of as parametrizing an *$\mathcal{F}$- or $\mathcal{D}$-prime-strip* that occurs in the *Hodge-Arakelov-theoretic evaluation* of [IUTchII]. In order to render the construction multiradial, it is necessary to replace such *holomorphic* $\mathcal{F}$-/$\mathcal{D}$-prime-strips by *mono-analytic* $\mathcal{F}^{\vdash}$-/$\mathcal{D}^{\vdash}$-prime-strips. In particular, as discussed above, one may construct, for each such $\mathcal{F}^{\vdash}$-/$\mathcal{D}^{\vdash}$-prime-strip, a collection of **log-shells** associated to the various $\underline{v} \in \underline{\mathbb{V}}$. Write $\mathbb{V}_{\mathbb{Q}}$ for the set of valuations of $\mathbb{Q}$. Then, in order to obtain objects that are *immune* to the various label indeterminacies discussed above, we consider, for each element $* \in \mathbb{S}^{\pm}_{j+1}$, and for each [say, for simplicity, *nonarchimedean*] $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$,

· the *direct sum of the* **log-shells** associated to the prime-strip labeled by the given element $* \in \mathbb{S}^{\pm}_{j+1}$ at the $\underline{v} \in \underline{\mathbb{V}}$ that lie over $v_{\mathbb{Q}}$;

we then form

· the **tensor product**, over the elements $* \in \mathbb{S}^{\pm}_{j+1}$, of these *direct sums*.

This collection of tensor products associated to $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ will be referred to as the **tensor packet** associated to the collection of prime-strips indexed by elements of $\mathbb{S}^{\pm}_{j+1}$. One may carry out this construction of the tensor packet either for *holomorphic* $\mathcal{F}$-/$\mathcal{D}$-prime-strips [cf. Proposition 3.1] or for *mono-analytic* $\mathcal{F}^{\vdash}$-/$\mathcal{D}^{\vdash}$-prime-strips [cf. Proposition 3.2].

The tensor packets associated to $\mathcal{D}^{\vdash}$-prime-strips will play a crucial role in the theory of §3, as **"multiradial mono-analytic containers"** for the principal objects of interest [cf. the discussion of Remark 3.12.2, (ii)], namely,

· the action of the **splitting monoids** of the **LGP-monoids** — i.e., the monoids generated by the **theta values** $\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\dots,l^*}$ — on the portion of the *tensor packets* just defined at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. Fig. I.4 below; Propositions 3.4, 3.5; the discussion of [IUTchII], Introduction];

· the action of copies "$(F^{\times}_{\mathrm{mod}})_j$" of [the multiplicative monoid of nonzero elements of] the **number field** F_{mod} labeled by $j = 1, \dots, l^*$ on the product, over $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, of the portion of the *tensor packets* just defined at $v_{\mathbb{Q}}$ [cf. Fig. I.5 below; Propositions 3.3, 3.7, 3.10].

$$
\begin{array}{ccccccccccc}
 & & \underline{\underline{q}}^{1} \curvearrowright & & & & \underline{\underline{q}}^{j^2} \curvearrowright & & & & \underline{\underline{q}}^{(l^*)^2} \curvearrowright \\
/^{\pm} & \hookrightarrow & /^{\pm}/^{\pm} & \hookrightarrow & \dots & \hookrightarrow & /^{\pm}/^{\pm} \dots /^{\pm} & \hookrightarrow & \dots & \hookrightarrow & /^{\pm}/^{\pm} \dots \dots /^{\pm} \\
\mathbb{S}^{\pm}_{1} & & \mathbb{S}^{\pm}_{1+1=2} & & & & \mathbb{S}^{\pm}_{j+1} & & & & \mathbb{S}^{\pm}_{1+l^*=l^{\pm}}
\end{array}
$$

Fig. I.4: Splitting monoids of LGP-monoids acting on tensor packets

$$\begin{array}{ccccccccccc}
 & & (F^{\times}_{\mathrm{mod}})_1 \curvearrowright & & & & (F^{\times}_{\mathrm{mod}})_j \curvearrowright & & & & (F^{\times}_{\mathrm{mod}})_{l^*} \curvearrowright \\
/^{\pm} & \hookrightarrow & /^{\pm}/^{\pm} & \hookrightarrow & \ldots & \hookrightarrow & /^{\pm}/^{\pm} \ldots /^{\pm} & \hookrightarrow & \ldots & \hookrightarrow & /^{\pm}/^{\pm} \ldots \ldots /^{\pm} \\
\mathbb{S}^{\pm}_{1} & & \mathbb{S}^{\pm}_{1+1=2} & & & & \mathbb{S}^{\pm}_{j+1} & & & & \mathbb{S}^{\pm}_{1+l^*=l^{\pm}}
\end{array}$$

Fig. I.5: Copies of $F^{\times}_{\mathrm{mod}}$ acting on tensor packets

Indeed, these [splitting monoids of] **LGP-monoids** and copies "$(F^{\times}_{\mathrm{mod}})_j$" of [the multiplicative monoid of nonzero elements of] the **number field** F_{mod} admit *natural embeddings into/actions on* the various *tensor packets* associated to *labeled* $\mathcal{F}$*-prime-strips* in each $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ of the log-theta-lattice. One then obtains **vertically coric** versions of these splitting monoids of LGP-monoids and labeled copies "$(F^{\times}_{\mathrm{mod}})_j$" of [the multiplicative monoid of nonzero elements of] the number field F_{mod} by applying suitable **Kummer isomorphisms** between

· *log-shells/tensor packets associated to [labeled]* $\mathcal{F}$*-prime-strips* and
· *log-shells/tensor packets associated to [labeled]* $\mathcal{D}$*-prime-strips.*

Finally, by passing to the

· *log-shells/tensor packets associated to [labeled]* $\mathcal{D}^{\vdash}$*-prime-strips*

— i.e., by *forgetting the* **arithmetic holomorphic structure** associated to a *specific vertical line* of the log-theta-lattice — one obtains the desired **multiradial representation**, i.e., description in terms that make sense from the point of view of an **alien** *arithmetic holomorphic structure*, of the **splitting monoids of LGP-monoids** and *labeled copies of the* **number field** F_{mod} discussed above. This passage to the multiradial representation is obtained by admitting the following *three types of* **indeterminacy**:

(Ind1): This is the indeterminacy that arises from the *automorphisms of* **processions of** $\mathcal{D}^{\vdash}$**-prime-strips** that appear in the multiradial representation — i.e., more concretely, from *permutation automorphisms* of the label sets $\mathbb{S}^{\pm}_{j+1}$ that appear in the processions discussed above, as well as from the *automorphisms of the* $\mathcal{D}^{\vdash}$*-prime-strips* that appear in these processions.

(Ind2): This is the ["non-(Ind1) portion" of the] indeterminacy that arises from the *automorphisms of the* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips* that appear in the Θ-/$\Theta^{\times\boldsymbol{\mu}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link — i.e., in particular, at [for simplicity] $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the $\widehat{\mathbb{Z}}^{\times}$**-indeterminacies** acting on local copies of "$\mathcal{O}^{\times\boldsymbol{\mu}}$" [cf. the above discussion].

(Ind3): This is the indeterminacy that arises from the **upper semi-compatibility** *of the* $\mathfrak{log}$*-Kummer correspondences* associated to the specific vertical line of the log-theta-lattice under consideration [cf. the above discussion].

A detailed description of this multiradial representation, together with the indeterminacies (Ind1), (Ind2) is given in Theorem 3.11, (i) [and summarized in Theorem A, (i), below; cf. also Fig. I.6 below].

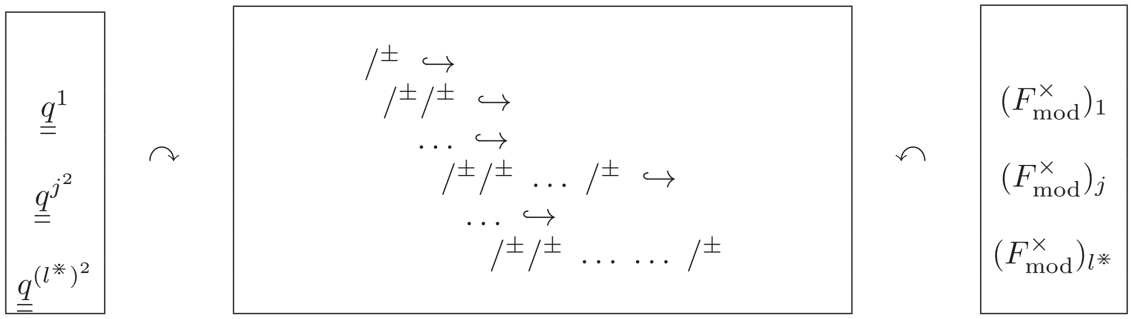

Fig. I.6: The full multiradial representation

One important property of the multiradial representation discussed above concerns the relationship between the three main components — i.e., roughly speaking, *log-shells*, *splitting monoids of LGP-monoids*, and *number fields* — of this multiradial representation and the **$\mathfrak{log}$-Kummer correspondence** of the specific *vertical line* of the log-theta-lattice under consideration. This property — which may be thought of as a sort of **"non-interference"**, or **"mutual compatibility"**, property — asserts that the multiplicative monoids constituted by the splitting monoids of LGP-monoids and copies of $F^{\times}_{\mathrm{mod}}$ *"do not interfere"*, relative to the various arrows that occur in the $\mathfrak{log}$-*Kummer correspondence*, with the *local units* at $\underline{v} \in \underline{\mathbb{V}}$ that give rise to the *log-shells*. In the case of splitting monoids of LGP-monoids, this *non-interference/mutual compatibility* property is, in essence, a formal consequence of the existence of the **canonical splittings** [up to roots of unity] of the *theta/Gaussian monoids* that appear into *unit group* and *value group* portions [cf. the discussion of [IUTchII], Introduction]. Here, we recall that, in the case of the theta monoids, these canonical splittings are, in essence, a formal consequence of the **constant multiple rigidity** property of mono-theta environments reviewed above. In the case of copies of F_{mod}, this *non-interference/mutual compatibility* property is, in essence, a formal consequence of the well-known fact in elementary algebraic number theory that any nonzero element of a number field that is **integral** at every valuation of the number field is necessarily a **root of unity**. These mutual compatibility properties are described in detail in Theorem 3.11, (ii), and summarized in Theorem A, (ii), below.

Another important property of the multiradial representation discussed above concerns the relationship between the three main components — i.e., roughly speaking, *log-shells*, *splitting monoids of LGP-monoids*, and *number fields* — of this multiradial representation and the **$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-links**, i.e., the *horizontal arrows* of the log-theta-lattice under consideration. This property — which may be thought of as a property of **compatibility** with the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link — asserts that the *cyclotomic rigidity isomorphisms* that appear in the Kummer theory surrounding the splitting monoids of LGP-monoids and copies of $F^{\times}_{\mathrm{mod}}$ are *immune* to the $\widehat{\mathbb{Z}}^{\times}$-*indeterminacies* that act on the copies of "$\mathcal{O}^{\times\boldsymbol{\mu}}$" that arise in the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-*prime-strips* that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. In the case of splitting monoids of LGP-monoids, this property amounts precisely to the *multiradiality* theory developed in §2 [cf. the above discussion], i.e., in essence, to the **mono-theta-theoretic cyclotomic rigidity** property reviewed in the above discussion. In the case of copies of $F^{\times}_{\mathrm{mod}}$, this property follows from the theory surrounding the construction of the cyclotomic rigidity isomorphisms discussed in [IUTchI], Example 5.1, (v). These compatibility properties are described in detail in Theorem 3.11, (iii), and summarized in Theorem A, (iii), below.

At this point, we pause to observe that although considerable attention has been devoted so far in the present series of papers, especially in [IUTchII], to the theory of *Gaussian monoids*, not so much attention has been devoted [i.e., outside of [IUTchI], §5; [IUTchII], Corollaries 4.7, 4.8] to [the multiplicative monoids constituted by] copies of $F^{\times}_{\mathrm{mod}}$. These copies of $F^{\times}_{\mathrm{mod}}$ enter into the theory of the *multiradial representation* discussed above in the form of various types of *global Frobenioids* in the following way. If one starts from the *number field* F_{mod}, one natural Frobenioid that can be associated to F_{mod} is the Frobenioid $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ of [stack-theoretic] *arithmetic line bundles* on [the spectrum of the ring of integers of] F_{mod} discussed in [IUTchI], Example 5.1, (iii) [cf. also Example 3.6 of the present paper]. From the point of view of the theory surrounding the *multiradial representation* discussed above, there are *two natural ways* to approach the construction of "$\mathcal{F}^{\circledast}_{\mathrm{mod}}$":

($\circledast_{\mathrm{MOD}}$) **(Rational Function Torsor Version)**: This approach consists of considering the category $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ of $F^{\times}_{\mathrm{mod}}$-*torsors* equipped with *trivializations* at each $\underline{v} \in \underline{\mathbb{V}}$ [cf. Example 3.6, (i), for more details].

($\circledast_{\mathfrak{mod}}$) **(Local Fractional Ideal Version)**: This approach consists of considering the category $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ of collections of *integral structures* on the various completions $K_{\underline{v}}$ at $\underline{v} \in \underline{\mathbb{V}}$ and morphisms between such collections of integral structures that arise from multiplication by elements of $F^{\times}_{\mathrm{mod}}$ [cf. Example 3.6, (ii), for more details].

Then one has *natural isomorphisms of Frobenioids*

$$\mathcal{F}^{\circledast}_{\mathrm{mod}} \quad \overset{\sim}{\to} \quad \mathcal{F}^{\circledast}_{\mathrm{MOD}} \quad \overset{\sim}{\to} \quad \mathcal{F}^{\circledast}_{\mathfrak{mod}}$$

that induce the respective *identity morphisms* $F^{\times}_{\mathrm{mod}} \to F^{\times}_{\mathrm{mod}} \to F^{\times}_{\mathrm{mod}}$ on the associated *rational function monoids* [cf. [FrdI], Corollary 4.10]. In particular, at first glance, $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ and $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ appear to be "essentially equivalent" objects.

On the other hand, when regarded from the point of view of the *multiradial representations* discussed above, these two constructions exhibit a number of significant differences — cf. Fig. I.7 below; the discussion of Remarks 3.6.2, 3.10.1. For instance, whereas the construction of ($\circledast_{\mathrm{MOD}}$) depends only on the **multiplicative** structure of $F^{\times}_{\mathrm{mod}}$, the construction of ($\circledast_{\mathfrak{mod}}$) involves the *module*, i.e., the **additive**, structure of the localizations $K_{\underline{v}}$. The global portion of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-*link* (respectively, the $\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-*link*) is, by definition [cf. Definition 3.8, (ii)], constructed by means of the *realification* of the Frobenioid that appears in the construction of ($\circledast_{\mathrm{MOD}}$) (respectively, ($\circledast_{\mathfrak{mod}}$)). This means that the construction of the global portion of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link — which is the version of the Θ-link that is in fact ultimately used in the theory of the multiradial representation — depends only on the *multiplicative* monoid structure of a copy of $F^{\times}_{\mathrm{mod}}$, together with the various valuation homomorphisms $F^{\times}_{\mathrm{mod}} \to \mathbb{R}$ associated to $\underline{v} \in \underline{\mathbb{V}}$. Thus, the *mutual compatibility* [discussed above] of copies of $F^{\times}_{\mathrm{mod}}$ with the $\mathfrak{log}$-*Kummer correspondence* implies that one may perform this construction of the global portion of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link in a fashion that is *immune* to the *"upper semi-compatibility" indeterminacy* (Ind3) [discussed above]. By contrast, the construction of $(\circledast_{\mathfrak{mod}})$ involves integral structures on the underlying local *additive* modules "$K_{\underline{v}}$", i.e., from the point of view of the multiradial representation, integral structures on *log-shells* and *tensor packets* of log-shells, which *are* subject to the "upper semi-compatibility" indeterminacy (Ind3) [discussed above]. In particular, the $\mathfrak{log}$-Kummer correspondence subjects the construction of $(\circledast_{\mathfrak{mod}})$ to *"substantial distortion"*. On the other hand, the essential role played by local integral structures in the construction of $(\circledast_{\mathfrak{mod}})$ enables one to compute the *global arithmetic degree* of the arithmetic line bundles constituted by objects of the category "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" in terms of **log-volumes** on **log-shells** and **tensor packets** of log-shells [cf. Proposition 3.9, (iii)]. This property of the construction of $(\circledast_{\mathfrak{mod}})$ will play a *crucial role* in deriving the **explicit estimates** for such log-volumes that are obtained in Corollary 3.12 [cf. Theorem B below].

$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$	$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$
biased toward **multiplicative** structures	biased toward **additive** structures
easily related to **value group/non-coric** portion "$(-)^{\Vdash\blacktriangleright}$" of $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link	easily related to **unit group/coric** portion "$(-)^{\vdash\times\boldsymbol{\mu}}$" of $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link, i.e., **mono-analytic log-shells**
admits **precise $\mathfrak{log}$-Kummer correspondence**	only admits **"upper semi-compatible" $\mathfrak{log}$-Kummer correspondence**
rigid, but not suited to explicit computation	subject to **substantial distortion**, but suited to **explicit estimates**

Fig. I.7: $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ versus $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$

Thus, in summary, the natural isomorphism $\mathcal{F}^{\circledast}_{\mathrm{MOD}} \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathfrak{mod}}$ discussed above plays the important role, in the context of the *multiradial representation* discussed above, of *relating*

- the **multiplicative** structure of the global number field F_{mod} to the **additive** structure of F_{mod},

· the **unit group/coric** portion "$(-)^{\vdash\times\boldsymbol{\mu}}$" of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link to the **value group/non-coric** portion "$(-)^{\Vdash\blacktriangleright}$" of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link.

Finally, in Corollary 3.12 [cf. also Theorem B below], we apply the *multiradial representation* discussed above to estimate certain *log-volumes* as follows. We begin by introducing some terminology [cf. Definition 3.8, (i)]. We shall refer to the object that arises in any of the versions [including *realifications*] of the global Frobenioid "$\mathcal{F}^{\circledast}_{\mathrm{mod}}$" discussed above — such as, for instance, the global realified Frobenioid that occurs in the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link — by considering the arithmetic divisor determined by the zero locus of the elements "$\underline{\underline{q}}_{\underline{v}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ as a *q-pilot object*. The **log-volume** of the ***q*-pilot object** will be denoted by

$$- \ |\log(\underline{\underline{q}})| \ \in \ \mathbb{R}$$

— so $|\log(\underline{\underline{q}})| > 0$ [cf. Corollary 3.12; Theorem B]. In a similar vein, we shall refer to the object that arises in the global realified Frobenioid that occurs in the *domain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link by considering the arithmetic divisor determined by the zero locus of the collection of *theta values* "$\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\ldots,l^*}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ as a Θ*-pilot object*. The **log-volume** of the **holomorphic hull** — cf. Remark 3.9.5, (i); Step (xi) of the proof of Corollary 3.12 — of the **union** of the collection of **possible images** of the **Θ-pilot object** in the **multiradial representation** — i.e., where we recall that these "possible images" are subject to the **indeterminacies** (Ind1), (Ind2), (Ind3) — will be denoted by

$$- \ |\log(\underline{\underline{\Theta}})| \ \in \ \mathbb{R} \bigcup \{+\infty\}$$

[cf. Corollary 3.12; Theorem B]. Here, the reader might find the use of the notation "$-$" and "$|\ldots|$" confusing [i.e., since this notation suggests that $- \ |\log(\underline{\underline{\Theta}})|$ is a *non-positive real number*, which would appear to imply that the possibility that $- \ |\log(\underline{\underline{\Theta}})| = +\infty$ may be excluded from the outset]. The reason for the use of this notation, however, is to express the point of view that $- \ |\log(\underline{\underline{\Theta}})|$ should be regarded as a *positive real multiple* of $- \ |\log(\underline{\underline{q}})|$ [i.e., which is indeed a *negative real number*!] plus a *possible error term*, which [*a priori*!] might be equal to $+\infty$. Then the content of Corollary 3.12, Theorem B may be summarized, roughly speaking [cf. Remark 3.12.1, (ii)], as a result concerning the

negativity *of the* **Θ-pilot log-volume** $|\log(\underline{\underline{\Theta}})|$

— i.e., where we write $|\log(\underline{\underline{\Theta}})| \ \stackrel{\mathrm{def}}{=} \ -(- \ |\log(\underline{\underline{\Theta}})|) \ \in \ \mathbb{R} \bigcup \{-\infty\}$. Relative to the analogy between the theory of the present series of papers and *complex/p-adic Teichmüller theory* [cf. [IUTchI], §I4], this result may be thought of as a statement to the effect that

> *"the pair consisting of a number field equipped with an elliptic curve is* **metrically hyperbolic***, i.e., has* **negative curvature***".*

That is to say, it may be thought of as a sort of analogue of the inequality

$$\chi_S \ = \ - \int_S d\mu_S \ < \ 0$$

arising from the classical **Gauss-Bonnet formula** on a hyperbolic Riemann surface of finite type S [where we write χ_S for the *Euler characteristic* of S and $d\mu_S$ for the Kähler metric on S determined by the *Poincaré metric* on the upper half-plane — cf. the discussion of Remark 3.12.3], or, alternatively, of the inequality

$$(1-p)(2g_X - 2) \ \leq \ 0$$

that arises by computing *global degrees of line bundles* in the context of the **Hasse invariant** that arises in p-adic Teichmüller theory [where X is a *smooth, proper hyperbolic curve* of genus g_X over the ring of Witt vectors of a perfect field of characteristic p which is *canonical* in the sense of p-adic Teichmüller theory — cf. the discussion of Remark 3.12.4, (v)].

The proof of Corollary 3.12 [i.e., Theorem B] is based on the following *fundamental observation*: the **multiradial representation** discussed above yields

two tautologically equivalent *ways to compute*
the **q-pilot log-volume** $-\ |\log(\underline{\underline{q}})|$

— cf. Fig. I.8 below; Step (xi) of the proof of Corollary 3.12. That is to say, suppose that one starts with the ***q*-pilot object** in the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater ${}^{1,0}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ at $(1,0)$, which we think of as being represented, via the approach of $(\circledast_{\mathfrak{mod}})$, by means of the action of the various $\underline{\underline{q}}_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, on the **log-shells** that arise, via the **$\mathfrak{log}$-link** ${}^{1,-1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{1,0}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$, from the various local "$\mathcal{O}^{\times \boldsymbol{\mu}}$'s" in the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater ${}^{1,-1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ at $(1,-1)$. Thus, if one considers the *value group* "$(-)^{\Vdash \blacktriangleright}$" and *unit group* "$(-)^{\vdash \times \boldsymbol{\mu}}$" portions of the *codomain* of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link ${}^{0,0}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} {}^{1,0}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ in the context of the *arithmetic holomorphic structure* of the vertical line $(1, \circ)$, this action on log-shells may be thought of as a somewhat **intricate "intertwining"** between these value group and unit group portions [cf. Remark 3.12.2, (ii)]. On the other hand, the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link ${}^{0,0}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}} {}^{1,0}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ constitutes a sort of **gluing isomorphism** between the *arithmetic holomorphic structures* associated to the vertical lines $(0, \circ)$ and $(1, \circ)$ that is based on

> **forgetting this intricate intertwining**, i.e., by working solely with *abstract isomorphisms of $\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strips.*

Thus, in order to relate the arithmetic holomorphic structures, say, at $(0,0)$ and $(1,0)$, one must apply the *multiradial representation* discussed above. That is to say, one starts by applying the theory of **bi-coric mono-analytic log-shells** given in Theorem 1.5. One then applies the **Kummer theory** surrounding the **splitting monoids of theta/Gaussian monoids** and copies of the **number field** F_{mod}, which allows one to pass from the **Frobenius-like** versions of various objects that appear in — i.e., that are necessary in order to consider — the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-*link* to the corresponding **étale-like** versions of these objects that appear in the *multiradial representation*. This passage from Frobenius-like versions to étale-like versions is referred to as the operation of **Kummer-detachment** [cf. Fig. I.8; Remark 1.5.4, (i)]. As discussed above, this operation of Kummer-detachment is possible precisely as a consequence of the **compatibility** of the multiradial representation with the *indeterminacies* (Ind1), (Ind2), (Ind3), hence, in particular, with the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$**-link**. Here, we recall that since the log-theta-lattice is, as discussed above, *far from commutative*, in order to represent the various *"*$\mathfrak{log}$*-link-conjugates"* at $(0, m)$ [for $m \in \mathbb{Z}$] in terms that may be understood from the point of view of the arithmetic holomorphic structure at $(1, 0)$, one must work [not only with the Kummer isomorphisms at a *single* $(0, m)$, but rather] with the **entire** $\mathfrak{log}$**-Kummer correspondence**. In particular, one must take into account the *indeterminacy* (Ind3). Once one completes the operation of Kummer-detachment so as to obtain *vertically coric* versions of objects on the vertical line $(0, \circ)$, one then passes to *multiradial objects*, i.e., to the "final form" of the *multiradial representation*, by taking into account [once again] the *indeterminacy* (Ind1), i.e., that arises from working with *[mono-analytic!]* $\mathcal{D}^{\vdash}$*-* [as opposed to $\mathcal{D}$-!] *prime-strips*. Finally, one computes the **log-volume** of the holomorphic hull of this "final form" multiradial representation of the Θ-pilot object — i.e., subject to the *indeterminacies* (Ind1), (Ind2), (Ind3)! — and concludes the desired estimates from the *tautological observation* that

> *the* **log-theta-lattice** *— and, in particular, the* **"gluing isomorphism"** *constituted by the* $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$**-link** *— were* **constructed** *precisely in such a way as to ensure that the computation of the log-volume of the holomorphic hull of the union of the collection of possible images of the Θ-pilot object [cf. the definition of* $|\log(\underline{\underline{\Theta}})|$*] necessarily* **amounts to a computation of** *[an upper bound for]* $|\log(\underline{\underline{q}})|$

multiradial representation at 0-column $(0, \circ)$	**permutation symmetry** of $\approx$ **étale-picture**	**multiradial representation** at 1-column $(1, \circ)$
Kummer-detachment via $\mathfrak{log}$-**Kummer** $\Uparrow$ **compatibly** with $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-**link**		**comparison** via $\Downarrow$ **hol. hull, log-vol.**
Θ**-pilot object** in $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater at $(0, 0)$	$(-)^{\Vdash \blacktriangleright}$-portion, $(-)^{\vdash \times \boldsymbol{\mu}}$-portion $\approx$ of $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$**-link**	$\boldsymbol{q}$**-pilot object** in $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater at $(1, 0)$

Fig. I.8: Two tautologically equivalent ways to compute the log-volume of the q-pilot object at $(1, 0)$

— cf. Fig. I.8; Step (xi) of the proof of Corollary 3.12. That is to say, the "gluing isomorphism" constituted by the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link relates two distinct *"arithmetic holomorphic structures"*, i.e., two distinct *copies* of conventional ring/scheme theory, that are glued together precisely by means of a relation that identifies the Θ-*pilot object* in the *domain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link with the *q-pilot object* in the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. Thus, once one sets up such an *apparatus*, the computation of the log-volume of the holomorphic hull of the union of possible images of the Θ-pilot object in the domain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link in terms of the q-pilot object in the codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link amounts — *tautologically*! — to the computation of the log-volume of the q-pilot object [in the codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link] in terms of *itself*, i.e., to a computation that reflects certain *intrinsic properties* of this q-pilot object. This is the content of Corollary 3.12 [i.e., Theorem B]. As discussed above, this sort of *"computation of intrinsic properties"* in the present context of a number field equipped with an elliptic curve may be regarded as analogous to the "computations of intrinsic properties" reviewed above in the classical complex and p-adic cases.

We conclude the present Introduction with the following summaries of the *main results* of the present paper.

Theorem A. (Multiradial Algorithms for Logarithmic Gaussian Procession Monoids) *Fix a collection of* **initial Θ-data** $(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$ *as in [IUTchI], Definition 3.1. Let*

$$\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n,m\in\mathbb{Z}}$$

be a **collection of distinct $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** *[relative to the given initial Θ-data] — which we think of as arising from a* LGP**-Gaussian log-theta-lattice** *[cf. Definition 3.8, (iii)]. For each $n \in \mathbb{Z}$, write*

$${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

for the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater determined, up to isomorphism, by the various ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, where $m \in \mathbb{Z}$, via the **vertical coricity** *of Theorem 1.5, (i) [cf. Remark 3.8.2].*

(i) **(Multiradial Representation)** *Write*

$${}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}}$$

for the **collection of data** *consisting of*

(a) **tensor packets** *of* **log-shells***;*

(b) **splitting monoids** *of* **LGP-monoids** *acting on the tensor packets of (a);*

(c) copies, labeled by $j \in \mathbb{F}^{\circledast}_l$, of [the multiplicative monoid of nonzero elements of] the **number field** F_{mod} *acting on the tensor packets of (a)*

[cf. Theorem 3.11, (i), (a), (b), (c), for more details] regarded up to **indeterminacies** *of the following two types:*

(Ind1) *the indeterminacies induced by the* **automorphisms** *of the* **procession of** $\mathcal{D}^{\vdash}$**-prime-strips** $\mathrm{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T)$ *that gives rise to the tensor packets of (a);*

(Ind2) *the ["non-*(Ind1) *portion" of the] indeterminacies that arise from the* **automorphisms of the** $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$**-prime-strips** *that appear in the* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$*-link, i.e., in particular, at [for simplicity]* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*, the* $\widehat{\mathbb{Z}}^{\times}$**-indeterminacies** *acting on local copies of "*$\mathcal{O}^{\times\boldsymbol{\mu}}$*"*

— cf. Theorem 3.11, (i), for more details. Then ${}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}}$ *may be constructed via an* **algorithm** *in the procession of* $\mathcal{D}^{\vdash}$*-prime-strips* $\mathrm{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T)$*, which is* **functorial** *with respect to isomorphisms of processions of* $\mathcal{D}^{\vdash}$*-prime-strips. For* $n, n' \in \mathbb{Z}$*, the* **permutation symmetries** *of the* **étale-picture** *discussed in [IUTchI], Corollary 6.10, (iii); [IUTchII], Corollary 4.11, (ii), (iii) [cf. also Corollary 2.3, (ii); Remarks 2.3.2 and 3.8.2, of the present paper], induce* **compatible poly-isomorphisms**

$$\mathrm{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T) \stackrel{\sim}{\to} \mathrm{Prc}({}^{n',\circ}\mathfrak{D}^{\vdash}_T); \quad {}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}} \stackrel{\sim}{\to} {}^{n',\circ}\mathfrak{R}^{\mathrm{LGP}}$$

which are, moreover, compatible with the **bi-coricity** *poly-isomorphisms*

$${}^{n,\circ}\mathcal{D}^{\vdash}_0 \stackrel{\sim}{\to} {}^{n',\circ}\mathcal{D}^{\vdash}_0$$

of Theorem 1.5, (iii) [cf. also [IUTchII], Corollaries 4.10, (iv); 4.11, (i)].

(ii) **(**$\mathfrak{log}$**-Kummer Correspondence)** *For* $n, m \in \mathbb{Z}$*, the inverses of the* **Kummer isomorphisms** *associated to the various* $\mathcal{F}$**-prime-strips** *and* **NF-bridges** *that appear in the* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *induce "inverse Kummer"* **isomorphisms** *between the* **vertically coric** *data (a), (b), (c) of (i) and the corresponding* **Frobenioid-theoretic** *data arising from each* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *[cf. Theorem 3.11, (ii), (a), (b), (c), for more details]. Moreover, as one varies* $m \in \mathbb{Z}$*, the corresponding* **Kummer isomorphisms** *[i.e., inverses of "inverse Kummer" isomorphisms] of* **splitting monoids** *of* **LGP-monoids** *[cf. (i), (b)] and labeled copies of the* **number field** F_{mod} *[cf. (i), (c)] are* **mutually compatible***, relative to the* $\mathfrak{log}$**-links** *of the* n*-th column of the* LGP*-Gaussian log-theta-lattice under consideration, in the sense that the only portions of the [Frobenioid-theoretic] domains of these Kummer isomorphisms that are possibly related to one another via the* $\mathfrak{log}$*-links consist of* **roots of unity** *in the domains of the* $\mathfrak{log}$*-links [multiplication by which corresponds, via the* $\mathfrak{log}$*-link, to an* **"addition by zero"** *indeterminacy, i.e., to* **no indeterminacy***!] — cf. Proposition 3.5, (ii), (c); Proposition 3.10, (ii); Theorem 3.11, (ii), for more details. On the other hand, the Kummer isomorphisms of* **tensor packets** *of* **log-shells** *[cf. (i), (a)] are subject to a certain* **"indeterminacy"** *as follows:*

(Ind3) *as one varies* $m \in \mathbb{Z}$*, these Kummer isomorphisms of tensor packets of log-shells are* **"upper semi-compatible"***, relative to the* $\mathfrak{log}$**-links** *of the*

n-th column of the LGP-*Gaussian log-theta-lattice under consideration, in a sense that involves certain* **natural inclusions "$\subseteq$"** *at $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$ and certain* **natural surjections "$\twoheadrightarrow$"** *at $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$ — cf. Proposition 3.5, (ii), (a), (b); Theorem 3.11, (ii), for more details.*

Finally, as one varies $m \in \mathbb{Z}$, these Kummer isomorphisms of tensor packets of log-shells are [precisely!] **compatible**, *relative to the* **$\mathfrak{log}$-links** *of the n-th column of the* LGP-*Gaussian log-theta-lattice under consideration, with the respective* **log-volumes** *[cf. Proposition 3.9, (iv)].*

(iii) **($\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-Link Compatibility)** *The various Kummer isomorphisms of (ii) satisfy compatibility properties with the various* **horizontal arrows** *— i.e., $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-links — of the* LGP-*Gaussian log-theta-lattice under consideration as follows: The* **tensor packets** *of* **log-shells** *[cf. (i), (a)] are compatible, relative to the relevant Kummer isomorphisms, with [the unit group portion "$(-)^{\vdash \times \boldsymbol{\mu}}$" of] the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. the indeterminacy "(Ind2)" of (i)]; we refer to Theorem 3.11, (iii), (a), (b), for more details. The identity automorphism on the objects that appear in the construction of the* **splitting monoids** *of* **LGP-monoids** *via mono-theta environments [cf. (i), (b)] is compatible, relative to the relevant Kummer isomorphisms and isomorphisms of mono-theta environments, with the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. the indeterminacy "(Ind2)" of (i)]; we refer to Theorem 3.11, (iii), (c), for more details. The identity automorphism on the objects that appear in the construction of the labeled copies of the* **number field** *F_{mod} [cf. (i), (c)] is compatible, relative to the relevant Kummer isomorphisms and cyclotomic rigidity isomorphisms [cf. the discussion of Remark 2.3.2; the constructions of [IUTchI], Example 5.1, (v)], with the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. the indeterminacy "(Ind2)" of (i); we refer to Theorem 3.11, (iii), (d), for more details.*

Theorem B. (Log-volume Estimates for Multiradially Represented Splitting Monoids of Logarithmic Gaussian Procession Monoids) *Suppose that we are in the situation of Theorem A. Write*

$$- |\log(\underline{\underline{\Theta}})| \ \in \ \mathbb{R} \bigcup \{+\infty\}$$

for the **procession-normalized mono-analytic log-volume** *[where the average is taken over $j \in \mathbb{F}_l^{\circledast}$ — cf. Remark 3.1.1, (ii), (iii), (iv); Proposition 3.9, (i), (ii); Theorem 3.11, (i), (a), for more details] of the* **holomorphic hull** *[cf. Remark 3.9.5, (i)] of the* **union** *of the* **possible images** *of a* **Θ-pilot object** *[cf. Definition 3.8, (i)], relative to the relevant* **Kummer isomorphisms** *[cf. Theorems A, (ii); 3.11, (ii)], in the* **multiradial representation** *of Theorems A, (i); 3.11, (i), which we regard as* **subject** *to the* **indeterminacies** (Ind1), (Ind2), (Ind3) *described in Theorems A, (i), (ii); 3.11, (i), (ii). Write*

$$- |\log(\underline{\underline{q}})| \ \in \ \mathbb{R}$$

for the **procession-normalized mono-analytic log-volume** *of the image of a* **$\boldsymbol{q}$-pilot object** *[cf. Definition 3.8, (i)], relative to the relevant* **Kummer isomorphisms** *[cf. Theorems A, (ii); 3.11, (ii)], in the* **multiradial representation** *of*

Theorems A, (i); 3.11, (i), which we do **not** *regard as subject to the indeterminacies* (Ind1), (Ind2), (Ind3) *described in Theorems A, (i), (ii); 3.11, (i), (ii). Here, we recall the definition of the symbol "$\triangle$" as the result of identifying the* **labels**

$$\text{“}0\text{”} \ \textit{and} \ \text{“}\langle \mathbb{F}_l^{\circledast} \rangle\text{”}$$

[cf. [IUTchII], Corollary 4.10, (i)]. In particular, $|\log(\underline{\underline{q}})| > 0$ *is easily computed in terms of the various* ***q*-parameters** *of the elliptic curve* E_F *[cf. [IUTchI], Definition 3.1, (b)] at* $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ $(\neq \emptyset)$*. Then it holds that* $-|\log(\underline{\underline{\Theta}})| \in \mathbb{R}$*, and*

$$-|\log(\underline{\underline{\Theta}})| \ \geq \ -|\log(\underline{\underline{q}})|$$

— i.e., $C_\Theta \geq -1$ *for any real number* $C_\Theta \in \mathbb{R}$ *such that* $-|\log(\underline{\underline{\Theta}})| \ \leq \ C_\Theta \cdot |\log(\underline{\underline{q}})|$.

Acknowledgements:

The research discussed in the present paper profited enormously from the generous support that the author received from the *Research Institute for Mathematical Sciences*, a Joint Usage/Research Center located in Kyoto University. At a personal level, I would like to thank *Fumiharu Kato*, *Akio Tamagawa*, *Go Yamashita*, *Mohamed Saïdi*, *Yuichiro Hoshi*, *Ivan Fesenko*, *Fucheng Tan*, *Emmanuel Lepage*, *Arata Minamide*, and *Wojciech Porowski* for many stimulating discussions concerning the material presented in this paper. Also, I feel deeply indebted to *Go Yamashita*, *Mohamed Saïdi*, and *Yuichiro Hoshi* for their meticulous reading of and numerous comments concerning the present paper. Finally, I would like to express my deep gratitude to *Ivan Fesenko* for his quite substantial efforts to disseminate — for instance, in the form of a survey that he wrote — the theory discussed in the present series of papers.

Notations and Conventions:

We shall continue to use the "Notations and Conventions" of [IUTchI], §0.

Section 1: The Log-theta-lattice

In the present §1, we discuss various enhancements to the theory of **log-shells**, as developed in [AbsTopIII]. In particular, we develop the theory of the **$\mathfrak{log}$-link** [cf. Definition 1.1; Propositions 1.2, 1.3], which, together with the $\Theta^{\times \boldsymbol{\mu}}$- and $\Theta^{\times \boldsymbol{\mu}}_{\text{gau}}$-links of [IUTchII], Corollary 4.10, (iii), leads naturally to the construction of the **log-theta-lattice**, an apparatus that is *central* to the theory of the present series of papers. We conclude the present §1 with a discussion of various **coric structures** associated to the log-theta-lattice [cf. Theorem 1.5].

In the following discussion, we assume that we have been given *initial* Θ*-data* as in [IUTchI], Definition 3.1. We begin by reviewing various aspects of the theory of *log-shells* developed in [AbsTopIII].

Definition 1.1. Let

$$^\dagger\mathfrak{F} = \{^\dagger\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

be an $\mathcal{F}$*-prime-strip* [relative to the given initial Θ-data — cf. [IUTchI], Definition 5.2, (i)]. Write

$$^\dagger\mathfrak{F}^{\vdash} = \{^\dagger\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}; \quad ^\dagger\mathfrak{F}^{\vdash\times\boldsymbol{\mu}} = \{^\dagger\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}; \quad ^\dagger\mathfrak{D} = \{^\dagger\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

for the associated $\mathcal{F}^{\vdash}$*-*, $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-*, $\mathcal{D}$*-prime-strips* [cf. [IUTchI], Remark 5.2.1, (ii); [IUTchII], Definition 4.9, (vi), (vii); [IUTchI], Remark 5.2.1, (i)]. Recall the *functorial algorithm* of [IUTchII], Corollary 4.6, (i), in the $\mathcal{F}$-prime-strip $^\dagger\mathfrak{F}$ for constructing the assignment $\Psi_{\text{cns}}(^\dagger\mathfrak{F})$ given by

$$\underline{\mathbb{V}}^{\text{non}} \ni \underline{v} \;\mapsto\; \Psi_{\text{cns}}(^\dagger\mathfrak{F})_{\underline{v}} \;\stackrel{\text{def}}{=}\; \left\{G_{\underline{v}}(^\dagger\Pi_{\underline{v}}) \;\curvearrowright\; \Psi_{^\dagger\mathcal{F}_{\underline{v}}}\right\}$$
$$\underline{\mathbb{V}}^{\text{arc}} \ni \underline{v} \;\mapsto\; \Psi_{\text{cns}}(^\dagger\mathfrak{F})_{\underline{v}} \;\stackrel{\text{def}}{=}\; \Psi_{^\dagger\mathcal{F}_{\underline{v}}}$$

— where the data in brackets "$\{-\}$" is to be regarded as being well-defined only up to a $^\dagger\Pi_{\underline{v}}$*-conjugacy indeterminacy* [cf. [IUTchII], Corollary 4.6, (i), for more details]. In the following, we shall write

$$(-)^{\underline{\text{gp}}} \;\stackrel{\text{def}}{=}\; (-)^{\text{gp}} \bigcup \{0\}$$

for the *formal union* with $\{0\}$ of the groupification $(-)^{\text{gp}}$ of a [multiplicatlvey written] monoid "$(-)$". Thus, by setting the product of all elements of $(-)^{\underline{\text{gp}}}$ with 0 to be equal to 0, one obtains a natural monoid structure on $(-)^{\underline{\text{gp}}}$.

(i) Let $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$. Write

$$(\Psi_{^\dagger\mathcal{F}_{\underline{v}}} \;\supseteq\; \Psi^{\times}_{^\dagger\mathcal{F}_{\underline{v}}} \;\to) \quad \Psi^{\sim}_{^\dagger\mathcal{F}_{\underline{v}}} \;\stackrel{\text{def}}{=}\; (\Psi^{\times}_{^\dagger\mathcal{F}_{\underline{v}}})^{\text{pf}}$$

for the *perfection* $(\Psi^{\times}_{^\dagger\mathcal{F}_{\underline{v}}})^{\text{pf}}$ of the submonoid of units $\Psi^{\times}_{^\dagger\mathcal{F}_{\underline{v}}}$ of $\Psi_{^\dagger\mathcal{F}_{\underline{v}}}$. Now let us recall from the theory of [AbsTopIII] [cf. [AbsTopIII], Definition 3.1, (iv); [AbsTopIII], Proposition 3.2, (iii), (v)] that the natural, algorithmically constructible *ind-topological field structure* on $\Psi^{\text{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ allows one to define a $p_{\underline{v}}$*-adic logarithm* on $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$, which, in turn, yields a *functorial algorithm* in the Frobenioid ${}^\dagger\mathcal{F}_{\underline{v}}$ for constructing an *ind-topological field structure* on $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$. Write

$$\Psi_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})} \subseteq \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$$

for the resulting *multiplicative monoid* of nonzero integers. Here, we observe that the resulting *diagram*

$$\Psi_{{}^\dagger\mathcal{F}_{\underline{v}}} \supseteq \Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}} \rightarrow \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} = \Psi^{\text{gp}}_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$$

is *compatible* with the various *natural actions* of ${}^\dagger\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$ on each of the [four] "Ψ's" appearing in this diagram. The pair $\{{}^\dagger\Pi_{\underline{v}} \curvearrowright \Psi_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}\}$ now determines a *Frobenioid*

$$\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})$$

[cf. [AbsTopIII], Remark 3.1.1; [IUTchI], Remark 3.3.2] — which is, in fact, *naturally isomorphic* to the Frobenioid ${}^\dagger\mathcal{F}_{\underline{v}}$, but which we wish to think of as being related to ${}^\dagger\mathcal{F}_{\underline{v}}$ via the *above diagram*. We shall denote this diagram by means of the notation

$${}^\dagger\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} \mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})$$

and refer to this relationship between ${}^\dagger\mathcal{F}_{\underline{v}}$ and $\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})$ as the **tautological $\mathfrak{log}$-link** *associated to* ${}^\dagger\mathcal{F}_{\underline{v}}$ [or, when ${}^\dagger\mathfrak{F}$ is fixed, *at* $\underline{v}$]. If $\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} {}^\ddagger\mathcal{F}_{\underline{v}}$ is any [poly-]isomorphism of Frobenioids, then we shall write

$${}^\dagger\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathcal{F}_{\underline{v}}$$

for the diagram obtained by post-composing the tautological $\mathfrak{log}$-link associated to ${}^\dagger\mathcal{F}_{\underline{v}}$ with the given [poly-]isomorphism $\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} {}^\ddagger\mathcal{F}_{\underline{v}}$ and refer to this relationship between ${}^\dagger\mathcal{F}_{\underline{v}}$ and ${}^\ddagger\mathcal{F}_{\underline{v}}$ as a **$\mathfrak{log}$-link** from ${}^\dagger\mathcal{F}_{\underline{v}}$ to ${}^\ddagger\mathcal{F}_{\underline{v}}$; when the given [poly-]isomorphism $\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} {}^\ddagger\mathcal{F}_{\underline{v}}$ is the *full poly-isomorphism*, then we shall refer to the resulting $\mathfrak{log}$-link as the *full* $\mathfrak{log}$*-link* from ${}^\dagger\mathcal{F}_{\underline{v}}$ to ${}^\ddagger\mathcal{F}_{\underline{v}}$. Finally, we recall from [AbsTopIII], Definition 3.1, (iv), that the image in $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ of the submonoid of $G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$*-invariants* of $\Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ constitutes a *compact topological module*, which we shall refer to as the *pre-log-shell*. Write $p^*_{\underline{v}} \overset{\text{def}}{=} p_{\underline{v}}$ when $p_{\underline{v}}$ is *odd* and $p^*_{\underline{v}} \overset{\text{def}}{=} p^2_{\underline{v}}$ when $p_{\underline{v}}$ is *even*. Then we shall refer to the result of multiplying the pre-log-shell by the factor $(p^*_{\underline{v}})^{-1}$ as the **log-shell**

$$\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}} \subseteq \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} = \Psi^{\text{gp}}_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$$

[cf. [AbsTopIII], Definition 5.4, (iii)]. In particular, by applying the natural, algorithmically constructible ind-topological field structure on $\Psi^{\text{gp}}_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$ [cf. [AbsTopIII], Proposition 3.2, (iii)], it thus follows that one may think of this *log-shell* as an object associated to the *codomain* of *any* [that is to say, not necessarily tautological!] $\mathfrak{log}$*-link*

$${}^\dagger\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathcal{F}_{\underline{v}}$$

— i.e., an object that is determined by the image of a certain portion [namely, the $G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}})$-*invariants* of $\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$] of the *domain* of this $\mathfrak{log}$-link.

(ii) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. For $N \in \mathbb{N}_{\geq 1}$, write $\Psi^{\boldsymbol{\mu}_N}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \subseteq \Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \subseteq \Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ for the subgroup of *N-th roots of unity* and $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ for the *[pointed] universal covering* of the topological group determined by the *groupification* $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ of the topological monoid $\Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$. Then one verifies immediately that one may think of the composite covering of topological groups

$$\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$$

— where the second "$\twoheadrightarrow$" is the natural surjection — as a *[pointed] universal covering* of $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$. That is to say, one may think of $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ as an *object constructed from* $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ [cf. also Remark 1.2.1, (i), below]. Now let us recall from the theory of [AbsTopIII] [cf. [AbsTopIII], Definition 4.1, (iv); [AbsTopIII], Proposition 4.2, (i), (ii)] that the natural, algorithmically constructible [i.e., starting from the collection of data ${}^{\dagger}\mathcal{F}_{\underline{v}}$ — cf. [IUTchI], Definition 5.2, (i), (b)] *topological field structure* on $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ allows one to define a *[complex archimedean] logarithm* on $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$, which, in turn, yields a *functorial algorithm* in the collection of data ${}^{\dagger}\mathcal{F}_{\underline{v}}$ [cf. [IUTchI], Definition 5.2, (i), (b)] for constructing a *topological field structure* on $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$, together with a $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$-*Kummer structure* on ${}^{\dagger}\mathbb{U}_{\underline{v}} \stackrel{\mathrm{def}}{=} {}^{\dagger}\mathcal{D}_{\underline{v}}$ [cf. [AbsTopIII], Definition 4.1, (iv); [IUTchII], Proposition 4.4, (i)]. Write

$$\Psi_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})} \subseteq \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$$

for the resulting *multiplicative monoid* of nonzero integral elements [i.e., elements of norm ≤ 1]. Here, we observe that the resulting *diagram*

$$\Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \subseteq \Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadleftarrow \Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} = \Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$$

is *compatible* [cf. the discussion of [AbsTopIII], Definition 4.1, (iv)] with the *co-holomorphicizations* determined by the *natural* $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$-*Kummer* [cf. [IUTchII], Proposition 4.4, (i)] and $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$-*Kummer* [cf. the above discussion] *structures* on ${}^{\dagger}\mathbb{U}_{\underline{v}}$. The triple of data consisting of the topological monoid $\Psi_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$, the Aut-holomorphic space ${}^{\dagger}\mathbb{U}_{\underline{v}}$, and the $\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$-Kummer structure on ${}^{\dagger}\mathbb{U}_{\underline{v}}$ discussed above determines a *collection of data* [i.e., as in [IUTchI], Definition 5.2, (i), (b)]

$$\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})$$

which is, in fact, *naturally isomorphic* to the collection of data ${}^{\dagger}\mathcal{F}_{\underline{v}}$, but which we wish to think of as being related to ${}^{\dagger}\mathcal{F}_{\underline{v}}$ via the *above diagram.* We shall denote this diagram by means of the notation

$${}^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} \mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})$$

and refer to this relationship between ${}^{\dagger}\mathcal{F}_{\underline{v}}$ and $\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})$ as the **tautological $\mathfrak{log}$-link** *associated to* ${}^{\dagger}\mathcal{F}_{\underline{v}}$ [or, when ${}^{\dagger}\mathfrak{F}$ is fixed, *at* $\underline{v}$]. If $\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}_{\underline{v}}$ is any [poly-]isomorphism of collections of data [i.e., as in [IUTchI], Definition 5.2, (i), (b)], then we shall write

$$^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{F}_{\underline{v}}$$

for the diagram obtained by post-composing the tautological $\mathfrak{log}$-link associated to $^{\dagger}\mathcal{F}_{\underline{v}}$ with the given [poly-]isomorphism $\mathfrak{log}(^{\dagger}\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}_{\underline{v}}$ and refer to this relationship between $^{\dagger}\mathcal{F}_{\underline{v}}$ and $^{\ddagger}\mathcal{F}_{\underline{v}}$ as a **$\mathfrak{log}$-link** from $^{\dagger}\mathcal{F}_{\underline{v}}$ to $^{\ddagger}\mathcal{F}_{\underline{v}}$; when the given [poly-]isomorphism $\mathfrak{log}(^{\dagger}\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}_{\underline{v}}$ is the *full poly-isomorphism*, then we shall refer to the resulting $\mathfrak{log}$-link as the *full* $\mathfrak{log}$*-link* from $^{\dagger}\mathcal{F}_{\underline{v}}$ to $^{\ddagger}\mathcal{F}_{\underline{v}}$. Finally, we recall from [AbsTopIII], Definition 4.1, (iv), that the submonoid of units $\Psi^{\times}_{^{\dagger}\mathcal{F}_{\underline{v}}} \subseteq \Psi_{^{\dagger}\mathcal{F}_{\underline{v}}}$ determines a *compact topological subquotient* of $\Psi^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}}$, which we shall refer to as the *pre-log-shell*. We shall refer to the $\Psi^{\times}_{\mathfrak{log}(^{\dagger}\mathcal{F}_{\underline{v}})}$-orbit of the [uniquely determined] closed line segment of $\Psi^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}}$ which is *preserved by multiplication by* ± 1 and whose endpoints differ by a *generator* of the kernel of the natural surjection $\Psi^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{^{\dagger}\mathcal{F}_{\underline{v}}}$ — or, equivalently, the $\Psi^{\times}_{\mathfrak{log}(^{\dagger}\mathcal{F}_{\underline{v}})}$-orbit of the result of *multiplying by* N the [uniquely determined] closed line segment of $\Psi^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}}$ which is *preserved by multiplication by* ± 1 and whose endpoints differ by a *generator* of the kernel of the natural surjection $\Psi^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}} \twoheadrightarrow \Psi^{\mathrm{gp}}_{^{\dagger}\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{^{\dagger}\mathcal{F}_{\underline{v}}}$ — as the **log-shell**

$$\mathcal{I}_{^{\dagger}\mathcal{F}_{\underline{v}}} \subseteq \Psi^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}} = \Psi^{\mathrm{gp}}_{\mathfrak{log}(^{\dagger}\mathcal{F}_{\underline{v}})}$$

[cf. [AbsTopIII], Definition 5.4, (v)]. Thus, one may think of the *log-shell* as an *object constructed from* $\Psi^{\mathrm{gp}}_{^{\dagger}\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{^{\dagger}\mathcal{F}_{\underline{v}}}$. Moreover, by applying the natural, algorithmically constructible topological field structure on $\Psi^{\mathrm{gp}}_{\mathfrak{log}(^{\dagger}\mathcal{F}_{\underline{v}})}$ $(= \Psi^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}})$, it thus follows that one may think of this *log-shell* as an object associated to the *codomain* of *any* [that is to say, not necessarily tautological!] $\mathfrak{log}$*-link*

$$^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{F}_{\underline{v}}$$

— i.e., an object that is determined by the image of a certain portion [namely, the *subquotient* $\Psi^{\times}_{^{\dagger}\mathcal{F}_{\underline{v}}}$ of $\Psi^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}}$] of the *domain* of this $\mathfrak{log}$-link.

(iii) Write

$$\underline{\mathfrak{log}}(^{\dagger}\mathfrak{F}) \stackrel{\mathrm{def}}{=} \left\{\underline{\mathfrak{log}}(^{\dagger}\mathcal{F}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \Psi^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}}\right\}_{\underline{v}\in\underline{\mathbb{V}}}$$

for the collection of ind-topological modules constructed in (i), (ii) above indexed by $\underline{v} \in \underline{\mathbb{V}}$ — where the group structure arises from the *additive* portion of the field structures on $\Psi^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}}$ discussed in (i), (ii); for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, we regard $\Psi^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}}$ as equipped with its natural $G_{\underline{v}}(^{\dagger}\Pi_{\underline{v}})$*-action*. Write

$$\mathfrak{log}(^{\dagger}\mathfrak{F}) \stackrel{\mathrm{def}}{=} \{\mathfrak{log}(^{\dagger}\mathcal{F}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$$

for the $\mathcal{F}$-prime-strip determined by the data $\mathfrak{log}(^{\dagger}\mathcal{F}_{\underline{v}})$ constructed in (i), (ii) for $\underline{v} \in \underline{\mathbb{V}}$. We shall denote by

$$^{\dagger}\mathfrak{F} \xrightarrow{\mathfrak{log}} \mathfrak{log}(^{\dagger}\mathfrak{F})$$

the collection of diagrams $\{{}^\dagger\mathcal{F}_{\underline{v}} \xrightarrow{\mathfrak{log}} \mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$ constructed in (i), (ii) for $\underline{v} \in \underline{\mathbb{V}}$ and refer to this relationship between ${}^\dagger\mathfrak{F}$ and $\mathfrak{log}({}^\dagger\mathfrak{F})$ as the **tautological $\mathfrak{log}$-link** *associated to* ${}^\dagger\mathfrak{F}$. If $\mathfrak{log}({}^\dagger\mathfrak{F}) \xrightarrow{\sim} {}^\ddagger\mathfrak{F}$ is any [poly-]isomorphism of $\mathcal{F}$-prime-strips, then we shall write

$$ {}^\dagger\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathfrak{F} $$

for the diagram obtained by post-composing the tautological $\mathfrak{log}$-link associated to ${}^\dagger\mathfrak{F}$ with the given [poly-]isomorphism $\mathfrak{log}({}^\dagger\mathfrak{F}) \xrightarrow{\sim} {}^\ddagger\mathfrak{F}$ and refer to this relationship between ${}^\dagger\mathfrak{F}$ and ${}^\ddagger\mathfrak{F}$ as a **$\mathfrak{log}$-link** from ${}^\dagger\mathfrak{F}$ to ${}^\ddagger\mathfrak{F}$; when the given [poly-]isomorphism $\mathfrak{log}({}^\dagger\mathfrak{F}) \xrightarrow{\sim} {}^\ddagger\mathfrak{F}$ is the *full poly-isomorphism*, then we shall refer to the resulting $\mathfrak{log}$-link as the *full* $\mathfrak{log}$*-link* from ${}^\dagger\mathfrak{F}$ to ${}^\ddagger\mathfrak{F}$. Finally, we shall write

$$ \mathcal{I}_{{}^\dagger\mathfrak{F}} \stackrel{\text{def}}{=} \{\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}} $$

for the collection of log-shells constructed in (i), (ii) for $\underline{v} \in \underline{\mathbb{V}}$ and refer to this collection as the **log-shell** *associated to* ${}^\dagger\mathfrak{F}$ and [by a slight abuse of notation]

$$ \mathcal{I}_{{}^\dagger\mathfrak{F}} \subseteq \underline{\mathfrak{log}}({}^\dagger\mathfrak{F}) $$

for the collection of natural inclusions indexed by $\underline{v} \in \underline{\mathbb{V}}$. In particular, [cf. the discussion of (i), (ii)], it thus follows that one may think of this *log-shell* as an object associated to the *codomain* of *any* [that is to say, not necessarily tautological!] $\mathfrak{log}$-*link*

$$ {}^\dagger\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathfrak{F} $$

— i.e., an object that is determined by the image of a certain portion [cf. the discussion of (i), (ii)] of the *domain* of this $\mathfrak{log}$-link.

(iv) Let $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$. Then observe that it follows immediately from the constructions of (i) that the *ind-topological modules with* $G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$*-action* $\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}} \subseteq \underline{\mathfrak{log}}({}^\dagger\mathcal{F}_{\underline{v}})$ may be constructed *solely* from the collection of data ${}^\dagger\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}$ [i.e., the portion of the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip ${}^\dagger\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}$ labeled by $\underline{v}$]. That is to say, in light of the definition of a $\times\boldsymbol{\mu}$*-Kummer structure* [cf. [IUTchII], Definition 4.9, (i), (ii), (iv), (vi), (vii)], these constructions only require the *perfection* "$(-)^{\text{pf}}$" *of the units* and are *manifestly unaffected* by the operation of forming the quotient by a torsion subgroup of the units. Write

$$ \mathcal{I}_{{}^\dagger\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}} \subseteq \underline{\mathfrak{log}}({}^\dagger\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) $$

for the resulting ind-topological modules with $G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$-action, *regarded as objects constructed from* ${}^\dagger\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}$.

(v) Let $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$. Then by applying the algorithms for constructing "$k^\sim(G)$", "$\mathcal{I}(G)$" given in [AbsTopIII], Proposition 5.8, (v), to the [object of the category "$\mathbb{TM}^\vdash$" of split topological monoids discussed in [IUTchI], Example 3.4, (ii), determined by the] *split Frobenioid* portion of the collection of data ${}^\dagger\mathcal{F}^\vdash_{\underline{v}}$, one obtains a *functorial algorithm* in the collection of data ${}^\dagger\mathcal{F}^\vdash_{\underline{v}}$ for constructing a *topological module* $\underline{\mathfrak{log}}({}^\dagger\mathcal{F}^\vdash_{\underline{v}})$ [i.e., corresponding to "$k^\sim(G)$"] and a *topological subspace* $\mathcal{I}_{{}^\dagger\mathcal{F}^\vdash_{\underline{v}}}$ [i.e., corresponding to "$\mathcal{I}(G)$"]. In fact, this functorial algorithm only makes use of the *unit portion* of this split Frobenioid, together with a *pointed universal covering* of this unit portion. Moreover, by arguing as in (ii), one may in fact regard this functorial algorithm as an algorithm that only makes use of the *quotient of this unit portion by its N-torsion subgroup*, for $N \in \mathbb{N}_{\geq 1}$, together with a *pointed universal covering* of this quotient. That is to say, this functorial algorithm may, in fact, be regarded as a *functorial algorithm* in the collection of data ${}^{\dagger}\mathcal{F}^{\vdash \times \boldsymbol{\mu}}_{\underline{v}}$ [cf. Remark 1.2.1, (i), below; [IUTchII], Definition 4.9, (v), (vi), (vii)]. Write

$$\mathcal{I}_{{}^{\dagger}\mathcal{F}^{\vdash \times \boldsymbol{\mu}}_{\underline{v}}} \quad \subseteq \quad \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash \times \boldsymbol{\mu}}_{\underline{v}})$$

for the resulting topological module equipped with a closed subspace, *regarded as objects constructed from* ${}^{\dagger}\mathcal{F}^{\vdash \times \boldsymbol{\mu}}_{\underline{v}}$.

(vi) Finally, just as in (iii), we shall write

$$\mathcal{I}_{{}^{\dagger}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}} \quad \stackrel{\mathrm{def}}{=} \quad \{\mathcal{I}_{{}^{\dagger}\mathcal{F}^{\vdash \times \boldsymbol{\mu}}_{\underline{v}}}\}_{\underline{v} \in \underline{\mathbb{V}}} \quad \subseteq \quad \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}) \quad \stackrel{\mathrm{def}}{=} \quad \{\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash \times \boldsymbol{\mu}}_{\underline{v}})\}_{\underline{v} \in \underline{\mathbb{V}}}$$

for the resulting collections of data constructed *solely* from the $\mathcal{F}^{\vdash \times \boldsymbol{\mu}}$-prime-strip ${}^{\dagger}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}$ [i.e., which we do *not* regard as objects constructed from ${}^{\dagger}\mathfrak{F}$!].

Remark 1.1.1.

(i) Thus, $\mathfrak{log}$-links may be thought of as **correspondences** between *certain portions* of the ind-topological monoids in the *domain* of the $\mathfrak{log}$-link and *certain portions* of the ind-topological monoids in the *codomain* of the $\mathfrak{log}$-link. Frequently, in the theory of the present paper, we shall have occasion to consider **"iterates"** of $\mathfrak{log}$-links. The $\mathfrak{log}$-links — i.e., correspondences between certain portions of the ind-topological monoids in the domains and codomains of the $\mathfrak{log}$-links — that appear in such iterates are to be understood as being *defined only on the [local]* **units** [cf. also (ii) below, in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$] that appear in the *domains of these* $\mathfrak{log}$*-links*. Thus, for instance, when considering [the nonzero elements of] a *global number field* embedded in an "idèlic" product [indexed by the set of all valuations of the number field] of localizations, we shall regard the $\mathfrak{log}$-links that appear as being *defined only on the product* [indexed by the set of all valuations of the number field] *of the groups of local units* that appear in the domains of these $\mathfrak{log}$-links. Indeed, in the theory of the present paper, the *only reason* for the introduction of $\mathfrak{log}$-links is to render possible

the *construction of the* **log-shells** *from the various [local]* **units**.

That is to say, the construction of log-shells does not require the use of the "non-unit" — i.e., the *local* and *global* **"value group"** — portions of the various monoids in the domain. Thus, when considering the effect of applying various iterates of $\mathfrak{log}$-links, it suffices, from the point of view of computing the effect of the construction of the log-shells from the local units, to consider the effect of such iterates on the various groups of local units that appear.

(ii) Suppose that we are in the situation of the discussion of *[local]* **units** in (i), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Then, when thinking of **Kummer structures** in terms of Aut-**holomorphic structures** and **co-holomorphicizations**, as in the discussion of [IUTchI], Remark 3.4.2, it is natural to regard the *[local] units* that appear as being, in fact, **"Aut-holomorphic semi-germs"**, that is to say,

- *projective systems of arbitrarily small neighborhoods* of the *[local]* **units** [i.e., of "$\mathbb{S}^1$" in "$\mathbb{C}^\times$", or, in the notation of [IUTchI], Example 3.4, (i); [IUTchI], Remark 3.4.2, of "$\mathcal{O}^\times(\mathcal{C}_{\underline{v}})$" in "$\mathcal{O}^\rhd(\mathcal{C}_{\underline{v}})^{\mathrm{gp}}$"], equipped with
- the Aut-*holomorphic structures* induced by resticting the ambient Aut-holomorphic structure [i.e., of "$\mathbb{C}^\times$", or, in the notation of [IUTchI], Example 3.4, (i); [IUTchI], Remark 3.4.2, of "$\mathcal{O}^\rhd(\mathcal{C}_{\underline{v}})^{\mathrm{gp}}$"],
- the *group structure [germ]* induced by resticting the ambient group structure [i.e., of "$\mathbb{C}^\times$", or, in the notation of [IUTchI], Example 3.4, (i); [IUTchI], Remark 3.4.2, of "$\mathcal{O}^\rhd(\mathcal{C}_{\underline{v}})^{\mathrm{gp}}$"], and
- a *choice of one of the two connected components* of the complement of the units in a sufficiently small neighborhood [i.e., determined by "$\mathcal{O}^\rhd_{\mathbb{C}} \setminus \mathbb{S}^1 \subseteq \mathbb{C}^\times \setminus \mathbb{S}^1$", or, in the notation of [IUTchI], Example 3.4, (i); [IUTchI], Remark 3.4.2, by "$\mathcal{O}^\rhd(\mathcal{C}_{\underline{v}}) \setminus \mathcal{O}^\times(\mathcal{C}_{\underline{v}}) \subseteq \mathcal{O}^\rhd(\mathcal{C}_{\underline{v}})^{\mathrm{gp}} \setminus \mathcal{O}^\times(\mathcal{C}_{\underline{v}})$"].

Indeed, one verifies immediately that such *"*Aut*-holomorphic semi-germs"* are rigid in the sense that they *do not admit any nontrivial holomorphic automorphisms.* In particular, by thinking of the *[local] units* as *"*Aut*-holomorphic semi-germs"* in this way, the approach to **Kummer structures** in terms of Aut-**holomorphic structures** and **co-holomorphicizations** discussed in [IUTchI], Remark 3.4.2, carries over without change [cf. [AbsTopIII], Corollary 2.3, (i)]. Moreover, in light of the well-known *discreteness* of the image of the *units of a number field* via the logarithms of the various archimedean valuations of the number field [cf., e.g., [Lang], p. 144, Theorem 5], it follows that the "idèlic" aspects discussed in (i) are also unaffected by thinking in terms of Aut-holomorphic semi-germs.

Remark 1.1.2.

(i) In the notation of Definition 1.1, (i), we observe that the **tautological $\mathfrak{log}$-link**

$$^\dagger\mathcal{F}_{\underline{v}} \ \xrightarrow{\mathfrak{log}}\ \mathfrak{log}(^\dagger\mathcal{F}_{\underline{v}})$$

satisfies the property that there is a **taulological identification** between the $^\dagger\Pi_{\underline{v}}$'s that appear in the data that gives rise to the *domain* [i.e., $\{^\dagger\Pi_{\underline{v}} \curvearrowright \Psi_{^\dagger\mathcal{F}_{\underline{v}}}\}$] and the data that gives rise to the *codomain* [i.e., $\{^\dagger\Pi_{\underline{v}} \curvearrowright \Psi_{\mathfrak{log}(^\dagger\mathcal{F}_{\underline{v}})}\}$] of the tautological $\mathfrak{log}$-link. By contrast, the $^\dagger\Pi_{\underline{v}}$ that appears in the data that gives rise to the *domain* of the **full $\mathfrak{log}$-link**

$$^\dagger\mathcal{F}_{\underline{v}} \ \xrightarrow{\mathfrak{log}}\ ^\ddagger\mathcal{F}_{\underline{v}}$$

is related to the $^\ddagger\Pi_{\underline{v}}$ [where we use analogous notational conventions for objects associated to $^\ddagger\mathcal{F}$ to the notational conventions already in force for objects associated to $^\dagger\mathcal{F}$] that appears in the data that gives rise to the *codomain* of the full $\mathfrak{log}$-link by means of a **full poly-isomorphism** $^\dagger\Pi_{\underline{v}} \xrightarrow{\sim} {}^\ddagger\Pi_{\underline{v}}$. In this situation,

> the *specific isomorphism* between the $^\dagger\Pi_{\underline{v}}$'s in the domain and codomain of the tautological $\mathfrak{log}$-link may be thought of as a sort of **specific "rigidifying path"** between the $^\dagger\Pi_{\underline{v}}$'s in the domain and codomain of the tautological $\mathfrak{log}$-link that is constructed precisely by using [in an essential

way!] *Frobenius-like monoids* that are related via the $p_{\underline{v}}$*-adic logarithm* [cf. the construction of Definition 1.1, (i)], i.e., by applying the *Galois-equivariance* of the power series defining the $p_{\underline{v}}$-adic logarithm to relate automorphisms of the monoid $\Psi_{{}^\dagger\mathcal{F}_{\underline{v}}}$ to [induced!] automorphisms of the monoid $\Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} = \Psi^{\mathrm{gp}}_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$.

Here, the use of the term "path" is intended to be in the spirit of the notion of a *path in the étale groupoid* [i.e., in the context of the classical theory of the *étale fundamental group*], except that, in the present context, we allow *arbitrary automorphism indeterminacies*, instead of just inner automorphism indeterminacies. In the present paper, we shall work mainly with the **full $\mathfrak{log}$-link** [i.e., not with the tautological $\mathfrak{log}$-link!] since, in the context of the *multiradial algorithms* to be developed in §3 below, it will be of *crucial importance* to be able to

> express the relationship between the *étale-like* ${}^{(-)}\Pi_{\underline{v}}$'s in the *domain* and *codomain* of the $\mathfrak{log}$-links that appear in *purely étale-like terms*, i.e., in a fashion that is [unlike the *specific "rigidifying path"* discussed above!] **free** of **any dependence** on the **Frobenius-like monoids** involved.

This is precisely what is achieved by working with the *"tautologically indeterminate isomorphism"* between ${}^{(-)}\Pi_{\underline{v}}$'s that underlies the *full* $\mathfrak{log}$*-link.*

(ii) An analogous discussion to that of (i) may be given in the situation of Definition 1.1, (ii), i.e., in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. We leave the routine details to the reader.

From the point of view of the present series of papers, the theory of [AbsTopIII] may be summarized as follows.

Proposition 1.2. **($\mathfrak{log}$-links Between $\mathcal{F}$-prime-strips)** *Let*

$${}^\dagger\mathfrak{F} = \{{}^\dagger\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}; \quad {}^\ddagger\mathfrak{F} = \{{}^\ddagger\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

be $\mathcal{F}$-prime-strips [relative to the given initial Θ-data — cf. [IUTchI], Definition 5.2, (i)] and

$${}^\dagger\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathfrak{F}$$

a $\mathfrak{log}$-link from ${}^\dagger\mathfrak{F}$ to ${}^\ddagger\mathfrak{F}$. Write ${}^\dagger\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}$, ${}^\ddagger\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}$ for the associated $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips [cf. [IUTchII], Definition 4.9, (vi), (vii)]; ${}^\dagger\mathfrak{D}$, ${}^\ddagger\mathfrak{D}$ for the associated $\mathcal{D}$-prime-strips [cf. [IUTchI], Remark 5.2.1, (i)]; ${}^\dagger\mathfrak{D}^{\vdash}$, ${}^\ddagger\mathfrak{D}^{\vdash}$ for the associated $\mathcal{D}^{\vdash}$-prime-strips [cf. [IUTchI], Definition 4.1, (iv)]. Also, let us recall the **diagrams**

$$\Psi_{{}^\dagger\mathcal{F}_{\underline{v}}} \supseteq \Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}} \to \underline{\mathfrak{log}}({}^\dagger\mathcal{F}_{\underline{v}}) = \Psi^{\mathrm{gp}}_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})} \xrightarrow{\sim} \Psi^{\mathrm{gp}}_{{}^\ddagger\mathcal{F}_{\underline{v}}} \qquad (*_{\mathrm{non}})$$

$$\Psi_{{}^\dagger\mathcal{F}_{\underline{v}}} \subseteq \Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}} \twoheadleftarrow \underline{\mathfrak{log}}({}^\dagger\mathcal{F}_{\underline{v}}) = \Psi^{\mathrm{gp}}_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})} \xrightarrow{\sim} \Psi^{\mathrm{gp}}_{{}^\ddagger\mathcal{F}_{\underline{v}}} \qquad (*_{\mathrm{arc}})$$

— where the $\underline{v}$ of $(*_{\mathrm{non}})$ (respectively, $(*_{\mathrm{arc}})$) belongs to $\underline{\mathbb{V}}^{\mathrm{non}}$ (respectively, $\underline{\mathbb{V}}^{\mathrm{arc}}$), and the [poly-]isomorphisms on the right are induced by the " $\xrightarrow{\mathfrak{log}}$ " — of Definition 1.1, (i), (ii).

(i) **(Coricity of Associated $\mathcal{D}$-Prime-Strips)** *The* $\mathfrak{log}$*-link* ${}^{\dagger}\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathfrak{F}$ *induces* **[poly-]isomorphisms**

$${}^{\dagger}\mathfrak{D} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}; \quad {}^{\dagger}\mathfrak{D}^{\vdash} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}^{\vdash}$$

between the associated $\mathcal{D}$*- and* $\mathcal{D}^{\vdash}$*-prime-strips. In particular, the [poly-]isomorphism* ${}^{\dagger}\mathfrak{D} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}$ *induced by* ${}^{\dagger}\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathfrak{F}$ *induces a [poly-]isomorphism*

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}) \xrightarrow{\sim} \Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})$$

between the collections of monoids equipped with auxiliary data of [IUTchII], Corollary 4.5, (i).

(ii) **(Simultaneous Compatibility with Ring Structures)** *At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*, the* **natural** ${}^{\dagger}\Pi_{\underline{v}}$**-actions** *on the "*$\Psi$*'s" appearing in the diagram* $(*_{\mathrm{non}})$ *are compatible with the ind-topological* **ring structures** *on* $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *and* $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$*. At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*, the* **co-holomorphicizations** *determined by the* **natural** $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$**-** *and* $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$ $(=\Psi^{\sim}_{{}^{\dagger}\mathcal{F}_{\underline{v}}})$**-Kummer structures** *on* ${}^{\dagger}\mathbb{U}_{\underline{v}}$ *— which [cf. the discussion of Definition 1.1, (ii)] are compatible with the diagram* $(*_{\mathrm{arc}})$ *— are compatible with the topological* **ring structures** *on* $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *and* $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$*.*

(iii) **(Simultaneous Compatibility with Log-volumes)** *At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*, the diagram* $(*_{\mathrm{non}})$ *is compatible with the* **natural** $p_{\underline{v}}$**-adic log-volumes** *[cf. [AbsTopIII], Proposition 5.7, (i), (c); [AbsTopIII], Corollary 5.10, (ii)] on the subsets of* ${}^{\dagger}\Pi_{\underline{v}}$*-invariants of* $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *and* $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$*. At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*, the diagram* $(*_{\mathrm{arc}})$ *is compatible with the* **natural angular log-volume** *[cf. Remark 1.2.1, (i), below; [AbsTopIII], Proposition 5.7, (ii); [AbsTopIII], Corollary 5.10, (ii)] on* $\Psi^{\times}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *and the* **natural radial log-volume** *[cf. [AbsTopIII], Proposition 5.7, (ii), (c); [AbsTopIII], Corollary 5.10, (ii)] on* $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$ *— cf. also Remark 1.2.1, (ii), below.*

(iv) **(Kummer theory)** *The* **Kummer isomorphisms**

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}) \xrightarrow{\sim} \Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}); \qquad \Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F}) \xrightarrow{\sim} \Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})$$

of [IUTchII], Corollary 4.6, (i), **fail** *to be* **compatible** *with the [poly-]isomorphism* $\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}) \xrightarrow{\sim} \Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})$ *of (i), relative to the diagrams* $(*_{\mathrm{non}})$*,* $(*_{\mathrm{arc}})$ *[and the notational conventions of Definition 1.1] — cf. [AbsTopIII], Corollary 5.5, (iv). [Here, we regard the diagrams* $(*_{\mathrm{non}})$*,* $(*_{\mathrm{arc}})$ *as diagrams that relate* $\Psi_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *and* $\Psi_{{}^{\ddagger}\mathcal{F}_{\underline{v}}}$*, via the [poly-]isomorphism* $\mathfrak{log}({}^{\dagger}\mathfrak{F}) \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}$ *that determines the* $\mathfrak{log}$*-link* ${}^{\dagger}\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathfrak{F}$*.]*

(v) **(Holomorphic Log-shells)** *At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*, the* **log-shell**

$$\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \subseteq \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}) \quad (\xrightarrow{\sim} \Psi^{\mathrm{gp}}_{{}^{\ddagger}\mathcal{F}_{\underline{v}}})$$

satisfies the following properties: (a_{non}) $\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *is* **compact***, hence of* **finite log-volume** *[cf. [AbsTopIII], Corollary 5.10, (i)];* (b_{non}) $\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ *contains the submonoid* *of* ${}^\dagger\Pi_{\underline{v}}$*-invariants of* $\Psi_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$ *[cf. [AbsTopIII], Definition 5.4, (iii)]; (c_{non})* $\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ *contains the image of the submonoid of* ${}^\dagger\Pi_{\underline{v}}$*-invariants of* $\Psi^\times_{{}^\dagger\mathcal{F}_{\underline{v}}}$. *At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, *the* **log-shell**

$$\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}} \subseteq \underline{\mathfrak{log}}({}^\dagger\mathcal{F}_{\underline{v}}) \quad (\xrightarrow{\sim} \quad \Psi^{\mathrm{gp}}_{{}^\ddagger\mathcal{F}_{\underline{v}}})$$

satisfies the following properties: (a_{arc}) $\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ *is* **compact**, *hence of* **finite radial log-volume** *[cf. [AbsTopIII], Corollary 5.10, (i)]; (b_{arc})* $\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ *contains* $\Psi_{\mathfrak{log}({}^\dagger\mathcal{F}_{\underline{v}})}$ *[cf. [AbsTopIII], Definition 5.4, (v)]; (c_{arc}) the image of* $\mathcal{I}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ *in* $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ *contains* $\Psi^\times_{{}^\dagger\mathcal{F}_{\underline{v}}}$ *[i.e., in essence, the pre-log-shell].*

(vi) **(Nonarchimedean Mono-analytic Log-shells)** *At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, *if we write* ${}^\dagger\mathcal{D}^\vdash_{\underline{v}} = \mathcal{B}({}^\dagger G_{\underline{v}})^0$ *for the portion of* ${}^\dagger\mathfrak{D}^\vdash$ *indexed by* $\underline{v}$ *[cf. the notation of [IUTchII], Corollary 4.5], then the algorithms for constructing "*$k^\sim(G)$*", "*$\mathcal{I}(G)$*" given in [AbsTopIII], Proposition 5.8, (ii), yield a* **functorial algorithm** *in the category* ${}^\dagger\mathcal{D}^\vdash_{\underline{v}}$ *for constructing an ind-topological module equipped with a continuous* ${}^\dagger G_{\underline{v}}$*-action*

$$\underline{\mathfrak{log}}({}^\dagger\mathcal{D}^\vdash_{\underline{v}}) \stackrel{\mathrm{def}}{=} \left\{ {}^\dagger G_{\underline{v}} \curvearrowright k^\sim({}^\dagger G_{\underline{v}}) \right\}$$

and a topological submodule — i.e., a **"mono-analytic log-shell"** *—*

$$\mathcal{I}_{{}^\dagger\mathcal{D}^\vdash_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathcal{I}({}^\dagger G_{\underline{v}}) \subseteq k^\sim({}^\dagger G_{\underline{v}})$$

equipped with a $p_{\underline{v}}$*-adic log-volume [cf. [AbsTopIII], Corollary 5.10, (iv)]. Moreover, there is a natural* **functorial algorithm** *[cf. the second display of [IUTchII], Corollary 4.6, (ii)] in the collection of data* ${}^\dagger\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}$ *[i.e., the portion of* ${}^\dagger\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}$ *labeled by* $\underline{v}$*] for constructing an* Ism**-orbit of isomorphisms** *[cf. [IUTchII], Example 1.8, (iv); [IUTchII], Definition 4.9, (i), (vii)]*

$$\underline{\mathfrak{log}}({}^\dagger\mathcal{D}^\vdash_{\underline{v}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^\dagger\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}})$$

of ind-topological modules [cf. Definition 1.1, (iv)], as well as a **functorial algorithm** *[cf. [AbsTopIII], Corollary 5.10, (iv), (c), (d); the fourth display of [IUTchII], Corollary 4.5, (ii); the final display of [IUTchII], Corollary 4.6, (i)] in the collection of data* ${}^\dagger\mathcal{F}_{\underline{v}}$ *for constructing* **isomorphisms**

$$\underline{\mathfrak{log}}({}^\dagger\mathcal{D}^\vdash_{\underline{v}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^\dagger\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^\dagger\mathcal{F}_{\underline{v}}) \quad (\xrightarrow{\sim} \quad \Psi^{\mathrm{gp}}_{{}^\ddagger\mathcal{F}_{\underline{v}}})$$

of ind-topological modules. The various isomorphisms of the last two displays are **compatible** *with one another, as well as with the respective* ${}^\dagger G_{\underline{v}}$*- and* $G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$*-actions [relative to the natural identification* ${}^\dagger G_{\underline{v}} = G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$ *that arises from regarding* ${}^\dagger\mathcal{D}^\vdash_{\underline{v}}$ *as an object constructed from* ${}^\dagger\mathcal{D}_{\underline{v}}$*], the respective* **log-shells**, *and the respective* **log-volumes** *on these log-shells.*

(vii) **(Archimedean Mono-analytic Log-shells)** *At* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, *the algorithms for constructing "*$k^\sim(G)$*", "*$\mathcal{I}(G)$*" given in [AbsTopIII], Proposition 5.8, (v), yield a* **functorial algorithm** *in* ${}^\dagger\mathcal{D}^\vdash_{\underline{v}}$ *[regarded as an object of the category*

"$\mathbb{TM}^{\vdash}$" of split topological monoids discussed in [IUTchI], Example 3.4, (ii)] for constructing a topological module

$$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \stackrel{\mathrm{def}}{=} k^{\sim}({}^{\dagger}G_{\underline{v}})$$

and a topological subspace — i.e., a **"mono-analytic log-shell"** *—*

$$\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathcal{I}({}^{\dagger}G_{\underline{v}}) \subseteq k^{\sim}({}^{\dagger}G_{\underline{v}})$$

equipped with angular and radial log-volumes [cf. [AbsTopIII], Corollary 5.10, (iv)]. Moreover, there is a natural **functorial algorithm** *[cf. the second display of [IUTchII], Corollary 4.6, (ii)] in the collection of data ${}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}$ for constructing a* **poly-isomorphism** *[i.e., an orbit of isomorphisms with respect to the* **independent** *actions of $\{\pm 1\}$ on each of the direct factors that occur in the construction of [AbsTopIII], Proposition 5.8, (v)]*

$$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}})$$

of topological modules [cf. Definition 1.1, (v)], as well as a **functorial algorithm** *[cf. [AbsTopIII], Corollary 5.10, (iv), (c), (d); the fourth display of [IUTchII], Corollary 4.5, (ii); the final display of [IUTchII], Corollary 4.6, (i)] in the collection of data ${}^{\dagger}\mathcal{F}_{\underline{v}}$ for constructing* **poly-isomorphisms** *[i.e., orbits of isomorphisms with respect to the* **independent** *actions of $\{\pm 1\}$ on each of the direct factors that occur in the construction of [AbsTopIII], Proposition 5.8, (v)]*

$$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}}) \quad (\stackrel{\sim}{\to} \quad \Psi^{\mathrm{gp}}_{{}^{\ddagger}\mathcal{F}_{\underline{v}}})$$

of topological modules. The various isomorphisms of the last two displays are **compatible** *with one another, as well as with the respective* **log-shells** *and the respective* **angular** *and* **radial log-volumes** *on these log-shells.*

(viii) **(Mono-analytic Log-shells)** *The various [poly-]isomorphisms of (vi), (vii) [cf. also Definition 1.1, (iii), (vi)] yield collections of [poly-]isomorphisms indexed by $\underline{v} \in \underline{\mathbb{V}}$*

$$\underline{\mathfrak{log}}({}^{\dagger}\mathfrak{D}^{\vdash}) \stackrel{\mathrm{def}}{=} \{\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}} \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}) \stackrel{\mathrm{def}}{=} \{\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$$

$$\mathcal{I}_{{}^{\dagger}\mathfrak{D}^{\vdash}} \stackrel{\mathrm{def}}{=} \{\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}} \stackrel{\sim}{\to} \mathcal{I}_{{}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}} \stackrel{\mathrm{def}}{=} \{\mathcal{I}_{{}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

$$\begin{aligned} \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{D}^{\vdash}) \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}) \stackrel{\sim}{\to} \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F}) &\stackrel{\mathrm{def}}{=} \{\underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}} \\ &\left(\stackrel{\sim}{\to} \quad \Psi^{\mathrm{gp}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F}) \stackrel{\mathrm{def}}{=} \{\Psi^{\mathrm{gp}}_{{}^{\ddagger}\mathcal{F}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}} \right) \end{aligned}$$

$$\mathcal{I}_{{}^{\dagger}\mathfrak{D}^{\vdash}} \stackrel{\sim}{\to} \mathcal{I}_{{}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}} \stackrel{\sim}{\to} \mathcal{I}_{{}^{\dagger}\mathfrak{F}} \stackrel{\mathrm{def}}{=} \{\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

— where, in the definition of "$\Psi^{\mathrm{gp}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F})$", we regard each $\Psi^{\mathrm{gp}}_{{}^{\ddagger}\mathcal{F}_{\underline{v}}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, as being equipped with its natural $G_{\underline{v}}({}^{\ddagger}\Pi_{\underline{v}})$-action [cf. the discussion at the beginning of Definition 1.1].

(ix) **(Coric Holomorphic Log-shells)** *Let* ${}^*\mathfrak{D}$ *be a* $\mathcal{D}$*-prime-strip; write*

$$\mathfrak{F}({}^*\mathfrak{D})$$

for the $\mathcal{F}$**-prime-strip** *naturally determined by* $\Psi_{\mathrm{cns}}({}^*\mathfrak{D})$ *[cf. [IUTchII], Remark 4.5.1, (i)]. Suppose that* ${}^\dagger\mathfrak{F} = {}^\ddagger\mathfrak{F} = \mathfrak{F}({}^*\mathfrak{D})$*, and that the given* $\mathfrak{log}$*-link* $\mathfrak{F}({}^*\mathfrak{D}) = {}^\dagger\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^\ddagger\mathfrak{F} = \mathfrak{F}({}^*\mathfrak{D})$ *is the* **full** $\mathfrak{log}$*-link. Then there exists a* **functorial algorithm** *in the* $\mathcal{D}$*-prime-strip* ${}^*\mathfrak{D}$ *for constructing a collection of topological subspaces — i.e., a collection of* **"coric holomorphic log-shells"** *—*

$$\mathcal{I}_{{}^*\mathfrak{D}} \quad \stackrel{\mathrm{def}}{=} \quad \mathcal{I}_{{}^\dagger\mathfrak{F}}$$

of the collection $\Psi^{\mathrm{gp}}_{\mathrm{cns}}({}^*\mathfrak{D})$*, which may be naturally identified with* $\Psi^{\mathrm{gp}}_{\mathrm{cns}}({}^\ddagger\mathfrak{F})$*, together with a collection of natural isomorphisms [cf. (viii); the fourth display of [IUTchII], Corollary 4.5, (ii)]*

$$\mathcal{I}_{{}^*\mathfrak{D}^\vdash} \quad \stackrel{\sim}{\to} \quad \mathcal{I}_{{}^*\mathfrak{D}}$$

— where we write ${}^*\mathfrak{D}^\vdash$ *for the* $\mathcal{D}^\vdash$*-prime-strip determined by* ${}^*\mathfrak{D}$*.*

(x) **(Frobenius-picture)** *Let* $\{{}^n\mathfrak{F}\}_{n\in\mathbb{Z}}$ *be a* **collection of distinct** $\mathcal{F}$**-prime-strips** *[relative to the given initial* Θ*-data — cf. [IUTchI], Definition 5.2, (i)] indexed by the integers. Write* $\{{}^n\mathfrak{D}\}_{n\in\mathbb{Z}}$ *for the associated* $\mathcal{D}$*-prime-strips [cf. [IUTchI], Remark 5.2.1, (i)] and* $\{{}^n\mathfrak{D}^\vdash\}_{n\in\mathbb{Z}}$ *for the associated* $\mathcal{D}^\vdash$*-prime-strips [cf. [IUTchI], Definition 4.1, (iv)]. Then the* **full** $\mathfrak{log}$**-links** ${}^n\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{(n+1)}\mathfrak{F}$*, for* $n \in \mathbb{Z}$*, give rise to an* **infinite chain**

$$\ldots \xrightarrow{\mathfrak{log}} {}^{(n-1)}\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^n\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{(n+1)}\mathfrak{F} \xrightarrow{\mathfrak{log}} \ldots$$

of $\mathfrak{log}$**-linked** $\mathcal{F}$**-prime-strips** *which induces chains of full poly-isomorphisms*

$$\ldots \stackrel{\sim}{\to} {}^n\mathfrak{D} \stackrel{\sim}{\to} {}^{(n+1)}\mathfrak{D} \stackrel{\sim}{\to} \ldots \qquad \textit{and} \qquad \ldots \stackrel{\sim}{\to} {}^n\mathfrak{D}^\vdash \stackrel{\sim}{\to} {}^{(n+1)}\mathfrak{D}^\vdash \stackrel{\sim}{\to} \ldots$$

on the associated $\mathcal{D}$*- and* $\mathcal{D}^\vdash$*-prime-strips [cf. (i)]. These chains may be represented symbolically as an* **oriented graph** $\vec{\Gamma}$ *[cf. [AbsTopIII], §0]*

$$\begin{array}{ccccccccc} \ldots & \to & \bullet & \to & \bullet & \to & \bullet & \to & \ldots \\ & \ldots & & \searrow & \downarrow & \swarrow & & \ldots & \\ & & & & \circ & & & & \end{array}$$

— i.e., where the horizontal arrows correspond to the " $\xrightarrow{\mathfrak{log}}$ *'s"; the "*$\bullet$*'s" correspond to the "*${}^n\mathfrak{F}$*"; the "*$\circ$*" corresponds to the "*${}^n\mathfrak{D}$*", identified up to isomorphism; the vertical/diagonal arrows correspond to the* **Kummer isomorphisms** *of (iv). This oriented graph* $\vec{\Gamma}$ *admits a natural action by* $\mathbb{Z}$ *[cf. [AbsTopIII], Corollary 5.5, (v)] — i.e., a* **translation symmetry** *— that fixes the* **"core"** $\circ$*, but it does* **not admit arbitrary permutation symmetries***. For instance,* $\vec{\Gamma}$ *does not admit an automorphism that switches two adjacent vertices, but leaves the remaining vertices fixed.*

Proof. The various assertions of Proposition 1.2 follow immediately from the definitions and the references quoted in the statements of these assertions. ○

Remark 1.2.1.

(i) Suppose that we are in the situation of Definition 1.1, (ii). Then at the level of *metrics* — i.e., which give rise to **angular log-volumes** as in Proposition 1.2, (iii) — we suppose that $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ is equipped with the metric obtained by descending the metric of $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}$, but we regard the object

$\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ [or $\Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{{}^\dagger\mathcal{F}_{\underline{v}}}$] as being equipped with a **"weight** N**"**

— i.e., which has the effect of ensuring that the **log-volume** of $\Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ is **equal** to that of $\Psi^{\times}_{{}^\dagger\mathcal{F}_{\underline{v}}}$. That is to say, this convention concerning "weights" ensures that working with $\Psi^{\mathrm{gp}}_{{}^\dagger\mathcal{F}_{\underline{v}}}/\Psi^{\boldsymbol{\mu}_N}_{{}^\dagger\mathcal{F}_{\underline{v}}}$ does not have any effect on various computations of log-volume.

(ii) Although, at first glance, the *compatibility* with *archimedean log-volumes* discussed in Proposition 1.2, (iii), appears to relate *"different objects"* — i.e., *angular* versus *radial* log-volumes — in the domain and codomain of the $\mathfrak{log}$-link under consideration, in fact, this compatibility property may be regarded as an **invariance** property — i.e., that relates *"similar objects"* in the domain and codomain of the $\mathfrak{log}$-link under consideration — by reasoning as follows. Let k be a *complex archimedean field.* Write $\mathcal{O}^{\times}_k \subseteq k$ for the group of elements of absolute value $= 1$ and $k^{\times} \subseteq k$ for the group of nonzero elements. In the following, we shall use the term *"metric on k"* to refer to a Riemannian metric on the real analytic manifold determined by k that is *compatible* with the two natural *almost complex structures* on this real analytic manifold and, moreover, is *invariant* with respect to arbitrary *additive translation* automorphisms of k. In passing, we note that any metric on k is also *invariant* with respect to *multiplication* by elements $\in \mathcal{O}^{\times}_k$. Next, let us observe that the metrics on k naturally form a *torsor* over $\mathbb{R}_{>0}$. In particular, if we write $k^{\times} \cong \mathcal{O}^{\times}_k \times \mathbb{R}_{>0}$ for the natural direct product decomposition, then one verifies immediately that

> *any metric on k is* **uniquely determined** *either by its restriction to $\mathcal{O}^{\times}_k \subseteq k$ or by its restriction to $\mathbb{R}_{>0} \subseteq k$.*

Thus, if one regards the *compatibility* property concerning angular and radial log-volumes discussed in Proposition 1.2, (iii), as a property concerning the *respective restrictions* of the corresponding *uniquely determined* **metrics** [i.e., the metrics corresponding to the respective standard norms on the complex archimedean fields under consideration — cf. [AbsTopIII], Proposition 5.7, (ii), (a)], then this compatibility property discussed in Proposition 1.2, (iii), may be regarded as a property that asserts the **invariance** of the respective natural metrics with respect to the "transformation" constituted by the $\mathfrak{log}$-link.

Remark 1.2.2. Before proceeding, we pause to consider the significance of the various properties discussed in Proposition 1.2, (v). For simplicity, we suppose that "${}^{\dagger}\mathfrak{F}$" is the $\mathcal{F}$-prime-strip that arises from the data constructed in [IUTchI], Examples 3.2, (iii); 3.3, (i); 3.4, (i) [cf. [IUTchI], Definition 5.2, (i)].

(i) Suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$. Thus, $K_{\underline{v}}$ [cf. the notation of [IUTchI], Definition 3.1, (e)] is a **mixed-characteristic nonarchimedean local field**. Write $k \stackrel{\mathrm{def}}{=} K_{\underline{v}}$, $\mathcal{O}_k \subseteq k$ for the ring of integers of k, $\mathcal{O}_k^{\times} \subseteq \mathcal{O}_k$ for the group of units, and $\log_k : \mathcal{O}_k^{\times} \to k$ for the $p_{\underline{v}}$*-adic logarithm*. Then, at a more concrete level — i.e., relative to the notation of the present discussion — the **log-shell** "$\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$" corresponds to the submodule

$$\mathcal{I}_k \stackrel{\mathrm{def}}{=} (p_{\underline{v}}^*)^{-1} \cdot \log_k(\mathcal{O}_k^{\times}) \ \subseteq \ k$$

— where $p_{\underline{v}}^* = p_{\underline{v}}$ if $p_{\underline{v}}$ is *odd*, $p_{\underline{v}}^* = p_{\underline{v}}^2$ if $p_{\underline{v}}$ is *even* — while the properties ($\mathrm{b}_{\mathrm{non}}$), ($\mathrm{c}_{\mathrm{non}}$) of Proposition 1.2, (v), correspond, respectively, to the evident **inclusions**

$$\mathcal{O}_k^{\triangleright} \ \stackrel{\mathrm{def}}{=} \ \mathcal{O}_k \setminus \{0\} \ \subseteq \ \mathcal{O}_k \ \subseteq \ \mathcal{I}_k; \quad \log_k(\mathcal{O}_k^{\times}) \ \subseteq \ \mathcal{I}_k$$

of subsets of k.

(ii) Suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Thus, $K_{\underline{v}}$ [cf. the notation of [IUTchI], Definition 3.1, (e)] is a **complex archimedean field**. Write $k \stackrel{\mathrm{def}}{=} K_{\underline{v}}$, $\mathcal{O}_k \subseteq k$ for the subset of elements of absolute value ≤ 1, $\mathcal{O}_k^{\times} \subseteq \mathcal{O}_k$ for the group of elements of absolute value $= 1$, and $\exp_k : k \to k^{\times}$ for the *exponential map*. Then, at a more concrete level — i.e., relative to the notation of the present discussion — the **log-shell** "$\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$" corresponds to the subset

$$\mathcal{I}_k \ \stackrel{\mathrm{def}}{=} \ \{a \in k \mid |a| \leq \pi\} \ \subseteq \ k$$

of elements of absolute value $\leq \pi$, while the properties ($\mathrm{b}_{\mathrm{arc}}$), ($\mathrm{c}_{\mathrm{arc}}$) of Proposition 1.2, (v), correspond, respectively, to the evident **inclusions**

$$\mathcal{O}_k^{\triangleright} \ \stackrel{\mathrm{def}}{=} \ \mathcal{O}_k \setminus \{0\} \ \subseteq \ \mathcal{O}_k \ \subseteq \ \mathcal{I}_k; \quad \mathcal{O}_k^{\times} \ \subseteq \ \exp_k(\mathcal{I}_k)$$

— where we note the slightly different roles played, in the archimedean [cf. the present (ii)] and nonarchimedean [cf. (i)] cases, by the exponential and logarithmic functions, respectively [cf. [AbsTopIII], Remark 4.5.2].

(iii) The diagram represented by the oriented graph $\vec{\Gamma}$ of Proposition 1.2, (x), is, of course, **far from commutative** [cf. Proposition 1.2, (iv)]! Ultimately, however, [cf. the discussion of Remark 1.4.1, (ii), below] we shall be interested in

(a) constructing **invariants** *with respect to the* $\mathbb{Z}$*-action on* $\vec{\Gamma}$ — i.e., in effect, constructing objects via functorial algorithms in the **coric** $\mathcal{D}$-prime-strips "${}^{n}\mathfrak{D}$" —

while, at the same time,

(b) *relating the corically constructed objects* of (a) to the non-coric "${}^{n}\mathfrak{F}$" via the various **Kummer isomorphisms** of Proposition 1.2, (iv).

That is to say, from the point of view of (a), (b), the content of the *inclusions* discussed in (i) and (ii) above may be interpreted, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, as follows:

> the **coric holomorphic log-shells** of Proposition 1.2, (ix), contain *not only* the images, via the Kummer isomorphisms [i.e., the vertical/diagonal arrows of $\vec{\Gamma}$], of the various "$\mathcal{O}^{\triangleright}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, *but also the* **images**, via the composite of the Kummer isomorphisms with the *various* **iterates** [cf. Remark 1.1.1] *of the* $\mathfrak{log}$*-link* [i.e., the horizontal arrows of $\vec{\Gamma}$], of the portions of the various "$\mathcal{O}^{\triangleright}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ on which these iterates are defined.

An analogous statement in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ may be formulated by adjusting the wording appropriately so as to accommodate the latter portion of this statement, which corresponds to a certain *surjectivity* — we leave the routine details to the reader. Thus, although the diagram [corresponding to] $\vec{\Gamma}$ fails to be *commutative*,

> the *coric holomorphic log-shells* involved exhibit a sort of **"upper semi-commutativity"** with respect to containing/surjecting onto *the various images arising from composites of arrows in* $\vec{\Gamma}$.

(iv) Note that although the diagram $\vec{\Gamma}$ admits a natural "upper semi-commutativity" interpretation as discussed in (iii) above, it **fails** *to admit a corresponding "***lower** *semi-commutativity" interpretation*. Indeed, such a "lower semi-commutativity" interpretation would amount to the existence of some sort of *collection of portions of the various "$\mathcal{O}^{\triangleright}$'s" involved* [cf. the discussion of (i), (ii) above] — i.e., a sort of **"core"** — that are mapped to one another *isomorphically* by the various maps "$\log_k$"/"$\exp_k$" [cf. the discussion of (i), (ii) above] in a fashion that is **compatible** with the various **Kummer isomorphisms** that appear in the diagram $\vec{\Gamma}$. On the other hand, it is difficult to see how to construct such a collection of portions of the various "$\mathcal{O}^{\triangleright}$'s" involved.

(v) Proposition 1.2, (iii), may be interpreted in the spirit of the discussion of (iii) above. That is to say, although the diagram corresponding to $\vec{\Gamma}$ fails to be *commutative*, it is nevertheless *"***commutative** *with respect to* **log-volumes***"*, in the sense discussed in Proposition 1.2, (iii). This "commutativity with respect to log-volumes" allows one to work with log-volumes in a fashion that is *consistent with all composites of the various arrows of* $\vec{\Gamma}$. Log-volumes will play an important role in the theory of §3, below, as a sort of *mono-analytic version* of the notion of the *degree of a global arithmetic line bundle* [cf. the theory of [AbsTopIII], §5].

(vi) As discussed in [AbsTopIII], §I3, the $\mathfrak{log}$-links of $\vec{\Gamma}$ may be thought of as a sort of **"juggling of $\boxplus$, $\boxtimes$"** [i.e., of the two combinatorial dimensions of the ring structure constituted by addition and multiplication]. The **"arithmetic holomorphic structure"** constituted by the coric $\mathcal{D}$-prime-strips is *immune to this juggling*, and hence may be thought as representing a sort of **quotient** *of the horizontal arrow portion of* $\vec{\Gamma}$ *by the action of* $\mathbb{Z}$ [cf. (iii), (a)] — i.e., at the level of abstract oriented graphs, as a sort of **"oriented copy of $\mathbb{S}^1$"**. That is to say, the horizontal arrow portion of $\vec{\Gamma}$ may be thought of as a sort of **"unraveling"** of this "oriented copy of $\mathbb{S}^1$", which is subject to the *"juggling of* $\boxplus$, $\boxtimes$*"* constituted by the $\mathbb{Z}$-action. Here, it is useful to recall that

(a) the *Frobenius-like structures* constituted by the monoids that appear in the horizontal arrow portion of $\vec{\Gamma}$ play the *crucial* role in the theory of the present series of papers of allowing one to construct such **"non-ring/scheme-theoretic filters"** as the **Θ-link** [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)].

By contrast,

(b) the *étale-like structures* constituted by the coric $\mathcal{D}$-prime-strips play the *crucial role* in the theory of the present series of papers of allowing one to construct objects that are capable of **"functorially permeating"** such non-ring/scheme-theoretic filters as the Θ-link [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)].

Finally, in order to *relate* the theory of (a) to the theory of (b), one must avail oneself of **Kummer theory** [cf. (iii), (b), above].

mono-anabelian coric *étale-like* structures	*invariant differential* $d\theta$ on $\mathbb{S}^1$
post-anabelian *Frobenius-like* structures	*coordinate functions* $\int_\bullet d\theta$ on $\vec{\Gamma}$

Fig. 1.1: Analogy with the differential geometry of $\mathbb{S}^1$

(vii) From the point of view of the discussion in (vi) above of the *"oriented copy of* $\mathbb{S}^1$*"* obtained by forming the quotient of the horizontal arrow portion of $\vec{\Gamma}$ by $\mathbb{Z}$, one may think of the *coric étale-like structures* of Proposition 1.2, (i) — as well as the various objects constructed from these coric étale-like structures via the various *mono-anabelian algorithms* discussed in [AbsTopIII] — as corresponding to the *"canonical invariant differential* $d\theta$*"* on $\mathbb{S}^1$ [which is, in particular, *invariant* with respect to the action of $\mathbb{Z}!$]. On the other hand, the various *post-anabelian Frobenius-like structures* obtained by *forgetting* the mono-anabelian algorithms applied to construct these objects — cf., e.g.., the "$\Psi_{\mathrm{cns}}({}^\dagger\mathfrak{F})$" that appear in the *Kummer isomorphisms* of Proposition 1.2, (iv) — may be thought of as *coordinate functions* on the horizontal arrow portion of $\vec{\Gamma}$ [which are *not* invariant with respect to the action of $\mathbb{Z}!$] of the form "$\int_\bullet d\theta$" obtained by integrating the invariant differential $d\theta$ along various paths of $\vec{\Gamma}$ that emanate from some fixed vertex "$\bullet$" of $\vec{\Gamma}$. This point of view is summarized in Fig. 1.1 above. Finally, we observe that this point of view is reminiscent of the discussion of [AbsTopIII], §I5, concerning the analogy between the theory of [AbsTopIII] and the construction of canonical coordinates via integration of Frobenius-invariant differentials in the classical p-adic theory.

Remark 1.2.3.

(i) Observe that, relative to the notation of Remark 1.2.2, (i), any **multiplicative indeterminacy** with respect to the action on $\mathcal{O}_k^{\triangleright}$ of some *subgroup* $H \subseteq \mathcal{O}_k^{\times}$ at some "$\bullet$" of the diagram $\vec{\Gamma}$ gives rise to an **additive indeterminacy** with respect to the action of $\log_k(H)$ on the copy of "$\mathcal{O}_k$" that corresponds to the subsequent "$\bullet$" of the diagram $\vec{\Gamma}$. In particular, if H consists of *roots of unity*, then $\log_k(H) = \{0\}$, so *the resulting additive indeterminacy ceases to exist*. This observation will play a *crucial role* in the theory of §3, below, when it is applied in the context of the *constant multiple rigidity* properties constituted by the *canonical splittings of theta and Gaussian monoids* discussed in [IUTchII], Proposition 3.3, (i); [IUTchII], Corollary 3.6, (iii) [cf. also [IUTchII], Corollary 1.12, (ii); the discussion of [IUTchII], Remark 1.12.2, (iv)].

(ii) In the theory of §3, below, we shall consider *global arithmetic line bundles*. This amounts, in effect, to considering **multiplicative translates** by $f \in F_{\mathrm{mod}}^{\times}$ of the product of the various "$\mathcal{O}_k^{\times}$" of Remark 1.2.2, (i), (ii), as $\underline{v}$ ranges over the elements of $\underline{\mathbb{V}}$. Such translates are *disjoint* from one another, *except* in the case where f is a *unit* at all $\underline{v} \in \underline{\mathbb{V}}$. By elementary algebraic number theory [cf., e.g., [Lang], p. 144, the proof of Theorem 5], this corresponds precisely to the case where f is a *root of unity*. In particular, to consider quotients by this *multiplicative* action by $F_{\mathrm{mod}}^{\times}$ at one "$\bullet$" of the diagram $\vec{\Gamma}$ [where we allow $\underline{v}$ to range over the elements of $\underline{\mathbb{V}}$] gives rise to an **additive indeterminacy** by *"logarithms of roots of unity"* at the subsequent "$\bullet$" of the diagram $\vec{\Gamma}$. In particular, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, *the resulting additive indeterminacy ceases to exist* [cf. the discussion of (i); Definition 1.1, (iv)]; at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, the resulting indeterminacy corresponds to considering certain quotients of the copies of "$\mathcal{O}_k^{\times}$" — i.e., of "$\mathbb{S}^1$" — that appear by *some finite subgroup* [cf. the discussion of Definition 1.1, (ii)]. These observations will be of use in the development of the theory of §3, below.

Remark 1.2.4.

(i) At this point, we pause to recall the important observation that the $\mathfrak{log}$-link is **incompatible** with the **ring structures** of $\Psi^{\mathrm{gp}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$ and $\Psi^{\mathrm{gp}}_{\mathfrak{log}({}^{\dagger}\mathcal{F}_{\underline{v}})}$ [cf. the notation of Proposition 1.2, (ii)], in the sense that it does not arise from a *ring homomorphism* between these two rings. The barrier constituted by this incompatibility between the ring structures on either side of the $\mathfrak{log}$-link is precisely what is referred to as the **"$\mathfrak{log}$-wall"** in the theory of [AbsTopIII] [cf. the discussion of [AbsTopIII], §I4]. This incompatibility with the respective ring structures implies that it is not possible, *a priori*, to transport objects whose structure depends on these ring structures via the $\mathfrak{log}$-link by invoking the principle of "transport of structure". From the point of view of the theory of the present series of papers, this means, in particular, that

> the $\mathfrak{log}$-wall is **incompatible** with **conventional scheme-theoretic basepoints**, which are defined by means of geometric points [i.e., *ring homomorphisms* of a certain type]

— cf. the discussion of [IUTchII], Remark 3.6.3, (i); [AbsTopIII], Remark 3.7.7, (i). In this context, it is useful to recall that étale fundamental groups — i.e., Galois groups — are defined as certain *automorphism groups of fields/rings*; in particular, the definition of such a Galois group "as a certain automorphism group of some ring structure" is *incompatible*, in a quite essential way, with the $\mathfrak{log}$-wall. In a similar vein, **Kummer theory**, which depends on the *multiplicative structure* of the ring under consideration, is also *incompatible*, in a quite essential way, with the $\mathfrak{log}$-wall [cf. Proposition 1.2, (iv)]. That is to say, in the context of the $\mathfrak{log}$-link,

> the only structure of interest that is manifestly **compatible** with the $\mathfrak{log}$-link [cf. Proposition 1.2, (i), (ii)] is the associated $\boldsymbol{\mathcal{D}}$**-prime-strip**

— i.e., the *abstract topological groups* [isomorphic to "$\Pi_{\underline{v}}$" — cf. the notation of [IUTchI], Definition 3.1, (e), (f)] at $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$ and *abstract* Aut-*holomorphic spaces* [isomorphic to "$\mathbb{U}_{\underline{v}}$" — cf. the notation of [IUTchII], Proposition 4.3] at $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$. Indeed, this observation is precisely the *starting point* of the theory of [AbsTopIII] [cf. the discussion of [AbsTopIII], §I1, §I4].

(ii) Other important examples of structures which are *incompatible with the* $\mathfrak{log}$*-wall* include

(a) the *additive structure* on the image of the Kummer map [cf. the discussion of [AbsTopIII], Remark 3.7.5];

(b) in the *"birational"* situation — i.e., where one replaces "$\Pi_{\underline{v}}$" by the absolute Galois group $\Pi^{\text{birat}}_{\underline{v}}$ of the *function field* of the *affine curve* that gave rise to $\Pi_{\underline{v}}$ — the datum of the *collection of closed points* that determines the affine curve [cf. [AbsTopIII], Remark 3.7.7, (ii)].

Note, for instance in the case of (b), when, say, for simplicity, $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \bigcap \underline{\mathbb{V}}^{\text{non}}$, that one may think of the additional datum under consideration as consisting of the natural outer surjection $\Pi^{\text{birat}}_{\underline{v}} \twoheadrightarrow \Pi_{\underline{v}}$ that arises from the *scheme-theoretic morphism* from the spectrum of the function field to the given affine curve. On the other hand, just as in the case of the discussion of scheme-theoretic basepoints in (i), the construction of such an object $\Pi^{\text{birat}}_{\underline{v}} \twoheadrightarrow \Pi_{\underline{v}}$ whose structure depends, in an essential way, on the scheme [i.e., ring!] structures involved necessarily *fails to be compatible with the* $\mathfrak{log}$*-link* [cf. the discussion of [AbsTopIII], Remark 3.7.7, (ii)].

(iii) One way to understand the *incompatibility* discussed in (ii), (b), is as follows. Write $\Delta^{\text{birat}}_{\underline{v}}$, $\Delta_{\underline{v}}$ for the respective kernels of the natural surjections $\Pi^{\text{birat}}_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$, $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$. Then if one forgets about the *scheme-theoretic* basepoints discussed in (i), $G_{\underline{v}}$, $\Delta^{\text{birat}}_{\underline{v}}$, and $\Delta_{\underline{v}}$ may be understood on *both* sides of the $\mathfrak{log}$-wall as *"some topological group"*, and each of the topological groups $\Delta^{\text{birat}}_{\underline{v}}$, $\Delta_{\underline{v}}$ may be understood on *both* sides of the $\mathfrak{log}$-wall as being equipped with *"some outer* $G_{\underline{v}}$*-action"* — cf. the two diagonal arrows of Fig. 1.2 below. On the other hand, the datum of a *particular outer surjection* $\Delta^{\text{birat}}_{\underline{v}} \twoheadrightarrow \Delta_{\underline{v}}$ [cf. the dotted line in Fig. 1.2] relating these two diagonal arrows — which depends, in an essential way, on the scheme [i.e., ring] structures involved! — necessarily *fails to be compatible with the* $\mathfrak{log}$*-link* [cf. the discussion of [AbsTopIII], Remark 3.7.7, (ii)]. This issue of *"triangular compatibility between independent indeterminacies"* is formally reminiscent of the issue of compatibility of outer homomorphisms discussed in [IUTchI], Remark 4.5.1, (i) [cf. also [IUTchII], Remark 2.5.2, (ii)].

indep.
bp. indet.
$\curvearrowright$

$\boxed{\text{nonarch. local abs. Galois group } G_{\underline{v}}}$

indep. bp. indet. $\curvearrowright$ $\swarrow$ $\curvearrowright$ $\searrow$ indep. bp. indet. $\curvearrowright$

$\boxed{\textbf{birational}\text{ geom. fund. gp. } \Delta_{\underline{v}}^{\mathrm{birat}}}$ $\overset{?}{\cdots\cdots\cdots\cdots\cdots}$ $\boxed{\textbf{affine}\text{ geom. fund. gp. } \Delta_{\underline{v}}}$

Fig. 1.2: Independent basepoint indeterminacies obstruct relationship between birational and affine geometric fundamental groups

Remark 1.2.5. The discussion in Remark 1.2.4 of the *incompatibility* of the $\mathfrak{log}$-*wall* with various structures that arise from *ring/scheme-theory* is closely related to the issue of avoiding the use of **fixed ring/scheme-theoretic reference models** in **mono-anabelian** construction algorithms [cf. the discussion of [IUTchI], Remark 3.2.1, (i); [AbsTopIII], §I4]. Put another way, at least in the context of the $\mathfrak{log}$-link [i.e., situations of the sort considered in [AbsTopIII], as well as in the present paper], mono-anabelian construction algorithms may be understood as

> algorithms whose **dependence** on data arising from such *fixed ring/scheme-theoretic reference models* is **"invariant"**, or **"coric"**, with respect to the action of $\mathfrak{log}$ on such models.

A substantial portion of [AbsTopIII], §3, is devoted precisely to the task of giving a *precise formulation* of this concept of *"invariance"* by means of such notions as *observables*, *families of homotopies*, and *telecores*. For instance, one approach to formulating the *failure* of the *ring structure of a fixed reference model* to be *"coric"* with respect to $\mathfrak{log}$ may be seen in [AbsTopIII], Corollary 3.6, (iv); [AbsTopIII], Corollary 3.7, (iv).

Proposition 1.3. **($\mathfrak{log}$-links Between $\Theta^{\pm\mathrm{ell}}$NF-Hodge Theaters)** *Let*

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}; \quad ^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

be $\Theta^{\pm \mathrm{ell}}$**NF-Hodge theaters** *[relative to the given initial* Θ*-data] — cf. [IUTchI], Definition 6.13, (i). Write* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$, ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *for the associated* $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$**NF-Hodge theaters** *— cf. [IUTchI], Definition 6.13, (ii). Then:*

(i) **(Construction of the $\mathfrak{log}$-Link)** *Fix an isomorphism*

$$\Xi : {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

of $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters. Let* ${}^{\dagger}\mathfrak{F}_{\square}$ *be one of the* $\mathcal{F}$*-prime-strips that appear in the* Θ*- and* $\Theta^{\pm}$*-bridges that constitute* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *— i.e., either one of the* $\mathcal{F}$**-prime-strips**

$${}^{\dagger}\mathfrak{F}_{>}, \quad {}^{\dagger}\mathfrak{F}_{\succ}$$

or one of the **constituent** $\mathcal{F}$**-prime-strips** *of the capsules*

$${}^{\dagger}\mathfrak{F}_{J}, \quad {}^{\dagger}\mathfrak{F}_{T}$$

[cf. [IUTchI], Definition 5.5, (ii); [IUTchI], Definition 6.11, (i)]. Write ${}^{\ddagger}\mathfrak{F}_{\square}$ *for the corresponding* $\mathcal{F}$*-prime-strip of* ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$*. Then the poly-isomorphism determined by* Ξ *between the* $\mathcal{D}$*-prime-strips associated to* ${}^{\dagger}\mathfrak{F}_{\square}$, ${}^{\ddagger}\mathfrak{F}_{\square}$ *uniquely determines a poly-isomorphism* $\mathfrak{log}({}^{\dagger}\mathfrak{F}_{\square}) \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}_{\square}$ *[cf. Definition 1.1, (iii); [IUTchI], Corollary 5.3, (ii)], hence a* $\mathfrak{log}$*-link* ${}^{\dagger}\mathfrak{F}_{\square} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathfrak{F}_{\square}$ *[cf. Definition 1.1, (iii)]. We shall denote by*

$${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

and refer to as a $\mathfrak{log}$**-link** *from* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *to* ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *the collection of data consisting of* Ξ*, together with the collection of* $\mathfrak{log}$*-links* ${}^{\dagger}\mathfrak{F}_{\square} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathfrak{F}_{\square}$*, as "*$\square$*" ranges over all possibilities for the* $\mathcal{F}$*-prime-strips in question. When* Ξ *is replaced by a poly-isomorphism* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$*, we shall also refer to the resulting collection of* $\mathfrak{log}$*-links [i.e., corresponding to each constituent isomorphism of the poly-isomorphism* Ξ*] as a* $\mathfrak{log}$**-link** *from* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *to* ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$*. When* Ξ *is the full poly-isomorphism, we shall refer to the resulting* $\mathfrak{log}$*-link as the* **full** $\mathfrak{log}$**-link**.

(ii) **(Coricity)** *Any* $\mathfrak{log}$*-link* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *induces [and may be thought of as "lying over"] a* **[poly-]isomorphism**

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

of $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters [and indeed coincides with the* $\mathfrak{log}$*-link constructed in (i) from this [poly-]isomorphism of* $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters].*

(iii) **(Further Properties of the $\mathfrak{log}$-Link)** *In the notation of (i), any* $\mathfrak{log}$*-link* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *satisfies, for each* $\mathcal{F}$*-prime-strip* ${}^{\dagger}\mathfrak{F}_{\square}$*, properties corresponding to the properties of Proposition 1.2, (ii), (iii), (iv), (v), (vi), (vii), (viii), (ix), i.e., concerning* **simultaneous compatibility** *with* **ring structures** *and* **log-volumes**, **Kummer theory**, *and* **log-shells**.

(iv) **(Frobenius-picture)** *Let* $\{{}^n\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n\in\mathbb{Z}}$ *be a* **collection of distinct** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *[relative to the given initial* Θ*-data] indexed by the integers. Write* $\{{}^n\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n\in\mathbb{Z}}$ *for the associated* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters. Then the* **full** $\mathfrak{log}$**-links** ${}^n\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*, for* $n \in \mathbb{Z}$*, give rise to an* **infinite chain**

$$\dots \xrightarrow{\mathfrak{log}} {}^{(n-1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} \dots$$

of $\mathfrak{log}$**-linked** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *which induces a chain of full poly-isomorphisms*

$$\dots \stackrel{\sim}{\to} {}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\sim}{\to} {}^{(n+1)}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \stackrel{\sim}{\to} \dots$$

on the associated $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters. These chains may be represented symbolically as an* **oriented graph** $\vec{\Gamma}$ *[cf. [AbsTopIII], §0]*

$$\begin{array}{ccccccccc} \dots & \to & \bullet & \to & \bullet & \to & \bullet & \to & \dots \\ & \dots & & \searrow & \downarrow & \swarrow & & \dots & \\ & & & & \circ & & & & \end{array}$$

— i.e., where the horizontal arrows correspond to the " $\xrightarrow{\mathfrak{log}}$ *'s"; the "*$\bullet$*'s" correspond to the "*${}^n\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*"; the "*$\circ$*" corresponds to the "*${}^n\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*", identified up to isomorphism; the vertical/diagonal arrows correspond to the* **Kummer isomorphisms** *implicit in the statement of (iii). This oriented graph* $\vec{\Gamma}$ *admits a natural action by* $\mathbb{Z}$ *[cf. [AbsTopIII], Corollary 5.5, (v)] — i.e., a* **translation symmetry** *— that fixes the* **"core"** $\circ$*, but it does* **not admit arbitrary permutation symmetries**. *For instance,* $\vec{\Gamma}$ *does not admit an automorphism that switches two adjacent vertices, but leaves the remaining vertices fixed.*

Proof. The various assertions of Proposition 1.3 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 1.3.1. Note that in Proposition 1.3, (i), it was necessary to carry out the given construction of the $\mathfrak{log}$-link first for a **single** Ξ [i.e., as opposed to a poly-isomorphism Ξ], in order to maintain *compatibility* with the crucial **"±-synchronization"** [cf. [IUTchI], Remark 6.12.4, (iii); [IUTchII], Remark 4.5.3, (iii)] inherent in the structure of a $\Theta^{\pm\mathrm{ell}}$-Hodge theater.

Remark 1.3.2. In the construction of Proposition 1.3, (i), the constituent $\mathcal{F}$-prime-strips ${}^\dagger\mathfrak{F}_t$, for $t \in T$, of the capsule ${}^\dagger\mathfrak{F}_T$ are considered without regard to the $\mathbb{F}_l^{\rtimes\pm}$*-symmetries* discussed in [IUTchII], Corollary 4.6, (iii). On the other hand, one verifies immediately that the $\mathfrak{log}$**-links** associated, in the construction of Proposition 1.3, (i), to these $\mathcal{F}$-prime-strips ${}^\dagger\mathfrak{F}_t$, for $t \in T$ — i.e., more precisely, associated to the **labeled collections of monoids** $\Psi_{\mathrm{cns}}({}^\dagger\mathfrak{F}_\succ)_t$ of [IUTchII], Corollary 4.6, (iii) — are in fact **compatible** with the $\mathbb{F}_l^{\rtimes\pm}$**-symmetrizing isomorphisms** discussed in [IUTchII], Corollary 4.6, (iii), hence also with the **conjugate synchronization** determined by these $\mathbb{F}_l^{\rtimes\pm}$-symmetrizing isomorphisms — cf. the discussion of Step

(vi) of the proof of Corollary 3.12 of §3 below. We leave the routine details to the reader.

Remark 1.3.3.

(i) In the context of Proposition 1.3 [cf. also the discussion of Remarks 1.2.4, 1.3.1, 1.3.2], it is of interest to *observe* that the relationship between the various **Frobenioid-theoretic** [i.e., *Frobenius-like*!] portions of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters in the *domain* and *codomain* of the $\mathfrak{log}$**-link** of Proposition 1.3, (i),

> *does* **not** *include any data — i.e., of the sort discussed in Remark 1.2.4, (ii), (a), (b); Remark 1.2.4, (iii) — that is* **incompatible***, relative to the relevant* **Kummer isomorphisms***, with the* **coricity** *property for étale-like structures given in Proposition 1.3, (ii).*

Indeed, this *observation* may be understood as a consequence of the fact [cf. Remarks 1.3.1, 1.3.2; [IUTchI], Corollary 5.3, (i), (ii), (iv); [IUTchI], Corollary 5.6, (i), (ii), (iii)] that these Frobenioid-theoretic portions of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters under consideration are completely [i.e., *fully faithfully*!] *controlled* [cf. the discussion of (ii) below for more details], via **functorial algorithms**, by the corresponding *étale-like* structures, i.e., structures that appear in the associated $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters, which satisfy the crucial **coricity** property of Proposition 1.3, (ii).

(ii) In the context of (i), it is of interest to recall that the global portion of the underlying Θ^{ell}-bridges is defined [cf. [IUTchI], Definition 6.11, (ii)] in such a way that it does not contain any global Frobenioid-theoretic data! In particular, the issue discussed in (i) concerns only the Frobenioid-theoretic portions of the following:

(a) the various $\mathcal{F}$**-prime-strips** that appear;
(b) the underlying **Θ-Hodge theaters** of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters under consideration;
(c) the global portion of the underlying **NF-bridges** of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters under consideration.

Here, the Frobenioid-theoretic data of (c) gives rise to **independent** [i.e., for corresponding portions of the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters in the *domain* and *codomain* of the $\mathfrak{log}$-*link*] **basepoints** with respect to the $\mathbb{F}_l^{\circledast}$**-symmetry** [cf. [IUTchI], Corollary 5.6, (iii); [IUTchI], Remark 6.12.6, (iii); [IUTchII], Remark 4.7.6]. On the other hand, the independent basepoints that arise from the Frobenioid-theoretic data of (b), as well as of the portion of (a) that lies in the underlying ΘNF-Hodge theater, do not cause any problems [i.e., from the point of view of the sort of *incompatibility* discussed in (i)] since this data is only subject to relationships *defined* by means of *full poly-isomorphisms* [cf. [IUTchI], Examples 4.3, 4.4]. That is to say, the $\mathcal{F}$-prime-strips that lie in the underlying $\Theta^{\pm \mathrm{ell}}$-Hodge theater constitute the most *delicate* [i.e., relative to the issue of independent basepoints!] portion of the Frobenioid-theoretic data of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater. This delicacy revolves around the *global synchronization of* $\pm$*-indeterminacies* in the underlying $\Theta^{\pm\mathrm{ell}}$-Hodge theater [cf. [IUTchI], Remark 6.12.4, (iii); [IUTchII], Remark 4.5.3, (iii)]. On the other hand, this delicacy does not in fact cause any problems [i.e., from the point of view of the sort of *incompatibility* discussed in (i)] since [cf. [IUTchI], Remark 6.12.4, (iii); [IUTchII], Remark 4.5.3, (iii)] the synchronizations of $\pm$-indeterminacies in the underlying $\Theta^{\pm\mathrm{ell}}$-Hodge theater are *defined* [not by means of *scheme-theoretic* relationships, but rather] by applying the **intrinsic structure** *of the underlying* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*-Hodge theater*, which satisfies the crucial **coricity** property of Proposition 1.3, (ii) [cf. the discussion of (i); Remarks 1.3.1, 1.3.2].

The diagrams discussed in the following Definition 1.4 will play a *central role* in the theory of the present series of papers.

Definition 1.4. We maintain the notation of Proposition 1.3 [cf. also [IUTchII], Corollary 4.10, (iii)]. Let $\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n,m\in\mathbb{Z}}$ be a *collection of distinct* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters* [relative to the given initial Θ-data] indexed by pairs of integers. Then we shall refer to either of the diagrams

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}} & {}^{n,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}} & {}^{n+1,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}} & {}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}} & {}^{n+1,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} & {}^{n,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} & {}^{n+1,m+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} & {}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} & {}^{n+1,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

— where the *vertical* arrows are the *full* $\mathfrak{log}$*-links*, and the *horizontal* arrows are the $\Theta^{\times\boldsymbol{\mu}}$- and $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$*-links* of [IUTchII], Corollary 4.10, (iii) — as the *log-theta-lattice*. We shall refer to the log-theta-lattice that involves the $\Theta^{\times\boldsymbol{\mu}}$- (respectively, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-)

links as *non-Gaussian* (respectively, *Gaussian*). Thus, either of these diagrams may be represented symbolically by an *oriented graph*

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

— where the "$\bullet$'s" correspond to the "${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$".

Remark 1.4.1.

(i) One fundamental property of the log-theta-lattices discussed in Definition 1.4 is the following:

> the various squares that appear in each of the log-theta-lattices discussed in Definition 1.4 are **far** *from being* **[1-]commutative**!

Indeed, whereas the *vertical* arrows in each log-theta-lattice are constructed by applying the various *logarithms* at $\underline{v} \in \underline{\mathbb{V}}$ — i.e., which are defined by means of power series that depend, in an essential way, on the *local ring structures* at $\underline{v} \in \underline{\mathbb{V}}$ — the *horizontal arrows* in each log-theta-lattice [i.e., the $\Theta^{\times\boldsymbol{\mu}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links] are *incompatible* with these local ring structures at $\underline{v} \in \underline{\mathbb{V}}$ in an essential way [cf. [IUTchII], Remark 1.11.2, (i), (ii)].

(ii) Whereas the horizontal arrows in each log-theta-lattice [i.e., the $\Theta^{\times\boldsymbol{\mu}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links] allow one, roughly speaking, to **identify** the respective "$\mathcal{O}^{\times\boldsymbol{\mu}}$'s" at [for simplicity] $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ on either side of the horizontal arrow [cf. [IUTchII], Corollary 4.10, (iv)], in order to avail oneself of the theory of **log-shells** — which will play an essential role in the *multiradial representation of the Gaussian monoids* to be developed in §3 below — it is necessary for the "$\bullet$" [i.e., $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater] in which one operates to appear as the *codomain* of a $\mathfrak{log}$-link, i.e., of a vertical arrow of the log-theta-lattice [cf. the discussion of [AbsTopIII], Remark 5.10.2, (iii)]. That is to say, from the point of view of the goal of constructing the multiradial representation of the Gaussian monoids that is to be developed in §3 below,

> each execution of a *horizontal arrow* of the log-theta-lattice necessarily obligates a subsequent execution of a *vertical arrow* of the log-theta-lattice.

On the other hand, in light of the *noncommutativity* observed in (i), this **"intertwining"** *of the horizontal and vertical arrows of the log-theta-lattice* means that the desired **multiradiality** — i.e., **simultaneous compatibility** with the arithmetic holomorphic structures on *both sides of a horizontal arrow* of the log-theta-lattice — can only be realized [cf. the discussion of Remark 1.2.2, (iii)] if one works with objects that are **invariant** *with respect to the vertical arrows* [i.e., with respect to the action of $\mathbb{Z}$ discussed in Proposition 1.3, (iv)], that is to say, with **"vertical cores"**, of the log-theta-lattice.

(iii) From the point of view of the analogy between the theory of the present series of papers and *p-adic Teichmüller theory* [cf. [AbsTopIII], §I5], the *vertical arrows* of the log-theta-lattice correspond to the *Frobenius morphism in positive characteristic*, whereas the *horizontal arrows* of the log-theta-lattice correspond to the *"transition from $p^n\mathbb{Z}/p^{n+1}\mathbb{Z}$ to $p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$"*, i.e., the *mixed characteristic extension structure of a ring of Witt vectors* [cf. [IUTchI], Remark 3.9.3, (i)]. These correspondences are summarized in Fig. 1.3 below. In particular, the "intertwining of horizontal and vertical arrows of the log-theta-lattice" discussed in (ii) above may be thought of as the analogue, in the context of the theory of the present series of papers, of the well-known "intertwining between the mixed characteristic extension structure of a ring of Witt vectors and the Frobenius morphism in positive characteristic" that appears in the classical p-adic theory.

horizontal arrows of the *log-theta-lattice*	**mixed characteristic** extension structure of a ring of *Witt vectors*
vertical arrows of the *log-theta-lattice*	the **Frobenius** morphism in *positive characteristic*

Fig. 1.3: Analogy between the log-theta-lattice and p-adic Teichmüller theory

Remark 1.4.2.

(i) The horizontal and vertical arrows of the log-theta-lattices discussed in Definition 1.4 share the common property of being *incompatible with the local ring structures*, hence, in particular, with the *conventional scheme-theoretic basepoints* on either side of the arrow in question [cf. the discussion of [IUTchII], Remark 3.6.3, (i)]. On the other hand, whereas the linking data of the *vertical arrows* [i.e., the $\mathfrak{log}$-link] is **rigid** and corresponds to a **single fixed, rigid arithmetic holomorphic structure** in which *addition* and *multiplication* are subject to *"rotations"* [cf. the discussion of [AbsTopIII], §I3], the linking data of the *horizontal arrows* [i.e., the $\Theta^{\times\boldsymbol{\mu}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links] — i.e., more concretely, the "$\mathcal{O}^{\times\boldsymbol{\mu}}$'s" at [for simplicity] $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ — is subject to a $\widehat{\mathbb{Z}}^{\times}$**-indeterminacy**, which has the effect of *obliterating the arithmetic holomorphic structure associated to a vertical line* of the log-theta-lattice [cf. the discussion of [IUTchII], Remark 1.11.2, (i), (ii)].

(ii) If, in the spirit of the discussion of [IUTchII], Remark 1.11.2, (ii), one attempts to *"force"* the horizontal arrows of the log-theta-lattice to be compatible with the arithmetic holomorphic structures on either side of the arrow by declaring — in the style of the $\mathfrak{log}$-link! — that these horizontal arrows induce an isomorphism of the respective "$\Pi_{\underline{v}}$'s" at [for simplicity] $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, then one must contend with a situation in which the "common arithmetic holomorphic structure rigidified by the isomorphic copies of $\Pi_{\underline{v}}$" is obliterated each time one takes into account the action of a nontrivial element of $\widehat{\mathbb{Z}}^\times$ [i.e., that arises from the $\widehat{\mathbb{Z}}^\times$-*indeterminacy* involved] on the corresponding "$\mathcal{O}^{\times\boldsymbol{\mu}}$". In particular, in order to keep track of the arithmetic holomorphic structure currently under consideration, one must, in effect, consider **paths** that record the sequence of "$\Pi_{\underline{v}}$-rigidifying" and "$\widehat{\mathbb{Z}}^\times$-indeterminacy" operations that one invokes. On the other hand, the horizontal lines of the log-theta-lattices given in Definition 1.4 amount, in effect, to *universal covering spaces* of the loops — i.e., *"unraveling paths of the loops"* [cf. the discussion of Remark 1.2.2, (vi)] — that occur as one invokes various series of "$\Pi_{\underline{v}}$-rigidifying" and "$\widehat{\mathbb{Z}}^\times$-indeterminacy" operations. Thus, in summary, any attempt as described above to *"force"* the horizontal arrows of the log-theta-lattice to be compatible with the arithmetic holomorphic structures on either side of the arrow does not result in any substantive simplification of the theory of the present series of papers. We refer the reader to [IUTchIV], Remark 3.6.3, for a discussion of a related topic.

We are now ready to state the *main result* of the present §1.

Theorem 1.5. **(Bi-cores of the Log-theta-lattice)** *Fix a collection of* **initial Θ-data**

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\text{bad}}_{\text{mod}},\ \underline{\epsilon})$$

as in [IUTchI], Definition 3.1. Then any **Gaussian log-theta-lattice** *corresponding to this collection of initial Θ-data [cf. Definition 1.4] satisfies the following properties:*

(i) **(Vertical Coricity)** *The vertical arrows of the Gaussian log-theta-lattice induce* **full poly-isomorphisms** *between the respective associated $\mathcal{D}$-$\Theta^{\pm\text{ell}}$NF-Hodge theaters*

$$\dots \ \overset{\sim}{\to}\ {}^{n,m}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}} \ \overset{\sim}{\to}\ {}^{n,m+1}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\text{ell}}\text{NF}} \ \overset{\sim}{\to}\ \dots$$

[cf. Proposition 1.3, (ii)]. Here, $n \in \mathbb{Z}$ is held fixed, while $m \in \mathbb{Z}$ is allowed to vary.

(ii) **(Horizontal Coricity)** *The horizontal arrows of the Gaussian log-theta-lattice induce* **full poly-isomorphisms** *between the respective associated $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips*

$$\dots \ \overset{\sim}{\to}\ {}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \ \overset{\sim}{\to}\ {}^{n+1,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \ \overset{\sim}{\to}\ \dots$$

[cf. [IUTchII], Corollary 4.10, (iv)]. Here, $m \in \mathbb{Z}$ is held fixed, while $n \in \mathbb{Z}$ is allowed to vary.

(iii) **(Bi-coric $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-Prime-Strips)** *For $n, m \in \mathbb{Z}$, write ${}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}$ for the $\mathcal{D}^{\vdash}$-prime-strip associated to the $\mathcal{F}^{\vdash}$-prime-strip ${}^{n,m}\mathfrak{F}^{\vdash}_{\triangle}$ labeled "$\triangle$" of the $\Theta^{\pm\text{ell}}$NF-Hodge theater ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$ [cf. [IUTchII], Corollary 4.10, (i)]; ${}^{n,m}\mathfrak{D}_{\succ}$ for the $\mathcal{D}$-prime-strip labeled "$\succ$" of the $\Theta^{\pm\text{ell}}$NF-Hodge theater ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$ [cf. [IUTchI],*

Definitions 6.11, (i), (iii); 6.13, (i)]. Let us **identify** *[cf. [IUTchII], Corollary 4.10, (i)] the collections of data*

$$\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{\succ})_0 \text{ and } \Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{\succ})_{\langle \mathbb{F}_l^* \rangle}$$

via the **isomorphism** *of the final display of [IUTchII], Corollary 4.5, (iii), and denote by*

$$\mathfrak{F}^{\vdash}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ})$$

the resulting $\mathcal{F}^{\vdash}$-prime-strip. [Thus, it follows immediately from the constructions involved — cf. the discussion of [IUTchII], Corollary 4.10, (i) — that there is a **natural identification isomorphism** $\mathfrak{F}^{\vdash}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ}) \xrightarrow{\sim} \mathfrak{F}^{\vdash}_{>}({}^{n,m}\mathfrak{D}_{>})$*, where we write $\mathfrak{F}^{\vdash}_{>}({}^{n,m}\mathfrak{D}_{>})$ for the $\mathcal{F}^{\vdash}$-prime-strip determined by $\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{>})$.] Write*

$$\mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ}), \quad \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ})$$

for the $\mathcal{F}^{\vdash\times}$-, $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips determined by $\mathfrak{F}^{\vdash}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ})$ [cf. [IUTchII], Definition 4.9, (vi), (vii)]. Thus, by applying the isomorphisms "$\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})^{\times}_{\underline{v}} \xrightarrow{\sim} \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D}^{\vdash})^{\times}_{\underline{v}}$", for $\underline{v} \in \underline{\mathbb{V}}$, of [IUTchII], Corollary 4.5, (ii), [it follows immediately from the definitions that] there exists a **functorial algorithm** *in the $\mathcal{D}^{\vdash}$-prime-strip ${}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}$ for constructing an $\mathcal{F}^{\vdash\times}$-prime-strip $\mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})$, together with a* **functorial algorithm** *in the $\mathcal{D}$-prime-strip ${}^{n,m}\mathfrak{D}_{\succ}$ for constructing a* **natural isomorphism**

$$\mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})$$

— i.e., in more intuitive terms, "$\mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ})$", hence also the associated $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip "$\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ})$", may be naturally regarded, up to isomorphism, as objects constructed from ${}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}$. Then the poly-isomorphisms of (i) [cf. Remark 1.3.2], (ii) induce, respectively, poly-isomorphisms of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips

$$\ldots \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m+1}\mathfrak{D}_{\succ}) \xrightarrow{\sim} \ldots$$

$$\ldots \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n+1,m}\mathfrak{D}^{\vdash}_{\triangle}) \xrightarrow{\sim} \ldots$$

— where we note that, relative to the natural isomorphisms of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips $\mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})$ discussed above, the collection of isomorphisms that constitute the poly-isomorphisms of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips of the first line of the display is, in general, **strictly smaller** *than the collection of isomorphisms that constitute the poly-isomorphisms of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips of the second line of the display [cf. the existence of non-scheme-theoretic automorphisms of absolute Galois groups of MLF's, as discussed in [AbsTopIII], §I3]; the poly-isomorphisms of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips of the second line of the display are* **not full** *[cf. [IUTchII], Remark 1.8.1]. In particular, by composing these isomorphisms, one obtains* **poly-isomorphisms** *of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips*

$$\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle})$$

for arbitrary $n', m' \in \mathbb{Z}$. That is to say, in more intuitive terms, the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip "${}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})$", regarded up to a certain class of isomorphisms, is an *invariant — which we shall refer to as* **"bi-coric"** *— of both the horizontal and the vertical arrows of the Gaussian log-theta-lattice. Finally, the Kummer isomorphisms "*$\Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{F}) \xrightarrow{\sim} \Psi_{\mathrm{cns}}({}^{\ddagger}\mathfrak{D})$*" of [IUTchII], Corollary 4.6, (i), determine* **Kummer isomorphisms**

$$ {}^{n,m}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle} \xrightarrow{\sim} \mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}) $$

which are **compatible** *with the poly-isomorphisms of (ii), as well as with the* $\times\boldsymbol{\mu}$*-Kummer structures at the* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ *of the various* $\mathcal{F}^{\vdash \times \boldsymbol{\mu}}$*-prime-strips involved [cf. [IUTchII], Definition 4.9, (vi), (vii)]; a similar compatibility holds for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ *[cf. the discussion of the final portion of [IUTchII], Definition 4.9, (v)].*

(iv) **(Bi-coric Mono-analytic Log-shells)** *The poly-isomorphisms that constitute the bi-coricity property discussed in (iii) induce* **poly-isomorphisms**

$$ \left\{\mathcal{I}_{{}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})\right\} \xrightarrow{\sim} \left\{\mathcal{I}_{{}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle})\right\} $$

$$ \left\{\mathcal{I}_{\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})} \subseteq \underline{\mathfrak{log}}(\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}))\right\} \xrightarrow{\sim} \left\{\mathcal{I}_{\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}({}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle})} \subseteq \underline{\mathfrak{log}}(\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}({}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle}))\right\} $$

for arbitrary $n, m, n', m' \in \mathbb{Z}$ *that are* **compatible** *with the* **natural poly-isomorphisms**

$$ \left\{\mathcal{I}_{{}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})\right\} \xrightarrow{\sim} \left\{\mathcal{I}_{\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})} \subseteq \underline{\mathfrak{log}}(\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}))\right\} $$

of Proposition 1.2, (viii). On the other hand, by applying the constructions of Definition 1.1, (i), (ii), to the collections of data "$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_0$*" and "*$\Psi_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}_{\succ})_{\langle \mathbb{F}^{*}_{l} \rangle}$*" used in [IUTchII], Corollary 4.10, (i), to construct* ${}^{n,m}\mathfrak{F}^{\vdash}_{\triangle}$ *[cf. Remark 1.3.2], one obtains a* **["holomorphic"] log-shell***, together with an enveloping "*$\underline{\mathfrak{log}}(-)$*" [cf. the pair "*$\mathcal{I}_{{}^{\dagger}\mathfrak{F}} \subseteq \underline{\mathfrak{log}}({}^{\dagger}\mathfrak{F})$*" of Definition 1.1, (iii)], which we denote by*

$$ \mathcal{I}_{{}^{n,m}\mathfrak{F}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n,m}\mathfrak{F}_{\triangle}) $$

[by means of a slight abuse of notation, since no $\mathcal{F}$*-prime-strip "*${}^{n,m}\mathfrak{F}_{\triangle}$*" has been defined!]. Then one has* **natural poly-isomorphisms**

$$ \left\{\mathcal{I}_{{}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})\right\} \xrightarrow{\sim} \left\{\mathcal{I}_{{}^{n,m}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n,m}\mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle})\right\} $$
$$ \xrightarrow{\sim} \left\{\mathcal{I}_{{}^{n,m}\mathfrak{F}_{\triangle}} \subseteq \underline{\mathfrak{log}}({}^{n,m}\mathfrak{F}_{\triangle})\right\} $$

[cf. the poly-isomorphisms obtained in Proposition 1.2, (viii)]; here, the first "$\xrightarrow{\sim}$*" may be regarded as being induced by the Kummer isomorphisms of (iii) and is* **compatible** *with the poly-isomorphisms induced by the poly-isomorphisms of (ii).*

(v) **(Bi-coric Mono-analytic Global Realified Frobenioids)** *Let* $n, m, n', m' \in \mathbb{Z}$*. Then the poly-isomorphisms of* $\mathcal{D}^{\vdash}$*-prime-strips* ${}^{n,m}\mathfrak{D}^{\vdash}_{\triangle} \xrightarrow{\sim} {}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle}$ *induced by the full poly-isomorphisms of (i), (ii) induce [cf. [IUTchII], Corollaries 4.5, (ii); 4.10, (v)] an isomorphism of collections of data*

$$ (\mathcal{D}^{\Vdash}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}),\ \mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{{}^{n,m}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) $$
$$ \xrightarrow{\sim} \quad (\mathcal{D}^{\Vdash}({}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle}),\ \mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle})) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{{}^{n',m'}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) $$

— *i.e., consisting of a Frobenioid, a bijection, and a collection of isomorphisms of topological monoids indexed by* $\underline{\mathbb{V}}$. *Moreover, this isomorphism of collections of data is* **compatible**, *relative to the horizontal arrows of the Gaussian log-theta-lattice [cf., e.g., the full poly-isomorphisms of (ii)], with the* $\mathbb{R}_{>0}$**-orbits** *of the isomorphisms of collections of data*

$$({}^{n,m}\mathcal{C}^{\Vdash}_{\triangle},\ \mathrm{Prime}({}^{n,m}\mathcal{C}^{\Vdash}_{\triangle}) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{{}^{n,m}\rho_{\triangle,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$
$$\xrightarrow{\sim} \quad (\mathcal{D}^{\Vdash}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}),\ \mathrm{Prime}(\mathcal{D}^{\Vdash}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{{}^{n,m}\rho_{\mathcal{D}^{\Vdash},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

obtained by applying the functorial algorithm discussed in the final portion of [IUTchII], Corollary 4.6, (ii) [cf. also the latter portions of [IUTchII], Corollary 4.10, (i), (v)].

Proof. The various assertions of Theorem 1.5 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 1.5.1.

(i) Note that the theory of **conjugate synchronization** developed in [IUTchII] [cf., especially, [IUTchII], Corollaries 4.5, (iii); 4.6, (iii)] plays an essential role in establishing the **bi-coricity** properties discussed in Theorem 1.5, (iii), (iv), (v) — i.e., at a more technical level, in constructing the objects equipped with a subscript "$\triangle$" that appear in Theorem 1.5, (iii); [IUTchII], Corollary 4.10, (i). That is to say, the conjugate synchronization determined by the various symmetrizing isomorphisms of [IUTchII], Corollaries 4.5, (iii); 4.6, (iii), may be thought of as a sort of **descent** mechanism that allows one to descend data that, *a priori*, is **label-dependent** [i.e., depends on the labels "$t \in \mathrm{LabCusp}^{\pm}(-)$"] to data that is **label-independent**. Here, it is important to recall that these labels depend, in an essential way, on the **"arithmetic holomorphic structures"** involved — i.e., at a more technical level, on the *geometric fundamental groups* involved — hence only make sense within a *vertical line* of the log-theta-lattice. That is to say, the significance of this transition from label-dependence to label-independence lies in the fact that this transition is precisely what allows one to construct objects that make sense in *horizontally adjacent* "•'s" of the log-theta-lattice, i.e., to construct *horizontally coric* objects [cf. Theorem 1.5, (ii); the second line of the fifth display of Theorem 1.5, (iii)]. On the other hand, in order to construct the horizontal arrows of the log-theta-lattice, it is necessary to work with **Frobenius-like structures** [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)]. In particular, in order to construct *vertically coric* objects [cf. the first line of the fifth display of Theorem 1.5, (iii)], it is necessary to pass to **étale-like** structures [cf. the discussion of Remark 1.2.4, (i)] by means of **Kummer isomorphisms** [cf. the final display of Theorem 1.5, (iii)]. Thus, in summary,

> the **bi-coricity** properties discussed in Theorem 1.5, (iii), (iv), (v) — i.e., roughly speaking, the bi-coricity of the various "$\mathcal{O}^{\times\boldsymbol{\mu}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ — may be thought of as a consequence of the *intricate interplay of various aspects* of the theory of **Kummer-compatible conjugate synchronization** established in [IUTchII], Corollaries 4.5, (iii); 4.6, (iii).

(ii) In light of the central role played by the theory of conjugate synchronization in the constructions that underlie Theorem 1.5 [cf. the discussion of (i)], it is of interest to examine in more detail to what extent the highly technically nontrivial theory of conjugate synchronization may be replaced by a simpler apparatus. One naive approach to this problem is the following. Let G be a *topological group* [such as one of the absolute Galois groups $G_{\underline{v}}$ associated to $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$]. Then one way to attempt to avoid the application of the theory of conjugate synchronization — which amounts, in essence, to the construction of a **diagonal embedding**

$$G \quad \hookrightarrow \quad G \times \ldots \times G$$

[cf. the notation "$\langle|\mathbb{F}_l|\rangle$", "$\langle\mathbb{F}_l^{\circledast}\rangle$" that appears in [IUTchII], Corollaries 3.5, 3.6, 4.5, 4.6] in a product of copies of G that, *a priori*, may only be identified with one another *up to conjugacy* [i.e., up to composition with an inner automorphism] — is to try to work, instead, with the $(G \times \ldots \times G)$**-conjugacy class** *of such a* **diagonal**. Here, to simplify the notation, let us assume that the above products of copies of G are, in fact, products of *two copies* of G. Then to *identify* the diagonal embedding $G \hookrightarrow G \times G$ with its $(G \times G)$-conjugates implies that one must consider **identifications**

$$(g, g) \quad \sim \quad (g, hgh^{-1}) \ = \ (g, [h, g] \cdot g)$$

[where $g, h \in G$] — i.e., one must identify (g, g) with the product of (g, g) with $(1, [h, g])$. On the other hand, the original purpose of working with distinct copies of G lies in considering **distinct Galois-theoretic Kummer classes** — corresponding to **distinct theta values** [cf. [IUTchII], Corollaries 3.5, 3.6] — at distinct components. That is to say, to identify elements of $G \times G$ that differ by a factor of $(1, [h, g])$ is **incompatible**, in an essential way, with the convention that such a factor $(1, [h, g])$ should correspond to distinct elements [i.e., "1" and "$[h, g]$"] at distinct components [cf. the discussion of Remark 1.5.3, (ii), below]. Here, we note that this incompatibility may be thought of as an essential consequence of the *highly nonabelian nature* of G, e.g., when G is taken to be a copy of $G_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$. Thus, in summary, this naive approach to replacing the theory of conjugate synchronization by a simpler apparatus is *inadequate* from the point of view of the theory of the present series of papers.

(iii) At a purely combinatorial level, the notion of conjugate synchronization is reminiscent of the **label synchronization** discussed in [IUTchI], Remark 4.9.2, (i), (ii). Indeed, both conjugate and label synchronization may be thought of as a sort of **combinatorial** *representation of the* **arithmetic holomorphic structure** associated to a single vertical line of the log-theta-lattice [cf. the discussion of [IUTchI], Remark 4.9.2, (iv)].

Remark 1.5.2.

(i) Recall that unlike the case with the action of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry on the various labeled copies of the absolute Galois group $G_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ [cf. [IUTchII], Corollaries 4.5, (iii); 4.6, (iii)], it is *not* possible to establish an analogous theory of **conjugate synchronization** in the case of the $\mathbb{F}_l^{\circledast}$-symmetry for *labeled copies of* $\overline{F}$ [cf. [IUTchII], Remark 4.7.2]. This is to say, the closest analogue of the conjugate synchronization obtained in the local case relative to the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is the action of the $\mathbb{F}_l^{\circledast}$-symmetry on *labeled copies of the subfields* $F_{\mathrm{mod}} \subseteq F_{\mathrm{sol}} \subseteq \overline{F}$ and the *pseudo-monoid of* ${}_\infty\kappa$*-coric rational functions*, i.e., as discussed in [IUTchII], Corollaries 4.7, (ii); 4.8, (ii). One consequence of this incompatibility of the $\mathbb{F}_l^{\circledast}$-symmetry with the full algebraic closure $\overline{F}$ of F_{mod} is that, as discussed in [IUTchI], Remark 5.1.5, the reconstruction of the **ring structure** on labeled copies of the subfield $F_{\mathrm{sol}} \subseteq \overline{F}$ subject to the $\mathbb{F}_l^{\circledast}$-symmetry [cf. [IUTchII], Corollaries 4.7, (ii); 4.8, (ii)], **fails** to be **compatible** with the various **localization** operations that occur in the structure of a $\mathcal{D}$**-**Θ**NF-Hodge theater**. This is one quite essential reason why it is not possible to establish **bi-coricity** properties for, say, "$F_{\mathrm{sol}}^{\times}$" [which we regard as being equipped with the *ring structure* on the union of "$F_{\mathrm{sol}}^{\times}$" with $\{0\}$ — without which the abstract pair "$\mathrm{Gal}(F_{\mathrm{sol}}/F_{\mathrm{mod}}) \curvearrowright F_{\mathrm{sol}}^{\times}$" consisting of an abstract module equipped with the action of an abstract topological group is *not very interesting*] that are analogous to the bi-coricity properties established in Theorem 1.5, (iii), for "$\mathcal{O}^{\times\boldsymbol{\mu}}$" [cf. the discussion of Remark 1.5.1, (i)]. From this point of view,

> the **bi-coric mono-analytic global realified Frobenioids** of Theorem 1.5, (v) — i.e., in essence, the notion of **"log-volume"** [cf. the point of view of Remark 1.2.2, (v)] — may be thought of as a sort of **"closest possible approximation"** to such a "bi-coric $F_{\mathrm{sol}}^{\times}$" [i.e., which does not exist].

Alternatively, from the point of view of the theory to be developed in §3 below,

> we shall apply the **bi-coric "**$\mathcal{O}^{\times\boldsymbol{\mu}}$**'s"** of Theorem 1.5, (iii) — i.e., in the form of the **bi-coric mono-analytic log-shells** of Theorem 1.5, (iv) — to construct **"multiradial containers"** for the labeled copies of F_{mod} discussed above by applying the **localization functors** discussed in [IUTchII], Corollaries 4.7, (iii); 4.8, (iii).

That is to say, such "multiradial containers" will play the role of a **transportation mechanism** for "$F_{\mathrm{mod}}^{\times}$" — up to *certain indeterminacies*! — between *distinct arithmetic holomorphic structures* [i.e., distinct vertical lines of the log-theta-lattice].

(ii) In the context of the discussion of *"multiradial containers"* in (i) above, we recall [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)] that, in general, **Kummer theory** plays a *crucial role* precisely in situations in which one performs constructions — such as, for instance, the construction of the Θ-, $\Theta^{\times\boldsymbol{\mu}}$-, or $\Theta_{\mathrm{gau}}^{\times\boldsymbol{\mu}}$-links — that are **"not bound to conventional scheme theory"**. That is to say, in the case of the labeled copies of "F_{mod}" discussed in (i), the **incompatibility** of **"solvable reconstructions"** of the **ring structure** with the **localization** operations that occur in a $\mathcal{D}$-ΘNF-Hodge theater [cf. [IUTchI], Remark 5.1.5] may be thought of as a reflection of the **dismantling** of the **global prime-tree structure** of a number field [cf. the discussion of [IUTchII], Remark 4.11.2, (iv)] that underlies the *construction of the* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* performed in [IUTchI], [IUTchII], hence, in particular, as a reflection of the requirement of establishing a **Kummer-compatible** theory of **conjugate synchronization** relative to the $\mathbb{F}_l^{\rtimes\pm}$-symmetry [cf. the discussion of Remark 1.5.1, (i)].

(iii) Despite the failure of labeled copies of "$F_{\mathrm{mod}}^{\times}$" to admit a natural *bi-coric structure* — a state of affairs that forces one to resort to the use of *"multiradial* *containers"* in order to transport such labeled copies of "$F_{\mathrm{mod}}^{\times}$" to alien arithmetic holomorphic structures [cf. the discussion of (i) above] — the global Frobenioids associated to copies of "$F_{\mathrm{mod}}^{\times}$" nevertheless possess important properties that are *not* satisfied, for instance, by the bi-coric global realified Frobenioids discussed in Theorem 1.5, (v) [cf. also [IUTchI], Definition 5.2, (iv); [IUTchII], Corollary 4.5, (ii); [IUTchII], Corollary 4.6, (ii)]. Indeed, unlike the objects contained in the *realified* global Frobenioids that appear in Theorem 1.5, (v), the objects contained in the global Frobenioids associated to copies of "$F_{\mathrm{mod}}^{\times}$" correspond to *genuine* **"conventional arithmetic line bundles"**. In particular, by applying the **ring structure** of the copies of "F_{mod}" under consideration, one can *push forward* such arithmetic line bundles so as to obtain *arithmetic vector bundles* over [the ring of rational integers] $\mathbb{Z}$ and then form *tensor products* of such arithmetic vector bundles. Such operations will play a key role in the theory of §3 below, as well as in the theory to be developed in [IUTchIV].

Remark 1.5.3.

(i) In [QuCnf] [cf. also [AbsTopIII], Proposition 2.6; [AbsTopIII], Corollary 2.7], a theory was developed concerning deformations of holomorphic structures on Riemann surfaces in which holomorphic structures are represented by means of **squares** or **rectangles** on the surface, while quasiconformal Teichmüller deformations of holomorphic structures are represented by **parallelograms** on the surface. That is to say, relative to suitable choices of local coordinates, quasiconformal Teichmüller deformations may be thought of as affine linear deformations in which one of the two underlying real dimensions of the Riemann surface is *dilated* by some factor $\in \mathbb{R}_{>0}$, while the other underlying real dimensions is *left undeformed*. From this point of view, the theory of **conjugate synchronization** — which may be regarded as a sort of **rigidity** that represents the *arithmetic holomorphic structure* associated to a vertical line of the log-theta-lattice [cf. the discussion given in [IUTchII], Remarks 4.7.3, 4.7.4, of the *uniradiality* of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry that underlies the phenomenon of conjugate synchronization] — may be thought of as a sort of **nonarchimedean arithmetic analogue** of the representation of holomorphic structures by means of squares/rectangles referred to above. That is to say, the *right angles* which are characteristic of squares/rectangles may be thought of as a sort of *synchronization* between the metrics of the two underlying real dimensions of a Riemann surface [i.e., metrics which, *a priori*, may differ by some *dilating* factor] — cf. Fig. 1.4 below. Here, we mention in passing that this point of view is reminiscent of the discussion of [IUTchII], Remark 3.6.5, (ii), in which the point of view is taken that the phenomenon of conjugate synchronization may be thought of as a reflection of the **coherence** of the **arithmetic holomorphic structures** involved.

(ii) Relative to the point of view discussed in (i), the approach described in Remark 1.5.1, (ii), to "avoiding conjugate synchronization by identifying the various conjugates of the diagonal embedding" corresponds — in light of the *highly nonabelian* nature of the groups involved! [cf. the discussion of Remark 1.5.1, (ii)] — to thinking of a holomorphic structure on a Riemann surface as an "equivalence class of holomorphic structures in the usual sense relative to the equivalence relation of differing by a Teichmüller deformation"! That is to say, such an [unconventional!]

approach to the definition of a holomorphic structure allows one to circumvent the issue of *rigidifying* the relationship between the metrics of the two underlying real dimensions of the Riemann surface — but only at the cost of rendering unfeasible any meaningful theory of "deformations of a holomorphic structure"!

(iii) The analogy discussed in (i) between conjugate synchronization [which arises from the $\mathbb{F}_l^{\rtimes\pm}$-symmetry!] and the representation of a complex holomorphic structure by means of squares/rectangles may also be applied to the **"κ-sol-conjugate synchronization"** [cf. the discussion of [IUTchI], Remark 5.1.5] given in [IUTchII], Corollary 4.7, (ii); [IUTchII], Corollary 4.8, (ii), between, for instance, the various labeled non-realified and realified global Frobenioids by means of the $\mathbb{F}_l^{*}$**-symmetry**. Indeed, this analogy is all the more apparent in the case of the *realified* global Frobenioids — which admit a natural $\mathbb{R}_{>0}$*-action*. Here, we observe in passing that, just as the theory of conjugate synchronization [via the $\mathbb{F}_l^{\rtimes\pm}$-symmetry] plays an essential role in the construction of the *local portions* of the $\Theta^{\times\boldsymbol{\mu}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}$-links given in [IUTchII], Corollary 4.10, (i), (ii), (iii),

> the **synchronization** of **global realified Frobenioids** by means of the $\mathbb{F}_l^{*}$**-symmetry** may be related — via the isomorphisms of Frobenioids of the second displays of [IUTchII], Corollary 4.7, (iii); [IUTchII], Corollary 4.8, (iii) [cf. also the discussion of [IUTchII], Remark 4.8.1] — to the construction of the *global realified Frobenioid portion* of the $\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}$**-link** given in [IUTchII], Corollary 4.10, (ii).

On the other hand, the synchronization involving the *non-realified* global Frobenioids may be thought of as a sort of *further rigidification* of the global realified Frobenioids. As discussed in Remark 1.5.2, (iii), this "further rigidification" will play an important role in the theory of §3 below.

$$
\begin{array}{cccccc}
G_{\underline{v}} & \dots & G_{\underline{v}} & G_{\underline{v}} & G_{\underline{v}} & \dots & G_{\underline{v}} \\
\curvearrowright & \dots & \curvearrowright & \curvearrowright & \curvearrowright & \dots & \curvearrowright \\
-l^{*} & \dots & -1 & 0 & 1 & \dots & l^{*}
\end{array}
$$

$$
\begin{array}{ccccccc}
 & \vdots & & & & & \\
\mathbb{R}_{>0} \curvearrowright & \vdots & & \nearrow & \searrow & & \\
 & \vdots & \nearrow & & & \searrow & \\
 & \dots & \dots & \dots & \dots & \dots & \curvearrowleft \text{id} \\
 & \vdots & \searrow & & & \nearrow & \\
 & \vdots & & \searrow & \nearrow & &
\end{array}
$$

Fig. 1.4: Analogy between conjugate synchronization and the representation of complex holomorphic structures via squares/rectangles

Remark 1.5.4.

(i) As discussed in [IUTchII], Remark 3.8.3, (iii), one of the main themes of the present series of papers is the goal of giving an **explicit description** of what **one** *arithmetic holomorphic structure* — i.e., one *vertical line of the log-theta-lattice* — looks like from the point of view of a **distinct** *arithmetic holomorphic structure* — i.e., another vertical line of the log-theta-lattice — that is only related to the original arithmetic holomorphic structure via some mono-analytic core, e.g., the various bi-coric structures discussed in Theorem 1.5, (iii), (iv), (v). Typically, the objects of interest that are constructed within the original arithmetic holomorphic structure are **Frobenius-like** structures [cf. the discussion of [IUTchII], Remark 3.6.2], which, as we recall from the discussion of Remark 1.5.2, (ii) [cf. also the discussion of [IUTchII], Remark 3.6.2, (ii)], are necessary in order to perform constructions — such as, for instance, the construction of the Θ-, $\Theta^{\times\boldsymbol{\mu}}$-, or $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links — that are **"not bound to conventional scheme theory"**. Indeed, the main example of such an object of interest consists precisely of the **Gaussian monoids** discussed in [IUTchII], §3, §4. Thus, the operation of describing such an object of interest from the point of view of a *distinct arithmetic holomorphic structure* may be broken down into *two steps*:

(a) passing from *Frobenius-like structures* to *étale-like structures* via various **Kummer isomorphisms**;

(b) transporting the resulting *étale-like structures* from one arithmetic holomorphic structure to another by means of various **multiradiality properties**.

In particular, the computation of what the object of interest looks like from the point of view of a distinct arithmetic holomorphic structure may be broken down into the computation of the **indeterminacies** or *"departures from rigidity"* that arise — i.e., the computation of *"what sort of* **damage** *is incurred to the object of interest"* — during the execution of each of these two steps (a), (b). We shall refer to the indeterminacies that arise from (a) as **Kummer-detachment indeterminacies** and to the indeterminacies that arise from (b) as **étale-transport indeterminacies**.

(ii) *Étale-transport indeterminacies* typically amount to the indeterminacies that occur as a result of the execution of various *"anabelian" or "group-theoretic" algorithms*. One fundamental example of such indeterminacies is constituted by the indeterminacies that occur in the context of Theorem 1.5, (iii), (iv), as a result of the existence of **automorphisms** of the various [copies of] local absolute Galois groups $G_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, which are *not of scheme-theoretic origin* [cf. the discussion of [AbsTopIII], §I3].

(iii) On the other hand, one important example, from the point of view of the theory of the present series of papers, of a *Kummer-detachment indeterminacy* is constituted by the **Frobenius-picture diagrams** given in Propositions 1.2, (x); 1.3, (iv) — i.e., the issue of *which path* one is to take from a particular "$\bullet$" to the coric "$\circ$". That is to say, despite the fact that these diagrams *fail to be commutative*, the **"upper semi-commutativity"** property satisfied by the **coric holomorphic**

log-shells involved [cf. the discussion of Remark 1.2.2, (iii)] may be regarded as a sort of computation, in the form of an *upper estimate*, of the Kummer-detachment indeterminacy in question. Another important example, from the point of view of the theory of the present series of papers, of a *Kummer-detachment indeterminacy* is given by the $\widehat{\mathbb{Z}}^{\times}$**-indeterminacies** discussed in Remark 1.4.2 [cf. also the *Kummer isomorphisms* of the final display of Theorem 1.5, (iii)].

Section 2: Multiradial Theta Monoids

In the present §2, we **globalize** the **multiradial** portion of the local theory of **theta monoids** developed in [IUTchII], §1, §3, at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ [cf., especially, [IUTchII], Corollary 1.12; [IUTchII], Proposition 3.4] so as to cover the theta monoids/Frobenioids of [IUTchII], Corollaries 4.5, (iv), (v); 4.6, (iv), (v), and explain how the resulting theory may be fit into the framework of the **log-theta-lattice** developed in §1.

In the following discussion, we assume that we have been given *initial* Θ*-data* as in [IUTchI], Definition 3.1. Let ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$ be a $\Theta^{\pm\text{ell}}$*NF-Hodge theater* [relative to the given initial Θ-data — cf. [IUTchI], Definition 6.13, (i)] and

$$\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}\}_{n,m\in\mathbb{Z}}$$

a *collection of distinct* $\Theta^{\pm\text{ell}}$*NF-Hodge theaters* [relative to the given initial Θ-data] indexed by pairs of integers, which we think of as arising from a *Gaussian log-theta-lattice*, as in Definition 1.4. We begin by reviewing the theory of theta monoids developed in [IUTchII].

Proposition 2.1. (Vertical Coricity and Kummer Theory of Theta Monoids) *We maintain the notation introduced above. Also, we shall use the notation* $\text{Aut}_{\mathcal{F}^{\Vdash}}(-)$ *to denote the group of automorphisms of the* $\mathcal{F}^{\Vdash}$*-prime-strip in parentheses. Then:*

(i) **(Vertically Coric Theta Monoids)** *In the notation of [IUTchII], Corollary 4.5, (iv), (v) [cf. also the assignment "0,* $\succ\ \mapsto\ >$*" of [IUTchI], Proposition 6.7], there are* **functorial algorithms** *in the* $\mathcal{D}$*- and* $\mathcal{D}^{\vdash}$*-prime-strips* ${}^{\dagger}\mathfrak{D}_{>}$*,* ${}^{\dagger}\mathfrak{D}^{\vdash}_{>}$ *associated to the* $\Theta^{\pm\text{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$ *for constructing collections of data indexed by* $\underline{\mathbb{V}}$

$$\underline{\mathbb{V}} \ni \underline{v} \ \mapsto\ \Psi_{\text{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}; \quad \underline{\mathbb{V}} \ni \underline{v} \ \mapsto\ {}_{\infty}\Psi_{\text{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}$$

as well as a global realified Frobenioid

$$\mathcal{D}^{\Vdash}_{\text{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>})$$

equipped with a **bijection** $\text{Prime}(\mathcal{D}^{\Vdash}_{\text{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>})) \overset{\sim}{\to} \underline{\mathbb{V}}$ *and corresponding local isomorphisms, for each* $\underline{v} \in \underline{\mathbb{V}}$*, as described in detail in [IUTchII], Corollary 4.5, (v). In particular, each isomorphism of the full poly-isomorphism induced [cf. Theorem 1.5, (i)] by a* **vertical** *arrow of the* **Gaussian log-theta-lattice** *under consideration induces a compatible collection of isomorphisms*

$$\Psi_{\text{env}}({}^{n,m}\mathfrak{D}_{>}) \ \overset{\sim}{\to}\ \Psi_{\text{env}}({}^{n,m+1}\mathfrak{D}_{>}); \quad {}_{\infty}\Psi_{\text{env}}({}^{n,m}\mathfrak{D}_{>}) \ \overset{\sim}{\to}\ {}_{\infty}\Psi_{\text{env}}({}^{n,m+1}\mathfrak{D}_{>})$$

$$\mathcal{D}^{\Vdash}_{\text{env}}({}^{n,m}\mathfrak{D}^{\vdash}_{>}) \ \overset{\sim}{\to}\ \mathcal{D}^{\Vdash}_{\text{env}}({}^{n,m+1}\mathfrak{D}^{\vdash}_{>})$$

— *where the final isomorphism of Frobenioids is compatible with the respective bijections involving "*$\mathrm{Prime}(-)$*", as well as with the respective local isomorphisms for each* $\underline{v} \in \underline{\mathbb{V}}$.

(ii) **(Kummer Isomorphisms)** *In the notation of [IUTchII], Corollary 4.6, (iv), (v), there are* **functorial algorithms** *in the* $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *for constructing collections of data indexed by* $\underline{\mathbb{V}}$

$$\underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ \Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}; \quad \underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$$

as well as a global realified Frobenioid

$$\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta})$$

equipped with a **bijection** $\mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta})) \xrightarrow{\sim} \underline{\mathbb{V}}$ *and corresponding local isomorphisms, for each* $\underline{v} \in \underline{\mathbb{V}}$*, as described in detail in [IUTchII], Corollary 4.6, (v). Moreover, there are* **functorial algorithms** *in* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ *for constructing* **Kummer isomorphisms**

$$\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta}) \ \xrightarrow{\sim} \ \Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}); \quad {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta}) \ \xrightarrow{\sim} \ {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$$

$$\mathcal{C}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}) \quad \xrightarrow{\sim} \quad \mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>})$$

— *where the final isomorphism of Frobenioids is compatible with the respective bijections involving "*$\mathrm{Prime}(-)$*", as well as with the respective local isomorphisms for each* $\underline{v} \in \underline{\mathbb{V}}$ — *with the data discussed in (i) [cf. [IUTchII], Corollary 4.6, (iv), (v)]. Finally, the collection of data* $\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$ *gives rise, in a natural fashion, to an* $\mathcal{F}^{\vdash}$*-prime-strip* $\mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$ *[cf. the* $\mathcal{F}^{\vdash}$*-prime-strip "*${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}}$*" of [IUTchII], Corollary 4.10, (ii)]; the global realified Frobenioid* $\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>})$*, equipped with the bijection* $\mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}^{\vdash}_{>})) \xrightarrow{\sim} \underline{\mathbb{V}}$ *and corresponding local isomorphisms, for each* $\underline{v} \in \underline{\mathbb{V}}$*, reviewed in (i), together with the* $\mathcal{F}^{\vdash}$*-prime-strip* $\mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$*, determine an* $\mathcal{F}^{\Vdash}$*-prime-strip* $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$ *[cf. the* $\mathcal{F}^{\Vdash}$*-prime-strip "*${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}$*" of [IUTchII], Corollary 4.10, (ii)]. In particular, the first and third Kummer isomorphisms of the above display may be interpreted as [compatible] isomorphisms*

$${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{env}} \ \xrightarrow{\sim} \ \mathfrak{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}); \quad {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \ \xrightarrow{\sim} \ \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$$

of $\mathcal{F}^{\vdash}$*-,* $\mathcal{F}^{\Vdash}$*-prime-strips.*

(iii) **(Kummer Theory at Bad Primes)** *The portion at* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *of the Kummer isomorphisms of (ii) is obtained by composing the Kummer isomorphisms of [IUTchII], Proposition 3.3, (i) — which, we recall, were defined by forming* **Kummer classes** *in the context of* **mono-theta environments** *that arise from* **tempered Frobenioids** — *with the isomorphisms on cohomology classes induced [cf. the upper left-hand portion of the first display of [IUTchII], Proposition 3.4, (i)] by the* **full poly-isomorphism of projective systems of mono-theta environments** *"*$\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}) \ \xrightarrow{\sim} \ \mathbb{M}^{\Theta}_{*}({}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})$*" [cf. [IUTchII], Proposition 3.4; [IUTchII], Remark 4.2.1, (iv)] between projective systems of mono-theta environments that arise from tempered Frobenioids [i.e., "*${}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$*"] and projective systems of mono-theta*

environments that arise from the tempered fundamental group [i.e., "$^{\dagger}\mathcal{D}_{>,\underline{v}}$"] — cf. the left-hand portion of the third display of [IUTchII], Corollary 3.6, (ii), in the context of the discussion of [IUTchII], Remark 3.6.2, (i). Here, each "isomorphism on cohomology classes" is induced by the isomorphism on **exterior cyclotomes**

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(^{\dagger}\mathcal{D}_{>,\underline{v}})) \quad \stackrel{\sim}{\to} \quad \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}))$$

determined by each of the isomorphisms that constitutes the full poly-isomorphism of projective systems of mono-theta environments discussed above. In particular, the **composite map**

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(^{\dagger}\mathcal{D}_{>,\underline{v}})) \otimes \mathbb{Q}/\mathbb{Z} \quad \to \quad (\Psi_{^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times\boldsymbol{\mu}}$$

obtained by composing the result of applying "$\otimes\ \mathbb{Q}/\mathbb{Z}$" to this isomorphism on exterior cyclotomes with the **natural inclusion**

$$\Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}(^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}})) \otimes \mathbb{Q}/\mathbb{Z} \quad \hookrightarrow \quad (\Psi_{^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times}$$

[cf. the notation of [IUTchII], Proposition 3.4, (i); the description given in [IUTchII], Proposition 1.3, (i), of the exterior cyclotome of a mono-theta environment that arises from a tempered Frobenioid] and the natural projection $(\Psi_{^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times} \twoheadrightarrow (\Psi_{^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}})^{\times\boldsymbol{\mu}}$ is equal to the **zero map**.

(iv) **(Kummer Theory at Good Nonarchimedean Primes)** *The* **unit** *portion at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ of the Kummer isomorphisms of (ii) is obtained [cf. [IUTchII], Proposition 4.2, (iv)] as the unit portion of a "labeled version" of the* **isomorphism of ind-topological monoids** *equipped with a* **topological group action** *— i.e., in the language of [AbsTopIII], Definition 3.1, (ii), the isomorphism of "***MLF-Galois $\mathbb{TM}$-pairs***" — discussed in [IUTchII], Proposition 4.2, (i) [cf. also [IUTchII], Remark 1.11.1, (i), (a); [AbsTopIII], Proposition 3.2, (iv)]. In particular, the portion at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ of the $\mathrm{Aut}_{\mathcal{F}^{\Vdash}}(^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}})$-***orbit** *of the second isomorphism of the final display of (ii) may be obtained as a "labeled version" of the* **"Kummer poly-isomorphism of semi-simplifications"** *given in the final display of [IUTchII], Proposition 4.2, (ii).*

(v) **(Kummer Theory at Archimedean Primes)** *The* **unit** *portion at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ of the Kummer isomorphisms of (ii) is obtained [cf. [IUTchII], Proposition 4.4, (iv)] as the unit portion of a "labeled version" of the* **isomorphism of topological monoids** *discussed in [IUTchII], Proposition 4.4, (i). In particular, the portion at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ of the $\mathrm{Aut}_{\mathcal{F}^{\Vdash}}(^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}})$-***orbit** *of the second isomorphism of the final display of (ii) may be obtained as a "labeled version" of the* **"Kummer poly-isomorphism of semi-simplifications"** *given in the final display of [IUTchII], Proposition 4.4, (ii) [cf. also [IUTchII], Remark 4.6.1].*

(vi) **(Compatibility with Constant Monoids)** *The definition of the* **unit portion** *of the* **theta monoids** *involved [cf. [IUTchII], Corollary 4.10, (iv)] gives rise to* **natural isomorphisms**

$$^{\dagger}\mathfrak{F}^{\vdash\times}_{\triangle} \ \stackrel{\sim}{\to}\ ^{\dagger}\mathfrak{F}^{\vdash\times}_{\mathrm{env}}; \quad \mathfrak{F}^{\vdash\times}_{\triangle}(^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}) \ \stackrel{\sim}{\to}\ \mathfrak{F}^{\vdash\times}_{\mathrm{env}}(^{\dagger}\mathfrak{D}_{>})$$

— *i.e., where the morphism induced on* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips by the first displayed isomorphism is precisely the isomorphism of the first display of [IUTchII], Corollary 4.10, (iv) — of the respective associated* $\mathcal{F}^{\vdash\times}$*-prime-strips [cf. the notation of Theorem 1.5, (iii), where the label "*n, m*" is replaced by the label "*$\dagger$*"]. Moreover, these natural isomorphisms are* **compatible** *with the* **Kummer isomorphisms** *of (ii) above and Theorem 1.5, (iii).*

Proof. The various assertions of Proposition 2.1 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 2.1.1. The theory of **mono-theta environments** [cf. Proposition 2.1, (iii)] will play a crucial role in the theory of the present §2 [cf. Theorem 2.2, (ii); Corollary 2.3, (iv), below] in the *passage from* **Frobenius-like** *to* **étale-like** *structures* [cf. Remark 1.5.4, (i), (a)] at *bad primes*. In particular, the various **rigidity** properties of mono-theta environments established in [EtTh] play a fundamental role in ensuring that the resulting *"Kummer-detachment indeterminacies"* [cf. the discussion of Remark 1.5.4, (i)] are sufficiently mild so as to allow the establishment of the various reconstruction algorithms of interest. For this reason, we pause to review the *main properties* of mono-theta environments established in [EtTh] [cf. [EtTh], Introduction] — namely,

(a) **cyclotomic rigidity**
(b) **discrete rigidity**
(c) **constant multiple rigidity**
(d) **isomorphism class compatibility**
(e) **Frobenioid structure compatibility**

— and the roles played by these main properties in the theory of the present series of papers. Here, we remark that "isomorphism class compatibility" [i.e., (d)] refers to compatibility with the convention that various objects of the tempered Frobenioids [and their associated base categories] under consideration are known only *up to isomorphism* [cf. [EtTh], Corollary 5.12; [EtTh], Remarks 5.12.1, 5.12.2]. In the Introduction to [EtTh], instead of referring to (d) in this form, we referred to the property of compatibility with the *topology of the tempered fundamental group*. In fact, however, this compatibility with the topology of the tempered fundamental group is a consequence of (d) [cf. [EtTh], Remarks 5.12.1, 5.12.2]. On the other hand, from the point of view of the present series of papers, the essential property of interest in this context is best understood as being the property (d).

(i) First, we recall that the significance, in the context of the theory of the present series of papers, of the *compatibility with the Frobenioid structure* of the tempered Frobenioids under consideration [i.e., (e)] — i.e., in particular, with the *monoidal portion*, equipped with its natural *Galois action*, of these Frobenioids — lies in the role played by this *"Frobenius-like"* monoidal portion in performing constructions — such as, for instance, the construction of the $\mathfrak{log}$-, Θ-, $\Theta^{\times\boldsymbol{\mu}}$-, or $\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}$-links — that are **"not bound to conventional scheme theory"**, but may be related, via **Kummer theory**, to various *étale-like structures* [cf. the discussions of Remark 1.5.4, (i); [IUTchII], Remark 3.6.2, (ii); [IUTchII], Remark 3.6.4, (ii), (v)].

(ii) Next, we consider *isomorphism class compatibility* [i.e., (d)]. As discussed above, this compatibility corresponds to regarding each of the various objects of the tempered Frobenioids [and their associated base categories] under consideration as being known only *up to isomorphism* [cf. [EtTh], Corollary 5.12; [EtTh], Remarks 5.12.1, 5.12.2]. As discussed in [IUTchII], Remark 3.6.4, (i), the significance of this property (d) in the context of the present series of papers lies in the fact that — unlike the case with the *projective systems* constituted by *Kummer towers* constructed from N-th power morphisms, which are compatible with only the *multiplicative*, but *not the additive structures* of the $p_{\underline{v}}$-adic local fields involved — *each individual object* in such a Kummer tower corresponds to a *single* field [i.e., as opposed to a projective system of multiplicative groups of fields]. This **field/ring structure** is necessary in order to apply the theory of the $\mathfrak{log}$**-link** developed in §1 — cf. the *vertical coricity* discussed in Proposition 2.1, (i). Note, moreover, that, unlike the $\mathfrak{log}$-, Θ-, $\Theta^{\times\boldsymbol{\mu}}$-, or $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-links, the N-th power morphisms that appear in a Kummer tower are **"algebraic"**, hence compatible with the conventional scheme theory surrounding the étale [or tempered] fundamental group. In particular, since the tempered Frobenioids under consideration may be constructed from such scheme-theoretic categories, the fundamental groups on either side of such an N-th power morphism may be related *up to an indeterminacy arising from an* **inner automorphism** of the tempered fundamental group [i.e., the "fundamental group" of the base category] under consideration — cf. the discussion of [IUTchII], Remark 3.6.3, (ii). On the other hand, the objects that appear in these Kummer towers necessarily arise from *nontrivial line bundles* [indeed, line bundles all of whose positive tensor powers are nontrivial!] on tempered coverings of a Tate curve — cf. the constructions underlying the Frobenioid-theoretic version of the mono-theta environment [cf. [EtTh], Proposition 1.1; [EtTh], Lemma 5.9]; the crucial role played by the *commutator "*$[-,-]$*"* in the theory of *cyclotomic rigidity* [i.e., (a)] reviewed in (iv) below. In particular, the extraction of various N-th roots in a Kummer tower necessarily leads to *mutually non-isomorphic line bundles*, i.e., mutually non-isomorphic objects in the Kummer tower. From the point of view of *reconstruction algorithms*, such non-isomorphic objects may be **naturally** — i.e., **algorithmically** — related to another only via **indeterminate isomorphisms** [cf. (d)!]. This point of view is precisely the starting point of the discussion of — for instance, *"constant multiple indeterminacy"* in — [EtTh], Remarks 5.12.2, 5.12.3.

(iii) Next, we recall that the significance of *constant multiple rigidity* [i.e., (c)] in the context of the present series of papers lies in the construction of the **canonical splittings of theta monoids** via **restriction to the zero section** discussed, for instance, in [IUTchII], Corollary 1.12, (ii); [IUTchII], Proposition 3.3, (i); [IUTchII], Remark 1.12.2, (iv) [cf. also Remark 1.2.3, (i), of the present paper].

(iv) Next, we review the significance of *cyclotomic rigidity* [i.e., (a)] in the context of the present series of papers. First, we recall that this cyclotomic rigidity is essentially a consequence of the *nondegenerate* nature of the *commutator "*$[-,-]$*"* of the theta groups involved [cf. the discussion of [EtTh], Introduction; [EtTh], Remark 2.19.2]. Put another way, since this commutator is quadratic in nature, one may think of this nondegenerate nature of the commutator as a statement to the effect that *"the degree of the commutator is precisely* 2*"*. At a more concrete level, the cyclotomic rigidity arising from a mono-theta environment consists of a certain specific isomorphism between the *interior* and *exterior cyclotomes* [cf. the discussion of [IUTchII], Definition 1.1, (ii); [IUTchII], Remark 1.1.1]. Put another way, one may think of this cyclotomic rigidity isomorphism as a sort of rigidification of a certain *"projective line of cyclotomes"*, i.e., the projectivization of the direct sum of the interior and exterior cyclotomes [cf. the computations that underlie [EtTh], Proposition 2.12]. In particular, this rigidification is fundamentally *nonlinear* in nature. Indeed, if one attempts to compose it with an N-th power morphism, then one is obliged to sacrifice constant multiple rigidity [i.e., (c)] — cf. the discussion of [EtTh], Remark 5.12.3. That is to say, the *distinguished nature* of the **"first power"** of the cyclotomic rigidity isomorphism is an important theme in the theory of [EtTh] [cf. the discussion of [EtTh], Remark 5.12.5; [IUTchII], Remark 3.6.4, (iii), (iv)]. The **multiradiality** of mono-theta-theoretic cyclotomic rigidity [cf. [IUTchII], Corollary 1.10] — which lies in stark contrast with the indeterminacies that arise when one attempts to give a multiradial formulation [cf. [IUTchII], Corollary 1.11; the discussion of [IUTchII], Remark 1.11.3] of the more classical *"MLF-Galois pair cyclotomic rigidity"* arising from local class field theory — will play a *central role* in the theory of the present §2 [cf. Theorem 2.2, Corollary 2.3 below].

(v) Finally, we review the significance of *discrete rigidity* [i.e., (b)] in the context of the present series of papers. First, we recall that, at a technical level, whereas cyclotomic rigidity may be regarded [cf. the discussion of (iv)] as a consequence of the fact that "the degree of the commutator is precisely 2", discrete rigidity may be regarded as a consequence of the fact that *"the degree of the commutator is* ≤ 2*"* [cf. the statements and proofs of [EtTh], Proposition 2.14, (ii), (iii)]. At a more concrete level, discrete rigidity assures one that one may restrict one's attentions to **$\mathbb{Z}$-multiples/powers** — as opposed to **$\widehat{\mathbb{Z}}$-multiples/powers** — of divisors, line bundles, and rational functions [such as, for instance, the *q-parameter*!] on the tempered coverings of a Tate curve that occur in the theory of [EtTh] [cf. [EtTh], Remark 2.19.4]. This prompts the following question:

> Can one develop a theory of $\widehat{\mathbb{Z}}$-*divisors/line bundles/rational functions* in, for instance, a parallel fashion to the way in which one considers *perfections* and *realifications* of Frobenioids in the theory of [FrdI]?

As far as the author can see at the time of writing, the answer to this question is *"no"*. Indeed, unlike the case with $\mathbb{Q}$ or $\mathbb{R}$, there is no notion of **positivity** [or negativity] in $\widehat{\mathbb{Z}}$. For instance, $-1 \in \widehat{\mathbb{Z}}$ may be obtained as a limit of positive integers. In particular, if one had a theory of $\widehat{\mathbb{Z}}$-divisors/line bundles/rational functions, then such a theory would necessarily require one to "confuse" positive [i.e., effective] and negative divisors, hence to work *birationally.* But to work birationally means, in particular, that one must sacrifice the conventional structure of isomorphisms [e.g., automorphisms] between line bundles — which plays an indispensable role, for instance, in the constructions underlying the Frobenioid-theoretic version of the mono-theta environment [cf. [EtTh], Proposition 1.1; [EtTh], Lemma 5.9; the crucial role played by the *commutator* *"*$[-,-]$*"* in the theory of *cyclotomic rigidity* [i.e., (a)] reviewed in (iv) above].

Remark 2.1.2.

(i) In the context of the discussion of Remark 2.1.1, (v), it is of interest to recall [cf. [IUTchII], Remark 4.5.3, (iii); [IUTchII], Remark 4.11.2, (iii)] that the essential role played, in the context of the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, by the *"global bookkeeping operations"* involving the **labels** of the evaluation points gives rise, in light of the **profinite** nature of the global étale fundamental groups involved, to a situation in which one must apply the "complements on tempered coverings" developed in [IUTchI], §2. That is to say, in the notation of the discussion given in [IUTchII], Remark 2.1.1, (i), of the various tempered coverings that occur at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, these "complements on tempered coverings" are applied precisely so as to allow one to restrict one's attention to the [*discrete*!] $\underline{\mathbb{Z}}$**-conjugates** — i.e., as opposed to [*profinite*!] $\widehat{\underline{\mathbb{Z}}}$-conjugates [where we write $\widehat{\underline{\mathbb{Z}}}$ for the profinite completion of $\underline{\mathbb{Z}}$] — of the theta functions involved. In particular, although such "evaluation-related issues", which will become relevant in the context of the theory of §3 below, do not play a role in the theory of the present §2, the role played by the theory of [IUTchI], §2, in the theory of the present series of papers may also be thought of as a sort of "discrete rigidity" — which we shall refer to as **"evaluation discrete rigidity"** — i.e., a sort of rigidity that is concerned with similar issues to the issues discussed in the case of *"mono-theta-theoretic discrete rigidity"* in Remark 2.1.1, (v), above.

(ii) Next, let us suppose that we are in the situation discussed in [IUTchII], Proposition 2.1. Fix $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Write $\Pi \stackrel{\mathrm{def}}{=} \Pi_{\underline{v}}$; $\widehat{\Pi}$ for the profinite completion of Π. Thus, we have *natural surjections* $\Pi \twoheadrightarrow l \cdot \underline{\mathbb{Z}}$ $(\subseteq \underline{\mathbb{Z}})$, $\widehat{\Pi} \twoheadrightarrow l \cdot \widehat{\underline{\mathbb{Z}}}$ $(\subseteq \widehat{\underline{\mathbb{Z}}})$. Write $\Pi^{\dagger} \stackrel{\mathrm{def}}{=} \widehat{\Pi} \times_{\widehat{\underline{\mathbb{Z}}}} \underline{\mathbb{Z}} \subseteq \widehat{\Pi}$. Next, we observe that from the point of view of the *evaluation points*, the evaluation discrete rigidity discussed in (i) corresponds to the issue of whether, relative to some arbitrarily chosen basepoint, the **"coordinates"** [i.e., element of the "torsor over $\underline{\mathbb{Z}}$" discussed in [IUTchII], Remark 2.1.1, (i)] of the evaluation point lie $\in \underline{\mathbb{Z}}$ or $\in \widehat{\underline{\mathbb{Z}}}$. Thus, if one is only concerned with the issue of arranging for these coordinates to lie $\in \underline{\mathbb{Z}}$, then one is led to pose the following question:

> Is it possible to simply use the *"partially tempered fundamental group"* $\Pi^{\dagger}$ instead of the "full" tempered fundamental group Π in the theory of the present series of papers?

The answer to this question is *"no"*. One way to see this is to consider the [easily verified] natural isomorphism

$$N_{\widehat{\Pi}}(\Pi^{\dagger})/\Pi^{\dagger} \ \stackrel{\sim}{\to} \ \widehat{\underline{\mathbb{Z}}}/\underline{\mathbb{Z}}$$

involving the *normalizer* $N_{\widehat{\Pi}}(\Pi^{\dagger})$ of $\Pi^{\dagger}$ in $\widehat{\Pi}$. One consequence of this isomorphism is that — unlike the tempered fundamental group Π [cf., e.g., [SemiAnbd], Theorems 6.6, 6.8] — the topological group $\Pi^{\dagger}$ *fails to satisfy various fundamental* **absolute anabelian** *properties* which play a *crucial role* in the theory of [EtTh], as well as in the present series of papers [cf., e.g., the theory of [IUTchII], §2]. At a more concrete level, unlike the case with the tempered fundamental group Π, the *profinite conjugacy indeterminacies* that act on $\Pi^{\dagger}$ give rise to $\widehat{\underline{\mathbb{Z}}}$*-translation* *indeterminacies* acting on the coordinates of the evaluation points involved. That is to say, in the case of Π, such $\widehat{\underline{\mathbb{Z}}}$-translation indeterminacies are avoided precisely by applying the "complements on tempered coverings" developed in [IUTchI], §2 — i.e., in a word, as a consequence of the *"highly anabelian nature"* of the [full!] tempered fundamental group Π.

Theorem 2.2. (Kummer-compatible Multiradiality of Theta Monoids) *Fix a collection of* **initial Θ-data**

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$$

as in [IUTchI], Definition 3.1. Let ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *be a* $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater** *[relative to the given initial* Θ*-data — cf. [IUTchI], Definition 6.13, (i)]. For* $\Box \in \{\Vdash, \Vdash\blacktriangleright\times\boldsymbol{\mu}, \vdash\times\boldsymbol{\mu}\}$*, write* $\mathrm{Aut}_{\mathcal{F}^{\Box}}(-)$ *for the group of automorphisms of the* $\mathcal{F}^{\Box}$*-prime-strip in parentheses [cf. [IUTchI], Definition 5.2, (iv); [IUTchII], Definition 4.9, (vi), (vii), (viii)].*

(i) **(Automorphisms of Prime-strips)** *The natural functors determined by assigning to an* $\mathcal{F}^{\Vdash}$*-prime-strip the associated* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*- and* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips [cf. [IUTchII], Definition 4.9, (vi), (vii), (viii)] and then composing with the natural isomorphisms of Proposition 2.1, (vi), determine natural homomorphisms*

$$\mathrm{Aut}_{\mathcal{F}^{\Vdash}}(\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})) \ \rightarrow\ \mathrm{Aut}_{\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}}(\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})) \ \twoheadrightarrow\ \mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}(\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}))$$

$$\mathrm{Aut}_{\mathcal{F}^{\Vdash}}({}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}) \ \rightarrow\ \mathrm{Aut}_{\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}}({}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}}) \ \twoheadrightarrow\ \mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}({}^{\dagger}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle})$$

— where the second arrows in each line are surjections — that are **compatible** *with the* **Kummer isomorphisms** *of Proposition 2.1, (ii), and Theorem 1.5, (iii) [cf. the final portions of Proposition 2.1, (iv), (v), (vi)].*

(ii) **(Kummer Aspects of Multiradiality at Bad Primes)** *Let* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*. Write*

$${}_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}} \ \subseteq\ {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}; \quad {}_{\infty}\Psi^{\perp}_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} \ \subseteq\ {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$$

for the submonoids corresponding to the respective **splittings** *[cf. [IUTchII], Corollaries 3.5, (iii); 3.6, (iii)], i.e., the submonoids generated by "*${}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*})$*" [cf. the notation of [IUTchII], Proposition 3.1, (i)] and the respective* **torsion subgroups***. Now consider the commutative diagram*

$$\begin{array}{ccccc}
{}_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}} & \supseteq & {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\boldsymbol{\mu}}_{\underline{v}} & \subseteq & {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\times}_{\underline{v}} \\
\downarrow & & \downarrow & & \downarrow \\
{}_{\infty}\Psi^{\perp}_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}} & \supseteq & {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\boldsymbol{\mu}}_{\underline{v}} & \subseteq & {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\times}_{\underline{v}}
\end{array}$$

$$\begin{array}{ccccc}
\twoheadrightarrow & {}_{\infty}\Psi_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})^{\times\boldsymbol{\mu}}_{\underline{v}} & \xrightarrow{\sim} & \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})^{\times\boldsymbol{\mu}}_{\underline{v}} \\
 & \downarrow & & \downarrow \\
\twoheadrightarrow & {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\times\boldsymbol{\mu}}_{\underline{v}} & \xrightarrow{\sim} & \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^{\dagger}\mathfrak{F}^{\vdash}_{\triangle})^{\times\boldsymbol{\mu}}_{\underline{v}}
\end{array}$$

— *where the inclusions "$\supseteq$", "$\subseteq$" are the natural inclusions; the surjections "$\twoheadrightarrow$" are the natural surjections; the superscript "$\boldsymbol{\mu}$" denotes the torsion subgroup; the superscript "$\times$" denotes the group of units; the superscript "$\times\boldsymbol{\mu}$" denotes the quotient "$(-)^\times/(-)^{\boldsymbol{\mu}}$"; the first four vertical arrows are the isomorphisms determined by the inverse of the second* **Kummer isomorphism** *of the third display of Proposition 2.1, (ii); ${}^\dagger\mathfrak{D}^\vdash_\triangle$ is as discussed in Theorem 1.5, (iii); ${}^\dagger\mathfrak{F}^\vdash_\triangle$ is as discussed in [IUTchII], Corollary 4.10, (i); the final vertical arrow is the inverse of the* **"Kummer poly-isomorphism"** *determined by the second displayed isomorphism of [IUTchII], Corollary 4.6, (ii); the final upper horizontal arrow is the* **poly-isomorphism** *determined by composing the isomorphism determined by the inverse of the second displayed natural isomorphism of Proposition 2.1, (vi), with the poly-automorphism of $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\dagger\mathfrak{D}^\vdash_\triangle)^{\times\boldsymbol{\mu}}_{\underline{v}}$ induced by the* **full poly-automorphism** *of the $\mathcal{D}^\vdash$-prime-strip ${}^\dagger\mathfrak{D}^\vdash_\triangle$; the final lower horizontal arrow is the poly-automorphism determined by the condition that the final square be commutative. This commutative diagram is compatible with the various group actions involved relative to the following diagram*

$$\begin{array}{ccccc} \Pi_{\underline{\underline{X}}}(\mathbb{M}^\Theta_*({}^\dagger\mathcal{D}_{>,\underline{v}})) & \twoheadrightarrow & G_{\underline{v}}(\mathbb{M}^\Theta_*({}^\dagger\mathcal{D}_{>,\underline{v}})) & = & G_{\underline{v}}(\mathbb{M}^\Theta_*({}^\dagger\mathcal{D}_{>,\underline{v}})) \\ & = & G_{\underline{v}}(\mathbb{M}^\Theta_*({}^\dagger\mathcal{D}_{>,\underline{v}})) & \stackrel{\sim}{\to} & G_{\underline{v}}(\mathbb{M}^\Theta_*({}^\dagger\mathcal{D}_{>,\underline{v}})) \end{array}$$

[cf. the notation of [IUTchII], Proposition 3.1; [IUTchII], Remark 4.2.1, (iv); [IUTchII], Corollary 4.5, (iv)] — where "$\twoheadrightarrow$" denotes the natural surjection; "$\stackrel{\sim}{\to}$" denotes the full poly-automorphism of $G_{\underline{v}}(\mathbb{M}^\Theta_*({}^\dagger\mathcal{D}_{>,\underline{v}}))$. Finally, each of the various composite maps

$$_\infty\Psi_{\mathrm{env}}({}^\dagger\mathfrak{D}_>)^{\boldsymbol{\mu}}_{\underline{v}} \;\to\; \Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\dagger\mathfrak{F}^\vdash_\triangle)^{\times\boldsymbol{\mu}}_{\underline{v}}$$

is equal to the **zero map** *[cf. ($b_{\underline{v}}$) below; the final portion of Proposition 2.1, (iii)]. In particular, the* **identity** *automorphism on the following objects is* **compatible**, *relative to the various natural morphisms involved [cf. the above commutative diagram], with the collection of automorphisms of $\Psi^{\mathrm{ss}}_{\mathrm{cns}}({}^\dagger\mathfrak{F}^\vdash_\triangle)^{\times\boldsymbol{\mu}}_{\underline{v}}$ induced by* **arbitrary automorphisms** $\in \mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}({}^\dagger\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_\triangle)$ *[cf. [IUTchII], Corollary 1.12, (iii); [IUTchII], Proposition 3.4, (i)]:*

($a_{\underline{v}}$) $_\infty\Psi^\perp_{\mathrm{env}}({}^\dagger\mathfrak{D}_>)_{\underline{v}} \supseteq {}_\infty\Psi_{\mathrm{env}}({}^\dagger\mathfrak{D}_>)^{\boldsymbol{\mu}}_{\underline{v}}$;

($b_{\underline{v}}$) $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta_*({}^\dagger\mathcal{D}_{>,\underline{v}}))\otimes\mathbb{Q}/\mathbb{Z}$ *[cf. the discussion of Proposition 2.1, (iii)], relative to the natural isomorphism* $\Pi_{\boldsymbol{\mu}}(\mathbb{M}^\Theta_*({}^\dagger\mathcal{D}_{>,\underline{v}}))\otimes\mathbb{Q}/\mathbb{Z} \stackrel{\sim}{\to} {}_\infty\Psi_{\mathrm{env}}({}^\dagger\mathfrak{D}_>)^{\boldsymbol{\mu}}_{\underline{v}}$ *of [IUTchII], Remark 1.5.2 [cf. ($a_{\underline{v}}$)];*

($c_{\underline{v}}$) the projective system of **mono-theta environments** $\mathbb{M}^\Theta_*({}^\dagger\mathcal{D}_{>,\underline{v}})$ *[cf. ($b_{\underline{v}}$)];*

($d_{\underline{v}}$) the splittings $_\infty\Psi^\perp_{\mathrm{env}}({}^\dagger\mathfrak{D}_>)_{\underline{v}} \twoheadrightarrow {}_\infty\Psi_{\mathrm{env}}({}^\dagger\mathfrak{D}_>)^{\boldsymbol{\mu}}_{\underline{v}}$ *[cf. ($a_{\underline{v}}$)] by means of restriction to* **zero-labeled evaluation points** *[cf. [IUTchII], Proposition 3.1, (i)].*

Proof. The various assertions of Theorem 2.2 follow immediately from the definitions and the references quoted in the statements of these assertions. ○

Remark 2.2.1. In light of the *central importance* of Theorem 2.2, (ii), in the theory of the present §2, we pause to examine the significance of Theorem 2.2, (ii), in more conceptual terms.

(i) In the situation of Theorem 2.2, (ii), let us write [for simplicity] $\Pi_{\underline{v}} \stackrel{\text{def}}{=} \Pi_{\underline{\underline{X}}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))$, $G_{\underline{v}} \stackrel{\text{def}}{=} G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))$, $\Pi_{\boldsymbol{\mu}} \stackrel{\text{def}}{=} \Pi_{\boldsymbol{\mu}}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))$ [cf. ($\mathrm{b}_{\underline{v}}$)]. Also, for simplicity, we write $(l \cdot \Delta_{\Theta}) \stackrel{\text{def}}{=} (l \cdot \Delta_{\Theta})(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))$ [cf. [IUTchII], Proposition 1.5, (iii)]. Here, we recall that in fact, $(l \cdot \Delta_{\Theta})$ may be thought of as an object *constructed from* $\Pi_{\underline{v}}$ [cf. [IUTchII], Proposition 1.4]. Then the projective system of mono-theta environments $\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}})$ [cf. ($\mathrm{c}_{\underline{v}}$)] may be thought of as a sort of *"amalgamation of* $\Pi_{\underline{v}}$ *and* $\Pi_{\boldsymbol{\mu}}$*"*, where the amalgamation is such that it allows the *reconstruction of the* **mono-theta-theoretic cyclotomic rigidity isomorphism**

$$(l \cdot \Delta_{\Theta}) \ \stackrel{\sim}{\to} \ \Pi_{\boldsymbol{\mu}}$$

[cf. [IUTchII], Proposition 1.5, (iii)] — i.e., not just the $\widehat{\mathbb{Z}}^{\times}$*-orbit* of this isomorphism!

(ii) Now, in the notation of (i), the *Kummer classes* $\in {}_{\infty}\Psi^{\perp}_{\text{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}$ [cf. ($\mathrm{a}_{\underline{v}}$)] constituted by the various *étale theta functions* may be thought of, for a suitable characteristic open subgroup $H \subseteq \Pi_{\underline{v}}$, as *twisted homomorphisms*

$$(\Pi_{\underline{v}} \supseteq) \ H \ \to \ \Pi_{\boldsymbol{\mu}}$$

whose restriction to $(l \cdot \Delta_{\Theta})$ coincides with the cyclotomic rigidity isomorphism $(l \cdot \Delta_{\Theta}) \ \stackrel{\sim}{\to} \ \Pi_{\boldsymbol{\mu}}$ discussed in (i). Then the essential content of Theorem 2.2, (ii), lies in the observation that

> since the **Kummer-theoretic link** between étale-like data and Frobenius-like data at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ is established by means of projective systems of **mono-theta environments** [cf. the discussion of Proposition 2.1, (iii)] — i.e., which do *not* involve the various monoids "$(-)^{\times\boldsymbol{\mu}}$"! — the **mono-theta-theoretic cyclotomic rigidity isomorphism** [i.e., *not just the* $\widehat{\mathbb{Z}}^{\times}$*-orbit* of this isomorphism!] is **immune** to the various automorphisms of the monoids "$(-)^{\times\boldsymbol{\mu}}$" which, from the point of view of the **multiradial formulation** to be discussed in Corollary 2.3 below, arise from isomorphisms of *coric data*.

Put another way, this "immunity" may be thought of as a sort of **decoupling** of the *"geometric"* [i.e., in the sense of the geometric fundamental group $\Delta_{\underline{v}} \subseteq \Pi_{\underline{v}}$] and *"base-field-theoretic"* [i.e., associated to the local absolute Galois group $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$] data which allows one to treat the exterior cyclotome $\Pi_{\boldsymbol{\mu}}$ — which, *a priori*, "looks base-field-theoretic" — as being part of the "geometric" data. From the point of view of the multiradial formulation to be discussed in Corollary 2.3 below [cf. also the discussion of [IUTchII], Remark 1.12.2, (vi)], this decoupling may be thought of as a sort of **splitting** into **purely radial** and **purely coric** components — i.e., with respect to which $\Pi_{\boldsymbol{\mu}}$ is *"purely radial"*, while the various monoids "$(-)^{\times\boldsymbol{\mu}}$" are *"purely coric"*.

(iii) Note that the immunity to automorphisms of the monoids "$(-)^{\times\boldsymbol{\mu}}$" discussed in (ii) lies in *stark contrast* to the $\widehat{\mathbb{Z}}^{\times}$**-indeterminacies** that arise in the case of the cyclotomic rigidity isomorphisms constructed from **MLF-Galois pairs** in a fashion that makes *essential use of the monoids "$(-)^{\times\boldsymbol{\mu}}$"*, as discussed in [IUTchII], Corollary 1.11; [IUTchII], Remark 1.11.3. In the following discussion, let us write "$\mathcal{O}^{\times\boldsymbol{\mu}}$" for the various monoids "$(-)^{\times\boldsymbol{\mu}}$" that occur in the situation of Theorem 2.2; also, we shall use similar notation "$\mathcal{O}^{\boldsymbol{\mu}}$", "$\mathcal{O}^{\times}$", "$\mathcal{O}^{\rhd}$", "$\mathcal{O}^{\mathrm{gp}}$", "$\mathcal{O}^{\widehat{\mathrm{gp}}}$" [cf. the notational conventions of [IUTchII], Example 1.8, (ii), (iii), (iv), (vii)]. Thus, we have a diagram

$$\begin{array}{ccccccccc}
\mathcal{O}^{\boldsymbol{\mu}} & \subseteq & \mathcal{O}^{\times} & \subseteq & \mathcal{O}^{\rhd} & \subseteq & \mathcal{O}^{\mathrm{gp}} & \subseteq & \mathcal{O}^{\widehat{\mathrm{gp}}} \\
& \searrow & \downarrow & & & & & & \\
& & \mathcal{O}^{\times\boldsymbol{\mu}} & & & & & &
\end{array}$$

of natural morphisms between monoids equipped with $\Pi_{\underline{v}}$-actions. Relative to this notation, the essential *input data* for the cyclotomic rigidity isomorphism constructed from an MLF-Galois pair is given by "$\mathcal{O}^{\rhd}$" [cf. [IUTchII], Corollary 1.11, (a)]. On the other hand — unlike the case with $\mathcal{O}^{\boldsymbol{\mu}}$ — a $\widehat{\mathbb{Z}}^{\times}$-indeterminacy acting on $\mathcal{O}^{\times\boldsymbol{\mu}}$ does not lie under an *identity action* on $\mathcal{O}^{\times}$! That is to say, a $\widehat{\mathbb{Z}}^{\times}$-indeterminacy acting on $\mathcal{O}^{\times\boldsymbol{\mu}}$ can only be *lifted naturally* to $\widehat{\mathbb{Z}}^{\times}$-indeterminacies on $\mathcal{O}^{\times}$, $\mathcal{O}^{\widehat{\mathrm{gp}}}$ [cf. Fig. 2.1 below; [IUTchII], Corollary 1.11, (a), in the case where one takes "Γ" to be $\widehat{\mathbb{Z}}^{\times}$; [IUTchII], Remark 1.11.3, (ii)]. In the presence of such $\widehat{\mathbb{Z}}^{\times}$-indeterminacies, one can only recover the $\widehat{\mathbb{Z}}^{\times}$*-orbit* of the MLF-Galois-pair-theoretic cyclotomic rigidity isomorphism.

$$\begin{array}{ccccc}
\widehat{\mathbb{Z}}^{\times} \curvearrowright & & \widehat{\mathbb{Z}}^{\times} \curvearrowright & & \widehat{\mathbb{Z}}^{\times} \curvearrowright \\
\end{array}$$
$$\boxed{\begin{array}{ccccccccc}
\mathcal{O}^{\times\boldsymbol{\mu}} & \twoheadleftarrow & \mathcal{O}^{\times} & \subseteq & \mathcal{O}^{\rhd} & \subseteq & \mathcal{O}^{\mathrm{gp}} & \subseteq & \mathcal{O}^{\widehat{\mathrm{gp}}} \\
& & (\supseteq \mathcal{O}^{\boldsymbol{\mu}}) & & & & & &
\end{array}}$$

Fig. 2.1: *Induced* $\widehat{\mathbb{Z}}^{\times}$*-indeterminacies* in the case of MLF-Galois pair cyclotomic rigidity

$$\begin{array}{ccc}
\mathrm{id} \curvearrowright & & \widehat{\mathbb{Z}}^{\times} \curvearrowright \\
\boxed{\Pi_{\boldsymbol{\mu}} \stackrel{\sim}{\to} \mathcal{O}^{\boldsymbol{\mu}}} & \to & \boxed{\mathcal{O}^{\times\boldsymbol{\mu}}}
\end{array}$$

Fig. 2.2: *Insulation from* $\widehat{\mathbb{Z}}^{\times}$*-indeterminacies* in the case of mono-theta-theoretic cyclotomic rigidity

(iv) Thus, in summary, [cf. Fig. 2.2 above]

> **mono-theta-theoretic cyclotomic rigidity** plays an essential role in the theory of the present §2 — and, indeed, in the theory of the present series of papers! — in that it serves to **insulate** the **étale theta function** from the $\widehat{\mathbb{Z}}^{\times}$**-indeterminacies** which act on the **coric log-shells** [i.e., the various monoids "$(-)^{\times \boldsymbol{\mu}}$"].

The techniques that underlie the resulting *multiradiality of theta monoids* [cf. Corollary 2.3 below], cannot, however, be applied immediately to the case of *Gaussian monoids*. That is to say, the corresponding multiradiality of Gaussian monoids, to be discussed in §3 below, requires one to apply the theory of *log-shells* developed in §1 [cf. [IUTchII], Remark 2.9.1, (iii); [IUTchII], Remark 3.4.1, (ii); [IUTchII], Remark 3.7.1]. On the other hand, as we shall see in §3 below, the multiradiality of Gaussian monoids **depends** *in an essential way* on the multiradiality of theta monoids discussed in the present §2 as a sort of *"essential first step"* constituted by the *decoupling* discussed in (ii) above. Indeed, if one tries to consider the **Kummer theory** of the **theta values** [i.e., the "$\underline{\underline{q}}_{v}^{j^2}$" — cf. [IUTchII], Remark 2.5.1, (i)] just as elements of the *base field* — i.e., *without availing oneself of the theory of the étale theta function* — then it is difficult to see how to *rigidify the cyclotomes* involved by any means other than the theory of *MLF-Galois pairs* discussed in (iii) above. But, as discussed in (iii) above, this approach to cyclotomic rigidity gives rise to $\widehat{\mathbb{Z}}^{\times}$*-indeterminacies* — i.e., to *confusion* between the theta values "$\underline{\underline{q}}_{v}^{j^2}$" and their $\widehat{\mathbb{Z}}^{\times}$*-powers*, which is unacceptable from the point of view of the theory of the present series of papers! For another approach to understanding the indispensability of the multiradiality of theta monoids, we refer to Remark 2.2.2 below.

Remark 2.2.2.

(i) One way to understand the very *special role* played by the **theta values** [i.e., the values of the theta function] in the theory of the present series of papers is to consider the following naive question:

> Can one develop a similar theory to the theory of the present series of papers in which one replaces the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-link
>
> $$\underline{\underline{q}} \quad \mapsto \quad \underline{\underline{q}}^{\begin{pmatrix} 1^2 \\ \vdots \\ (l^*)^2 \end{pmatrix}}$$
>
> [cf. [IUTchII], Remark 4.11.1] by a correspondence of the form
>
> $$\underline{\underline{q}} \ \mapsto \ \underline{\underline{q}}^{\lambda}$$
>
> — where λ is some *arbitrary positive integer*?

The answer to this question is *"no"*. Indeed, such a correspondence does not come equipped with the extensive **multiradiality** machinery — such as **mono-theta-theoretic cyclotomic rigidity** and the **splittings** determined by **zero-labeled**

evaluation points — that has been developed for the étale theta function [cf. the discussion of Step (vi) of the proof of Corollary 3.12 of §3 below]. For instance, the lack of mono-theta-theoretic cyclotomic rigidity means that one does not have an apparatus for **insulating** the **Kummer classes** of such a correspondence from the $\widehat{\mathbb{Z}}^\times$-indeterminacies that act on the various monoids "$(-)^{\times\boldsymbol{\mu}}$" [cf. the discussion of Remark 2.2.1, (iv)]. The splittings determined by zero-labeled evaluation points also play an essential role in **decoupling** these monoids "$(-)^{\times\boldsymbol{\mu}}$" — i.e., the **coric log-shells** — from the **"purely radial"** [or, put another way, "value group"] portion of such a correspondence "$\underline{\underline{q}} \mapsto \underline{\underline{q}}^\lambda$" [cf. the discussion of (iii) below; Remark 2.2.1, (ii); [IUTchII], Remark 1.12.2, (vi)]. Note, moreover, that if one tries to realize such a multiradial splitting via *evaluation* — i.e., in accordance with the principle of **"Galois evaluation"** [cf. the discussion of [IUTchII], Remark 1.12.4] — for a correspondence "$\underline{\underline{q}} \mapsto \underline{\underline{q}}^\lambda$" by, for instance, taking λ to be *one of the "j^2"* [where j is a positive integer] that appears as a value of the étale theta function, then one must contend with issues of **symmetry** between the zero-labeled evaluation point and the evaluation point corresponding to λ — i.e., symmetry issues that are resolved in the theory of the present series of papers by means of the theory surrounding the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** [cf. the discussion of [IUTchII], Remarks 2.6.2, 3.5.2]. As discussed in [IUTchII], Remark 2.6.3, this sort of situation leads to numerous *conditions on the collection of evaluation points* under consideration. In particular, ultimately, it is difficult to see how to construct a theory as in the present series of papers for any collection of evaluation points other than the collection that is in fact adopted in the definition of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link.

(ii) As discussed in Remark 2.2.1, (iv), we shall be concerned, in §3 below, with developing multiradial formulations for Gaussian monoids. These multiradial formulations will be subject to certain *indeterminacies*, which — although *sufficiently mild* to allow the execution of the *volume computations* that will be the subject of [IUTchIV] — are, nevertheless, *substantially more severe* than the indeterminacies that occur in the multiradial formulation given for theta monoids in the present §2 [cf. Corollary 2.3 below]. Indeed, the indeterminacies in the multiradial formulation given for theta monoids in the present §2 — which essentially consist of *multiplication by roots of unity* [cf. [IUTchII], Proposition 3.1, (i)] — are *essentially negligible* and may be regarded as a consequence of the highly nontrivial **Kummer theory** surrounding **mono-theta environments** [cf. Proposition 2.1, (iii); Theorem 2.2, (ii)], which, as discussed in Remark 2.2.1, (iv), cannot be mimicked for "theta values regarded just as elements of the base field". That is to say, the quite **exact** nature of the multiradial formulation for theta monoids — i.e., which contrasts sharply with the somewhat **approximate** nature of the multiradial formulation for Gaussian monoids to be developed in §3 — constitutes another *important ingredient* of the theory of the present paper that one must sacrifice if one attempts to work with correspondences $\underline{\underline{q}} \mapsto \underline{\underline{q}}^\lambda$ as discussed in (i), i.e., correspondences which do not come equipped with the extensive multiradiality machinery that arises as a consequence of the theory of the *étale theta function* developed in [EtTh].

(iii) One way to understand the significance, in the context of the discussions of (i) and (ii) above, of the **multiradial coric/radial decouplings** furnished by the splittings determined by the zero-labeled evaluation points is as follows. Ultimately, in order to establish, in §3 below, multiradial formulations for Gaussian monoids, it will be of crucial importance to pass from the **Frobenius-like theta monoids** that appear in the *domain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link to **vertically coric étale-like** objects by means of **Kummer theory** [cf. the discussions of Remarks 1.2.4, (i); 1.5.4, (i), (iii)], in the context of the relevant **$\mathfrak{log}$-Kummer correspondences**, as discussed, for instance, in Remark 3.12.2, (iv), (v), below [cf. also [IUTchII], Remark 1.12.2, (iv)]. On the other hand, in order to obtain formulations expressed in terms that are meaningful from the point of view of the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link, it is necessary [cf. the discussion of Remark 3.12.2, (iv), (v), below] to relate this **Kummer theory** of **theta monoids** in the *domain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link to the Kummer theory constituted by the $\times\boldsymbol{\mu}$-Kummer structures that appear in the **horizontally coric** portion of the data that constitutes the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$-link [cf. Theorem 1.5, (ii)]. This is precisely what is achieved by the **Kummer-compatibility** of the multiradial splitting via *evaluation* — i.e., in accordance with the principle of **"Galois evaluation"** [cf. the discussion of [IUTchII], Remark 1.12.4]. This state of affairs [cf., especially, the two displays of [IUTchII], Corollary 1.12, (ii); the final arrow of the diagram "$(\dagger_{\boldsymbol{\mu},\times\boldsymbol{\mu}})$" of [IUTchII], Corollary 1.12, (iii)] is illustrated in Fig. 2.3 below.

$$
\begin{array}{ccc}
\mathrm{id} \curvearrowright & & \mathrm{Aut}(G),\ \mathrm{Ism} \curvearrowright \\
\boxed{\ {}_{\infty}\underline{\underline{\theta}} \ \curvearrowleft \ \Pi \ \hookleftarrow \ \Pi/\Delta\ } & \to & \boxed{\ G \ \curvearrowright \ \mathcal{O}^{\times\boldsymbol{\mu}}\ } \\
\bigcap & & \| \\
\mathrm{id} \curvearrowright & & \mathrm{Aut}(G),\ \mathrm{Ism} \curvearrowright \\
\boxed{\ \mathcal{O}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}} \ \curvearrowleft \ \Pi \ \hookleftarrow \ \Pi/\Delta\ } & \begin{matrix}\to \\ \vdots \\ \to\end{matrix} & \boxed{\ G \ \curvearrowright \ \mathcal{O}^{\times\boldsymbol{\mu}}\ } \\
 & & \\
{}_{\infty}\underline{\underline{\theta}} & \mapsto & 1 \ \in \ \mathcal{O}^{\times\boldsymbol{\mu}}
\end{array}
$$

Fig. 2.3: *Kummer-compatible* splittings via *evaluation* at zero-labeled evaluation points [i.e., "$\Pi \hookleftarrow \Pi/\Delta$"]

Here, the *multiple arrows* [i.e., indicated by means of the "$\to$'s" separated by vertical dots] in the *lower portion* of the diagram correspond to the fact that the "$\mathcal{O}^{\times}$" on the left-hand side of this lower portion is related to the "$\mathcal{O}^{\times\boldsymbol{\mu}}$" on the right-hand side via an Ism-*orbit* of morphisms; the analogous arrow in the *upper portion* of the diagram consists of a *single arrow* [i.e., "$\to$"] and corresponds to the fact that the restriction of the multiple arrows in the lower portion of the diagram to "${}_{\infty}\underline{\underline{\theta}}$" amounts to a single arrow, i.e., precisely as a consequence of the fact that ${}_{\infty}\underline{\underline{\theta}} \ \mapsto \ 1 \ \in \ \mathcal{O}^{\times\boldsymbol{\mu}}$ [cf. the situation illustrated in Fig. 2.2]. On the other hand, the "Π/Δ's" on the left-hand side of both the upper and the lower portions of the diagram are related to the "G's" on the right-hand side via the unique tautological $\mathrm{Aut}(G)$-orbit of isomorphisms. Thus, from the point of view of Fig. 2.3, the crucial **Kummer-compatibility** discussed above may be understood as the statement that

> the **multiradial** structure [cf. the lower portion of Fig. 2.3] on the "theta monoid $\mathcal{O}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}$" furnished by the **splittings** via **Galois evaluation** into **coric/radial** components is **compatible** with the relationship between the respective **Kummer theories** of the "$\mathcal{O}^{\times}$" portion of "$\mathcal{O}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}$" [on the left] and the coric "$\mathcal{O}^{\times \boldsymbol{\mu}}$" [on the right].

This state of affairs lies in *stark contrast* to the situation that arises in the case of a naive correspondence of the form "$\underline{\underline{q}} \mapsto \underline{\underline{q}}^{\lambda}$" as discussed in (i): That is to say, in the case of such a naive correspondence, the corresponding arrows "$\rightarrow$" of the analogue of Fig. 2.3 map

$$\underline{\underline{q}}^{\lambda} \quad \mapsto \quad 1 \ \in \ \mathcal{O}^{\times \boldsymbol{\mu}}$$

and hence are **fundamentally incompatible** with passage to **Kummer classes**, i.e., since the Kummer class of $\underline{\underline{q}}^{\lambda}$ in a suitable cohomology group of Π/Δ is *by no means* mapped, via the poly-isomorphism $\Pi/\Delta \stackrel{\sim}{\rightarrow} G$, to the *trivial element* of the relevant cohomology group of G.

We conclude the present §2 with the following **multiradial** interpretation [cf. [IUTchII], Remark 4.1.1, (iii); [IUTchII], Remark 4.3.1] — in the spirit of the *étale-picture of $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters* of [IUTchII], Corollary 4.11, (ii) — of the theory surrounding Theorem 2.2.

Corollary 2.3. (Étale-picture of Multiradial Theta Monoids) *In the notation of Theorem 2.2, let*

$$\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}\}_{n,m \in \mathbb{Z}}$$

be a **collection of distinct $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters** *[relative to the given initial Θ-data] — which we think of as arising from a* **Gaussian log-theta-lattice** *[cf. Definition 1.4]. Write ${}^{n,m}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ for the $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theater associated to ${}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$. Consider the* **radial environment** *[cf. [IUTchII], Example 1.7, (ii)] defined as follows. We define a collection of* **radial data**

$$^{\dagger}\mathfrak{R} \ = \ ({}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}, \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}), {}^{\dagger}\mathfrak{R}^{\mathrm{bad}}, \mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}), \mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \stackrel{\sim}{\rightarrow} \mathfrak{F}^{\vdash \times \boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}))$$

to consist of

($a_{\mathfrak{R}}$) a $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$;

($b_{\mathfrak{R}}$) the $\mathcal{F}^{\Vdash}$-prime-strip $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$ associated to ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ [cf. Proposition 2.1, (ii)];

($c_{\mathfrak{R}}$) *the data* ($a_{\underline{v}}$), ($b_{\underline{v}}$), ($c_{\underline{v}}$), ($d_{\underline{v}}$) *of Theorem 2.2, (ii), for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, *which we denote by* ${}^{\dagger}\mathfrak{R}^{\mathrm{bad}}$;

($d_{\mathfrak{R}}$) *the* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strip* $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})$ *associated to* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *[cf. Theorem 1.5, (iii)]*;

($e_{\mathfrak{R}}$) *the* **full poly-isomorphism** *of* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips* $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})$.

We define a morphism between two collections of radial data ${}^{\dagger}\mathfrak{R} \rightarrow {}^{\ddagger}\mathfrak{R}$ *[where we apply the evident notational conventions with respect to "*$\dagger$*" and "*$\ddagger$*"] to consist of data as follows:*

($a_{\mathrm{Mor}_{\mathfrak{R}}}$) *an isomorphism of* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$;

($b_{\mathrm{Mor}_{\mathfrak{R}}}$) *the isomorphism of* $\mathcal{F}^{\Vdash}$*-prime-strips* $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\ddagger}\mathfrak{D}_{>})$ *induced by the isomorphism of* ($a_{\mathrm{Mor}_{\mathfrak{R}}}$);

($c_{\mathrm{Mor}_{\mathfrak{R}}}$) *the isomorphism between collections of data* ${}^{\dagger}\mathfrak{R}^{\mathrm{bad}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{R}^{\mathrm{bad}}$ *induced by the isomorphism of* ($a_{\mathrm{Mor}_{\mathfrak{R}}}$);

($d_{\mathrm{Mor}_{\mathfrak{R}}}$) *an isomorphism of* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips* $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\ddagger}\mathfrak{D}^{\vdash}_{\triangle})$;

($e_{\mathrm{Mor}_{\mathfrak{R}}}$) *we observe that the isomorphisms of* ($b_{\mathrm{Mor}_{\mathfrak{R}}}$) *and* ($d_{\mathrm{Mor}_{\mathfrak{R}}}$) *are necessarily compatible with the poly-isomorphisms of* ($e_{\mathfrak{R}}$) *for "*$\dagger$*", "*$\ddagger$*".*

We define a collection of **coric data**

$$ {}^{\dagger}\mathfrak{C} \;=\; ({}^{\dagger}\mathfrak{D}^{\vdash}, \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}^{\vdash})) $$

to consist of

($a_{\mathfrak{C}}$) *a* $\mathcal{D}^{\vdash}$*-prime-strip* ${}^{\dagger}\mathfrak{D}^{\vdash}$;

($b_{\mathfrak{C}}$) *the* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strip* $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}^{\vdash})$ *associated to* ${}^{\dagger}\mathfrak{D}^{\vdash}$ *[cf. [IUTchII], Corollary 4.5, (ii); [IUTchII], Definition 4.9, (vi), (vii)].*

We define a morphism between two collections of coric data ${}^{\dagger}\mathfrak{C} \rightarrow {}^{\ddagger}\mathfrak{C}$ *[where we apply the evident notational conventions with respect to "*$\dagger$*" and "*$\ddagger$*"] to consist of data as follows:*

($a_{\mathrm{Mor}_{\mathfrak{C}}}$) *an isomorphism of* $\mathcal{D}^{\vdash}$*-prime-strips* ${}^{\dagger}\mathfrak{D}^{\vdash} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}^{\vdash}$;

($b_{\mathrm{Mor}_{\mathfrak{C}}}$) *an isomorphism of* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips* $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\dagger}\mathfrak{D}^{\vdash}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}({}^{\ddagger}\mathfrak{D}^{\vdash})$ *that induces the isomorphism* ${}^{\dagger}\mathfrak{D}^{\vdash} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}^{\vdash}$ *on associated* $\mathcal{D}^{\vdash}$*-prime-strips of* ($a_{\mathrm{Mor}_{\mathfrak{C}}}$).

The **radial algorithm** *is given by the assignment*

$$ \begin{aligned} {}^{\dagger}\mathfrak{R} \;=\; ({}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}, \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}), {}^{\dagger}\mathfrak{R}^{\mathrm{bad}}, \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}), \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})) \\ \mapsto \;\; {}^{\dagger}\mathfrak{C} \;=\; ({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle}, \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\dagger}\mathfrak{D}^{\vdash}_{\triangle})) \end{aligned} $$

— together with the assignment on morphisms determined by the data of $(d_{\mathrm{Mor}_{\mathfrak{R}}})$*. Then:*

(i) The functor associated to the radial algorithm defined above is **full** *and* **essentially surjective**. *In particular, the radial environment defined above is* **multiradial**.

(ii) Each $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* ${}^{n,m}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*, for* $n, m \in \mathbb{Z}$*, defines, in an evident way, an associated collection of radial data* ${}^{n,m}\mathfrak{R}$*. The poly-isomorphisms induced by the* **vertical** *arrows of the* **Gaussian log-theta-lattice** *under consideration [cf. Theorem 1.5, (i)] induce poly-isomorphisms of radial data* $\ldots \xrightarrow{\sim} {}^{n,m}\mathfrak{R} \xrightarrow{\sim} {}^{n,m+1}\mathfrak{R} \xrightarrow{\sim} \ldots$*. Write*

$${}^{n,\circ}\mathfrak{R}$$

for the collection of radial data obtained by identifying the various ${}^{n,m}\mathfrak{R}$*, for* $m \in \mathbb{Z}$*, via these poly-isomorphisms and* ${}^{n,\circ}\mathfrak{C}$ *for the collection of coric data associated, via the radial algorithm defined above, to the radial data* ${}^{n,\circ}\mathfrak{R}$*. In a similar vein, the* **horizontal** *arrows of the Gaussian log-theta-lattice under consideration induce full poly-isomorphisms* $\ldots \xrightarrow{\sim} {}^{n,m}\mathfrak{D}^{\vdash}_{\triangle} \xrightarrow{\sim} {}^{n+1,m}\mathfrak{D}^{\vdash}_{\triangle} \xrightarrow{\sim} \ldots$ *of* $\mathcal{D}^{\vdash}$*-prime-strips [cf. Theorem 1.5, (ii)]. Write*

$${}^{\circ,\circ}\mathfrak{C}$$

for the collection of coric data obtained by identifying the various ${}^{n,\circ}\mathfrak{C}$*, for* $n \in \mathbb{Z}$*, via these poly-isomorphisms. Thus, by applying the radial algorithm defined above to each* ${}^{n,\circ}\mathfrak{R}$*, for* $n \in \mathbb{Z}$*, we obtain a diagram — i.e., an* **étale-picture of radial data** *— as in Fig. 2.4 below. This diagram satisfies the important property of admitting* **arbitrary permutation symmetries** *among the spokes [i.e., the labels* $n \in \mathbb{Z}$*] and is* **compatible***, in the evident sense, with the étale-picture of* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters of [IUTchII], Corollary 4.11, (ii).*

$$
\begin{array}{ccccc}
 & & \boxed{\begin{array}{c}\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>}) \\ + \ {}^{n,\circ}\mathfrak{R}^{\mathrm{bad}} + \ldots\end{array}} & & \\
\ldots & & | & & \ldots \\
\boxed{\begin{array}{c}\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{n',\circ}\mathfrak{D}_{>}) \\ + \ {}^{n',\circ}\mathfrak{R}^{\mathrm{bad}} + \ldots\end{array}} & — & \boxed{\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{\circ,\circ}\mathfrak{D}^{\vdash}_{\triangle})} & — & \boxed{\begin{array}{c}\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{n'',\circ}\mathfrak{D}_{>}) \\ + \ {}^{n'',\circ}\mathfrak{R}^{\mathrm{bad}} + \ldots\end{array}} \\
\ldots & & | & & \ldots \\
 & & \boxed{\begin{array}{c}\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{n''',\circ}\mathfrak{D}_{>}) \\ + \ {}^{n''',\circ}\mathfrak{R}^{\mathrm{bad}} + \ldots\end{array}} & &
\end{array}
$$

Fig. 2.4: Étale-picture of radial data

(iii) The [poly-]isomorphisms of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips of/induced by* $(e_{\mathfrak{R}})$, $(b_{\mathrm{Mor}_{\mathfrak{R}}})$, $(d_{\mathrm{Mor}_{\mathfrak{R}}})$ *[cf. also* $(e_{\mathrm{Mor}_{\mathfrak{R}}})$*] are* **compatible**, *relative to the* **Kummer isomorphisms** *of Proposition 2.1, (ii) [cf. also Proposition 2.1, (vi)], and Theorem 1.5, (iii), with the poly-isomorphisms — arising from the* **horizontal arrows** *of the Gaussian log-theta-lattice — of Theorem 1.5, (ii).*

(iv) The **algorithmic construction** *of the isomorphisms* $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>}) \overset{\sim}{\to} \mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\ddagger}\mathfrak{D}_{>})$, ${}^{\dagger}\mathfrak{R}^{\mathrm{bad}} \overset{\sim}{\to} {}^{\ddagger}\mathfrak{R}^{\mathrm{bad}}$ *of* $(b_{\mathrm{Mor}_{\mathfrak{R}}})$, $(c_{\mathrm{Mor}_{\mathfrak{R}}})$, *as well as of the* **Kummer isomorphisms** *and* **poly-isomorphisms of projective systems of mono-theta environments** *discussed in Proposition 2.1, (ii), (iii) [cf. also Proposition 2.1, (vi); the second display of Theorem 2.2, (ii)], and Theorem 1.5, (iii), (v), are* **compatible** *[cf. the final portions of Theorems 1.5, (v); 2.2, (ii)] with the* **horizontal arrows** *of the Gaussian log-theta-lattice [cf., e.g., the full poly-isomorphisms of Theorem 1.5, (ii)], in the sense that these constructions are* **stabilized/equivariant/functorial** *with respect to arbitrary automomorphisms of the domain and codomain of these horizontal arrows of the Gaussian log-theta-lattice.*

Proof. The various assertions of Corollary 2.3 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 2.3.1.

(i) In the context of the étale-picture of Fig. 2.4, it is of interest to recall the point of view of the discussion of [IUTchII], 1.12.5, (i), (ii), concerning the analogy between **étale-pictures** in the theory of the present series of papers and the **polar coordinate representation** of the **classical Gaussian integral**.

(ii) The étale-picture discussed in Corollary 2.3, (ii), may be thought of as a sort of **canonical splitting** of the portion of the **Gaussian log-theta-lattice** under consideration that involves **theta monoids** [cf. the discussion of [IUTchI], §I1, preceding Theorem A].

(iii) The portion of the **multiradiality** discussed in Corollary 2.3, (iv), at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ corresponds, in essence, to the multiradiality discussed in [IUTchII], Corollary 1.12, (iii); [IUTchII], Proposition 3.4, (i).

Remark 2.3.2. A similar result to Corollary 2.3 may be formulated concerning the **multiradiality** properties satisfied by the **Kummer theory** of ${}_{\infty}\kappa$**-coric structures** as discussed in [IUTchII], Corollary 4.8. That is to say, the Kummer theory of the **localization poly-morphisms**

$$\left\{ \{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}\}_j \quad \to \quad {}^{\dagger}\mathbb{M}_{\infty\kappa v_j} \subseteq {}^{\dagger}\mathbb{M}_{\infty\kappa\times v_j} \right\}_{\underline{v}\in\underline{\mathbb{V}}}$$

discussed in [IUTchII], Corollary 4.8, (iii), is based on the **cyclotomic rigidity isomorphisms** for ${}_{\infty}\kappa$**-coric structures** discussed in [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi), (viii) [cf. also the discussion of [IUTchII], Corollary 4.8, (i)], which satisfy **"insulation"** properties analogous to the properties discussed in Remark 2.2.1 in the case of *mono-theta-theoretic cyclotomic rigidity.*

Moreover, the reconstruction of ${}_\infty\kappa$**-coric structures** from ${}_\infty\kappa\times$**-structures** via restriction of Kummer classes

$$^{\ddagger}\mathbb{M}_{\infty\kappa v_j} \subseteq {}^{\ddagger}\mathbb{M}_{\infty\kappa\times v_j} \to {}^{\ddagger}\mathbb{M}^{\times}_{\infty\kappa\times v_j} \xrightarrow{\sim} {}^{\ddagger}\mathbb{M}^{\times}_{v_j}$$

as discussed in [IUTchI], Definition 5.2, (vi), (viii) — i.e., a reconstruction in accordance with the principle of **Galois evaluation** [cf. [IUTchII], Remark 1.12.4] — may be regarded as a **decoupling** into

· **radial** [i.e., $\{\pi_1^{\kappa\text{-sol}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}\}_j$; ${}^{\dagger}\mathbb{M}_{\infty\kappa v_j}$; ${}^{\ddagger}\mathbb{M}_{\infty\kappa v_j}$] and
· **coric** [i.e., the quotient of ${}^{\ddagger}\mathbb{M}^{\times}_{\infty\kappa\times v_j} \xrightarrow{\sim} {}^{\ddagger}\mathbb{M}^{\times}_{v_j}$ by its torsion subgroup]

components, i.e., in an entirely analogous fashion to the *mono-theta-theoretic* case discussed in Remark 2.2.2, (iii). The **Galois evaluation** that gives rise to the *theta values* "$\underline{\underline{q}}_{v}^{j^2}$" in the case of theta monoids corresponds to the construction via Galois evaluation of the **monoids** "${}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{mod}}$", i.e., via the operation of **restricting Kummer classes** associated to elements of ${}_\infty\kappa$**-coric structures**, as discussed in [IUTchI], Example 5.1, (v); [IUTchII], Corollary 4.8, (i) [cf. also [IUTchI], Definition 5.2, (vi), (viii)]. We leave the routine details of giving a formulation in the style of Corollary 2.3 to the reader.

Remark 2.3.3. In the context of Remark 2.3.2, it is of interest to compare and contrast the **multiradiality** properties that hold in the **theta** [cf. Remarks 2.2.1, 2.2.2; Corollary 2.3] and **number field** [cf. Remark 2.3.2] cases, as follows.

(i) One important *similarity* between the *theta* and *number field* cases lies in the establishment of **multiradiality** properties, i.e., such as the **radial/coric decoupling** discussed in Remarks 2.2.2, (iii); 2.3.2, by using the **geometric dimension** of the elliptic curve under consideration as a sort of

> **"multiradial geometric container"** for the **radial arithmetic data** of interest, i.e., **theta values** "$\underline{\underline{q}}_{v}^{j^2}$" or copies of the **number field** "F_{mod}".

That is to say, in the theta case, the theory of **theta functions** on **Tate curves** as developed in [EtTh] furnishes such a geometric container for the theta values, while in the number field case, the absolute anabelian interpretation developed in [AbsTopIII] of the theory of **Belyi maps** as **Belyi cuspidalizations** [cf. [IUTchI], Remark 5.1.4] furnishes such a geometric container for copies of F_{mod}. In this context, another important similarity is the passage from such a geometric container to the radial arithmetic data of interest by means of **Galois evaluation** [cf. Remark 2.2.2, (i), (iii); Remark 2.3.2].

(ii) One important theme of the present series of papers is the point of view of *dismantling the* **two underlying combinatorial dimensions** of [the ring of integers of] a number field — cf. the discussion of Remark 3.12.2 below. As discussed in [IUTchI], Remark 6.12.3 [cf. also [IUTchI], Remark 6.12.6], this dismantling may be compared to the dismantling of the **single complex holomorphic dimension** of the **upper half-plane** into **two underlying real dimensions**. If one considers this dismantling from such a classical point of view, then one is tempted to attempt to understand the dismantling into two underlying real dimensions, by, in effect,

base-changing from $\mathbb{R}$ to $\mathbb{C}$, so as to obtain **two-dimensional complex holomorphic objects**, which we regard as being equipped with some sort of **descent data** arising from the base-change from $\mathbb{R}$ to $\mathbb{C}$.

Translating this approach back into the case of number fields, one obtains a situation in which one attempts to understand the dismantling of the two underlying combinatorial dimensions of [the ring of integers of] a number field by working with *two-dimensional scheme-theoretic data* — i.e., such as an **elliptic curve** over [a suitable localization of the ring of integers of] a **number field** — equipped with *"suitable descent data"*. From this point of view, one may think of

> the **"multiradial geometric containers"** discussed in (i) as a sort of *realization* of such **two-dimensional scheme-theoretic data**,

and of

> the accompanying **Galois evaluation** operations, i.e., the **multiradial representations** up to certain **mild indeterminacies** obtained in Theorem 3.11, below [cf. also the discussion of Remark 3.12.2, below], as a sort of *realization* of the corresponding **"suitable descent data"**.

This sort of interpretation is reminiscent of the interpretation of **multiradiality** in terms of **parallel transport** via a **connection** as discussed in [IUTchII], Remark 1.7.1, and the closely related interpretation given in the discussion of [IUTchII], Remark 1.9.2, (iii), of the **tautological approach** to multiradiality in terms of **PD-envelopes** in the style of the p-adic theory of the crystalline site.

(iii) Another fundamental *similarity* between the *theta* and *number field* cases may be seen in the fact that the associated **Galois evaluation** operations — i.e., that give rise to the theta values "$\underline{\underline{q}}_{\underline{v}}^{j^2}$" [cf. [IUTchII], Corollary 3.6] or copies of the number field "F_{mod}" [cf. [IUTchII], Corollary 4.8, (i), (ii)] — are performed in the context of the $\mathfrak{log}$**-link**, which depends, in a quite essential way, on the **arithmetic holomorphic** [i.e., **ring!**] structures of the various local fields involved — cf., for instance, the discussion of the relevant $\mathfrak{log}$**-Kummer correspondences** in Remark 3.12.2, (iv), (v), below. On the other hand, one fundamental *difference* between the *theta* and *number field* cases may be observed in the fact that whereas

- the output data in the theta case — i.e., the **theta values** "$\underline{\underline{q}}_{\underline{v}}^{j^2}$" — **depends**, in an essential way, on the **labels** $j \in \mathbb{F}_l^{\circledast}$,
- the output data in the number field case — i.e., the copies of the **number field** "F_{mod}" — is **independent** of these labels $j \in \mathbb{F}_l^{\circledast}$.

In this context, let us recall that these labels $j \in \mathbb{F}_l^{\circledast}$ correspond, in essence, to collections of *cuspidal inertia groups* [cf. [IUTchI], Definition 4.1, (ii)] of the *local geometric fundamental groups* that appear [i.e., in the notation of the discussion of Remark 2.2.2, (iii), the subgroup "Δ ($\subseteq \Pi$)" of the local arithmetic fundamental group Π]. On the other hand, let us recall that, in the context of these local arithmetic fundamental groups Π, the **arithmetic holomorphic structure** also depends, in an essential way, on the geometric fundamental group portion [i.e.,

"$\Delta \subseteq \Pi$"] of Π [cf., e.g., the discussion of [AbsTopIII], Theorem 1.9, in [IUTchI], Remark 3.1.2, (ii); the discussion of [AbsTopIII], §I3]. In particular, it is a quite *nontrivial fact* that

> the **Galois evaluation** and **Kummer theory** in the **theta** case may be performed [cf. [IUTchII], Corollary 3.6] in a consistent fashion that is **compatible** with **both** the **labels** $j \in \mathbb{F}_l^{\circledast}$ [cf. also the associated **symmetries** discussed in [IUTchII], Corollary 3.6, (i)] and the **arithmetic holomorphic structures** involved

— i.e., both of which depend on "Δ" in an essential way. By contrast,

> the corresponding **Galois evaluation** and **Kummer theory** operations in the **number field** case are performed [cf. [IUTchII], Corollary 4.8, (i), (ii)] in a way that is **compatible** with the **arithmetic holomorphic structures** involved, but yields output data [i.e., copies of the number field "F_{mod}"] that is **free** of any **dependence** on the *labels* $j \in \mathbb{F}_l^{\circledast}$.

Of course, the **global realified Gaussian Frobenioids** constructed in [IUTchII], Corollary 4.6, (v), which also play an important role in the theory of the present series of papers, involve global data that **depends**, in an essential way, on the **labels** $j \in \mathbb{F}_l^{\circledast}$, but this dependence occurs only in the context of **global realified Frobenioids**, i.e., which [cf. the notation "$\Vdash$" as it is used in [IUTchI], Definition 5.2, (iv); [IUTchII], Definition 4.9, (viii), as well as in Definition 2.4, (iii), below] are **mono-analytic** in nature [i.e., do *not* depend on the *arithmetic holomorphic structure* of copies of the number field "F_{mod}"].

(iv) In the context of the observations of (iii), we make the further observation that it is a *highly nontrivial* fact that the construction algorithm for the **mono-theta-theoretic cyclotomic rigidity isomorphism** applied in the theta case admits $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** [cf. the discussion of [IUTchII], Remark 1.1.1, (v); [IUTchII], Corollary 3.6, (i)] in a fashion that is **consistent** with the **dependence** of the **theta values** on the labels $j \in \mathbb{F}_l^{\circledast}$. As discussed in [IUTchII], Remark 1.1.1, (v), this state of affairs differs quite substantially from the state of affairs that arises in the case of the approach to cyclotomic rigidity taken in [IUTchI], Example 5.1, (v), which is based on a rather *"straightforward"* or *"naive"* utilization of the **Kummer classes** of **rational functions**. That is to say, the "highly nontrivial" fact just observed in the theta case would amount, from the point of view of this "naive Kummer approach" to cyclotomic rigidity, to the existence of a **rational function** [or, alternatively, a collection of rational functions *without "labels"*] that is **invariant** [up to, say, multiples by roots of unity] with respect to the $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** that appear, but nevertheless attains values on some $\mathbb{F}_l^{\rtimes\pm}$-orbit of points that have **distinct valuations** at **distinct points** — a situation that is clearly **self-contradictory**!

(v) One way to appreciate the *nontriviality* of the "highly nontrivial" fact observed in (iv) is as follows. One possible approach to realizing the apparently "self-contradictory" state of affairs constituted by a "*symmetric* rational function with *non-symmetric* values" consists of replacing the local arithmetic fundamental group "Π" [cf. the notation of the discussion of (iii)] by some suitable **closed subgroup of infinite index** of Π. That is to say, if one works with such infinite index closed subgroups of Π, then the possibility arises that the Kummer classes of those rational functions that constitute the *obstruction to symmetry* in the case of some given rational function of interest [i.e., at a more concrete level, the rational functions that arise as *quotients* of the given rational function by its $\mathbb{F}_l^{\rtimes\pm}$*-conjugates*] *vanish* upon *restriction* to such infinite index closed subgroups of Π. On the other hand, this approach has the following *"fundamental deficiencies"*, both of which relate to an apparently **fatal lack of compatibility** with the **arithmetic holomorphic structures** involved:

· It is not clear that the **absolute anabelian** results of [AbsTopIII], §1 — i.e., which play a *fundamental role* in the theory of the present series of papers — admit generalizations to the case of such infinite index closed subgroups of Π.

· The vanishing of **Kummer classes** of certain rational functions that occurs when one *restricts* to such infinite index closed subgroups of Π will not, in general, be compatible with the **ring structures** involved [i.e., of the rings/fields of rational functions that appear].

In particular, this approach does not appear to be likely to give rise to a meaningful theory.

(vi) Another possible approach to realizing the apparently "self-contradictory" state of affairs constituted by a "*symmetric* rational function with *non-symmetric* values" consists of working with **distinct rational functions**, i.e., one **symmetric** rational function [or collection of rational functions] for constructing *cyclotomic rigidity isomorphisms* via the Kummer-theoretic approach of [IUTchI], Example 5.1, (v), and one **non-symmetric** rational function to which one applies *Galois evaluation* operations to construct the analogue of "theta values". On the other hand, this approach has the following *"fundamental deficiency"*, which again relates to a sort of **fatal lack of compatibility** with the **arithmetic holomorphic structures** involved: The crucial **absolute anabelian** results of [AbsTopIII], §1 [cf. also the discussion of [IUTchI], Remark 3.1.2, (ii), (iii)], depend, in an essential way, on the use of **numerous cyclotomes** [i.e., copies of "$\widehat{\mathbb{Z}}(1)$"] — which, for simplicity, we shall denote by

$$\boldsymbol{\mu}^{*}_{\mathrm{et}}$$

in the present discussion — that arise from the various **cuspidal inertia groups** at the **cusps** "$*$" of [the various cuspidalizations associated to] the hyperbolic curve under consideration. These cyclotomes "$\boldsymbol{\mu}^{*}_{\mathrm{et}}$" [i.e., for various cusps "$*$"] may be **naturally identified** with one another, i.e., via the *natural isomorphisms* of [AbsTopIII], Proposition 1.4, (ii); write

$$\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$$

for the cyclotome resulting from this natural identification. Moreover, since the various [pseudo-]monoids constructed by applying these anabelian results are constructed as sub[-pseudo-]monoids of first [group] cohomology modules with coefficients in the *cyclotome* $\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$, it follows [cf. the discussion of [IUTchII], Remark 1.5.2] that the *cyclotome*

$$\boldsymbol{\mu}_{\mathrm{Fr}}$$

determined by [i.e., the cyclotome obtained by applying $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, -)$ to the *torsion subgroup* of] such a [pseudo-]monoid may be **tautologically identified** — i.e., whenever the [pseudo-]monoid under consideration is regarded [not just as an abstract *"Frobenius-like"* [pseudo-]monoid, but rather] as the *"étale-like"* output data of an **anabelian construction** of the sort just discussed — with the cyclotome $\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$. In the context of the relevant $\mathfrak{log}$**-Kummer correspondences** [i.e., as discussed in Remark 3.12.2, (iv), (v), below; Theorem 3.11, (ii), below], we shall work with various **Kummer isomorphisms** between such Frobenius-like and étale-like versions of various [pseudo-]monoids, i.e., in the notation of the final display of Proposition 1.3, (iv), between various objects associated to the **Frobenius-like** "$\bullet$'s" and corresponding objects associated to the **étale-like** "$\circ$". Now so long as one regards these various Frobenius-like "$\bullet$'s" and the étale-like "$\circ$" as **distinct labels** for corresponding objects, the diagram constituted by the relevant $\mathfrak{log}$-Kummer correspondence does **not** *result in any "vicious circles" or* **"loops"**. On the other hand, ultimately in the theory of §3 [cf., especially, the final portion of Theorem 3.11, (iii), (c), (d), below; the proof of Corollary 3.12 below], we shall be interested in applying the theory to the task of constructing algorithms to describe objects of interest of *one arithmetic holomorphic structure* in terms of some **alien** *arithmetic holomorphic structure* [cf. Remark 3.11.1] by means of **"multiradial containers"** [cf. Remark 3.12.2, (ii)]. These multiradial containers arise from *étale-like* versions of objects, but are ultimately applied as *containers* for *Frobenius-like* versions of objects. That is to say,

> in order for *such multiradial containers to function as* **containers**, it is necessary to contend with the consequences of **identifying** the **Frobenius-like** and **étale-like** versions of various objects under consideration, e.g., in the context of the above discussion, of identifying $\boldsymbol{\mu}_{\mathrm{Fr}}$ with $\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$.

On the other hand, let us recall that the approach to constructing *cyclotomic rigidity isomorphisms* associated to rational functions via the Kummer-theoretic approach of [IUTchI], Example 5.1, (v), amounts in effect [i.e., in the context of the above discussion], to **"identifying"** various "$\boldsymbol{\mu}^{*}_{\mathrm{et}}$'s" with various "sub-cyclotomes" of "$\boldsymbol{\mu}_{\mathrm{Fr}}$" via morphisms that differ from the usual *natural identification* precisely by *multiplication* by the **order** [$\in \mathbb{Z}$] at "$*$" of the **zeroes/poles** of the rational function under consideration. That is to say,

> to execute such a **cyclotomic rigidity isomorphism construction** in a situation *subject to the* **further identification** *of* $\boldsymbol{\mu}_{\mathrm{Fr}}$ *with* $\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$ [which, we recall, was obtained by identifying the various "$\boldsymbol{\mu}^{*}_{\mathrm{et}}$'s"!] does indeed result — at least in an *a priori* sense! — in **"vicious circles"/"loops"**

[cf. the discussion of [IUTchIV], Remark 3.3.1, (i); the reference to this discussion in [IUTchI], Remark 4.3.1, (ii)]. That is to say, in order to *avoid* any possible *contradictions* that might arise from such "vicious circles"/"loops", it is necessary to work with objects that are **"invariant"**, or **"coric"**, with respect to such "vicious circles"/"loops", i.e., to regard

> the *cyclotome* $\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$ as being **subject** to **indeterminacies** with respect to **multiplication** by elements of the *submonoid*
> $$\mathbb{I}^{\mathrm{ord}} \quad \subseteq \quad \pm\mathbb{N}_{\geq 1} \quad \stackrel{\mathrm{def}}{=} \quad \mathbb{N}_{\geq 1} \times \{\pm 1\}$$
> generated by the **orders** [$\in \mathbb{Z}$] of the **zeroes/poles** of the rational function(s) that appear in the cyclotomic rigidity isomorphism construction under consideration.

In the following discussion, we shall also write $\mathbb{I}^{\mathrm{ord}}_{\geq 1} \subseteq \mathbb{N}_{\geq 1}$, $\mathbb{I}^{\mathrm{ord}}_{\pm} \subseteq \{\pm 1\}$ for the respective images of $\mathbb{I}^{\mathrm{ord}}$ via the natural projections to $\mathbb{N}_{\geq 1}$, $\{\pm 1\}$. This sort of indeterminacy is **fundamentally incompatible**, for *numerous reasons*, with any sort of construction that purports to be analogous to the construction of the "theta values" in the theory of the present series of papers, i.e., at least whenever the resulting *indeterminacy submonoid* $\mathbb{I}^{\mathrm{ord}} \subseteq \pm\mathbb{N}_{\geq 1}$ is *nontrivial.* For instance, it follows immediately, by considering the effect of *independent indeterminacies* of this type on *valuations* at *distinct* $\underline{v} \in \underline{\mathbb{V}}$, that such independent indeterminacies are **incompatible** with the **"product formula"** [i.e., with the structure of the *global realified Frobenioids* involved — cf. [IUTchI], Remark 3.5.1, (ii)]. Here, we observe that *this sort of indeterminacy does not occur in the* **theta** *case* [cf. Fig. 2.5 below] — i.e., the resulting *indeterminacy submonoid*

$$(\pm\mathbb{N}_{\geq 1} \ \supseteq) \quad \mathbb{I}^{\mathrm{ord}} \quad = \quad \{1\}$$

— precisely as a consequence of the fact [which is closely related to the *symmetry* properties discussed in [IUTchII], Remark 1.1.1, (v)] that

> *the order [$\in \mathbb{Z}$] of the* **zeroes/poles** *of the* **theta function** *at* **every cusp** *is* **equal to 1**

[cf. [EtTh], Proposition 1.4, (i); [IUTchI], Remark 3.1.2, (ii), (iii)] — a state of affairs that can *never* occur in the case of an *algebraic rational function* [i.e., since the *sum* of the orders [$\in \mathbb{Z}$] of the zeroes/poles of an algebraic rational function is always equal to 0]! On the other hand, in the **number field** case [cf. Fig. 2.6 below], the portion of the indeterminacy under consideration that is *constituted by* $\mathbb{I}^{\mathrm{ord}}_{\geq 1}$ is *avoided* precisely [cf. the discussion of [IUTchI], Example 5.1, (v)] by applying the property

$$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} \ = \ \{1\}$$

[cf. also the discussion of (vii) below!], which has the effect of **isolating** the $\widehat{\mathbb{Z}}^{\times}$-*torsor of interest* [i.e., some *specific* isomorphism between cyclotomes] from the subgroup of $\mathbb{Q}_{>0}$ generated by $\mathbb{I}^{\mathrm{ord}}_{\geq 1}$. This technique for avoiding the indeterminacy constituted by $\mathbb{I}^{\mathrm{ord}}_{\geq 1}$ **remains valid** even after the identification discussed above of $\boldsymbol{\mu}_{\mathrm{Fr}}$ with $\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$. By contrast, the portion of the indeterminacy under consideration that is *constituted by* $\mathbb{I}^{\mathrm{ord}}_{\pm}$ is *avoided* in the construction of [IUTchI], Example 5.1, (v), precisely by applying the fact that the *inverse* of a nonconstant *κ-coric rational function* is *never κ-coric* [cf. the discussion of [IUTchI], Remark 3.1.7, (i)] — a technique that **depends**, in an essential way, on **distinguishing** cusps "$*$" at which the orders [$\in \mathbb{Z}$] of the zeroes/poles of the rational function(s) under consideration are **distinct**. In particular, this technique is **fundamentally incompatible** with the identification discussed above of $\boldsymbol{\mu}_{\mathrm{Fr}}$ with $\boldsymbol{\mu}^{\forall}_{\mathrm{et}}$. That is to say, in summary,

> in the **number field** case, in order to regard **étale-like** versions of objects as **containers** for **Frobenius-like** versions of objects, it is necessary to regard the relevant **cyclotomic rigidity isomorphisms** — hence also the **output data** of interest in the **number field** case, i.e., copies of [the union with $\{0\}$ of] the group "$F^{\times}_{\mathrm{mod}}$" — as being **subject** to an **indeterminacy** constituted by [possible] multiplication by $\{\pm 1\}$.

This does not result in any additional technical obstacles, however, since the **output data** of interest in the **number field** case — i.e., copies of [the union with $\{0\}$ of] the group "$F^{\times}_{\mathrm{mod}}$" — is [*unlike the case with the theta values* "$\underline{\underline{q}}_{\underline{v}}^{j^2}$"!] **stabilized** by the action of $\{\pm 1\}$

— cf. the discussion of Remark 3.11.4 below. Moreover, we observe in passing, in the context of the Galois evaluation operations in the number field case, that the copies of [the group] "$F^{\times}_{\mathrm{mod}}$" are constructed **globally** and in a fashion compatible with the $\mathbb{F}^{\circledast}_{l}$**-symmetry** [cf. [IUTchII], Corollary 4.8, (i), (ii)], hence, in particular, in a fashion that does not require the establishment of *compatibility* properties [e.g., relating to the *"product formula"*] between constructions at distinct $\underline{v} \in \underline{\mathbb{V}}$.

$$\begin{array}{cccccccccc} & +1 & +1 & +1 & +1 & & +1 & +1 & +1 & +1 & \\ \dots & * & * & * & * & \dots & * & * & * & * & \dots \end{array}$$

Fig. 2.5: Orders [$\in \mathbb{Z}$] of zeroes/poles of the *theta function* at the cusps "$*$"

$$\begin{array}{cccccccccc} & 0 & 0 & +8 & -5 & & -6 & +3 & 0 & 0 & \\ \dots & * & * & * & * & \dots & * & * & * & * & \dots \end{array}$$

Fig. 2.6: Orders [$\in \mathbb{Z}$] of zeroes/poles of an *algebraic rational function* at the cusps "$*$"

(vii) In the context of the discussion of (vi), we observe that the *indeterminacy* issues discussed in (vi) may be thought of as a sort of **"multiple cusp version"** of the **"N-th power versus first power"** and **"linearity"** issues discussed in [IUTchII], Remark 3.6.4, (iii). Also, in this context, we recall from the discussion at the beginning of Remark 2.1.1 that the theory of **mono-theta-theoretic cyclotomic rigidity** satisfies the important property of being **compatible** with the **topology** of the **tempered fundamental group**. Such a compatibility contrasts sharply with the cyclotomic rigidity algorithms discussed in [IUTchI], Example 5.1, (v), which *depend* [cf. the discussion of (vi) above!], in an essential way, on the property

$$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$$

— i.e., which is **fundamentally incompatible** *with the* **topology** *of the profinite groups involved* [as can be seen, for instance, by considering the fact that $\mathbb{N}_{\geq 1}$ forms a **dense** subset of $\widehat{\mathbb{Z}}$]. This close relationship between **cyclotomic rigidity** and [a sort of] **discrete rigidity** [i.e., the property of the above display] is reminiscent of the discussion given in [IUTchII], Remark 2.8.3, (ii), of such a relationship in the case of mono-theta environments.

(viii) In the context of the discussion of (vi), (vii), we observe that the *indeterminacy* issues discussed in (vi) also occur in the case of the cyclotomic rigidity algorithms discussed in [IUTchI], Definition 5.2, (vi), i.e., in the context of **mixed-characteristic local fields**. On the other hand, [cf. [IUTchII], Proposition 4.2, (i)] these algorithms in fact yield the *same cyclotomic rigidity isomorphism* as the cyclotomic rigidity isomorphisms that are applied in [AbsTopIII], Proposition 3.2, (iv) [i.e., the cyclotomic rigidity isomorphisms discussed in [AbsTopIII], Proposition 3.2, (i), (ii); [AbsTopIII], Remark 3.2.1]. Moreover, these cyclotomic rigidity isomorphisms discussed in [AbsTopIII] are **manifestly compatible** with the **topology** of the profinite groups involved. From the point of view of the discussion of (vi), this sort of *"de facto" compatibility with the topology of the profinite groups involved* may be thought of as a reflection of the fact that these cyclotomic rigidity isomorphisms discussed in [AbsTopIII] amount, in essence, to applying the approach to cyclotomic rigidity by considering the *Kummer theory* of *algebraic rational functions* [i.e., the approach of (vi), or, alternatively, of [IUTchI], Example 5.1, (v)], in the case where the algebraic rational functions are taken to be the **uniformizers** — i.e., "rational functions" [any one of which is well-defined up to a unit] with precisely **one zero** of **order 1** and **no poles** [cf. the discussion of the theta function in (vi)!] — of the **mixed-characteristic local field** under consideration. Put another way, this sort of "de facto" compatibility may be regarded as a reflection of the fact that, unlike *number fields* [i.e., "NF's"] or *one-dimensional function fields* [i.e., "one-dim. FF's"], *mixed-characteristic local fields* [i.e., "MLF's"] are equipped with a **uniquely determined "canonical valuation"** — a situation that is reminiscent of the fact that the order [$\in \mathbb{Z}$] of the *zeroes/poles* of the *theta function* at *every cusp* is *equal to* 1 [i.e., the fact that "the set of *equivalences classes of cusps* relative to the equivalence relationship on cusps determined by considering the order [$\in \mathbb{Z}$] of the zeroes/poles of the theta function is of *cardinality one*"]. From the point of view of *"geometric containers"* discussed in (i) and (ii), this state of affairs may be summarized as follows:

> the **indeterminacy** issues that occur in the context of the discussion of **cyclotomic rigidity isomorphisms** in (vi) exhibit **similar qualitative** behavior in the
>
> **MLF/mono-theta** ($\longleftrightarrow$ **one valuation/cusp**)
>
> [i.e., where the expression "one cusp" is to be understood as referring to "one equivalence class of cusps", as discussed above] cases, as well as in the
>
> **NF/one-dim. FF** ($\longleftrightarrow$ **global** collection of **valuations/cusps**)
>
> cases.

Put another way, at least at the level of the *theory of valuations*,

> the theory of **theta** functions (respectively, **one-dimensional function fields**) serves as an accurate "**qualitative geometric model**" of the theory of **mixed-characteristic local fields** (respectively, **number fields**).

Finally, we observe that in this context, the crucial property "$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$" that occurs in the discussion of the *number field/one-dimensional function field* cases is *highly reminiscent* of the *global nature of number fields* [i.e., such as $\mathbb{Q}$! — cf. the discussion of Remark 3.12.1, (iii), below].

(ix) The comparison given in (viii) of the special properties satisfied by the **theta function** with the corresponding properties of the **algebraic rational functions** that appear in the **number field** case is reminiscent of the analogy discussed in [IUTchI], Remark 6.12.3, (iii), with the **classical upper half-plane**. That is to say, the *eigenfunction* for the *additive symmetries* of the upper half-plane [i.e., which corresponds to the *theta* case]

$$q \stackrel{\mathrm{def}}{=} e^{2\pi i z}$$

Aspect of the theory	*Theta case*	*Number field case*
multiradial geometric container	**theta functions** on **Tate curves**	**Belyi maps/ cuspidalizations**
radial arithmetic data via **Galois evaluation**	**theta values** "$\underline{\underline{q}}^{j^2}_{\underline{v}}$"	copies of **number field** "F_{mod}" $\left(\supseteq F^{\times}_{\mathrm{mod}} \curvearrowleft \{\pm 1\}\right)$
Galois evaluation output data **dependence** on "Δ"	**simultaneously dependent** on **labels**, **holomorphic str.**	**indep.** of **labels**, **dependent** on **holomorphic str.**
cyclotomic rigidity isomorphism	**compatible** with $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, **tempered topology**	**incompatible** with $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, **profinite topology**
approach to eliminating **cyclo. rig. isom. indeterminacies**	order $[\in \mathbb{Z}]$ of **zeroes/poles** of **theta function** at **every cusp = 1**	$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$, **non-invertibility** of nonconstant κ**-coric** rational functions
qualitative geometric model for **arithmetic**	**MLF/mono-theta** ($\longleftrightarrow$ **one valuation/cusp**) analogy	**NF/one-dim. FF** ($\longleftrightarrow$ **global** collection of **valuations/cusps**) analogy
analogy with **eigenfunctions** for **symmetries** of **upper half-plane**	**highly transcendental** function in z: $q \stackrel{\mathrm{def}}{=} e^{2\pi i z}$	**algebraic rational** function of z: $w \stackrel{\mathrm{def}}{=} \frac{z-i}{z+i}$

Fig. 2.7: Comparison between the *theta* and *number field* cases

is **highly transcendental** in the coordinate z, whereas the *eigenfunction* for the *multiplicative symmetries* of the upper half-plane [i.e., which corresponds to the *number field* case]

$$w \stackrel{\text{def}}{=} \tfrac{z-i}{z+i}$$

is an **algebraic rational function** in the coordinate z.

(x) The various properties discussed above in the *theta* and *number field* cases are summarized in Fig. 2.7 above.

Remark 2.3.4. Before proceeding, it is perhaps of interest to review once more the essential content of [EtTh] in light of the various observations made in Remark 2.3.3.

(i) The starting point of the relationship between the theory of [EtTh] and the theory of the present series of papers lies [cf. the discussion of Remark 2.1.1, (i); [IUTchII], Remark 3.6.2, (ii)] in the various **non-ring/scheme-theoretic filters** [i.e., $\mathfrak{log}$-links and various types of Θ-links] between distinct ring/scheme theories that are constructed in the present series of papers. Such non-scheme-theoretic filters may only be constructed by making use of **Frobenius-like** structures. On the other hand, **étale-like structures** are important in light of their ability to relate structures on opposite sides of such non-scheme-theoretic filters. Then **Kummer theory** is applied to relate corresponding Frobenius-like and étale-like structures. Moreover, it is crucial that this Kummer theory be conducted in a **multiradial** fashion. This is achieved by means of certain **radial/coric decouplings**, by making use of **multiradial geometric containers**, as discussed in Remark 2.3.3, (i), (ii). That is to say, it is necessary to make use of such multiradial geometric containers and then to pass to theta values or number fields by means of **Galois evaluation**, since direct use of such theta values or number fields results in a Kummer theory that does **not** satisfy the desired multiradiality properties [cf. Remarks 2.2.1, 2.3.2].

(ii) The most naive approach to the Kummer theory of the *"functions"* that are to be used as *"multiradial geometric containers"* may be seen in the approach involving **algebraic rational functions** on the various algebraic curves under consideration, i.e., in the fashion of [IUTchI], Example 5.1, (v) [cf. also [IUTchI], Definition 5.2, (vi)]. On the other hand, in the context of the **local** theory at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, this approach suffers from the *fatal drawback* of being **incompatible** with the **profinite topology** of the profinite fundamental groups involved [cf. the discussion of Remark 2.3.3, (vi), (vii), (viii); Figs. 2.5, 2.6]. Thus, in order to maintain compatibility with the profinite/tempered topology of the profinite/tempered fundamental groups involved, one is obliged to work with the **Kummer theory** of **theta functions**, **truncated modulo** N. On the other hand, the naive approach to this sort of [truncated modulo N] Kummer theory of theta functions suffers from the fatal drawback of being **incompatible** with **discrete rigidity** [cf. Remark 2.1.1, (v)]. This incompatibility with discrete rigidity arises from a lack of *"shifting automorphisms"* as in [EtTh], Proposition 2.14, (ii) [cf. also [EtTh], Remark 2.14.3], and is closely related to the **incompatibility** of this naive approach with the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** [cf. the discussion of [IUTchII], Remark 1.1.1, (iv), (v)]. In order to surmount such incompatibilities, one is obliged to consider not the Kummer theory of theta functions in the naive sense, but rather, so to speak, the **Kummer theory** of [the **first Chern classes** of] the **line bundles** associated to theta functions [cf. the discussion of [IUTchII], Remark 1.1.1, (v)]. Thus, in summary:

[truncated] Kummer theory of **theta** [*not algebraic rational!*] **functions**
$\implies$ **compatible** with **profinite/tempered topologies**;

[truncated] Kummer theory of [**first Chern classes** of] **line bundles**
[*not rational functions!*]
$\implies$ **compatible** with **discrete rigidity,** $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**.

(iii) To consider the "[truncated] Kummer theory of line bundles [associated to the theta function]" amounts, in effect, to considering the [partially truncated] *arithmetic fundamental group* of the $\mathbb{G}_m$-torsor determined by such a line bundle in a fashion that is **compatible** with the various **tempered Frobenioids** and **tempered fundamental groups** under consideration. Such a "[partially truncated] arithmetic fundamental group" corresponds precisely to the *"topological group"* portion of the data that constitutes a mono-theta or bi-theta environment [cf. [EtTh], Definition 2.13, (ii), (a); [EtTh], Definition 2.13, (iii), (a)]. In the context of the theory of theta functions, such "[partially truncated] arithmetic fundamental groups" are equipped with *two natural distinguished [classes of] sections*, namely, **theta sections** and **algebraic sections**. If one thinks of the [partially truncated] arithmetic fundamental groups under consideration as being equipped *neither* with data corresponding to theta sections *nor* with data corresponding to algebraic sections, then the resulting mathematical object is necessarily subject to *indeterminacies* arising from *multiplication by constant units* [i.e., "$\mathcal{O}^\times$" of the base local field], hence, in particular, suffers from the drawback of being *incompatible* with *constant multiple rigidity* [cf. Remark 2.1.1, (iii)]. On the other hand, if one thinks of the [partially truncated] arithmetic fundamental groups under consideration as being equipped *both* with data corresponding to theta sections *and* with data corresponding to algebraic sections, then the resulting mathematical object suffers from the *same lack of symmetries* as the [truncated] Kummer theory of theta functions [cf. the discussion of (ii)], hence, in particular, is *incompatible* with *discrete rigidity* [cf. Remark 2.1.1, (v)]. Finally, if one thinks of the [partially truncated] arithmetic fundamental groups under consideration as being equipped *only* with data corresponding to *algebraic sections* [i.e., but not with data corresponding to theta sections!], then the resulting mathematical object is not equipped with sufficient data to apply the *crucial commutator property* of [EtTh], Proposition 2.12 [cf. also the discussion of [EtTh], Remark 2.19.2], hence, in particular, is *incompatible* with *cyclotomic rigidity* [cf. Remark 2.1.1, (iv)]. That is to say, it is only by thinking of the [partially truncated] arithmetic fundamental groups under consideration as being equipped *only* with data corresponding to *theta sections* [i.e., but not with data corresponding to algebraic sections!] — i.e., in short, by working with **mono-theta environments** — that one may achieve a situation that is **compatible** with the **tempered topology** of the tempered fundamental groups involved, the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, and **all three** types of **rigidity** [cf. the initial portion of Remark 2.1.1; [IUTchII], Remark 3.6.4, (ii)]. Thus, in summary:

working **neither** *with theta sections* **nor** *with algebraic sections* $\implies$

incompatible with **constant multiple rigidity**!

working with **bi-theta environments**, i.e.,
working simultaneously with **both** *theta sections* **and** *algebraic sections* $\Longrightarrow$
incompatible with **discrete rigidity, $\mathbb{F}_l^{\rtimes\pm}$-symmetry**!

working with *algebraic sections* **but not** *theta sections* $\Longrightarrow$
incompatible with **cyclotomic rigidity**!

working with **mono-theta environments**, i.e.,
working with *theta sections* **but not** *algebraic sections* $\Longrightarrow$
compatible with **tempered topology**, **$\mathbb{F}_l^{\rtimes\pm}$-symmetry**, **all three rigidities**!

(iv) Finally, we note that the approach of [EtTh] to the theory of theta functions differs substantially from *more conventional approaches* to the theory of theta functions such as

· the classical **function-theoretic** approach via **explicit series representations**, i.e., as given at the beginning of the Introduction to [IUTchII] [cf. also [EtTh], Proposition 1.4], and

· the **representation-theoretic** approach, i.e., by considering irreducible representations of **theta groups**.

Both of these more conventional approaches depend, in an essential way, on the **ring structures** — i.e., on both the **additive** and the **multiplicative** structures — of the various rings involved. [Here, we recall that explicit series are constructed precisely by adding and multiplying various functions on some space, whereas representations are, in effect, modules over suitable rings, hence, by definition, involve both additive and multiplicative structures.] In particular, although these more conventional approaches are well-suited to many situations in which one considers *"the"* theta function in some *fixed model of scheme/ring theory*, they are **ill-suited** to the situations treated in the present series of papers, i.e., where one must consider theta functions that appear in various **distinct ring/scheme theories**, which [cf. the discussion of (i)] may only be related to one another by means of suitable *Frobenius-like* and *étale-like* structures such as **tempered Frobenioids** and **tempered fundamental groups**. Here, we recall that these tempered Frobenioids correspond essentially to **multiplicative monoid** structures arising from the various rings of functions that appear, whereas tempered fundamental groups correspond to various **Galois actions**. That is to say, consideration of such multiplicative monoid structures and Galois actions is **compatible** with the **dismantling** of the additive and multiplicative structures of a ring, i.e., as considered in the present series of papers [cf. the discussion of Remark 3.12.2 below].

Definition 2.4.

(i) Let

$$^{\ddagger}\mathfrak{F}^{\vdash} \quad = \quad \{^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

be an $\mathcal{F}^{\vdash}$*-prime-strip*. Then recall from the discussion of [IUTchII], Definition 4.9, (ii), that at each $\underline{w} \in \underline{\mathbb{V}}^{\text{bad}}$, the splittings of the split Frobenioid ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$ determine *submonoids* "$\mathcal{O}^{\perp}(-) \subseteq \mathcal{O}^{\rhd}(-)$", as well as *quotient monoids* "$\mathcal{O}^{\perp}(-) \twoheadrightarrow \mathcal{O}^{\blacktriangleright}(-)$" [i.e., by forming the quotient of "$\mathcal{O}^{\perp}(-)$" by its torsion subgroup]. In a similar vein, for each $\underline{w} \in \underline{\mathbb{V}}^{\text{good}}$, the splitting of the split Frobenioid determined by [indeed, "constituted by", when $\underline{w} \in \underline{\mathbb{V}}^{\text{good}} \bigcap \underline{\mathbb{V}}^{\text{non}}$ — cf. [IUTchI], Definition 5.2, (ii)] ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{w}}$ determines a *submonoid "*$\mathcal{O}^{\perp}(-) \subseteq \mathcal{O}^{\rhd}(-)$*"* whose subgroup of units is *trivial* [cf. [IUTchII], Definition 4.9, (iv), when $\underline{w} \in \underline{\mathbb{V}}^{\text{good}} \bigcap \underline{\mathbb{V}}^{\text{non}}$]; in this case, we set $\mathcal{O}^{\blacktriangleright}(-) \stackrel{\text{def}}{=} \mathcal{O}^{\perp}(-)$. Write

$${}^{\ddagger}\mathfrak{F}^{\vdash\perp} \;=\; \{{}^{\ddagger}\mathcal{F}^{\vdash\perp}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}; \quad {}^{\ddagger}\mathfrak{F}^{\vdash\blacktriangleright} \;=\; \{{}^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

for the collections of data obtained by replacing the split Frobenioid portion of each ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}$ by the *Frobenioids* determined, respectively, by the subquotient monoids "$\mathcal{O}^{\perp}(-) \subseteq \mathcal{O}^{\rhd}(-)$", "$\mathcal{O}^{\blacktriangleright}(-)$" just defined.

(ii) We define [in the spirit of [IUTchII], Definition 4.9, (vii)] an $\mathcal{F}^{\vdash\perp}$*-prime-strip* to be a collection of data

$${}^{*}\mathfrak{F}^{\vdash\perp} = \{{}^{*}\mathcal{F}^{\vdash\perp}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

that satisfies the following conditions: (a) if $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, then ${}^{*}\mathcal{F}^{\vdash\perp}_{\underline{v}}$ is a *Frobenioid* that is isomorphic to ${}^{\ddagger}\mathcal{F}^{\vdash\perp}_{\underline{v}}$ [cf. (i)]; (b) if $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, then ${}^{*}\mathcal{F}^{\vdash\perp}_{\underline{v}}$ consists of a Frobenioid and an object of $\mathbb{TM}^{\vdash}$ [cf. [IUTchI], Definition 5.2, (ii)] such that ${}^{*}\mathcal{F}^{\vdash\perp}_{\underline{v}}$ is isomorphic to ${}^{\ddagger}\mathcal{F}^{\vdash\perp}_{\underline{v}}$. In a similar vein, we define an $\mathcal{F}^{\vdash\blacktriangleright}$*-prime-strip* to be a collection of data

$${}^{*}\mathfrak{F}^{\vdash\blacktriangleright} = \{{}^{*}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

that satisfies the following conditions: (a) if $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, then ${}^{*}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$ is a *Frobenioid* that is isomorphic to ${}^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$ [cf. (i)]; (b) if $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, then ${}^{*}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$ consists of a Frobenioid and an object of $\mathbb{TM}^{\vdash}$ [cf. [IUTchI], Definition 5.2, (ii)] such that ${}^{*}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$ is isomorphic to ${}^{\ddagger}\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$. A *morphism of* $\mathcal{F}^{\vdash\perp}$- (respectively, $\mathcal{F}^{\vdash\blacktriangleright}$-) *prime-strips* is defined to be a collection of isomorphisms, indexed by $\underline{\mathbb{V}}$, between the various constituent objects of the prime-strips [cf. [IUTchI], Definition 5.2, (iii)].

(iii) We define [in the spirit of [IUTchII], Definition 4.9, (viii)] an $\mathcal{F}^{\Vdash\perp}$*-prime-strip* to be a collection of data

$${}^{*}\mathfrak{F}^{\Vdash\perp} \;=\; ({}^{*}\mathcal{C}^{\Vdash},\ \text{Prime}({}^{*}\mathcal{C}^{\Vdash}) \stackrel{\sim}{\to} \underline{\mathbb{V}},\ {}^{*}\mathfrak{F}^{\vdash\perp},\ \{{}^{*}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

satisfying the conditions (a), (b), (c), (d), (e), (f) of [IUTchI], Definition 5.2, (iv), for an $\mathcal{F}^{\Vdash}$-prime-strip, except that the portion of the collection of data constituted by an $\mathcal{F}^{\vdash}$-prime-strip is replaced by an $\mathcal{F}^{\vdash\perp}$-prime-strip. [We leave the routine details to the reader.] In a similar vein, we define an $\mathcal{F}^{\Vdash\blacktriangleright}$*-prime-strip* to be a collection of data

$${}^{*}\mathfrak{F}^{\Vdash\blacktriangleright} \;=\; ({}^{*}\mathcal{C}^{\Vdash},\ \text{Prime}({}^{*}\mathcal{C}^{\Vdash}) \stackrel{\sim}{\to} \underline{\mathbb{V}},\ {}^{*}\mathfrak{F}^{\vdash\blacktriangleright},\ \{{}^{*}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

satisfying the conditions (a), (b), (c), (d), (e), (f) of [IUTchI], Definition 5.2, (iv), for an $\mathcal{F}^{\Vdash}$-prime-strip, except that the portion of the collection of data constituted by an $\mathcal{F}^{\vdash}$-prime-strip is replaced by an $\mathcal{F}^{\vdash\blacktriangleright}$-prime-strip. [We leave the routine details to the reader.] A *morphism of* $\mathcal{F}^{\Vdash\perp}$- (respectively, $\mathcal{F}^{\Vdash\blacktriangleright}$-) *prime-strips* is defined to be an isomorphism between collections of data as discussed above.

Remark 2.4.1.

(i) Thus, by applying the constructions of Definition 2.4, (i), to the [underlying $\mathcal{F}^{\vdash}$-prime-strips associated to the] $\mathcal{F}^{\Vdash}$-prime-strips "$\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$" that appear in Corollary 2.3, one may regard the multiradiality of Corollary 2.3, (i), as implying a corresponding **multiradiality** assertion concerning the associated $\mathcal{F}^{\Vdash\perp}$-prime-strips "$\mathfrak{F}^{\Vdash\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})$".

(ii) Suppose that we are in the situation discussed in (i). Then at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, the submonoids "$\mathcal{O}^{\perp}(-) \subseteq \mathcal{O}^{\rhd}(-)$" may be regarded, in a natural way [cf. Proposition 2.1, (ii); Theorem 2.2, (ii)], as *submonoids* of the monoids "${}_{\infty}\Psi^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{D}_{>})_{\underline{v}}$" of Theorem 2.2, (ii), ($\mathrm{a}_{\underline{v}}$). Moreover, the resulting inclusion of monoids is **compatible** with the **multiradiality** discussed in (i) and the multiradiality of the data "${}^{\dagger}\mathfrak{R}^{\mathrm{bad}}$" of Corollary 2.3, ($\mathrm{c}_{\mathfrak{R}}$), that is implied by the multiradiality of Corollary 2.3, (i).

Remark 2.4.2.

(i) One verifies immediately that, just as one may associate to an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* a *pilot object* in the global realified Frobenioid portion of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip [cf. [IUTchII], Definition 4.9, (viii)], one may associate to an $\mathcal{F}^{\Vdash\blacktriangleright}$-*prime-strip* a **pilot object** in the global realified Frobenioid portion of the $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strip [i.e., in the notation of the final display of Definition 2.4, (iii), the global realified Frobenioid ${}^{*}\mathcal{C}^{\Vdash}$ of the $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strip ${}^{*}\mathfrak{F}^{\Vdash\blacktriangleright}$].

(ii) For $\underline{v} \in \underline{\mathbb{V}}$ lying over $v \in \mathbb{V}_{\mathrm{mod}}$ and $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathbb{V}(\mathbb{Q})$, write

· $r_{\underline{v}} \stackrel{\mathrm{def}}{=} [(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}] \cdot \log(p_{\underline{v}}) \in \mathbb{R}$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$,

· $r_{\underline{v}} \stackrel{\mathrm{def}}{=} [(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}] \cdot \mathrm{ord}_{\underline{v}}(\underline{\underline{q}}_{\underline{v}}) \cdot \log(p_{\underline{v}}) \in \mathbb{R}$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$

— where, if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, then $\mathrm{ord}_{\underline{v}} : K^{\times}_{\underline{v}} \to \mathbb{Q}$ denotes the natural $p_{\underline{v}}$-adic valuation normalized so that $\mathrm{ord}_{\underline{v}}(p_{\underline{v}}) = 1$, and $\underline{\underline{q}}_{\underline{v}}$ is as in [IUTchI], Example 3.2, (iv);

$$r^{\blacktriangleright}_{\underline{v}} \stackrel{\mathrm{def}}{=} -\frac{r_{\underline{v}}}{\sum\limits_{\underline{w}\in\underline{\mathbb{V}}^{\mathrm{bad}}} r_{\underline{w}}}$$

[cf. the constructions of [IUTchI], Example 3.5; [IUTchI], Remark 3.5.1; the discussion of *weights* in Remark 3.1.1, (ii), below].

(iii) In the notation of (ii), let M be any **ordered monoid** isomorphic [as an ordered monoid] to $\mathbb{R}$ [endowed with the usual additive and order structures]. Then M naturally determines a *collection of data*

$$(M,\ \{M_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ \{\rho_{M_{\underline{v}}} : M_{\underline{v}} \stackrel{\sim}{\to} M\}_{\underline{v}\in\underline{\mathbb{V}}})$$

as follows: for each $\underline{v} \in \underline{\mathbb{V}}$, we take $M_{\underline{v}}$ to be a copy of M and $\rho_{M_{\underline{v}}} : M_{\underline{v}} \xrightarrow{\sim} M$ to be the isomorphism of monoids [that reverses the ordering!] given by *multiplying* by $r^{\blacktriangleright}_{\underline{v}} \in \mathbb{R}$.

(iv) In the notation of (ii), (iii), suppose, further, that we have been a **negative element**

$$\eta_M \in M$$

[i.e., an element < 0], which we shall refer to as a **pilot element**. Then, since, for $\underline{v} \in \underline{\mathbb{V}}$, $M_{\underline{v}}$ is defined to be a copy of M, η_M determines an element $\eta_{M_{\underline{v}}} \in M_{\underline{v}}$. Thus, the *pair* (M, η_M) naturally determines a *collection of data*

$$(M,\ \{M^{\blacktriangleright}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ \{\rho_{M^{\blacktriangleright}_{\underline{v}}} : M^{\blacktriangleright}_{\underline{v}} \hookrightarrow M\}_{\underline{v}\in\underline{\mathbb{V}}})$$

as follows: for each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, we take $M^{\blacktriangleright}_{\underline{v}} \subseteq M_{\underline{v}}$ to be the submonoid [isomorphic to $\mathbb{N}$] generated by $\eta_{M_{\underline{v}}}$ and $\rho_{M^{\blacktriangleright}_{\underline{v}}} : M^{\blacktriangleright}_{\underline{v}} \hookrightarrow M$ to be the restriction of $\rho_{M_{\underline{v}}}$ to $M^{\blacktriangleright}_{\underline{v}}$; for each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, we take $M^{\blacktriangleright}_{\underline{v}} \subseteq M_{\underline{v}}$ to be the submonoid [isomorphic to $\mathbb{R}_{\geq 0}$] given by the elements ≤ 0 and $\rho_{M^{\blacktriangleright}_{\underline{v}}} : M^{\blacktriangleright}_{\underline{v}} \hookrightarrow M$ to be the restriction of $\rho_{M_{\underline{v}}}$ to $M^{\blacktriangleright}_{\underline{v}}$. In particular, it follows immediately from the construction of this data that

$$\rho_{M^{\blacktriangleright}_{\underline{v}}}(\eta_{M_{\underline{v}}}) \ = \ r^{\blacktriangleright}_{\underline{v}} \cdot \eta_M$$

for each $\underline{v} \in \underline{\mathbb{V}}$.

(v) Now we observe that the constructions of (iii) and (iv) allow one to give a sort of *"converse"* to the construction of the *pilot object* in (i). Indeed, consider the $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strip ${}^*\mathfrak{F}^{\Vdash\blacktriangleright}$ in the final display of Definition 2.4, (iii). Next, observe that the **"Picard group"** constructions "$\mathrm{Pic}_{\Phi}(-)$" and "$\mathrm{Pic}_{\mathcal{C}}(-)$" of [FrdI], Theorem 5.1, (i), applied to any object of the global realified Frobenioid ${}^*\mathcal{C}^{\Vdash}$ yield canonically isomorphic groups for *any object* of ${}^*\mathcal{C}^{\Vdash}$. In particular, it makes sense to speak of "$\mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$". Moreover, it follows from [FrdI], Theorem 6.4, (i), (ii), that $\mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$ is equipped with a canonical structure of *ordered monoid*, with respect to which it is isomorphic to $\mathbb{R}$ [endowed with the usual additive and order structures]. Relative to this structure of ordered monoid, the *pilot object* discussed in (i) [cf. also the discussion of [IUTchII], Definition 4.9, (viii)] determines a **negative element** $\eta_{{}^*\mathcal{C}^{\Vdash}} \in \mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$. Thus, one verifies immediately, by recalling the various definitions involved, that the *collection of data* "$(M, \{M_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \{\rho_{M_{\underline{v}}} : M_{\underline{v}} \xrightarrow{\sim} M\}_{\underline{v}\in\underline{\mathbb{V}}})$" constructed in (iii) from "M" is already sufficient to reconstruct, i.e., by taking $M \stackrel{\mathrm{def}}{=} \mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$, the *collection of data*

$$({}^*\mathcal{C}^{\Vdash},\ \mathrm{Prime}({}^*\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}})$$

[cf. the notation of the final display of Definition 2.4, (iii)], while the *collection of data* "$(M, \{M^{\blacktriangleright}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \{\rho_{M^{\blacktriangleright}_{\underline{v}}} : M^{\blacktriangleright}_{\underline{v}} \hookrightarrow M\}_{\underline{v}\in\underline{\mathbb{V}}})$" constructed in (iv) from the pair "(M, η_M)" is sufficient to reconstruct, i.e., by taking $M \stackrel{\mathrm{def}}{=} \mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$ and $\eta_M \stackrel{\mathrm{def}}{=} \eta_{{}^*\mathcal{C}^{\Vdash}}$, the *collection of data*

$$({}^*\mathcal{C}^{\Vdash},\ \mathrm{Prime}({}^*\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{\Phi_{{}^*\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ \{{}^*\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

where, for $\underline{v} \in \underline{\mathbb{V}}$, we write $\Phi_{{}^*\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}}$ for the [constant!] *divisor monoid* [i.e., in effect, a single monoid isomorphic to $\mathbb{N}$ or $\mathbb{R}_{\geq 0}$] determined by the Frobenioid structure [cf. [FrdI], Corollary 4.11, (iii); [FrdII], Theorem 1.2, (i)] on ${}^*\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}$ [cf. the notation of the final display of Definition 2.4, (i)].

(vi) One immediate consequence of the discussion of (v) is the following:

If one starts from $M = \mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$, then the resulting *collection of data*

$$({}^*\mathcal{C}^{\Vdash},\ \mathrm{Prime}({}^*\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}})$$

yields a **common container**, namely, the Frobenioid ${}^*\mathcal{C}^{\Vdash}$ [regarded as an object reconstructed from $M = \mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$!], in which **distinct choices** of the [*negative*!] **pilot element** $\in M = \mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$ — hence also the data

$$({}^*\mathcal{C}^{\Vdash},\ \mathrm{Prime}({}^*\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{\Phi_{{}^*\mathcal{F}^{\vdash\blacktriangleright}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ \{{}^*\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

[which may be thought of as a sort of *"further rigidification"* on ${}^*\mathcal{C}^{\Vdash}$] reconstructed from such distinct choices of pilot element — may be **compared** with one another.

By contrast, if one attempts to compare the constructions of (v) applied to **positive** and **negative** "$\eta_M \in M$" [i.e., which amounts to *reversing the order structure* on M!], then already the corresponding Frobenioids "${}^*\mathcal{C}^{\Vdash}$" [i.e., attached to the same *group* "$\mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$", but with *reversed order structures*!] involve *pre-steps* [i.e., in effect, the category-theoretic version of the notion of an *inclusion of line bundles* — cf. [FrdI], Definition 1.2, (iii)] going in *opposite directions.* That is to say, such Frobenioids may only be compared with one another if they are embedded in some sort of larger **ambient category** in which *the pre-steps are rendered invertible*; but this already implies that all objects arising from such Frobenioids become *isomorphic* in the ambient category. That is to say, working in such a larger ambient category already renders any sort of argument that requires one to *distinguish distinct elements* of $\mathrm{Pic}({}^*\mathcal{C}^{\Vdash})$ — i.e., *distinct arithmetic degrees/heights* of arithmetic line bundles — *meaningless* [cf. the discussion of *positivity* in Remark 2.1.1, (v)].

Section 3: Multiradial Logarithmic Gaussian Procession Monoids

In the present §3, we apply the theory developed thus far in the present series of papers to give [cf. Theorem 3.11 below] **multiradial algorithms** for a slightly modified version of the **Gaussian monoids** discussed in [IUTchII], §4. This modification revolves around the combinatorics of *processions*, as developed in [IUTchI], §4, §5, §6, and is necessary in order to establish the desired multiradiality. At a more concrete level, these combinatorics require one to apply the theory of **tensor packets** [cf. Propositions 3.1, 3.2, 3.3, 3.4, 3.7, 3.9, below]. Finally, we observe in Corollary 3.12 that these multiradial algorithms give rise to certain **estimates** concerning the **log-volumes** of the **logarithmic Gaussian procession monoids** that occur. This observation forms the starting point of the theory to be developed in [IUTchIV].

In the following discussion, we assume that we have been given *initial* Θ*-data* as in [IUTchI], Definition 3.1. Also, we shall write

$$\mathbb{V}_{\mathbb{Q}} \overset{\mathrm{def}}{=} \mathbb{V}(\mathbb{Q})$$

[cf. [IUTchI], §0] and apply the notation of Definition 1.1 of the present paper. We begin by discussing the theory of **tensor packets**, which may be thought of as a sort of *amalgamation* of the theory of *log-shells* developed in §1 with the theory of *processions* developed in [IUTchI], §4, §5, §6.

Proposition 3.1. **(Local Holomorphic Tensor Packets)** *Let*

$$\{{}^{\alpha}\mathfrak{F}\}_{\alpha \in A} \;=\; \Big\{\{{}^{\alpha}\mathcal{F}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}\Big\}_{\alpha \in A}$$

be an ***n*-capsule***, with index set A, of $\mathcal{F}$-prime-strips [relative to the given initial Θ-data — cf. [IUTchI], §0; [IUTchI], Definition 5.2, (i)]. Then [cf. the notation of Definition 1.1, (iii)] for $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}$, by considering* **invariants** *with respect to the natural action of various open subgroups of the topological group ${}^{\alpha}\Pi_{\underline{v}}$, one may regard $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$ as an* **inductive limit** *of* **topological modules***, each of which is of finite dimension over $\mathbb{Q}_{v_{\mathbb{Q}}}$; we shall refer to the correspondence*

$$\mathbb{V}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \;\mapsto\; \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \;\overset{\mathrm{def}}{=}\; \bigoplus_{\underline{\mathbb{V}} \,\ni\, \underline{v} \,\mid\, v_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$$

as the **[1-]tensor packet** *associated to the $\mathcal{F}$-prime-strip ${}^{\alpha}\mathfrak{F}$ and to the correspondence*

$$\mathbb{V}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \;\mapsto\; \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) \;\overset{\mathrm{def}}{=}\; \bigotimes_{\alpha \in A} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \;=\; \bigoplus_{\{\underline{v}_{\alpha}\}_{\alpha \in A}} \Big\{ \bigotimes_{\alpha \in A} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}_{\alpha}}) \Big\}$$

— where the tensor products are to be understood as tensor products of ind-topological modules [i.e., as discussed above], and the direct sum is over all collections $\{\underline{v}_{\alpha}\}_{\alpha \in A}$ of [not necessarily distinct!] elements $\underline{v}_{\alpha} \in \underline{\mathbb{V}}$ lying over $v_{\mathbb{Q}}$ and indexed by $\alpha \in A$ —

as the $[n\text{-}]$**tensor packet** *associated to the collection of* $\mathcal{F}$*-prime-strips* $\{{}^{\alpha}\mathfrak{F}\}_{\alpha \in A}$. *Then:*

(i) **(Ring Structures)** *The* **ind-topological field structures** *on the various* $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$ *[cf. Definition 1.1, (i), (ii), (iii)], for* $\alpha \in A$, *determine an* **ind-topological ring structure** *on* $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$ *with respect to which* $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$ *may be regarded as an inductive limit of* **direct sums of ind-topological fields**. *Such decompositions as direct sums of ind-topological fields are uniquely determined by the ind-topological ring structure on* $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$ *and, moreover, are compatible, for* $\alpha \in A$, *with the natural action of the topological group* ${}^{\alpha}\Pi_{\underline{v}}$ *[where* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}$*] on the direct summand with subscript* $\underline{v}$ *of the factor labeled* α.

(ii) **(Integral Structures)** *Fix elements* $\alpha \in A$, $\underline{v} \in \underline{\mathbb{V}}$, $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ *such that* $\underline{v} \mid v_{\mathbb{Q}}$. *Relative to the tensor product in the above definition of* $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$, *write*

$$\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}}) \otimes \Big\{ \bigotimes_{\beta \in A \setminus \{\alpha\}} \underline{\mathfrak{log}}({}^{\beta}\mathcal{F}_{v_{\mathbb{Q}}}) \Big\} \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$$

for the ind-topological submodule determined by the tensor product of the factors labeled by $\beta \in A \setminus \{\alpha\}$ *with the tensor product of the direct summand with subscript* $\underline{v}$ *of the factor labeled* α. *Then* $\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ *forms a direct summand of the ind-topological ring* $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$; $\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ *may be regarded as an inductive limit of* **direct sums of ind-topological fields**; *such decompositions as direct sums of ind-topological fields are uniquely determined by the ind-topological ring structure on* $\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$. *Moreover, by forming the tensor product with "1's" in the factors labeled by* $\beta \in A \setminus \{\alpha\}$, *one obtains a* **natural injective homomorphism** *of ind-topological rings*

$$\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}}) \to \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$$

that, for suitable choices [which are, in fact, cofinal] of objects appearing in the inductive limit descriptions given above for the domain and codomain, induces an **isomorphism** *of such an object in the domain onto each of the direct summand ind-topological fields of the object in the codomain. In particular, the integral structure*

$$\overline{\Psi}_{\mathfrak{log}({}^{\alpha}\mathcal{F}_{\underline{v}})} \stackrel{\mathrm{def}}{=} \Psi_{\mathfrak{log}({}^{\alpha}\mathcal{F}_{\underline{v}})} \bigcup \{0\} \subseteq \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$$

[cf. the notation of Definition 1.1, (i), (ii)] determines **integral structures** *on each of the direct summand ind-topological fields that appear in the inductive limit descriptions of* $\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$, $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$.

Proof. The various assertions of Proposition 3.1 follow immediately from the definitions and the references quoted in the statements of these assertions [cf. also Remark 3.1.1, (i), below]. ○

Remark 3.1.1.

(i) Let $\underline{v} \in \underline{\mathbb{V}}$. In the notation of [IUTchI], Definition 3.1, write $k \stackrel{\mathrm{def}}{=} K_{\underline{v}}$; let $\overline{k}$ be an algebraic closure of k. Then, roughly speaking, in the notation of Proposition 3.1,

$$\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}}) \stackrel{\sim}{\to} \overline{k}; \quad \overline{\Psi}_{\mathfrak{log}({}^{\alpha}\mathcal{F}_{\underline{v}})} \stackrel{\sim}{\to} \mathcal{O}_{\overline{k}};$$

$$\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}}) \quad \stackrel{\sim}{\to} \quad \bigotimes \; \overline{k} \quad \stackrel{\sim}{\to} \quad \varinjlim \; \bigoplus \; \overline{k} \quad \supseteq \quad \varinjlim \; \bigoplus \; \mathcal{O}_{\overline{k}}$$

— i.e., one verifies immediately that each ind-topological field $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$ is isomorphic to $\overline{k}$; each $\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ is a topological tensor product [say, over $\mathbb{Q}$] of copies of $\overline{k}$, hence may be described as an inductive limit of direct sums of copies of $\overline{k}$; each $\overline{\Psi}_{\mathfrak{log}({}^{\alpha}\mathcal{F}_{\underline{v}})}$ is a copy of the set [i.e., a ring, when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$] of integers $\mathcal{O}_{\overline{k}} \subseteq \overline{k}$. In particular, the "integral structures" discussed in the final portion of Proposition 3.1, (ii), correspond to copies of $\mathcal{O}_{\overline{k}}$ contained in copies of $\overline{k}$.

(ii) Ultimately, for $\underline{v} \in \underline{\mathbb{V}}$, we shall be interested [cf. Proposition 3.9, (i), (ii), below] in considering **log-volumes** on the portion of $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$ corresponding to $K_{\underline{v}}$. On the other hand, let us recall that we do not wish to consider *all* of the valuations in $\mathbb{V}(K)$. That is to say, we wish to restrict ourselves to considering the subset $\underline{\mathbb{V}} \subseteq \mathbb{V}(K)$, equipped with the *natural bijection* $\underline{\mathbb{V}} \stackrel{\sim}{\to} \mathbb{V}_{\mathrm{mod}}$ [cf. [IUTchI], Definition 3.1, (e)], which we wish to think of as a sort of *"local analytic section"* [cf. the discussion of [IUTchI], Remark 4.3.1, (i)] of the natural morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$ [or, perhaps more precisely, $\mathrm{Spec}(K) \to \mathrm{Spec}(F_{\mathrm{mod}})$]. In particular, it will be necessary to consider these log-volumes on the portion of $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}})$ corresponding to $K_{\underline{v}}$ relative to the **weight** $[K_{\underline{v}} : (F_{\mathrm{mod}})_v]^{-1}$, where we write $v \in \mathbb{V}_{\mathrm{mod}}$ for the element determined [via the natural bijection just discussed] by $\underline{v}$ [cf. the discussion of [IUTchI], Example 3.5, (i), (ii), (iii), where similar factors appear]. When, moreover, we consider direct sums over all $\underline{v} \in \underline{\mathbb{V}}$ lying over a given $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ as in the case of $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$, it will be convenient to use the **normalized weight**

$$\frac{1}{[K_{\underline{v}} : (F_{\mathrm{mod}})_v] \cdot \Big(\sum_{\mathbb{V}_{\mathrm{mod}} \ni w | v_{\mathbb{Q}}} [(F_{\mathrm{mod}})_w : \mathbb{Q}_{v_{\mathbb{Q}}}] \Big)}$$

— i.e., *normalized* so that multiplication by $p_{v_{\mathbb{Q}}}$ affects log-volumes by addition or subtraction [that is to say, depending on whether $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$ or $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$] of the quantity $\log(p_{v_{\mathbb{Q}}}) \in \mathbb{R}$. In a similar vein, when we consider log-volumes on the portion of $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$ corresponding to the tensor product of various $K_{\underline{v}_{\alpha}}$, where $\underline{\mathbb{V}} \ni \underline{v}_{\alpha} \mid v_{\mathbb{Q}}$, it will be necessary to consider these log-volumes relative to the **weight**

$$\frac{1}{\prod_{\alpha \in A} [K_{\underline{v}_{\alpha}} : (F_{\mathrm{mod}})_{v_{\alpha}}]}$$

— where we write $v_{\alpha} \in \mathbb{V}_{\mathrm{mod}}$ for the element determined by $\underline{v}_{\alpha}$. When, moreover, we consider direct sums over all possible choices for the data $\{\underline{v}_{\alpha}\}_{\alpha \in A}$, it will be convenient to use the **normalized weight**

$$\frac{1}{\Big(\prod_{\alpha \in A} [K_{\underline{v}_{\alpha}} : (F_{\mathrm{mod}})_{v_{\alpha}}] \Big) \cdot \Big\{ \sum_{\{w_{\alpha}\}_{\alpha \in A}} \Big(\prod_{\alpha \in A} [(F_{\mathrm{mod}})_{w_{\alpha}} : \mathbb{Q}_{v_{\mathbb{Q}}}] \Big) \Big\}}$$

— where the sum is over all collections $\{w_{\alpha}\}_{\alpha \in A}$ of [not necessarily distinct!] elements $w_{\alpha} \in \mathbb{V}_{\mathrm{mod}}$ lying over $v_{\mathbb{Q}}$ and indexed by $\alpha \in A$. Again, these normalized weights are *normalized* so that multiplication by $p_{v_{\mathbb{Q}}}$ affects log-volumes by addition or subtraction [that is to say, depending on whether $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$ or $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$] of the quantity $\log(p_{v_{\mathbb{Q}}}) \in \mathbb{R}$.

(iii) In the discussion to follow, we shall, for simplicity, use the term *"measure space"* to refer to a *locally compact Hausdorff topological space* whose topology admits a *countable basis*, and which is equipped with a *complete Borel measure* in the sense of [Royden], Chapter 11, §1; [Royden], Chapter 14, §1. In particular, one may speak of the *product measure space* [cf. [Royden], Chapter 12, §4] of any finite nonempty collection of measure spaces. Then observe that *care must be exercised* when considering the various *weighted sums of log-volumes* discussed in (ii), since, unlike, for instance, the *log-volumes* discussed in [item (a) of] [AbsTopIII], Proposition 5.7, (i), (ii),

> such **weighted sums of log-volumes** do **not**, in general, arise as some positive real multiple of the [natural] logarithm of a **"volume"** or **"measure"** in the usual sense of measure theory.

In particular, when considering *direct sums* of the sort that appear in the second or third displays of the statement of Proposition 3.1, although it is *clear from the definitions* how to compute a weighted sum of log-volumes of the sort discussed in (ii) in the case of a *region* that arises as a *direct product* of, say, compact subsets of positive measure in each of the direct summands [i.e., since the volume/measure of such a compact subset may be computed as the *infimum* of the volume/measure of the compact open subsets that contain it], it is *not immediately clear* from the definitions how to compute such a weighted sum of log-volumes in the case of *more general regions*. In the following, for ease of reference, let us refer to such a

> *region* that arises as a *direct product* of compact subsets of positive measure in each of the direct summands as a **direct product region**

and to a

> *region* that arises as a *direct product* of relatively compact subsets in each of the direct summands as a **direct product pre-region**.

Then we observe in the remainder of the present Remark 3.1.1 that although, in the present series of papers,

> the regions that will actually be **of interest** in the development of the theory are, in fact, **direct product [pre-]regions**, in which case the computation of weighted sums of log-volumes is completely straightforward [cf. also the discussion of Remark 3.9.7, (ii), (iii), below],

in fact,

> **weighted sums of log-volumes** of the sort discussed in (ii) may be computed for, say, **arbitrary Borel sets** by applying the elementary construction discussed in (iv) below.

Here, in the context of the situation discussed in the final portion of (ii), we note that this construction in (iv) below is applied relative to the following given data:

- the finite set "V" is taken to be the direct product

$$\prod_{\alpha \in A} \underline{\mathbb{V}}_{v_{\mathbb{Q}}} \quad (\overset{\sim}{\to} \prod_{\alpha \in A} (\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}})$$

[where the subscript "$v_{\mathbb{Q}}$" denotes the fiber over $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$];

· for "$v \in V$", the cardinality "N_v" is taken to be the product that appears in the discussion of (ii)

$$\prod_{\alpha \in A} [K_{\underline{v}_\alpha} : (F_{\mathrm{mod}})_{v_\alpha}]$$

[where we think of "$v \in V$" as a collection $\{\underline{v}_\alpha\}_{\alpha \in A}$ of elements of $\underline{\mathbb{V}}_{v_{\mathbb{Q}}}$ that lies over a collection $\{v_\alpha\}_{\alpha \in A}$ of elements of $(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}$], while "$M_v$" is taken to be the [*radial*, if $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}{}^{\mathrm{arc}}$] portion of the direct summand in the third display of the statement of Proposition 3.1 indexed by $v \in V$ that corresponds to the tensor product of the $\{K_{\underline{v}_\alpha}\}_{\alpha \in A}$.

[By the *"radial"* portion of a topological tensor product of a finite collection of complex archimedean fields, we mean the *direct product of the copies of* $\mathbb{R}_{>0}$ that arise by forming the *quotients by the units* [i.e., copies of $\mathbb{S}^1$] of each of the complex archimedean fields that appears in the *direct sum of fields* [cf. (i)] that arises from such a topological tensor product.] Then one verifies immediately that, in the case of *"direct product regions"* [as discussed above], the result of multiplying the [natural] logarithm of the *"$\mathcal{E}$-weighted measure $\mu_{\mathcal{E}}(-)$"* of (iv) by a suitable normalization factor [i.e., a suitable positive real number] yields the *weighted sums of log-volumes* discussed in (ii).

(iv) Let V a *nonempty finite set*; $\mathcal{E} \stackrel{\mathrm{def}}{=} \{E_v\}_{v \in V}$ a *collection of nonempty finite sets*; $\mathcal{M} \stackrel{\mathrm{def}}{=} \{(M_v, \mu_v)\}_{v \in V}$ a collection of nonempty *measure spaces* [cf. the discussion of (iii) above]. For $v \in V$, write

$$E \stackrel{\mathrm{def}}{=} \prod_{v' \in V} E_{v'}; \quad E_{\neq v} \stackrel{\mathrm{def}}{=} \prod_{V \ni v' \neq v} E_{v'};$$
$$E \times V \twoheadrightarrow W \stackrel{\mathrm{def}}{=} \coprod_{v' \in V} E_{\neq v'} \times \{v'\} \twoheadrightarrow V$$

— where the first arrow "$\twoheadrightarrow$" is defined by the condition that, for $v' \in V$, it restricts to the natural projection $E \times \{v'\} \twoheadrightarrow E_{\neq v'} \times \{v'\}$ on $E \times \{v'\}$; the second arrow "$\twoheadrightarrow$" is defined by the condition that, for $v' \in V$, it restricts to the natural projection $E_{\neq v'} \times \{v'\} \twoheadrightarrow \{v'\}$ on $E_{\neq v'} \times \{v'\}$. If $W \ni w \mapsto v \in V$ via the natural surjection $W \twoheadrightarrow V$ just discussed, then write $(M_w, \mu_w) \stackrel{\mathrm{def}}{=} (M_v, \mu_v)$. If Z is a subset of W or V, then we shall write

$$M_Z \stackrel{\mathrm{def}}{=} \prod_{z \in Z} M_z; \quad M_{E \times V} \stackrel{\mathrm{def}}{=} \prod_{(e,v) \in E \times V} M_v = \prod_{e \in E} M_V;$$
$$(M_{E \times V} \supseteq) \ M_{E * V} \stackrel{\mathrm{def}}{=} \Big\{ \{m_{e,v}\}_{(e,v) \in E \times V} \mid m_{e',v} = m_{e'',v},$$
$$\forall (e', e'') \in E \times_{E_{\neq v}} E \subseteq E \times E \Big\} \stackrel{\sim}{\rightarrow} M_W$$

— where the bijection $M_{E*V} \stackrel{\sim}{\rightarrow} M_W$ is the map induced by the various natural projections $E \twoheadrightarrow E_{\neq v}$ that constitute the natural projection $E \times V \twoheadrightarrow W$; this bijection $M_{E*V} \stackrel{\sim}{\rightarrow} M_W$ is easily verified to be a *homeomorphism*. Thus, M_W,

M_V, and $M_{E\times V}$ are equipped with natural *product measure space* structures; the bijection $M_{E*V} \xrightarrow{\sim} M_W$, together with the measure space structure on M_W, induces a *measure space* structure on M_{E*V}. In particular, if $S \subseteq M_V$ is any *Borel set*, then the product

$$\prod_{e\in E} S \subseteq M_{E\times V}$$

is a Borel set of $M_{E\times V}$; the intersection of this product with M_{E*V}

$$S_E \stackrel{\text{def}}{=} \left\{\prod_{e\in E} S\right\} \bigcap M_{E*V} \subseteq M_{E*V}$$

is a Borel set of M_{E*V} ($\xrightarrow{\sim} M_W$). Thus, in summary, for any *Borel set* $S \subseteq M_V$, one may speak of the *"$\mathcal{E}$-weighted measure"*

$$\mu_{\mathcal{E}}(S) \in \mathbb{R}_{\geq 0} \bigcup \{+\infty\}$$

of S, i.e., the *measure*, relative to the *measure space* structure of M_{E*V} ($\xrightarrow{\sim} M_W$), of S_E. Since, moreover, one verifies immediately that the above construction is *functorial* with respect to isomorphisms of the given data $(V, \mathcal{E}, \mathcal{M})$, it follows immediately that, in fact, $\mu_{\mathcal{E}}(-)$ is *completely determined* by the *cardinalities* $\mathcal{N} \stackrel{\text{def}}{=} \{N_v\}_{v\in V}$ of the finite sets $\mathcal{E} = \{E_v\}_{v\in V}$, i.e., by the data $(V, \mathcal{N}, \mathcal{M})$. Finally, we observe that when $S \subseteq M_V$ is a **"direct product region"** [cf. the discussion of (iii)], i.e., a set of the form $\prod_{v\in V} S_v$, where $S_v \subseteq M_v$ is a compact subset of positive measure, then a straightforward computation reveals that

$$\tfrac{1}{N_{\mathcal{E}}} \cdot \mu_{\mathcal{E}}^{\log}(S) = \sum_{v\in V} \tfrac{1}{N_v} \cdot \mu_v^{\log}(S_v)$$

— where we write $N_{\mathcal{E}} = \prod_{v\in V} N_v$, and each superscript "log" denotes the natural logarithm of the corresponding quantity without a superscript.

Remark 3.1.2. The constructions involving **local holomorphic tensor packets** given in Proposition 3.1 may be applied to the capsules that appear in the various **$\mathcal{F}$-prime-strip processions** obtained by considering the evident $\mathcal{F}$-prime-strip analogues [cf. [IUTchI], Remark 5.6.1; [IUTchI], Remark 6.12.1] of the **holomorphic processions** discussed in [IUTchI], Proposition 4.11, (i); [IUTchI], Proposition 6.9, (i).

Proposition 3.2. **(Local Mono-analytic Tensor Packets)** *Let*

$$\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha\in A} = \left\{\{{}^{\alpha}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\right\}_{\alpha\in A}$$

be an $\boldsymbol{n}$**-capsule***, with index set A, of $\mathcal{D}^{\vdash}$-prime-strips [relative to the given initial Θ-data — cf. [IUTchI], §0; [IUTchI], Definition 4.1, (iii)]. Then [cf. the notation of Proposition 1.2, (vi), (vii)] we shall refer to the correspondence*

$$\mathbb{V}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \mapsto \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \stackrel{\text{def}}{=} \bigoplus_{\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$$

as the **[1-]tensor packet** *associated to the* $\mathcal{D}^{\vdash}$*-prime-strip* ${}^{\alpha}\mathfrak{D}^{\vdash}$ *and to the correspondence*

$$\mathbb{V}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \mapsto \underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \stackrel{\text{def}}{=} \bigotimes_{\alpha \in A} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$$

— where the tensor product is to be understood as a tensor product of ind-topological modules — as the **[n-]tensor packet** *associated to the collection of* $\mathcal{D}^{\vdash}$*-prime-strips* $\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha\in A}$*. For* $\alpha \in A$, $\underline{v} \in \underline{\mathbb{V}}$, $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ *such that* $\underline{v} \mid v_{\mathbb{Q}}$*, we shall write*

$$\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$$

for the ind-topological submodule determined by the tensor product of the factors labeled by $\beta \in A \setminus \{\alpha\}$ *with the tensor product of the direct summand with subscript* $\underline{v}$ *of the factor labeled* α *[cf. Proposition 3.1, (ii)]. If the capsule of* $\mathcal{D}^{\vdash}$*-prime-strips* $\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha\in A}$ *arises from a capsule of* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$**-prime-strips**

$$\{{}^{\alpha}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}\}_{\alpha\in A} = \left\{\{{}^{\alpha}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\right\}_{\alpha\in A}$$

[relative to the given initial Θ*-data — cf. [IUTchI], §0; [IUTchII], Definition 4.9, (vii)], then we shall use similar notation to the notation just introduced concerning* $\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha\in A}$ *to denote objects associated to* $\{{}^{\alpha}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}\}_{\alpha\in A}$*, i.e., by* **replacing** *"*$\mathcal{D}^{\vdash}$*" in the above notational conventions by "*$\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*" [cf. also the notation of Proposition 1.2, (vi), (vii)]. Then:*

(i) **(Mono-analytic/Holomorphic Compatibility)** *Suppose that the capsule of* $\mathcal{D}^{\vdash}$*-prime-strips* $\{{}^{\alpha}\mathfrak{D}^{\vdash}\}_{\alpha\in A}$ *arises from the capsule of* $\mathcal{F}$*-prime-strips* $\{{}^{\alpha}\mathfrak{F}\}_{\alpha\in A}$ *of Proposition 3.1; write* $\{{}^{\alpha}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}\}_{\alpha\in A}$ *for the capsule of* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips associated to* $\{{}^{\alpha}\mathfrak{F}\}_{\alpha\in A}$*. Then the poly-isomorphisms "*$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^{\dagger}\mathcal{F}_{\underline{v}})$*" of Proposition 1.2, (vi), (vii), induce* **natural poly-isomorphisms of ind-topological modules**

$$\underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{v_{\mathbb{Q}}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}); \quad \underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^{A}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{v_{\mathbb{Q}}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$$

$$\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \xrightarrow{\sim} \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$$

between the various **"mono-analytic" tensor packets** *of the present Proposition 3.2 and the* **"holomorphic" tensor packets** *of Proposition 3.1.*

(ii) **(Integral Structures)** *If* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}^{\text{non}}_{\mathbb{Q}}$*, then the* **mono-analytic log-shells** *"*$\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}}$*" of Proposition 1.2, (vi), determine [i.e., by forming suitable direct sums and tensor products] topological submodules*

$$\mathcal{I}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}); \qquad \mathcal{I}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}); \qquad \mathcal{I}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \subseteq \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$$

— which may be regarded as **integral structures** *on the* $\mathbb{Q}$*-spans of these submodules. If* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}^{\text{arc}}_{\mathbb{Q}}$*, then by regarding the* **mono-analytic log-shell** *"*$\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}}$*" of Proposition 1.2, (vii), as the "closed unit ball" of a Hermitian metric on "*$\underline{\mathfrak{log}}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$*", and considering the induced direct sum Hermitian metric on* $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*,*

together with the induced tensor product Hermitian metric on $\underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*, one obtains Hermitian metrics on* $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*,* $\underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*, and* $\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$*, whose associated* **closed unit balls**

$$\mathcal{I}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}); \qquad \mathcal{I}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}); \qquad \mathcal{I}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}}) \subseteq \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$$

may be regarded as **integral structures** *on* $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*,* $\underline{\mathfrak{log}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$*, and* $\underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$*, respectively. For arbitrary* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$*, we shall denote by "*$\mathcal{I}^{\mathbb{Q}}((-))$*" the* $\mathbb{Q}$**-span** *of "*$\mathcal{I}((-))$*"; also, we shall apply this notation involving "*$\mathcal{I}((-))$*", "*$\mathcal{I}^{\mathbb{Q}}((-))$*" with "*$\mathcal{D}^{\vdash}$*"* **replaced** *by "*$\mathcal{F}$*" or "*$\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*" for the various objects obtained from the "*$\mathcal{D}^{\vdash}$*-versions" discussed above by applying the* **natural poly-isomorphisms** *of (i).*

Proof. The various assertions of Proposition 3.2 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 3.2.1. The issue of **estimating the discrepancy** between the **holomorphic** *integral structures* of Proposition 3.1, (ii), and the **mono-analytic** *integral structures* of Proposition 3.2, (ii), will form one of the main topics to be discussed in [IUTchIV] — cf. also Remark 3.9.1 below.

Remark 3.2.2. The constructions involving **local mono-analytic tensor packets** given in Proposition 3.2 may be applied to the capsules that appear in the various $\mathcal{D}^{\vdash}$**-prime-strip processions** — i.e., **mono-analytic processions** — discussed in [IUTchI], Proposition 4.11, (ii); [IUTchI], Proposition 6.9, (ii).

Proposition 3.3. **(Global Tensor Packets)** *Let*

$${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

be a $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater** *[relative to the given initial* Θ*-data] — cf. [IUTchI], Definition 6.13, (i). Thus,* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *determines* Θ*NF- and* $\Theta^{\pm\mathrm{ell}}$*-Hodge theaters* ${}^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}}$*,* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$ *as in [IUTchII], Corollary 4.8. Let* $\{{}^{\alpha}\mathfrak{F}\}_{\alpha\in A}$ *be an n-capsule of* $\mathcal{F}$*-prime-strips as in Proposition 3.1. Suppose, further, that A is a* **subset** *of the index set J that appears in the* Θ*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}}$*, and that, for each* $\alpha \in A$*, we are given a* $\mathfrak{log}$**-link**

$${}^{\alpha}\mathfrak{F} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathfrak{F}_{\alpha}$$

— i.e., a poly-isomorphism of $\mathcal{F}$*-prime-strips* $\mathfrak{log}({}^{\alpha}\mathfrak{F}) \overset{\sim}{\to} {}^{\dagger}\mathfrak{F}_{\alpha}$ *[cf. Definition 1.1, (iii)]. Next, recall the field* ${}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}$ *discussed in [IUTchII], Corollary 4.8, (i); thus, one also has, for* $j \in J$*, a labeled version* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_j$ *of this field [cf. [IUTchII], Corollary 4.8, (ii)]. We shall refer to*

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_A \overset{\mathrm{def}}{=} \bigotimes_{\alpha\in A} ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha}$$

— *where the tensor product is to be understood as a tensor product of modules — as the* **global [n-]tensor packet** *associated to the subset $A \subseteq J$ and the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$.*

(i) **(Ring Structures)** *The* **field structure** *on the various $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha}$, for $\alpha \in A$, determine a* **ring structure** *on $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{A}$ with respect to which $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{A}$ decomposes, uniquely, as a* **direct sum of number fields**. *Moreover, the various* **localization functors** *"$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{j} \quad \to \quad {}^{\dagger}\mathfrak{F}_{j}$" considered in [IUTchII], Corollary 4.8, (iii), determine, by composing with the given* **$\mathfrak{log}$-links**, *a* **natural injective localization ring homomorphism**

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{A} \quad \to \quad \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) \ \stackrel{\mathrm{def}}{=} \ \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$$

to the product of the local holomorphic tensor packets considered in Proposition 3.1.

(ii) **(Integral Structures)** *Fix an element $\alpha \in A$. Then by forming the tensor product with "1's" in the factors labeled by $\beta \in A \setminus \{\alpha\}$, one obtains a* **natural ring homomorphism**

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha} \ \to \ ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{A}$$

that induces an **isomorphism** *of the domain onto a subfield of each of the direct summand number fields of the codomain. For each $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, this homomorphism is* **compatible**, *in the evident sense, relative to the* **localization** *homomorphism of (i), with the natural homomorphism of ind-topological rings considered in Proposition 3.1, (ii). Moreover, for each $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$, the composite of the above displayed homomorphism with the component at $v_{\mathbb{Q}}$ of the localization homomorphism of (i) maps the ring of integers of the number field $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha}$ into the submodule constituted by the* **integral structure** *on $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$ considered in Proposition 3.1, (ii); for each $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$, the composite of the above displayed homomorphism with the component at $v_{\mathbb{Q}}$ of the localization homomorphism of (i) maps the set of archimedean integers [i.e., elements of absolute value ≤ 1 at all archimedean primes] of the number field $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha}$ into the direct product of subsets constituted by the* **integral structures** *considered in Proposition 3.1, (ii), on the various direct summand ind-topological fields of $\underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$.*

Proof. The various assertions of Proposition 3.3 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 3.3.1. One may perform analogous constructions to the constructions of Proposition 3.3 for the fields "$\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_{j}$" of [IUTchII], Corollary 4.7, (ii) [cf. also the localization functors of [IUTchII], Corollary 4.7, (iii)], constructed from the associated *$\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$. These constructions are *compatible* with the corresponding constructions of Proposition 3.3, in the evident sense, relative to the various *labeled Kummer-theoretic isomorphisms* of [IUTchII], Corollary 4.8, (ii). We leave the routine details to the reader.

Remark 3.3.2.

(i) One may consider the **image** of the **localization** homomorphism of Proposition 3.3, (i), in the case of the various **local holomorphic tensor packets** arising from **processions**, as discussed in Remark 3.1.2. Indeed, at the level of the *labels* involved, this is immediate in the case of the *"$\mathbb{F}_l^{\circledast}$-processions"* of [IUTchI], Proposition 4.11, (i). On the other hand, in the case of the *"$|\mathbb{F}_l|$-processions"* of [IUTchI], Proposition 6.9, (i), this may be achieved by applying the *identifying isomorphisms* between the *zero label* $0 \in |\mathbb{F}_l|$ and the *diagonal label* $\langle \mathbb{F}_l^{\circledast} \rangle$ *associated to* $\mathbb{F}_l^{\circledast}$ discussed in [the final display of] [IUTchII], Corollary 4.6, (iii) [cf. also [IUTchII], Corollary 4.8, (ii)].

(ii) In a similar vein, one may *compose* the *"$\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater version"* discussed in Remark 3.3.1 of the **localization** homomorphism of Proposition 3.3, (i), with the product over $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ of the inverses of the upper right-hand displayed isomorphisms at $v_{\mathbb{Q}}$ of Proposition 3.2, (i), and then consider the **image** of this composite morphism in the case of the various **local mono-analytic tensor packets** arising from **processions**, as discussed in Remark 3.2.2. Just as in the holomorphic case discussed in (i), in the case of the *"$|\mathbb{F}_l|$-processions"* of [IUTchI], Proposition 6.9, (ii), this obliges one to apply the *identifying isomorphisms* between the *zero label* $0 \in |\mathbb{F}_l|$ and the *diagonal label* $\langle \mathbb{F}_l^{\circledast} \rangle$ *associated to* $\mathbb{F}_l^{\circledast}$ discussed in [the final display of] [IUTchII], Corollary 4.5, (iii).

(iii) The various *images of global tensor packets* discussed in (i) and (ii) above may be *identified* — i.e., in light of the *injectivity* of the homomorphisms applied to construct these images — with the *global tensor packets* themselves. These **local holomorphic/local mono-analytic global tensor packet images** will play a *central role* in the development of the theory of the present §3 [cf., e.g., Proposition 3.7, below].

Remark 3.3.3. The $\mathfrak{log}$**-shifted** nature of the **localization** homomorphism of Proposition 3.3, (i), will play a crucial role in the development of the theory of present §3 — cf. the discussion of [IUTchII], Remark 4.8.2, (i), (iii).

$$
\begin{array}{ccccccccccc}
 & & \underline{\underline{q}}^{1} \curvearrowright & & & & \underline{\underline{q}}^{j^2} \curvearrowright & & & & \underline{\underline{q}}^{(l^{\circledast})^2} \curvearrowright \\
/^{\pm} & \hookrightarrow & /^{\pm}/^{\pm} & \hookrightarrow & \dots & \hookrightarrow & /^{\pm}/^{\pm} \ \dots \ /^{\pm} & \hookrightarrow & \dots & \hookrightarrow & /^{\pm}/^{\pm} \ \dots \ \dots \ /^{\pm} \\
\mathbb{S}^{\pm}_{1} & & \mathbb{S}^{\pm}_{1+1=2} & & & & \mathbb{S}^{\pm}_{j+1} & & & & \mathbb{S}^{\pm}_{1+l^{\circledast}=l^{\pm}}
\end{array}
$$

Fig. 3.1: Splitting monoids of LGP-monoids acting on tensor packets

Proposition 3.4. (Local Packet-theoretic Frobenioids)

(i) **(Single Packet Monoids)** *In the situation of Proposition 3.1, fix elements $\alpha \in A$, $\underline{v} \in \underline{\mathbb{V}}$, $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ such that $\underline{v} \mid v_{\mathbb{Q}}$. Then the operation of forming the* **image** *via the natural homomorphism $\underline{\mathfrak{log}}({}^{\alpha}\mathcal{F}_{\underline{v}}) \to \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ [cf. Proposition 3.1, (ii)]*

of the monoid $\Psi_{\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})}$ *[cf. the notation of Definition 1.1, (i), (ii)], together with its submonoid of units* $\Psi^{\times}_{\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})}$ *and realification* $\Psi^{\mathbb{R}}_{\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})}$, *determines monoids*

$$\Psi_{\mathfrak{log}(^{A,\alpha}\mathcal{F}_{\underline{v}})}, \quad \Psi^{\times}_{\mathfrak{log}(^{A,\alpha}\mathcal{F}_{\underline{v}})}, \quad \Psi^{\mathbb{R}}_{\mathfrak{log}(^{A,\alpha}\mathcal{F}_{\underline{v}})}$$

— which are equipped with $G_{\underline{v}}(^{\alpha}\Pi_{\underline{v}})$**-actions** *when* $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$ *and, in the case of the first displayed monoid, with a pair consisting of an* Aut**-holomorphic orbispace** *and a* **Kummer structure** *when* $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$. *We shall think of these monoids as [possibly realified]* **subquotients** *of*

$$\underline{\mathfrak{log}}(^{A,\alpha}\mathcal{F}_{\underline{v}})$$

that **act** *[multiplicatively] on suitable [possibly realified] subquotients of* $\underline{\mathfrak{log}}(^{A,\alpha}\mathcal{F}_{\underline{v}})$. *In particular, when* $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, *the first displayed monoid, together with its* $^{\alpha}\Pi_{\underline{v}}$*-action, determine a* **Frobenioid** *equipped with a natural isomorphism to* $\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})$; *when* $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, *the first displayed monoid, together with its* Aut*-holomorphic orbispace and Kummer structure, determine a collection of data equipped with a natural isomorphism to* $\mathfrak{log}(^{\alpha}\mathcal{F}_{\underline{v}})$.

(ii) **(Local Logarithmic Gaussian Procession Monoids)** *Let*

$$^{\ddagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \quad \xrightarrow{\mathfrak{log}} \quad ^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$$

be a $\mathfrak{log}$**-link** *of* $\Theta^{\pm\text{ell}}$**NF-Hodge theaters** *as in Proposition 1.3, (i) [cf. also the situation of Proposition 3.3]. Consider the* $\mathcal{F}$**-prime-strip processions** *that arise as the* $\mathcal{F}$*-prime-strip analogues [cf. Remark 3.1.2; [IUTchI], Remark 6.12.1] of the* **holomorphic processions** *discussed in [IUTchI], Proposition 6.9, (i), when the functor of [IUTchI], Proposition 6.9, (i), is applied to the* $\Theta^{\pm}$**-bridges** *associated to* $^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$, $^{\ddagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$; *we shall refer to such processions as "†-" or "‡-" processions. Here, we recall that for* $j \in \{1, \ldots, l^{*}\}$, *the index set of the* $(j+1)$*-capsule that appears in such a procession is denoted* $\mathbb{S}^{\pm}_{j+1}$. *Then by applying the various constructions of* **"single packet monoids"** *given in (i) in the case of the various capsules of* $\mathcal{F}$*-prime-strips that appear in a holomorphic ‡-procession — i.e., more precisely, in the case of the* **label** $j \in \{1, \ldots, l^{*}\}$ *[which we shall occasionally identify with its image in* $\mathbb{F}^{*}_{l} \subseteq |\mathbb{F}_{l}|$*] that appears in the* $(j+1)$**-capsule** *of the ‡-procession — to the pull-backs, via the* **poly-isomorphisms** *that appear in the definition [cf. Definition 1.1, (iii)] of the given* $\mathfrak{log}$**-link**, *of the [collections of] monoids equipped with actions by* **topological groups** *when* $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$ *and* **splittings** *[up to torsion, when* $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$*]* $\Psi_{\mathcal{F}_{\text{gau}}}(^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$, $_{\infty}\Psi_{\mathcal{F}_{\text{gau}}}(^{\dagger}\mathcal{HT}^{\Theta})_{\underline{v}}$ *of [IUTchII], Corollary 4.6, (iv), for* $\underline{v} \in \underline{\mathbb{V}}$, *one obtains a* **functorial algorithm** *in the* $\mathfrak{log}$**-link of** $\Theta^{\pm\text{ell}}$**NF-Hodge theaters** $^{\ddagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}} \xrightarrow{\mathfrak{log}} {^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}}$ *for constructing [collections of] monoids equipped with actions by* **topological groups** *when* $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$ *and* **splittings** *[up to torsion, when* $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$*]*

$$\underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ \Psi_{\mathcal{F}_{\text{LGP}}}(^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}})_{\underline{v}}; \qquad \underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ _{\infty}\Psi_{\mathcal{F}_{\text{LGP}}}(^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}})_{\underline{v}}$$

— which we refer to as **"[local] LGP-monoids"**, *or "logarithmic Gaussian procession monoids" [cf. Fig. 3.1 above]. Here, we note that the notation "*$(^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}})$*"*

constitutes a slight abuse of notation. Also, we note that this functorial algorithm requires one to apply the **compatibility** *of the given* $\mathfrak{log}$*-link with the* $\mathbb{F}_l^{\rtimes\pm}$**-symmetrizing isomorphisms** *involved [cf. Remark 1.3.2]. For* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*, the component labeled* $j \in \{1, \ldots, l^*\}$ *of the submonoid of* **Galois invariants** *[cf. (i)] of the* **entire** *LGP-monoid* $\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}$ *is a subset of*

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})$$

[i.e., where the notation "$;\ddagger$*" denotes the result of applying the discussion of (i) to the case of* $\mathcal{F}$*-prime-strips labeled "*$\ddagger$*"; cf. also the notational conventions of Proposition 3.2, (ii)] that acts multiplicatively on* $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})$ *[cf. the constructions of [IUTchII], Corollary 3.6, (ii)]. For any* $\underline{v} \in \underline{\mathbb{V}}$*, the component labeled* $j \in \{1, \ldots, l^*\}$ *of the submodule of* **Galois invariants** *[cf. (i) when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*; this Galois action is trivial when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*] of the* **unit portion** $\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})^{\times}_{\underline{v}}$ *of such an LGP-monoid is a subset of*

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})$$

[cf. the discussion of (i); the notational conventions of Proposition 3.2, (ii)] that acts multiplicatively on $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}})$ *[cf. the constructions of [IUTchII], Corollary 3.6, (ii); [IUTchII], Proposition 4.2, (iv); [IUTchII], Proposition 4.4, (iv)].*

Proof. The various assertions of Proposition 3.4 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Proposition 3.5. (Kummer Theory and Upper Semi-compatibility for Vertically Coric Local LGP-Monoids) *Let* $\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n,m\in\mathbb{Z}}$ *be a* **collection of distinct** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *[relative to the given initial* Θ*-data] — which we think of as arising from a* **Gaussian log-theta-lattice** *[cf. Definition 1.4]. For each* $n \in \mathbb{Z}$*, write*

$${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

for the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater determined, up to isomorphism, by the various* ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*, where* $m \in \mathbb{Z}$*, via the* **vertical coricity** *of Theorem 1.5, (i).*

(i) **(Vertically Coric Local LGP-Monoids and Associated Kummer Theory)** *Write*

$$\mathfrak{F}({}^{n,\circ}\mathfrak{D}_{\succ})_t$$

for the $\mathcal{F}$*-prime-strip associated [cf. [IUTchII], Remark 4.5.1, (i)] to the labeled collection of monoids "*$\Psi_{\mathrm{cns}}({}^{n,\circ}\mathfrak{D}_{\succ})_t$*" of [IUTchII], Corollary 4.5, (iii) [i.e., where we take "*$\dagger$*" to be "*$n, \circ$*"]. Recall the constructions of Proposition 3.4, (ii), involving* $\mathcal{F}$*-prime-strip processions. Then by applying these constructions to the* $\mathcal{F}$*-prime-strips "*$\mathfrak{F}({}^{n,\circ}\mathfrak{D}_{\succ})_t$*" and the various full* $\mathfrak{log}$*-links associated [cf. the discussion of Proposition 1.2, (ix)] to these* $\mathcal{F}$*-prime-strips — which we consider in a fashion* **compatible** *with the* $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** *involved [cf. Remark 1.3.2; Proposition*

3.4, (ii)] — we obtain a **functorial algorithm** *in the* $\mathcal{D}$**-**$\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater** ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *for constructing [collections of] monoids*

$$\underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ \Psi_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}; \qquad \underline{\mathbb{V}} \ni \underline{v} \ \mapsto \ {}_{\infty}\Psi_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}$$

equipped with actions by **topological groups** *when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ *and* **splittings** *[up to torsion, when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*] — which we refer to as* **"vertically coric [local] LGP-monoids"***. For each* $n, m \in \mathbb{Z}$*, this functorial algorithm is* **compatible** *[in the evident sense] with the functorial algorithm of Proposition 3.4, (ii) — i.e., where we take "*$\dagger$*" to be "*n, m*" and "*$\ddagger$*" to be "*$n, m-1$*" — relative to the* **Kummer isomorphisms** *of labeled data*

$$\Psi_{\mathrm{cns}}({}^{n,m'}\mathfrak{F}_{\succ})_t \quad \stackrel{\sim}{\to} \quad \Psi_{\mathrm{cns}}({}^{n,\circ}\mathfrak{D}_{\succ})_t$$

of [IUTchII], Corollary 4.6, (iii), and the evident identification, for $m' = m, m-1$*, of* ${}^{n,m'}\mathfrak{F}_t$ *[i.e., the* $\mathcal{F}$*-prime-strip that appears in the associated* $\Theta^{\pm}$*-bridge] with the* $\mathcal{F}$*-prime-strip associated to* $\Psi_{\mathrm{cns}}({}^{n,m'}\mathfrak{F}_{\succ})_t$*. In particular, for each* $n, m \in \mathbb{Z}$*, we obtain* **Kummer isomorphisms** *of [collections of] monoids*

$$\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}} \ \stackrel{\sim}{\to} \ \Psi_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}$$

$${}_{\infty}\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}} \ \stackrel{\sim}{\to} \ {}_{\infty}\Psi_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}$$

equipped with actions by **topological groups** *when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ *and* **splittings** *[up to torsion, when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*], for* $\underline{v} \in \underline{\mathbb{V}}$*.*

(ii) **(Upper Semi-compatibility)** *The Kummer isomorphisms of the final two displays of (i) are* **"upper semi-compatible"** *— cf. the discussion of "upper semi-commutativity" in Remark 1.2.2, (iii) — with the various* $\mathfrak{log}$**-links of** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** ${}^{n,m-1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *[where* $m \in \mathbb{Z}$*] of the Gaussian log-theta-lattice under consideration in the following sense. Let* $j \in \{0, 1, \ldots, l^*\}$*. Then:*

(a) **(Nonarchimedean Primes)** *For* $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$*, the topological module*

$$\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{v_{\mathbb{Q}}})$$

— i.e., that arises from applying the constructions of Proposition 3.4, (ii) [where we allow "j*" to be 0], in the* **vertically coric** *context of (i) above [cf. also the notational conventions of Proposition 3.2, (ii)] —* **contains** *the images of the submodules of* **Galois invariants** *[where we recall the Galois actions that appear in the data of [IUTchII], Corollary 4.6, (i), (iii)] of the* **groups of units** $(\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})_{|t|})^{\times}_{\underline{v}}$*, for* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}$ *and* $|t| \in \{0, \ldots, j\}$*, via* **both**

(1) the tensor product, over such $|t|$*, of the [relevant]* **Kummer isomorphisms** *of (i),* **and**

(2) the tensor product, over such $|t|$*, of the pre-composite of these Kummer isomorphisms with the* m'*-th* **iterates** *[cf. Remark*

1.1.1] of the $\mathfrak{log}$**-links**, *for* $m' \geq 1$, *of the* n*-th column of the Gaussian log-theta-lattice under consideration [cf. the discussion of Remark 1.2.2, (i), (iii)].*

(b) **(Archimedean Primes)** *For* $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$, *the closed unit ball*

$$\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{v_{\mathbb{Q}}})$$

— i.e., that arises from applying the constructions of Proposition 3.4, (ii) [where we allow "j" to be 0], in the **vertically coric** *context of (i) above [cf. also the notational conventions of Proposition 3.2, (ii)] —* **contains** *the image, via the tensor product, over* $|t| \in \{0, \ldots, j\}$, *of the [relevant]* **Kummer isomorphisms** *of (i), of* **both**

(1) the **groups of units** $(\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})_{|t|})^{\times}_{\underline{v}}$, *for* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}$, **and**

(2) the **closed balls of radius** π *inside* $(\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})_{|t|})^{\mathrm{gp}}_{\underline{v}}$ *[cf. the notational conventions of Definition 1.1], for* $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}$.

Here, we recall from the discussion of Remark 1.2.2, (ii), (iii), that, if we regard each $\mathfrak{log}$*-link as a correspondence that only concerns the* **units** *that appear in its domain [cf. Remark 1.1.1], then a closed ball as in (2) contains, for each* $m' \geq 1$, *a subset that* **surjects**, *via the* m'*-th* **iterate** *of the* $\mathfrak{log}$**-link** *of the* n*-th column of the Gaussian log-theta-lattice under consideration, onto the subset of the group of units* $(\Psi_{\mathrm{cns}}({}^{n,m-m'}\mathfrak{F}_{\succ})_{|t|})^{\times}_{\underline{v}}$ *on which this iterate is defined.*

(c) **(Bad Primes)** *Let* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$; *suppose that* $j \neq 0$. *Recall that the various monoids "*$\Psi_{\mathcal{F}_{\mathrm{LGP}}}(-)_{\underline{v}}$*", "*${}_{\infty}\Psi_{\mathcal{F}_{\mathrm{LGP}}}(-)_{\underline{v}}$*" constructed in Proposition 3.4, (ii), as well as the monoids "*$\Psi_{\mathrm{LGP}}(-)_{\underline{v}}$*", "*${}_{\infty}\Psi_{\mathrm{LGP}}(-)_{\underline{v}}$*" constructed in (i) above, are equipped with natural* **splittings up to torsion**. *Write*

$$\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}(-)_{\underline{v}} \subseteq \Psi_{\mathcal{F}_{\mathrm{LGP}}}(-)_{\underline{v}}; \quad {}_{\infty}\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}(-)_{\underline{v}} \subseteq {}_{\infty}\Psi_{\mathcal{F}_{\mathrm{LGP}}}(-)_{\underline{v}}$$

$$\Psi^{\perp}_{\mathrm{LGP}}(-)_{\underline{v}} \subseteq \Psi_{\mathrm{LGP}}(-)_{\underline{v}}; \quad {}_{\infty}\Psi^{\perp}_{\mathrm{LGP}}(-)_{\underline{v}} \subseteq {}_{\infty}\Psi_{\mathrm{LGP}}(-)_{\underline{v}}$$

for the submonoids corresponding to these splittings [cf. the submonoids "$\mathcal{O}^{\perp}(-) \subseteq \mathcal{O}^{\triangleright}(-)$*" discussed in Definition 2.4, (i), in the case of "*$\Psi^{\perp}$*"; the notational conventions of Theorem 2.2, (ii), in the case of "*${}_{\infty}\Psi^{\perp}$*"]. [Thus, the subgroup of units of "*$\Psi^{\perp}$*" consists of the* $2l$*-torsion subgroup of "*Ψ*", while the subgroup of units of "*${}_{\infty}\Psi^{\perp}$*" contains the entire torsion subgroup of "*${}_{\infty}\Psi$*".] Then, as* m **ranges** *over the elements of* $\mathbb{Z}$, *the actions, via the [relevant]* **Kummer isomorphisms** *of (i), of the various monoids* $\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}$ $(\subseteq {}_{\infty}\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}})$ *on the ind-topological modules*

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{\underline{v}}) \subseteq \underline{\mathfrak{log}}({}^{\mathbb{S}^{\pm}_{j+1},j}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{\underline{v}})$$

[where $j = 1, \ldots, l^{*}$*] — i.e., that arise from applying the constructions of Proposition 3.4, (ii), in the* **vertically coric** *context of (i) above [cf. also the notational conventions of Proposition 3.2, (ii)] — are* **mutually**

compatible, *relative to the* $\mathfrak{log}$**-links** *of the* n*-th column of the Gaussian log-theta-lattice under consideration, in the sense that the only portions of these actions that are possibly related to one another via these* $\mathfrak{log}$*-links are the* **indeterminacies** *with respect to* **multiplication by roots of unity** *in the domains of the* $\mathfrak{log}$*-links, that is to say, indeterminacies at* m *that correspond, via the* $\mathfrak{log}$*-link, to* **"addition by zero"** *— i.e., to* **no indeterminacy**! *— at* $m+1$.

Now let us think of the submodules of **Galois invariants** *[cf. the discussion of Proposition 3.4, (ii)] of the various* **groups of units**, *for* $\underline{v} \in \underline{\mathbb{V}}$,

$$(\Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})_{|t|})^{\times}_{\underline{v}}, \quad \Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})^{\times}_{\underline{v}}$$

and the **splitting monoids**, *for* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$,

$$\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}$$

as acting on various portions of the modules, for $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$,

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{v_{\mathbb{Q}}})$$

not *via a* **single Kummer isomorphism** *as in (i) — which* **fails to be compatible** *with the* $\mathfrak{log}$**-links** *of the Gaussian log-theta-lattice! — but rather via the* **totality** *of the various pre-composites of Kummer isomorphisms with* **iterates** *[cf. Remark 1.1.1] of the* $\mathfrak{log}$**-links** *of the Gaussian log-theta-lattice — i.e., precisely as was described in detail in (a), (b), (c) above [cf. also the discussion of Remark 3.11.4 below]. Thus, one obtains a sort of* **"**$\mathfrak{log}$**-Kummer correspondence"** *between the* **totality**, *as* m *ranges over the elements of* $\mathbb{Z}$, *of the various groups of units and splitting monoids just discussed [i.e., which are labeled by "*n, m*"] and their* **actions** *[as just described] on the "*$\mathcal{I}^{\mathbb{Q}}$*" labeled by "*$n, \circ$*" which is* **invariant** *with respect to the* **translation symmetries** *[cf. Proposition 1.3, (iv)] of the* n*-th column of the Gaussian log-theta-lattice [cf. the discussion of Remark 1.2.2, (iii)].*

Proof. The various assertions of Proposition 3.5 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Example 3.6. Concrete Representations of Global Frobenioids. Before proceeding, we pause to take a closer look at the Frobenioid "${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}$" of [IUTchI], Example 5.1, (iii), i.e., more concretely speaking, the Frobenioid of *arithmetic line bundles on the stack* "S_{mod}" of [IUTchI], Remark 3.1.5. Let us write

$$\mathcal{F}^{\circledast}_{\mathrm{mod}}$$

for the Frobenioid "${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}}$" of [IUTchI], Example 5.1, (iii), in the case where the data denoted by the label "†" arises [in the evident sense] from data as discussed in [IUTchI], Definition 3.1. In the following discussion, we shall use the notation of [IUTchI], Definition 3.1.

(i) **(Rational Function Torsor Version)** For each $\underline{v} \in \underline{\mathbb{V}}$, the valuation on $K_{\underline{v}}$ determined by $\underline{v}$ determines a *group homomorphism* $\beta_{\underline{v}} : F^{\times}_{\mathrm{mod}} \to K^{\times}_{\underline{v}}/\mathcal{O}^{\times}_{K_{\underline{v}}}$ [cf. Remark 3.6.1 below]. Then let us define a *category* $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ as follows. An *object* $\mathcal{T} = (T, \{t_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$ of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ consists of a collection of data

(a) an $F^{\times}_{\mathrm{mod}}$*-torsor* T;

(b) for each $\underline{v} \in \underline{\mathbb{V}}$, a *trivialization* $t_{\underline{v}}$ of the torsor $T_{\underline{v}}$ obtained from T by executing the "change of structure group" operation determined by the homomorphism $\beta_{\underline{v}}$

subject to the condition that there exists an element $t \in T$ such that $t_{\underline{v}}$ *coincides* with the trivialization of $T_{\underline{v}}$ determined by t *for all but finitely many* $\underline{v}$. An *elementary morphism* $\mathcal{T}_1 = (T_1, \{t_{1,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) \to \mathcal{T}_2 = (T_2, \{t_{2,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$ between objects of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ is defined to be an isomorphism $T_1 \xrightarrow{\sim} T_2$ of $F^{\times}_{\mathrm{mod}}$-torsors which is *integral* at each $\underline{v} \in \underline{\mathbb{V}}$, i.e., maps the trivialization $t_{1,\underline{v}}$ to an element of the $\mathcal{O}^{\triangleright}_{K_{\underline{v}}}$*-orbit* of $t_{2,\underline{v}}$. There is an evident notion of *composition of elementary morphisms*, as well as an evident notion of *tensor powers* $\mathcal{T}^{\otimes n}$, for $n \in \mathbb{Z}$, of an object $\mathcal{T}$ of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$. A *morphism* $\mathcal{T}_1 = (T_1, \{t_{1,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) \to \mathcal{T}_2 = (T_2, \{t_{2,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$ between objects of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ is defined to consist of a positive integer n and an elementary morphism $(\mathcal{T}_1)^{\otimes n} \to \mathcal{T}_2$. There is an evident notion of *composition of morphisms.* Thus, $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ forms a *category.* In fact, one verifies immediately that, from the point of view of the theory of Frobenioids developed in [FrdI], [FrdII], $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ admits a natural *Frobenioid* structure [cf. [FrdI], Definition 1.3], for which the *base category* is the category with precisely one arrow. Relative to this Frobenioid structure, the elementary morphisms are precisely the *linear morphisms*, and the positive integer "n" that appears in the definition of a morphism of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ is the *Frobenius degree* of the morphism. Moreover, by associating to an arithmetic line bundle on S_{mod} the $F^{\times}_{\mathrm{mod}}$-torsor determined by restricting the line bundle to the generic point of S_{mod} and the local trivializations at $\underline{v} \in \underline{\mathbb{V}}$ determined by the various local integral structures, one verifies immediately that there exists a *natural isomorphism of Frobenioids*

$$\mathcal{F}^{\circledast}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathrm{MOD}}$$

that induces the *identity morphism* $F^{\times}_{\mathrm{mod}} \to F^{\times}_{\mathrm{mod}}$ on the associated *rational function monoids* [cf. [FrdI], Corollary 4.10].

(ii) **(Local Fractional Ideal Version)** Let us define a *category* $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ as follows. An *object*

$$\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ consists of a collection of *"fractional ideals"* $J_{\underline{v}} \subseteq K_{\underline{v}}$ for each $\underline{v} \in \underline{\mathbb{V}}$ — i.e., a finitely generated nonzero $\mathcal{O}_{K_{\underline{v}}}$-submodule of $K_{\underline{v}}$ when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$; a positive real multiple of $\mathcal{O}_{K_{\underline{v}}} \stackrel{\mathrm{def}}{=} \{\lambda \in K_{\underline{v}} \mid |\lambda| \leq 1\} \subseteq K_{\underline{v}}$ when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ — such that $J_{\underline{v}} = \mathcal{O}_{K_{\underline{v}}}$ *for all but finitely many* $\underline{v}$. If $\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ is an object of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$, then for any element $f \in F^{\times}_{\mathrm{mod}}$, one obtains an object $f \cdot \mathcal{J} = \{f \cdot J_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ by multiplying each of the fractional ideals $J_{\underline{v}}$ by f. Moreover, if $\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ is an object of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$, then for any $n \in \mathbb{Z}$, there is an evident notion of the n-th tensor power $\mathcal{J}^{\otimes n}$ of $\mathcal{J}$. An *elementary morphism* $\mathcal{J}_1 = \{J_{1,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \to \mathcal{J}_2 = \{J_{2,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ between objects of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ is defined to be an element $f \in F^{\times}_{\text{mod}}$ that is *integral* with respect to $\mathcal{J}_1$ and $\mathcal{J}_2$ in the sense that $f \cdot J_{1,\underline{v}} \subseteq J_{2,\underline{v}}$ for each $\underline{v} \in \underline{\mathbb{V}}$. There is an evident notion of *composition of elementary morphisms*. A *morphism* $\mathcal{J}_1 = \{J_{1,\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}} \to \mathcal{J}_2 = \{J_{2,\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ between objects of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ is defined to consist of a positive integer n and an elementary morphism $(\mathcal{J}_1)^{\otimes n} \to \mathcal{J}_2$. There is an evident notion of *composition of morphisms*. Thus, $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ forms a *category*. In fact, one verifies immediately that, from the point of view of the theory of Frobenioids developed in [FrdI], [FrdII], $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ admits a natural *Frobenioid* structure [cf. [FrdI], Definition 1.3], for which the *base category* is the category with precisely one arrow. Relative to this Frobenioid structure, the elementary morphisms are precisely the *linear morphisms*, and the positive integer "n" that appears in the definition of a morphism of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ is the *Frobenius degree* of the morphism. Moreover, by associating to an object $\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ the arithmetic line bundle on S_{mod} obtained from the trivial arithmetic line bundle on S_{mod} by modifying the integral structure of the trivial line bundle at $\underline{v} \in \underline{\mathbb{V}}$ in the fashion prescribed by $J_{\underline{v}}$, one verifies immediately that there exists a *natural isomorphism of Frobenioids*

$$\mathcal{F}^{\circledast}_{\mathfrak{mod}} \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\text{mod}}$$

that induces the *identity morphism* $F^{\times}_{\text{mod}} \to F^{\times}_{\text{mod}}$ on the associated *rational function monoids* [cf. [FrdI], Corollary 4.10].

(iii) By composing the isomorphisms of Frobenioids of (i) and (ii), one thus obtains a *natural isomorphism of Frobenioids*

$$\mathcal{F}^{\circledast}_{\mathfrak{mod}} \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\text{MOD}}$$

that induces the *identity morphism* $F^{\times}_{\text{mod}} \to F^{\times}_{\text{mod}}$ on the associated *rational function monoids* [cf. [FrdI], Corollary 4.10]. One verifies immediately that although the above isomorphism of Frobenioids is not necessarily determined by the condition that it induce the identity morphism on $F^{\times}_{\text{mod}}$, the induced isomorphism between the respective *perfections* [hence also on *realifications*] of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$, $\mathcal{F}^{\circledast}_{\text{MOD}}$ is *completely determined* by this condition.

Remark 3.6.1. Note that, as far as the various constructions of Example 3.6, (i), are concerned, the various homomorphisms $\beta_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}$, may be thought of, alternatively, as a collection of

$$\textit{subquotients of the perfection } (F^{\times}_{\text{mod}})^{\text{pf}} \text{ of } F^{\times}_{\text{mod}}$$

— each of which is equipped with a *submonoid of "nonnegative elements"* — that are *completely determined* by the *ring structure* of the field F_{mod} [i.e., equipped with its structure as the *field of moduli* of X_F].

Remark 3.6.2.

(i) In the theory to be developed below, we shall be interested in relating certain Frobenioids — which will, in fact, be isomorphic to the *realification* of $\mathcal{F}^{\circledast}_{\text{mod}}$ — that lie on opposite sides of [a certain enhanced version of] the $\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}$**-link** to one another. In particular, at the level of objects of the Frobenioids involved, it only makes sense to work with **isomorphism classes** of objects that are preserved by the isomorphisms of Frobenioids that appear. Here, we note that the isomorphism classes of the sort of Frobenioids that appear in this context are determined by the **divisor** and **rational function monoids** of the [model] Frobenioid in question [cf. the constructions given in [FrdI], Theorem 5.2, (i), (ii)]. In this context, we observe that the rational function monoid $F^{\times}_{\mathrm{mod}}$ of $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ satisfies the following *fundamental property*:

[the union with $\{0\}$ of] $F^{\times}_{\mathrm{mod}}$ admits a natural **additive structure**.

In this context, we note that this property is *not* satisfied by

(a) the rational function monoids of the *perfection* or *realification* of $\mathcal{F}^{\circledast}_{\mathrm{mod}}$

(b) subgroups $\Gamma \subseteq F^{\times}_{\mathrm{mod}}$ — such as, for instance, the trivial subgroup $\{1\}$ or the subgroup of *S-units*, for $S \subseteq \mathbb{V}_{\mathrm{mod}}$ a nonempty finite subset — that do not arise as the multiplicative group of some subfield of F_{mod} [cf. [AbsTopIII], Remark 5.10.2, (iv)].

The significance of this fundamental property is that it allows one to represent the objects of $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ **additively**, i.e., as *modules* — cf. the point of view of Example 3.6, (ii). At a more concrete level, if, in the notation of (b), one considers the result of *"adding"* two elements of a Γ-*torsor* [cf. the point of view of Example 3.6, (i)!], then the resulting "sum" can only be rendered meaningful, relative to the given Γ-torsor, if Γ is *additively closed*. The **additive representation** of objects of $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ will be of *crucial importance* in the theory of the present series of papers since it will allow us to relate objects of $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ on opposite sides of [a certain enhanced version of] the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{gau}}$-link to one another — which, a priori, are only related to one another at the level of *realifications* in a **multiplicative** fashion — by means of **["additive"] mono-analytic log-shells** [cf. the discussion of [IUTchII], Remark 4.7.2].

(ii) One way to understand the content of the discussion of (i) is as follows: whereas

the construction of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ depends on the **additive** structure of $F^{\times}_{\mathrm{mod}}$

in an essential way,

the construction of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ is strictly **multiplicative** in nature.

Indeed, the construction of $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ given in Example 3.6, (i), is essentially the same as the construction of $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ given in [FrdI], Example 6.3 [i.e., in effect, in [FrdI], Theorem 5.2, (i)]. From this point of view, it is natural to **identify** $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ with $\mathcal{F}^{\circledast}_{\mathrm{mod}}$ via the natural isomorphism of Frobenioids of Example 3.6, (i). We shall often do this in the theory to be developed below.

Proposition 3.7. (Global Packet-theoretic Frobenioids)

(i) **(Single Packet Rational Function Torsor Version)** *In the notation of Proposition 3.3: For each $\alpha \in A$, there is an* **algorithm** *for constructing, as discussed in Example 3.6, (i) [cf. also Remark 3.6.1], from the [number] field given by the* **image**

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_{\alpha}$$

of the composite

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{\alpha} \;\to\; ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_{A} \;\to\; \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})$$

of the homomorphisms of Proposition 3.3, (i), (ii), a **Frobenioid** $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$, *together with a* **natural isomorphism of Frobenioids**

$$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha} \;\xrightarrow{\sim}\; ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$$

[cf. the notation of [IUTchII], Corollary 4.8, (ii)] that induces the **tautological** *isomorphism* $({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_{\alpha}$ *on the associated rational function monoids [cf. Example 3.6, (i)]. We shall often use this isomorphism of Frobenioids to* **identify** $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha}$ *with* $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$ *[cf. Remark 3.6.2, (ii)]. Write* $({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_{\alpha}$ *for the* **realification** *of* $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$.

(ii) **(Single Packet Local Fractional Ideal Version)** *In the notation of Propositions 3.3, 3.4: For each* $\alpha \in A$, *there is an* **algorithm** *for constructing, as discussed in Example 3.6, (ii), from the [number] field* $({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_{\alpha} \stackrel{\mathrm{def}}{=} ({}^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_{\alpha}$ *[cf. (i)] and the* **Galois invariants** *of the local monoids*

$$\Psi_{\mathfrak{log}({}^{A,\alpha}\mathcal{F}_{\underline{v}})} \;\subseteq\; \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$$

for $\underline{v} \in \underline{\mathbb{V}}$ *of Proposition 3.4, (i) — i.e., so the corresponding local "fractional ideal* $J_{\underline{v}}$*" of Example 3.6, (ii), is a* **subset** *[indeed a submodule when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*] of* $\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ *whose* $\mathbb{Q}$*-span is equal to* $\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ *[cf. the notational conventions of Proposition 3.2, (ii)] — a* **Frobenioid** $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha}$, *together with* **natural isomorphisms of Frobenioids**

$$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha} \;\xrightarrow{\sim}\; ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha}; \quad ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha} \;\xrightarrow{\sim}\; ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_{\alpha}$$

that induce the **tautological** *isomorphisms* $({}^{\dagger}\mathbb{M}^{\circledast}_{\mathfrak{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{mod}})_{\alpha}$, $({}^{\dagger}\mathbb{M}^{\circledast}_{\mathfrak{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_{\alpha}$ *on the associated rational function monoids [cf. the natural isomorphism of Frobenioids of (i); Example 3.6, (ii), (iii)]. Write* $({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_{\alpha}$ *for the* **realification** *of* $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha}$.

(iii) **(Global Realified** LGP**-Frobenioids)** *In the notation of Proposition 3.4: By applying the composites of the isomorphisms of Frobenioids "*${}^{\dagger}\mathcal{C}^{\Vdash}_{j} \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_{j}$*" of [IUTchII], Corollary 4.8, (iii), with the realifications "*$({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_{\alpha} \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_{\alpha}$*" of the isomorphisms of Frobenioids of (i) above to the global realified Frobenioid portion* ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{gau}}$ *of the* $\mathcal{F}^{\Vdash}$*-prime-strip* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$ *of [IUTchII], Corollary 4.10, (ii) [cf. Remarks 1.5.3, (iii); 3.3.2, (i)], one obtains a* **functorial algorithm** *in the* **$\mathfrak{log}$-link of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *of Proposition 3.4, (ii), for constructing a Frobenioid*

$$\mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$$

— which we refer to as a **"global realified LGP-Frobenioid"**. *Here, we note that the notation "*$({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$*" constitutes a slight abuse of notation. In particular, the global realified Frobenioid* ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}} \stackrel{\mathrm{def}}{=} \mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$, *together with*

the collection of data $\Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$ *constructed in Proposition 3.4, (ii), give rise, in a natural fashion, to an* $\mathcal{F}^{\Vdash}$**-prime-strip**

$$ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}} \ = \ ({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}}, \ \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \ {}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{LGP}}, \ \{{}^{\dagger}\rho_{\mathrm{LGP},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) $$

— cf. the construction of the $\mathcal{F}^{\Vdash}$*-prime-strip* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$ *in [IUTchII], Corollary 4.10, (ii) — together with a* **natural isomorphism**

$$ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}} \quad \xrightarrow{\sim} \quad {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}} $$

of $\mathcal{F}^{\Vdash}$*-prime-strips [i.e., that arises* **tautologically** *from the construction of* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$*!].*

(iv) **(Global Realified $\mathfrak{lgp}$-Frobenioids)** *In the situation of (iii) above, write* $\Psi_{\mathcal{F}_{\mathfrak{lgp}}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}) \stackrel{\mathrm{def}}{=} \Psi_{\mathcal{F}_{\mathrm{LGP}}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$, ${}^{\dagger}\mathfrak{F}^{\vdash}_{\mathfrak{lgp}} \stackrel{\mathrm{def}}{=} {}^{\dagger}\mathfrak{F}^{\vdash}_{\mathrm{LGP}}$*. Then by replacing, in the construction of (iii), the isomorphisms "*$({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_\alpha \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_\alpha$*" by the natural isomorphisms "*$({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}})_\alpha \xrightarrow{\sim} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_\alpha$*" [cf. (ii)], one obtains a* **functorial algorithm** *in the* **$\mathfrak{log}$-link of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *of Proposition 3.4, (ii), for constructing a Frobenioid*

$$ \mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}) $$

— which we refer to as a **"global realified $\mathfrak{lgp}$-Frobenioid"** *— as well as an* $\mathcal{F}^{\Vdash}$**-prime-strip**

$$ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}} \ = \ ({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}, \ \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \ {}^{\dagger}\mathfrak{F}^{\vdash}_{\mathfrak{lgp}}, \ \{{}^{\dagger}\rho_{\mathfrak{lgp},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) $$

— where we write ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}} \stackrel{\mathrm{def}}{=} \mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$ *— together with* **tautological isomorphisms**

$$ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}} \quad \xrightarrow{\sim} \quad {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}} \quad \xrightarrow{\sim} \quad {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}} $$

of $\mathcal{F}^{\Vdash}$*-prime-strips [cf. (iii)].*

(v) **(Realified Product Embeddings and Non-realified Global Frobenioids)** *The constructions of* $\mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$, $\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})$ *given in (iii) and (iv) above give rise to a commutative diagram of categories*

$$ \begin{array}{ccc} \mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}) & \hookrightarrow & \prod_{j\in\mathbb{F}^{*}_{l}} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_j \\ \downarrow & & \downarrow \\ \mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}) & \hookrightarrow & \prod_{j\in\mathbb{F}^{*}_{l}} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_j \end{array} $$

— where the horizontal arrows are **embeddings** *that arise tautologically from the constructions of (iii) and (iv) [cf. [IUTchII], Remark 4.8.1, (i)]; the vertical arrows are* **isomorphisms***; the left-hand vertical arrow arises from the second isomorphism that appears in the final display of (iv); the right-hand vertical arrow is the product of the* **realifications** *of copies of the inverse of the second isomorphism that appears in the final display of (ii). In particular, by applying the definition of* $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j$ *—*

i.e., in terms of **local fractional ideals** *[cf. (ii)] — together with the products of* **realification functors**

$$\prod_{j \in \mathbb{F}_l^*} ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j \;\to\; \prod_{j \in \mathbb{F}_l^*} ({}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_j$$

[cf. [FrdI], Proposition 5.3], one obtains an **algorithm** *for constructing, in a fashion compatible [in the evident sense] with the* **local isomorphisms** $\{{}^{\dagger}\rho_{\mathfrak{lgp},\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$, $\{{}^{\dagger}\rho_{\mathrm{LGP},\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ *of (iii) and (iv), objects of the [global!] categories* $\mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})$, $\mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})$ *from the local fractional ideals generated by elements of the* **monoids** *[cf. (iv); Proposition 3.4, (ii)]*

$$\Psi_{\mathcal{F}_{\mathfrak{lgp}}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})_{\underline{v}}$$

for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$.

Proof. The various assertions of Proposition 3.7 follow immediately from the definitions and the references quoted in the statements of these assertions. ○

Definition 3.8.

(i) In the situation of Proposition 3.7, (iv), (v), write $\Psi^{\perp}_{\mathcal{F}_{\mathfrak{lgp}}}(-)_{\underline{v}} \stackrel{\mathrm{def}}{=} \Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}(-)_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. the notation of Proposition 3.5, (ii), (c)]. Then we shall refer to the object of

$$\prod_{j \in \mathbb{F}_l^*} ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_j \quad \text{or} \quad \prod_{j \in \mathbb{F}_l^*} ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j$$

— as well as its *realification*, regarded as an object of ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}} = \mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})$ or ${}^{\dagger}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}} = \mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})$ [cf. Proposition 3.7, (iii), (iv), (v)] — determined by any collection, indexed by $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, of generators up to torsion of the monoids $\Psi^{\perp}_{\mathcal{F}_{\mathfrak{lgp}}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}})_{\underline{v}}$ as a **Θ-pilot object** [cf. also Remark 3.8.1 below]. We shall refer to the object of the [global realified] Frobenioid

$${}^{\dagger}\mathcal{C}^{\Vdash}_{\triangle}$$

of [IUTchII], Corollary 4.10, (i), determined by any collection, indexed by $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, of generators up to torsion of the splitting monoid associated to the split Frobenioid ${}^{\dagger}\mathcal{F}^{\vdash}_{\triangle,\underline{v}}$ [i.e., the data indexed by $\underline{v}$ of the $\mathcal{F}^{\vdash}$-prime-strip ${}^{\dagger}\mathfrak{F}^{\vdash}_{\triangle}$ of [IUTchII], Corollary 4.10, (i)] — that is to say, at a more concrete level, determined by the "$\underline{\underline{q}}_{\underline{v}}$", for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. the notation of [IUTchI], Example 3.2, (iv)] — as a ***q*-pilot object** [cf. also Remark 3.8.1 below].

(ii) Let

$${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

be a $\mathfrak{log}$-*link of* $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters* and

$$ {}^{*}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} $$

a $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater* [all relative to the given initial Θ-data]. Recall the $\mathcal{F}^{\Vdash}$-prime-strip

$$ {}^{*}\mathfrak{F}^{\Vdash}_{\triangle} $$

constructed from ${}^{*}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ in [IUTchII], Corollary 4.10, (i). Following the notational conventions of [IUTchII], Corollary 4.10, (iii), let us write ${}^{*}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle}$ (respectively, ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}}$; ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$) for the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip associated to the $\mathcal{F}^{\Vdash}$-prime-strip ${}^{*}\mathfrak{F}^{\Vdash}_{\triangle}$ (respectively, ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$; ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$) [cf. Proposition 3.7, (iii), (iv); [IUTchII], Definition 4.9, (viii); the functorial algorithm described in [IUTchII], Definition 4.9, (vi)]. Then — in the style of [IUTchII], Corollary 4.10, (iii) — we shall refer to the full poly-isomorphism of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}} \overset{\sim}{\to} {}^{*}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle}$ as the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link**

$$ {}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} \quad {}^{*}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} $$

from ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ to ${}^{*}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$, relative to the $\mathfrak{log}$-link ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$, and to the full poly-isomorphism of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathfrak{lgp}} \overset{\sim}{\to} {}^{*}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle}$ as the $\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$**-link**

$$ {}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}} \quad {}^{*}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} $$

from ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ to ${}^{*}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$, relative to the $\mathfrak{log}$-link ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$.

(iii) Let $\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}\}_{n,m\in\mathbb{Z}}$ be a *collection of distinct* $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters* [relative to the given initial Θ-data] indexed by pairs of integers. Then we shall refer to the first (respectively, second) diagram

$$ \begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\ \dots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n,m+1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n+1,m+1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} & \dots \\ & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\ \dots & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} & {}^{n+1,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} & \dots \\ & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\ & & \vdots & & \vdots & & \end{array} $$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}} & {}^{n,m+1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}} & {}^{n+1,m+1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
\dots & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}} & {}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}} & {}^{n+1,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} & \xrightarrow{\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}} & \dots \\
 & & \uparrow \mathfrak{log} & & \uparrow \mathfrak{log} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

— where the *vertical* arrows are the *full* $\mathfrak{log}$*-links*, and the *horizontal* arrow of the first (respectively, second) diagram from ${}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ to ${}^{n+1,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ is the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$- (respectively, $\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-) link from ${}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ to ${}^{n+1,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$, relative to the full $\mathfrak{log}$-link ${}^{n,m-1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ [cf. (ii)] — as the [LGP**-Gaussian**] (respectively, [$\mathfrak{lgp}$**-Gaussian**]) **log-theta-lattice**. Thus, [cf. Definition 1.4] either of these diagrams may be represented symbolically by an *oriented graph*

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

— where the "$\bullet$'s" correspond to the "${}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$".

Remark 3.8.1. The LGP-Gaussian and $\mathfrak{lgp}$-Gaussian log-theta-lattices are, of course, closely related, but, in the theory to be developed below, we shall mainly be interested in the LGP**-Gaussian log-theta-lattice** [for reasons to be explained in Remark 3.10.1, (ii), below]. On the other hand, our computation of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link will involve the $\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-link, as well as related **Θ-pilot objects**, in an essential way. Here, we note, for future reference, that both the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$- and the $\Theta^{\times \boldsymbol{\mu}}_{\mathfrak{lgp}}$-link map **$\Theta$-pilot objects** to **$q$-pilot objects**. Also, we observe that this terminology of *"Θ-pilot/q-pilot objects"* is *consistent* with the notion of a **"pilot object"** associated to a $\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strip, as defined in [IUTchII], Definition 4.9, (viii).

Remark 3.8.2. One verifies immediately that the *main results* obtained so far concerning Gaussian log-theta-lattices — namely, Theorem 1.5, Proposition 2.1, Corollary 2.3 [cf. also Remark 2.3.2], and Proposition 3.5 — generalize immediately [indeed, "formally"] to the case of LGP- or $\mathfrak{lgp}$-**Gaussian log-theta-lattices**. Indeed, the substantive content of these results concerns portions of the log-theta-lattices involved that are *substantively unaffected* by the transition from "Gaussian" to "LGP- or $\mathfrak{lgp}$-Gaussian".

Remark 3.8.3. In the definition of the various **horizontal arrows** of the **log-theta-lattices** discussed in Definition 3.8, (iii), it may appear to the reader, at first glance, that, instead of working with $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips, it might in fact be sufficient to replace the unit [i.e., $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip] portions of these prime-strips by the associated **log-shells** [cf. Proposition 1.2, (vi), (vii)], on which, at nonarchimedean $\underline{v} \in \underline{\mathbb{V}}$, the associated local Galois groups act *trivially.* In fact, however, this is *not* the case. That is to say, the *nontrivial Galois action* on the local unit portions of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips involved is necessary in order to consider the **Kummer theory** [cf. Proposition 3.5, (i), (ii), as well as Proposition 3.10, (i), (iii); Theorem 3.11, (iii), (c), (d), below] of the various local and global objects for which the log-shells serve as **"multiradial containers"** [cf. the discussion of Remark 1.5.2]. Here, we recall that this Kummer theory plays a crucial role in the theory of the present series of papers in relating corresponding *Frobenius-like* and *étale-like* objects [cf. the discussion of Remark 1.5.4, (i)].

Proposition 3.9. **(Log-volume for Packets and Processions)**

(i) **(Local Holomorphic Packets)** *In the situation of Proposition 3.2, (i), (ii): Suppose that $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$, $\alpha \in A$. Then the $p_{v_{\mathbb{Q}}}$-adic log-volume on each of the direct summand $p_{v_{\mathbb{Q}}}$-adic fields of $\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$, $\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$, and $\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ — cf. the direct sum decompositions of Proposition 3.1, (i), together with the discussion of* **normalized weights** *in Remark 3.1.1, (ii), (iii), (iv) — determines [cf. [AbsTopIII], Proposition 5.7, (i)]* **log-volumes**

$$\mu^{\log}_{\alpha,v_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})) \rightarrow \mathbb{R}; \quad \mu^{\log}_{A,v_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})) \rightarrow \mathbb{R}$$

$$\mu^{\log}_{A,\alpha,\underline{v}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})) \rightarrow \mathbb{R}$$

— where we write "$\mathfrak{M}(-)$" for the set of **nonempty compact open subsets** *of "$(-)$" — such that the log-volume of each of the* **"local holomorphic" integral structures** *of Proposition 3.1, (ii) — i.e., the elements*

$$\mathcal{O}_{{}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}); \quad \mathcal{O}_{{}^{A}\mathcal{F}_{v_{\mathbb{Q}}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}); \quad \mathcal{O}_{{}^{A,\alpha}\mathcal{F}_{\underline{v}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$$

of "$\mathfrak{M}(-)$" given by the integral structures discussed in Proposition 3.1, (ii), on each of the direct summand $p_{v_{\mathbb{Q}}}$-adic fields — is equal to **zero**. *Here, we assume that these log-volumes are normalized so that multiplication of an element of "$\mathfrak{M}(-)$" by $p_{\underline{v}}$ corresponds to adding the quantity $-\log(p_{\underline{v}}) \in \mathbb{R}$; we shall refer to this normalization as the* **packet-normalization**. *Suppose that $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{arc}}$, $\alpha \in A$. Then the sum of the radial log-volumes on each of the direct summand complex archimedean fields of $\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$, $\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$, and $\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ — cf. the direct sum decompositions of Proposition 3.1, (i), together with the discussion of* **normalized**

weights *in Remark 3.1.1, (ii), (iii), (iv) — determines [cf. [AbsTopIII], Proposition 5.7, (ii)]* **log-volumes**

$$\mu^{\log}_{\alpha,v_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})) \rightarrow \mathbb{R}; \quad \mu^{\log}_{A,v_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})) \rightarrow \mathbb{R}$$

$$\mu^{\log}_{A,\alpha,\underline{v}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})) \rightarrow \mathbb{R}$$

— where we write "$\mathfrak{M}(-)$" for the set of **compact closures of nonempty open subsets** *of "$(-)$" — such that the log-volume of each of the* **"local holomorphic" integral structures** *of Proposition 3.1, (ii) — i.e., the elements*

$$\mathcal{O}_{{}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}); \quad \mathcal{O}_{{}^{A}\mathcal{F}_{v_{\mathbb{Q}}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}); \quad \mathcal{O}_{{}^{A,\alpha}\mathcal{F}_{\underline{v}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$$

of "$\mathfrak{M}(-)$" given by the products of the integral structures discussed in Proposition 3.1, (ii), on each of the direct summand complex archimedean fields — is equal to **zero**. *Here, we assume that these log-volumes are normalized so that multiplication of an element of "$\mathfrak{M}(-)$" by $e = 2.71828\ldots$ corresponds to adding the quantity $1 = \log(e) \in \mathbb{R}$; we shall refer to this normalization as the* **packet-normalization**. *In both the nonarchimedean and archimedean cases, "$\mu^{\log}_{A,v_{\mathbb{Q}}}$" is* **invariant** *with respect to* **permutations** *of A. Finally, when working with collections of capsules in a procession, as in Proposition 3.4, (ii), we obtain, in both the nonarchimedean and archimedean cases, log-volumes on the products of the "$\mathfrak{M}(-)$" associated to the various capsules under consideration, which we normalize by taking the* **average**, *over the various capsules under consideration; we shall refer to this normalization as the* **procession-normalization** *[cf. Remark 3.9.3 below].*

(ii) **(Mono-analytic Compatibility)** *In the situation of Proposition 3.2, (i), (ii): Suppose that $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. Then by applying the $p_{v_{\mathbb{Q}}}$-adic log-volume, when $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$, or the radial log-volume, when $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$, on the* **mono-analytic log-shells** *"$\mathcal{I}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}}$" of Proposition 1.2, (vi), (vii), (viii), and adjusting appropriately [cf. Remark 3.9.1 below for more details] to account for the* **discrepancy** *between the* **"local holomorphic" integral structures** *of Proposition 3.1, (ii), and the* **"mono-analytic" integral structures** *of Proposition 3.2, (ii), one obtains [by a slight abuse of notation]* **log-volumes**

$$\mu^{\log}_{\alpha,v_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})) \rightarrow \mathbb{R}; \quad \mu^{\log}_{A,v_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})) \rightarrow \mathbb{R}$$

$$\mu^{\log}_{A,\alpha,\underline{v}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}})) \rightarrow \mathbb{R}$$

— where "$\mathfrak{M}(-)$" is as in (i) above — which are **compatible** *with the log-volumes obtained in (i), relative to the* **natural poly-isomorphisms** *of Proposition 3.2, (i). In particular, these log-volumes may be constructed via a* **functorial algorithm** *from the $\mathcal{D}^{\vdash}$-prime-strips under consideration. If one considers the* **mono-analyticization** *[cf. [IUTchI], Proposition 6.9, (ii)] of a holomorphic procession as in Proposition 3.4, (ii), then taking the average, as in (i) above, of the* **packet-normalized** *log-volumes of the above display gives rise to* **procession-normalized** *log-volumes, which are compatible, relative to the natural poly-isomorphisms of Proposition 3.2, (i), with the procession-normalized log-volumes of (i). Finally, by* **replacing** *"$\mathcal{D}^{\vdash}$" by "$\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$" [cf. also the discussion of Proposition 1.2, (vi),*

(vii), (viii)], one obtains a similar theory of log-volumes for the various objects associated to the mono-analytic log-shells "$\mathcal{I}_{\dagger\mathcal{F}^{\vdash}_{\underline{v}}\times\mu}$", which is **compatible** *with the theory obtained for "$\mathcal{D}^{\vdash}$" relative to the various* **natural poly-isomorphisms** *of Proposition 3.2, (i).*

(iii) **(Global Compatibility)** *In the situation of Proposition 3.7, (i), (ii): Write*

$$\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) \stackrel{\text{def}}{=} \prod_{v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}} \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) \subseteq \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) = \prod_{v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}} \underline{\mathfrak{log}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$$

and

$$\mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})) \subseteq \prod_{v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}} \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}))$$

for the subset of elements whose components, indexed by $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, have **zero log-volume** *[cf. (i)] for all but finitely many $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. Then, by adding the log-volumes of (i) [all but finitely many of which are zero!] at the various $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, one obtains a* **global log-volume**

$$\mu^{\log}_{A,\mathbb{V}_{\mathbb{Q}}} : \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})) \to \mathbb{R}$$

which is **invariant** *with respect to* **multiplication** *by elements of*

$$({}^{\dagger}\mathbb{M}^{\circledast}_{\mathfrak{mod}})_{\alpha} = ({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_{\alpha} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})$$

as well as with respect to **permutations** *of A, and, moreover, satisfies the following property concerning [the elements of "$\mathfrak{M}(-)$" determined by] objects "$\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$" of $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha}$ [cf. Example 3.6, (ii); Proposition 3.7, (ii)]: the* **global log-volume** *$\mu^{\log}_{A,\mathbb{V}_{\mathbb{Q}}}(\mathcal{J})$ is equal to the* **degree** *of the* **arithmetic line bundle** *determined by $\mathcal{J}$ [cf. the discussion of Example 3.6, (ii); the natural isomorphism $({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha} \stackrel{\sim}{\to} ({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}})_{\alpha}$ of Proposition 3.7, (ii)], relative to a* **suitable normalization**.

(iv) **($\mathfrak{log}$-link Compatibility)** *Let $\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n,m\in\mathbb{Z}}$ be a* **collection of distinct $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** *[relative to the given initial Θ-data] — which we think of as arising from an* LGP**-Gaussian log-theta-lattice** *[cf. Definition 3.8, (iii)]. Then [cf. also the discussion of Remark 3.9.4 below]:*

(a) For $n, m \in \mathbb{Z}$, the **log-volumes** *constructed in (i), (ii), (iii) above determine log-volumes on the various "$\mathcal{I}^{\mathbb{Q}}((-))$" that appear in the construction of the* **local/global** LGP**-/$\mathfrak{lgp}$-monoids/Frobenioids** *that appear in the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$ constructed in Proposition 3.7, (iii), (iv), relative to the* **$\mathfrak{log}$-link** *${}^{n,m-1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.*

(b) At the level of the $\mathbb{Q}$-spans of **log-shells** *"$\mathcal{I}^{\mathbb{Q}}((-))$" that arise from the various* **$\mathcal{F}$-prime-strips** *involved, the log-volumes of (a) indexed by (n, m) are* **compatible** *— in the sense discussed in Propositions 1.2, (iii); 1.3, (iii) — with the corresponding log-volumes indexed by $(n, m-1)$, relative to the $\mathfrak{log}$-link ${}^{n,m-1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\mathfrak{log}} {}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.*

Proof. The various assertions of Proposition 3.9 follow immediately from the definitions and the references quoted in the statements of these assertions. ◯

Remark 3.9.1. In the spirit of the *explicit descriptions* of Remark 3.1.1, (i) [cf. also Remark 1.2.2, (i), (ii)], we make the following observations.

(i) Suppose that $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\text{non}}$. Write $\{\underline{v}_1, \dots, \underline{v}_{n_{v_{\mathbb{Q}}}}\}$ for the [distinct!] elements of $\underline{\mathbb{V}}$ that lie over $v_{\mathbb{Q}}$. For each $i = 1, \dots, n_{v_{\mathbb{Q}}}$, set $k_i \stackrel{\text{def}}{=} K_{\underline{v}_i}$; write $\mathcal{O}_{k_i} \subseteq k_i$ for the ring of integers of k_i,

$$\mathcal{I}_i \stackrel{\text{def}}{=} (p_{v_{\mathbb{Q}}}^*)^{-1} \cdot \log_{k_i}(\mathcal{O}_{k_i}^{\times}) \ \subseteq \ k_i$$

— where $p_{v_{\mathbb{Q}}}^* = p_{\underline{v}}$ if $p_{v_{\mathbb{Q}}}$ is *odd*, $p_{v_{\mathbb{Q}}}^* = p_{v_{\mathbb{Q}}}^2$ if $p_{v_{\mathbb{Q}}}$ is *even* — cf. Remark 1.2.2, (i). Then, roughly speaking, in the notation of Proposition 3.9, (i), the **mono-analytic integral structures** of Proposition 3.2, (ii), in

$$\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \ \xrightarrow{\sim} \ \bigoplus_{i=1}^{n_{v_{\mathbb{Q}}}} k_i; \quad \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) \ \xrightarrow{\sim} \ \bigotimes_{\alpha \in A} \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$$

are given by

$$\mathcal{I}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \ \xrightarrow{\sim} \ \bigoplus_{i=1}^{n_{v_{\mathbb{Q}}}} \mathcal{I}_i; \quad \mathcal{I}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) \ \xrightarrow{\sim} \ \bigotimes_{\alpha \in A} \mathcal{I}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$$

while the **local holomorphic integral structures**

$$\mathcal{O}_{{}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}} \ \subseteq \ \mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}); \quad \mathcal{O}_{{}^{A}\mathcal{F}_{v_{\mathbb{Q}}}} \ \subseteq \ \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$$

of Proposition 3.9, (i), in the ind-topological rings $\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$, $\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$ — both of which are *direct sums* of finite extensions of $\mathbb{Q}_{p_{v_{\mathbb{Q}}}}$ — are given by the *subrings of integers* in $\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$, $\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$. Thus, by applying the formula of the final display of [AbsTopIII], Proposition 5.8, (iii), for the log-volume of $\mathcal{I}_i$, [one verifies easily that] one may compute the *log-volumes*

$$\mu_{\alpha, v_{\mathbb{Q}}}^{\log}(\mathcal{I}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})), \quad \mu_{A, v_{\mathbb{Q}}}^{\log}(\mathcal{I}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}))$$

entirely in terms of the given **initial Θ-data**. We leave the routine details to the reader.

(ii) Suppose that $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\text{arc}}$. Write $\{\underline{v}_1, \dots, \underline{v}_{n_{v_{\mathbb{Q}}}}\}$ for the [distinct!] elements of $\underline{\mathbb{V}}$ that lie over $v_{\mathbb{Q}}$. For each $i = 1, \dots, n_{v_{\mathbb{Q}}}$, set $k_i \stackrel{\text{def}}{=} K_{\underline{v}_i}$; write $\mathcal{O}_{k_i} \stackrel{\text{def}}{=} \{\lambda \in k_i \mid |\lambda| \leq 1\} \subseteq k_i$ for the "set of integers" of k_i,

$$\mathcal{I}_i \stackrel{\text{def}}{=} \pi \cdot \mathcal{O}_{k_i} \ \subseteq \ k_i$$

— cf. Remark 1.2.2, (ii). Then, roughly speaking, in the notation of Proposition 3.9, (i), the **discrepancy** between the **mono-analytic integral structures** of

Proposition 3.2, (ii), determined by the $\mathcal{I}(^\dagger\mathcal{F}_{\underline{v}_i}) \xrightarrow{\sim} \mathcal{I}_i \subseteq k_i$ and the **local holomorphic integral structures**

$$\mathcal{O}_{^\alpha\mathcal{F}_{v_\mathbb{Q}}} \subseteq \mathcal{I}^\mathbb{Q}(^\alpha\mathcal{F}_{v_\mathbb{Q}}) \xrightarrow{\sim} \bigoplus_{i=1}^{n_{v_\mathbb{Q}}} k_i$$

$$\mathcal{O}_{^A\mathcal{F}_{v_\mathbb{Q}}} \subseteq \mathcal{I}^\mathbb{Q}(^A\mathcal{F}_{v_\mathbb{Q}}) \xrightarrow{\sim} \bigotimes_{\alpha \in A} \mathcal{I}^\mathbb{Q}(^\alpha\mathcal{F}_{v_\mathbb{Q}})$$

of Proposition 3.9, (i), in the topological rings $\mathcal{I}^\mathbb{Q}(^\alpha\mathcal{F}_{v_\mathbb{Q}})$, $\mathcal{I}^\mathbb{Q}(^A\mathcal{F}_{v_\mathbb{Q}})$ — both of which are direct sums of complex archimedean fields — determined by taking the product [relative to this direct sum decomposition] of the respective *"subsets of integers"* may be computed entirely in terms of the given **initial Θ-data**, by applying the following two [easily verified] observations:

(a) Equip $\mathbb{C}$ with its standard Hermitian metric, i.e., the metric determined by the complex norm. This metric on $\mathbb{C}$ determines a tensor product metric on $\mathbb{C} \otimes_\mathbb{R} \mathbb{C}$, as well as a direct sum metric on $\mathbb{C} \oplus \mathbb{C}$. Then, relative to these metrics, any *isomorphism of topological rings* [i.e., arising from the Chinese remainder theorem]

$$\mathbb{C} \otimes_\mathbb{R} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \oplus \mathbb{C}$$

is **compatible** with these **metrics**, up to a factor of 2, i.e., the metric on the right-hand side corresponds to 2 times the metric on the left-hand side.

(b) Relative to the notation of (a), the **direct sum decomposition** $\mathbb{C} \oplus \mathbb{C}$, together with its Hermitian metric, is **preserved**, relative to the displayed isomorphism of (a), by the operation of conjugation on either of the two copies of "$\mathbb{C}$" that appear in $\mathbb{C} \otimes_\mathbb{R} \mathbb{C}$, as well as by the operations of multiplying by ± 1 or $\pm\sqrt{-1}$ via either of the two copies of "$\mathbb{C}$" that appear in $\mathbb{C} \otimes_\mathbb{R} \mathbb{C}$.

We leave the routine details to the reader.

(iii) The computation of the discrepancy between local holomorphic and mono-analytic integral structures will be discussed in more detail in [IUTchIV], §1.

Remark 3.9.2. In the situation of Proposition 3.9, (iii), one may construct **["mono-analytic"] algorithms** for recovering the **subquotient** of the *perfection* of $(^\dagger\mathbb{M}^\circledast_{\mathfrak{mod}})_\alpha = (^\dagger\mathbb{M}^\circledast_{\mathrm{MOD}})_\alpha$ associated to $\underline{w} \in \underline{\mathbb{V}}$ [cf. Remark 3.6.1], together with the *submonoid of "nonnegative elements"* of such a subquotient, by considering the effect of multiplication by elements of $(^\dagger\mathbb{M}^\circledast_{\mathfrak{mod}})_\alpha = (^\dagger\mathbb{M}^\circledast_{\mathrm{MOD}})_\alpha$ on the *log-volumes* defined on the various $\mathcal{I}^\mathbb{Q}(^{A,\alpha}\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} \mathcal{I}^\mathbb{Q}(^{A,\alpha}\mathcal{D}^\vdash_{\underline{v}})$ [cf. Proposition 3.9, (ii)].

Remark 3.9.3. With regard to the **procession-normalizations** discussed in Proposition 3.9, (i), (ii), the reader might wonder the following: Is it possible to work with more general **weighted averages**, i.e., as opposed to just *averages*, in the usual sense, over the capsules that appear in the procession?

The answer to this question is *"no"*. Indeed, in the situation of Proposition 3.4, (ii), for $j \in \{1, \ldots, l^*\}$, the *packet-normalized* log-volume corresponding to the capsule with index set $\mathbb{S}^{\pm}_{j+1}$ may be thought of as a log-volume that arises from *"any one of the log-shells whose label* $\in \{0, 1, \ldots, j\}$*"*. In particular, if $j', j_1, j_2 \in \{1, \ldots, l^*\}$, and $j' \leq j_1, j_2$, then log-volumes corresponding to the **same log-shell** labeled j' might give rise to packet-normalized log-volumes corresponding to *either* of [the capsules with index sets] $\mathbb{S}^{\pm}_{j_1+1}$, $\mathbb{S}^{\pm}_{j_2+1}$. That is to say, in order for the resulting notion of a *procession-normalized log-volume* to be **compatible** with the appearance of the component labeled j' in various distinct capsules of the procession — i.e., compatible with the various **inclusion morphisms** of the procession! — one has no choice but to assign the *same weights* to [the capsules with index sets] $\mathbb{S}^{\pm}_{j_1+1}$, $\mathbb{S}^{\pm}_{j_2+1}$.

Remark 3.9.4. The **$\mathfrak{log}$-link compatibility** of **log-volumes** discussed in Proposition 3.9, (iv), may be *formulated somewhat more explicitly* by applying various elementary observations, as follows.

(i) Let (M, μ_M) be a *measure space* [i.e., in the sense of the discussion of Remark 3.1.1, (iii)]. We shall say that a subset $S \subseteq M$ is *pre-ample* if S is a relatively compact Borel set, that a pre-ample subset $S \subseteq M$ is *ample* if $\mu_M(S) > 0$, and that (M, μ_M) is *ample* if there exists an ample subset of M. In the following, for the sake of simplicity, we assume that (M, μ_M) is *ample*. Also, to simplify the notation, we shall often denote the dependence of objects constructed from the *pair* (M, μ_M) by means of the notation "(M)" [i.e., as opposed to "(M, μ_M)"]. Write

$$\mathrm{Sub}(M)$$

for the *set of pre-ample subsets* of M and

$$\mathrm{Fn}(M)$$

for the *set of Borel measurable functions* $f : M \to \mathbb{R}_{\geq 0}$ such that the image $f(M) \subseteq \mathbb{R}_{\geq 0}$ of f is a *finite set*, and, moreover, $M \supseteq f^{-1}(\mathbb{R}_{>0}) \in \mathrm{Sub}(M)$. Observe that $\mathrm{Fn}(M)$ is equipped with a *natural monoid structure* [induced by the natural monoid structure on $\mathbb{R}_{\geq 0}$], as well as a *natural action* by $\mathbb{R}_{>0}$ [induced by the natural action by multiplication of $\mathbb{R}_{>0}$ on $\mathbb{R}_{\geq 0}$]. By assigning to an element $S \in \mathrm{Sub}(M)$ the *characteristic function* $\chi_S : M \to \mathbb{R}_{\geq 0}$ [i.e., which is $= 1$ on S and $= 0$ on $M \setminus S$], we shall regard $\mathrm{Sub}(M)$ as a *subset* of $\mathrm{Fn}(M)$. Note that *integration over* M, relative to the *measure* μ_M, determines an $\mathbb{R}_{>0}$*-equivariant surjection*

$$\int_M : \mathrm{Fn}(M) \twoheadrightarrow \mathbb{R}_{\geq 0}$$

whose restriction to $\mathrm{Sub}(M)$ maps $\mathrm{Sub}(M) \ni S \mapsto \mu_M(S) \in \mathbb{R}_{\geq 0}$. In particular, if we write $\mathrm{Fn}\mathbb{R}\mathrm{ss}_M : \mathrm{Fn}(M) \twoheadrightarrow \mathbb{R}\mathrm{ss}(M)$ for the natural map to the *quotient set* of $\mathrm{Fn}(M)$ [i.e., the set of equivalence classes of elements of $\mathrm{Fn}(M)$] determined by $\int_M$

[so $\mathbb{R}\mathrm{ss}(M)$ also admits a *natural monoid structure*, as well as a *natural action* by $\mathbb{R}_{>0}$], then we obtain a *natural* $\mathbb{R}_{>0}$*-equivariant isomorphism of monoids*

$$\int_M^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(M) \overset{\sim}{\to} \mathbb{R}_{\geq 0}$$

such that $\int_M = \int_M^{\mathbb{R}\mathrm{ss}} \circ \mathrm{Fn}\mathbb{R}\mathrm{ss}_M$. Here,

we wish to think of **integration** $\int_M$ — and hence of the **quotient**

$$\mathrm{Fn}\mathbb{R}\mathrm{ss}_M : \mathrm{Fn}(M) \twoheadrightarrow \mathbb{R}\mathrm{ss}(M)$$

— as a sort of **"realified semi-simplification"** of (M, μ_M)

[i.e., roughly in the spirit of the *Grothendieck group* associated to an additive category], that is to say, a *quotient* in the category of *commutative monoids with* $\mathbb{R}_{>0}$*-action*, whose restriction to $\mathrm{Sub}(M) \subseteq \mathrm{Fn}(M)$

· *identifies* $S_1, S_2 \in \mathrm{Sub}(M)$ such that $\mu_M(S_1) = \mu_M(S_2)$ [such as, for instance, *additive translates* of an element $S \in \mathrm{Sub}(M)$ relative to an *additive structure* on M with respect to which μ_M is *invariant*];
· *maps* $S_1 \cup S_2 \in \mathrm{Sub}(M)$ to the *sum* [relative to the *monoid structure* on the *quotient*] of the images of $S_1, S_2 \in \mathrm{Sub}(M)$ whenever $S_1, S_2 \in \mathrm{Sub}(M)$ are *disjoint* [i.e., as subsets of M].

We shall refer to a subset $E \subseteq \mathrm{Fn}(M)$ as *ample* if $(\mathbb{R}_{\geq 0} \supseteq)\ \mathbb{R}_{>0} \cap \int_M(E) \neq \emptyset$. Thus, if, for instance, $S \in \mathrm{Sub}(M)$ is *ample* and *compact*, then the pair $(S, \mu_M|_S)$ obtained by restricting μ_M to S is an *ample measure space* that determines *compatible natural inclusions* $\mathrm{Sub}(S) \hookrightarrow \mathrm{Sub}(M)$, $\mathrm{Fn}(S) \hookrightarrow \mathrm{Fn}(M)$ [the latter of which is defined by *extension by zero*] — which we shall use to regard $\mathrm{Sub}(S)$, $\mathrm{Fn}(S)$ as subsets of $\mathrm{Sub}(M)$, $\mathrm{Fn}(M)$, respectively — such that the subsets $\mathrm{Sub}(S), \mathrm{Fn}(S) \subseteq \mathrm{Fn}(M)$ are *ample*. If $E \subseteq \mathrm{Fn}(M)$ is *ample*, then the image of E in $\mathbb{R}\mathrm{ss}(M)$ determines a *natural subset* $\mathbb{R}\mathrm{ss}(E) \subseteq \mathbb{R}\mathrm{ss}(M)$, whose $\mathbb{R}_{\geq 0}$-orbit $\mathbb{R}_{\geq 0} \cdot \mathbb{R}\mathrm{ss}(E)$ is equal to $\mathbb{R}\mathrm{ss}(M)$. In particular, if $S \in \mathrm{Sub}(M)$ is *ample* and *compact*, then we obtain *natural* $\mathbb{R}_{>0}$*-equivariant isomorphisms of monoids*

$$\mathbb{R}\mathrm{ss}(S) \overset{\sim}{\to} \mathbb{R}_{\geq 0} \cdot \mathbb{R}\mathrm{ss}(\mathrm{Fn}(S)) \overset{\sim}{\to} \mathbb{R}\mathrm{ss}(M)$$

— where, the notation "$\mathbb{R}_{\geq 0} \cdot \mathbb{R}\mathrm{ss}(\mathrm{Fn}(S))$" is intended relative to the interpretation of $\mathrm{Fn}(S)$ as a subset of $\mathrm{Fn}(M)$ — such that the *composite isomorphism* $\mathbb{R}\mathrm{ss}(S) \overset{\sim}{\to} \mathbb{R}\mathrm{ss}(M)$ is *compatible* with the isomorphisms $\int_S^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(S) \overset{\sim}{\to} \mathbb{R}_{\geq 0}$, $\int_M^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(M) \overset{\sim}{\to} \mathbb{R}_{\geq 0}$. Finally, we observe that if (M_1, μ_{M_1}) and (M_2, μ_{M_2}) are *ample measure spaces*, then the *product measure space* $(M_1 \times M_2, \mu_{M_1 \times M_2})$ is also an ample measure space; moreover, there is a *natural map*

$$\mathrm{Sub}(M_1) \times \mathrm{Sub}(M_2) \to \mathrm{Sub}(M_1 \times M_2)$$

that maps $(S_1, S_2) \mapsto S_1 \times S_2$ and induces a *natural* $\mathbb{R}_{>0}$*-equivariant isomorphism of monoids*

$$\mathbb{R}\mathrm{ss}(M_1) \otimes \mathbb{R}\mathrm{ss}(M_2) \overset{\sim}{\to} \mathbb{R}\mathrm{ss}(M_1 \times M_2)$$

that is *compatible* with the isomorphisms $\int_{M_1}^{\mathbb{R}\mathrm{ss}} \otimes \int_{M_2}^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(M_1) \otimes \mathbb{R}\mathrm{ss}(M_2) \xrightarrow{\sim} \mathbb{R}_{\geq 0}$, $\int_{M_1 \times M_2}^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(M_1 \times M_2) \xrightarrow{\sim} \mathbb{R}_{\geq 0}$. [Here, we observe that there is a natural notion of *"tensor product of monoids isomorphic to* $\mathbb{R}_{\geq 0}$*"* since such a monoid may be thought of, by passing to the groupification of such a monoid, as a *one-dimensional* $\mathbb{R}$*-vector space equipped with a subset [which forms a* $\mathbb{R}_{>0}$*-torsor] of "positive elements"*.]

(ii) One *very rough* approach to understanding the $\mathfrak{log}$*-link compatibility* of *log-volumes* is the following. Suppose that instead of knowing this property, one only knows that

> each application of the $\mathfrak{log}$-link has the effect of **dilating volumes** by a factor $\lambda \in \mathbb{R}_{>0} \setminus \{1\}$.

[Here, relative to the notation of (i), we observe that this sort of situation in which *volumes* are *dilated* in a nontrivial fashion may be seen in the following example:

> Suppose that $M \stackrel{\mathrm{def}}{=} \mathbb{Q}_p$, for some prime number p, equipped with the [additive] Haar measure $\mu_{\mathbb{Q}_p}$ normalized so that $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ has measure 1, so $(\mathbb{Q}_p, \mu_{\mathbb{Q}_p})$ is an *ample measure space* in the sense of (i). Then *multiplication by* p induces a bijection $\alpha_p : \mathbb{Q}_p \xrightarrow{\sim} \mathbb{Q}_p$. Moreover, α_p induces *compatible bijections* $\mathrm{Sub}(\alpha_p) : \mathrm{Sub}(\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{Sub}(\mathbb{Q}_p)$, $\mathrm{Fn}(\alpha_p) : \mathrm{Fn}(\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{Fn}(\mathbb{Q}_p)$, $\mathbb{R}\mathrm{ss}(\alpha_p) : \mathbb{R}\mathrm{ss}(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{R}\mathrm{ss}(\mathbb{Q}_p)$. On the other hand, [*unlike* the situation discussed in (i) concerning the "composite isomorphism $\mathbb{R}\mathrm{ss}(S) \xrightarrow{\sim} \mathbb{R}\mathrm{ss}(M)$"!] in the present context, $\mathbb{R}\mathrm{ss}(\alpha_p)$ is *not compatible* with the isomorphisms $\int_{\mathbb{Q}_p}^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{R}_{\geq 0}$ in the domain and codomain of $\mathbb{R}\mathrm{ss}(\alpha_p)$, i.e., it is only compatible *up to a factor* p^{-1} $(\neq 1)$!]

Then in order to *compute log-volumes* in a fashion that is *consistent* with the various arrows [i.e., both Kummer isomorphisms and $\mathfrak{log}$-links!] of the "systems" constituted by the $\mathfrak{log}$**-Kummer correspondences** discussed in Proposition 3.5, (ii), it would be necessary to regard the various "log-volumes" computed as only giving rise to well-defined elements [not $\in \mathbb{R}$, but rather]

$$\in \quad \mathbb{R}/\mathbb{Z} \cdot \log(\lambda) \quad (\cong \ \mathbb{S}^1)$$

— a situation which is *not acceptable*, relative to the goal of obtaining *log-volume estimates* [i.e., as in Corollary 3.12 below] for the various objects for which log-shells serve as *"multiradial containers"* [cf. the discussion of Remark 1.5.2; the content of Theorem 3.11 below].

(iii) In the following discussion, we use the notation of Remark 1.2.2, (i). Thus, we regard k as being equipped with the [additive] Haar measure μ_k normalized so that $\mu_k(\mathcal{O}_k) = 1$ [cf. [AbsTopIII], Proposition 5.7, (i)]. Then (k, μ_k) is an *ample measure space* in the sense of (i); $\mathcal{O}_k^\times \subseteq k$ is an *ample subset*; for any *compact ample subset* $S \subseteq \mathcal{O}_k^\times$ on which $\log_k : \mathcal{O}_k^\times \to k$ is *injective*, we have $\mu_k(S) = \mu_k(\log_k(S))$ [cf. [AbsTopIII], Proposition 5.7, (i), (c)]. In particular, by applying the *formalism* of **realified semi-simplifications** introduced in (i), we conclude that the diagram

$$\begin{array}{ccccccc} k & \supseteq & \mathcal{O}_k^\times & \xrightarrow{\log_k} & \log_k(\mathcal{O}_k^\times) & \subseteq & k \\ & & \cup & & \cup & & \\ & & S & \xrightarrow{\sim} & \log_k(S) & & \end{array}$$

induces a *commutative diagram*

$$\begin{array}{ccccccc}
\mathbb{R}\mathrm{ss}(k) & \overset{\sim}{\leftarrow} & \mathbb{R}\mathrm{ss}(\mathcal{O}_k^\times) & \overset{\sim}{\leftarrow} & \mathbb{R}\mathrm{ss}(S) & \overset{\sim}{\rightarrow} & \mathbb{R}\mathrm{ss}(\log_k(S)) \\
\downarrow \int_k^{\mathbb{R}\mathrm{ss}} & & \downarrow \int_{\mathcal{O}_k^\times}^{\mathbb{R}\mathrm{ss}} & & \downarrow \int_S^{\mathbb{R}\mathrm{ss}} & & \downarrow \int_{\log_k(S)}^{\mathbb{R}\mathrm{ss}} \\
\mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0}
\end{array}$$

$$\begin{array}{ccccc}
\overset{\sim}{\rightarrow} & \mathbb{R}\mathrm{ss}(\log_k(\mathcal{O}_k^\times)) & \overset{\sim}{\rightarrow} & \mathbb{R}\mathrm{ss}(k) \\
& \downarrow \int_{\log_k(\mathcal{O}_k^\times)}^{\mathbb{R}\mathrm{ss}} & & \downarrow \int_k^{\mathbb{R}\mathrm{ss}} \\
= & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0}
\end{array}$$

— where the vertical arrows are $\mathbb{R}_{>0}$-*equivariant isomorphisms of monoids*, and the *composite [$\mathbb{R}_{>0}$-equivariant] isomorphism [of monoids]* $\mathbb{R}\mathrm{ss}(\mathcal{O}_k^\times) \overset{\sim}{\rightarrow} \mathbb{R}\mathrm{ss}(\log_k(\mathcal{O}_k^\times))$ is easily verified to be *independent* of the choice of the *compact ample subset* $S \subseteq \mathcal{O}_k^\times$. [Also, we observe that it is easily verified that there *exist* compact ample subsets $S \subseteq \mathcal{O}_k^\times$ for which the induced map $S \twoheadrightarrow \log_k(S)$ is *injective*.] One may then compose this diagram with the bijection

$$\log : \mathbb{R}_{\geq 0} \overset{\sim}{\rightarrow} \mathbb{R} \cup \{-\infty\}$$

determined by the *natural logarithm* and then multiply by a *suitable normalization factor* $\in \mathbb{R}_{>0}$ to conclude that

> *the diagram*
> $$k \supseteq \mathcal{O}_k^\times \overset{\log_k}{\longrightarrow} \log_k(\mathcal{O}_k^\times) \subseteq k$$
> *induces* $\mathbb{R}_{>0}$**-equivariant isomorphisms of monoids** *on the respective* **realified semi-simplifications** *"*$\mathbb{R}\mathrm{ss}(-)$*", all of which are* **compatible** *with the* **log-volume** *maps on each of the "*$\mathbb{R}\mathrm{ss}(-)$*'s", i.e., which restrict to the "usual log-volume maps" on the respective "*$\mathrm{Sub}(-)$*'s", relative to the natural maps "*$\mathrm{Sub}(-) \rightarrow \mathbb{R}\mathrm{ss}(-)$*".*

This is one way to formulate the **$\mathfrak{log}$-link compatibility** of **log-volumes** discussed in Proposition 3.9, (iv), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$. Finally, we observe that this $\mathfrak{log}$-*link compatibility* with *log-volumes* is itself *compatible* with passing to **finite extensions** of k [or, more generally, $\mathbb{Q}_{p_{\underline{v}}}$], as follows. Let $k_1 \subseteq k_2$ be finite field extensions of $\mathbb{Q}_{p_{\underline{v}}}$. We shall use analogous notation for objects associated to k_1 and k_2 to the notation that was used above for objects associated to k. Then observe that since $\mathcal{O}_{k_2}$ is a *finite free* $\mathcal{O}_{k_1}$*-module of rank* $[k_2 : k_1]$, it follows that the [additive] *compact topological group* $\mathcal{O}_{k_2}$ is isomorphic to the product of $[k_2 : k_1]$ copies of the [additive] compact topological group $\mathcal{O}_{k_1}$. In particular, since the Haar measure of a compact topological group is *invariant* with respect to arbitrary automorphisms of the topological group, we thus conclude [cf. the discussion of *product measure spaces* in (i)] that the *inclusion of topological fields* $k_1 \hookrightarrow k_2$ induces *natural* $\mathbb{R}_{>0}$-*equivariant isomorphisms of monoids*

$$\mathbb{R}\mathrm{ss}(k_1)^{\otimes[k_2:k_1]} \overset{\sim}{\leftarrow} \mathbb{R}\mathrm{ss}(\mathcal{O}_{k_1})^{\otimes[k_2:k_1]} \overset{\sim}{\rightarrow} \mathbb{R}\mathrm{ss}(\mathcal{O}_{k_2}) \overset{\sim}{\rightarrow} \mathbb{R}\mathrm{ss}(k_2)$$

such that the *composite isomorphism* $\mathbb{R}\mathrm{ss}(k_1)^{\otimes[k_2:k_1]} \overset{\sim}{\rightarrow} \mathbb{R}\mathrm{ss}(k_2)$ is *compatible* with the $\mathbb{R}_{>0}$-equivariant isomorphisms of monoids

$$\left(\int_{k_1}^{\mathbb{R}\mathrm{ss}}\right)^{\otimes[k_2:k_1]} : \mathbb{R}\mathrm{ss}(k_1)^{\otimes[k_2:k_1]} \overset{\sim}{\rightarrow} \mathbb{R}_{\geq 0}$$

and $\int_{k_2}^{\mathbb{R}\mathrm{ss}} : \mathbb{R}\mathrm{ss}(k_2) \xrightarrow{\sim} \mathbb{R}_{\geq 0}$.

(iv) In the notation of (iii), suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$; write $\underline{\underline{q}} \stackrel{\mathrm{def}}{=} \underline{\underline{q}}_{\underline{v}}$. Thus, we have a *submonoid*

$$\mathcal{O}_k^{\times} \times \underline{\underline{q}}^{\mathbb{N}} \ \subseteq \ k$$

of the underlying multiplicative monoid of k. Then the various arrows of the $\mathfrak{log}$-**Kummer correspondence** discussed in Proposition 3.5, (ii), may be thought of, from the point of view of a *vertically coric étale holomorphic* copy of "k" [i.e., a copy *labeled "$n, \circ$"*, as in Proposition 3.5, (i)], as corresponding to the *operations*

$$\begin{aligned} k \ &\rightsquigarrow \ \mathcal{O}_k^{\times} \times \underline{\underline{q}}^{\mathbb{N}} \ (\subseteq k) \\ &\rightsquigarrow \ \mathcal{O}_k^{\times} \ (\subseteq k) \\ &\rightsquigarrow \ \log_k(\mathcal{O}_k^{\times}) \ (\subseteq k) \ \rightsquigarrow \ k \end{aligned}$$

— i.e., of passing first from k to the multiplicative submonoid $\mathcal{O}_k^{\times} \times \underline{\underline{q}}^{\mathbb{N}}$, then to the multiplicative submonoid $\mathcal{O}_k^{\times}$, then applying $\log_k$ to obtain an additive submonoid of k, and finally passing from this submonoid back to k itself. Then the **log-volume compatibility** discussed in (iii) may be understood, in the context of the **$\mathfrak{log}$-Kummer correspondence**, as the statement that

> the *operations* of the above display induce $\mathbb{R}_{>0}$-*equivariant isomorphisms of monoids*
>
> $$\mathbb{R}\mathrm{ss}(k) \ \xrightarrow{\sim} \ \mathbb{R}\mathrm{ss}(\mathcal{O}_k^{\times}) \ \xrightarrow{\sim} \ \mathbb{R}\mathrm{ss}(\log_k(\mathcal{O}_k^{\times})) \ \xrightarrow{\sim} \ \mathbb{R}\mathrm{ss}(k)$$
>
> that are *compatible* with the respective [normalized] *log-volume maps* to $\mathbb{R} \cup \{-\infty\}$ [cf. the discussion of (iii)], in such a way as to *avoid any interference*, up to multiplication by roots of unity, with the submonoid $\underline{\underline{q}}^{\mathbb{N}} \subseteq k$, which induces, by applying the [normalized] *log-volume* to the image of $\mathcal{O}_k \subseteq k$ via multiplication by elements of this submonoid, an embedding
>
> $$\mathbb{N} \ \hookrightarrow \ \mathbb{R}\mathrm{ss}(k) \ \xrightarrow{\sim} \ \mathbb{R} \cup \{-\infty\}$$
>
> that maps $\mathbb{N} \ni 1 \ \mapsto -\log(\underline{\underline{q}}) \in \mathbb{R}$

[where we write $\log(\underline{\underline{q}}) \stackrel{\mathrm{def}}{=} \mathrm{ord}_{\underline{v}}(\underline{\underline{q}}_{\underline{v}}) \cdot \log(p_{\underline{v}}) \in \mathbb{R}$ — cf. the notation of Remark 2.4.2, (ii)]. A similar interpretation of *log-volume compatibility* in the context of the $\mathfrak{log}$-*Kummer correspondence* may be given in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ by simply omitting the portion of the above discussion concerning "$\underline{\underline{q}}$".

(v) In the notation of Remark 3.9.1, (i), we observe that the discussion of (iii), (iv), may be extended to *topological tensor products* of the form

$$k_{i_A} \ \stackrel{\mathrm{def}}{=} \ \bigotimes_{\alpha \in A} \ k_{i_\alpha}$$

— where $i_\alpha \in \{1, \ldots, n_{v_{\mathbb{Q}}}\}$, for each $\alpha \in A$, and we regard k_{i_A} as being equipped with the [additive] Haar measure normalized [cf. Proposition 3.9, (i)] so that the *ring of integers* $\mathcal{O}_{k_{i_A}} \subseteq k_{i_A}$ [i.e., the integral structure discussed in Proposition 3.1, (ii)] has Haar measure $= 1$. Indeed, each of the *direct summand fields* of k_{i_A} [cf. Proposition 3.1, (i)] may be taken to be a [finite extension of a] "k" as in (iii), (iv). In particular, the measure space k_{i_A} may be regarded as a *product measure space* of [finite extensions of] "k" as in (iii), (iv), so one may extend (iii), (iv) to k_{i_A} by applying (iii), (iv) to *each factor* of this product measure space [cf. the discussion of *product measure spaces* in (i)]. [We leave the routine details to the reader.] On the other hand, in this context, it is also of interest to observe that it follows immediately from the discussion of *compatibility* with *finite extensions* in (iii), together with the discussion of *product measure spaces* in (i), that, for each $\alpha \in A$, the natural structure of k_{i_A} as a k_{i_α}-*algebra* determines *natural* $\mathbb{R}_{>0}$-*equivariant isomorphisms of monoids*

$$\mathbb{R}\mathrm{ss}(\mathbb{Q}_{p_{\underline{v}}})^{\otimes d_A} \ \stackrel{\sim}{\to}\ \mathbb{R}\mathrm{ss}(k_{i_\alpha})^{\otimes d_\alpha} \ \stackrel{\sim}{\to}\ \mathbb{R}\mathrm{ss}(k_{i_A})$$

— where we write $d_\alpha \stackrel{\mathrm{def}}{=} \prod_{\beta \in A \setminus \{\alpha\}} [k_{i_\beta} : \mathbb{Q}_{p_{\underline{v}}}]$, $d_A \stackrel{\mathrm{def}}{=} d_\alpha \cdot [k_{i_\alpha} : \mathbb{Q}_{p_{\underline{v}}}]$. In particular,

> the $\mathfrak{log}$**-link compatibility** of **log-volumes** [as discussed above] for the *realified semi-simplification*
> $$\mathbb{R}\mathrm{ss}(k_{i_A})$$
> of the *topological tensor product* k_{i_A} may be understood, for any $\alpha \in A$, as the [functorially induced!] d_α-*th tensor power* of the $\mathfrak{log}$-*link compatibility* of *log-volumes* for the *realified semi-simplification*
> $$\mathbb{R}\mathrm{ss}(k_{i_\alpha})$$
> of k_{i_α} or, alternatively/equivalently, as the [functorially induced!] d_A-*th tensor power* of the $\mathfrak{log}$-*link compatibility* of *log-volumes* for the *realified semi-simplification*
> $$\mathbb{R}\mathrm{ss}(\mathbb{Q}_{p_{\underline{v}}})$$
> of $\mathbb{Q}_{p_{\underline{v}}}$

— where we note that the latter "alternative/equivalent" approach has the virtue of being *independent* of the choice of $\alpha \in A$.

(vi) In the following discussion, we use the notation of Remark 1.2.2, (ii). We regard the *complex archimedean field* k as being equipped with the *standard Euclidean metric* [cf. the discussion of *"metrics"* in Remark 1.2.1, (ii)], with respect to which $\mathcal{O}_k^\times \subseteq k$ has length 2π. This metric on k thus determines *measures* $\mu_{|k|}$ on $|k| \stackrel{\mathrm{def}}{=} k/\mathcal{O}_k^\times$ and $\mu_{\mathcal{O}_k^\times}$ on $\mathcal{O}_k^\times \subseteq k$ [cf. the situation discussed in [AbsTopIII], Proposition 5.7, (ii)] such that $(|k|, \mu_{|k|})$ and $(\mathcal{O}_k^\times, \mu_{\mathcal{O}_k^\times})$ are *ample measure spaces* in the sense of (i). Moreover, by

(a) thinking of $\mathcal{O}_k^\times$ as a *union* of *closed arcs* [i.e., whose *interiors* are *disjoint*] of measure $\mu_{\mathcal{O}_k^\times}(-) < \epsilon$, for some positive real number ϵ,

(b) considering *additive translates* of such *closed arcs* that map one of the endpoints of the arc to $0 \in k$,

(c) *projecting such additive translates* via the natural surjection $k \twoheadrightarrow |k|$, and

(d) passing to the *limit* $\epsilon \to 0$,

one verifies immediately that we obtain, by applying the *formalism* of **realified semi-simplifications** introduced in (i), a *natural* $\mathbb{R}_{>0}$*-equivariant isomorphism of monoids* $\rho_k : \mathbb{R}\mathrm{ss}(\mathcal{O}_k^\times) \xrightarrow{\sim} \mathbb{R}\mathrm{ss}(|k|)$, together with a *commutative diagram*

$$\begin{array}{ccccccc}
\mathbb{R}\mathrm{ss}(|k|) & \xleftarrow{\sim} & \mathbb{R}\mathrm{ss}(\mathcal{O}_k^\times) & \xleftarrow{\sim} & \mathbb{R}\mathrm{ss}(\log_k(\mathcal{O}_k^\times)) & \xrightarrow{\sim} & \mathbb{R}\mathrm{ss}(|k|) \\
\downarrow \int_{|k|}^{\mathbb{R}\mathrm{ss}} & & \downarrow \int_{\mathcal{O}_k^\times}^{\mathbb{R}\mathrm{ss}} & & \downarrow \int_{\log_k(\mathcal{O}_k^\times)}^{\mathbb{R}\mathrm{ss}} & & \downarrow \int_{|k|}^{\mathbb{R}\mathrm{ss}} \\
\mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0} & = & \mathbb{R}_{\geq 0}
\end{array}$$

— where the vertical arrows are $\mathbb{R}_{>0}$*-equivariant isomorphisms of monoids*; the first "$\xleftarrow{\sim}$" is ρ_k; we regard $\log_k(\mathcal{O}_k^\times) \stackrel{\mathrm{def}}{=} \exp_k^{-1}(\mathcal{O}_k^\times)$ as being equipped with the *measure* $\mu_{\log_k(\mathcal{O}_k^\times)}$ [such that $(\log_k(\mathcal{O}_k^\times), \mu_{\log_k(\mathcal{O}_k^\times)})$ is an *ample measure space*] obtained by pulling back $\mu_{|k|}$ via the *homeomorphism* $\log_k(\mathcal{O}_k^\times) \xrightarrow{\sim} |k|$ induced by restricting the natural surjection $k \twoheadrightarrow |k|$ to $\log_k(\mathcal{O}_k^\times) \subseteq k$; the second "$\xleftarrow{\sim}$" is the natural $\mathbb{R}_{>0}$-equivariant isomorphism of monoids naturally induced [i.e., by considering *ample* $S \subseteq \log_k(\mathcal{O}_k^\times)$ that map *bijectively* to $\exp_k(S) \subseteq \mathcal{O}_k^\times$ — cf. [AbsTopIII], Proposition 5.7, (ii), (c)] by the *universal covering map* $\exp_k|_{\log_k(\mathcal{O}_k^\times)} : \log_k(\mathcal{O}_k^\times) \to \mathcal{O}_k^\times$; the "$\xrightarrow{\sim}$" is the natural $\mathbb{R}_{>0}$-equivariant isomorphism of monoids induced by the *homeomorphism* $\log_k(\mathcal{O}_k^\times) \xrightarrow{\sim} |k|$. One may then compose this diagram with the bijection

$$\log : \mathbb{R}_{\geq 0} \xrightarrow{\sim} \mathbb{R} \cup \{-\infty\}$$

determined by the *natural logarithm* and then multiply by a *suitable normalization factor* $\in \mathbb{R}_{>0}$ to conclude that

> *the diagram*
>
> $$|k| \twoheadleftarrow k \supseteq \mathcal{O}_k^\times \xleftarrow{\exp_k} \log_k(\mathcal{O}_k^\times) \subseteq k \twoheadrightarrow |k|$$
>
> *induces* $\mathbb{R}_{>0}$**-equivariant isomorphisms of monoids** *on the respective* **realified semi-simplifications** *"*$\mathbb{R}\mathrm{ss}(-)$*" of* $|k|$, $\mathcal{O}_k^\times$, $\log_k(\mathcal{O}_k^\times)$, *and* $|k|$; *each of these isomorphisms is* **compatible** *with the* **log-volume** *map on "*$\mathbb{R}\mathrm{ss}(-)$*", i.e., which restricts to the "usual radial/angular log-volume map" on "*$\mathrm{Sub}(-)$*" [that is to say, the map uniquely determined by the radial/angular log-volume map of [AbsTopIII], Proposition 5.7, (ii), (a)] relative to the natural map "*$\mathrm{Sub}(-) \to \mathbb{R}\mathrm{ss}(-)$*".*

This is one way to formulate the $\mathfrak{log}$**-link compatibility** of **log-volumes** discussed in Proposition 3.9, (iv), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. One verifies immediately that one also has analogues for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ of (iv), (v). [We leave the routine details to the reader.]

Remark 3.9.5. In situations that involve consideration of *various sorts of regions* [cf. the discussion of Remarks 3.1.1, (iii), (iv); 3.9.4] to which the *log-volume* may be applied, it is often of use to consider the notion of the *holomorphic hull* of a region.

(i) Suppose that we are in the situation of Proposition 3.9, (i). Let

$$^{\alpha}\mathcal{U} \subseteq \mathcal{I}^{\mathbb{Q}}(^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) \quad \text{(respectively, } ^{A}\mathcal{U} \subseteq \mathcal{I}^{\mathbb{Q}}(^{A}\mathcal{F}_{v_{\mathbb{Q}}}); \quad ^{A,\alpha}\mathcal{U} \subseteq \mathcal{I}^{\mathbb{Q}}(^{A,\alpha}\mathcal{F}_{\underline{v}}))$$

be a subset that contains a relatively compact subset whose *log-volume* [cf. the discussion of Remark 3.1.1, (iii), (iv), as well as Remark 3.9.7, (ii), below] is *finite* [i.e., $> -\infty$]. If $^{\alpha}\mathcal{U}$ (respectively, $^{A}\mathcal{U}$; $^{A,\alpha}\mathcal{U}$) is *relatively compact*, then we define the **holomorphic hull** of $^{\alpha}\mathcal{U}$ (respectively, $^{A}\mathcal{U}$; $^{A,\alpha}\mathcal{U}$) to be the smallest subset of the form

$$^{\alpha}\mathcal{H} \stackrel{\mathrm{def}}{=} \lambda \cdot \mathcal{O}_{^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}} \quad \text{(respectively, } ^{A}\mathcal{H} \stackrel{\mathrm{def}}{=} \lambda \cdot \mathcal{O}_{^{A}\mathcal{F}_{v_{\mathbb{Q}}}}; \quad ^{A,\alpha}\mathcal{H} \stackrel{\mathrm{def}}{=} \lambda \cdot \mathcal{O}_{^{A,\alpha}\mathcal{F}_{\underline{v}}})$$

— where, relative to the direct sum decomposition of $\mathcal{I}^{\mathbb{Q}}((-))$ as a direct sum of fields [cf. the discussion of Proposition 3.9, (i)], $\lambda \in \mathcal{I}^{\mathbb{Q}}((-))$ is an element such that each *component* of λ [i.e., relative to this direct sum decomposition] is *nonzero* — that *contains* $^{\alpha}\mathcal{U}$ (respectively, $^{A}\mathcal{U}$; $^{A,\alpha}\mathcal{U}$). If $^{\alpha}\mathcal{U}$ (respectively, $^{A}\mathcal{U}$; $^{A,\alpha}\mathcal{U}$) is *not relatively compact*, then we define the **holomorphic hull** of $^{\alpha}\mathcal{U}$ (respectively, $^{A}\mathcal{U}$; $^{A,\alpha}\mathcal{U}$) to be $\mathcal{I}^{\mathbb{Q}}(^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$ (respectively, $\mathcal{I}^{\mathbb{Q}}(^{A}\mathcal{F}_{v_{\mathbb{Q}}})$; $\mathcal{I}^{\mathbb{Q}}(^{A,\alpha}\mathcal{F}_{\underline{v}})$). One verifies immediately that the holomorphic hull is well-defined [under the conditions stated].

(ii) In the remainder of the discussion of the present Remark 3.9.5, for the sake of simplicity, we shall refer to "holomorphic hulls" as *"hulls"*. Write

$$\mathbb{P} \stackrel{\mathrm{def}}{=} \{P \subseteq \mathcal{I}^{\mathbb{Q}}((-)) \mid P \text{ is a } \textit{direct product region} \text{ [cf. Remark 3.1.1, (iii)]}\};$$
$$\mathbb{H} \stackrel{\mathrm{def}}{=} \{H \subsetneq \mathcal{I}^{\mathbb{Q}}((-)) \mid H \text{ is a } \textit{hull} \text{ [cf. (i)]}\}$$

— where the argument "$(-)$" is "$^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}$", "$^{A}\mathcal{F}_{v_{\mathbb{Q}}}$", or "$^{A,\alpha}\mathcal{F}_{\underline{v}}$" [cf. (i)], and we observe that $\mathbb{H} \subseteq \mathbb{P}$. Then it is essentially a *tautology* that the operation of *forming the* **hull** discussed in (i)

$$^{\square}\mathcal{U} \mapsto {}^{\square}\mathcal{H}$$

— where "$\square$" is "α", "A", or "A, α" — determines a map

$$\phi : \mathbb{P} \to \mathbb{H}$$

that may be **characterized uniquely** by the following properties

(P1) $\phi(H) = H$, for any $H \in \mathbb{H}$;
(P2) $P \subseteq \phi(P)$, for any $P \in \mathbb{P}$;
(P3) $\phi(P_1) \subseteq \phi(P_2)$, for any $P_1, P_2 \in \mathbb{P}$ such that $P_1 \subseteq P_2$.

Indeed, since, as is easily verified, any intersection of elements of $\mathbb{H}$ which is of *finite log-volume* necessarily determines an element of $\mathbb{H}$, it follows formally from (P1), (P2), (P3) that

$$\phi(P) = \bigcap_{\mathbb{H} \ni H \supseteq P} H$$

for any $P \in \mathbb{P}$. Put another way, this map ϕ may be thought of as a sort of **adjoint**, or **push forward** in the opposite direction, of the inclusion $\mathbb{H} \subseteq \mathbb{P}$. Alternatively, ϕ may be thought of as a sort of **canonical splitting** of the inclusion $\mathbb{H} \subseteq \mathbb{P}$, or, in the spirit of the discussion of Remark 3.9.4, as a sort of **integration** operation. The *compatibility* [cf. (P2), (P3)] of ϕ with the *pre-order structure* on $\mathbb{P}$ determined by inclusion of direct product regions will play an *important role* in the context of various *log-volume estimates* of regions.

(iii) Now we consider the various *log-volumes* $\mu^{\log}_{(-)}$ [where the argument "$(-)$" is "$\alpha, v_{\mathbb{Q}}$", "$A, v_{\mathbb{Q}}$", or "$A, \alpha, \underline{v}$" — cf. (ii)] of Proposition 3.9, (i) [cf. also Remark 3.1.1, (iii)], in the context of the discussion of (ii). In the following, for the sake of simplicity, we shall denote "$\mu^{\log}_{(-)}$" by $\mu^{\log}$. For $P \in \mathbb{P}$, write

$$\Phi(P) \stackrel{\mathrm{def}}{=} \{H \in \mathbb{H} \mid \phi(P) \supseteq H, \ (\mu^{\log}(\phi(P)) \geq) \ \mu^{\log}(H) \geq \mu^{\log}(P)\} \ \subseteq \ \mathbb{H};$$

$$\Xi(P) \stackrel{\mathrm{def}}{=} \{H \in \mathbb{H} \mid \phi(P) \supseteq H, \ (\mu^{\log}(\phi(P)) \geq) \ \mu^{\log}(H) = \mu^{\log}(P)\} \ \subseteq \ \Phi(P);$$

$$H_{\Phi(P)} \stackrel{\mathrm{def}}{=} \bigcup_{H \in \Phi(P)} H \ \subseteq \ \phi(P); \quad H_{\Xi(P)} \stackrel{\mathrm{def}}{=} \bigcup_{H \in \Xi(P)} H \ \subseteq \ H_{\Phi(P)} \ \subseteq \ \phi(P).$$

Thus, one may think of elements $\in \Phi(P)$ or $\in \Xi(P)$ as

"log-volume approximations" of P by means of *hulls* $\in \mathbb{H}$.

If one thinks of distinct elements $\in \Phi(P)$ or $\in \Xi(P)$ — i.e., of the issue of constructing a **"log-volume hull-approximant"** of P — as a sort of *indeterminacy* [i.e., in the assignment to P of a *specific element* $\in \mathbb{H}$!], then

> this **indeterminacy** is **compact**, i.e., in the sense that all possible choices of an element $\in \Phi(P)$ or $\in \Xi(P)$ are contained in the *compact set* $\phi(P) \in \mathbb{H}$.

Indeed, developing the theory in such a way that

all the **indeterminacies** that occur in the theory are **compact**

is in some sense one important theme in the present series of papers. Note that this *compactness* would *not be valid* if, in the definition of $\Phi(-)$ or $\Xi(-)$, one *omits* the condition "$H \subseteq \phi(P)$".

(iv) In the context of (iii), we observe that

$$\phi(P) \in \Phi(P), \text{ so } \phi(P) = H_{\Phi(P)},$$

but the issue of whether or not $\Xi(P) = \emptyset$ is *not so immediate*. Indeed:

(Ξ1) If *either* of the following conditions is satisfied, then it is easily verified that $\Xi(P) \neq \emptyset$:

(Ξ^{non}1) if we write K^{cl} for the *Galois closure* of K over $\mathbb{Q}$, then the *residue field extension degree* of each valuation $\in \mathbb{V}(K^{\mathrm{cl}})$ that divides $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$ is $= 1$, and, moreover, $\mu^{\log}(P) = \mu^{\log}(Q)$, for some $Q \in \mathbb{P}$ which is a $\mathbb{Z}_{p_{v_{\mathbb{Q}}}}$*-submodule* of $\mathcal{I}^{\mathbb{Q}}((-))$;

(Ξ^{arc}1) $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$.

(Ξ2) If one allows the $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ in the present discussion to *vary*, and one considers *global situations* [i.e., which necessarily involve the *unique valuation* $\in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$!] as in Proposition 3.9, (iii), then it is easily verified that the *global analogue* of "$\Xi(P)$" is *nonempty*.

On the other hand, *in general, it is not so clear whether or not* $\Xi(P) \neq \emptyset$. In this context, it is also of interest to observe that if $P \in \mathbb{H}$, then

$$\{\phi(P)\} = \Phi(P) = \Xi(P), \text{ so } P = \phi(P) = H_{\Phi(P)} = H_{\Xi(P)},$$

but in general, even in the situation of (Ξ1), the *inclusion* $P \subseteq \phi(P) = H_{\Phi(P)}$, as well as the induced *inequality of log-volumes* $\mu^{\log}(P) \leq \mu^{\log}(H_{\Phi(P)}) = \mu^{\log}(\phi(P))$, is *strict*. Indeed, for instance,

(Ξ3) in the situation of (Ξ1), if $(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}$ [cf. the notational conventions of Remark 3.1.1, (iii)] and A are of *cardinality* ≥ 2, then one verifies easily that there exist $P \in \mathbb{P}$ for which $\mu^{\log}(P) < \mu^{\log}(H_{\Xi(P)})$ $(\leq \mu^{\log}(H_{\Phi(P)}) = \mu^{\log}(\phi(P)))$.

This sort of phenomenon may be seen in the following *example*:

Let p be a prime number. Write $I \stackrel{\mathrm{def}}{=} \mathbb{Q}_p \times \mathbb{Q}_p$;

$$H_0 \stackrel{\mathrm{def}}{=} \mathbb{Z}_p \times \mathbb{Z}_p \subseteq I; \quad H_1 \stackrel{\mathrm{def}}{=} (p^{-1} \cdot \mathbb{Z}_p) \times (p \cdot \mathbb{Z}_p) \subseteq I;$$
$$P \stackrel{\mathrm{def}}{=} H_0 \cup (\{p^{-1}\} \times \mathbb{Z}_p) \subseteq I; \quad H_P \stackrel{\mathrm{def}}{=} (p^{-1} \cdot \mathbb{Z}_p) \times \mathbb{Z}_p \subseteq I$$

— where we think of I as being equipped with the *Haar measure* μ_I normalized so that $\mu_I(H_0) = 1$. Thus, p corresponds to "$p_{v_{\mathbb{Q}}}$" such that "$(\mathbb{V}_{\mathrm{mod}})_{v_{\mathbb{Q}}}$" is of *cardinality* 2; I corresponds to "$\mathcal{I}^{\mathbb{Q}}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$"; P corresponds to "P"; H_P corresponds to "$\phi(P)$"; H_0 and H_1 correspond to elements of "$\Xi(P)$", so $H_0 \cup H_1$ corresponds to a subset of "$H_{\Xi(P)}$", hence also a subset of "$H_{\Phi(P)} = \phi(P)$". Then

$$\mu_I(P) = \mu_I(H_0) = 1 < 2 - p^{-1} = \mu_I(H_0 \cup H_1)$$

— i.e., the *inequality of [log-]volumes* in question is *strict*. In fact, by considering various *translates* of H_0, H_1 by *automorphisms* of the $\mathbb{Z}_p$-module H_P, one verifies immediately that H_P corresponds not only to "$\phi(P) = H_{\Phi(P)}$", but also to "$H_{\Xi(P)}$". That is to say, this is a situation in which one has "$H_{\Xi(P)} = H_{\Phi(P)} = \phi(P)$", hence also "$\mu^{\log}(P) < \mu^{\log}(H_{\Xi(P)}) = \mu^{\log}(H_{\Phi(P)}) = \mu^{\log}(\phi(P))$".

(v) Let E be a set, $S \subseteq E$ a *proper subset* of E of *cardinality* ≥ 2 [so $S \neq \emptyset \neq E \setminus S$]. Write

$$(E \twoheadrightarrow) \; E \,\overline{\wedge}\, S \stackrel{\mathrm{def}}{=} (E \setminus S) \coprod \{S\}$$

[i.e., "*E upper S*"] for the *set-theoretic quotient* of E by S, i.e., the quotient of E obtained by identifying the elements of S and leaving $E \setminus S$ unaffected. Write $\overline{\wedge}_S \stackrel{\mathrm{def}}{=} \{S\} \subseteq E \,\overline{\wedge}\, S$. Then observe that

any set-theoretic map

$$(E \supseteq) \; S_1 \to S_2 \; (\subseteq E)$$

between *nonempty subsets* $S_1, S_2 \subseteq S$ $(\subseteq E)$ induces, upon passing to the *quotient* $E \twoheadrightarrow E \,\overline{\wedge}\, S$, the *identity map*

$$(E \,\overline{\wedge}\, S \supseteq) \; \overline{\wedge}_S \to \overline{\wedge}_S \; (\subseteq E \,\overline{\wedge}\, S)$$

> between the *images* [i.e., both of which are equal to $\barwedge_S$!] of S_1, S_2 in $E \barwedge S$, hence *lies over* the *identity map* $E \barwedge S \to E \barwedge S$ on $E \barwedge S$.

Moreover, this map may be *"extended"* to the case where S_i [for $i \in \{1, 2\}$] is *empty* if this S_i is treated as a *"formal intersection"* [cf. our hypothesis that the *cardinality* of S is ≥ 2] — i.e., a *"category-theoretic formal fiber product, or inverse system, over E"* — of some collection of *nonempty subsets* of S. That is to say, such an inverse system induces, upon passing to the *quotient* $E \twoheadrightarrow E \barwedge S$, a system that consists of identity maps between copies of $\barwedge_S$. In particular,

> if one thinks in terms of such *formal inverse systems*, then *"formal empty sets"* $\subseteq S$ ($\subseteq E$) also map to $\barwedge_S \subseteq E \barwedge S$.

Finally, we observe that the above discussion may be thought of as an

> **abstract set-theoretic formalization** of the notions of **upper semi-commutativity/semi-compatibility**, as discussed in Remark 1.2.2, (iii); Remark 1.5.4, (iii); Proposition 3.5, (ii)

— i.e., where [cf. the notational conventions of Propositions 3.2, (ii); 3.5, (ii)] one takes the $S \subseteq E$ of the present discussion to be "$\mathcal{I}((-)) \subseteq \mathcal{I}^{\mathbb{Q}}((-))$", and we observe that, in the context of upper semi-commutativity/semi-compatibility, the *empty set* always arises as an intersection between a nonempty set and the *domain of definition* [cf. the discussion of Remark 1.1.1] of the *"set-theoretic logarithm map"* under consideration.

(vi) Let us return to the discussion of (ii), (iii), (iv). Let $P \in \mathbb{P}$. Then let us observe that

> the *abstract set-theoretic "$\barwedge$-formalism"* of (v) — i.e., where one takes "$S \subseteq E$" to be $\phi(P) \subseteq \mathcal{I}^{\mathbb{Q}}((-))$ — yields a *convenient tool* for **identifying** P with its various **log-volume hull-approximants** $\in \Phi(P)$ or $\in \Xi(P)$ [all of which are nonempty subsets of $\phi(P) \in \mathbb{H}$ — cf. the discussion of (iii)], i.e., of passing to a **quotient** in which the **indeterminacy** discussed in (iii) is **eliminated**.

Moreover, one verifies easily that this *identification* is achieved in such a way that *images* of *distinct* $H_1, H_2 \in \mathbb{H}$ map to the *same subset* of $\mathcal{I}^{\mathbb{Q}}((-)) \barwedge \phi(P)$ if and only if $H_1, H_2 \subseteq \phi(P)$. That is to say, the *equivalence relation* on $\mathbb{H}$ induced by the quotient map $\mathcal{I}^{\mathbb{Q}}((-)) \twoheadrightarrow \mathcal{I}^{\mathbb{Q}}((-)) \barwedge \phi(P)$ is the *"expected equivalence relation"*

$$\mathbb{H} \ni H_1 \ \sim \ H_2 \in \mathbb{H} \qquad \Longleftrightarrow \qquad H_1, H_2 \subseteq \phi(P)$$

on $\mathbb{H}$. Finally, we observe that the discussion of the present (vi) may be applied not only to *single elements* $P \in \mathbb{P}$, but also to *bounded families of elements* $P_B \stackrel{\mathrm{def}}{=} \{P_\beta\}_{\beta \in B}$ indexed by some index set B [i.e., collections of elements $P_\beta \in \mathbb{P}$ such that $\cup_{\beta \in B} P_\beta \subseteq \mathcal{I}^{\mathbb{Q}}((-))$ is *relatively compact*], by taking "$S \subseteq E$" in the discussion of (v) to be

$$\phi(P_B) \ \stackrel{\mathrm{def}}{=} \ \bigcap_{\mathbb{H} \ni H \supseteq P_\beta, \ \forall \beta \in B} H \quad \subseteq \quad \mathcal{I}^{\mathbb{Q}}((-))$$

[cf. the representation of $\phi(P)$ as an intersection in (ii)].

(vii) The operation of forming the **hull** will play a *crucial role* in the context of Corollary 3.12 below, for the following reason:

> the *output of "possible images"* [cf. the statement of Corollary 3.12] that arises from applying the *multiradial algorithms* of Theorem 3.11 below **cannot** be **directly compared** [i.e., at least in any *a priori* sense] to the *objects* in the *local* and *global Frobenioids* that appear in the *codomain* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. Definition 3.8, (i), (ii); [IUTchII], Definition 4.9, (viii)] determined by the **arithmetic line bundle** that gives rise to the ***q*-pilot object**.

The *obstructions* to performing such a *comparison* may be *eliminated* in the following way [cf., especially, the display of (Ob5)]:

(Ob1) **$\mathcal{O}^{\times}$-Indeterminacies acting on tensor packets of log-shells**: The various *"possible images"* that occur as the *output* of the *multiradial algorithms* under consideration are *regions* — i.e., in essence, elements $\in \mathbb{P}$ — contained in *tensor packets* of *log-shells* $\mathcal{I}_k$ [where, for simplicity, we apply the notational conventions of Remark 1.2.2, (i), at *nonarchimedean* valuations]. By contrast, the *arithmetic line bundle* that gives rise to the *q-pilot object* arises, locally, as an *ideal*, i.e., an $\mathcal{O}_k$*-submodule*, contained in the $\mathcal{O}_k$*-module* $\mathcal{O}_k$, which, to avoid confusion, we denote by $\mathcal{O}_k^{\mathrm{mdl}}$. Here, we observe that

> *unlike* the *ring* $\mathcal{O}_k$, the $\mathcal{O}_k$*-module* $\mathcal{O}_k^{\mathrm{mdl}}$ does **not** admit a **canonical generator** [i.e., a canonical element corresponding to the element $1 \in \mathcal{O}_k$]; by contrast, $\mathcal{I}_k \subseteq k$ *can only be defined* by using the **ring structure** of $\mathcal{O}_k$ and is **not**, in general, **stabilized** by the natural action [via multiplication] by $\mathcal{O}_k^{\times}$.

That is to say, $\mathcal{O}_k^{\mathrm{mdl}}$ only admits a *"canonical generator"* up to an *indeterminacy* given by *multiplication by* $\mathcal{O}_k^{\times}$, i.e., an indeterminacy that does *not stabilize* $\mathcal{I}_k$.

(Ob2) **From arbitrary regions to arithmetic vector bundles, i.e., hulls**: Thus, by passing from an *arbitrary given region* $\in \mathbb{P}$ to the associated **hull** $\phi(P) \in \mathbb{H}$, we obtain a region $\phi(P) \in \mathbb{H}$ that is *stabilized* by the natural action of $\mathcal{O}_k^{\times}$ [cf. (Ob1)] and, moreover, [unlike an arbitrary element $\in \mathbb{P}$!] may be regarded as defining the *local* portion of a *global* **arithmetic vector bundle** relative to the ring structure labeled by some $\alpha \in A$ [i.e., which is typically taken, when $0 \in A \subseteq |\mathbb{F}_l|$, to be the *zero label* $0 \in |\mathbb{F}_l|$].

(Ob3) **From arithmetic vector bundles to arithmetic line bundles via "$\det_{\otimes M}(-)$"**: Moreover, by forming the **determinant** of the *arithmetic vector bundle* constituted by a *hull* $\in \mathbb{H}$, one obtains an **arithmetic line bundle**, i.e., which does indeed yield *objects* in the *local* and [by allowing $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, $\underline{v} \in \underline{\mathbb{V}}$ to *vary*] *global Frobenioids* that appear in the *codomain* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, hence may be **compared**, in a meaningful way, to the objects determined by the *arithmetic line bundle* that gives rise to the ***q*-pilot object**. Here, we observe that:

(Ob3-1) **Weighted tensor powers/determinant**: In fact, when forming such a *"determinant"*, it is necessary to perform the following operations:

(Ob3-1-1) In order to obtain a *"determinant"* that is *consistent* with the computation of the **log-volume** by means of certain **weighted sums** [cf. the discussion of Remark 3.1.1], it is necessary to work with **suitable positive tensor powers** [i.e., corresponding to the *weights* — cf. the various products of "N_v's" in the discussion of the final portion of Remark 3.1.1, (iv)] of the determinant line bundles corresponding to the various *direct summands* [as in the second and third displays of Proposition 3.1] of the tensor packet of log-shells "$\mathcal{I}^{\mathbb{Q}}((-))$".

(Ob3-1-2) In order to obtain a *"determinant"* that is *consistent* with the **normalization** of the **log-volume** given by "$\mathcal{O}_{(-)}$" [cf. Proposition 3.9, (i)], it is necessary to *tensor* the "determinant" of (Ob3-1-1) with the *inverse* of the "determinant" [in the sense of (Ob3-1-1)] of the *structure sheaf* [i.e., "$\mathcal{O}_{(-)}$"], which may be thought of as a sort of adjustment to take into account the *ramification* that occurs in the various local fields involved.

[We leave the routine details to the reader.]

(Ob3-2) **Positive tensor powers of the determinant**: In the context of (Ob3-1), we observe that there is *no particular reason* to require that the various *exponents* [i.e., which correspond to *weights* — cf. the various products of "N_v's" in the discussion of the final portion of Remark 3.1.1, (iv)] of these "suitable positive tensor powers" are necessarily *relatively prime*. In particular, the resulting *"determinant"* might in fact be more accurately described as a *"determinant raised to some positive tensor power"*. In the following, we shall denote this operation of forming the *"determinant raised to some positive tensor power"* by means of the notation

$$\text{“}\det_{\otimes M}(-)\text{”}$$

— where M denotes the [uniquely determined] positive integer [cf. the positive integer "$N_{\mathcal{E}}$" that appears in the final portion of the discussion of Remark 3.1.1, (iv)] such that this operation "$\det_{\otimes M}(-)$" maps [the result of tensoring the "$\mathcal{O}_{(-)}$" of Proposition 3.9, (i), with] an *arithmetic line bundle* to the ***M*-th tensor power** of the *arithmetic line bundle*. Thus, for instance, by taking M to be *sufficiently large* [in the "multiplicative sense", i.e., "sufficiently divisible"], we may, for the sake of simplicity, assume [cf. the "stack-theoretic twists" at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, arising from the structure of the stack-theoretic quotient discussed in [IUTchI], Remark 3.1.5] that the *localization* at each $\underline{v} \in \underline{\mathbb{V}}$ of any *arithmetic line bundle* that appears as the *output* of the operation $\det_{\otimes M}(-)$ is always *trivial*.

(Ob3-3) **Determinants and log-volumes**: Finally, we observe in passing that since [cf. the situation discussed in Proposition 3.9, (iii)] the *arithmetic degree* of such an *arithmetic line bundle* may be interpreted, by working with *suitable normalization factors*, as the *log-volume* of the *original arithmetic vector bundle* [i.e., to which the operation $\det_{\otimes M}(-)$ was applied — cf. (Ob3-1), (Ob3-2); the discussion of Remark 3.1.1], *this intermediate step of applying* $\det_{\otimes M}(-)$ *may be omitted* in discussions in which one is only interested in *computing log-volumes*.

(Ob4) **Positive tensor powers of arithmetic line bundles**: From the point of view of the *original goal* [cf. the discussion at the beginning of the present (vii)] of obtaining objects that may be *compared* to the objects in the *local* and *global Frobenioids* that appear in the *codomain* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link determined by the *arithmetic line bundle* that gives rise to the *q-pilot object*, we thus conclude from (Ob3) that

> applying the operation $\det_{\otimes M}(-)$ yields objects that may indeed be **compared** to the objects in the *local* and *global Frobenioids* that appear in the *codomain* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link determined by the **arithmetic line bundle** that gives rise to the $\boldsymbol{M}$**-th tensor power** of the $\boldsymbol{q}$**-pilot object**.

Since, however, the *internal structure* of these *local* and *global Frobenioids* [as well of as the *localization functors* that relate local to global Frobenioids] remains *unaffected*, the **latter "slightly modified goal"** [i.e., of *comparison* with *M-th tensor powers* of objects that arise from the q-pilot object, as opposed to the *"original goal"* of comparison with objects that arise from the original q-pilot object] *does* **not** *result in any substantive problems* such as, for instance, an *indeterminacy* arising from *confusion* between a given arithmetic line bundle and its M-th tensor power [i.e., an indeterminacy analogous to the indeterminacy involving "$\mathbb{I}^{\mathrm{ord}}$" discussed in Remark 2.3.3, (vi)]. One way to understand this situation is as follows:

(Ob4-1) **From non-tensor-power to tensor-power Frobenioids via naive Frobenius functors**: One may think of the *local* and *global Frobenioids* [as well as of the *localization functors* that relate local to global Frobenioids] that appear in the *"slightly modified goal"* as **"$\boldsymbol{M}$-th tensor power versions"** of the *local* and *global Frobenioids* that appear in the *"original goal"*. That is to say, one may think of these *"tensor-power Frobenioids"* as *copies* of the *"non-tensor-power Frobenioids"* obtained by applying the **naive Frobenius functor of degree $\boldsymbol{M}$** of [FrdI], Proposition 2.1, (i). In particular, we conclude [i.e., from [FrdI], Proposition 2.1, (i)] that the *non-tensor-power Frobenioids completely determine the tensor-power Frobenioids*.

(Ob4-2) **From tensor-power to non-tensor-power Frobenioids via tensor power roots**: Alternatively, one may think of the *non-tensor-power Frobenioids* [i.e., that appear in the *"original goal"*] as being obtained from the *tensor-power Frobenioids* [i.e., that appear in the *"slightly modified goal"*] by **"extracting $\boldsymbol{M}$-th power roots"**. Since the *rational function monoids* [cf. [FrdI], Theorem 5.2, (ii)] that give rise to the various *local Frobenioids* under consideration [cf. [IUTchII], Definition 4.9, (vi), (vii), (viii)] are *not divisible* at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the *tensor-power Frobenioids only determine the non-tensor-power Frobenioids up to certain twists*. Of course, these *twists* may be eliminated [cf. (Ob4-1)!] simply by applying the *naive Frobenius functor of degree* M.

(Ob4-3) **Tensor-power-twist indeterminacies**: In particular, if one thinks of the *output* of the crucial operation $\det_{\otimes M}(-)$ [cf. (Ob3)] as lying in the *tensor-power Frobenioids*, then one may always **"reconstruct"** the *non-tensor-power Frobenioids* from the *tensor-power Frobenioids* simply by considering *new copies of the tensor-power Frobenioids* which are related to the *given copies of tensor-power Frobenioids* by applying the *naive Frobenius functor of degree* M whose *domain* is the *new copies*, and whose *codomain* is the *given copies*. On the other hand, these *reconstructed non-tensor-power Frobenioids*, though completely determined up to isomorphism, are only related to one another, when regarded over the *given copies of tensor-power Frobenioids*, up to **certain twists** — i.e., up to a **"tensor-power-twist indeterminacy"** — as discussed in (Ob4-2). Since, however, we shall ultimately [e.g., in the context of Corollary 3.12] only be interested in *estimates of log-volumes*, such *tensor-power-twist indeterminacies* will *not* have any substantive effect on our computations [i.e., of log-volumes — cf. the discussion of (Ob3-3)].

(Ob5) **Independence of the "indeterminacy of possibilities"**: The issue of selecting a **specific element** in some collection of *"possible regions"* $\in \mathbb{P}$ that appears in the *output* of the *multiradial algorithm* is an issue that is **internal** to the algorithm. In particular, in order to **compare**, in a meaningful way, the output of the algorithm to some object — i.e., such as the *arithmetic line bundle* that gives rise to the *q-pilot object* — that is essentially **external** to the algorithm, it is necessary to work with objects that are **independent** of the **choice** of such a specific element/possibility. This may be achieved by

working with the **hull** [cf. the discussion of (Ob1), (Ob2), (Ob3), (Ob4)]

$$\phi(P_B)$$

associated to the *[bounded] collection of possible regions* $P_B \stackrel{\mathrm{def}}{=} \{P_\beta\}_{\beta \in B}$ [cf. the discussion in the final portion of (vi)] that appears as the *output* of the *multiradial algorithms* under consideration and applying the **abstract set-theoretic $\overline{\curlywedge}$-formalism** of (v) [cf. also (vi)].

Here, we observe that this $\overline{\curlywedge}$-formalism of (v) may be applied *not only* to $\phi(P_B)$ *but also* [cf. the discussion of (Ob3)] to $\det_{\otimes M}(\phi(P_B))$ and $\mu^{\log}(\phi(P_B))$ [and in a *compatible* fashion].

(Ob6) **Hull-approximants for the log-volume of a given region**: Since one is ultimately interested in *estimating log-volumes* [cf. the discussion of (iii), (iv)], it is tempting to consider *simply replacing a given region* $P \in \mathbb{P}$ by $\mu^{\log}(P)$. On the other hand, in order to obtain objects *comparable* with the *q-pilot object* [cf. (Ob1), (Ob2), (Ob3), (Ob4), (Ob5)], one is obliged to work with *hulls* $\in \mathbb{H}$ [cf. also the discussion of (Ob7) below]. This state of affairs suggests working with, for instance, $\Xi(P) \subseteq \mathbb{H}$, i.e., with **hull-approximants** for $\mu^{\log}(P)$ [cf. the discussion of (iii), (vi)]. In this context, it is useful to recall [cf. the discussion of (iv)] that, in general, it is not so clear whether or not $\Xi(P) = \emptyset$. This already makes it more natural to consider $\Phi(P) \subseteq \mathbb{H}$ [cf. the discussion of (iii)], i.e., as opposed to $\Xi(P) \subseteq \mathbb{H}$. On the other hand, the issue of **independence** of the **choice** of a *specific possibility internal to the algorithm* under consideration [cf. (Ob5)] already means that one must consider $\mu^{\log}(H_{\Phi(P)})$ or $\mu^{\log}(H_{\Xi(P)})$, as opposed to $\mu^{\log}(P)$, which, in general, *may be* $> \mu^{\log}(P)$ *and indeed* $= \mu^{\log}(\phi(P))$ [cf. the discussion of (iv)].

(Ob7) **Compatibility with $\mathfrak{log}$-Kummer correspondences**: In (Ob6), the discussion of the issue of *simply replacing a given region* $P \in \mathbb{P}$ by $\mu^{\log}(P)$ — i.e., put another away, of passing to the **quotient** [cf. the discussion of Remark 3.9.4, as well as of (viii) below] given by taking the **log-volume** — was subject to the **constraint** that one must construct, i.e., by working with *hulls* [cf. (Ob1), (Ob2), (Ob3), (Ob4), (Ob5)], objects that may be *meaningfully compared* to *objects* in the *local* and *global Frobenioids* that appear in the *codomain* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. This *constraint* prompts the following question:

> Why is it that one cannot simply adopt **log-volumes** as the **"ultimate stage for comparison"** — that is to say, *without* passing through hulls or objects in the local and global Frobenioids referred to above?

At a more technical level, this question may be reformulated as follows:

> Why is it that one cannot **eliminate** the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strip portion* [cf. [IUTchII], Definition 4.9, (vii)] — i.e., in more concrete terms, for, say, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$,
>
> the **local Galois groups** "$G_{\underline{v}}$" and **units** "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$"
>
> [cf. the notation of the discussion surrounding [IUTchI], Fig. I1.2; here and in the following discussion, we regard "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" as being equipped with the *auxiliary structure*, i.e., a *collection of submodules* [cf. [IUTchII], Definition 4.9, (i)] or *system of compatible surjections* [cf. [IUTchII], Definition 4.9, (v)], with which it is equipped in the definition of an $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip] — from the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link?

[Closely related issues are discussed in (ix), (x) below.] The essential reason for this may be understood as follows:

(Ob7-1) **Local Galois groups**: The local Galois groups "$G_{\underline{v}}$" [for, say, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$] satisfy the important property of being **invariant**, up to isomorphism, with respect to the transformations constituted by the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$- and $\mathfrak{log}$-links — cf. the **vertical** and **horizontal coricity** properties discussed in Theorem 1.5, (i), (ii), as well as the discussion of [IUTchII], Remark 3.6.2, (ii). These coricity properties play a *fundamental role* in the theory of the present paper, i.e., by allowing one to relate, via these coricity properties, objects on either side of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$- and $\mathfrak{log}$-links which do *not* satisfy such invariance properties. In particular, the theory of the present series of papers *cannot* function properly if the local Galois groups are eliminated from the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link.

(Ob7-2) **Units**: Thus, it remains to consider what happens if one eliminates the *[Frobenius-like!] units* [but *not* the *local Galois groups* — cf. (Ob7-1)!] from the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. This amounts to replacing the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link by the *associated* $\mathcal{F}^{\Vdash\blacktriangleright}$*-prime-strips* [cf. Definition 2.4, (iii)]. Of course, since one still has the *local Galois groups*, one can consider the *étale-like units* "$\mathcal{O}^{\times\boldsymbol{\mu}}(G_{\underline{v}})$" [i.e., "$\mathcal{O}^{\times\boldsymbol{\mu}}(G)$", in the case where one takes "G" to be $G_{\underline{v}}$] of [IUTchII], Example 1.8, (iv). On the other hand, these *étale-like units* **differ fundamentally** from their *Frobenius-like counterparts* in the following respect:

- The **Frobenius-like units** "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" in the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link are [*tautologically*!] related **only** to the Frobenius-like units at the **same vertical coordinate** [i.e., in a vertical column of the log-theta-lattice] as the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link under consideration, i.e., **not** to the Frobenius-like units at other vertical coordinates in this vertical column. In particular, these Frobenius-like units arise from the **same underlying multiplicative** structure [i.e., of the **ring structure** determined, on various Frobenius-like multiplicative monoids, by the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater to which they belong] as the local and global [*Frobenius-like*!] **value group** portion of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip under consideration. Put another way, the **splittings** of unit group and value group portions that appear in the *intrinsic structure* of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips under consideration [cf. [IUTchII], Definition 4.9, (vi), (viii)] are **consistent** with the *underlying multiplicative* structure of the *ring structure* determined [on various Frobenius-like multiplicative monoids] by the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater under consideration.

- By contrast, the **étale-like counterparts** of these Frobenius-like units are *constrained* by their *vertical coricity* [cf. Theorem 1.5, (i)] to be related, via the relevant $\mathfrak{log}$-*Kummer correspondences*, **simultaneously** to the corresponding Frobenius-like units at **every verti-**

> **cal coordinate** in a vertical column of the log-theta-lattice as the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link under consideration. In particular, the relationship between these étale-like units and the corresponding Frobenius-like units at various vertical coordinates in the vertical column under consideration is **subject** to the action of arbitrary iterates of the $\mathfrak{log}$**-link**, hence to a **complicated confusion** between the *unit group* and *value group* portions at various vertical coordinates of this vertical column. This *complicated confusion* is **inconsistent** with the *intrinsic structure* of the $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips under consideration [cf. Definition 2.4, (iii)], that is to say, with treating the *local and global value group portions* of these $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips as objects that are *not subject* to any *constraints* in their relationship to the *étale-like units*, i.e., to the *local Galois group portions* of these $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips. Put another way, if one regards the *étale-like units* as the **sole access route**, from the point of view of the Frobenius-like units in the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link under consideration, to the Frobenius-like units in the *domain* of this $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, then one obtains a situation in which the data in the $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips [i.e., *"non-mutually constrained local/global value group portions and local Galois groups*] is **"over-constrained/over-determined"**.

Thus, in summary, one **cannot eliminate** the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strip portion* [cf. [IUTchII], Definition 4.9, (vii)] — i.e., in more concrete terms, for, say, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the *local Galois groups* "$G_{\underline{v}}$" and *units* "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" — from the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. One *important consequence* of the fact that the local Galois group and unit portions are indeed included in the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link is the [*"proper functioning"*, as described in the present paper, of the] theory of $\mathfrak{log}$**-Kummer correspondences** and **log-shells** — which serve as **"multiradial containers"** [cf. Remarks 1.5.2, 2.3.3, 2.3.4, 3.8.3] — both of which play a *central role* in the present paper.

(Ob8) **Vertical shifts in the output data**: One *important consequence* of the theory of $\mathfrak{log}$-*Kummer correspondences* lies in the fact that it allows one to **transport/relate** [i.e., by applying the theory of $\mathfrak{log}$-*Kummer correspondences*!] the *output* of the *multiradial algorithms* under consideration to **different vertical coordinates** within a vertical column of the log-theta-lattice. In fact,

> this **output** — even if one works with *hulls* [cf. (Ob1), (Ob2), (Ob3), (Ob4), (Ob5)]! — yields, *a priori*, objects in local and global Frobenioids that **differ**, strictly speaking, from the corresponding [*multiplicative*!] local and global Frobenioids that

appear in the **input data** of the algorithm [i.e., the *codomain* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link — cf. Definition 3.8, (i), (ii)] by a **"vertical shift"** in the log-theta-lattice, i.e., more concretely, by an application of the $\mathfrak{log}$**-link** [that is to say, which produces *additive* log-shells from the *multiplicative* "$\mathcal{O}^{\times\boldsymbol{\mu}}$'s" in the *input data*].

In particular, it is precisely by applying the theory of $\mathfrak{log}$-*Kummer correspondences* that we will ultimately be able to obtain *objects* [i.e., objects in local and global Frobenioids] arising from the *output* of the *multiradial algorithms* under consideration that may indeed be **meaningfully compared** with objects in the local and global Frobenioids that appear in the *input data* of the algorithm [cf. Step (xi-d) of the proof of Corollary 3.12 below]. On the other hand, in this context, it is important to note that since such *comparable objects* are obtained by applying the $\mathfrak{log}$-Kummer correspondence, the local and global Frobenioids to which these *comparable objects* belong are necessarily **subject** to the indeterminacies of the relevant $\mathfrak{log}$-Kummer correspondence, i.e., in more concrete terms, to *arbitrary iterates* of the $\mathfrak{log}$**-link** [cf. the discussion of the final portion of Remark 3.12.2, (v)].

(Ob9) **Hulls in the context of the $\mathfrak{log}$-link and log-volumes**: In the context of the discussion of the final portion of (Ob8), we observe that the operation of *passing to* **realified semi-simplications** [cf. Remark 3.9.4, (iii), (iv), (v), (vi)] in situations where one considers the $\mathfrak{log}$-*link compatibility* of the *log-volume*, is a **quotient operation** on both the *domain* and the *codomain* of the $\mathfrak{log}$-link that induces a **natural bijection** between **log-volumes** of **hulls** in the domain and codomain of the $\mathfrak{log}$-link. That is to say, the fact that this *quotient operation* [i.e., of passing to *realified semi-simplifications*] induces such a *natural bijection* is *not affected* — i.e., unlike the situation considered in (Ob1), (Ob2), (Ob3), (Ob4), (Ob5)! — by the fact that the operation of *passing to realified semi-simplications* [cf. Remark 3.9.4, (iii), (iv), (v), (vi)] involves, at various intermediate steps, the use of various **regions** which are **not hulls**. The *fundamental qualitative difference* between the present situation, on the one hand, and the situation considered in (Ob1), (Ob2), (Ob3), (Ob4), (Ob5) [i.e., which required the formation of *hulls*!], on the other, may be understood as follows:

(Ob9-1) **Formal indeterminacies acting on comparable objects**: Once the passage to **comparable** objects via $\det_{\otimes M}(-)$ of a suitable **hull** has been achieved [cf. the discussion of (Ob5)], the various **formal, or stack-theoretic/diagram-theoretic, indeterminacies** that arose from this passage to comparable objects — i.e.,

- the *tensor-power-twist indeterminacies* of (Ob4-3),
- the application of the $\overline{\wedge}$-*formalism* in (Ob5), and
- the *indeterminacy* with respect to application of *arbitrary iterates* of the $\mathfrak{log}$-*link* of (Ob8)

— have **no effect** on the **comparability** of the objects obtained in (Ob5). That is to say, these indeterminacies function solely as *compatibility conditions* that must be satisfied [e.g., by applying the theory of *realified semi-simplications*, as developed in Remark 3.9.4, (iii), (iv), (v), (vi)] when passing to *"coarse/set-theoretic invariants"* such as the *log-volume*.

(Ob9-2) **Non-explicit relationships between comparable and non-comparable objects**: By contrast, the situation discussed in (Ob1), (Ob2), (Ob3), (Ob4), (Ob5) was one in which — until the *"final conclusion"* of this discussion in (Ob5) — *comparable objects had not yet been obtained*. Put another way, prior to this *"final conclusion"*, the *precise relationship* between the *non-comparable objects* that occurred as the *a priori output* of the *multiradial algorithms* under consideration, on the one hand, and *comparable objects*, on the other, had **not** yet been **explicitly computed**.

Closely related issues are discussed in (ix) below.

(viii) In the context of (vi), (vii), it is of interest to observe that, just as in the case of the operations of

(sQ1) **Kummer-detachment**, i.e., passing from *Frobenius-like* [that is to say, strictly speaking, Frobenius-like structures that contain certain *étale-like structures*] to ["purely"] *étale-like* structures [cf. Remark 1.5.4, (i), as well as the *vertical arrows* of the *commutative diagram* of Remark 3.10.2 below], and

(sQ2) **Galois evaluation** [cf. [IUTchII], Remark 1.12.4, as well as the *horizontal arrows* of the *commutative diagram* of Remark 3.10.2 below],

the operations of

(sQ3) passing from more general regions to **positive tensor powers** of **determinants** of **hulls** and then applying the **abstract set-theoretic $\bar{\wedge}$-formalism** of (v) [cf. the discussion of (Ob1), (Ob2), (Ob3), (Ob4), (Ob5), (Ob6), (Ob7)],

(sQ4) adjusting the **vertical shifts** [i.e., in the vertical column of the log-theta-lattice corresponding to the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link under consideration] in the output of the multiradial algorithm by applying the **$\mathfrak{log}$-Kummer correspondence** [cf. the discussion of (Ob8)], as well as of

(sQ5) passing to **log-volumes** [cf. (Ob3), (Ob4), (Ob6), (Ob7), (Ob9)], via the formalism of **realified semi-simplifications** discussed in Remark 3.9.4,

may be regarded as **intricate (sub)quotient** — or [cf. the discussion of (ii)] **push forward/splitting/integration** — operations. Indeed, from this point of view, the content of the *entire theory of the present series of papers* may be regarded as the development of

a suitable collection of **(sub)quotient operation algorithms** for constructing a

relatively simple, concrete (sub)quotient

of the complicated apparatus constituted by the full **log-theta-lattice**.

The *goal* of this construction of *(sub)quotient operation algorithms* — i.e., of the *entire theory of the present series of papers* — may then be understood as

the **computation** of the **projection**, via the resulting *relatively simple, concrete (sub)quotient*, of

the **"Θ-intertwining"** [i.e., the structure on an abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip as the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip arising from the Θ-*pilot object* appearing in the *domain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-*link* of Definition 3.8, (ii)]

onto structures arising from the *vertical column* in the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-*link*, that is to say, where

the **"*q*-intertwining"** [i.e., the structure on an abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip as the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip arising from the *q-pilot object* appearing in the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-*link* of Definition 3.8, (ii)]

is *in force* [cf. the discussion of Remark 3.12.2, (ii), below].

This *computation*, when suitably interpreted, amounts, *essentially tautologically*, to the **inequality** of Corollary 3.12 below. Here, we observe that each of these (sub)quotient operations (sQ1), (sQ2), (sQ3), (sQ4), (sQ5) may be understood as an operation whose purpose is to **simplify** the quite complicated apparatus constituted by the full *log-theta-lattice* by allowing the introduction of various **indeterminacies**. Put another way, the **nontriviality** of these various (sub)quotient operations lies

in the **very delicate balance** between **minimizing** the **indeterminacies** that *arise from passing to a quotient*, while at the same time ensuring **compatibility** with the structures that exist *prior to formation of the quotient*.

Indeed:

· In the case of (sQ1), i.e., the case of *Kummer-detachment indeterminacies*, this delicate balance is discussed in detail in Remarks 1.5.4, 2.1.1, 2.2.1, 2.2.2, 2.3.3, as well as Remark 3.10.1, (ii), (iii), below.

· In the case of (sQ2), i.e., the case of *Galois evaluation*, the delicate issue of *compatibility* with *Kummer theory* is discussed in [IUTchII], Remark 1.12.4.

· In the case of (sQ3), i.e., the case of passing to *hulls*, various delicate issues — such as, for instance, the delicate issues of *tensor-power-twist indeterminacies* [cf. (Ob4-3)], the $\overline{\wedge}$-*formalism* [cf. (Ob5)], and *compatibility* with $\mathfrak{log}$-*Kummer correspondences* [cf. (Ob7)] — are discussed in (Ob1), (Ob2), (Ob3), (Ob4), (Ob5), (Ob6), (Ob7) [cf. also (ix), (x) below].

· In the case of (sQ4), the adjustment of *vertical shifts* via $\mathfrak{log}$-*Kummer correspondences* results in an indeterminacy with respect to application of *arbitrary iterates* of the $\mathfrak{log}$**-link**, i.e., in the vertical column of the log-theta-lattice corresponding to the *codomain* of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link under consideration [cf. (Ob8)].

· In the case of (sQ5), i.e., the case of passing to *log-volumes*, various subtleties surrounding the *compatibility* of the log-volume with the $\mathfrak{log}$-*link* are discussed in detail in Remark 3.9.4, as well as in (vii) of the present Remark 3.9.5 [cf., especially, the discussion of (Ob9)].

Finally, in this context, we observe that, in light of the **rigidity** of **étale-like** structures [cf. the discussion of [IUTchII], Remark 3.6.2, (ii)], i.e., at a more concrete level, of objects constructed via **anabelian algorithms**, the construction of *suitable subquotients* of the *étale-like* portion of the *log-theta-lattice* — that is to say, as in the case of (sQ2), (sQ3), (sQ4), (sQ5) — is a particularly **nontrivial** issue.

(ix) In the context of the discussion of (vii) [cf., especially, (Ob7), (Ob8), (Ob9)], (viii), it is important to observe that there is a *fundamental qualitative difference* between (sQ3), (sQ4), on the one hand, and (sQ5), on the other:

(cQ3) **Compatibility of (sQ3) with $\mathcal{F}^{\vdash \times \boldsymbol{\mu}}$-prime-strip data**: The fact that [in the notation of (Ob1), (Ob2)] *hulls* $\in \mathbb{H}$ are *stabilized* by multiplication by elements of $\mathcal{O}_k$ implies that, by taking [a suitable positive tensor power of] the determinant [cf. (Ob3)], they determine *objects* [i.e., the "$\det_{\otimes M}(\phi(P_B))$" of (Ob5)] in the *local* and [by allowing $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, $\underline{v} \in \underline{\mathbb{V}}$ to *vary*] *global Frobenioids* that appear in the *codomain* $\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-*prime-strip* of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link. In particular, by considering suitable *pull-back morphisms* in these *local Frobenioids* [i.e., which correspond to base-change morphisms in conventional scheme theory — cf. [FrdI], Definition 1.3, (i)], one obtains *objects* equipped with natural faithful actions by the *local Galois groups* "$G_{\underline{v}}$" and *units* "$\mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" [cf. the notation of (Ob7)], i.e., the data that corresponds to the $\mathcal{F}^{\vdash \times \boldsymbol{\mu}}$-prime-strip portion of the $\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link. Moreover, as discussed in (vi), the *quotient induced on* $\mathbb{H}$ by the *set-theoretic* $\overline{\wedge}$-*formalism* of (v) [cf. the display of (Ob5)] may be understood as corresponding to the consideration of the *"$\overline{\wedge}$-category"* consisting of

($\overline{\wedge}^{\mathrm{lc}}_1$) *objects* in the *local Frobenioid* under consideration equipped with a *"structure poly-morphism"* to the *original object arising from a hull*, i.e., the "$\det_{\otimes M}(\phi(P_B))$" of (Ob5), given by the $\mathrm{Aut}(\det_{\otimes M}(\phi(P_B)))$-orbit of a *linear morphism* in the *local Frobenioid* [cf. [FrdI], Definition 1.2, (i)] to $\det_{\otimes M}(\phi(P_B))$ and

($\overline{\wedge}^{\mathrm{lc}}_2$) *morphisms* between such objects that are compatible with the *structure poly-morphism*.

[Alternatively, one could consider a slightly modified version of this "$\overline{\wedge}$-category" by restricting the objects to be objects that arise from *hull-approximants for the log-volume*, i.e., as in the discussion of (Ob6).] By considering suitable *pull-back morphisms* in this $\overline{\wedge}$-*category*, we again obtain *objects* equipped with *mutually compatible* [i.e., relative to varying the object within the $\overline{\wedge}$-category] natural faithful actions by the *local Galois groups* "$G_{\underline{v}}$" and *units* "$\mathcal{O}^{\times \boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" [cf. the notation of (Ob7)], i.e., the data that corresponds to the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip portion of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. Next, we observe that one may consider *categories of "local-global $\overline{\wedge}$-collections of objects"*, i.e., categories whose *objects* are collections consisting of

($\overline{\wedge}_1^{\text{lc-gl}}$) a *"local"* object in the $\overline{\wedge}$-*category* at each $\underline{v} \in \underline{\mathbb{V}}$,

($\overline{\wedge}_2^{\text{lc-gl}}$) a *"global"* object in the *global realified Frobenioid* of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip under consideration, and

($\overline{\wedge}_3^{\text{lc-gl}}$) *localization isomorphisms* between the image of the *local* object at each $\underline{v} \in \underline{\mathbb{V}}$ in the realification of the local Frobenioid at $\underline{v}$ and the localization of the *global object* at the element $\in$ Prime$(-)$ of the global realified Frobenioid corresponding to $\underline{v}$

[and whose *morphisms* are compatible collections of morphisms between the respective portions of the data $\overline{\wedge}_1^{\text{lc-gl}}$ and $\overline{\wedge}_2^{\text{lc-gl}}$] — cf. the discussion of the [closely related] *functors* in the final displays of [FrdII], Example 5.6, (iii), (iv). In particular, just as the *tensor-power-twist indeterminacies* of (sQ3) [cf. (Ob4-3)] and the *indeterminacy* with respect to application of *arbitrary iterates* of the $\mathfrak{log}$-*link* of (sQ4) [cf. (Ob8)] may be understood as **"formal, or stack-theoretic/diagram-theoretic, quotients"** [i.e., as opposed to *"coarse/set-theoretic quotients"* given by *set-theoretic invariants* such as the *log-volume* — cf. the discussion of (Ob9-1)], the *pair* consisting of

($\overline{\wedge}_1^{\text{fQ}}$) such a *category of "local-global $\overline{\wedge}$-collections"* [cf. $\overline{\wedge}_1^{\text{lc-gl}}$, $\overline{\wedge}_2^{\text{lc-gl}}$, $\overline{\wedge}_3^{\text{lc-gl}}$] and

($\overline{\wedge}_2^{\text{fQ}}$) the analogous *category of "local-global collections"*, i.e., where the "$\overline{\wedge}$-category" at each $\underline{v} \in \underline{\mathbb{V}}$ [cf. $\overline{\wedge}_1^{\text{lc-gl}}$] is replaced by the original *local Frobenioid* [portion of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip under consideration] at each $\underline{v} \in \underline{\mathbb{V}}$,

may also be regarded as the *"formal, or stack-theoretic, quotient"* corresponding to the operation of considering "$\overline{\wedge}_2^{\text{fQ}}$ *modulo* $\overline{\wedge}_1^{\text{fQ}}$".

(cQ4) **Compatibility of (sQ4) with $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip data**: Since the *adjustment of vertical shifts* in (sQ4) is obtained precisely by applying the $\mathfrak{log}$-*Kummer correspondence*, this adjustment operation is *tautologically compatible* [cf. the *vertical coricity* of Theorem 1.5, (i)] with *suitable isomorphisms* between the *local Galois groups* "$G_{\underline{v}}$" and the *étale-like units* "$\mathcal{O}^{\times\boldsymbol{\mu}}(G_{\underline{v}})$" [cf. the notation of (Ob7-1), (Ob7-2)] that appear. Alternatively, this adjustment operation is *tautologically compatible* with *suitable isomorphisms* between the *local Galois groups* "$G_{\underline{v}}$" and the *Frobenius-like units* "$\mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$" [cf. the notation of (Ob7)], so long as one allows for an **indeterminacy** with respect to application of *arbitrary iterates* of the $\mathfrak{log}$**-link** [cf. the discussion of (Ob8)].

(iQ5) **Incompatibility of (sQ5) with $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip data**: By contrast, unlike the situation with (sQ3), (sQ4), passing to *log-volumes* [i.e., (sQ5)] amounts precisely to *forgetting* the *local Galois groups* and *Frobenius-like units*, i.e., the data that corresponds to the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip portion of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. the discussion of (Ob7)].

Here, we recall that (sQ3), (sQ4), (sQ5) all occur *within the vertical column* of the log-theta-lattice corresponding to the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link under consideration. In particular, the various *local Galois groups* "$G_{\underline{v}}$" are all equipped with *rigidifications* as *quotients* of [isomorphs of] "$\Pi_{\underline{v}}$" [cf. the notation of the discussion surrounding [IUTchI], Fig. I1.2]. Put another way [cf. also the discussion of (Ob9)]:

· One may think of the *compatibility* properties (cQ3), (cQ4) as a sort of **arithmetic holomorphicity** [relative to the vertical column under consideration] or, alternatively, as a sort of *compatibility* with the **$\mathfrak{log}$-Kummer correspondence** of this vertical column. This point of view is reminiscent of the use of the descriptive *"holomorphic"* in the term *"holomorphic hull"*.

· Conversely, one may think of the *incompatibility* property (iQ5) as corresponding to the operation of *forgetting* this arithmetic holomorphic structure or, alternatively, as a sort of *incompatibility* with the $\mathfrak{log}$-*Kummer correspondence* of this vertical column.

From the point of view of the *computation* of the *projection* of the Θ-*intertwining* onto the *q-intertwining* discussed in (viii), this *fundamental qualitative difference* — i.e., (cQ3), (cQ4) versus (iQ5) — has a *very substantive consequence*:

It is precisely by *passing through (sQ3), (sQ4)* — i.e., *before applying (sQ5)!* [cf. also the discussion of (Ob7)] — that the **chain** of *poly-isomorphisms of* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* [i.e., including the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip portion of these $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips!] that

· begins with the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip arising from the ***q*-pilot object** in the **codomain** of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link,

· passes through the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link** to the **domain** of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link,

· passes through the various *poly-isomorphisms of* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* [cf. the diagram of Remark 3.10.2 below; the discussion of *"IPL"* in Remark 3.11.1, (iii), below] induced by **(sQ1), (sQ2)**, and

· finally, passes through **(sQ3), (sQ4)**, which satisfy the *compatibility* property with the $\mathfrak{log}$-*Kummer correspondence* discussed above [i.e., (cQ3), (cQ4)]

forms a **closed loop**, i.e., up to the introduction of the **"formal quotient indeterminacies"** discussed in (cQ3), (cQ4) [cf. also the discussion of (Ob9-1)].

In this context, we observe that a *non-closed loop* would yield a situation from which *no nontrivial conclusions may be drawn*, for essentially the *same reason* [that no nontrivial conclusions may be drawn] as in the case of the *"distinct labels approach"* of Remark 3.11.1, (vii), below [cf. also the discussion of (Ob9-2); Remark 3.12.2, (ii), (c^{itw}), (c^{toy}), below]. That it to say, it is only by constructing such a *closed loop* that one can *complete* the *computation* of the *projection* [that is to say, as discussed in (viii)] of the Θ-*intertwining* onto the *q-intertwining*, i.e.,

> *complete the computation of the* **Θ-intertwining** *structure, up to suitable indeterminacies, on a* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip** *that is* **constrained** *to be subject to the* $\boldsymbol{q}$**-intertwining**.

Here, we recall from the discussion of (Ob7) [cf. also (x) below for a discussion of a related topic] that the *construction* of this sort of mathematical structure — i.e.,

> a $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip** that is **simultaneously equipped** with **two intertwinings**, namely, the Θ-*intertwining*, up to indeterminacies, and the *q-intertwining*

— *cannot* be achieved if one omits various subportions of the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip portion of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip. It is this *computation/construction* that will allow us, in Corollary 3.12 below, to conclude *nontrivial*, albeit *essentially tautological*, consequences from the theory of the present series of papers, such as the *inequality* of Corollary 3.12 [cf. Substeps (xi-d), (xi-e), (xi-f), (xi-g) of the proof of Corollary 3.12; Fig. 3.8 below]. Put another way, if one attempts to *skip either (sQ3) or (sQ4)* and *pass directly* from (sQ2) to (sQ4) or (sQ5) [or from (sQ3) to (sQ5)], then the *resulting chain* of poly-isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips *no longer forms a closed loop*, and one can *no longer* conclude any *nontrivial* consequences from the theory of the present series of papers.

(x) In the context of the discussion of (vii), (viii), (ix), it is of interest to observe that it is **not possible** [at least in any immediate sense!] to

> work with regions $\in \mathbb{P}$ that do not necessarily belong to $\mathbb{H}$ — and hence *avoid the operation of passing to the hull*! — by replacing the *local and global Frobenioids* [i.e., categories of local and global *arithmetic line bundles*] that appear in the definition of an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip [cf. [IUTchII], Definition 4.9, (viii)] by *"more general* **categories of regions** $\in \mathbb{P}$*"*.

Indeed, any sort of category of regions $\in \mathbb{P}$ necessarily requires consideration of the **multi-dimensional underlying space** of $\mathcal{I}^{\mathbb{Q}}((-))$ [cf. (ii)], i.e., in essence, an **additive module** of **rank** > 1. Put another way, the only natural way to relate various *"lines"* [i.e., rank 1 submodules] within this space to one another is by invoking the *additive structure* of this module. On the other hand, since the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link is **not compatible** with the **additive structures** in its domain and codomain, it is of *crucial importance* that the categories that are *glued together* via the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link be **purely multiplicative** in nature, i.e., independent, at least in an *a priori* sense, of the *additive structures* in the domain and codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. In particular, one must, in effect, work with **arithmetic line bundles** [which — unlike *arithmetic vector bundles of rank* > 1! — may indeed be defined in a way that only uses the *multiplicative* structures of the rings involved] — cf. the discussion of (Ob1), (Ob2), (Ob3), (Ob4). Of course, instead of working [as we in fact do in the present series of papers] with arithmetic line bundles over F_{mod}, up to certain "stack-theoretic twists" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. [IUTchI], Remark 3.1.5], where we work with local arithmetic line bundles over $K_{\underline{v}}$ [which are necessary in order to accommodate the use of various powers of $\underline{\underline{q}}_{\underline{v}}$! — cf. [IUTchI], Example 3.2, (iv)], one could instead consider working with *arithmetic line bundles over* $\mathbb{Q}$. Relative to the arithmetic line bundles over F_{mod} or $K_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, that in fact appear in the present series of papers, working with arithmetic line bundles over $\mathbb{Q}$ amounts, in effect, to applying some sort of

> **norm**, or **determinant** operation, from F_{mod} down to $\mathbb{Q}$ or, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, from $K_{\underline{v}}$ down to $\mathbb{Q}_{p_{\underline{v}}}$

[followed by tensor product with a certain fixed arithmetic line bundle on $\mathbb{Q}$ or $\mathbb{Q}_{p_{\underline{v}}}$, in order to take into account the *ramification* of F_{mod} over $\mathbb{Q}$ or $K_{\underline{v}}$ over $\mathbb{Q}_{p_{\underline{v}}}$ — cf. the discussion of (Ob3-1-2)]. On the other hand, if we write $G_{p_{\underline{v}}} \subseteq \mathrm{Gal}(\overline{F}/\mathbb{Q})$ for the unique decomposition group of $p_{\underline{v}}$ that contains $G_{\underline{v}}$, then one verifies immediately that the fact that $G_{p_{\underline{v}}}$ does **not** admit a **splitting**

$$\text{“}G_{p_{\underline{v}}} \ \stackrel{\sim}{\to}\ G_{\underline{v}} \times (G_{p_{\underline{v}}}/G_{\underline{v}})\text{”}$$

implies that this sort of *norm operation* from $K_{\underline{v}}$ down to $\mathbb{Q}_{p_{\underline{v}}}$ **cannot** be extended, in any meaningful sense, to any sort of **Galois-equivariant** [i.e., $G_{\underline{v}}$-equivariant] operation on algebraic closures of $K_{\underline{v}}$ and $\mathbb{Q}_{p_{\underline{v}}}$. Since the *faithful action* of $G_{\underline{v}}$ on the *unit portion* of the local Frobenioids in an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip plays a *central role* [cf. the discussion of (Ob7)] in the theory of **$\mathfrak{log}$-Kummer correspondences** and **log-shells** [which play a *central role* in the present paper!], the *incompatibility* of any sort of *norm operation* with the *local Galois group* $G_{\underline{v}}$ makes such a norm operation *fundamentally unsuited* for defining the *gluings* that constitute the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link.

Remark 3.9.6. In the context of Proposition 3.9, (iii), (iv) [cf. also Remark 3.9.4], we make the following observation. The *$\mathfrak{log}$-link compatibility* of Proposition 3.9, (iv) [cf. also Proposition 1.2, (iii); Proposition 1.3, (iii); Remark 3.9.4] amounts to a *coincidence of log-volumes* — *not* of *arbitrary* regions that appear in the domain and codomain of the $\mathfrak{log}$-link, but rather — of *certain types of “sufficiently small” regions* that appear in the domain and codomain of the $\mathfrak{log}$-link. On the other hand, the *“product formula”* — i.e., at a more concrete level, the *“ratios of conversion”* [cf. [IUTchI], Remark 3.5.1, (ii)] between *log-volumes* at distinct $\underline{v} \in \underline{\mathbb{V}}$ — may be formulated [without loss of generality!] in terms of such “sufficiently small” regions. Thus, in summary, we conclude that

> the **$\mathfrak{log}$-link compatibility** of Proposition 3.9, (iv), implies a **compatibility** of **“product formulas”**, i.e., of **“ratios of conversion”** between *log-volumes* at distinct $\underline{v} \in \underline{\mathbb{V}}$, in the domain and codomain of the $\mathfrak{log}$-link.

In particular, in the context of Proposition 3.9, (iii), we conclude that Proposition 3.9, (iv), implies a **compatibility** between **global arithmetic degrees** in the domain and codomain of the $\mathfrak{log}$-link.

Remark 3.9.7. When computing *log-volumes of various regions* of the sort considered in Proposition 3.9, it is useful to keep the following *elementary observations* in mind:

(i) In the context of Proposition 3.9, (iii), the defining condition “**zero log-volume** for all but finitely many $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$” for

$$\mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})) \ \subseteq \ \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}))$$

that is imposed on the various components indexed by $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ of the direct product of the above display may be satisfied by considering elements of this direct product such that for *all but finitely many* of the elements $w_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$ for which $p_{w_{\mathbb{Q}}}$ is **unramified** in K, the component at $w_{\mathbb{Q}}$ is given by $\mathcal{I}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$. Indeed, for each such $w_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$, the subset

$$\mathcal{O}_{(-)} = \mathcal{I}((-)) \subseteq \mathcal{I}^{\mathbb{Q}}((-))$$

[cf. the notation of Proposition 3.2, (ii); Proposition 3.9, (i); the final sentence of [AbsTopIII], Definition 5.4, (iii)] has *zero log-volume*. Finally, in the context of Proposition 3.9, (ii), we observe that, for each such $w_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}^{\mathrm{non}}$, the subset $\mathcal{I}((-)) \subseteq \mathcal{I}^{\mathbb{Q}}((-))$ is a *mono-analytic invariant*, which, moreover, [cf. Remark 3.9.5, (i)] is equal to its own *holomorphic hull*.

(ii) In the context of Proposition 3.9, (i), (ii), we observe that one may consider the **log-volume** of **more general**, say, **relatively compact subsets** $E \subseteq \mathcal{I}^{\mathbb{Q}}((-))$ [cf. the discussion of Remark 3.1.1, (iii)] than the sets which belong to $\mathfrak{M}(\mathcal{I}^{\mathbb{Q}}((-)))$, i.e.,

> simply by *defining* the *log-volume of* E to be the **infimum** of the log-volumes of the sets $E^* \in \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}((-)))$ such that $E \subseteq E^*$.

This definition means that one must allow for the possibility that the *log-volume of* E is $-\infty$. Alternatively [and *essentially equivalently*!], one can treat such E by thinking of such an E as corresponding to the

inverse system of $E^* \in \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}((-)))$ such that $E \subseteq E^*$.

Here, when E is a **direct product pre-region**, it is natural to consider instead the inverse system of **direct product regions** $E^* \in \mathfrak{M}(\mathcal{I}^{\mathbb{Q}}((-)))$ such that $E \subseteq E^*$ [cf. the discussion of Remark 3.1.1, (iii)]. This *approach via inverse systems* of regions each of which has *finite log-volume* has the advantage that it allows one to always work with **finite log-volumes**.

(iii) In a similar vein, in the context of Proposition 3.9, (iii), we observe that one may consider the **log-volume** of **more general collections of relatively compact subsets** [cf. the discussion of Remark 3.1.1, (iii)] than the collections of sets of the sort considered in the discussion of (i) above. Indeed, if

$$\{E_{v_{\mathbb{Q}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})\}_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}}$$

is a collection of subsets such that, for *some* collection of sets $\{E^*_{v_{\mathbb{Q}}}\}_{v_{\mathbb{Q}}}$ of the sort considered in the discussion of (i), it holds that $E_{v_{\mathbb{Q}}} \subseteq E^*_{v_{\mathbb{Q}}}$, for each $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, then one may

> simply *define* the *log-volume of* $\{E_{v_{\mathbb{Q}}}\}_{v_{\mathbb{Q}}}$ to be the **infimum** of the log-volumes of the collections of sets $\{E^*_{v_{\mathbb{Q}}}\}_{v_{\mathbb{Q}}}$ of the sort considered in the discussion of (i) above such that $E_{v_{\mathbb{Q}}} \subseteq E^*_{v_{\mathbb{Q}}}$, for each $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$

[in which case one must allow for the possibility that the *log-volume of* E is $-\infty$]; alternatively [and *essentially equivalently*!], one may think of such a collection $\{E_{v_{\mathbb{Q}}}\}_{v_{\mathbb{Q}}}$ as corresponding to the

inverse system of collections $\{E^*_{v_{\mathbb{Q}}}\}_{v_{\mathbb{Q}}}$ of the sort considered in the discussion of (i) above such that $E_{v_{\mathbb{Q}}} \subseteq E^*_{v_{\mathbb{Q}}}$, for each $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$

[an approach that has the advantage that it allows one to always work with **finite log-volumes**]. Here, in the case where each $E_{v_{\mathbb{Q}}}$, for $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, is a **direct product pre-region**, it is natural to consider instead inverse systems $\{E^*_{v_{\mathbb{Q}}}\}_{v_{\mathbb{Q}}}$ as above such that each $E^*_{v_{\mathbb{Q}}}$, for $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, is a **direct product region** [cf. the discussion of Remark 3.1.1, (iii)].

Proposition 3.10. (Global Kummer Theory and Non-interference with Local Integers) *Let* $\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n,m\in\mathbb{Z}}$ *be a* **collection of distinct** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *[relative to the given initial* Θ*-data] — which we think of as arising from an* LGP**-Gaussian log-theta-lattice** *[cf. Definition 3.8, (iii); Proposition 3.5; Remark 3.8.2]. For each* $n \in \mathbb{Z}$*, write*

$${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

for the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater determined, up to isomorphism, by the various* ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*, where* $m \in \mathbb{Z}$*, via the* **vertical coricity** *of Theorem 1.5, (i) [cf. Remark 3.8.2].*

(i) **(Vertically Coric Global** LGP**-,** $\mathfrak{lgp}$**-Frobenioids and Associated Kummer Theory)** *Recall the constructions of various global Frobenioids in Proposition 3.7, (i), (ii), (iii), (iv), in the context of* $\mathcal{F}$*-prime-strip processions. Then by applying these constructions to the* $\mathcal{F}$*-prime-strips "*$\mathfrak{F}({}^{n,\circ}\mathfrak{D}_{\succ})_t$*" [cf. the notation of Proposition 3.5, (i)] and the various full* $\mathfrak{log}$*-links associated [cf. the discussion of Proposition 1.2, (ix)] to these* $\mathcal{F}$*-prime-strips — which we consider in a fashion* **compatible** *with the* $\mathbb{F}_l^{\rtimes\pm}$**-symmetries** *involved [cf. Remark 1.3.2; Proposition 3.4, (ii)] — we obtain* **functorial algorithms** *in the* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater** ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *for constructing [number]* **fields***,* **monoids***, and* **Frobenioids** *equipped with natural isomorphisms*

$$\begin{aligned}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha \;&=\; \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha \\ \supseteq\; \mathbb{M}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha \;&=\; \mathbb{M}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha\end{aligned}$$

$$\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha \;\supseteq\; \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

$$\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha \;\overset{\sim}{\to}\; \mathcal{F}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha \;\overset{\sim}{\to}\; \mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

[cf. the number fields, monoids, and Frobenioids "$\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j \supseteq \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$*", "*$\mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})_j$*" of [IUTchII], Corollary 4.7, (ii)] for* $\alpha \in A$*, where* A *is a subset of* J *[cf. Proposition 3.3], as well as* $\mathcal{F}^{\Vdash}$**-prime-strips** *equipped with natural isomorphisms*

$$\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{gau}} \;\overset{\sim}{\to}\; \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}} \;\overset{\sim}{\to}\; \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}}$$

— [all of] which we shall refer to as being **"vertically coric"***. For each* $n, m \in \mathbb{Z}$*, these functorial algorithms are* **compatible** *[in the evident sense] with the ["non-vertically coric"] functorial algorithms of Proposition 3.7, (i), (ii), (iii), (iv) —*

i.e., where [in Proposition 3.7, (iii), (iv)] we take "†" to be "n, m" and "‡" to be "$n, m-1$" — relative to the **Kummer isomorphisms** *of labeled data*

$$\Psi_{\mathrm{cns}}({}^{n,m'}\mathfrak{F}_{\succ})_t \quad \xrightarrow{\sim} \quad \Psi_{\mathrm{cns}}({}^{n,\circ}\mathfrak{D}_{\succ})_t$$

$$({}^{n,m'}\mathbb{M}^{\circledast}_{\mathrm{mod}})_j \xrightarrow{\sim} \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{D}^{\circledcirc})_j; \quad ({}^{n,m'}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_j \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{D}^{\circledcirc})_j$$

[cf. [IUTchII], Corollary 4.6, (iii); [IUTchII], Corollary 4.8, (ii)] and the evident identification, for $m' = m, m-1$, of ${}^{n,m'}\mathfrak{F}_t$ [i.e., the $\mathcal{F}$-prime-strip that appears in the associated $\Theta^{\pm}$-bridge] with the $\mathcal{F}$-prime-strip associated to $\Psi_{\mathrm{cns}}({}^{n,m'}\mathfrak{F}_{\succ})_t$ [cf. Proposition 3.5, (i)]. In particular, for each $n, m \in \mathbb{Z}$, we obtain **"Kummer isomorphisms"** *of fields, monoids, Frobenioids, and $\mathcal{F}^{\Vdash}$-prime-strips*

$$({}^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_\alpha \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha; \quad ({}^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_\alpha \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

$$({}^{n,m}\mathbb{M}^{\circledast}_{\mathfrak{mod}})_\alpha \xrightarrow{\sim} \mathbb{M}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha; \quad ({}^{n,m}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_\alpha \xrightarrow{\sim} \mathbb{M}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

$$({}^{n,m}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_\alpha \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha; \quad ({}^{n,m}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_\alpha \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

$$({}^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_\alpha \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha; \quad ({}^{n,m}\mathbb{M}^{\circledast}_{\mathrm{mod}})_\alpha \xrightarrow{\sim} \mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

$$({}^{n,m}\mathcal{F}^{\circledast}_{\mathrm{mod}})_\alpha \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha; \quad {}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{gau}} \xrightarrow{\sim} \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{gau}}$$

$${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}} \xrightarrow{\sim} \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}; \quad {}^{n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}} \xrightarrow{\sim} \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}}$$

that are **compatible** *with the various equalities, natural inclusions, and natural isomorphisms discussed above.*

(ii) **(Non-interference with Local Integers)** *In the notation of Propositions 3.2, (ii); 3.4, (i); 3.7, (i), (ii); 3.9, (iii), we have*

$$({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_\alpha \bigcap \prod_{\underline{v}\in\underline{\mathbb{V}}} \Psi_{\mathfrak{log}({}^{A,\alpha}\mathcal{F}_{\underline{v}})} = ({}^{\dagger}\mathbb{M}^{\circledast\boldsymbol{\mu}}_{\mathrm{MOD}})_\alpha$$

$$\Big(\subseteq \prod_{\underline{v}\in\underline{\mathbb{V}}} \mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}}) = \prod_{v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}} \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) = \mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})\Big)$$

— where we write $({}^{\dagger}\mathbb{M}^{\circledast\boldsymbol{\mu}}_{\mathrm{MOD}})_\alpha \subseteq ({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_\alpha$ for the [finite] subgroup of torsion elements, i.e., **roots of unity***; for $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, we identify the product $\prod_{\underline{\mathbb{V}}\ni\underline{v}|v_{\mathbb{Q}}} \mathcal{I}^{\mathbb{Q}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$ with $\mathcal{I}^{\mathbb{Q}}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}})$. Now let us think of the various* **groups**

$$({}^{n,m}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_j$$

[of nonzero elements of a number field] as acting on various portions of the modules

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{\mathbb{V}_{\mathbb{Q}}})$$

[i.e., where the subscript "$\mathbb{V}_{\mathbb{Q}}$" denote the direct product over $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ — cf. the notation of Proposition 3.5, (ii)] **not** *via a* **single Kummer isomorphism** *as in (i), but rather via the* **totality** *of the various pre-composites of Kummer isomorphisms with* **iterates** *[cf. Remark 1.1.1] of the* **$\mathfrak{log}$-links** *of the* LGP*-Gaussian*

log-theta-lattice — where we observe that these actions are **mutually compatible** *up to [harmless!]* **"identity indeterminacies"** *at an adjacent "m", precisely as a consequence of the equality of the first display of the present (ii) [cf. the discussion of Remark 1.2.3, (ii); the discussion of Definition 1.1, (ii), concerning quotients by "$\Psi^{\boldsymbol{\mu}_N}_{{}^{\dagger}\mathcal{F}_{\underline{v}}}$" at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$; the discussion of Definition 1.1, (iv), at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$] — cf. also the discussion of Remark 3.11.4 below. Thus, one obtains a sort of* **"$\mathfrak{log}$-Kummer correspondence"** *between the* **totality**, *as m ranges over the elements of $\mathbb{Z}$, of the various groups [of nonzero elements of a number field] just discussed [i.e., which are labeled by "n, m"] and their actions [as just described] on the "$\mathcal{I}^{\mathbb{Q}}$" labeled by "$n, \circ$" which is* **invariant** *with respect to the* **translation symmetries** *[cf. Proposition 1.3, (iv)] of the n-th column of the* LGP*-Gaussian log-theta-lattice [cf. the discussion of Remark 1.2.2, (iii)].*

(iii) **(Frobenioid-theoretic $\mathfrak{log}$-Kummer Correspondences)** *The relevant Kummer isomorphisms of (i) induce, via the "$\mathfrak{log}$-Kummer correspondence" of (ii) [cf. also Proposition 3.7, (i); Remarks 3.6.1, 3.9.2],* **isomorphisms of Frobenioids**

$$({}^{n,m}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_\alpha \ \overset{\sim}{\to}\ \mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{H}\mathcal{T}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

$$({}^{n,m}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_\alpha \ \overset{\sim}{\to}\ \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{H}\mathcal{T}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_\alpha$$

that are **mutually compatible**, *as m varies over the elements of $\mathbb{Z}$, with the $\mathfrak{log}$-links of the* LGP*-Gaussian log-theta-lattice. Moreover, these compatible isomorphisms of Frobenioids, together with the relevant Kummer isomorphisms of (i), induce, via the* **global** *"$\mathfrak{log}$-Kummer correspondence" of (ii) and the* **splitting monoid** *portion of the "$\mathfrak{log}$-Kummer correspondence" of Proposition 3.5, (ii),* **isomorphisms of associated $\mathcal{F}^{\Vdash\perp}$-prime-strips** *[cf. Definition 2.4, (iii)]*

$${}^{n,m}\mathfrak{F}^{\Vdash\perp}_{\mathrm{LGP}} \ \overset{\sim}{\to}\ \mathfrak{F}^{\Vdash\perp}({}^{n,\circ}\mathcal{H}\mathcal{T}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}$$

that are **mutually compatible**, *as m varies over the elements of $\mathbb{Z}$, with the $\mathfrak{log}$-links of the* LGP*-Gaussian log-theta-lattice.*

Proof. The various assertions of Proposition 3.10 follow immediately from the definitions and the references quoted in the statements of these assertions. Here, we observe that the computation of the *intersection* of the first display of (ii) is an immediate consequence of the well-known fact that the set of nonzero elements of a number field that are *integral* at all of the places of the number field consists of the set of *roots of unity* contained in the number field [cf. the discussion of Remark 1.2.3, (ii); [Lang], p. 144, the proof of Theorem 5]. ○

Remark 3.10.1.

(i) Note that the $\mathfrak{log}$-*Kummer correspondence* of Proposition 3.10, (ii), induces isomorphisms of Frobenioids as in the first display of Proposition 3.10, (iii), precisely because the construction of "$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_\alpha$" only involves the group "$({}^{\dagger}\mathbb{M}^{\circledast}_{\mathrm{MOD}})_\alpha$", together with the collection of *subquotients* of its perfection indexed by $\underline{\mathbb{V}}$ [cf. Proposition 3.7, (i); Remarks 3.6.1, 3.9.2]. By contrast, the construction of "$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_\alpha$" also involves the *local monoids* "$\Psi_{\mathfrak{log}({}^{A,\alpha}\mathcal{F}_{\underline{v}})} \subseteq \underline{\mathfrak{log}}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$" in an essential way [cf.

Proposition 3.7, (ii)]. These local monoids are subject to a somewhat more complicated "$\mathfrak{log}$-Kummer correspondence" [cf. Proposition 3.5, (ii)] that revolves around *"upper semi-compatibility"*, i.e., in a word, *one-sided inclusions*, as opposed to precise equalities. The imprecise nature of such one-sided inclusions is *incompatible* with the construction of "$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha}$". In particular, one cannot construct $\mathfrak{log}$-*link-compatible* isomorphisms of Frobenioids for "$({}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_{\alpha}$" as in the first display of Proposition 3.10, (iii).

(ii) The **precise compatibility** of "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$" with the $\mathfrak{log}$-links of the LGP-Gaussian log-theta-lattice [cf. the discussion of (i); the first "mutual compatibility" of Proposition 3.10, (iii)] makes it more suited [i.e., by comparison to "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$"] to the task of computing the **Kummer-detachment indeterminacies** [cf. Remark 1.5.4, (i), (iii)] that arise when one attempts to pass from the *Frobenius-like structures* constituted by the global portion of the domain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-*links* of the LGP-Gaussian log-theta-lattice to corresponding *étale-like structures.* That is to say, the mutual compatibility of the isomorphisms

$$^{n,m}\mathfrak{F}^{\Vdash\perp}_{\mathrm{LGP}} \ \stackrel{\sim}{\to}\ \mathfrak{F}^{\Vdash\perp}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}$$

of the second display of Proposition 3.10, (iii), asserts, in effect, that such *Kummer-detachment indeterminacies do not arise.* This is precisely the reason why we wish to work with the LGP-, as opposed to the $\mathfrak{lgp}$-, Gaussian log-theta lattice [cf. Remark 3.8.1]. On the other hand, the essentially **multiplicative** nature of "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$" [cf. Remark 3.6.2, (ii)] makes it *ill-suited* to the task of computing the **étale-transport indeterminacies** [cf. Remark 1.5.4, (i), (ii)] that occur as one passes between distinct arithmetic holomorphic structures on opposite sides of a $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link.

(iii) By contrast, whereas the **additive nature** of the local modules [i.e., local fractional ideals] that occur in the construction of "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" renders "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" ill-suited to the computation of Kummer-detachment indeterminacies [cf. the discussion of (i), (ii)], the *close relationship* [cf. Proposition 3.9, (i), (ii), (iii)] of these local modules to the **mono-analytic log-shells** that are **coric** with respect to the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. Theorem 1.5, (iv); Remark 3.8.2] renders "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" *well-suited* to the computation of the **étale-transport indeterminacies** that occur as one passes between distinct arithmetic holomorphic structures on opposite sides of a $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. That is to say, although various *distortions* of these local modules arise as a result of both [the Kummer-detachment indeterminacies constituted by] the local **"upper semi-compatibility"** of Proposition 3.5, (ii), and [the étale-transport indeterminacies constituted by] the **discrepancy** *between* **local holomorphic** *and* **mono-analytic** *integral structures* [cf. Remark 3.9.1, (i), (ii)], one may nevertheless compute — i.e., if one takes into account the various distortions that occur, **"estimate"** — the **global arithmetic degrees** of objects of "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" by computing **log-volumes** [cf. Proposition 3.9, (iii)], which are **bi-coric**, i.e., coric with respect to both the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-links [cf. Proposition 3.9, (ii)] and the $\mathfrak{log}$-links [cf. Proposition 3.9, (iv)] of the LGP-Gaussian log-theta-lattice. This **computability** is precisely the topic of Corollary 3.12 below. On the other hand, the issue of obtaining *concrete estimates* will be treated in [IUTchIV].

$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$/<u>LGP-*structures*</u>	$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$/<u>$\mathfrak{lgp}$-*structures*</u>
biased toward **multiplicative** structures	biased toward **additive** structures
easily related to **value group/non-coric** portion "$(-)^{\Vdash\blacktriangleright}$" of $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link	easily related to **unit group/coric** portion "$(-)^{\vdash\times\boldsymbol{\mu}}$" of $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-link, i.e., **mono-analytic log-shells**
admits **precise $\mathfrak{log}$-Kummer correspondence**	only admits **"upper semi-compatible" $\mathfrak{log}$-Kummer correspondence**
rigid, but not suited to explicit computation	subject to **substantial distortion**, but suited to **explicit estimates**

Fig. 3.2: $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$/LGP-structures versus $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$/$\mathfrak{lgp}$-structures

(iv) The various properties of "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$" and "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" discussed in (i), (ii), (iii) above are summarized in Fig. 3.2 above. In this context, it is of interest to observe that the natural isomorphisms of Frobenioids

$$\mathcal{F}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\alpha} \ \stackrel{\sim}{\to}\ \mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\alpha}$$

as well as the resulting isomorphisms of $\mathcal{F}^{\Vdash}$-prime-strips

$$\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}} \ \stackrel{\sim}{\to}\ \mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}}$$

of Proposition 3.10, (i), play the *highly nontrivial* role of relating [cf. the discussion of [IUTchII], Remark 4.8.2, (i)] the "*multiplicatively biased* $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$" to the "*additively biased* $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" by means of the **global ring structure** of the number field $\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\alpha} = \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\alpha}$. A similar statement holds concerning the tautological isomorphism of $\mathcal{F}^{\Vdash}$-prime-strips ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}} \stackrel{\sim}{\to} {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$ of Proposition 3.7, (iv).

Remark 3.10.2. In the context of the various *Kummer isomorphisms* discussed in the final display of Proposition 3.10, (i), it is useful to recall that the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}}$, ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$ that appear in the definition of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-*links* in Definition 3.8, (ii), were constructed from the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip ${}^{\dagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}}$ [associated to the $\mathcal{F}^{\Vdash}$-prime-strip ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}}$] of [IUTchII], Corollary 4.10, (ii), in a fashion that we review as follows. First, we remark that, in the present discussion, it is convenient for us to think of ourselves as working with objects arising from the LGP-*Gaussian log-theta-lattice* of Definition 3.8, (iii) [so "†" will be replaced by "(n, m)" or "$(n, \circ)$"]. Now recall, from the theory developed so far in the present series of papers, that we have a *commutative diagram of* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips*

$$\begin{array}{ccccccc}
\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}({}^{n,\circ})_{\mathrm{env}} & \overset{\sim}{\to} & \mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}({}^{n,\circ})_{\mathrm{gau}} & \overset{\sim}{\to} & \mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}({}^{n,\circ})_{\mathrm{LGP}} & \overset{\sim}{\to} & \mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}({}^{n,\circ})_{\mathfrak{lgp}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
{}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}} & \overset{\sim}{\to} & {}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{gau}} & \overset{\sim}{\to} & {}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}} & \overset{\sim}{\to} & {}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathfrak{lgp}}
\end{array}$$

— where

- for simplicity, we use the abbreviated version "${}^{n,\circ}$" of the notation "${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$" of Proposition 3.10, (i);

- the *first vertical arrow* is the induced $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip version of the *Kummer isomorphism* [whose codomain includes an argument "${}^{\dagger}\mathfrak{D}_{>}$", which we denote here by "${}^{n,\circ}$"] of the final display of Proposition 2.1, (ii) [cf. also Proposition 2.1, (iii), (iv), (v)];

- the *second, third, and fourth vertical arrows* are the induced $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip versions of the *Kummer isomorphisms* of the final display of Proposition 3.10, (i);

- the *first lower horizontal arrow* is the induced $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip version of the *evaluation isomorphism* of the final display of [IUTchII], Corollary 4.10, (ii);

- the *second and third lower horizontal arrows* are the induced $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip versions of the *tautological isomorphisms* of the final displays of Proposition 3.7, (iii), (iv);

- the *first upper horizontal arrow* is the induced $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip version of the *étale-like evaluation isomorphism* implicit in the construction [via [IUTchII], Corollary 4.6, (iv), (v)] of the evaluation isomorphism of the final display of [IUTchII], Corollary 4.10, (ii);

- the *second and third upper horizontal arrows* are the induced $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip versions of the *natural isomorphisms* of the second display of Proposition 3.10, (i).

That is to say, in summary,

> the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}}$, ${}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$ that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-, $\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$-*links* of Definition 3.8, (iii), were *constructed from the* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* ${}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}}$ and *related to this* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strip* ${}^{n,m}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{env}}$ via the lower horizontal arrows of the above commutative diagram; moreover, each of these *lower horizontal arrows* may be constructed by *conjugating the corresponding upper horizontal arrow* by the relevant *Kummer isomorphisms*, i.e., by the *vertical arrows* in the diagram.

We are now ready to discuss the *main theorem* of the present series of papers.

Theorem 3.11. (Multiradial Algorithms via LGP-Monoids/Frobenioids) *Fix a collection of* **initial Θ-data**

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}},\ \underline{\epsilon})$$

as in [IUTchI], Definition 3.1. Let

$$\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n,m\in\mathbb{Z}}$$

be a **collection of distinct** $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** *[relative to the given initial* Θ*-data] — which we think of as arising from an* LGP**-Gaussian log-theta-lattice** *[cf. Definition 3.8, (iii)]. For each* $n \in \mathbb{Z}$*, write*

$${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

for the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater determined, up to isomorphism, by the various* ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*, where* $m \in \mathbb{Z}$*, via the* **vertical coricity** *of Theorem 1.5, (i) [cf. Remark 3.8.2].*

(i) **(Multiradial Representation)** *Consider the* **procession of** $\mathcal{D}^{\vdash}$**-prime-strips** $\mathrm{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T)$

$$\{{}^{n,\circ}\mathfrak{D}^{\vdash}_0\} \hookrightarrow \{{}^{n,\circ}\mathfrak{D}^{\vdash}_0,\ {}^{n,\circ}\mathfrak{D}^{\vdash}_1\} \hookrightarrow \ \dots\ \hookrightarrow \{{}^{n,\circ}\mathfrak{D}^{\vdash}_0,\ {}^{n,\circ}\mathfrak{D}^{\vdash}_1,\ \dots,\ {}^{n,\circ}\mathfrak{D}^{\vdash}_{l^*}\}$$

obtained by applying the natural functor of [IUTchI], Proposition 6.9, (ii), to [the $\mathcal{D}$-$\Theta^{\pm}$*-bridge associated to]* ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$*. Consider also the following data:*

(a) for $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$*,* $j \in |\mathbb{F}_l|$*, the* **topological modules** *and* **mono-analytic integral structures**

$$\mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \ \subseteq\ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}});\quad \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}}) \ \subseteq\ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}})$$

— where the notation "$;n,\circ$*" denotes the result of applying the construction in question to the case of* $\mathcal{D}^{\vdash}$*-prime-strips labeled "*$n,\circ$*" — of Proposition 3.2, (ii) [cf. also the notational conventions of Proposition 3.4, (ii)], which we regard as equipped with the* **procession-normalized mono-analytic log-volumes** *of Proposition 3.9, (ii);*

(b) for $\underline{\mathbb{V}}^{\mathrm{bad}} \ni \underline{v}$*, the* **splitting monoid**

$$\Psi^{\perp}_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}}$$

of Proposition 3.5, (ii), (c) [cf. also the notation of Proposition 3.5, (i)], which we regard — via the natural poly-isomorphisms

$$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}}) \ \xrightarrow{\sim}\ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}({}^{n,\circ}\mathfrak{D}_{\succ})_{\underline{v}}) \ \xrightarrow{\sim}\ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j}\mathcal{F}({}^{n,\circ}\mathfrak{D}_{\succ})_{\underline{v}})$$

for $j \in \mathbb{F}_l^{\circledast}$ *[cf. Proposition 3.2, (i), (ii)] — as a* **subset** *of*

$$\prod_{j \in \mathbb{F}_l^{\circledast}} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}, j; n,\circ}\mathcal{D}^{\vdash}_{\underline{v}})$$

equipped with a(n) [multiplicative] **action** *on* $\prod_{j \in \mathbb{F}_l^{\circledast}} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}, j; n,\circ}\mathcal{D}^{\vdash}_{\underline{v}})$*;*

(c) for $j \in \mathbb{F}_l^{\circledast}$*, the* **number field**

$$\begin{aligned} \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j &= \overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j \\ &\subseteq \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}; n,\circ}\mathcal{D}^{\vdash}_{\mathbb{V}_{\mathbb{Q}}}) \stackrel{\mathrm{def}}{=} \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}; n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) \end{aligned}$$

[cf. the natural poly-isomorphisms discussed in (b); Proposition 3.9, (iii); Proposition 3.10, (i)], together with **natural isomorphisms** *between the associated* **global non-realified/realified Frobenioids**

$$\mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j \stackrel{\sim}{\to} \mathcal{F}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j$$

$$\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j \stackrel{\sim}{\to} \mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j$$

[cf. Proposition 3.10, (i)], whose associated **"global degrees"** *may be computed by means of the* **log-volumes** *of (a) [cf. Proposition 3.9, (iii)].*

Write

$${}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}}$$

for the **collection of data** *(a), (b), (c) regarded up to* **indeterminacies** *of the following two types:*

(Ind1) *the indeterminacies induced by the* **automorphisms** *of the* **procession of** $\mathcal{D}^{\vdash}$**-prime-strips** $\mathrm{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T)$*;*

(Ind2) *for each* $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$ *(respectively,* $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$*), the indeterminacies induced by the action of* **independent** *copies of* Ism *[cf. Proposition 1.2, (vi)] (respectively, copies of each of the automorphisms of order* 2 *whose orbit constitutes the poly-automorphism discussed in Proposition 1.2, (vii)) on each of the* **direct summands** *of the* $j+1$ **factors** *appearing in the tensor product used to define* $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1}; n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$ *[cf. (a) above; Proposition 3.2, (ii)] — where we recall that the cardinality of the collection of direct summands is equal to the cardinality of the set of* $\underline{v} \in \underline{\mathbb{V}}$ *that lie over* $v_{\mathbb{Q}}$*.*

Then ${}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}}$ *may be constructed via an* **algorithm** *in the procession of* $\mathcal{D}^{\vdash}$*-prime-strips* $\mathrm{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T)$ *that is* **functorial** *with respect to isomorphisms of processions of* $\mathcal{D}^{\vdash}$*-prime-strips. For* $n, n' \in \mathbb{Z}$*, the* **permutation symmetries** *of the* **étale-picture** *discussed in [IUTchI], Corollary 6.10, (iii); [IUTchII], Corollary 4.11, (ii), (iii) [cf. also Corollary 2.3, (ii); Remarks 2.3.2 and 3.8.2, of the present paper], induce* **compatible poly-isomorphisms**

$$\mathrm{Prc}({}^{n,\circ}\mathfrak{D}^{\vdash}_T) \stackrel{\sim}{\to} \mathrm{Prc}({}^{n',\circ}\mathfrak{D}^{\vdash}_T); \quad {}^{n,\circ}\mathfrak{R}^{\mathrm{LGP}} \stackrel{\sim}{\to} {}^{n',\circ}\mathfrak{R}^{\mathrm{LGP}}$$

which are, moreover, compatible with the poly-isomorphisms

$$ {}^{n,\circ}\mathfrak{D}_0^{\vdash} \xrightarrow{\sim} {}^{n',\circ}\mathfrak{D}_0^{\vdash} $$

induced by the **bi-coricity** *poly-isomorphisms of Theorem 1.5, (iii) [cf. also [IUTchII], Corollaries 4.10, (iv); 4.11, (i)].*

(ii) **($\mathfrak{log}$-Kummer Correspondence)** *For $n, m \in \mathbb{Z}$, the* **Kummer isomorphisms** *of labeled data*

$$ \Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{F}_{\succ})_t \xrightarrow{\sim} \Psi_{\mathrm{cns}}({}^{n,\circ}\mathfrak{D}_{\succ})_t $$

$$ \{\pi_1^{\kappa\text{-sol}}({}^{n,m}\mathcal{D}^{\circledast}) \curvearrowright {}^{n,m}\mathbb{M}^{\circledast}_{\infty\kappa}\}_j \xrightarrow{\sim} \{\pi_1^{\kappa\text{-sol}}({}^{n,\circ}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^{n,\circ}\mathcal{D}^{\circledcirc})\}_j $$

$$ ({}^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}})_j \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{n,\circ}\mathcal{D}^{\circledcirc})_j $$

— where $t \in \mathrm{LabCusp}^{\pm}({}^{n,\circ}\mathfrak{D}_{\succ})$ — of [IUTchII], Corollary 4.6, (iii); [IUTchII], Corollary 4.8, (i), (ii) [cf. also Propositions 3.5, (i); 3.10, (i), of the present paper] induce **isomorphisms** *between the* **vertically coric** *data (a), (b), (c) of (i) [which we regard, in the present (ii), as data which has* **not yet** *been subjected to the indeterminacies* (Ind1), (Ind2) *discussed in (i)] and the corresponding data arising from each $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, i.e.:*

(a) for $\underline{\mathbb{V}} \ni \underline{v} \mid v_{\mathbb{Q}}$, $j \in |\mathbb{F}_l|$, isomorphisms with **local mono-analytic tensor packets** *and their $\mathbb{Q}$-spans*

$$ \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};n,m}\mathcal{F}_{v_{\mathbb{Q}}}) \xrightarrow{\sim} \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};n,m}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{v_{\mathbb{Q}}}) \xrightarrow{\sim} \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) $$

$$ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,m}\mathcal{F}_{v_{\mathbb{Q}}}) \xrightarrow{\sim} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,m}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{v_{\mathbb{Q}}}) \xrightarrow{\sim} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}) $$

$$ \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},j;n,m}\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},j;n,m}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \xrightarrow{\sim} \mathcal{I}({}^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}}) $$

$$ \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;n,m}\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;n,m}\mathcal{F}^{\vdash\times\boldsymbol{\mu}}_{\underline{v}}) \xrightarrow{\sim} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1},j;n,\circ}\mathcal{D}^{\vdash}_{\underline{v}}) $$

[cf. Propositions 3.2, (i), (ii); 3.4, (ii); 3.5, (i)], all of which are **compatible** *with the respective* **log-volumes** *[cf. Proposition 3.9, (ii)];*

(b) for $\underline{\mathbb{V}}^{\mathrm{bad}} \ni \underline{v}$, isomorphisms of **splitting monoids**

$$ \Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}} \xrightarrow{\sim} \Psi^{\perp}_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}} $$

[cf. Proposition 3.5, (i); Proposition 3.5, (ii), (c)];

(c) for $j \in \mathbb{F}_l^{*}$, isomorphisms of **number fields** *and* **global non-realified/ realified Frobenioids**

$$ ({}^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}})_j \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j; \quad ({}^{n,m}\overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}})_j \xrightarrow{\sim} \overline{\mathbb{M}}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j $$

$$ ({}^{n,m}\mathcal{F}^{\circledast}_{\mathrm{MOD}})_j \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j; \quad ({}^{n,m}\mathcal{F}^{\circledast}_{\mathfrak{mod}})_j \xrightarrow{\sim} \mathcal{F}^{\circledast}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j $$

$$ ({}^{n,m}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}})_j \xrightarrow{\sim} \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j; \quad ({}^{n,m}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}})_j \xrightarrow{\sim} \mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_j $$

*which are compatible with the respective natural isomorphisms between "*MOD*"- and "*$\mathfrak{mod}$*"-subscripted versions [cf. Proposition 3.10, (i)]; here, the isomorphisms of the third line of the display induce isomorphisms of the* **global realified Frobenioid** *portions*

$$ {}^{n,m}\mathcal{C}^{\Vdash}_{\mathrm{LGP}} \xrightarrow{\sim} \mathcal{C}^{\Vdash}_{\mathrm{LGP}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}); \quad {}^{n,m}\mathcal{C}^{\Vdash}_{\mathfrak{lgp}} \xrightarrow{\sim} \mathcal{C}^{\Vdash}_{\mathfrak{lgp}}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}) $$

of the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}$, ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$, and $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}}$ [cf. Propositions 3.7, (iii), (iv), (v); 3.10, (i)].

Moreover, as one varies $m \in \mathbb{Z}$, the various isomorphisms of (b) and of the first line in the first display of (c) are **mutually compatible** *with one another, relative to the* **$\mathfrak{log}$-links** *of the n-th column of the* LGP*-Gaussian log-theta-lattice under consideration, in the sense that the only portions of the domains of these isomorphisms that are possibly related to one another via the $\mathfrak{log}$-links consist of* **roots of unity** *in the domains of the $\mathfrak{log}$-links [multiplication by which corresponds, via the $\mathfrak{log}$-link, to an* **"addition by zero"** *indeterminacy, i.e., to* **no indeterminacy***!] — cf. Proposition 3.5, (ii), (c); Proposition 3.10, (ii). This mutual compatibility of the isomorphisms of the first line in the first display of (c) implies a corresponding* **mutual compatibility** *between the isomorphisms of the second and third lines in the first display of (c) that* **involve the subscript "**MOD**"** *[but* **not** *between the isomorphisms that involve the subscript "$\mathfrak{mod}$"! — cf. Proposition 3.10, (iii); Remark 3.10.1]. On the other hand, the isomorphisms of (a) are subject to a certain* **"indeterminacy"** *as follows:*

(Ind3) *as one varies $m \in \mathbb{Z}$, the isomorphisms of (a) are* **"upper semi-compatible"***, relative to the* **$\mathfrak{log}$-links** *of the n-th column of the* LGP*-Gaussian log-theta-lattice under consideration, in a sense that involves certain* **natural inclusions "$\subseteq$"** *at $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$ and certain* **natural surjections "$\twoheadrightarrow$"** *at $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{arc}}_{\mathbb{Q}}$ — cf. Proposition 3.5, (ii), (a), (b), for more details.*

Finally, as one varies $m \in \mathbb{Z}$, the isomorphisms of (a) are [precisely!] **compatible***, relative to the* **$\mathfrak{log}$-links** *of the n-th column of the* LGP*-Gaussian log-theta-lattice under consideration, with the respective* **log-volumes** *[cf. Proposition 3.9, (iv)].*

(iii) **($\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-Link Compatibility)** *The various Kummer isomorphisms of (ii) satisfy compatibility properties with the various* **horizontal arrows** *— i.e., $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-links — of the* LGP*-Gaussian log-theta-lattice under consideration as follows:*

*(a) The first Kummer isomorphism of the first display of (ii) induces — by applying the $\mathbb{F}^{\rtimes\pm}_l$-***symmetry** *of the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ — a* **Kummer isomorphism** *${}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,\circ}\mathfrak{D}^{\vdash}_{\triangle})$ [cf. Theorem 1.5, (iii)]. Relative to this Kummer isomorphism, the full poly-isomorphism of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips*

$$ \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,\circ}\mathfrak{D}^{\vdash}_{\triangle}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n+1,\circ}\mathfrak{D}^{\vdash}_{\triangle}) $$

is **compatible** *with the full poly-isomorphism of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips*

$$ {}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \xrightarrow{\sim} {}^{n+1,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} $$

induced [cf. Theorem 1.5, (ii)] by the **horizontal arrows** *of the* LGP-*Gaussian log-theta-lattice under consideration [cf. Theorem 1.5, (iii)].*

(b) The $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{env}}$, $\mathfrak{F}^{\Vdash}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>})$ [cf. Proposition 2.1, (ii)] that appear **implicitly** *in the construction of the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}$, ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}}$ [cf. (ii), (b), (c), above; Proposition 3.4, (ii); Proposition 3.7, (iii), (iv); [IUTchII], Corollary 4.6, (iv), (v); [IUTchII], Corollary 4.10, (ii)] admit* **natural isomorphisms** *of associated $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips ${}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \xrightarrow{\sim} {}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}$, $\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}({}^{n,\circ}\mathfrak{D}^{\vdash}_{\triangle}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>})$ [cf. Proposition 2.1, (vi)]. Relative to these natural isomorphisms and to the Kummer isomorphism discussed in (a) above, the full poly-isomorphism of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips*

$$\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}({}^{n,\circ}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{env}}({}^{n+1,\circ}\mathfrak{D}_{>})$$

is **compatible** *with the full poly-isomorphism of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips*

$${}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \xrightarrow{\sim} {}^{n+1,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}$$

induced [cf. Theorem 1.5, (ii)] by the **horizontal arrows** *of the* LGP-*Gaussian log-theta-lattice under consideration [cf. Corollary 2.3, (iii)].*

(c) Recall the data "${}^{n,\circ}\mathfrak{R}$" [cf. Corollary 2.3, (ii)] associated to the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ — data which appears **implicitly** *in the construction of the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}$, ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathfrak{lgp}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathfrak{lgp}}$ [cf. (ii), (b), (c), above; Proposition 3.4, (ii); Proposition 3.7, (iii), (iv); [IUTchII], Corollary 4.6, (iv), (v); [IUTchII], Corollary 4.10, (ii)]. This data that arises from ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ is related to corresponding data that arises from the projective system of mono-theta environments associated to the tempered Frobenioids of the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{n,m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ via the* **Kummer isomorphisms** *and* **poly-isomorphisms of projective systems of mono-theta environments** *discussed in Proposition 2.1, (ii), (iii) [cf. also Proposition 2.1, (vi); the second display of Theorem 2.2, (ii)] and Theorem 1.5, (iii) [cf. also (a), (b) above], (v). The* **algorithmic construction** *of these Kummer isomorphisms and poly-isomorphisms of projective systems of mono-theta environments, as well as of the poly-isomorphism*

$${}^{n,\circ}\mathfrak{R} \xrightarrow{\sim} {}^{n+1,\circ}\mathfrak{R}$$

induced by any **permutation symmetry** *of the* **étale-picture** *[cf. the final portion of (i) above; Corollary 2.3, (ii); Remark 3.8.2] ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\sim} {}^{n+1,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ is* **compatible** *with the* **horizontal arrows** *of the* LGP-*Gaussian log-theta-lattice under consideration, e.g., with the full poly-isomorphism of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips*

$${}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \xrightarrow{\sim} {}^{n+1,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}$$

induced [cf. Theorem 1.5, (ii)] by these horizontal arrows [cf. Corollary 2.3, (iv)], in the sense that these constructions are **stabilized/equivariant/functorial** *with respect to arbitrary automomorphisms of the domain and codomain of these horizontal arrows of the* LGP-*Gaussian log-theta-lattice. Finally, the* **algorithmic construction** *of the poly-isomorphisms of the first display above, the various related Kummer isomorphisms, and the various* **evaluation** *maps implicit in the portion of the* **$\mathfrak{log}$-Kummer correspondence** *discussed in (ii), (b), are* **compatible** *with the* **horizontal arrows** *of the* LGP-*Gaussian log-theta-lattice under consideration, i.e., up to the indeterminacies* (Ind1), (Ind2), (Ind3) *described in (i), (ii) [cf. also the discussion of Remark 3.11.4 below], in the sense that these constructions are* **stabilized/equivariant/functorial** *with respect to arbitrary automomorphisms of the domain and codomain of these horizontal arrows of the* LGP-*Gaussian log-theta-lattice.*

(d) *The* **algorithmic construction** *of the Kummer isomorphisms of the first display of (ii) [cf. also (a), (b) above; the* **gluing** *discussed in [IUTchII], Corollary 4.6, (iv); the Kummer compatibilities discussed in [IUTchII], Corollary 4.8, (iii); the relationship to the notation of [IUTchI], Definition 5.2, (vi), (viii), referred to in [IUTchII], Propositions 4.2, (i), and 4.4, (i)], as well as of the poly-isomorphisms between the data*

$$\left[\begin{array}{l}\{\pi_1^{\kappa\text{-sol}}({}^{n,\circ}\mathcal{D}^{\circledast}) \ \curvearrowright \ \mathbb{M}^{\circledast}_{\infty\kappa}({}^{n,\circ}\mathcal{D}^{\circledcirc})\}_j \\ \qquad \to \quad \mathbb{M}_{\infty\kappa v}({}^{n,\circ}\mathcal{D}_{\underline{v}_j}) \ \subseteq \ \mathbb{M}_{\infty\kappa\times v}({}^{n,\circ}\mathcal{D}_{\underline{v}_j})\end{array}\right]_{\underline{v}\in\underline{\mathbb{V}}}$$

$$\overset{\sim}{\to} \quad \left[\begin{array}{l}\{\pi_1^{\kappa\text{-sol}}({}^{n+1,\circ}\mathcal{D}^{\circledast}) \ \curvearrowright \ \mathbb{M}^{\circledast}_{\infty\kappa}({}^{n+1,\circ}\mathcal{D}^{\circledcirc})\}_j \\ \qquad \to \quad \mathbb{M}_{\infty\kappa v}({}^{n+1,\circ}\mathcal{D}_{\underline{v}_j}) \ \subseteq \ \mathbb{M}_{\infty\kappa\times v}({}^{n+1,\circ}\mathcal{D}_{\underline{v}_j})\end{array}\right]_{\underline{v}\in\underline{\mathbb{V}}}$$

[i.e., of the second line of the first display of [IUTchII], Corollary 4.7, (iii)] induced by any **permutation symmetry** *of the* **étale-picture** *[cf. the final portion of (i) above; Corollary 2.3, (ii); Remark 3.8.2]* ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \overset{\sim}{\to} {}^{n+1,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *are* **compatible** *[cf. the discussion of Remark 2.3.2] with the full poly-isomorphism of* $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$*-prime-strips*

$${}^{n,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} \ \overset{\sim}{\to} \ {}^{n+1,m}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}$$

induced [cf. Theorem 1.5, (ii)] by the **horizontal arrows** *of the* LGP-*Gaussian log-theta-lattice under consideration, in the sense that these constructions are* **stabilized/equivariant/functorial** *with respect to arbitrary automomorphisms of the domain and codomain of these horizontal arrows of the* LGP-*Gaussian log-theta-lattice. Finally, the* **algorithmic construction** *of the poly-isomorphisms of the first display above, the various related Kummer isomorphisms, and the various* **evaluation** *maps implicit in the portion of the* **$\mathfrak{log}$-Kummer correspondence** *discussed in (ii), (c), are* **compatible** *with the* **horizontal arrows** *of the* LGP-*Gaussian log-theta-lattice under consideration, i.e., up to the indeterminacies* (Ind1), (Ind2), (Ind3) *described in (i), (ii) [cf. also the discussion of Remark 3.11.4 below], in the sense that these constructions* *are* **stabilized/equivariant/functorial** *with respect to arbitrary automorphisms of the domain and codomain of these horizontal arrows of the* LGP-*Gaussian log-theta-lattice.*

Proof. The various assertions of Theorem 3.11 follow immediately from the definitions and the references quoted in the statements of these assertions — cf. also the various related observations of Remarks 3.11.1, 3.11.2, 3.11.3, 3.11.4 below. ◯

Remark 3.11.1.

(i) One way to summarize the content of Theorem 3.11 is as follows:

Theorem 3.11 gives an **algorithm** for describing, up to certain relatively **mild indeterminacies**, the **LGP-monoids** [cf. Fig. 3.1] — i.e., in essence, the **theta values**

$$\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*}$$

— which are constructed relative to the **scheme/ring structure**, i.e., **"arithmetic holomorphic structure"**, associated to *one* vertical line [i.e., "$(n, \circ)$" for some *fixed* $n \in \mathbb{Z}$] in the LGP-Gaussian log-theta-lattice under consideration, in terms of the *a priori* **alien** arithmetic holomorphic structure of *another* vertical line [i.e., "$(n + 1, \circ)$"] in the LGP-Gaussian log-theta-lattice under consideration [cf., especially, the final portion of Theorem 3.11, (i), concerning **functoriality** and *compatibility* with the **permutation symmetries** of the **étale-picture**].

This point of view is consistent with the point of view of the discussion of Remark 1.5.4; [IUTchII], Remark 3.8.3, (iii).

(ii) Although the various versions of the Θ-*link* are defined [cf. Definition 3.8, (ii)] as *gluings* of

the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* whose associated pilot object [cf. [IUTchII], Definition 4.9, (viii)] is some sort of Θ-*pilot object* in the *domain* of the Θ-link

to

the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* whose associated pilot object is some sort of *q-pilot object* in the *codomain* of the Θ-link,

in fact it is not difficult to see that the theory developed in the present series of papers remains **essentially unaffected**

even if one **replaces** this q-pilot $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip in the *codomain of the* Θ-*link* by **some other** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip**

such as, for instance, the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip whose associated pilot object is the q^λ-*pilot object* [i.e., the λ-th power of the q-pilot object, for some positive integer $\lambda > 1$] — cf. the discussion of Remark 3.12.1, (ii), below. One way to formulate this observation is as follows: The Θ-*link compatibility* described in Theorem 3.11, (iii), may be interpreted as an assertion to the effect that the **functorial construction**

algorithm for the Θ-*pilot object* up to certain *mild indeterminacies* [i.e., (Ind1), (Ind2), (Ind3)] that is given in Theorem 3.11 may be regarded as

> an algorithm whose **input data** is an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip** [i.e., the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip that appears in the *codomain of the* Θ-*link*], and whose **functoriality** is with respect to **arbitrary isomorphisms** of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strips* that appear as input data of the algorithm.

From the point of view of the **gluing** given by the Θ-*link*, this *functoriality* in the *input data* given by an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strip* may be interpreted in the following way:

> this **functoriality** allows one to regard the **functorial construction algorithm** for the Θ-*pilot object* up to certain *mild indeterminacies* that is given in Theorem 3.11 as an algorithm with respect to which the *codomain* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* of the Θ-link [together with the other $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters in the *same vertical line* of the log-theta-lattice as this *codomain* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater*] — i.e., in effect, the $\boldsymbol{q}$**-pilot** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip**, equipped with the *rigidification* determined by the **arithmetic holomorphic structure** constituted by this *vertical line* of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters — is **"coric"**, i.e., "remains **invariant**"/"may be regarded as being held **fixed**" throughout the execution of the various operations of the algorithm.

This interpretation will play a *crucial role* in the application of Theorem 3.11 to Corollary 3.12 below.

(iii) On the other hand, the **étale-picture permutation symmetries** discussed in the final portion of Theorem 3.11, (i) [cf. also the references to these symmetries in Theorem 3.11, (iii), (c), (d)], may be interpreted as follows: The **output data** of the **functorial construction algorithm** of Theorem 3.11 consists of a representation of the data of Theorem 3.11, (i), (b), (c) [cf. also Theorem 3.11, (iii), (c), (d)], up to certain *mild indeterminacies* on the *mono-analytic étale-like log-shells* of Theorem 3.11, (i), (a), that satisfies the following *properties*:

· **(Input prime-strip link (IPL))** This output data is constructed in such a way that it is **linked/related**, via full poly-isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strips** induced by operations in the algorithm, to the **input data prime-strip**, i.e., the **"coric"/"fixed"** $\boldsymbol{q}$**-pilot** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip**, equipped with its *rigidifying* **arithmetic holomorphic structure** [cf. the discussion of (ii)]. In particular, we note that *each of these "intermediate"* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-*prime-strips* that appears in the construction may *itself* be taken to be **both**

> the **input data** of the **functorial algorithm** of Theorem 3.11 [cf. the discussion of (ii)]

and

> [by applying the full poly-isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that link/relate it to the q-pilot $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip] the **input data** for the **Kummer theory** surrounding the $\boldsymbol{q}$**-pilot object** $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip** in its rigidifying $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater [cf. the discussion of (ii)].

At a more *explicit level*, the linking isomorphisms of "intermediate" $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips are given by *composing*

· the *inverses* of the *first two lower horizontal arrows* of the commutative diagram of Remark 3.10.2, followed by
· the *first vertical arrow* of this diagram — corresponding to the **Kummer theory** portion of Theorem 3.11, (iii), (c), (d) — followed by
· the *three upper horizontal arrows* of the diagram — corresponding to the **evaluation map** portion of Theorem 3.11, (iii), (c), (d).

Here, we observe that the *final evaluation map* portion of this composite involves a construction of the Θ-*pilot object* up to certain indeterminacies [i.e., (Ind1), (Ind2), (Ind3)], which, by applying the discussion of Remark 2.4.2, (v), (vi), may be *interpreted* — *provided* that certain *sign conditions* [cf. the discussion of Remark 2.4.2, (iv), (vi)] are satisfied, and one takes into account the considerations discussed in Remarks 3.9.6 [concerning the *product formula*], 3.9.7 [concerning *inverse systems of direct product regions*] — as a construction of the *global realified Frobenioid* portion of an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip, together with *various possibilities* [corresponding to the indeterminacies] for the *"further rigidification"* determined by the pilot object.

· **(Simultaneous holomorphic expressibility (SHE))** The construction of this output data, as well as the output data itself, is expressed in terms that are **simultaneously** valid/executable/well-defined relative to **both**

the **arithmetic holomorphic structure** that gives rise to the **Θ-pilot object** in the *domain* of the Θ-link — i.e., in more technical language, in terms of/as a **function** of structures in the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater in the domain of the Θ-link —

and

the **arithmetic holomorphic structure** that gives rise to the **input data prime-strip** [i.e., such as the q-pilot $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip, as discussed in (ii)] in the *codomain* of the Θ-link — i.e., in more technical language, in terms of/as a **function** of structures in the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater in the codomain of the Θ-link.

In passing, we observe that this property "SHE" may be understood, in a slightly more concrete way, as corresponding to the fact that the **chain** of (sub)quotients considered in Remark 3.9.5, (viii), (ix), forms a **closed loop**.

These two *fundamental properties* of the output data of the algorithm of Theorem 3.11 will play a *central role* in the application of Theorem 3.11 to Corollary 3.12 below. In the context of these two fundamental properties, it is interesting to observe that, relative to the analogy between *multiradiality* and *crystals/connections* [cf. [IUTchII], Remark 1.7.1; [IUTchII], Remark 1.9.2, (ii), (iii)],

> the distinction between *abstract* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* and various *specific realizations of such* $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* [e.g., arising from the structure of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater]

may be understood as corresponding to

> the distinction between *reduced characteristic* p *schemes* [where p is a prime number] and *thickenings of such schemes over* $\mathbb{Z}_p$

in the context of p-adic crystals.

(iv) The **SHE** property discussed in (iii) may be thought of as a sort of *"parallel transport"* mechanism for the Θ-pilot object [cf. the analogy between *multiradiality* and *connections*, as discussed in [IUTchII], Remark 1.7.1; [IUTchII], Remark 1.9.2, (ii)], up to certain mild indeterminacies, from the [arithmetic holomorphic structure represented by the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater in the] *domain* of the Θ-link to the [arithmetic holomorphic structure represented by the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater in the] *codomain* of the Θ-link. On the other hand, in this context, it is important to observe that:

· **(Algorithmic parallel transport (APT))** This parallel transport mechanism does **not** consist of a simple instance of transport of *some set-theoretic region* [such as the region in the tensor packet of log-shells determined by the Θ-*pilot object* in the *domain* of the Θ-link] *via some set-theoretic function*. Rather, it consists of a **construction algorithm** that is **simultaneously valid/executable/well-defined** with respect to the arithmetic holomorphic structures in the domain and codomain of the Θ-link [cf. the discussion of (iii)].

[In this context, it is important to remember that although this *construction algorithm* may yield, as *output*, various *"possible regions"*, such possible regions **cannot** necessarily be **directly compared** with various structures in the *codomain* of the Θ-link. That is to say, such *comparisons* typically require the application of further techniques, as discussed in Remark 3.9.5, (vii).] In particular, if one takes the point of view — *as will be done in Corollary 3.12 below!* — that one is only interested in considering the **qualitative logical aspects/consequences** of the *construction algorithm* of Theorem 3.11, then:

· **(Hidden internal structures (HIS))** One may [and, indeed, it is often useful to] regard this *construction algorithm* of Theorem 3.11 as a construction algorithm for producing **"some sort of output data"** satisfying various properties [cf. (iii)] associated to **"some sort of input data"** [cf. (ii)] and **forget** that this construction algorithm of Theorem 3.11 has *anything to do with* **theta functions** *[e.g., the theory of [EtTh]] or* **theta values** *[i.e., the* $\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*}$*]*!

That is to say, theta functions/theta values may be regarded as **HIS** of the construction algorithm of Theorem 3.11 — somewhat like the internal structure of the *CPU or operating system of a computer*! — i.e., internal structures whose technical details are [of course, of *crucial importance* from the point of view of the *actual functioning of the construction algorithm*, but nonetheless] *irrelevant* or *uninteresting* from the point of view of the *"end user"*, who is only interested in applying the construction algorithm to certain *input data* to obtain certain *output data*. [Here, we observe in passing that, relative to this analogy with the internal structure of the *CPU or operating system of a computer*, the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-prime-strips* that occur in the Θ-link may be thought of as a sort **connecting cable**, i.e., of the sort that is used to link distinct computers via the **internet**. That is to say, despite the fact that such a connecting cable may have a *very simple internal structure* by comparison to the computers that it connects, the connection that it furnishes has *highly nontrivial consequences* [e.g., as in the case of the *internet*!] — cf. the discussion in (iii) of the **input prime-strip link (IPL)** and the analogy with **crystals/connections**.] On the other hand, we observe that, unlike Corollary 3.12 below, which only concerns *qualitative logical aspects/consequences* of the construction algorithm of Theorem 3.11, the **explicit computation** to be performed in [IUTchIV], §1, of the **log-volumes** that occur in the statement of Corollary 3.12 makes *essential use* of the way in which **theta values** occur in the construction algorithm of Theorem 3.11.

(v) Thus, in summary, the above discussion yields a slightly different, and in some sense more detailed, way [by comparison to (i)] to summarize the content of the construction algorithm of Theorem 3.11 [cf. also the discussion of Remark 3.12.2, (ii), below]: The *functorial construction algorithm* of Theorem 3.11 is an algorithm whose

- **input data** consists solely of an $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strip**, regarded up to **isomorphism** [cf. (ii)], and whose
- **output data** consists of certain data that is **linked/related**, via full poly-isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-prime-strips** induced by operations in the algorithm, to the **input data prime-strip**, and, moreover, whose construction algorithm may be expressed in terms that are **simultaneously** valid/executable/well-defined relative to **both** the *arithmetic holomorphic structure* that gives rise to the Θ*-pilot object* in the *domain* of the Θ-link **and** the *arithmetic holomorphic structure* that gives rise to the **input data prime-strip** [i.e., such as the q-pilot $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip].

This construction algorithm of Theorem 3.11 makes *crucial use* of certain **HIS** such as *theta functions* and *theta values*, but these HIS may be **ignored**, if one is only interested in the qualitative logical aspects/consequences of the input and output data of the algorithm.

(vi) In the context of the **input prime-strip link (IPL)** and **simultaneous holomorphic expressibility (SHE)** properties discussed in (iii), it is perhaps of interest to consider what happens in the case of the very *simple, naive example* discussed in Remark 2.2.2, (i). That is to say, suppose that one considers the **"naive version"** of the **Θ-link** given by a correspondence of the form

$$\underline{\underline{q}} \mapsto \underline{\underline{q}}^{\lambda}$$

— where $\lambda > 1$ is a *positive integer* — relative to a **single** arithmetic holomorphic structure, i.e., in effect, *ring structure* "R". [Here, we remark that, unlike the situation considered in the discussion of (ii), where "$\underline{\underline{q}}^{\lambda}$" appears in the *codomain* of some modified version of the Θ-link, the "$\underline{\underline{q}}^{\lambda}$" in the present discussion appears in the *domain* of some modified version of the Θ-link.] Then the *very definition* of this *naive version* of the Θ-link yields an *explicit construction algorithm* for "$\underline{\underline{q}}^{\lambda}$", namely, as the λ-th power of "$\underline{\underline{q}}$". That is to say, this [essentially tautological!] explicit construction algorithm for "$\underline{\underline{q}}^{\lambda}$" satisfies the **SHE** property considered in (iii) in the sense that

> the *tautological construction algorithm* given by taking "the λ-th power of $\underline{\underline{q}}$" may be regarded as *simultaneously executable* relative to *both* the arithmetic holomorphic structure [i.e., in effect, ring structure] that gives rise to "$\underline{\underline{q}}$" *and* the arithmetic holomorphic structure [i.e., in effect, ring structure] that gives rise to "$\underline{\underline{q}}^{\lambda}$".

On the other hand, we observe that this sort of [essentially tautological!] SHE property is achieved as the cost of **sacrificing** the establishment of the analogue of the **IPL** property of (iii), in the sense that

> if one restricts oneself to considering "$\underline{\underline{q}}$" and "$\underline{\underline{q}}^{\lambda}$" inside the **fixed container** constituted by the given **arithmetic holomorphic structure** [i.e., in effect, ring structure "R"] that gives rise to "$\underline{\underline{q}}$", then the tautological construction algorithm considered above does **not induce** any sort of **identification** between "$\underline{\underline{q}}$" and "$\underline{\underline{q}}^{\lambda}$".

(vii) We maintain the notation of (vi). One may then approach the issue of establishing the analogue of the **IPL** property of (iii) by introducing a **formal symbol** "$*$" [corresponding to the abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the Θ-link] and then considering one of the following *two approaches*:

- **(Distinct labels)** It is essentially a tautology that in order to consider **both** of the assignments $* \mapsto \underline{\underline{q}}$ and $* \mapsto \underline{\underline{q}}^{\lambda}$ **simultaneously** [i.e., in order to establish the analogue of the IPL property of (iii)!], it is necessary to introduce **distinct labels** "†" and "‡" for the arithmetic holomorphic structures [i.e., in effect, ring structures] that give rise to "$\underline{\underline{q}}$" and "$\underline{\underline{q}}^{\lambda}$", respectively. That is to say, it is a tautology that one may consider the assignments

$$* \mapsto {}^{\ddagger}\underline{\underline{q}}^{\lambda}, \quad * \mapsto {}^{\dagger}\underline{\underline{q}}$$

  simultaneously and **without** introducing any **inconsistencies**. On the other hand, this approach via the introduction of *tautologically distinct labels* — which may be summarized via the diagram

$$\begin{array}{cccc} * & \mapsto & {}^{\ddagger}\underline{\underline{q}}^{\lambda} \in & {}^{\ddagger}R \\ & & & \vdots \\ \| & & & ?? \\ & & & \vdots \\ * & \mapsto & {}^{\dagger}\underline{\underline{q}} \in & {}^{\dagger}R \end{array}$$

IPL: holds
SHE: ??

  — has the *drawback* that it is by no means clear, at least in any *a priori* sense, how to establish the analogue of the **SHE** property of (iii), since it is by no means clear, at least in any *a priori* sense, *how to "compute" the relationship between the "†" and "‡" arithmetic holomorphic structures [i.e., in effect, ring structures].*

· **(Forced identification of arithmetic holomorphic structures)** Of course, one may then attempt to remedy the *drawback* that appeared in the *distinct labels approach* by simply **arbitrarily identifying** the "†" and "‡" arithmetic holomorphic structures [i.e., in effect, ring structures], that is to say, by simply **deleting/forgetting** the distinct labels "†" and "‡". This approach — which may be summarized via the diagram

$$\begin{array}{ccccc} * & \mapsto & \underline{\underline{q}}^{\lambda} \in & & R \\ \vdots & & & & \\ ?? & & & & \| \\ \vdots & & & & \\ * & \mapsto & \underline{\underline{q}} \in & & R \end{array}$$

IPL: ??
SHE: holds

— allows one to apply the [tautological!] *construction algorithm* discussed in (vi). On the other hand, this approach has the *drawback* that, in order to consider the assignments

$$* \mapsto \underline{\underline{q}}^{\lambda}, \quad * \mapsto \underline{\underline{q}}$$

simultaneously and **consistently** [i.e., in order to establish the analogue of the IPL property of (iii)!], one is led [at least in the absence of more sophisticated machinery!] to regard "$\underline{\underline{q}}$" as being **only well-defined up to possible confusion with "$\underline{\underline{q}}^{\lambda^n}$"**, for some indeterminate $n \in \mathbb{Z}$. That is to say, in summary, this approach gives rise to a sort of **"uninteresting/trivial multiradial representation** of "$\underline{\underline{q}}^{\lambda}$" via

$$\text{“}\{\underline{\underline{q}}^{\lambda^n}\}_{n \in \mathbb{Z}}\text{”}$$

— which [despite being uninteresting/trivial!] does indeed satisfy the *formal analogues* of the *IPL* and *SHE* properties of (iii).

(viii) We conclude our discussion of the *simple, naive examples* discussed in (vi) and (vii) by considering the *relationship* between these simple, naive examples and the *theory of the present series of papers.* We begin by observing that the *"trivial multiradial representation* $\{\underline{\underline{q}}^{\lambda^n}\}_{n \in \mathbb{Z}}$*"* discussed in (vii) is, on the one hand, of interest, in the context of the *IPL* and *SHE* properties of (iii), in that it constitutes a **useful elementary "toy model"** for considering the **qualitative logical aspects** of these fundamental properties satisfied by the *multiradial construction algorithm* of Theorem 3.11. On the other hand, this "trivial multiradial representation" is **useless** from the point of view of applications such as the log-volume estimates given in Corollary 3.12 below [cf. the discussion of the final portion of (iv)] for the following reasons: This **"trivial multiradial representation $\{\underline{\underline{q}}^{\lambda^n}\}_{n \in \mathbb{Z}}$"** is obtained by

· allowing for **indeterminacies** in the **value group** portion [i.e., "$\underline{\underline{q}}^{\mathbb{Z}}$"] of the data under consideration,

· while the **unit group** portion [i.e., the "$\mathcal{O}^{\times \boldsymbol{\mu}}$'s" associated to the local fields that appear] of the data under consideration is held **rigid** [i.e., not subject to indeterminacies];

· only working with the **multiplicative structure** constituted by the **value group** portion of the rings involved, and

· **ignoring** issues related to the **additive structure** of the rings involved, especially, issues related to the **intertwining** between the *additive* and *multiplicative* structures of these rings [cf. the discussion of Remark 3.12.2, (ii), below].

By contrast, the *log-volume estimates* of Corollary 3.12 below rely, in an essential way, on the fact that in the **multiradial construction algorithm** of Theorem 3.11:

· the **value group** portions of the data under consideration [i.e., the $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips associated to the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the definition of the Θ-link] are held **rigid** [i.e., are not subject to indeterminacies],

· while the **unit group** portions of the data under consideration [i.e., the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips associated to the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the definition of the Θ-link] are subject to the **indeterminacies** (Ind1), (Ind2), (Ind3);

· the multiradial construction algorithm makes use, via the **$\mathfrak{log}$-Kummer correspondence**, of the structure of the **intertwining** between the **additive** and **multiplicative** structures of the rings involved [cf. the discussion of Remark 3.12.2, (ii), (iii), (iv), (v), below].

Finally, we observe that the technique of assigning **distinct labels** that appears in the *distinct labels approach* discussed in (vii) is *formalized* in the theory of the present series of papers by means of the notion of **Frobenius-like structures**, i.e., at a more concrete level, mathematical objects that, at least *a priori*, only make sense within the $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater labeled "(n, m)"* [where $n, m \in \mathbb{Z}$] of the *log-theta-lattice*. The problem of relating objects arising from $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters with *distinct labels "(n, m)"* is then resolved in the present series of papers — **not** by means of **"forced identification"** [i.e., in the style of the discussion of (vii)] of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters with distinct labels, but rather — by considering the **permutation symmetries** [i.e., of the sort discussed in the final portion of Theorem 3.11, (i)] satisfied by **étale-like structures**. Here, it is perhaps useful to recall that the *fundamental model* for such permutation symmetries is, in the notation of [IUTchII], Example 1.8, (i),

$$\Pi \quad \longrightarrow \quad G \quad \longleftarrow \quad \Pi$$

— where the arrows "$\longrightarrow$" and "$\longleftarrow$" denote the poly-morphism given by composing the natural surjection $\Pi \twoheadrightarrow \Pi/\Delta$ with the full poly-isomorphism $\Pi/\Delta \xrightarrow{\sim} G$, and we observe that the diagram of this display admits a *permutation symmetry* that switches these two arrows "$\longrightarrow$" and "$\longleftarrow$".

Remark 3.11.2.

(i) In Theorem 3.11, (i), we do not apply the *formalism* or *language* developed in [IUTchII], §1, for discussing multiradiality. Nevertheless, the approach taken in Theorem 3.11, (i) — i.e., by regarding the collection of data (a), (b), (c) up to the indeterminacies given by (Ind1), (Ind2) — to constructing **"multiradial representations"** amounts, in essence, to a special case of the **tautological** approach to constructing multiradial environments discussed in [IUTchII], Example 1.9, (ii). That is to say, this tautological approach is applied to the **vertically coric** constructions of Proposition 3.5, (i); 3.10, (i), which, *a priori*, are *uniradial* in the sense that they depend, in an essential way, on the **arithmetic holomorphic structure** constituted by a *particular* vertical line — i.e., "$(n, \circ)$" for some *fixed* $n \in \mathbb{Z}$ — in the LGP-Gaussian log-theta-lattice under consideration.

(ii) One important underlying aspect of the *tautological approach to multiradiality* discussed in (i) is the treatment of the various **labels** that occur in the **multiplicative** and **additive combinatorial Teichmüller theory** associated to the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ under consideration [cf. the theory of [IUTchI], §4, §6]. The various transitions between types of labels is illustrated in Fig. 3.3 below. Here, we recall that:

(a) the passage from the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** to **labels** $\in \mathbb{F}_l$ forms the content of the associated $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-*Hodge theater* [cf. [IUTchI], Remark 6.6.1];

(b) the passage from **labels** $\in \mathbb{F}_l$ to **labels** $\in |\mathbb{F}_l|$ forms the content of the *functorial algorithm* of [IUTchI], Proposition 6.7;

(c) the passage from **labels** $\in |\mathbb{F}_l|$ to $\pm$**-processions** forms the content of [IUTchI], Proposition 6.9, (ii);

(d) the passage from the $\mathbb{F}_l^{\circledast}$**-symmetry** to **labels** $\in \mathbb{F}_l^{\circledast}$ forms the content of the associated $\mathcal{D}$-Θ*NF-Hodge theater* [cf. [IUTchI], Remark 4.7.2, (i)];

(e) the passage from **labels** $\in \mathbb{F}_l^{\circledast}$ to $\circledast$**-processions** forms the content of [IUTchI], Proposition 4.11, (ii);

(f) the compatibility between $\circledast$**-processions** and $\pm$**-processions**, relative to the natural inclusion of **labels** $\mathbb{F}_l^{\circledast} \hookrightarrow |\mathbb{F}_l|$, forms the content of [IUTchI], Proposition 6.9, (iii).

Here, we observe in passing that, in order to perform these various *transitions*, it is absolutely necessary to work with *all of the labels in* $\mathbb{F}_l$ *or* $|\mathbb{F}_l|$, i.e., one does not have the option of *"arbitrarily omitting certain of the labels"* [cf. the discussion of [IUTchII], Remark 2.6.3; [IUTchII], Remark 3.5.2]. Also, in this context, it is important to note that there is a fundamental difference between the **labels** $\in \mathbb{F}_l, |\mathbb{F}_l|, \mathbb{F}_l^{\circledast}$ — which are essentially **arithmetic holomorphic** in the sense that they depend, in an essential way, on the various local and global *arithmetic fundamental groups* involved — and the **index sets of the mono-analytic** $\pm$**-processions** that appear in the multiradial representation of Theorem 3.11, (i). Indeed, these index sets are just *"naked sets"* which are determined, up to isomorphism, by their *cardinality.* In particular,

> *the construction of these index sets is* **independent** *of the various* **arithmetic holomorphic structures** *involved.*

Indeed, it is precisely this property of these index sets that renders them suitable for use in the construction of the *multiradial representations* of Theorem 3.11, (i). As discussed in [IUTchI], Proposition 6.9, (i), for $j \in \{0, \ldots, l^{\ast}\}$, there are precisely $j+1$ *possibilities* for the "element labeled j" in the index set of cardinality $j+1$; this leads to a total of $(l^{\ast}+1)! = l^{\pm}!$ *possibilities* for the "label identification" of elements of index sets of capsules appearing in the mono-analytic $\pm$-processions of Theorem 3.11, (i). Finally, in this context, it is of interest to recall that the *"rougher approach to symmetrization"* that arises when one works with *mono-analytic processions* is *["downward"] compatible* with the *finer* arithmetically holomorphic approach to symmetrization that arises from the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** [cf. [IUTchII], Remark 3.5.3; [IUTchII], Remark 4.5.2, (ii); [IUTchII], Remark 4.5.3, (ii)].

$$
\begin{tikzcd}
\boxed{\mathbb{F}_l^{\rtimes\pm}\text{-symmetry}} \arrow[d, Rightarrow] & & \boxed{\mathbb{F}_l^{\ast}\text{-symmetry}} \arrow[d, Rightarrow] \\
\boxed{\text{labels} \in \mathbb{F}_l} \arrow[r, Rightarrow] & \boxed{\text{labels} \in |\mathbb{F}_l|} \arrow[d, Rightarrow] & \boxed{\text{labels} \in \mathbb{F}_l^{\ast}} \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
& \boxed{\pm\text{-procession}} & \boxed{\ast\text{-procession}} \arrow[l, Rightarrow]
\end{tikzcd}
$$

Fig. 3.3: Transitions from symmetries to labels to processions in a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater

(iii) Observe that the **"Kummer isomorphism of global realified Frobenioids"** that appears in the theory of [IUTchII], §4 — i.e., more precisely, the various versions of the isomorphism of Frobenioids "${}^{\ddagger}\mathcal{C}^{\Vdash} \xrightarrow{\sim} \mathcal{D}^{\Vdash}({}^{\ddagger}\mathfrak{D}^{\vdash})$" discussed in [IUTchII], Corollary 4.6, (ii), (v) — is constructed by considering isomorphisms between **local value groups** obtained by forming the **quotient** of the multiplicative groups associated to the various local fields that appear by the subgroups of **local units** [cf. [IUTchII], Propositions 4.2, (ii); 4.4, (ii)]. In particular, such "Kummer isomorphisms" *fail* to give rise to a **"$\mathfrak{log}$-Kummer correspondence"**, i.e., they fail to satisfy **mutual compatibility** properties of the sort discussed in the final portion of Theorem 3.11, (ii). Indeed, as discussed in Remark 1.2.3, (i) [cf. also [IUTchII], Remark 1.12.2, (iv)], at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the operation of forming a *multiplicative* quotient by *local units* corresponds, on the opposite side of the $\mathfrak{log}$-link, to forming an *additive* quotient by the submodule obtained as the *$p_{\underline{v}}$-adic logarithm of these local units*. This is precisely why, in the context of Theorem 3.11, (ii), we wish to work with the global non-realified/realified Frobenioids "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$", "$\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}$" that arise from copies of "F_{mod}" which satisfy a "$\mathfrak{log}$-Kummer correspondence", as described in the final portion of Theorem 3.11, (ii) [cf. the discussion of Remark 3.10.1]. On the other hand, the *pathologies/indeterminacies* that arise from working with global arithmetic line bundles by means of various *local data* at $\underline{v} \in \underline{\mathbb{V}}$ in the context of the $\mathfrak{log}$-*link* are *formalized* via the theory of the global Frobenioids "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$", together with the **"upper semi-compatibility"** of local units discussed in the final portion of Theorem 3.11, (ii) [cf. also the discussion of Remark 3.10.1].

(iv) In the context of the discussion of *global realified Frobenioids* given in (iii), we observe that, in the case of the global realified Frobenioids [constructed by means of "$\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}$"!] that appear in the $\mathcal{F}^{\Vdash}$-prime-strips ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$, $\mathfrak{F}^{\Vdash}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\mathrm{LGP}}$ [cf. Theorem 3.11, (ii), (c)], the various **localization functors** that appear [i.e., the various "${}^{\ddagger}\rho_{\underline{v}}$" of [IUTchI], Definition 5.2, (iv); cf. also the isomorphisms of the second display of [IUTchII], Corollary 4.6, (v)] may be **reconstructed**, in the spirit of the discussion of Remark 3.9.2, "by considering the *effect of multiplication* by elements of the *[non-realified] global monoids* under consideration on the *log-volumes* of the various local mono-analytic tensor packets that appear". [We leave the routine details to the reader.] This reconstructibility, together with the *mutual incompatibilities* observed in (iii) above that arise when one attempts to work *simultaneously* with *log-shells* and with the *splitting monoids* of the $\mathcal{F}^{\Vdash}$-prime-strip ${}^{n,m}\mathfrak{F}^{\Vdash}_{\mathrm{LGP}}$ at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, are the primary reasons for our omission of explicit mention of the splitting monoids at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ [which in fact appear as part of the data "${}^{n,\circ}\mathfrak{R}$" considered in the discussion of Theorem 3.11, (iii), (c)] from the statement of Theorem 3.11 [cf. Theorem 3.11, (i), (b); Theorem 3.11, (ii), (b); Theorem 3.11, (iii), (c), in the case of $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$].

Remark 3.11.3. Before proceeding, we pause to discuss the relationship between the **$\mathfrak{log}$-Kummer correspondence** of Theorem 3.11, (ii), and the **$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link compatibility** of Theorem 3.11, (iii).

(i) First, we recall [cf. Remarks 1.4.1, (i); 3.8.2] that the various squares that appear in the *[*LGP-*Gaussian] log-theta-lattice* are *far from being [*1-*]commutative*! On the other hand, the **bi-coricity** of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-*prime-strips* and *mono-analytic log-shells* discussed in Theorem 1.5, (iii), (iv), may be intepreted as the statement that

> the various squares that appear in the *[*LGP-*Gaussian] log-theta-lattice* **are** in fact **[1-]commutative** with respect to [the portion of the data associated to each "•" in the log-theta-lattice that is constituted by] these bi-coric $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-*prime-strips* and *mono-analytic log-shells*.

(ii) Next, let us observe that in order to relate both the *unit* and *value group* portions of the domain and codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link corresponding to *adjacent vertical lines* — i.e., $(n-1,*)$ and $(n,*)$ — of the [LGP-Gaussian] log-theta-lattice to one another,

> it is necessary to relate these **unit** and **value group** portions to one another by means of a **single** $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link**, i.e., from $(n-1,m)$ to (n,m).

That is to say, from the point of view of constructing the various LGP-*monoids* that appear in the multiradial representation of Theorem 3.11, (i), one is tempted to work with correspondences between *value groups* on adjacent vertical lines that lie in a **vertically once-shifted** position — i.e., say, at $(n-1, m)$ and (n, m) — relative to the correspondence between *unit groups* on adjacent vertical lines, i.e., say, at $(n-1, m-1)$ and $(n, m-1)$. On the other hand, such an approach *fails*, at least from an *a priori* point of view, precisely on account of the *noncommutativity* discussed in (i). Finally, we observe that in order to relate both unit and value groups by means of a *single* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$*-link*,

> it is necessary to avail oneself of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$*-link compatibility* properties discussed in Theorem 3.11, (iii) — i.e., of the theory of §2 and [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi), (viii) — so as to **insulate** the **cyclotomes** that appear in the **Kummer theory** surrounding the **étale theta function** and κ**-coric functions** from the $\mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}(-)$**-indeterminacies** that act on the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips involved as a result of the application of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link

— cf. the discussion of Remarks 2.2.1, 2.3.2.

(iii) As discussed in (ii) above, a "vertically once-shifted" approach to relating units on adjacent vertical lines *fails* on account of the *noncommutativity* discussed in (i). Thus, one natural approach to treating the units in a "vertically once-shifted" fashion — which, we recall, is necessary in order to relate the LGP-*monoids* on adjacent vertical lines to one another! — is to apply the **bi-coricity** *of mono-analytic log-shells* discussed in (i). On the other hand, to take this approach means that one must work in a *framework* that allows one to *relate* [cf. the discussion of Remark 1.5.4, (i)] the "Frobenius-like" structure constituted by the *Frobenioid-theoretic* units [i.e., which occur in the domain and codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link] to corresponding étale-like structures **simultaneously** via both

(a) the usual **Kummer isomorphisms** — i.e., so as to be compatible with the application of the compatibility properties of Theorem 3.11, (iii), as discussed in (ii) — and

(b) the **composite** of the usual Kummer isomorphisms with [a single iterate of] the **$\mathfrak{log}$-link** — i.e., so as to be compatible with the bi-coric treatment of mono-analytic log-shells [as well as the closely related construction of LGP-monoids] proposed above.

Such a framework may only be realized if one relates Frobenius-like structures to étale-like structures in a fashion that is **invariant** with respect to pre-composition with various iterates of the **$\mathfrak{log}$-link** [cf. the final portions of Propositions 3.5, (ii); 3.10, (ii)]. This is precisely what is achieved by the $\mathfrak{log}$*-Kummer correspondences* of the final portion of Theorem 3.11, (ii).

(iv) The discussion of (i), (ii), (iii) above may be summarized as follows: The **$\mathfrak{log}$-Kummer correspondences** of the final portion of Theorem 3.11, (ii), allow one to

(a) relate both the **unit** and the **value group** portions of the domain and codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link corresponding to *adjacent vertical lines* of the [LGP-Gaussian] log-theta-lattice to one another, in a fashion that

(b) **insulates** the **cyclotomes/Kummer theory** surrounding the **étale theta function** and **κ-coric functions** involved from the $\mathrm{Aut}_{\mathcal{F}^{\vdash\times\mu}}(-)$**-indeterminacies** that act on the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips involved as a result of the application of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. Theorem 3.11, (iii)], and, moreover,

(c) is **compatible** with the **bi-coricity** of the **mono-analytic log-shells** [cf. Theorem 1.5, (iv)], hence also with the operation of relating the LGP**-monoids** that appear in the multiradial representation of Theorem 3.11, (i), corresponding to *adjacent vertical lines* of the [LGP-Gaussian] log-theta-lattice to one another.

These observations will play a *key role* in the proof of Corollary 3.12 below.

Remark 3.11.4. In the context of the *compatibility* discussed in the final portion of Theorem 3.11, (iii), (c), (d), we make the following observations.

(i) First of all, we observe that consideration of the $\mathfrak{log}$**-Kummer correspondence** in the context of the **compatibility** discussed in the final portion of Theorem 3.11, (iii), (c), (d), amounts precisely to **forgetting the labels** of the various **Frobenius-like** "$\bullet$'s" [cf. the notation of the final display of Proposition 1.3, (iv)], i.e., to **identifying** data associated to these Frobenius-like "$\bullet$'s" with the corresponding data associated to the **étale-like** "$\circ$". In particular, [cf. the discussion of Theorem 3.11, (ii), preceding the statement of (Ind3)] *multiplication* of the data considered in Theorem 3.11, (ii), (b), (c), by *roots of unity* must be *"identified"* with the *identity automorphism.* Put another way, this data of Theorem 3.11, (ii), (b), (c), may only be considered **up to multiplication by roots of unity**. Thus, for instance, it only makes sense to consider **orbits** *of this data relative to multiplication by roots of unity* [i.e., as opposed to *specific elements* within such orbits]. This does not cause any problems in the case of the **theta values** considered in Theorem 3.11, (ii), (b), precisely because the theory developed so far was *formulated* precisely in such a way as to be **invariant** with respect to such **indeterminacies** [i.e., multiplication of the theta values by $2l$-th roots of unity — cf. the *left-hand portion* of Fig. 3.4 below]. In the case of the **number fields** [i.e., copies of F_{mod}] considered in Theorem 3.11, (ii), (c), the resulting indeterminacies do not cause any problems precisely because, in the theory of the present series of papers, ultimately one is only interested in the **global Frobenioids** [i.e., copies of "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$" and "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" and their realifications] associated to these number fields by means of constructions that only involve

· **local data**, together with
· the **entire set** — i.e., which, unlike *specific elements* of this set, is **stabilized** by multiplication by roots of unity of the number field [cf. the *left-hand portion* of Fig. 3.5 below] — constituted by the **number field** under consideration

[cf. the constructions of Example 3.6, (i), (ii); the discussion of Remark 3.9.2]. In this context, we recall from the discussion of Remark 2.3.3, (vi), that the operation of **forgetting the labels** of the various **Frobenius-like** "$\bullet$'s" also gives rise to various **indeterminacies** in the **cyclotomic rigidity isomorphisms** applied in the $\mathfrak{log}$-Kummer correspondence. On the other hand, in the case of the **theta values** considered in Theorem 3.11, (ii), (b), we recall from this discussion of

Remark 2.3.3, (vi), that such indeterminacies are in fact **trivial** [cf. the *right-hand portion* of Fig. 3.4 below]. In the case of the **number fields** [i.e., copies of F_{mod}] considered in Theorem 3.11, (ii), (c), we recall from this discussion of Remark 2.3.3, (vi), that such cyclotomic rigidity isomorphism indeterminacies amount to a possible indeterminacy of **multiplication by** ± 1 on copies of the multiplicative group $F_{\mathrm{mod}}^{\times}$ [cf. the *right-hand portion* of Fig. 3.5 below], i.e., indeterminacies which do not cause any problems, again, precisely as a consequence of the fact that such indeterminacies **stabilize** the **entire set** [i.e., as opposed to *specific elements* of this set] constituted by the **number field** under consideration. Finally, in this context, we observe [cf. the discussion at the beginning of Remark 2.3.3, (viii)] that, in the case of the various **local data** at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ that appears in Theorem 3.11, (ii), (a), and gives rise to the **holomorphic log-shells** that serve as **containers** for the data considered in Theorem 3.11, (ii), (b), (c), the corresponding **cyclotomic rigidity isomorphism indeterminacies** are in fact **trivial**. Indeed, this *triviality* may be understood as a consequence of the fact the following observation: Unlike the case with the cyclotomic rigidity isomorphisms that are applied in the context of the **geometric containers** [cf. the discussion of Remark 2.3.3, (i)] that appear in the case of the data of Theorem 3.11, (ii), (b), (c), i.e., which give rise to **"vicious circles"/"loops"** consisting of identification morphisms that differ from the usual *natural identification* by multiplication by elements of the *submonoid* $\mathbb{I}^{\mathrm{ord}} \subseteq \pm\mathbb{N}_{\geq 1}$ [cf. the discussion of Remark 2.3.3, (vi)],

> the cyclotomic rigidity isomorphisms that are applied in the context of this **local data** — even when subject to the *various identifications* arising from **forgetting the labels** of the various **Frobenius-like** "$\bullet$'s"! — only give rise to **natural isomorphisms** between **"geometric" cyclotomes** arising from the *geometric fundamental group* and **"arithmetic" cyclotomes** arising from copies of the *absolute Galois group* of the base [local] field [cf. [AbsTopIII], Corollary 1.10, (c); [AbsTopIII], Proposition 3.2, (i), (ii); [AbsTopIII], Remark 3.2.1].

That is to say, **no "vicious circles"/"loops"** arise since there is *never any confusion* between such "geometric" and "arithmetic" cyclotomes. [A similar phenomenon may be observed at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ with regard to the *Kummer structures* considered in [IUTchI], Example 3.4, (i).] Thus, in summary,

> the various **indeterminacies** that, *a priori*, might arise in the context of the portions of the **$\mathfrak{log}$-Kummer correspondence** that appear in the final portion of Theorem 3.11, (iii), (c), (d), are in fact **"invisible"**, i.e., they have **no substantive effect** on the objects under consideration

[cf. also the discussion of (ii) below]. This is precisely the sense in which the **"compatibility"** stated in the final portion of Theorem 3.11, (iii), (c), (d), is to be understood.

(ii) In the context of the discussion of (i), we make the following *observation*:

> the discussion in (i) of **indeterminacies** that, *a priori*, might arise in the context of the portions of the **$\mathfrak{log}$-Kummer correspondence** that appear in the final portion of Theorem 3.11, (iii), (c), (d), is **complete**, i.e., there are *no further possible indeterminacies* that might appear.

Indeed, this *observation* is a consequence of the "general nonsense" observation [cf., e.g., the discussion of [FrdII], Definition 2.1, (ii)] that, in general, **"Kummer isomorphisms"** are **completely determined** by the following data:

(a) **isomorphisms** between the respective **cyclotomes** under consideration;
(b) the **Galois action** on roots of elements of the monoid under consideration.

That is to say, the **compatibility** of all of the various constructions that appear with the actions of the relevant **Galois groups** [or arithmetic fundamental groups] is **tautological**, so there is no possibility that further indeterminacies might arise with respect to the data of (b). On the other hand, the effect of the indeterminacies that might arise with respect to the data of (a) was precisely the content of the latter portion of the discussion of (i) [i.e., of the discussion of Remark 2.3.3, (vi), (viii)].

(iii) In the context of the discussion of (i), we observe that the **"invisible indeterminacies"** discussed in (i) in the case of the data considered in Theorem 3.11, (ii), (b), (c), may be thought of as a sort of *analogue* for this data of the **indeterminacy** (Ind3) [cf. the discussion of the final portion of Theorem 3.11, (ii)] to which the data of Theorem 3.11, (ii), (a), is subject. By contrast, the **multiradiality** and **radial/coric decoupling** discussed in Remarks 2.3.2, 2.3.3 [cf. also Theorem 3.11, (iii), (c), (d)] may be understood as asserting precisely that the **indeterminacies** (Ind1)**,** (Ind2) discussed in Theorem 3.11, (i), which act, essentially, on the data of Theorem 3.11, (ii), (a), have **no effect** on the **geometric containers** [cf. the discussion of Remark 2.3.3, (i)] that underlie [i.e., prior to execution of the relevant **evaluation** operations] the data considered in Theorem 3.11, (ii), (b), (c).

$$\boldsymbol{\mu}_{2l} \quad \curvearrowright \quad \left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\ldots,l^*} \quad \curvearrowleft \quad \{1\} \quad (\subseteq \pm\mathbb{N}_{\geq 1})$$

Fig. 3.4: Invisible indeterminacies acting on *theta values*

$$\boldsymbol{\mu}(F_{\mathrm{mod}}^{\times}) \quad \curvearrowright \quad F_{\mathrm{mod}}^{\times} \quad \curvearrowleft \quad \{\pm 1\} \quad (\subseteq \pm\mathbb{N}_{\geq 1})$$

Fig. 3.5: Invisible indeterminacies acting on *copies of* $F_{\mathrm{mod}}^{\times}$

The following result may be thought of as a relatively *concrete consequence* of the somewhat abstract content of Theorem 3.11.

Corollary 3.12. (Log-volume Estimates for Θ-Pilot Objects) *Suppose that we are in the situation of Theorem 3.11. Write*

$$-\ |\log(\underline{\underline{\Theta}})| \ \in \ \mathbb{R} \bigcup \{+\infty\}$$

for the **procession-normalized mono-analytic log-volume** *[i.e., where the average is taken over* $j \in \mathbb{F}_l^{*}$ *— cf. Remark 3.1.1, (ii), (iii), (iv); Proposition 3.9, (i), (ii); Theorem 3.11, (i), (a)] of the* **holomorphic hull** *[cf. Remark 3.9.5, (i)] of the* **union** *of the* **possible images** *of a* Θ**-pilot object** *[cf. Definition 3.8, (i)],* *relative to the relevant* **Kummer isomorphisms** *[cf. Theorem 3.11, (ii)], in the* **multiradial representation** *of Theorem 3.11, (i), which we regard as* **subject** *to the* **indeterminacies** (Ind1), (Ind2), (Ind3) *described in Theorem 3.11, (i), (ii). Write*

$$-\ |\log(\underline{\underline{q}})|\ \in\ \mathbb{R}$$

for the **procession-normalized mono-analytic log-volume** *of the image of a* $\boldsymbol{q}$**-pilot object** *[cf. Definition 3.8, (i)], relative to the relevant* **Kummer isomorphisms** *[cf. Theorem 3.11, (ii)], in the* **multiradial representation** *of Theorem 3.11, (i), which we do* **not** *regard as subject to the indeterminacies* (Ind1), (Ind2), (Ind3) *described in Theorem 3.11, (i), (ii). Here, we recall the definition of the symbol "$\triangle$" as the result of identifying the* **labels**

$$\text{"}0\text{" and "}\langle\mathbb{F}_l^{*}\rangle\text{"}$$

[cf. [IUTchII], Corollary 4.10, (i)]. In particular, $|\log(\underline{\underline{q}})| > 0$ *is easily computed in terms of the various* $\boldsymbol{q}$**-parameters** *of the elliptic curve* E_F *[cf. [IUTchI], Definition 3.1, (b)] at* $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ $(\neq \emptyset)$*. Then it holds that* $-\ |\log(\underline{\underline{\Theta}})| \in \mathbb{R}$*, and*

$$-\ |\log(\underline{\underline{\Theta}})|\ \geq\ -\ |\log(\underline{\underline{q}})|$$

— i.e., $C_\Theta \geq -1$ *for any real number* $C_\Theta \in \mathbb{R}$ *such that* $-\ |\log(\underline{\underline{\Theta}})|\ \leq\ C_\Theta \cdot |\log(\underline{\underline{q}})|$.

Proof. We begin by observing that, since $|\log(\underline{\underline{q}})| > 0$, we may *assume without loss of generality* in the remainder of the proof that

$$-\ |\log(\underline{\underline{\Theta}})|\ <\ 0$$

whenever $-\ |\log(\underline{\underline{\Theta}})| \in \mathbb{R}$ [i.e., since an inequality $-\ |\log(\underline{\underline{\Theta}})|\ \geq\ 0$ would imply that $-\ |\log(\underline{\underline{\Theta}})|\ \geq\ 0\ >\ -|\log(\underline{\underline{q}})|$]. Now suppose that we are in the situation of Theorem 3.11. For $n \in \mathbb{Z}$, write

$${}^{n,\circ}\mathcal{U}\ \ \stackrel{\text{def}}{=}\ \ \left\{{}^{n,\circ}\mathcal{U}_{j,v_{\mathbb{Q}}}\right\}_{j\in|\mathbb{F}_l|,v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}}\ \ \subseteq\ \ {}^{n,\circ}\mathcal{U}^{\mathbb{Q}}\ \ \stackrel{\text{def}}{=}\ \ \left\{{}^{n,\circ}\mathcal{U}^{\mathbb{Q}}_{j,v_{\mathbb{Q}}}\right\}_{j\in|\mathbb{F}_l|,v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}}$$

[where we observe that the "$\subseteq$" constitutes a slight abuse of notation] for the collection of subsets ${}^{n,\circ}\mathcal{U}_{j,v_{\mathbb{Q}}} \subseteq {}^{n,\circ}\mathcal{U}^{\mathbb{Q}}_{j,v_{\mathbb{Q}}} \stackrel{\text{def}}{=} \mathcal{I}^{\mathbb{Q}}(\mathbb{S}^{\pm}_{j+1};{}^{n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$ [cf. Theorem 3.11, (i), (a)] given by the various **unions**, for $j \in |\mathbb{F}_l|$ and $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$, of the **possible images** of a Θ**-pilot object** [cf. Definition 3.8, (i)], relative to the relevant **Kummer isomorphisms** [cf. Theorem 3.11, (ii)], in the **multiradial representation** of Theorem 3.11, (i), which we regard as **subject** to the **indeterminacies** (Ind1), (Ind2), (Ind3) described in Theorem 3.11, (i), (ii);

$${}^{n,\circ}\overline{\mathcal{U}}\ \ =\ \ \left\{{}^{n,\circ}\overline{\mathcal{U}}_{j,v_{\mathbb{Q}}}\right\}_{j\in|\mathbb{F}_l|,v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}}\ \ \subseteq\ \ {}^{n,\circ}\mathcal{U}^{\mathbb{Q}}\ \ =\ \ \left\{{}^{n,\circ}\mathcal{U}^{\mathbb{Q}}_{j,v_{\mathbb{Q}}}\right\}_{j\in|\mathbb{F}_l|,v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}}$$

[where we observe that the "$\subseteq$" constitutes a slight abuse of notation] for the collection of subsets ${}^{n,\circ}\overline{\mathcal{U}}_{j,v_{\mathbb{Q}}} \subseteq {}^{n,\circ}\mathcal{U}^{\mathbb{Q}}_{j,v_{\mathbb{Q}}} = \mathcal{I}^{\mathbb{Q}}(\mathbb{S}^{\pm}_{j+1};{}^{n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$ [cf. Theorem 3.11, (i), (a)] given by the various **holomorphic hulls** [cf. Remark 3.9.5, (i)] of the subsets

${}^{n,\circ}\mathcal{U}_{j,v_{\mathbb{Q}}} \subseteq \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$, relative to the arithmetic holomorphic structure labeled "$n, \circ$". Here, we observe that one concludes easily from the [easily verified] *compactness* of the ${}^{1,\circ}\mathcal{U}_{j,v_{\mathbb{Q}}}$ [where $j \in |\mathbb{F}_l|$, $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$], together with the definition of the *log-volume*, that the quantity $-\,|\log(\underline{\underline{\Theta}})|$ is *finite*, hence *negative* [by our *assumption* at the beginning of the present proof!]. In particular, we observe [cf. Remark 2.4.2, (iv), (v), (vi); Remark 3.9.6; Remark 3.9.7; the discussion of "IPL" in Remark 3.11.1, (iii)] that

> we may restrict our attention to **possible images** of a **Θ-pilot object** that correspond to data [i.e., collections of regions] that may be interpreted as an **$\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strip**.

Now we proceed to review precisely what is achieved by the various portions of Theorem 3.11 and, indeed, by the theory developed thus far in the present series of papers. This review leads naturally to an interpretation of the theory that gives rise to the *inequality* asserted in the statement of Corollary 3.12. For ease of reference, we divide our discussion into *steps*, as follows.

(i) In the following discussion, we concentrate on a *single arrow* — i.e., a *single* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$*-link*

$$ {}^{0,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}} {}^{1,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} $$

— of the [LGP-Gaussian] log-theta-lattice under consideration. This arrow consists of the *full poly-isomorphism* of $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips

$$ {}^{0,0}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\mathrm{LGP}} \overset{\sim}{\rightarrow} {}^{1,0}\mathfrak{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}_{\triangle} $$

[cf. Definition 3.8, (ii)]. This poly-isomorphism may be thought of as consisting of a **"unit portion"** constituted by the associated [full] poly-isomorphism of $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips

$$ {}^{0,0}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{LGP}} \overset{\sim}{\rightarrow} {}^{1,0}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle} $$

and a **"value group portion"** constituted by the associated [full] poly-isomorphism of $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips

$$ {}^{0,0}\mathfrak{F}^{\Vdash\blacktriangleright}_{\mathrm{LGP}} \overset{\sim}{\rightarrow} {}^{1,0}\mathfrak{F}^{\Vdash\blacktriangleright}_{\triangle} $$

[cf. Definition 2.4, (iii)]. This value group portion of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link maps Θ*-pilot objects* of ${}^{0,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ to *q-pilot objects* of ${}^{1,0}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ [cf. Remark 3.8.1].

(ii) Whereas the units of the Frobenioids that appear in the $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strip ${}^{0,0}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{LGP}}$ are subject to $\mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}(-)$-indeterminacies [i.e., "(Ind1), (Ind2)" — cf. Theorem 3.11, (iii), (a), (b)], the **cyclotomes** that appear in the Kummer theory surrounding the **étale theta function** and **κ-coric functions**, i.e., which give rise to the "value group portion" ${}^{0,0}\mathfrak{F}^{\Vdash\blacktriangleright}_{\mathrm{LGP}}$, are **insulated** from these $\mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}(-)$**-indeterminacies** — cf. Theorem 3.11, (iii), (c), (d); the discussion of Remark 3.11.3, (iv); Fig. 3.6 below. Here, we recall that in the case of the étale theta function, this follows from the theory of §2, i.e., in essence, from the **cyclotomic rigidity of mono-theta environments**, as discussed in [EtTh]. On the other hand, in the case of κ-coric functions, this follows from the algorithms discussed in [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi), (viii).

	Θ-related objects	*NF-related objects*
require *mono-analytic containers*, **Kummer theory incompatible** with (Ind1), (Ind2)	local LGP-monoids [cf. Proposition 3.4, (ii)]	copies of F_{mod} [cf. Proposition 3.7, (i)]
independent of *mono-analytic containers*, **Kummer theory compatible** with (Ind1), (Ind2) [cf. Remark 2.3.3]	étale theta function, mono-theta environments [cf. Corollary 2.3]	global ${}_\infty\kappa$-coric, local ${}_\infty\kappa$-, ${}_\infty\kappa\times$-coric structures [cf. Remark 2.3.2]

Fig. 3.6: Relationship of theta- and number field-related objects to mono-analytic containers

(iii) In the following discussion, it will be of crucial importance to relate **simultaneously** both the **unit** and the **value group** portions of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link(s) involved on the 0-*column* [i.e., the vertical line indexed by 0] of the log-theta-lattice under consideration to the corresponding unit and value group portions on the 1-*column* [i.e., the vertical line indexed by 1] of the log-theta-lattice under consideration. On the other hand, if one attempts to relate the *unit* portions via one $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [say, from $(0, m)$ to $(1, m)$] and the *value group* portions via another $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [say, from $(0, m')$ to $(1, m')$, for $m' \neq m$], then the **non-commutativity** of the log-theta-lattice renders it practically impossible to obtain conclusions that require one to relate both the unit and the value group portions *simultaneously* [cf. the discussion of Remark 3.11.3, (i), (ii)]. This is precisely why we concentrate on a **single** $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link** [cf. (i)].

(iv) The issue discussed in (iii) is relevant in the context of the present discussion for the following reason. Ultimately, we wish to apply the **bi-coricity** of the **units** [cf. Theorem 1.5, (iii), (iv)] in order to compute the 0-*column* Θ-*pilot object* in terms of the arithmetic holomorphic structure of the 1-*column*. In order to do this, one must work with *units* that are **vertically once-shifted** [i.e., lie at $(n, m-1)$] relative to the value group structures involved [i.e., which lie at (n, m)] — cf. the discussion of Remark 3.11.3, (ii). The solution to the problem of simultaneously accommodating these apparently *contradictory requirements* — i.e., "vertical shift" vs. "impossibility of vertical shift" [cf. (iii)] — is given precisely by working, on the 0-column, with structures that are **invariant** with respect to

vertical shifts [i.e., "$(0,m) \mapsto (0,m+1)$"] of the log-theta-lattice [cf. the discussion surrounding Remark 1.2.2, (iii), (a)] such as **vertically coric structures** [i.e., indexed by "$(n,\circ)$"] that are related to the *"Frobenius-like"* structures which are *not* vertically coric by means of the **$\mathfrak{log}$-Kummer correspondences** of Theorem 3.11, (ii). Here, we note that this "solution" may be implemented only at the cost of admitting the *"indeterminacy"* constituted by the **upper semi-compatibility** of (Ind3).

(v) Thus, we begin our computation of the 0-*column* Θ-*pilot object* in terms of the arithmetic holomorphic structure of the 1-*column* by relating the units on the 0- and 1-columns by means of the **unit portion**

$$^{0,0}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\mathrm{LGP}} \overset{\sim}{\to} {}^{1,0}\mathfrak{F}^{\vdash\times\boldsymbol{\mu}}_{\triangle}$$

of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link from $(0,0)$ to $(1,0)$ [cf. (i)] and then applying the **bi-coricity** of the **units** of Theorem 1.5, (iii), (iv). In particular, the **mono-analytic log-shell** interpretation of this bi-coricity given in Theorem 1.5, (iv), will be applied to regard these mono-analytic log-shells as **"multiradial mono-analytic containers"** [cf. the discussion of Remark 1.5.2, (i), (ii), (iii)] for the various [local and global] value group structures that constitute the Θ-pilot object on the 0-column — cf. Fig. 3.6 above. [Here, we observe that the parallel treatment of *"theta-related"* and *"number field-related"* objects is reminiscent of the discussion of [IUTchII], Remark 4.11.2, (iv).] That is to say, we will relate the various Frobenioid-theoretic [i.e., "Frobenius-like" — cf. Remark 1.5.4, (i)]

- *local units* at $\underline{v} \in \underline{\mathbb{V}}$,
- *splitting monoids* at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, and
- *global Frobenioids*

indexed by $(0,m)$, for $m \in \mathbb{Z}$, to the *vertically coric* [i.e., indexed by "$(0,\circ)$"] versions of these bi-coric mono-analytic containers by means of the **$\mathfrak{log}$-Kummer correspondences** of Theorem 3.11, (ii), (a), (b), (c) — i.e., by *varying* the *"Kummer input index"* $(0,m)$ along the 0-column.

(vi) In the context of (v), it is useful to recall that the $\mathfrak{log}$-Kummer correspondences of Theorem 3.11, (ii), (b), (c), are obtained precisely as a consequence of the **splittings**, up to roots of unity, of the relevant monoids into **unit** and **value group** portions constructed by applying the **Galois evaluation** operations discussed in Remarks 2.2.2, (iii) [in the case of Theorem 3.11, (ii), (b)], and 2.3.2 [in the case of Theorem 3.11, (ii), (c)]. Moreover, we recall that the Kummer theory surrounding the local LGP-monoids of Proposition 3.4, (ii), depends, in an essential way, on the theory of [IUTchII], §3 [cf., especially, [IUTchII], Corollaries 3.5, 3.6], which, in turn, depends, in an essential way, on the Kummer theory surrounding **mono-theta environments** established in [EtTh]. Thus, for instance, we recall that the **discrete rigidity** established in [EtTh] is applied so as to avoid working, in the tempered Frobenioids that occur, with **"$\widehat{\mathbb{Z}}$-divisors/line bundles"** [i.e., "$\widehat{\mathbb{Z}}$-completions" of $\mathbb{Z}$-modules of divisors/line bundles], which are *fundamentally incompatible* with conventional notions of divisors/line bundles, hence, in particular, with *mono-theta-theoretic cyclotomic rigidity* [cf. Remark 2.1.1, (v)]. Also, we recall that **"isomorphism class compatibility"** — i.e., in the terminology of

[EtTh], *"compatibility with the* **topology** *of the* **tempered fundamental group***"* [cf. the discussion at the beginning of Remark 2.1.1] — allows one to apply the Kummer theory of mono-theta environments [i.e., the theory of [EtTh]] relative to the **ring-theoretic basepoints** that occur on either side of the **$\mathfrak{log}$-link** [cf. Remarks 2.1.1, (ii), and 2.3.3, (vii); [IUTchII], Remark 3.6.4, (i)], for instance, in the context of the $\mathfrak{log}$-*Kummer correspondence for the splitting monoids of local LGP-monoids*, whose construction depends, in an essential way [cf. the theory of [IUTchII], §3, especially, [IUTchII], Corollaries 3.5, 3.6], on the **conjugate synchronization** arising from the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**. That is to say,

> it is precisely by establishing this conjugate synchronization arising from the $\mathbb{F}_l^{\rtimes\pm}$-symmetry relative to *these basepoints* that occur on either side of the $\mathfrak{log}$-link that one is able to conclude the crucial **compatibility** *of this conjugate synchronization with the* $\mathfrak{log}$-*link* [cf. Remark 1.3.2].

A similar observation may be made concerning the *MLF-Galois pair* approach to the *cyclotomic rigidity isomorphism* that is applied at $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \bigcap \underline{\mathbb{V}}^{\text{non}}$ [cf. [IUTchII], Corollary 1.11, (a); [IUTchII], Remark 1.11.1, (i), (a); [IUTchII], Proposition 4.2, (i); [AbsTopIII], Proposition 3.2, (iv), as well as Remark 2.3.3, (viii), of the present paper], which amounts, in essence, to

> computations involving the Galois cohomology groups of *various subquotients — such as torsion subgroups [i.e., roots of unity] and associated value groups — of the [multiplicative] module of nonzero elements* of an algebraic closure of the mixed characteristic local field involved

[cf. the proof of [AbsAnab], Proposition 1.2.1, (vii)] — i.e., algorithms that are *manifestly compatible with the* **topology** *of the profinite groups involved* [cf. the discussion of Remark 2.3.3, (viii)], in the sense that they do not require one to pass to *Kummer towers* [cf. the discussion of [IUTchII], Remark 3.6.4, (i)], which are *fundamentally incompatible* with the *ring structure* of the fields involved. Here, we note in passing that the corresponding property for $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$ [cf. [IUTchII], Proposition 4.4, (i)] holds as a consequence of the interpretation discussed in [IUTchI], Remark 3.4.2, of **Kummer structures** in terms of **co-holomorphicizations**. On the other hand, the approaches to cyclotomic rigidity just discussed for $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ and $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$ *differ quite fundamentally* from the approach to cyclotomic rigidity taken in the case of *[global] number fields* in the algorithms described in [IUTchI], Example 5.1, (v); [IUTchI], Definition 5.2, (vi), (viii), which depend, in an essential way, on the property

$$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$$

— i.e., which is **fundamentally incompatible** *with the* **topology** *of the profinite groups involved* [cf. the discussion of Remark 2.3.3, (vi), (vii), (viii)] in the sense that it clearly cannot be obtained as some sort of limit of corresponding properties of $(\mathbb{Z}/N\mathbb{Z})^{\times}$! Nevertheless, with regard to uni-/multi-radiality issues, this approach to cyclotomic rigidity in the case of the number fields resembles the theory of mono-theta-theoretic cyclotomic rigidity at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ in that it admits a *natural* **multiradial** *formulation* [cf. Theorem 3.11, (iii), (d); the discussion of Remarks 2.3.2, 3.11.3], in sharp contrast to the essentially **uniradial** nature of the approach to cyclotomic rigidity via MLF-Galois pairs at $\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \bigcap \underline{\mathbb{V}}^{\text{non}}$ [cf. the discussion of [IUTchII], Remark 1.11.3]. These observations are summarized in Fig. 3.7 below.

Finally, we recall that [one verifies immediately that] the various approaches to cyclotomic rigidity just discussed are *mutually compatible* in the sense that they yield the same cyclotomic rigidity isomorphism in any setting in which more than one of these approaches may be applied.

Approach to cyclotomic rigidity	*Uni-/multi-radiality*	*Compatibility with $\mathbb{F}_l^{\rtimes\pm}$-symmetry, profinite/tempered topologies, ring structures, $\mathfrak{log}$-link*
mono-theta environments	**multiradial**	**compatible**
MLF-Galois pairs, via Brauer groups	**uniradial**	**compatible**
number fields, via $\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^\times = \{1\}$	**multiradial**	**incompatible**

Fig. 3.7: Three approaches to cyclotomic rigidity

(vii) In the context of the discussion in the final portion of (vi), it is of interest to recall that the constructions underlying the crucial **bi-coricity** theory of Theorem 1.5, (iii), (iv), depend, in an essential way, on the **conjugate synchronization** arising from the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, which allows one to relate the local monoids and Galois groups at distinct labels $\in |\mathbb{F}_l|$ to one another in a fashion that is *simultaneously compatible* both with

· the **vertically coric** structures and **Kummer theory** that give rise to the $\mathfrak{log}$**-Kummer correspondences** of Theorem 3.11, (ii),

and with

· the property of **distinguishing** [i.e., not identifying] data indexed by **distinct labels** $\in |\mathbb{F}_l|$

— cf. the discussion of Remark 1.5.1, (i), (ii). Since, moreover, this crucial conjugate synchronization is *fundamentally incompatible* with the $\mathbb{F}_l^{\circledast}$**-symmetry**, it is necessary to work with these two symmetries *separately*, as was done in [IUTchI], §4, §5, §6 [cf. [IUTchII], Remark 4.7.6]. Here, it is useful to recall that the $\mathbb{F}_l^{\circledast}$-symmetry also plays a crucial role, in that it allows one to *"descend to F_{mod}" at the level of* **absolute Galois groups** [cf. [IUTchII], Remark 4.7.6]. On the other hand, *both* the $\mathbb{F}_l^{\rtimes\pm}$- and $\mathbb{F}_l^{\circledast}$-symmetries share the property of being compatible with the **vertical coricity** and relevant **Kummer isomorphisms** of the 0-column — cf. the $\mathfrak{log}$**-Kummer correspondences** of Theorem 3.11, (ii), (b) [in the case of the

$\mathbb{F}_l^{\rtimes\pm}$-symmetry], (c) [in the case of the $\mathbb{F}_l^{\circledast}$-symmetry]. Here, we recall that the vertically coric versions of both the $\mathbb{F}_l^{\rtimes\pm}$- and the $\mathbb{F}_l^{\circledast}$-symmetries depend, in an essential way, on the **arithmetic holomorphic structure** of the 0-column, hence give rise to **multiradial** structures via the **tautological** approach to constructing such structures discussed in Remark 3.11.2, (i), (ii).

(viii) In the context of (vii), it is useful to recall that in order to construct the $\mathbb{F}_l^{\rtimes\pm}$-symmetry, it is necessary to make use of **global $\pm$-synchronizations** of various local $\pm$-indeterminacies. Since the local tempered fundamental groups at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ do not extend to a "global tempered fundamental group", these global $\pm$-synchronizations give rise to **profinite conjugacy indeterminacies** in the vertically coric construction of the LGP-monoids [i.e., the theta values at torsion points] given in [IUTchII], §2, which are *resolved* by applying the theory of [IUTchI], §2 — cf. the discussion of [IUTchI], Remark 6.12.4, (iii); [IUTchII], Remark 4.5.3, (iii); [IUTchII], Remark 4.11.2, (iii).

(ix) In the context of (vii), it is also useful to recall the important role played, in the theory of the present series of papers, by the various "copies of F_{mod}", i.e., more concretely, in the form of the various copies of the **global Frobenioids** "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$", "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" and their realifications. That is to say, the **ring structure** of the global field F_{mod} allows one to bridge the gap — i.e., furnishes a **translation apparatus** — between the **multiplicative** structures constituted by the global realified Frobenioids related via the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link and the **additive** representations of these global Frobenioids that arise from the "mono-analytic containers" furnished by the *mono-analytic log-shells* [cf. (v)]. Here, the **precise compatibility** of the ingredients for "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$" with the $\mathfrak{log}$-Kummer correspondence renders "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$" better suited to describing the relation to the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. Remark 3.10.1, (ii)]. On the other hand, the local portion of "$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$" — i.e., which is subject to **"upper semi-compatibility"** [cf. (Ind3)], hence only *"approximately compatible"* with the $\mathfrak{log}$-Kummer correspondence — renders it better suited to **explicit estimates** of global arithmetic degrees, by means of **log-volumes** [cf. Remark 3.10.1, (iii)].

(x) Thus, one may summarize the discussion thus far as follows. The theory of **"Kummer-detachment"** — cf. Remarks 1.5.4, (i); 2.1.1; 3.10.1, (ii), (iii) — furnished by Theorem 3.11, (ii), (iii), allows one to relate the **Frobenioid-theoretic** [i.e., "Frobenius-like"] structures that appear in the domain [i.e., at $(0,0)$] of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link** [cf. (i)] to the **multiradial representation** described in Theorem 3.11, (i), (a), (b), (c), but only at the cost of introducing the **indeterminacies**

(Ind1) — which may be thought of as arising from the requirement of *compatibility* with the **permutation symmetries** of the **étale-picture** [cf. Theorem 3.11, (i)];

(Ind2) — which may be thought of as arising from the requirement of *compatibility* with the $\mathrm{Aut}_{\mathcal{F}^{\vdash\times\boldsymbol{\mu}}}(-)$**-indeterminacies** that act on the domain/codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link** [cf. (ii); Theorem 3.11, (i), (iii)], i.e., with the **horizontal arrows** of the log-theta-lattice;

(Ind3) — which may be thought of as arising from the requirement of *compatibility* with the $\mathfrak{log}$**-Kummer correspondences** of Theorem 3.11, (ii), i.e., with the **vertical arrows** of the log-theta-lattice.

The various indeterminacies (Ind1), (Ind2), (Ind3) to which the multiradial representation is subject may be thought of as data that describes some sort of **"formal quotient"**, like the **"fine moduli spaces"** that appear in algebraic geometry. In this context, the **procession-normalized mono-analytic log-volumes** [i.e., where the average is taken over $j \in \mathbb{F}_l^{\circledast}$] of Theorem 3.11, (i), (a), (c), furnish a means of constructing a sort of associated **"coarse space"** or **"inductive limit"** [of the "inductive system" constituted by this "formal quotient"] — i.e., in the sense that [one verifies immediately — cf. Proposition 3.9, (ii) — that] the resulting log-volumes $\in \mathbb{R}$ are **invariant** with respect to the indeterminacies (Ind1), (Ind2), and have the effect of converting the indeterminacy (Ind3) into an **inequality** [from above]. Moreover, the **$\mathfrak{log}$-link compatibility** of the various log-volumes that appear [cf. Proposition 3.9, (iv); the final portion of Theorem 3.11, (ii)] ensures that these log-volumes are *compatible* with [the portion of the "formal quotient"/"inductive system" constituted by] the various arrows [i.e., *Kummer isomorphisms* and $\mathfrak{log}$-*links*] of the **$\mathfrak{log}$-Kummer correspondence** of Theorem 3.11, (ii). Here, we note that the *averages over* $j \in \mathbb{F}_l^{\circledast}$ that appear in the definition of the procession-normalized volumes involved may be thought of as a consequence of the **$\mathbb{F}_l^{\circledast}$-symmetry** acting on the labels of the theta values that give rise to the LGP-monoids — cf. also the definition of the symbol "$\triangle$" in [IUTchII], Corollary 4.10, (i), via the *identification of the symbols* "0" and "$\langle \mathbb{F}_l^{\circledast} \rangle$"; the discussion of Remark 3.9.3. Also, in this context, it is of interest to observe that the various **tensor products** that appear in the various local mono-analytic tensor packets that arise in the multiradial representation of Theorem 3.11, (i), (a), have the effect of **identifying** the operation of "multiplication by elements of $\mathbb{Z}$" — and hence also the effect on **log-volumes** of such multiplication operations! — at *different labels* $\in \mathbb{F}_l^{\circledast}$.

(xi) For ease of reference, we divide this *step* into *substeps*, as follows.

(xi-a) Consider a ***q*-pilot object** at $(1, 0)$, which we think of — relative to the relevant copy of "$\mathcal{F}_{\mathfrak{mod}}^{\circledast}$" — in terms of the **holomorphic log-shells** constructed at $(1, 0)$ [cf. the discussion of Remark 3.12.2, (iv), (v), below]. Then the **$\Theta_{\mathrm{LGP}}^{\times\boldsymbol{\mu}}$-link** from $(0, 0)$ to $(1, 0)$ may be interpreted as a sort of **gluing isomorphism** that relates the **arithmetic holomorphic structure** — i.e., the *"conventional ring/scheme-theory"* — at $(1, 0)$ to the arithmetic holomorphic structure at $(0, 0)$ in such a way that the **Θ-pilot object** at $(0, 0)$ [thought of as an object of the relevant global realified Frobenioid] corresponds to the ***q*-pilot object** at $(1, 0)$ [cf. (i); the discussion of Remark 3.12.2, (ii), below].

(xi-b) On the other hand, the multiradial construction algorithm of Theorem 3.11, which was summarized in the discussion of (x), yields a construction of a *collection of possibilities of output data* contained in

$$({}^{0,\circ}\mathcal{U}^{\mathbb{Q}} \supseteq) \quad {}^{0,\circ}\mathcal{U} \quad \overset{\sim}{\to} \quad {}^{1,\circ}\mathcal{U} \quad (\subseteq {}^{1,\circ}\mathcal{U}^{\mathbb{Q}})$$

— where the isomorphism "$\overset{\sim}{\to}$" arises from the **permutation symmetries** discussed in the final portion of Theorem 3.11, (i) — that satisfies the **input primestrip link (IPL)** and **simultaneous holomorphic expressibility (SHE)** properties discussed in Remark 3.11.1, (iii), (iv), (v) [cf. also the discussion of *"possible* *images"* at the beginning of the present proof]. Here, with regard to (IPL), we observe that the $\mathcal{F}^{\Vdash\blacktriangleright}$*-prime-strip* portion of the **link/relationship** of this collection of possibilities of output data to the **input data (**$\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-)prime-strip** [cf. Remark 3.11.1, (ii)] consists precisely of **(full poly-)isomorphisms of** $\mathcal{F}^{\Vdash\blacktriangleright}$**-prime-strips**, while the corresponding *link/relationship* for $\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$-prime-strips is *somewhat more complicated*, as a result of the *indeterminacies* (Ind1), (Ind2), (Ind3). Also, in this context, we observe that, although the multiradial construction algorithm of Theorem 3.11 in fact involves the Θ-*pilot object* at $(0,0)$, in the present discussion of Step (xi), we shall *only be concerned* with **qualitative logical aspects/consequences** of this construction algorithm, i.e., with the

- **input prime-strip link (IPL)**,
- **simultaneous holomorphic expressibility (SHE)**, and
- **algorithmic parallel transport (APT)**

properties discussed in Remark 3.11.1, (iii), (iv), (v). That is to say, we shall take the point of view of **"temporarily forgetting"** — cf. the discussion of **hidden internal structures (HIS)** in Remark 3.11.1, (iv) — the fact that the multiradial construction algorithm of Theorem 3.11 in fact involves Θ-*pilot objects*, *theta functions/values*, *mono-theta environments*. Alternatively, in the discussion to follow, we shall, roughly speaking, think of the multiradial construction algorithm of Theorem 3.11 as

> **"some"** algorithm that transforms a certain type of **input data** into a certain type of **output data** and, moreover, satisfies **certain properties (IPL)** and **(SHE)**.

(xi-c) Thus, the discussion of the (IPL) and (SHE) properties in (xi-b) may be summarized as follows:

> The *multiradial construction algorithm of Theorem* 3.11 yields a collection of possibilities of output data in ${}^{1,\circ}\mathcal{U}$ ($\subseteq {}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$) that are **linked/related** [cf. (IPL)], via **isomorphisms of** $\mathcal{F}^{\Vdash\blacktriangleright}$**-prime-strips**, to the representation [via the $\mathfrak{log}$-Kummer correspondence in the 1-column] of the $\boldsymbol{q}$**-pilot object** at $(1,0)$ on ${}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$, and, moreover, whose construction may be expressed entirely relative to the **arithmetic holomorphic structure** in the 1-column [cf. (SHE)].

Here, we recall that, in more concrete language, this "*arithmetic holomorphic structure* in the 1-column" amounts, in essence, to the **ring structure** labeled "$1,\circ$". Moreover, by slightly enlarging the collection of possibilities of output data under consideration by working with the **holomorphic hull** ${}^{1,\circ}\overline{\mathcal{U}}$ ($\supseteq {}^{1,\circ}\mathcal{U}$), we obtain output data that is expressed — *not* in terms of regions contained in various *tensor products of local fields* labeled "$1,\circ$" [i.e., more concretely, various isomorphs of "$K_{\underline{v}}$", for $\underline{v} \in \underline{\mathbb{V}}$], but rather — in terms of **localizations of arithmetic vector bundles** over certain *local rings* labeled "$1,\circ$" [i.e., more concretely, various isomorphs of "$\mathcal{O}_{K_{\underline{v}}}$", for $\underline{v} \in \underline{\mathbb{V}}$] — cf. the discussion of Remarks 3.9.5, (vii), (Ob1), (Ob2), (Ob5); 3.12.2, (v), below. Such an expression in terms of *"localizations of arithmetic vector bundles"* is necessary in order to render the output data in a form that is **comparable** to the representation of the $\boldsymbol{q}$**-pilot object** [i.e., which arises from a certain *arithmetic line bundle*] at $(1,0)$ on ${}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$.

(xi-d) The discussion of (xi-c) thus yields the following conclusion:

> The *multiradial construction algorithm* of Theorem 3.11, followed by formation of the *holomorphic hull*, yields a collection of possibilities of output data in ${}^{1,\circ}\overline{\mathcal{U}}$ that are **linked/related** [cf. (IPL)], via **isomorphisms of $\mathcal{F}^{\Vdash\blacktriangleright}$-prime-strips**, to the representation [via the $\mathfrak{log}$-*Kummer correspondence* in the 1-*column*] of the ***q*-pilot object** at $(1,0)$ on ${}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$, and, moreover, whose construction may be expressed entirely relative to **localizations of arithmetic vector bundles** over **rings** that arise in the **arithmetic holomorphic structure** in the 1-column [cf. (SHE)].

Here, we observe that these "localizations of arithmetic vector bundles" are [unlike the *arithmetic line bundle* that gives rise to the *q-pilot object*] of *rank* > 1. Moreover, the q-pilot object is defined at the level of *realifications* of Frobenioids of [global] arithmetic line bundles. Thus, it is only by forming [a suitable positive tensor power of] the *determinant* of the localizations of arithmetic vector bundles mentioned in the above display [cf. Remark 3.9.5, (vii), (Ob3), (Ob4)] and then applying the [suitably *normalized*, with respect to $j \in |\mathbb{F}_l|$] **log-volume** to various regions — i.e., the *region* ${}^{1,\circ}\overline{\mathcal{U}}$ and the *region* that arises from the representation of the q-pilot object at $(1,0)$ on ${}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$ — in ${}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$ [cf. Remark 3.9.5, (vii), (Ob3), (Ob4), (Ob6), (Ob7), (Ob9)], that we are able to obtain **completely comparable objects** [cf. Remarks 3.9.5, (vii), (Ob5), (Ob6), (Ob7), (Ob8), (Ob9); 3.9.5, (viii), (ix)], namely,

$$\mathbb{R}_{\leq -|\log(\underline{\underline{\Theta}})|} \stackrel{\mathrm{def}}{=} \{\lambda \in \mathbb{R} \mid \lambda \leq -|\log(\underline{\underline{\Theta}})|\} \ \subseteq \ \mathbb{R}; \quad -|\log(\underline{\underline{q}})| \in \mathbb{R}$$

— where we recall that, by definition, $-|\log(\underline{\underline{\Theta}})|$ is the [*negative* — cf. the discussion of *"possible images"* at the beginning of the present proof] log-volume of ${}^{1,\circ}\overline{\mathcal{U}}$, while $-|\log(\underline{\underline{q}})|$ is the log-volume of the region that arises from the representation of the q-pilot object at $(1,0)$ on ${}^{1,\circ}\mathcal{U}^{\mathbb{Q}}$. In this context, it is useful to recall from Proposition 3.9, (iii) [cf. also the discussion of Remarks 3.9.2, 3.9.6], that **global arithmetic degrees** of objects of global realified Frobenioids may be interpreted as **log-volumes** [cf. also the discussion of Remarks 1.5.2, (iii); 3.10.1, (iv), as well as of Remark 3.12.2, (v), below]. Finally, in this context, we observe [cf. the first display of the present (xi-d)] that it is of *crucial importance* to apply the **$\mathfrak{log}$-Kummer correspondence** in the **1-column** [cf. the discussion of $\mathfrak{log}$-*Kummer correspondences* in Remark 3.9.5, (vii), (Ob7), (Ob8); Remark 3.9.5, (viii), (sQ4); Remark 3.9.5, (ix); the final portion of Remark 3.9.5, (x); the discussion of the final portion of Remark 3.12.2, (v), below], in order to *rectify the* **vertical shift/mismatch** [cf. the discussion of (iii), (iv) in the case of the 0-*column*] between the **unit portion** of ${}^{1,0}\mathfrak{F}_{\triangle}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$ and the **log-shells** arising from [the image via the relevant Kummer isomorphisms of] this unit portion, which give rise to the *tensor packets of log-shells* that constitute ${}^{1,\circ}\mathcal{U}$.

(xi-e) Next, let us recall that the relationship, i.e., that arises by applying the *log-volume to the pilot-object*, between the **pilot-object log-volume** $-|\log(\underline{\underline{q}})| \in \mathbb{R}$ and the **input data ($\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-)prime-strip** is precisely the relationship **prescribed/imposed** by the **arithmetic holomorphic structure** in the 1-column, i.e., via the representation of the input data ($\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-)prime-strip on ${}^{1,\circ}\mathcal{U}$ relative to this 1-column arithmetic holomorphic structure. That is to say, *"expressibility relative to the arithmetic holomorphic structure in the* 1*-column"* [cf. (SHE)] amounts *precisely* to

> *"expressibility via operations that are valid/executable/well-defined even when* **subject** *to the* **condition** *that the* **pilot-object log-volume** *associated to the input data* ($\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$*-)prime-strip [which is, of course, linked/ related, via isomorphisms of* $\mathcal{F}^{\Vdash\blacktriangleright}$*-prime-strips, to the possible output data* $\mathcal{F}^{\Vdash\blacktriangleright}$*-prime-strips!] be equal to the* **fixed value** $-|\log(\underline{\underline{q}})| \in \mathbb{R}$*".*

In particular, the discussion of (xi-d) thus yields the following conclusion:

> The *multiradial construction algorithm* of Theorem 3.11, followed by formation of the *holomorphic hull* and application of the *log-volume*, yields a collection of **possible log-volumes of pilot-object output data**
>
> $$\mathbb{R}_{\leq -|\log(\underline{\underline{\Theta}})|} \subseteq \mathbb{R}$$
>
> that are **linked/related** [cf. (IPL)], via **isomorphisms of** $\mathcal{F}^{\Vdash\blacktriangleright}$**-prime-strips**, to the **pilot-object log-volume**
>
> $$-|\log(\underline{\underline{q}})| \in \mathbb{R}$$
>
> of the **input data (**$\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-)prime-strip** [cf. (SHE)].

(xi-f) Thus, we conclude from (xi-e) that

> the construction of the subset $\mathbb{R}_{\leq -|\log(\underline{\underline{\Theta}})|} \subseteq \mathbb{R}$ of **possible pilot-object log-volumes** of output data is **subject** to the **condition** that this construction of output data possibilities constitutes, in particular, a construction [perhaps only up to some sort of *"approximation"*, as a result of various *indeterminacies*] of the **pilot-object log-volume** of the **input data (**$\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$**-)prime-strip**, namely, $-|\log(\underline{\underline{q}})| \in \mathbb{R}$.

The inclusion $-|\log(\underline{\underline{q}})| \in \mathbb{R}_{\leq -|\log(\underline{\underline{\Theta}})|}$, hence also the inequality

$$-|\log(\underline{\underline{q}})| \leq -|\log(\underline{\underline{\Theta}})| \in \mathbb{R}$$

— i.e., the conclusion that $C_\Theta \geq -1$ for any $C_\Theta \in \mathbb{R}$ such that $-|\log(\underline{\underline{\Theta}})| \leq C_\Theta \cdot |\log(\underline{\underline{q}})|$ — in the statement of Corollary 3.12, then follows formally.

(xi-g) Thus, in summary,

> *the* **multiradial construction algorithm** *of Theorem 3.11, followed by formation of the* **holomorphic hull** *and application of the* **log-volume**, *yields* **two tautologically equivalent ways to compute** *the log-volume of the q-pilot object at* $(1,0)$ — cf. Fig. 3.8 below.

multiradial representation at 0-column $(0, \circ)$	**permutation symmetry** of $\approx$ **étale-picture**	**multiradial representation** at 1-column $(1, \circ)$
Kummer-detachment via $\mathfrak{log}$-**Kummer** $\Uparrow$ **compatibly** with $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-**link**		**comparison** via $\Downarrow$ **hol. hull, log-vol.**
Θ-pilot object in $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater at $(0,0)$	$(-)^{\Vdash\blacktriangleright}$-portion, $(-)^{\vdash\times\boldsymbol{\mu}}$-portion $\approx$ of $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link**	***q*-pilot object** in $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater at $(1,0)$

Fig. 3.8: Two tautologically equivalent ways to compute the log-volume of the q-pilot object at $(1,0)$

(xi-h) In this context, it is useful to recall that the above argument depends, in an essential way [cf. the discussion of (ii), (vi)], on the theory of [EtTh], which *does not admit any evident generalization* to the case of N-th tensor powers of Θ-pilot objects, for $N \geq 2$. That is to say, the *log-volume* of such an N-th tensor power of a Θ-pilot object must always be computed as the result of *multiplying the log-volume of the original* Θ*-pilot object by* N — cf. Remark 2.1.1, (iv); [IUTchII], Remark 3.6.4, (iii), (iv). In particular, although the analogue of the above argument for such an N-th tensor power would lead to **sharper inequalities** than the inequalities obtained here, it is difficult to see how to obtain such sharper inequalities via a routine generalization of the above argument. In fact, as we shall see in [IUTchIV], these sharper inequalities are known to be **false** [cf. [IUTchIV], Remark 2.3.2, (ii)].

(xii) In the context of the argument of (xi), it is useful to observe the important role played by the **global** realified Frobenioids that appear in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link. That is to say, since ultimately one is only concerned with the computation of *log-volumes*, it might appear, at first glance, that it is possible to dispense with the use of such global Frobenioids and instead work only with the various *local Frobenioids*, for $\underline{v} \in \underline{\mathbb{V}}$, that are directly related to the computation of log-volumes. On the other hand, observe that since the isomorphism of [local or global!] Frobenioids arising from the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link only preserves **isomorphism classes of objects** of these Frobenioids [cf. the discussion of Remark 3.6.2, (i)], to work only with local Frobenioids means that one must contend with the **indeterminacy** of not knowing whether, for instance, such a local Frobenioid object at some $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ corresponds to a *given open submodule* of the log-shell at $\underline{v}$ or to, say, the $p_{\underline{v}}^N$*-multiple of this submodule*, for $N \in \mathbb{Z}$. Put another way, one must contend with the indeterminacy arising from the fact that, unlike the case with the global Frobenioids "$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$", "$\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}$", objects of the various local Frobenioids that arise admit **endomorphisms** *which are* **not automorphisms**. This indeterminacy has the effect of rendering *meaningless* any attempt to perform a precise log-volume computation as in (xi). ○

Remark 3.12.1.

(i) In [IUTchIV], we shall be concerned with obtaining *more explicit upper bounds* on $-|\log(\underline{\underline{\Theta}})|$, i.e., *estimates* "$C_\Theta$" as in the statement of Corollary 3.12.

(ii) It is not difficult to verify that, for $\lambda \in \mathbb{Q}_{>0}$, one may obtain a similar theory to the theory developed in the present series of papers [cf. the discussion of Remark 3.11.1, (ii)] for *"generalized* $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$*-links"* of the form

$$\underline{\underline{q}}^{\lambda} \quad \mapsto \quad \underline{\underline{q}}^{\begin{pmatrix} 1^2 \\ \vdots \\ (l^*)^2 \end{pmatrix}}$$

— i.e., so the theory developed in the present series of papers corresponds to the case of $\lambda = 1$. This sort of "generalized $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link" is roughly reminiscent of — but by no means equivalent to! — the sort of issues considered in the discussion of Remark 2.2.2, (i). Here, we observe that raising to the λ-th power on the "$\underline{\underline{q}}$ side" *differs quite fundamentally* from raising to the λ-th power on the "$\underline{\underline{q}}^{(1^2 \ldots (l^*)^2)}$ side", an issue that is discussed briefly [in the case of $\lambda = N$] in the final portion of Step (xi) of the proof of Corollary 3.12. That is to say, "generalized $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-links" as in the above display *differ fundamentally* both from the situation of Remark 2.2.2, (i), and the situation discussed in the final portion of Step (xi) of the proof of Corollary 3.12 in that the theory of the **first power** of the **étale theta function** is left unchanged [i.e., relative to the theory developed in the present series of papers] — cf. the discussion of Remark 2.2.2, (i); Step (xi) of the proof of Corollary 3.12. At any rate, in the case of "generalized $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-links" as in the above display, one may apply the same arguments as the arguments used to prove Corollary 3.12 to conclude the *inequality*

$$C_\Theta \geq -\lambda$$

— i.e., which is *sharper*, for $\lambda < 1$, than the inequality obtained in Corollary 3.12 in the case of $\lambda = 1$. In fact, however, such sharper inequalities will not be of interest to us, since, in [IUTchIV], our estimates for the upper bound C_Θ will be *sufficiently rough as to be unaffected* by adding a constant of absolute value ≤ 1.

(iii) In the context of the discussion of (ii) above, it is of interest to note that the **multiradial** theory of **mono-theta-theoretic cyclotomic rigidity**, and, in particular, the theory of the **first power** of the **étale theta function**, may be regarded as a theory that concerns a sort of **"canonical profinite volume"** on the elliptic curves under consideration associated to the **first power** of the ample line bundle corresponding to the étale theta function. This point of view is also of interest in the context of the discussion of various approaches to *cyclotomic rigidity* summarized in Fig. 3.7 [cf. also the discussion of Remark 2.3.3]. Indeed, the elementary fact "$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$", which plays a key role in the **multiradial** algorithms for cyclotomic rigidity isomorphisms in the **number field** case [cf. [IUTchI], Example 5.1, (v), as well as the discussion of Remarks 2.3.2, 2.3.3 of the present paper], may be regarded as an immediate consequence of an easy interpretation of the *product formula* in terms of the *geometry* of the *domain* in the archimedean completion of the number field $\mathbb{Q}$ determined by the inequality "≤ 1", i.e., a domain which may be thought of as a sort of concrete geometric representation of a **"canonical unit of volume"** of the number field $\mathbb{Q}$.

Remark 3.12.2.

(i) One of the main themes of the present series of papers is the issue of *dismantling the* **two underlying combinatorial dimensions** of a number field — cf. Remarks 1.2.2, (vi), of the present paper, as well as [IUTchI], Remarks 3.9.3, 6.12.3, 6.12.6; [IUTchII], Remarks 4.7.5, 4.7.6, 4.11.2, 4.11.3, 4.11.4. The principle examples of this topic may be summarized as follows:

(a) splittings of various monoids into **unit** and **value group** portions;

(b) separating the "$\mathbb{F}_l$" arising from the l-torsion points of the elliptic curve — which may be thought of as a sort of *"finite approximation"* of $\mathbb{Z}$! — into a [*multiplicative*] $\mathbb{F}_l^{\circledast}$**-symmetry** — which may also be thought of as corresponding to the *global arithmetic* portion of the arithmetic fundamental groups involved — and a(n) [*additive*] $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** — which may also be thought of as corresponding to the *geometric* portion of the arithmetic fundamental groups involved;

(c) separating the ring structures of the various **global number fields** that appear into their respective underlying **additive** structures — which may be related directly to the various *log-shells* that appear — and their respective underlying **multiplicative** structures — which may be related directly to the various *Frobenioids* that appear.

From the point of view of Theorem 3.11, example (a) may be seen in the **"non-interference"** properties that underlie the **$\mathfrak{log}$-Kummer correspondences** of Theorem 3.11, (ii), (b), (c), as well as in the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link compatibility** properties discussed in Theorem 3.11, (ii), (c), (d).

(ii) On the other hand, another important theme of the present §3 consists of the issue of **"reassembling"** *these two dismantled combinatorial dimensions* by means of the **multiradial mono-analytic containers** furnished by the **mono-analytic log-shells** — cf. Fig. 3.6 — i.e., of exhibiting the extent to which these two dismantled combinatorial dimensions **cannot be separated** from one another, at least in the case of the **Θ-pilot object**, by describing the **"structure of the intertwining"** between these two dimensions that existed prior to their separation. From this point of view, one may think of the **multiradial representations** discussed in Theorem 3.11, (i) [cf. also Theorem 3.11, (ii), (iii)], as the *final output* of this "reassembling procedure" for Θ-pilot objects. From the point of view of example (a) of the discussion of (i), this "reassembling procedure" allows one to **compute/estimate the value group portions** of various monoids of arithmetic interest in terms of the **unit group portions** of these monoids. It is precisely these estimates that give rise to the **inequality** obtained in Corollary 3.12. That is to say, from the point of view of *dismantling/reassembling the intertwining between* *value group and unit group portions*, the argument of the proof of Corollary 3.12 may be summarized as follows:

(a$^{\mathrm{itw}}$) When considered from the point of view of **log-volumes** of **Θ-pilot** and ***q*-pilot** objects, the correspondence of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link [i.e., that sends Θ-pilot objects to q-pilot objects] may seem a bit **"mysterious"** or even, at first glance, *"self-contradictory"* to some readers.

(b$^{\mathrm{itw}}$) On the other hand, this correspondence of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link is made possible by the fact that one works with Θ-pilot or q-pilot objects in terms of **"sufficiently weakened data"** [namely, the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the definition of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link], i.e., data that is "sufficiently weak" that *one can no longer distinguish between Θ-pilot and q-pilot objects*.

(c$^{\mathrm{itw}}$) Thus, if one thinks of the $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strips that appear in the domain and codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link as a **"single abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip"** that is *regarded/only known up to isomorphism*, then the issue of *which log-volume* such an abstract $\mathcal{F}^{\Vdash\blacktriangleright\times\boldsymbol{\mu}}$-prime-strip corresponds to [cf. (a$^{\mathrm{itw}}$)] is precisely the issue of "**which intertwining** *between* **value group** *and* **unit group** *portions"* one considers, i.e., the issue of *"which arithmetic holomorphic structure"* [of the arithmetic holomorphic structures that appear in the domain and codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link] that one works in. Put another way, it is essentially a *tautological consequence* of the fact that these two arithmetic holomorphic structures in the *domain* and *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link are **distinguished** from one another that the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link yields a situation in which *both* the **Θ-intertwining** [i.e., the intertwining associated to the Θ-pilot object in the domain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link] and the ***q*-intertwining** [i.e., the intertwining associated to the q-pilot object in the codomain of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link] are **simultaneously valid**, i.e.,

$$\Big(q\text{-intertwining holds}\Big) \ \wedge \ \Big(\Theta\text{-intertwining holds}\Big)$$

— cf. the discussion of the *"distinct labels approach"* in Remark 3.11.1, (vii).

(d$^{\mathrm{itw}}$) On the other hand, from the point of view of the analogy between **multiradiality** and the classical theory of **parallel transport** via **connections** [cf. [IUTchII], Remark 1.7.1], the **multiradial representation** of Theorem 3.11 [cf. also the discussion of Remark 3.11.1, especially Remark 3.11.1, (ii), (iii)] asserts that, up to the *relatively mild "monodromy"* constituted by the **indeterminacies** (Ind1), (Ind2), (Ind3), one may **"parallel transport"** or **"confuse"** the Θ-pilot object in the **domain** of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, i.e., the Θ-pilot object represented relative to its *"native intertwining/arithmetic holomorphic structure"*, with the Θ-pilot object represented relative to the *"alien intertwining/arithmetic holomorphic structure"* in the **codomain** of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link.

(e$^{\mathrm{itw}}$) In particular, one may **fix** the arithmetic holomorphic structure of the *codomain* of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, i.e., the *"native intertwining/arithmetic holomorphic structure"* associated to the *q-pilot object* in the *codomain* of the

$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link, and then, by *applying* ($\mathrm{d}^{\mathrm{itw}}$) and working up to the *indeterminacies* (Ind1), (Ind2), (Ind3) [cf. also the subtleties discussed in (iv), (v) below; Remark 3.9.5, (vii), (viii), (ix)], **construct** the *"native intertwining/arithmetic holomorphic structure"* associated to the **Θ-pilot object** in the **domain** of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link as a mathematical structure that is **intrinsically associated** to the underlying structure of — hence, in particular, **simultaneously with/without invalidating** the conditions imposed by — the *"native intertwining/arithmetic holomorphic structure"* associated to the ***q*-pilot object** in the **codomain** of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link [cf. the discussion of Remark 3.11.1, especially Remark 3.11.1, (ii), (iii)]. Indeed, this point of view is precisely the point of view that is taken in the proof of Corollary 3.12 [cf., especially, Step (xi)].

($\mathrm{f}^{\mathrm{itw}}$) One way of summarizing the situation described in ($\mathrm{e}^{\mathrm{itw}}$) is in terms of *logical relations* as follows. The **multiradial representation** of Theorem 3.11 [cf. also the discussion of Remark 3.11.1] may be thought of [cf. the *first* "$\Longrightarrow$" of the following display] as an algorithm for constructing, **up to suitable indeterminacies** [cf. the discussion of ($\mathrm{e}^{\mathrm{itw}}$)], the **"$\Theta$-intertwining"** as a mathematical structure that is **intrinsically associated** to the underlying structure of — hence, in particular, **simultaneously with/without invalidating** [cf. the *logical relator "AND"*, i.e., "$\wedge$"] the conditions imposed by — the **"*q*-intertwining"**, while holding the **"single abstract $\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strip"** of the discussion of ($\mathrm{b}^{\mathrm{itw}}$), ($\mathrm{c}^{\mathrm{itw}}$) **fixed**, i.e., in symbols:

$$\Big(q\text{-itw.}\Big) \quad \Longrightarrow \quad \Big(q\text{-itw.}\Big) \wedge \Big(\Theta\text{-itw./indets.}\Big) \quad \Longrightarrow \quad \Big(\Theta\text{-itw./indets.}\Big)$$

— where the *second* "$\Longrightarrow$" of the above display is *purely formal*; "itw." and "/indets." are to be understood, respectively as abbreviations for "intertwining holds" and "up to suitable indeterminacies". Here, we observe that

> the "$\wedge$" of the above display may be regarded as the *"image"* of, hence, in particular, as a *consequence* of, the "$\wedge$" in the display of ($\mathrm{c}^{\mathrm{itw}}$), via the various **(sub)quotient operations** discussed in Remark 3.9.5, (viii), i.e., whose subtle *compatibility properties* allow one to conclude the "$\wedge$" of the above display from the "$\wedge$" in the display of ($\mathrm{c}^{\mathrm{itw}}$).

Thus, at the level of *logical relations*,

> the ***q*-intertwining**, hence also the *log-volume* of the *q-pilot object* in the codomain of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link, may be thought of as a **special case** of the **Θ-intertwining**, i.e., at a more concrete level, of the *log-volume* of the Θ*-pilot object* in the domain of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link, regarded up to *suitable indeterminacies.*

Corollary 3.12 then follows, *essentially formally.*

Alternatively, from the point of view of *"[very rough!] toy models"*, i.e., whose goal lies solely in representing certain *overall qualitative aspects* of a situation, one may think of the discussion of ($\mathrm{a}^{\mathrm{itw}}$) $\sim$ ($\mathrm{f}^{\mathrm{itw}}$) given above in the following terms:

($\mathrm{a}^{\mathrm{toy}}$) Consider **two distinct copies** ${}^{q}\mathbb{R}$ and ${}^{\Theta}\mathbb{R}$ of the **topological field of real numbers** $\mathbb{R}$, equipped with labels "q" and "Θ", together with an *abstract symbol* "$*$" and *assignments*

$$\lambda_q : \; * \;\mapsto\; {}^{q}(-h) \in {}^{q}\mathbb{R}, \qquad \lambda_\Theta : \; * \;\mapsto\; {}^{\Theta}(-2h) \in {}^{\Theta}\mathbb{R},$$

— where, in the present discussion, we shall write "${}^{q}(-)$", "${}^{\Theta}(-)$" to denote the respective elements/subsets of ${}^{q}\mathbb{R}$, ${}^{\Theta}\mathbb{R}$ determined by an element/subset "$(-)$" of $\mathbb{R}$; $h \in \mathbb{R}_{>0}$ is a positive real number that we are interested in **bounding** from above. If one **forgets** the *distinct labels* "q" and "Θ", then these two assignments λ_q, λ_Θ are **mutually incompatible** and *cannot be considered simultaneously*, i.e., they *contradict* one another [in the sense that $\mathbb{R} \ni -h \neq -2h \in \mathbb{R}$].

($\mathrm{b}^{\mathrm{toy}}$) One aspect of the situation of ($\mathrm{a}^{\mathrm{toy}}$) that renders the *simultaneous consideration* of the two assignments λ_q, λ_Θ *valid* — i.e., at the level of *logical relations*,

$$\Big(* \;\mapsto\; {}^{q}(-h) \in {}^{q}\mathbb{R}\Big) \;\wedge\; \Big(* \;\mapsto\; {}^{\Theta}(-2h) \in {}^{\Theta}\mathbb{R}\Big)$$

— is the use of the **abstract symbol** "$*$", i.e., which is, *a priori*, entirely **unrelated** to any copies of $\mathbb{R}$ [such as ${}^{q}\mathbb{R}$, ${}^{\Theta}\mathbb{R}$].

($\mathrm{c}^{\mathrm{toy}}$) The other aspect of the situation of ($\mathrm{a}^{\mathrm{toy}}$) that renders the *simultaneous consideration* of the two assignments λ_q, λ_Θ *valid*— i.e., at the level of *logical relations*,

$$\Big(* \;\mapsto\; {}^{q}(-h) \in {}^{q}\mathbb{R}\Big) \;\wedge\; \Big(* \;\mapsto\; {}^{\Theta}(-2h) \in {}^{\Theta}\mathbb{R}\Big)$$

— is the use of the **distinct labels** "q", "Θ" for the copies of $\mathbb{R}$ that appear in the assignments λ_q, λ_Θ.

($\mathrm{d}^{\mathrm{toy}}$) Now let us consider an *alternative approach to constructing* the assignment λ_Θ: We construct λ_Θ as the **"assignment with indeterminacies"**

$$\lambda_\Theta^{\mathrm{Ind}} : \; * \;\mapsto\; {}^{\Theta}\mathbb{R}_{\leq -2h+\epsilon} \subseteq {}^{\Theta}\mathbb{R}$$

— where $\mathbb{R}_{\leq -2h+\epsilon} \stackrel{\mathrm{def}}{=} \{x \in \mathbb{R} \mid x \leq -2h+\epsilon\} \subseteq \mathbb{R}$; $\epsilon \in \mathbb{R}_{>0}$ is some positive number.

($\mathrm{e}^{\mathrm{toy}}$) Now suppose that one **verifies** that one may **construct** the *"assignment with indeterminacies"* $\lambda_\Theta^{\mathrm{Ind}}$ of ($\mathrm{d}^{\mathrm{toy}}$) as a mathematical structure that is **intrinsically associated** to the underlying structure of the assignment λ_q — hence, in particular, **simultaneously with/without invalidating** the conditions imposed by — the assignment λ_q, even if one *forgets the labels* "q", "Θ" that were appended to copies of $\mathbb{R}$, i.e., even if one *identifies* ${}^{q}\mathbb{R}$, ${}^{\Theta}\mathbb{R}$, in the usual way, with $\mathbb{R}$ [cf. the properties (IPL), (SHE) of Remark 3.11.1, (iii)]. That is to say, we suppose that one can show that the assignments determined, respectively, by λ_q, $\lambda_\Theta^{\mathrm{Ind}}$, by identifying copies of $\mathbb{R}$, namely,

$$* \mapsto -h \in \mathbb{R}, \qquad * \mapsto \mathbb{R}_{\leq -2h+\epsilon} \subseteq \mathbb{R}$$

— where the latter assignment may be considered as the assignment that maps $*$ to *"some [undetermined] element* $\in \mathbb{R}_{\leq -2h+\epsilon}$*"* — are such that one may construct the latter assignment as a mathematical structure that is **intrinsically associated** to — hence, in particular, **simultaneously with/without invalidating** the conditions imposed by — the former assignment. Here, we note that it is *not particularly relevant* that "$\mathbb{R}_{\leq -2h+\epsilon}$" arose as some sort of *"perturbation via indeterminacies of* $2h$*"* [cf. the property (HIS) of Remark 3.11.1, (iv)].

($\mathrm{f}^{\mathrm{toy}}$) The discussion of ($\mathrm{e}^{\mathrm{toy}}$) may be summarized at the level of *logical relations* [cf. the displays of ($\mathrm{b}^{\mathrm{toy}}$), ($\mathrm{c}^{\mathrm{toy}}$)] as follows:

$$\Big(* \mapsto -h\Big) \implies \Big(* \mapsto -h\Big) \wedge \Big(* \mapsto \mathbb{R}_{\leq -2h+\epsilon}\Big) \implies \Big(* \mapsto \mathbb{R}_{\leq -2h+\epsilon}\Big)$$

— that is to say, "$* \mapsto -h$" may be regarded as a **special case** of "$* \mapsto \mathbb{R}_{\leq -2h+\epsilon}$", which, in turn, may be regarded as a *"version with indeterminacies"* of "$* \mapsto -2h$". One then concludes formally that $-h \in \mathbb{R}_{\leq -2h+\epsilon}$ and hence that

$$-h \leq -2h + \epsilon, \quad \text{i.e.,} \quad h \leq \epsilon$$

— that is to say, the *desired upper bound on* h.

(iii) One fundamental aspect of the theory that renders possible the "reassembling procedure" discussed in (ii) [cf. the discussion of Step (iv) of the proof of Corollary 3.12] is the **"juggling of $\boxplus$, $\boxtimes$"** [cf. the discussion of Remark 1.2.2, (vi)] effected by the **$\mathfrak{log}$-links**, i.e., the **vertical arrows** of the log-theta-lattice. This "juggling of $\boxplus$, $\boxtimes$" may be thought of as a sort of combinatorial way of representing the **arithmetic holomorphic structure** associated to a **vertical line** of the log-theta-lattice. Indeed, at *archimedean primes*, this juggling amounts essentially to *multiplication by* $\pm i$, which is a well-known method [cf. the notion of an *"almost complex structure"*!] for representing holomorphic structures in the classical theory of differential manifolds. On the other hand, it is important to recall in this context that this "juggling of $\boxplus$, $\boxtimes$" is precisely what gives rise to the **upper semi-compatibility** indeterminacy (Ind3) [cf. Proposition 3.5, (ii); Remark 3.10.1, (i)].

(iv) In the context of the discussion of (ii), (iii), it is of interest to **compare**, in the cases of the **0-** and **1-columns** of the log-theta-lattice, the way in which the theory of **$\mathfrak{log}$-Kummer correspondences** associated to a vertical column of the log-theta-lattice is applied in the proof of Corollary 3.12, especially in Steps (x) and (xi). We begin by observing that the *vertical column* [i.e., 0- or 1-column] under consideration may be depicted [*"horizontally"*!] in the fashion of the diagram of the third display of Proposition 1.3, (iv)

$$\begin{array}{ccccccccc}
 & & & & \bullet_0 & & & & \\
 & & & & \| & & & & \\
\ldots & \to & \bullet & \to & \bullet & \to & \bullet & \to & \ldots \\
 & \ldots & & \searrow & \downarrow & \swarrow & & \ldots & \\
 & & & & \circ & & & &
\end{array}$$

— where the "$\bullet_0$" in the first line of the diagram denotes the portion with *vertical coordinate* 0 [i.e., the portion at $(0, 0)$ or $(1, 0)$] of the vertical column under consideration. As discussed in Step (iii) of the proof of Corollary 3.12, since the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link is **fundamentally incompatible** with the **distinct arithmetic holomorphic structures** — i.e., **ring structures** — that exist in the 0- and 1-columns, one is obliged to work with the **Frobenius-like** versions of the *unit group* and *value group* portions of monoids arising from "$\bullet_0$" in the definition of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link precisely in order to avoid the need to contend, in the definition of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link, with the issue of describing the **"structure of the intertwining"** [cf. the discussion of (ii)] between these unit group and value group portions determined by the *distinct arithmetic holomorphic structures* — i.e., *ring structures* — that exist in the 0- and 1-columns. On the other hand, one is *also* obliged to work with the **étale-like** "$\circ$" versions of various objects since it is precisely these *vertically coric* versions that allow one to **access**, i.e., by serving as **containers** [cf. the discussion of (ii)] for, the other "$\bullet$'s" in the vertical column under consideration. That is to say, although the various **Kummer isomorphisms** that relate various portions of the Frobenius-like "$\bullet_0$" to the corresponding portions of the étale-like "$\circ$" may at first give the impression that either "$\bullet_0$" or "$\circ$" is superfluous or unnecessary in the theory, in fact

> *both "$\bullet_0$" and "$\circ$"* play an **essential** and **by no means superfluous** role in the theory of the vertical columns of the log-theta-lattice.

This aspect of the theory is essentially the same in the case of both the 0- and the 1-columns. The **$\mathfrak{log}$-link compatibility** of the various **log-volumes** that appear [cf. the discussion of Step (x) of the proof of Corollary 3.12; Proposition 3.9, (iv); the final portion of Theorem 3.11, (ii)] is another aspect of the theory that is essentially the same in the case of both the 0- and the 1-columns. Also, although the discussion of the **"non-interference"** properties that underlie the **$\mathfrak{log}$-Kummer correspondences** of Theorem 3.11, (ii), (b), (c), was only given expicitly, in effect, in the case of the 0-column, i.e., concerning **Θ-pilot objects**, entirely similar "non-interference" properties hold for **$\boldsymbol{q}$-pilot objects**. [Indeed, this may be seen, for instance, by applying the *same arguments* as the arguments that were applied in the case of Θ-pilot objects, or, for instance, by *specializing* the non-interference properties obtained for Θ-pilot objects to the index "$j = 1$" as in the discussion of *"pivotal distributions"* in [IUTchI], Example 5.4, (vii).] These *similarities* between the 0- and 1-columns are summarized in the upper portion of Fig. 3.9 below.

(v) In the discussion of (iv), we highlighted various *similarities* between the 0- and 1-columns of the log-theta-lattice in the context of Steps (x), (xi) of the proof of Corollary 3.12. By contrast, one *significant difference* between the theory of $\mathfrak{log}$-Kummer correspondences in the 0- and 1-columns is

> the **lack** of analogues for ***q*-pilot objects** of the crucial **multiradiality** properties summarized in Theorem 3.11, (iii), (c)

Aspect of the theory	*0-column/ Θ-pilot objects*	*1-column/ q-pilot objects*
essential role of *both* "$\bullet_0$" *and* "$\circ$"	similar	similar
$\mathfrak{log}$-link compatibility of **log-volumes**	similar	similar
"non-interference" properties of **$\mathfrak{log}$-Kummer correspondences**	similar	similar
multiradiality properties of Θ-/q-pilot objects	hold	do *not* hold
treatment of **log-shells/ unit group** portions	used as **mono-analytic containers** for **regions**	**tautological** documenting device for **logarithmic** relationship betw. ring structures
resulting **indeterminacies** acting on **log-shells**	(Ind1), (Ind2), (Ind3)	absorbed by applying **holomorphic hulls**, **log-volumes**

Fig. 3.9: Similarities and differences, in the context of the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$-link, between the 0- and 1-columns of the log-theta-lattice

— i.e., in effect, the lack of an analogue for the q-pilot objects of the theory of *rigidity properties* developed in [EtTh] [cf. the discussion of Remark 2.2.2, (i)]. Another *significant difference* between the theory of $\mathfrak{log}$-Kummer correspondences in the 0- and 1-columns lies in the way in which the associated *vertically coric holomorphic* **log-shells** [cf. Proposition 1.2, (ix)] are treated in their relationship to the **unit group** portions of monoids that occur in the various "•'s" of the $\mathfrak{log}$-Kummer correspondence. That is to say, in the case of the **0-column**, these log-shells are used as **containers** [cf. the discussion of (ii)] for the various **regions** [i.e., subsets] arising from these unit group portions via various composites of arrows in the $\mathfrak{log}$-Kummer correspondence. This approach has the *advantage* of admitting an *interpretation* — i.e., in terms of subsets of **mono-analytic log-shells** — that makes sense even relative to the *distinct arithmetic holomorphic structures* that appear in the 1-column of the log-theta-lattice [cf. Remark 3.11.1]. On the other hand, it has the *drawback* that it gives rise to the **upper semi-compatibility** indeterminacy (Ind3) discussed in the final portion of Theorem 3.11, (ii). By contrast,

> in the case of the **1-column**, since the associated **arithmetic holomorphic structure** is held **fixed** and *regarded* [cf. the discussion of Step (xi) of the proof of Corollary 3.12] as the *standard* with respect to which constructions arising from the 0-column are to be *computed*, there is **no need** [i.e., in the case of the 1-column] to require that the constructions applied **admit mono-analytic interpretations**.

That is to say, in the case of the 1-column, the various unit group portions of monoids at the various "•'s" simply serve as a means of *documenting the* **"logarithmic"** *relationship* [cf. the definition of the $\mathfrak{log}$-link given in Definition 1.1, (i), (ii)!] between the *ring structures* in the domain and codomain of the $\mathfrak{log}$-link. These ring structures give rise to the local copies of sets of integral elements "$\mathcal{O}$" with respect to which the "$\mathfrak{mod}$" *versions* [cf. Example 3.6, (ii)] of *categories of arithmetic line bundles* are defined at the various "•'s". Since the objects of these categories of arithmetic line bundles are **not equipped** with **local trivializations** at the various $\underline{v} \in \underline{\mathbb{V}}$ [cf. the discussion of **isomorphism classes of objects** of Frobenioids in Remark 3.6.2, (i)],

> **regions** in log-shells may only be related to such categories of arithmetic line bundles at the expense of allowing for an **indeterminacy** with respect to **"$\mathcal{O}^\times$"-multiples** at each $\underline{v} \in \underline{\mathbb{V}}$.

It is precisely this indeterminacy that necessitates the introduction, in Step (xi) of the proof of Corollary 3.12, of **holomorphic hulls**, i.e., which have the effect of *absorbing* this indeterminacy [cf. the discussion of Remark 3.9.5, (vii), (viii), (ix), (x), for more details]. Finally, in Step (xi) of the proof of Corollary 3.12,

> the **indeterminacy** in the *specification of a particular member* of the collection of ring structures just discussed — i.e., arising from the *choice of a particular composite* of arrows in the $\mathfrak{log}$-Kummer correspondence that is used to specify a **particular ring structure** among its various "logarithmic conjugates" — is **absorbed** by passing to **log-volumes**

— i.e., by applying the **$\mathfrak{log}$-link compatibility** [cf. (iv)] of the various log-volumes associated to these ring structures [cf. the discussion of Remark 3.9.5, (vii), (viii), (ix), (x), for more details]. Thus, unlike the case of the 0-column, where the *mono-analytic* interpretation via *regions* of mono-analytic log-shells gives rise only to *upper bounds* on log-volumes, the approach just discussed in the case of the 1-column — i.e., which makes essential use of the **ring structures** that are available as a consequence of the fact that the **arithmetic holomorphic structure** is held **fixed** — gives rise to **precise equalities** [i.e., not just inequalities!] concerning log-volumes. These *differences* between the 0- and 1-columns are summarized in the lower portion of Fig. 3.9.

Remark 3.12.3.

(i) Let S be a *hyperbolic Riemann surface of finite type* of genus g_S with r_S punctures. Write $\chi_S \stackrel{\text{def}}{=} -(2g_S - 2 + r_S)$ for the *Euler characteristic* of S and $d\mu_S$ for the Kähler metric on S [i.e., the $(1,1)$-form] determined by the *Poincaré metric* on the upper half-plane. Recall the *analogy* discussed in [IUTchI], Remark 4.3.3, between the theory of **log-shells**, which plays a key role in the theory developed in the present series of papers, and the **classical metric geometry of hyperbolic Riemann surfaces**. Then, relative to this analogy, the **inequality** obtained in Corollary 3.12 may be regarded as corresponding to the inequality

$$\chi_S \;=\; - \int_S \, d\mu_S \;<\; 0$$

— i.e., in essence, a statement of the **hyperbolicity** of S — arising from the classical **Gauss-Bonnet formula**, together with the **positivity** of $d\mu_S$. Relative to the analogy between *real analytic Kähler metrics* and *ordinary Frobenius liftings* discussed in [pOrd], Introduction, §2 [cf. also the discussion of [pTeich], Introduction, §0], the *local* property constituted by this positivity of $d\mu_S$ may be thought of as corresponding to the [*local* property constituted by the] *Kodaira-Spencer isomorphism of an indigenous bundle* — i.e., which gives rise to the *ordinarity* of the corresponding Frobenius lifting on the ordinary locus — in the p-adic theory. As discussed in [AbsTopIII], §I5, these properties of indigenous bundles in the p-adic theory may be thought of as corresponding, in the theory of *log-shells*, to the *"maximal incompatibility"* between the various *Kummer isomorphisms* and the *corically constructed data* of the Frobenius-picture of Proposition 1.2, (x). On the other hand, it is just this "maximal incompatibility" that gives rise to the *"upper semi-commutativity"* discussed in Remark 1.2.2, (iii), i.e., [from the point of view of the theory of the present §3] the **upper semi-compatibility** indeterminacy (Ind3) of Theorem 3.11, (ii), that underlies the **inequality** of Corollary 3.12 [cf. Step (x) of the proof of Corollary 3.12].

(ii) The *"metric aspect"* of Corollary 3.12 discussed in (i) is reminiscent of the analogy between the theory of the present series of papers and *classical complex Teichmüller theory* [cf. the discussion of [IUTchI], Remark 3.9.3] in the following sense:

> Just as *classical complex Teichmüller theory* is concerned with relating distinct holomorphic structures in a sufficiently **canonical** way as to **minimize** the resulting **conformality distortion**, the **canonical** nature of the algorithms discussed in Theorem 3.11 for relating **alien arithmetic holomorphic structures** [cf. Remark 3.11.1] gives rise to a relatively **strong estimate** of the **[log-]volume distortion** [cf. Corollary 3.12] resulting from such a deformation of the arithmetic holomorphic structure.

Remark 3.12.4. In light of the discussion of Remark 3.12.3, it is of interest to reconsider the analogy between the theory of the present series of papers and the *p-adic Teichmüller theory* of [*p*Ord], [*p*Teich], in the context of Theorem 3.11, Corollary 3.12.

(i) First, we observe that the **splitting monoids** at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$ [cf. Theorem 3.11, (i), (b); Theorem 3.11, (ii), (b)] may be regarded as analogous to the **canonical coordinates** of p-adic Teichmüller theory [cf., e.g., [*p*Teich], Introduction, §0.9] that are constructed over the *ordinary locus* of a canonical curve. In particular, it is natural to regard the **bad primes** $\in \underline{\mathbb{V}}^{\text{bad}}$ as corresponding to the **ordinary** locus of a canonical curve and the **good primes** $\in \underline{\mathbb{V}}^{\text{good}}$ as corresponding to the **supersingular** locus of a canonical curve. This point of view is reminiscent of the discussion of [IUTchII], Remark 4.11.4, (iii).

(ii) On the other hand, the **bi-coric mono-analytic log-shells** — i.e., the various local "$\mathcal{O}^{\times \boldsymbol{\mu}}$" — that appear in the tensor packets of Theorem 3.11, (i), (a); Theorem 3.11, (ii), (a), may be thought of as corresponding to the **[multiplicative!] Teichmüller representatives** associated to the various Witt rings that appear in p-adic Teichmüller theory. Within a *fixed arithmetic holomorphic structure*, these *mono-analytic* log-shells arise from **"local holomorphic units"** — i.e., "$\mathcal{O}^{\times}$" — which are subject to the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**. These "local holomorphic units" may be thought of as corresponding to the **positive characteristic ring structures** on [the positive characteristic reductions of] Teichmüller representatives. Here, the **uniradial**, i.e., **"non-multiradial"**, nature of these "local holomorphic units" [cf. the discussion of [IUTchII], Remark 4.7.4, (ii); [IUTchII], Figs. 4.1, 4.2] may be regarded as corresponding to the *mixed characteristic nature of Witt rings*, i.e., the **incompatibility** of Teichmüller representatives with the **additive structure** of Witt rings.

(iii) The set $\mathbb{F}_l^{\circledast}$ of $l^{\circledast}$ **"theta value labels"**, which plays an important role in the theory of the present series of papers, may be thought of as corresponding to the **"factor of p"** that appears in the *"mod p/p^2 portion"*, i.e., the gap separating the *"mod p"* and *"mod p^2"* portions, of the rings of Witt vectors that occur in the p-adic theory. From this point of view, one may think of the *procession-normalized volumes* obtained by taking **averages over** $j \in \mathbb{F}_l^{\circledast}$ [cf. Corollary 3.12] as corresponding to the operation of **dividing by p** to relate the "mod p/p^2 portion" of the Witt vectors to the "mod p portion" of the Witt vectors [i.e., the characteristic p theory]. In this context, the **multiradial representation** of Theorem 3.11, (i), by means of *mono-analytic log-shells labeled by elements of* $\mathbb{F}_l^{\circledast}$ may be thought of as corresponding to the **derivative** of the **canonical Frobenius lifting** on a canonical curve in the p-adic theory [cf. the discussion of [AbsTopIII], §I5] in the sense that this multiradial representation may be regarded as a sort of **comparison** of the **canonical splitting monoids** discussed in (i) to the **"absolute constants"** [cf. the discussion of (ii)] constituted by the **bi-coric** *mono-analytic log-shells.* This *"absolute comparison"* is precisely what results in the **indeterminacies** (Ind1), (Ind2) of Theorem 3.11, (i).

(iv) In the context of the discussion of (iii), we note that the set of labels $\mathbb{F}_l^{\circledast}$ may, alternatively, be thought of as corresponding to the **infinitesimal moduli** of the positive characteristic curve under consideration in the p-adic theory [cf. the discussion of [IUTchII], Remark 4.11.4, (iii), (d)]. That is to say, the *"deformation dimension"* constituted by the *horizontal dimension* of the log-theta-lattice in the theory of the present series of papers or by the deformations modulo various powers of p in the p-adic theory [cf. Remark 1.4.1, (iii); Fig. 1.3] is **highly canonical** in nature, hence may be thought of as being equipped with a natural isomorphism to the **"absolute moduli"** — i.e., so to speak, the *"moduli over* $\mathbb{F}_1$*"* — of the *given number field equipped with an elliptic curve*, in the theory of the present series of papers, or of the *given positive characteristic hyperbolic curve equipped with a nilpotent ordinary indigenous bundle*, in p-adic Teichmüller theory.

<u>*Inter-universal Teichmüller theory*</u>	<u>*p-adic Teichmüller theory*</u>
splitting monoids at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$	**canonical coordinates** on the *ordinary locus*
bad primes $\in \underline{\mathbb{V}}^{\mathrm{bad}}$	**ordinary** locus of a can. curve
good primes $\in \underline{\mathbb{V}}^{\mathrm{good}}$	**supersing.** locus of a can. curve
mono-analytic log-shells "$\mathcal{O}^{\times \boldsymbol{\mu}}$"	**[multiplicative!] Teich. reps.**
uniradial "local hol. units $\mathcal{O}^{\times}$" subject to $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**	**pos. char. ring structures** on [pos. char. reductions of] Teich. reps.
set of **"theta value labels"** $\mathbb{F}_l^{*}$	**factor p** in mod p/p^2 portion of Witt vectors
multiradial rep. via $\mathbb{F}_l^{*}$-labeled **mono-analytic log-shells** [cf. (Ind1), (Ind2), (Ind3)]	**derivative** of the **canonical Frobenius lifting**
set of **"theta value labels"** $\mathbb{F}_l^{*}$	implicit **"absolute moduli/$\mathbb{F}_1$"**
inequality arising from **upper semi-compatibility** [cf. (Ind3)]	**inequality** arising from interference between **Frobenius conjugates**

Fig. 3.10: The analogy between inter-universal Teichmüller theory and p-adic Teichmüller theory

(v) Let A be the ring of Witt vectors of a perfect field k of positive characteristic p; X a *smooth, proper hyperbolic curve* over A of genus g_X which is **canonical** in the sense of p-adic Teichmüller theory; $\widehat{X}$ the p-adic formal scheme associated to X; $\widehat{U} \subseteq \widehat{X}$ the *ordinary locus* of $\widehat{X}$. Write ω_{X_k} for the canonical bundle of $X_k \stackrel{\text{def}}{=} X \times_A k$. Then when [cf. the discussion of (iii)] one computes the **derivative** of the **canonical Frobenius lifting** $\Phi : \widehat{U} \to \widehat{U}$ on $\widehat{U}$, one must contend with *"interference phenomena"* between the various *copies* of some positive characteristic algebraic geometry set-up — i.e., at a more concrete level, the various *Frobenius conjugates* "t^{p^n}" [where t is a local coordinate on X_k] associated to various $n \in \mathbb{N}_{\geq 1}$. In particular, this derivative only yields [upon dividing by p] an *inclusion* [i.e., not an isomorphism!] of line bundles

$$\omega_{X_k} \hookrightarrow \Phi^* \omega_{X_k}$$

— also known as the **"[square] Hasse invariant"** [cf. [pOrd], Chapter II, Proposition 2.6; the discussion of "generalities on ordinary Frobenius liftings" given in [pOrd], Chapter III, §1]. Thus, at the level of *global degrees of line bundles*, we obtain an *inequality* [i.e., not an equality!]

$$(1-p)(2g_X - 2) \leq 0$$

— which may be thought of as being, in essence, a statement of the **hyperbolicity** of X [cf. the inequality of the display of Remark 3.12.3, (i)]. Since the *"Frobenius conjugate dimension"* [i.e., the "n" that appears in "t^{p^n}"] in the p-adic theory corresponds to the *vertical dimension* of the log-theta-lattice in the theory of the present series of papers [cf. Remark 1.4.1, (iii); Fig. 1.3], we thus see that the inequality of the above display in the p-adic case arises from circumstances that are *entirely analogous* to the circumstances — i.e., the **upper semi-compatibility** indeterminacy (Ind3) of Theorem 3.11, (ii) — that underlie the **inequality** of Corollary 3.12 [cf. Step (x) of the proof of Corollary 3.12; the discussion of Remark 3.12.3, (i)].

(vi) The analogies of the above discussion are summarized in Fig. 3.10 above.

Bibliography

[Lang] S. Lang, *Algebraic number theory*, Addison-Wesley Publishing Co. (1970).

[*p*Ord] S. Mochizuki, A Theory of Ordinary p-adic Curves, *Publ. Res. Inst. Math. Sci.* **32** (1996), pp. 957-1151.

[*p*Teich] S. Mochizuki, *Foundations of p-adic Teichmüller Theory*, AMS/IP Studies in Advanced Mathematics **11**, American Mathematical Society/International Press (1999).

[QuCnf] S. Mochizuki, Conformal and quasiconformal categorical representation of hyperbolic Riemann surfaces, *Hiroshima Math. J.* **36** (2006), pp. 405-441.

[SemiAnbd] S. Mochizuki, Semi-graphs of Anabelioids, *Publ. Res. Inst. Math. Sci.* **42** (2006), pp. 221-322.

[FrdI] S. Mochizuki, The Geometry of Frobenioids I: The General Theory, *Kyushu J. Math.* **62** (2008), pp. 293-400.

[FrdII] S. Mochizuki, The Geometry of Frobenioids II: Poly-Frobenioids, *Kyushu J. Math.* **62** (2008), pp. 401-460.

[EtTh] S. Mochizuki, The Étale Theta Function and its Frobenioid-theoretic Manifestations, *Publ. Res. Inst. Math. Sci.* **45** (2009), pp. 227-349.

[AbsTopIII] S. Mochizuki, Topics in Absolute Anabelian Geometry III: Global Reconstruction Algorithms, *J. Math. Sci. Univ. Tokyo* **22** (2015), pp. 939-1156.

[IUTchI] S. Mochizuki, *Inter-universal Teichmüller Theory I: Construction of Hodge Theaters*, RIMS Preprint **1756** (August 2012), to appear in *Publ. Res. Inst. Math. Sci.*

[IUTchII] S. Mochizuki, *Inter-universal Teichmüller Theory II: Hodge-Arakelov-theoretic Evaluation*, RIMS Preprint **1757** (August 2012), to appear in *Publ. Res. Inst. Math. Sci.*

[IUTchIV] S. Mochizuki, *Inter-universal Teichmüller Theory IV: Log-volume Computations and Set-theoretic Foundations*, RIMS Preprint **1759** (August 2012), to appear in *Publ. Res. Inst. Math. Sci.*

[Royden] H. L. Royden, *Real Analysis, Second Edition*, The Macmillan Publishing Co. (1968).

Updated versions of preprints are available at the following webpage:

`http://www.kurims.kyoto-u.ac.jp/~motizuki/papers-english.html`

INTER-UNIVERSAL TEICHMÜLLER THEORY IV: LOG-VOLUME COMPUTATIONS AND SET-THEORETIC FOUNDATIONS

Shinichi Mochizuki

April 2020

Abstract. The present paper forms the fourth and final paper in a series of papers concerning **"inter-universal Teichmüller theory"**. In the first three papers of the series, we introduced and studied the theory surrounding the **log-theta-lattice**, a *highly non-commutative* two-dimensional diagram of *"miniature models of conventional scheme theory"*, called $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters*, that were associated, in the first paper of the series, to certain data, called *initial* Θ*-data*. This data includes an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$. Consideration of various properties of the log-theta-lattice led naturally to the establishment, in the third paper of the series, of **multiradial algorithms** for constructing **"splitting monoids of LGP-monoids"**. Here, we recall that "multiradial algorithms" are algorithms that make sense from the point of view of an **"alien arithmetic holomorphic structure"**, i.e., the ring/scheme structure of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater related to a given $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater by means of a *non-ring/scheme-theoretic* horizontal arrow of the log-theta-lattice. In the present paper, estimates arising from these multiradial algorithms for splitting monoids of LGP-monoids are applied to verify various **diophantine results** which imply, for instance, the so-called **Vojta Conjecture** for hyperbolic curves, the **ABC Conjecture**, and the **Szpiro Conjecture** for elliptic curves. Finally, we examine — albeit from an extremely *naive/non-expert* point of view! — the *foundational/set-theoretic* issues surrounding the *vertical* and *horizontal arrows* of the log-theta-lattice by introducing and studying the basic properties of the notion of a **"species"**, which may be thought of as a sort of formalization, via set-theoretic formulas, of the intuitive notion of a *"type of mathematical object"*. These foundational issues are closely related to the central role played in the present series of papers by various results from **absolute anabelian geometry**, as well as to the idea of **gluing together** *distinct models of conventional scheme theory*, i.e., in a fashion that lies outside the framework of conventional scheme theory. Moreover, it is precisely these foundational issues surrounding the vertical and horizontal arrows of the log-theta-lattice that led naturally to the introduction of the term **"inter-universal"**.

Contents:

Introduction

The present paper forms the fourth and final paper in a series of papers concerning **"inter-universal Teichmüller theory"**. In the first three papers, [IUTchI], [IUTchII], and [IUTchIII], of the series, we introduced and studied the theory surrounding the **log-theta-lattice** [cf. the discussion of [IUTchIII], Introduction], a *highly non-commutative* two-dimensional diagram of *"miniature models of conventional scheme theory"*, called $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters*, that were associated, in the first paper [IUTchI] of the series, to certain data, called *initial* Θ*-data*. This data includes an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$ [cf. [IUTchI], §I1]. Consideration of various properties of the log-theta-lattice leads naturally to the establishment of **multiradial algorithms** for constructing **"splitting monoids of LGP-monoids"** [cf. [IUTchIII], Theorem A]. Here, we recall that "multiradial algorithms" [cf. the discussion of the Introductions to [IUTchII], [IUTchIII]] are algorithms that make sense from the point of view of an **"alien arithmetic holomorphic structure"**, i.e., the ring/scheme structure of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater related to a given $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater by means of a *non-ring/scheme-theoretic* horizontal arrow of the log-theta-lattice. In the final portion of [IUTchIII], by applying these multiradial algorithms for splitting monoids of LGP-monoids, we obtained *estimates* for the *log-volume* of these LGP-monoids [cf. [IUTchIII], Theorem B]. In the present paper, these estimates will be applied to verify various **diophantine results**.

In §1 of the present paper, we start by discussing various *elementary estimates* for the *log-volume* of various *tensor products* of the modules obtained by applying the *p-adic logarithm* to the *local units* — i.e., in the terminology of [IUTchIII], *"tensor packets of log-shells"* [cf. the discussion of [IUTchIII], Introduction] — in terms of various well-known invariants, such as *differents*, associated to a *mixed-characteristic nonarchimedean local field* [cf. Propositions 1.1, 1.2, 1.3, 1.4]. We then discuss similar — but technically much simpler! — log-volume estimates in the case of *complex archimedean local fields* [cf. Proposition 1.5]. After reviewing a certain classical estimate concerning the *distribution of prime numbers* [cf. Proposition 1.6], as well as some elementary general nonsense concerning *weighted averages* [cf. Proposition 1.7] and well-known elementary facts concerning *elliptic curves* [cf. Proposition 1.8], we then proceed to *compute explicitly*, in more elementary language, the quantity that was estimated in [IUTchIII], Theorem B. These computations yield a *quite strong/explicit* **diophantine inequality** [cf. Theorem 1.10] concerning elliptic curves that are in **"sufficiently general position"**, so that one may apply the general theory developed in the first three papers of the series.

In §2 of the present paper, after reviewing another classical estimate concerning the *distribution of prime numbers* [cf. Proposition 2.1, (ii)], we then proceed to apply the theory of [GenEll] to **reduce** various diophantine results concerning an **arbitrary** *elliptic curve over a number field* to results of the type obtained in Theorem 1.10 concerning elliptic curves that are in **"sufficiently general position"** [cf. Corollary 2.2]. This reduction allows us to derive the following result [cf. Corollary 2.3], which constitutes the **main application** of the *"inter-universal Teichmüller theory"* developed in the present series of papers.

Theorem A. **(Diophantine Inequalities)** *Let X be a smooth, proper, geometrically connected curve over a number field; $D \subseteq X$ a reduced divisor; $U_X \stackrel{\text{def}}{=} X \backslash D$; d a positive integer; $\epsilon \in \mathbb{R}_{>0}$ a positive real number. Write ω_X for the canonical sheaf on X. Suppose that U_X is a* **hyperbolic curve***, i.e., that the degree of the line bundle $\omega_X(D)$ is* **positive***. Then, relative to the notation of [GenEll] [reviewed in the discussion preceding Corollary 2.2 of the present paper], one has an* **inequality of "bounded discrepancy classes"**

$$\text{ht}_{\omega_X(D)} \lesssim (1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D)$$

of functions on $U_X(\overline{\mathbb{Q}})^{\leq d}$ — i.e., the function $(1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D) - \text{ht}_{\omega_X(D)}$ is bounded below by a **constant** *on $U_X(\overline{\mathbb{Q}})^{\leq d}$ [cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), of the present paper].*

Thus, Theorem A asserts an *inequality* concerning the *canonical height* [i.e., "$\text{ht}_{\omega_X(D)}$"], the *logarithmic different* [i.e., "log-diff_X"], and the *logarithmic conductor* [i.e., "log-cond_D"] of points of the curve U_X valued in number fields whose extension degree over $\mathbb{Q}$ is $\leq d$. In particular, the so-called **Vojta Conjecture** for hyperbolic curves, the **ABC Conjecture**, and the **Szpiro Conjecture** for elliptic curves all follow as special cases of Theorem A. We refer to [Vjt] for a detailed exposition of these conjectures.

Finally, in §3, we examine — albeit from an extremely *naive/non-expert* point of view! — certain **foundational issues** underlying the theory of the present series of papers. Typically in mathematical discussions [i.e., by mathematicians who are not equipped with a detailed knowledge of the theory of foundations!] — such as, for instance, the theory developed in the present series of papers! — one defines various **"types of mathematical objects"** [i.e., such as groups, topological spaces, or schemes], together with a notion of *"morphisms"* between two particular examples of a specific type of mathematical object [i.e., morphisms between groups, between topological spaces, or between schemes]. Such objects and morphisms [typically] determine a *category*. On the other hand, if one restricts one's attention to such a category, then one must keep in mind the fact that the structure of the category — i.e., which consists *only of a collection of objects and morphisms satisfying certain properties!* — does not include any mention of the various sets and conditions satisfied by those sets that give rise to the "type of mathematical object" under consideration. For instance, the data consisting of the underlying set of a group, the group multiplication law on the group, and the properties satisfied by this group multiplication law *cannot be recovered* [at least in an *a priori* sense!] from the structure of the "*category* of groups". Put another way, although the notion of a "type of mathematical object" may give rise to a "category of such objects", the notion of a "type of mathematical object" is much *stronger* — in the sense that it involves much more *mathematical structure* — than the notion of a category. Indeed, a given "type of mathematical object" may have a *very complicated internal structure*, but may give rise to a category equivalent to a *one-morphism category* [i.e., a category with precisely one morphism]; in particular, in such cases, the structure of the associated category does not retain any information of interest concerning the internal structure of the "type of mathematical object" under consideration.

In Definition 3.1, (iii), we formalize this intuitive notion of a "type of mathematical object" by defining the notion of a **species** as, roughly speaking, a *collection of set-theoretic formulas* that gives rise to a category in any given *model of set theory* [cf. Definition 3.1, (iv)], but, unlike any *specific* category [e.g., of groups, etc.] is **not confined** to any **specific model of set theory**. In a similar vein, by working with *collections of set-theoretic formulas*, one may define a species-theoretic analogue of the notion of a *functor*, which we refer to as a **mutation** [cf. Definition 3.3, (i)]. Given a diagram of mutations, one may then define the notion of a "mutation that extracts, from the diagram, a certain portion of the types of mathematical objects that appear in the diagram that is *invariant* with respect to the mutations in the diagram"; we refer to such a mutation as a **core** [cf. Definition 3.3, (v)].

One fundamental example, in the context of the present series of papers, of a diagram of mutations is the usual set-up of **[absolute] anabelian geometry** [cf. Example 3.5 for more details]. That is to say, one begins with the *species* constituted by schemes satisfying certain conditions. One then considers the *mutation*

$$X \quad \rightsquigarrow \quad \Pi_X$$

that associates to such a scheme X its étale fundamental group Π_X [say, considered up to inner automorphisms]. Here, it is important to note that the codomain of this mutation is the *species* constituted by topological groups [say, considered up to inner automorphisms] that satisfy certain conditions which *do not include* any information concerning *how the group is related* [for instance, via some sort of étale fundamental group mutation] *to a scheme*. The notion of an **anabelian reconstruction algorithm** may then be formalized as a *mutation* that forms a *"mutation-quasi-inverse"* to the fundamental group mutation.

Another fundamental example, in the context of the present series of papers, of a diagram of mutations arises from the *Frobenius morphism* in positive characteristic scheme theory [cf. Example 3.6 for more details]. That is to say, one fixes a prime number p and considers the *species* constituted by reduced quasi-compact schemes of characteristic p and quasi-compact morphisms of schemes. One then considers the *mutation* that associates

$$S \quad \rightsquigarrow \quad S^{(p)}$$

to such a scheme S the scheme $S^{(p)}$ with the same topological space, but whose regular functions are given by the p-th powers of the regular functions on the original scheme. Thus, the domain and codomain of this mutation are given by the same species. One may also consider a *log scheme* version of this example, which, at the level of monoids, corresponds, in essence, to assigning

$$M \quad \rightsquigarrow \quad p \cdot M$$

to a torsion-free abelian monoid M the submonoid $p \cdot M \subseteq M$ determined by the image of multiplication by p. Returning to the case of schemes, one may then observe that the well-known constructions of the **perfection** and the **étale site**

$$S \quad \rightsquigarrow \quad S^{\mathrm{pf}}; \quad S \quad \rightsquigarrow \quad S_{\text{ét}}$$

associated to a reduced scheme S of characteristic p give rise to **cores** of the diagram obtained by considering iterates of the **"Frobenius mutation"** just discussed.

This last example of the *Frobenius mutation* and the associated core constituted by the *étale site* is of particular importance in the context of the present series of papers in that it forms the *"intuitive prototype"* that underlies the theory of the **vertical** and **horizontal** lines of the **log-theta-lattice** [cf. the discussion of Remark 3.6.1, (i)]. One notable aspect of this example is the [evident!] fact that the *domain* and *codomain* of the Frobenius mutation are given by the *same species*. That is to say, despite the fact that in the *construction* of the scheme $S^{(p)}$ [cf. the notation of the preceding paragraph] from the scheme S, the scheme $S^{(p)}$ is *"subordinate"* to the scheme S, the domain and codomain species of the resulting Frobenius mutation *coincide*, hence, in particular, are *on a par with one another*. This sort of situation served, for the author, as a sort of model for the **$\mathfrak{log}$- and $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-links** of the log-theta-lattice, which may be formulated as *mutations* between the *species* constituted by the notion of a $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater*. That is to say, although in the *construction* of either the $\mathfrak{log}$- or the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link, the domain and codomain $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters are by no means on a "par" with one another, the domain and codomain $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters of the resulting $\mathfrak{log}$-/$\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-links are regarded as objects of the *same species*, hence, in particular, completely *on a par with one another*. This sort of **"relativization"** of **distinct models** *of conventional scheme theory over* $\mathbb{Z}$ via the notion of a $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater [cf. Fig. I.1 below; the discussion of "gluing together" such models of conventional scheme theory in [IUTchI], §I2] is one of the *most characteristic features* of the theory developed in the present series of papers and, in particular, lies [tautologically!] *outside the framework of conventional scheme theory over* $\mathbb{Z}$. That is to say, in the framework of conventional scheme theory over $\mathbb{Z}$, if one starts out with schemes over $\mathbb{Z}$ and constructs from them, say, by means of geometric objects such as the *theta function* on a Tate curve, some sort of Frobenioid that is isomorphic to a Frobenioid associated to $\mathbb{Z}$, then — unlike, for instance, the case of the *Frobenius morphism* in positive characteristic scheme theory —

> there is no way, within the framework of conventional scheme theory, to treat the newly constructed Frobenioid *"as if it is the Frobenioid associated to* $\mathbb{Z}$*, relative to some* **new** *version/model of conventional scheme theory"*.

...	non-scheme- / theoretic link	one model of conventional scheme theory over $\mathbb{Z}$	non-scheme- / theoretic link	another model of conventional scheme theory over $\mathbb{Z}$	non-scheme- / theoretic link	...

Fig. I.1: Relativized models of conventional scheme theory over $\mathbb{Z}$

If, moreover, one thinks of $\mathbb{Z}$ as being constructed, in the usual way, via *axiomatic set theory*, then one may interpret the *"absolute"* — i.e., *"tautologically* *unrelativizable"* — nature of conventional scheme theory over $\mathbb{Z}$ at a *purely set-theoretic level.* Indeed, from the point of view of the *"$\in$-structure"* of axiomatic set theory, there is *no way to treat* sets constructed at *distinct levels* of this $\in$-structure as being on a par with one another. On the other hand, if one focuses not on the level of the $\in$-structure to which a set belongs, but rather on *species*, then the notion of a species allows one to relate — i.e., to treat on a par with one another — objects belonging to the species that arise from sets constructed at distinct levels of the $\in$-structure. That is to say,

> the notion of a **species** allows one to **"simulate $\in$-loops"** *without violating the axiom of foundation of axiomatic set theory*

— cf. the discussion of Remark 3.3.1, (i).

As one constructs sets at *new levels* of the $\in$-structure of some model of axiomatic set theory — e.g., as one travels along *vertical* or *horizontal lines* of the *log-theta-lattice*! — one typically encounters *new schemes*, which give rise to *new Galois categories*, hence to *new Galois or étale fundamental groups*, which may only be constructed if one allows oneself to consider *new basepoints*, relative to *new universes.* In particular, one must continue to *extend the universe*, i.e., to *modify the model of set theory*, relative to which one works. Here, we recall in passing that such "extensions of universe" are possible on account of an **existence axiom** concerning **universes**, which is apparently attributed to the *"Grothendieck school"* and, moreover, cannot, apparently, be obtained as a consequence of the conventional ZFC axioms of axiomatic set theory [cf. the discussion at the beginning of §3 for more details]. On the other hand, ultimately in the present series of papers [cf. the discussion of [IUTchIII], Introduction], we wish to obtain **algorithms** for constructing various objects that arise in the context of the *new schemes/universes* discussed above — i.e., at *distant* $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters* of the log-theta-lattice — that *make sense* from the point of view of the *original schemes/universes* that occurred at the outset of the discussion. Again, the fundamental tool that makes this possible, i.e., that allows one to express constructions in the new universes in terms that makes sense in the original universe is precisely

> the **species-theoretic formulation** — i.e., the formulation via **set-theoretic formulas** that *do not depend on* **particular choices** *invoked in* **particular universes** — of the constructions of interest

— cf. the discussion of Remarks 3.1.2, 3.1.3, 3.1.4, 3.1.5, 3.6.2, 3.6.3. This is the point of view that gave rise to the term **"inter-universal"**. At a more concrete level, this "inter-universal" contact between constructions in distant models of conventional scheme theory in the log-theta-lattice is realized by considering [the *étale-like structures* given by] the various Galois or étale fundamental groups that occur as [the "type of mathematical object", i.e., *species* constituted by] **abstract topological groups** [cf. the discussion of Remark 3.6.3, (i); [IUTchI], §I3]. These abstract topological groups give rise to **vertical** or **horizontal cores** of the log-theta-lattice [cf. the discussion of [IUTchIII], Introduction; [IUTchIII], Theorem 1.5, (i), (ii)]. Moreover, once one obtains cores that are sufficiently *"nondegenerate"*, or *"rich in structure"*, so as to serve as *containers* for the *non-coric* portions of the various mutations [e.g., vertical and horizontal arrows of the log-theta-lattice] under consideration, then one may construct the desired algorithms, or **descriptions**, of these **non-coric portions** in terms of **coric containers**, up to certain *relatively mild* **indeterminacies** [i.e., which reflect the non-coric nature of these non-coric portions!] — cf. the illustration of this sort of situation given in Fig. I.2 below; Remark 3.3.1, (iii); Remark 3.6.1, (ii). In the context of the log-theta-lattice, this is precisely the sort of situation that was achieved in [IUTchIII], Theorem A [cf. the discussion of [IUTchIII], Introduction].

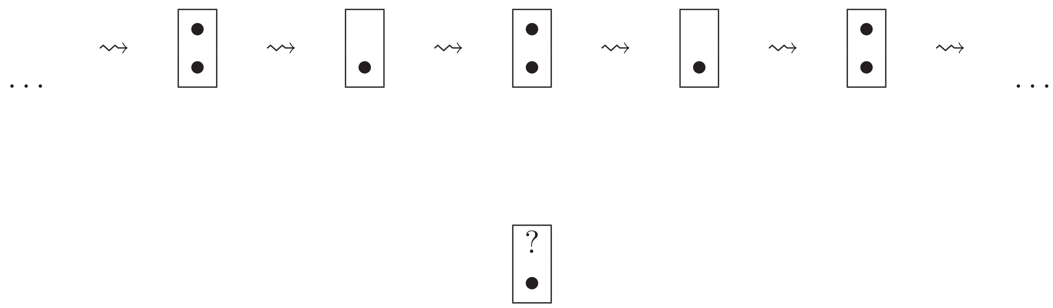

Fig. I.2: A coric container underlying a sequence of mutations

In the context of the above discussion of set-theoretic aspects of the theory developed in the present series of papers, it is of interest to note the following observation, relative to the analogy between the theory of the present series of papers and ***p*-adic Teichmüller theory** [cf. the discussion of [IUTchI], §I4]. If, instead of working *species-theoretically*, one attempts to document *all of the possible choices* that occur in various *newly introduced universes* that occur in a construction, then one finds that one is obliged to work with sets, such as sets obtained via **set-theoretic exponentiation**, of **very large cardinality**. Such sets of large cardinality are reminiscent of the **exponentially large** denominators that occur if one attempts to *p-adically formally integrate an arbitrary connection* as opposed to a **canonical crystalline connection** of the sort that occurs in the context of the **canonical liftings** of p-adic Teichmüller theory [cf. the discussion of Remark 3.6.2, (iii)]. In this context, it is of interest to recall the computations of [Finot], which assert, roughly speaking, that the canonical liftings of p-adic Teichmüller theory may, in certain cases, be characterized as liftings *"of minimal complexity"* in the sense that their Witt vector coordinates are given by *polynomials of* **minimal degree**.

Finally, we observe that although, in the above discussion, we concentrated on the *similarities*, from an *"inter-universal"* point of view, between the *vertical* and *horizontal* arrows of the log-theta-lattice, there is one important *difference* between these vertical and horizontal arrows: namely,

· whereas the copies of the *full arithmetic fundamental group* — i.e., in particular, the copies of the **geometric fundamental group** — on either side of a **vertical** arrow are **identified** with one another,

· in the case of a **horizontal** arrow, only the **Galois groups** *of the local base fields* on either side of the arrow are identified with one another

— cf. the discussion of Remark 3.6.3, (ii). One way to understand the reason for this difference is as follows. In the case of the *vertical* arrows — i.e., the $\mathfrak{log}$-*links*, which, in essence, amount to the various *local p-adic logarithms* — in order to *construct* the $\mathfrak{log}$-link, it is necessary to make use, in an essential way, of the **local ring structures** at $\underline{v} \in \underline{\mathbb{V}}$ [cf. the discussion of [IUTchIII], Definition 1.1, (i), (ii)], which may only be reconstructed from the *full arithmetic fundamental group*. By contrast, in order to construct the horizontal arrows — i.e., the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-*links* — this local ring structure is *unnecessary*. On the other hand, in order to construct the horizontal arrows, it is necessary to work with structures that, up to isomorphism, are *common* to both the *domain* and the *codomain* of the arrow. Since the construction of the domain of the $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link **depends**, in an essential way, on the *Gaussian monoids*, i.e., on the **labels** $\in \mathbb{F}_l^{\circledast}$ for the **theta values**, which are constructed from the *geometric fundamental group*, while the codomain only involves monoids arising from the local *q-parameters* "$\underline{\underline{q}}_{\underline{v}}$" [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], which are constructed in a fashion that is **independent** of these **labels**, in order to obtain an isomorphism between structures arising from the domain and codomain, it is necessary to restrict one's attention to the *Galois groups of the local base fields*, which are *free of any dependence on these labels*.

Acknowledgements:

The research discussed in the present paper profited enormously from the generous support that the author received from the *Research Institute for Mathematical Sciences*, a Joint Usage/Research Center located in Kyoto University. At a personal level, I would like to thank *Fumiharu Kato*, *Akio Tamagawa*, *Go Yamashita*, *Mohamed Saïdi*, *Yuichiro Hoshi*, *Ivan Fesenko*, *Fucheng Tan*, *Emmanuel Lepage*, *Arata Minamide*, and *Wojciech Porowski* for many stimulating discussions concerning the material presented in this paper. Also, I feel deeply indebted to *Go Yamashita*, *Mohamed Saïdi*, and *Yuichiro Hoshi* for their meticulous reading of and numerous comments concerning the present paper. In addition, I would like to thank *Kentaro Sato* for useful comments concerning the set-theoretic and foundational aspects of the present paper, as well as *Vesselin Dimitrov* and *Akshay Venkatesh* for useful comments concerning the analytic number theory aspects of the present paper. Finally, I would like to express my deep gratitude to *Ivan Fesenko* for his quite substantial efforts to disseminate — for instance, in the form of a survey that he wrote — the theory discussed in the present series of papers.

Notations and Conventions:

We shall continue to use the "Notations and Conventions" of [IUTchI], §0.

Section 1: Log-volume Estimates

In the present §1, we perform various *elementary* **local computations** concerning nonarchimedean and archimedean local fields which allow us to obtain **more explicit versions** [cf. Theorem 1.10 below] of the **log-volume estimates** for **Θ-pilot objects** obtained in [IUTchIII], Corollary 3.12.

In the following, if $\lambda \in \mathbb{R}$, then we shall write

$$\lceil \lambda \rceil \text{ (respectively, } \lfloor \lambda \rfloor)$$

for the *smallest* (respectively, *largest*) $n \in \mathbb{Z}$ such that $n \geq \lambda$ (respectively, $n \leq \lambda$). Also, we shall write "$\log(-)$" for the *natural logarithm* of a positive real number.

Proposition 1.1. (Multiple Tensor Products and Differents) *Let p be a prime number, I a finite set of cardinality ≥ 2, $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$. Write $\overline{R} \subseteq \overline{\mathbb{Q}}_p$ for the ring of integers of $\overline{\mathbb{Q}}_p$ and* $\mathrm{ord} : \overline{\mathbb{Q}}_p^\times \to \mathbb{Q}$ *for the natural p-adic valuation on $\overline{\mathbb{Q}}_p$, normalized so that* $\mathrm{ord}(p) = 1$*; for $\lambda \in \mathbb{Q}$, we shall write p^λ for "some" [unspecified] element of $\overline{\mathbb{Q}}_p$ such that* $\mathrm{ord}(p^\lambda) = \lambda$. *For $i \in I$, let $k_i \subseteq \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$; write $R_i \stackrel{\mathrm{def}}{=} \mathcal{O}_{k_i} = \overline{R} \bigcap k_i$ for the ring of integers of k_i and $\mathfrak{d}_i \in \mathbb{Q}_{\geq 0}$ for the order [i.e., "*$\mathrm{ord}(-)$*"] of any generator of the* **different ideal** *of R_i over $\mathbb{Z}_p$. Also, for any nonempty subset $E \subseteq I$, let us write*

$$R_E \stackrel{\mathrm{def}}{=} \bigotimes_{i \in E} R_i; \qquad \mathfrak{d}_E \stackrel{\mathrm{def}}{=} \sum_{i \in E} \mathfrak{d}_i$$

— where the tensor product is over $\mathbb{Z}_p$. Fix an element $* \in I$; write $I^* \stackrel{\mathrm{def}}{=} I \setminus \{*\}$. Then

$$p^{\mathfrak{d}_{I^*}} \cdot (R_I)^\sim \quad \subseteq \quad R_I \quad \subseteq \quad (R_I)^\sim$$

— where we write "$(-)^\sim$" for the **normalization** *of the [reduced] ring in parentheses in its ring of fractions, and we observe that it follows immediately from the definition of the "normalization" that the notation on the left-hand side of the first inclusion of the above display is well-defined for suitable "$p^{\mathfrak{d}_{I^*}}$" [such as products of elements $p^{\mathfrak{d}_i} \in R_i$, for $i \in I^*$] and independent of the choice of such suitable "$p^{\mathfrak{d}_{I^*}}$".*

Proof. Let us regard R_I as an *R_*-algebra* in the evident fashion. It is immediate from the definitions that $R_I \subseteq (R_I)^\sim$. Now observe that

$$\overline{R} \otimes_{R_*} R_I \quad \subseteq \quad \overline{R} \otimes_{R_*} (R_I)^\sim \quad \subseteq \quad (\overline{R} \otimes_{R_*} R_I)^\sim$$

— where $(\overline{R} \otimes_{R_*} R_I)^\sim$ decomposes as a *direct sum* of finitely many copies of $\overline{R}$. In particular, one verifies immediately, in light of the fact the $\overline{R}$ is *faithfully flat* over R_*, that to complete the proof of Proposition 1.1, it suffices to verify that

$$p^{\mathfrak{d}_{I^*}} \cdot (\overline{R} \otimes_{R_*} R_I)^\sim \quad \subseteq \quad \overline{R} \otimes_{R_*} R_I$$

— where we observe that it follows immediately from the definition of the "normalization" that the notation on the left-hand side of the inclusion of the above display is well-defined and independent of the choice of "$p^{\mathfrak{d}_{I^*}}$". On the other hand, it follows immediately from *induction on the cardinality of I* that to verify this last inclusion, it suffices to verify the inclusion in the case where I is of *cardinality two*. But in this case, the desired inclusion follows immediately from the *definition of the different ideal*. This completes the proof of Proposition 1.1. $\bigcirc$

Proposition 1.2. (Differents and Logarithms) *We continue to use the notation of Proposition 1.1. For $i \in I$, write e_i for the* **ramification index** *of k_i over $\mathbb{Q}_p$;*

$$a_i \stackrel{\text{def}}{=} \frac{1}{e_i} \cdot \lceil \frac{e_i}{p-2} \rceil \textit{ if } p > 2, \quad a_i \stackrel{\text{def}}{=} 2 \textit{ if } p = 2; \qquad b_i \stackrel{\text{def}}{=} \lfloor \frac{\log(p \cdot e_i/(p-1))}{\log(p)} \rfloor - \frac{1}{e_i}.$$

Thus,

$$\textit{if } p > 2 \textit{ and } e_i \leq p-2, \textit{ then } a_i = \frac{1}{e_i} = -b_i.$$

For any nonempty subset $E \subseteq I$, let us write

$$\log_p(R_E^{\times}) \stackrel{\text{def}}{=} \bigotimes_{i \in E} \log_p(R_i^{\times}); \qquad a_E \stackrel{\text{def}}{=} \sum_{i \in E} a_i; \qquad b_E \stackrel{\text{def}}{=} \sum_{i \in E} b_i$$

— where the tensor product is over $\mathbb{Z}_p$; we write "$\log_p(-)$" for the p-adic logarithm. For $\lambda \in \frac{1}{e_i} \cdot \mathbb{Z}$, we shall write $p^{\lambda} \cdot R_i$ for the fractional ideal of R_i generated by any element "p^{λ}" of k_i such that $\mathrm{ord}(p^{\lambda}) = \lambda$. *Let*

$$\phi: \ \log_p(R_I^{\times}) \otimes \mathbb{Q}_p \ \stackrel{\sim}{\to} \ \log_p(R_I^{\times}) \otimes \mathbb{Q}_p$$

be an **automorphism** *of the finite dimensional $\mathbb{Q}_p$-vector space $\log_p(R_I^{\times}) \otimes \mathbb{Q}_p$ that induces an automorphism of the submodule $\log_p(R_I^{\times})$. Then:*

(i) We have:

$$p^{a_i} \cdot R_i \quad \subseteq \quad \log_p(R_i^{\times}) \quad \subseteq \quad p^{-b_i} \cdot R_i$$

— where the "$\subseteq$'s" are **equalities** *when $p > 2$ and $e_i \leq p-2$.*

(ii) We have:

$$\begin{aligned} \phi(p^{\lambda} \cdot (R_I)^{\sim}) \quad &\subseteq \quad p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^{\times}) \\ &\subseteq \quad p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I} \cdot (R_I)^{\sim} \end{aligned}$$

for any $\lambda \in \frac{1}{e_i} \cdot \mathbb{Z}$, $i \in I$. [Here, we observe that, just as in Proposition 1.1, it follows immediately from the definition of the "normalization" that the notation of the above display is well-defined and independent of the various choices involved.] In particular, $\phi((R_I)^{\sim}) \subseteq p^{-\lceil \mathfrak{d}_I + a_I \rceil} \cdot \log_p(R_I^{\times}) \subseteq p^{-\lceil \mathfrak{d}_I + a_I \rceil - b_I} \cdot (R_I)^{\sim}$.

(iii) Suppose that $p > 2$, and that $e_i \leq p - 2$ for all $i \in I$. Then we have:

$$\phi(p^\lambda \cdot (R_I)^\sim) \quad \subseteq \quad p^{\lambda - \mathfrak{d}_I - 1} \cdot (R_I)^\sim$$

for any $\lambda \in \frac{1}{e_i} \cdot \mathbb{Z}$, $i \in I$. [Here, we observe that, just as in Proposition 1.1, it follows immediately from the definition of the "normalization" that the notation of the above display is well-defined and independent of the various choices involved.] In particular, $\phi((R_I)^\sim) \subseteq p^{-\mathfrak{d}_I - 1} \cdot (R_I)^\sim$.

(iv) If $p > 2$ and $e_i = 1$ for all $i \in I$, then $\phi((R_I)^\sim) \subseteq (R_I)^\sim$.

Proof. Since $a_i > \frac{1}{p-1}$, $\frac{p^{b_i + \frac{1}{e_i}}}{e_i} > \frac{1}{p-1}$ [cf. the definition of "$\lceil - \rceil$", "$\lfloor - \rfloor$"!], assertion (i) follows immediately from the well-known theory of the *p-adic logarithm* and *exponential* maps [cf., e.g., [Kobl], p. 81]. Next, we consider assertion (ii). Observe that it follows from the first displayed inclusion [*of R_I-modules!*] of Proposition 1.1 that

$$p^{\mathfrak{d}_I + a_I} \cdot (R_I)^\sim \quad \subseteq \quad \bigotimes_{i \in I} p^{a_i} \cdot R_i \quad \Big(\subseteq \quad R_I \quad = \quad \bigotimes_{i \in I} R_i \Big)$$

and hence that

$$\begin{aligned} p^\lambda \cdot (R_I)^\sim \quad &\subseteq \quad p^{\lambda - \mathfrak{d}_I - a_I} \cdot p^{\mathfrak{d}_I + a_I} \cdot (R_I)^\sim \\ &\subseteq \quad p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot p^{\mathfrak{d}_I + a_I} \cdot (R_I)^\sim \\ &\subseteq \quad p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^\times) \quad \subseteq \quad p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I} \cdot (R_I)^\sim \end{aligned}$$

— where, in the passage to the third and fourth inclusions following "$p^\lambda \cdot (R_I)^\sim$", we apply assertion (i). [Here, we observe that, just as in Proposition 1.1, it follows immediately from the definition of the "normalization" that the notation of the above two displays is well-defined and independent of the various choices involved.] Thus, assertion (ii) follows immediately from the fact that ϕ induces an automorphism of the submodule $\log_p(R_I^\times)$. Assertion (iii) follows from assertion (ii), together with the fact that if $p > 2$ and $e_i \leq p - 2$ for all $i \in I$, then we have $a_I = -b_I$, which implies that $\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I \geq \lambda - \mathfrak{d}_I - a_I - 1 - b_I \geq \lambda - \mathfrak{d}_I - 1$. Assertion (iv) follows from assertion (ii), together with the fact that if $p > 2$ and $e_i = 1$ for all $i \in I$, then we have $\mathfrak{d}_I = 0$, $a_I = -b_I \in \mathbb{Z}$. This completes the proof of Proposition 1.2. ○

Proposition 1.3. **(Estimates of Differents)** *We continue to use the notation of Proposition 1.2. Suppose that $k_0 \subseteq k_i$ is a subfield that contains $\mathbb{Q}_p$. Write $R_0 \stackrel{\text{def}}{=} \mathcal{O}_{k_0}$ for the ring of integers of k_0, $\mathfrak{d}_0$ for the order [i.e., "$\text{ord}(-)$"] of any generator of the different ideal of R_0 over $\mathbb{Z}_p$, e_0 for the ramification index of k_0 over $\mathbb{Q}_p$, $e_{i/0} \stackrel{\text{def}}{=} e_i / e_0$ ($\in \mathbb{Z}$), $[k_i : k_0]$ for the degree of the extension k_i/k_0, n_i for the unique nonnegative integer such that $[k_i : k_0]/p^{n_i}$ is an integer prime to p. Then:*

(i) We have:

$$\mathfrak{d}_i \geq \mathfrak{d}_0 + (e_{i/0} - 1)/(e_{i/0} \cdot e_0) = \mathfrak{d}_0 + (e_{i/0} - 1)/e_i$$

— *where the "$\geq$" is an* **equality** *if k_i is* **tamely ramified** *over k_0.*

(ii) Suppose that k_i is a finite Galois extension of a subfield $k_1 \subseteq k_i$ such that $k_0 \subseteq k_1$, and k_1 is tamely ramified over k_0. Then we have: $\mathfrak{d}_i \leq \mathfrak{d}_0 + n_i + 1/e_0$.

Proof. By replacing k_0 by an *unramified* extension of k_0 contained in k_i, we may assume without loss of generality in the following discussion that k_i is a *totally ramified* extension of k_0. First, we consider assertion (i). Let π_0 be a uniformizer of R_0. Then there exists an isomorphism of R_0-algebras $R_0[x]/(f(x)) \overset{\sim}{\to} R_i$, where $f(x) \in R_0[x]$ is a monic polynomial which is $\equiv x^{e_{i/0}} \pmod{\pi_0}$, that maps $x \mapsto \pi_i$ for some uniformizer π_i of R_i. Thus, the different $\mathfrak{d}_i$ may be computed as follows:

$$\begin{aligned}\mathfrak{d}_i - \mathfrak{d}_0 &= \mathrm{ord}(f'(\pi_i)) \geq \min(\mathrm{ord}(\pi_0), \mathrm{ord}(e_{i/0} \cdot \pi_i^{e_{i/0}-1})) \\ &\geq \min\left(\frac{1}{e_0}, \mathrm{ord}(\pi_i^{e_{i/0}-1})\right) = \min\left(\frac{1}{e_0}, \frac{e_{i/0}-1}{e_{i/0} \cdot e_0}\right) = \frac{e_{i/0}-1}{e_i}\end{aligned}$$

— where, for $\lambda, \mu \in \mathbb{R}$ such that $\lambda \geq \mu$, we define $\min(\lambda, \mu) \overset{\mathrm{def}}{=} \mu$. When k_i is *tamely ramified* over k_0, one verifies immediately that the inequalities of the above display are, in fact, equalities. This completes the proof of assertion (i).

Next, we consider assertion (ii). We apply *induction* on n_i. Since assertion (ii) follows immediately from assertion (i) when $n_i = 0$, we may assume that $n_i \geq 1$, and that assertion (ii) has been verified for smaller "n_i". By replacing k_1 by some tamely ramified extension of k_1 contained in k_i, we may assume without loss of generality that $\mathrm{Gal}(k_i/k_1)$ is a p-group. Since p-groups are solvable, and k_i is a *totally ramified* extension of k_0, it follows that there exists a subextension $k_1 \subseteq k_* \subseteq k_i$ such that k_i/k_* and k_*/k_1 are Galois extensions of degree p and p^{n_i-1}, respectively. Write $R_* \overset{\mathrm{def}}{=} \mathcal{O}_{k_*}$ for the ring of integers of k_*, $\mathfrak{d}_*$ for the order [i.e., "$\mathrm{ord}(-)$"] of any generator of the different ideal of R_* over $\mathbb{Z}_p$, and e_* for the ramification index of k_* over $\mathbb{Q}_p$. Thus, by the induction hypothesis, it follows that $\mathfrak{d}_* \leq \mathfrak{d}_0 + n_i - 1 + 1/e_0$. To verify that $\mathfrak{d}_i \leq \mathfrak{d}_0 + n_i + 1/e_0$, it suffices to verify that $\mathfrak{d}_i \leq \mathfrak{d}_0 + n_i + 1/e_0 + \epsilon$ for any positive real number ϵ. Thus, let us *fix* a positive real number ϵ. Then by possibly enlarging k_i and k_1, we may also assume without loss of generality that the tamely ramified extension k_1 of k_0 contains a *primitive p-th root of unity*, and, moreover, that the ramification index e_1 of k_1 over $\mathbb{Q}_p$ satisfies the inequality $e_1 \geq p/\epsilon$ [so $e_* \geq e_1 \geq p/\epsilon$]. Thus, k_i is a *Kummer extension* of k_*. In particular, there exists an *inclusion* of R_*-algebras $R_*[x]/(f(x)) \hookrightarrow R_i$, where $f(x) \in R_*[x]$ is a monic polynomial which is of the form $f(x) = x^p - \varpi_*$ for some element ϖ_* of R_* satisfying the estimates $0 \leq \mathrm{ord}(\varpi_*) \leq \frac{p-1}{e_*}$, that maps $x \mapsto \varpi_i$ for some element ϖ_i of R_i satisfying the estimates $0 \leq \mathrm{ord}(\varpi_i) \leq \frac{p-1}{p \cdot e_*}$. Now we compute:

$$\begin{aligned}\mathfrak{d}_i &\leq \mathrm{ord}(f'(\varpi_i)) + \mathfrak{d}_* \leq \mathrm{ord}(p \cdot \varpi_i^{p-1}) + \mathfrak{d}_0 + n_i - 1 + 1/e_0 \\ &= (p-1) \cdot \mathrm{ord}(\varpi_i) + \mathfrak{d}_0 + n_i + 1/e_0 \leq \frac{(p-1)^2}{p \cdot e_*} + \mathfrak{d}_0 + n_i + 1/e_0 \\ &\leq \frac{p}{e_*} + \mathfrak{d}_0 + n_i + 1/e_0 \leq \mathfrak{d}_0 + n_i + 1/e_0 + \epsilon\end{aligned}$$

— thus completing the proof of assertion (ii). $\bigcirc$

Remark 1.3.1. Similar estimates to those discussed in Proposition 1.3 may be found in [Ih], Lemma A.

Proposition 1.4. (Nonarchimedean Normalized Log-volume Estimates) *We continue to use the notation of Proposition 1.2. Also, for* $i \in I$, *write* $R_i^{\boldsymbol{\mu}} \subseteq R_i^{\times}$ *for the torsion subgroup of* $R_i^{\times}$, $R_i^{\times\boldsymbol{\mu}} \stackrel{\mathrm{def}}{=} R_i^{\times}/R_i^{\boldsymbol{\mu}}$, p^{f_i} *for the cardinality of the residue field of* k_i, *and* p^{m_i} *for the order of the* p*-primary component of* $R_i^{\boldsymbol{\mu}}$. *Thus, the order of* $R_i^{\boldsymbol{\mu}}$ *is equal to* $p^{m_i} \cdot (p^{f_i} - 1)$. *Then:*

(i) The **log-volumes** *constructed in [AbsTopIII], Proposition 5.7, (i), on the various finite extensions of* $\mathbb{Q}_p$ *contained in* $\overline{\mathbb{Q}}_p$ *may be suitably* **normalized** *[i.e., by dividing by the degree of the finite extension] so as to yield a notion of log-volume*

$$\mu^{\log}(-)$$

defined on compact open subsets of finite extensions of $\mathbb{Q}_p$ *contained in* $\overline{\mathbb{Q}}_p$, *valued in* $\mathbb{R}$, *and normalized so that* $\mu^{\log}(R_i) = 0$, $\mu^{\log}(p \cdot R_i) = -\log(p)$, *for each* $i \in I$. *Moreover, by applying the fact that tensor products of finitely many finite extensions of* $\mathbb{Q}_p$ *over* $\mathbb{Z}_p$ *decompose, naturally, as direct sums of finitely many finite extensions of* $\mathbb{Q}_p$, *we obtain a notion of log-volume — which, by abuse of notation, we shall also denote by "*$\mu^{\log}(-)$*" — defined on* **compact open subsets of such tensor products**, *valued in* $\mathbb{R}$, *and normalized so that* $\mu^{\log}((R_E)^\sim) = 0$, $\mu^{\log}(p \cdot (R_E)^\sim) = -\log(p)$, *for any nonempty set* $E \subseteq I$.

(ii) We have:

$$\mu^{\log}(\log_p(R_i^{\times})) = -\Big(\frac{1}{e_i} + \frac{m_i}{e_i f_i}\Big) \cdot \log(p)$$

[cf. [AbsTopIII], Proposition 5.8, (iii)].

(iii) Let $I^* \subseteq I$ *be a subset such that for each* $i \in I \setminus I^*$, *it holds that* $p - 2 \geq e_i$ (≥ 1). *Then for any* $\lambda \in \frac{1}{e_{i^\dagger}} \cdot \mathbb{Z}$, $i^\dagger \in I$, *we have inclusions* $\phi(p^{\lambda} \cdot (R_I)^\sim) \subseteq p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^{\times}) \subseteq p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I} \cdot (R_I)^\sim$ *and inequalities*

$$\mu^{\log}(p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^{\times})) \ \leq \ \Big(-\lambda + \mathfrak{d}_I + 1 + 4 \cdot |I^*|/p\Big) \cdot \log(p);$$

$$\mu^{\log}(p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I} \cdot (R_I)^\sim) \ \leq \ \Big(-\lambda + \mathfrak{d}_I + 1\Big) \cdot \log(p) \ + \ \sum_{i \in I^*} \{3 + \log(e_i)\}$$

— where we write "$|(-)|$*" for the cardinality of the set "*$(-)$*". Moreover,* $\mathfrak{d}_I + a_I \geq |I|$ *if* $p > 2$; $\mathfrak{d}_I + a_I \geq 2 \cdot |I|$ *if* $p = 2$.

(iv) If $p > 2$ *and* $e_i = 1$ *for all* $i \in I$, *then* $\phi((R_I)^\sim) \subseteq (R_I)^\sim$, *and* $\mu^{\log}((R_I)^\sim) = 0$.

Proof. Assertion (i) follows immediately from the definitions. Next, we consider assertion (ii). We begin by *observing* that every compact open subset of $R_i^{\times\boldsymbol{\mu}}$ may be covered by a finite collection of compact open subsets of $R_i^{\times\boldsymbol{\mu}}$ that arise as images of compact open subsets of $R_i^{\times}$ that map *injectively* to $R_i^{\times\boldsymbol{\mu}}$. In particular, by applying this *observation*, we conclude that the log-volume on $R_i^{\times}$ determines, in a natural way, a log-volume on the quotient $R_i^{\times} \twoheadrightarrow R_i^{\times\boldsymbol{\mu}}$. Moreover, in light of the compatibility of the log-volume with "$\log_p(-)$" [cf. [AbsTopIII], Proposition 5.7, (i), (c)], it follows immediately that $\mu^{\log}(\log_p(R_i^{\times})) = \mu^{\log}(R_i^{\times\boldsymbol{\mu}})$. Thus, it suffices to compute $e_i \cdot f_i \cdot \mu^{\log}(R_i^{\times\boldsymbol{\mu}}) = e_i \cdot f_i \cdot \mu^{\log}(R_i^{\times}) - \log(p^{m_i} \cdot (p^{f_i} - 1))$. On the other hand, it follows immediately from the basic properties of the log-volume [cf. [AbsTopIII], Proposition 5.7, (i), (a)] that $e_i \cdot f_i \cdot \mu^{\log}(R_i^{\times}) = \log(1 - p^{-f_i})$, so $e_i \cdot f_i \cdot \mu^{\log}(R_i^{\times\boldsymbol{\mu}}) = -(f_i + m_i) \cdot \log(p)$, as desired. This completes the proof of assertion (ii).

The inclusions of assertion (iii) follow immediately from Proposition 1.2, (ii). When $p = 2$, the fact that $\mathfrak{d}_I + a_I \geq 2 \cdot |I|$ follows immediately from the definition of "$\mathfrak{d}_i$" and "a_i" in Propositions 1.1, 1.2. When $p > 2$, it follows immediately from the definition of "a_i" in Proposition 1.2 that $a_i \geq 1/e_i$, for all $i \in I$; thus, since $\mathfrak{d}_i \geq (e_i - 1)/e_i$ for all $i \in I$ [cf. Proposition 1.3, (i)], we conclude that $\mathfrak{d}_i + a_i \geq 1$ for all $i \in I$, and hence that $\mathfrak{d}_I + a_I \geq |I|$, as asserted in the statement of assertion (iii). Next, let us observe that $\frac{1}{p-2} \leq \frac{4}{p}$ for $p \geq 3$; $\frac{p}{p-1} \leq 2$ for $p \geq 2$; $\frac{2}{p} \leq \frac{1}{\log(p)}$ for $p \geq 2$. Thus, it follows immediately from the definition of a_i, b_i in Proposition 1.2 that $a_i - \frac{1}{e_i} \leq \frac{4}{p} \leq \frac{2}{\log(p)}$, $(b_i + \frac{1}{e_i}) \cdot \log(p) \leq \log(2e_i) \leq 1 + \log(e_i)$ for $i \in I$; $a_i = \frac{1}{e_i} = -b_i$ for $i \in I \setminus I^*$. On the other hand, by assertion (i), we have $\mu^{\log}(R_I) \leq \mu^{\log}((R_I)^{\sim}) = 0$; by assertion (ii), we have $\mu^{\log}(\log_p(R_i^{\times})) \leq -\frac{1}{e_i} \cdot \log(p)$. Now we compute:

$$\begin{aligned}
\mu^{\log}(p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^{\times})) &\leq \Big(-\lambda + \mathfrak{d}_I + a_I + 1 \Big) \cdot \log(p) \ + \ \mu^{\log}(\log_p(R_I^{\times})) \\
&= \Big(-\lambda + \mathfrak{d}_I + a_I + 1 \Big) \cdot \log(p) \\
&\quad + \Big\{ \sum_{i \in I} \mu^{\log}(\log_p(R_i^{\times})) \Big\} \ + \ \mu^{\log}(R_I) \\
&\leq \Big\{ -\lambda + \mathfrak{d}_I + 1 + \sum_{i \in I} (a_i - \frac{1}{e_i}) \Big\} \cdot \log(p) \\
&\leq \Big(-\lambda + \mathfrak{d}_I + 1 + 4 \cdot |I^*|/p \Big) \cdot \log(p); \\
\mu^{\log}(p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I} \cdot (R_I)^{\sim}) &\leq \Big(-\lambda + \mathfrak{d}_I + a_I + b_I + 1 \Big) \cdot \log(p) \\
&\leq \Big(-\lambda + \mathfrak{d}_I + 1 \Big) \cdot \log(p) \ + \ \sum_{i \in I^*} \{3 + \log(e_i)\}
\end{aligned}$$

— thus completing the proof of assertion (iii). Assertion (iv) follows immediately from assertion (i) and Proposition 1.2, (iv). ◯

Proposition 1.5. **(Archimedean Metric Estimates)** *In the following, we shall regard the* **complex archimedean field** $\mathbb{C}$ *as being equipped with its standard* **Hermitian metric**, *i.e., the metric determined by the complex norm. Let us refer to as the* **primitive automorphisms** *of* $\mathbb{C}$ *the group of automorphisms [of order 8] of the underlying metrized real vector space of* $\mathbb{C}$ *generated by the operations of* **complex conjugation** *and* **multiplication** *by* ± 1 *or* $\pm\sqrt{-1}$.

(i) **(Direct Sum vs. Tensor Product Metrics)** *The metric on* $\mathbb{C}$ *determines a tensor product metric on* $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$*, as well as a direct sum metric on* $\mathbb{C} \oplus \mathbb{C}$*. Then, relative to these metrics, any* **isomorphism of topological rings** *[i.e., arising from the Chinese remainder theorem]*

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \ \stackrel{\sim}{\to} \ \mathbb{C} \oplus \mathbb{C}$$

is **compatible** *with these* **metrics***, up to a factor of* 2*, i.e., the metric on the right-hand side corresponds to* 2 *times the metric on the left-hand side. [Thus, lengths differ by a factor of* $\sqrt{2}$*.]*

(ii) **(Direct Sum vs. Tensor Product Automorphisms)** *Relative to the notation of (i), the* **direct sum decomposition** $\mathbb{C} \oplus \mathbb{C}$*, together with its Hermitian metric, is* **preserved***, relative to the displayed isomorphism of (i), by the automorphisms of* $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ *induced by the various* **primitive automorphisms** *of the two copies of "*$\mathbb{C}$*" that appear in the tensor product* $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$*.*

(iii) **(Direct Sums and Tensor Products of Multiple Copies)** *Let* I*,* V *be nonempty finite sets, whose cardinalities we denote by* $|I|$*,* $|V|$*, respectively. Write*

$$M \ \stackrel{\text{def}}{=} \ \bigoplus_{v \in V} \mathbb{C}_v$$

for the direct sum of copies $\mathbb{C}_v \stackrel{\text{def}}{=} \mathbb{C}$ *of* $\mathbb{C}$ *labeled by* $v \in V$*, which we regard as equipped with the direct sum metric, and*

$$M_I \ \stackrel{\text{def}}{=} \ \bigotimes_{i \in I} M_i$$

for the tensor product over $\mathbb{R}$ *of copies* $M_i \stackrel{\text{def}}{=} M$ *of* M *labeled by* $i \in I$*, which we regard as equipped with the* **tensor product metric** *[cf. the constructions of [IUTchIII], Proposition 3.2, (ii)]. Then the topological ring structure on each* $\mathbb{C}_v$ *determines a* **topological ring structure** *on* M_I *with respect to which* M_I *admits a unique* **direct sum decomposition** *as a direct sum of*

$$2^{|I|-1} \cdot |V|^{|I|}$$

copies of $\mathbb{C}$ *[cf. [IUTchIII], Proposition 3.1, (i)]. The* **direct sum metric** *on* M_I *— i.e., the metric determined by the natural metrics on these copies of* $\mathbb{C}$ *— is equal to*

$$2^{|I|-1}$$

times the original tensor product metric on M_I*. Write*

$$B_I \quad \subseteq \quad M_I$$

for the **"integral structure"** *[cf. the constructions of [IUTchIII], Proposition 3.1, (ii)] given by the direct product of the* **unit balls** *of the copies of* $\mathbb{C}$ *that occur in the direct sum decomposition of* M_I*. Then the tensor product metric on* M_I*, the direct sum decomposition of* M_I*, the direct sum metric on* M_I*, and the integral* *structure $B_I \subseteq M_I$ are* **preserved** *by the automorphisms of M_I induced by the various* **primitive automorphisms** *of the direct summands "$\mathbb{C}_v$" that appear in the factors "M_i" of the tensor product M_I.*

(iv) **(Tensor Product of Vectors of a Given Length)** *Suppose that we are in the situation of (iii). Fix $\lambda \in \mathbb{R}_{>0}$. Then*

$$M_I \quad \ni \quad \bigotimes_{i \in I} m_i \quad \in \quad \lambda^{|I|} \cdot B_I$$

for any collection of elements $\{m_i \in M_i\}_{i \in I}$ such that the component of m_i in each direct summand "$\mathbb{C}_v$" of M_i is of **length** *λ.*

Proof. Assertions (i) and (ii) are discussed in [IUTchIII], Remark 3.9.1, (ii), and may be verified by means of routine and elementary arguments. Assertion (iii) follows immediately from assertions (i) and (ii). Assertion (iv) follows immediately from the various definitions involved. ◯

Proposition 1.6. **(The Prime Number Theorem)** *If n is a positive integer, then let us write p_n for the n-th smallest prime number. [Thus, $p_1 = 2$, $p_2 = 3$, and so on.] Then there exists an integer n_0 such that it holds that*

$$n \;\leq\; \tfrac{4p_n}{3 \cdot \log(p_n)}$$

for all $n \geq n_0$. In particular, there exists a positive real number η_{prm} such that

$$\sum_{p \leq \eta} 1 \quad \leq \quad \tfrac{4\eta}{3 \cdot \log(\eta)}$$

— where the sum ranges over the prime numbers $p \leq \eta$ — for all positive real $\eta \geq \eta_{\text{prm}}$.

Proof. Relative to our notation, the *Prime Number Theorem* [cf., e.g., [DmMn], §3.10] implies that

$$\lim_{n \to \infty} \frac{n \cdot \log(p_n)}{p_n} \;=\; 1$$

— i.e., in particular, that for some positive integer n_0, it holds that

$$\frac{\log(p_n)}{p_n} \;\leq\; \frac{4}{3} \cdot \frac{1}{n}$$

for all $n \geq n_0$. The final portion of Proposition 1.6 follows formally. ◯

Proposition 1.7. **(Weighted Averages)** *Let E be a nonempty finite set, n a positive integer. For $e \in E$, let $\lambda_e \in \mathbb{R}_{>0}$, $\beta_e \in \mathbb{R}$. Then, for any $i = 1, \ldots, n$, we have:*

$$\frac{\sum_{\vec{e} \in E^n} \beta_{\vec{e}} \cdot \lambda_{\Pi \vec{e}}}{\sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}}} \quad = \quad \frac{\sum_{\vec{e} \in E^n} n \cdot \beta_{e_i} \cdot \lambda_{\Pi \vec{e}}}{\sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}}} \quad = \quad n \cdot \beta_{\text{avg}}$$

— *where we write* $\beta_{\mathrm{avg}} \stackrel{\mathrm{def}}{=} \beta_E/\lambda_E$, $\beta_E \stackrel{\mathrm{def}}{=} \sum_{e\in E} \beta_e \cdot \lambda_e$, $\lambda_E \stackrel{\mathrm{def}}{=} \sum_{e\in E} \lambda_e$,

$$\beta_{\vec{e}} \stackrel{\mathrm{def}}{=} \sum_{j=1}^{n} \beta_{e_j}; \qquad \lambda_{\Pi\vec{e}} \stackrel{\mathrm{def}}{=} \prod_{j=1}^{n} \lambda_{e_j}$$

for any n*-tuple* $\vec{e} = (e_1, \ldots, e_n) \in E^n$ *of elements of* E.

Proof. We begin by observing that

$$\lambda_E^n = \sum_{\vec{e}\in E^n} \lambda_{\Pi\vec{e}}\ ; \qquad \beta_E \cdot \lambda_E^{n-1} = \sum_{\vec{e}\in E^n} \beta_{e_i} \cdot \lambda_{\Pi\vec{e}}$$

for any $i = 1, \ldots, n$. Thus, summing over i, we obtain that

$$n \cdot \beta_E \cdot \lambda_E^{n-1} = \sum_{\vec{e}\in E^n} \beta_{\vec{e}} \cdot \lambda_{\Pi\vec{e}} = \sum_{\vec{e}\in E^n} n \cdot \beta_{e_i} \cdot \lambda_{\Pi\vec{e}}$$

and hence that

$$\begin{aligned} n \cdot \beta_{\mathrm{avg}} = n \cdot \beta_E \cdot \lambda_E^{n-1}/\lambda_E^n &= \Big(\sum_{\vec{e}\in E^n} \beta_{\vec{e}} \cdot \lambda_{\Pi\vec{e}}\Big) \cdot \Big(\sum_{\vec{e}\in E^n} \lambda_{\Pi\vec{e}}\Big)^{-1} \\ &= \Big(\sum_{\vec{e}\in E^n} n \cdot \beta_{e_i} \cdot \lambda_{\Pi\vec{e}}\Big) \cdot \Big(\sum_{\vec{e}\in E^n} \lambda_{\Pi\vec{e}}\Big)^{-1} \end{aligned}$$

as desired. ○

Remark 1.7.1. In Theorem 1.10 below, we shall apply Proposition 1.7 to compute various **packet-normalized log-volumes** of the sort discussed in [IUTchIII], Proposition 3.9, (i) — i.e., log-volumes normalized by means of the **normalized weights** discussed in [IUTchIII], Remark 3.1.1, (ii). Here, we recall that the normalized weights discussed in [IUTchIII], Remark 3.1.1, (ii), were computed relative to the *non-normalized log-volumes* of [AbsTopIII], Proposition 5.8, (iii), (vi) [cf. the discussion of [IUTchIII], Remark 3.1.1, (ii); [IUTchI], Example 3.5, (iii)]. By contrast, in the discussion of the present §1, our computations are performed relative to *normalized log-volumes* as discussed in Proposition 1.4, (i). In particular, it follows that the *weights* $[K_{\underline{v}} : (F_{\mathrm{mod}})_v]^{-1}$, where $\underline{\mathbb{V}} \ni \underline{v} \mid v \in \mathbb{V}_{\mathrm{mod}}$, of the discussion of [IUTchIII], Remark 3.1.1, (ii), must be *replaced* — i.e., when one works with *normalized log-volumes* as in Proposition 1.4, (i) — by the **weights**

$$[K_{\underline{v}} : \mathbb{Q}_{v_{\mathbb{Q}}}] \cdot [K_{\underline{v}} : (F_{\mathrm{mod}})_v]^{-1} = [(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}]$$

— where $\mathbb{V}_{\mathrm{mod}} \ni v \mid v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. This means that the *normalized weights* of the final display of [IUTchIII], Remark 3.1.1, (ii), must be replaced, when one works with *normalized log-volumes* as in Proposition 1.4, (i), by the **normalized weights**

$$\frac{\Big(\prod_{\alpha\in A} [(F_{\mathrm{mod}})_{v_\alpha} : \mathbb{Q}_{v_{\mathbb{Q}}}]\Big)}{\sum_{\{w_\alpha\}_{\alpha\in A}} \Big(\prod_{\alpha\in A} [(F_{\mathrm{mod}})_{w_\alpha} : \mathbb{Q}_{v_{\mathbb{Q}}}]\Big)}$$

— where the sum is over all collections $\{w_\alpha\}_{\alpha \in A}$ of [not necessarily distinct!] elements $w_\alpha \in \mathbb{V}_{\mathrm{mod}}$ lying over $v_{\mathbb{Q}}$ and indexed by $\alpha \in A$. Thus, in summary, when one works with *normalized log-volumes* as in Proposition 1.4, (i), the appropriate *normalized weights* are given by the expressions

$$\frac{\lambda_{\Pi \vec{e}^\dagger}}{\sum_{\vec{e} \in E^n} \lambda_{\Pi \vec{e}}}$$

[where $\vec{e}^\dagger \in E^n$] that appear in Proposition 1.7. Here, one takes "E" to be the set of elements of $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$ lying over a fixed $v_{\mathbb{Q}}$; one takes "n" to be the cardinality of A, so that one can write $A = \{\alpha_1, \ldots, \alpha_n\}$ [where the α_i are *distinct*]; if $e \in E$ corresponds to $\underline{v} \in \underline{\mathbb{V}}$, $v \in \mathbb{V}_{\mathrm{mod}}$, then one takes

$$\text{"}\lambda_e\text{"} \stackrel{\mathrm{def}}{=} [(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}] \in \mathbb{R}_{>0}$$

and "β_e" to be a normalized log-volume of some compact open subset of $K_{\underline{v}}$.

Before proceeding, we review some *well-known elementary facts* concerning *elliptic curves*. In the following, we shall write $\mathcal{M}_{\mathrm{ell}}$ for the *moduli stack of elliptic curves* over $\mathbb{Z}$ and

$$\mathcal{M}_{\mathrm{ell}} \subseteq \overline{\mathcal{M}}_{\mathrm{ell}}$$

for the *natural compactification* of $\mathcal{M}_{\mathrm{ell}}$, i.e., the moduli stack of one-dimensional semi-abelian schemes over $\mathbb{Z}$. Also, if R is a $\mathbb{Z}$-algebra, then we shall write $(\mathcal{M}_{\mathrm{ell}})_R \stackrel{\mathrm{def}}{=} \mathcal{M}_{\mathrm{ell}} \times_{\mathbb{Z}} R$, $(\overline{\mathcal{M}}_{\mathrm{ell}})_R \stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}_{\mathrm{ell}} \times_{\mathbb{Z}} R$.

Proposition 1.8. **(Torsion Points of Elliptic Curves)** *Let k be a perfect field, $\overline{k}$ an algebraic closure of k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$.*

(i) **("Serre's Criterion")** *Let $l \geq 3$ be a* **prime number** *that is* **invertible** *in k; suppose that $\overline{k} = k$. Let A be an* **abelian variety** *over k, equipped with a polarization λ. Write $A[l] \subseteq A(k)$ for the group of l-torsion points of $A(k)$. Then the natural map*

$$\phi : \quad \mathrm{Aut}_k(A, \lambda) \quad \to \quad \mathrm{Aut}(A[l])$$

from the group of automorphisms of the polarized abelian variety (A, λ) over k to the group of automorphisms of the abelian group $A[l]$ is **injective***.*

(ii) Let $E_{\overline{k}}$ be an **elliptic curve** *over $\overline{k}$ with origin $\epsilon_E \in E(\overline{k})$. For n a positive integer, write $E_{\overline{k}}[n] \subseteq E_{\overline{k}}(\overline{k})$ for the module of n-torsion points of $E_{\overline{k}}(\overline{k})$ and*

$$\mathrm{Aut}_{\overline{k}}(E_{\overline{k}}) \quad \subseteq \quad \mathrm{Aut}_k(E_{\overline{k}})$$

for the respective groups of ϵ_E-preserving automorphisms of the $\overline{k}$-scheme $E_{\overline{k}}$ and the k-scheme $E_{\overline{k}}$. Then we have a **natural exact sequence**

$$1 \longrightarrow \mathrm{Aut}_{\overline{k}}(E_{\overline{k}}) \longrightarrow \mathrm{Aut}_k(E_{\overline{k}}) \longrightarrow G_k$$

— where the image $G_E \subseteq G_k$ of the homomorphism $\mathrm{Aut}_k(E_{\overline{k}}) \to G_k$ *is* **open** *— and a* **natural representation**

$$\rho_n : \ \mathrm{Aut}_k(E_{\overline{k}}) \ \to \ \mathrm{Aut}(E_{\overline{k}}[n])$$

on the n-torsion points of $E_{\overline{k}}$. The finite extension k_E of k determined by G_E is the **minimal field of definition** *of $E_{\overline{k}}$, i.e., the field generated over k by the* **j-invariant** *of $E_{\overline{k}}$. Finally, if $H \subseteq G_k$ is any closed subgroup, which corresponds to an extension k_H of k, then the datum of a* **model** *of $E_{\overline{k}}$ over k_H [i.e., descent data for $E_{\overline{k}}$ from $\overline{k}$ to k_H] is equivalent to the datum of a* **section** *of the homomorphism* $\mathrm{Aut}_k(E_{\overline{k}}) \to G_k$ *over H. In particular, the homomorphism* $\mathrm{Aut}_k(E_{\overline{k}}) \to G_k$ *admits a section over G_E.*

(iii) In the situation of (ii), suppose further that $\mathrm{Aut}_{\overline{k}}(E_{\overline{k}}) = \{\pm 1\}$. *Then the representation ρ_2* **factors** *through G_E and hence defines a* **natural representation** $G_E \ \to \ \mathrm{Aut}(E_{\overline{k}}[2])$.

(iv) In the situation of (ii), suppose further that $l \geq 3$ is a **prime number** *that is* **invertible** *in k, and that $E_{\overline{k}}$ descends to elliptic curves E'_k and E''_k over k, all of whose l-torsion points are* **rational** *over k. Then E'_k is* **isomorphic** *to E''_k over k.*

(v) In the situation of (ii), suppose further that k is a **complete discrete valuation field** *with ring of integers $\mathcal{O}_k$, that $l \geq 3$ is a* **prime number** *that is* **invertible** *in $\mathcal{O}_k$, and that $E_{\overline{k}}$ descends to an elliptic curve E_k over k, all of whose l-torsion points are* **rational** *over k. Then E_k has* **semi-stable reduction** *over $\mathcal{O}_k$ [i.e., extends to a semi-abelian scheme over $\mathcal{O}_k$].*

(vi) In the situation of (iii), suppose further that 2 is **invertible** *in k, that $G_E = G_k$, and that the representation $G_E \ \to \ \mathrm{Aut}(E_{\overline{k}}[2])$ is* **trivial**. *Then $E_{\overline{k}}$ descends to an elliptic curve E_k over k which is defined by means of the* **Legendre form** *of the Weierstrass equation [cf., e.g., the statement of Corollary 2.2, below]. If, moreover, k is a* **complete discrete valuation field** *with ring of integers $\mathcal{O}_k$ such that 2 is* **invertible** *in $\mathcal{O}_k$, then E_k has* **semi-stable reduction** *over $\mathcal{O}_{k'}$ [i.e., extends to a semi-abelian scheme over $\mathcal{O}_{k'}$] for some finite extension $k' \subseteq \overline{k}$ of k such that $[k' : k] \leq 2$; if E_k has* **good reduction** *over $\mathcal{O}_{k'}$ [i.e., extends to an abelian scheme over $\mathcal{O}_{k'}$], then one may in fact take k' to be k.*

(vii) In the situation of (ii), suppose further that k is a **complete discrete valuation field** *with ring of integers $\mathcal{O}_k$, that $E_{\overline{k}}$ descends to an elliptic curve E_k over k, and that n is* **invertible** *in $\mathcal{O}_k$. If E_k has* **good reduction** *over $\mathcal{O}_k$ [i.e., extends to an abelian scheme over $\mathcal{O}_k$], then the action of G_k on $E_{\overline{k}}[n]$ is* **unramified**. *If E_k has* **bad multiplicative reduction** *over $\mathcal{O}_k$ [i.e., extends to a non-proper semi-abelian scheme over $\mathcal{O}_k$], then the kernel of the action of G_k on $E_{\overline{k}}[n]$ determines a* **tamely ramified** *extension of k whose ramification index over k divides n.*

Proof. First, we consider assertion (i). Suppose that ϕ is *not injective*. Since $\mathrm{Aut}_k(A, \lambda)$ is well-known to be *finite* [cf., e.g., [Milne], Proposition 17.5, (a)], we thus conclude that there exists an $\alpha \in \mathrm{Ker}(\phi)$ of order $n \neq 1$. We may assume without loss of generality that n is *prime*. Now we follow the argument of [Milne], Proposition 17.5, (b). Since α acts trivially on $A[l]$, it follows immediately that the endomorphism of A given by $\alpha - \mathrm{id}_A$ [where id_A denotes the identity automorphism of A] may be written in the form $l \cdot \beta$, for β an endomorphism of A over k. Write $T_l(A)$ for the *l-adic Tate module* of A. Since $\alpha^n = \mathrm{id}_A$, it follows that the eigenvalues of the action of α on $T_l(A)$ are *n-th roots of unity*. On the other hand, the eigenvalues of the action of β on $T_l(A)$ are *algebraic integers* [cf. [Milne], Theorem 12.5]. We thus conclude that each eigenvalue ζ of the action of α on $T_l(A)$ is an n-th root of unity which, as an algebraic integer, is $\equiv 1 \pmod{l}$ [where $l \geq 3$], hence $= 1$. Since $\alpha^n = \mathrm{id}_A$, it follows that α acts on $T_l(A)$ as a *semi-simple matrix* which is also *unipotent*, hence equal to the *identity matrix*. But this implies that $\alpha = \mathrm{id}_A$ [cf. [Milne], Theorem 12.5]. This contradiction completes the proof of assertion (i).

Next, we consider assertion (ii). Since $E_{\overline{k}}$ is *proper* over $\overline{k}$, it follows [by considering the space of global sections of the structure sheaf of $E_{\overline{k}}$] that any automorphism of the scheme $E_{\overline{k}}$ lies over an automorphism of $\overline{k}$. This implies the existence of a *natural exact sequence* and *natural representation* as in the statement of assertion (ii). The relationship between k_E and the *j-invariant* of $E_{\overline{k}}$ follows immediately from the well-known theory of the j-invariant of an elliptic curve [cf., e.g., [Silv], Chapter III, Proposition 1.4, (b), (c)]. The final portion of assertion (ii) concerning *models* of $E_{\overline{k}}$ follows immediately from the definitions. This completes the proof of assertion (ii). Assertion (iii) follows immediately from the fact that $\{\pm 1\}$ acts *trivially* on $E_{\overline{k}}[2]$.

Next, we consider assertion (iv). First, let us observe that it follows immediately from the final portion of assertion (ii) that a model E^*_k of $E_{\overline{k}}$ over k all of whose l-torsion points are *rational* over k corresponds to a closed subgroup $H^* \subseteq \mathrm{Aut}_k(E_{\overline{k}})$ that lies in the kernel of ρ_l and, moreover, maps isomorphically to G_k. On the other hand, it follows from assertion (i) that the restriction of ρ_l to $\mathrm{Aut}_{\overline{k}}(E_{\overline{k}}) \subseteq \mathrm{Aut}_k(E_{\overline{k}})$ is *injective*. Thus, the closed subgroup $H^* \subseteq \mathrm{Aut}_k(E_{\overline{k}})$ is *uniquely determined* by the condition that it lie in the kernel of ρ_l and, moreover, map isomorphically to G_k. This completes the proof of assertion (iv).

Next, we consider assertion (v). First, let us observe that, by considering *l-level structures*, we obtain a *finite covering* of $S \to (\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}[\frac{1}{l}]}$ which is *étale* over $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{Z}[\frac{1}{l}]}$ and *tamely ramified* over the divisor at infinity. Then it follows from assertion (i) that the algebraic stack S is in fact a *scheme*, which is, moreover, *proper* over $\mathbb{Z}[\frac{1}{l}]$. Thus, it follows from the valuative criterion for properness that any k-valued point of S determined by E_k — where we observe that such a point necessarily exists, in light of our assumption that the l-torsion points of E_k are *rational* over k — extends to an $\mathcal{O}_k$-valued point of S, hence also of $\overline{\mathcal{M}}_{\mathrm{ell}}$, as desired. This completes the proof of assertion (v).

Next, we consider assertion (vi). Since $G_E = G_k$, it follows from assertion (ii) that $E_{\overline{k}}$ descends to an elliptic curve E_k over k. Our assumption that the representation $G_k = G_E \to \mathrm{Aut}(E_{\overline{k}}[2])$ of assertion (iii) is *trivial* implies that the 2-torsion points of E_k are *rational* over k. Thus, by considering suitable *global sections* of tensor powers of the line bundle on E_k determined by the origin *on which the automorphism "-1" of E_k acts via multiplication by* ± 1 [cf., e.g., [Harts], Chapter IV, the proof of Proposition 4.6], one concludes immediately that a suitable [possibly trivial] *twist* E'_k of E_k over k [i.e., such that E'_k and E_k are isomorphic over some quadratic extension k' of k] may be defined by means of the *Legendre form* of the Weierstrass equation. Now suppose that k is a *complete discrete valuation field* with ring of integers $\mathcal{O}_k$ such that 2 is *invertible* in $\mathcal{O}_k$, and that E_k is defined by means of the *Legendre form* of the Weierstrass equation. Then the fact that E_k has *semi-stable reduction* over $\mathcal{O}_{k'}$ for some finite extension $k' \subseteq \overline{k}$ of k such that $[k' : k] \leq 2$ follows from the explicit computations of the proof of [Silv], Chapter VII, Proposition 5.4, (c). These explicit computations also imply that if E_k has *good reduction* over $\mathcal{O}_{k'}$, then one may in fact take k' to be k. This completes the proof of assertion (vi).

Assertion (vii) follows immediately from [NerMod], §7.4, Theorem 5, in the case of *good reduction* and from [NerMod], §7.4, Theorem 6, in the case of *bad multiplicative reduction.* ◯

We are now ready to apply the elementary computations discussed above to give *more explicit log-volume estimates for* Θ*-pilot objects*. We begin by recalling some notation and terminology from [GenEll], §1.

Definition 1.9. Let F be a *number field* [i.e., a finite extension of the rational number field $\mathbb{Q}$], whose *set of valuations* we denote by $\mathbb{V}(F)$. Thus, $\mathbb{V}(F)$ decomposes as a disjoint union $\mathbb{V}(F) = \mathbb{V}(F)^{\text{non}} \bigcup \mathbb{V}(F)^{\text{arc}}$ of *nonarchimedean* and *archimedean* valuations. If $v \in \mathbb{V}(F)$, then we shall write F_v for the *completion* of F at v; if $v \in \mathbb{V}(F)^{\text{non}}$, then we shall write e_v for the *ramification index* of F_v over $\mathbb{Q}_{p_v}$, f_v for the *residue field degree* of F_v over $\mathbb{Q}_{p_v}$, and q_v for the cardinality of the *residue field* of F_v.

(i) An *[*$\mathbb{R}$*-]arithmetic divisor* $\mathfrak{a}$ on F is defined to be a finite formal sum

$$\sum_{v \in \mathbb{V}(F)} c_v \cdot v$$

— where $c_v \in \mathbb{R}$, for all $v \in \mathbb{V}(F)$. Here, we shall refer to the set

$$\mathrm{Supp}(\mathfrak{a})$$

of $v \in \mathbb{V}(F)$ such that $c_v \neq 0$ as the *support of* $\mathfrak{a}$; if all of the c_v are ≥ 0, then we shall say that the arithmetic divisor is *effective*. Thus, the [$\mathbb{R}$-]arithmetic divisors on F naturally form a group $\mathrm{ADiv}_{\mathbb{R}}(F)$. The assignment

$$\mathbb{V}(F)^{\text{non}} \ni v \mapsto \log(q_v); \quad \mathbb{V}(F)^{\text{arc}} \ni v \mapsto 1$$

determines a homomorphism

$$\deg_F : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$$

which we shall refer to as the *degree* map. If $\mathfrak{a} \in \mathrm{ADiv}_{\mathbb{R}}(F)$, then we shall refer to

$$\underline{\deg}(\mathfrak{a}) \quad \stackrel{\text{def}}{=} \quad \frac{1}{[F : \mathbb{Q}]} \cdot \deg_F(\mathfrak{a})$$

as the *normalized degree* of $\mathfrak{a}$. Thus, for any finite extension K of F, we have

$$\underline{\deg}(\mathfrak{a}|_K) = \underline{\deg}(\mathfrak{a})$$

— where we write $\underline{\deg}(\mathfrak{a}|_K)$ for the normalized degree of the pull-back $\mathfrak{a}|_K \in \mathrm{ADiv}_{\mathbb{R}}(K)$ [defined in the evident fashion] of $\mathfrak{a}$ to K.

(ii) Let $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathbb{V}(\mathbb{Q})$, $E \subseteq \mathbb{V}(F)$ a nonempty set of elements lying over $v_{\mathbb{Q}}$. If $\mathfrak{a} = \sum_{v \in \mathbb{V}(F)} c_v \cdot v \in \mathrm{ADiv}_{\mathbb{R}}(F)$, then we shall write

$$\mathfrak{a}_E \stackrel{\mathrm{def}}{=} \sum_{v \in E} c_v \cdot v \ \in \ \mathrm{ADiv}_{\mathbb{R}}(F); \quad \underline{\deg}_E(\mathfrak{a}) \stackrel{\mathrm{def}}{=} \frac{\deg(\mathfrak{a}_E)}{\sum_{v \in E} [F_v : \mathbb{Q}_{v_{\mathbb{Q}}}]}$$

for the portion of $\mathfrak{a}$ *supported in* E and the *"normalized E-degree"* of $\mathfrak{a}$, respectively. Thus, for any finite extension K of F, we have

$$\underline{\deg}_{E|_K}(\mathfrak{a}|_K) = \underline{\deg}_E(\mathfrak{a})$$

— where we write $E|_K \subseteq \mathbb{V}(K)$ for the set of valuations lying over valuations $\in E$.

Theorem 1.10. (Log-volume Estimates for Θ-Pilot Objects) *Fix a collection of* **initial Θ-data** *as in [IUTchI], Definition 3.1. Suppose that we are in the situation of [IUTchIII], Corollary 3.12, and that the elliptic curve E_F has* **good reduction** *at every valuation $\in \mathbb{V}(F)^{\mathrm{good}} \bigcap \mathbb{V}(F)^{\mathrm{non}}$ that does* **not divide** *$2l$. In the notation of [IUTchI], Definition 3.1, let us write $d_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} [F_{\mathrm{mod}} : \mathbb{Q}]$, $(1 \leq)$ e_{mod} $(\leq d_{\mathrm{mod}})$ for the* **maximal ramification index** *of F_{mod} [i.e., of valuations $\in \mathbb{V}^{\mathrm{non}}_{\mathrm{mod}}$] over $\mathbb{Q}$, $d^*_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} 2^{12} \cdot 3^3 \cdot 5 \cdot d_{\mathrm{mod}}$, $e^*_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} 2^{12} \cdot 3^3 \cdot 5 \cdot e_{\mathrm{mod}}$ $(\leq d^*_{\mathrm{mod}})$, and*

$$F_{\mathrm{mod}} \quad \subseteq \quad F_{\mathrm{tpd}} \stackrel{\mathrm{def}}{=} F_{\mathrm{mod}}(\ E_{F_{\mathrm{mod}}}[2]\ ) \quad \subseteq \quad F$$

for the **"tripodal"** *intermediate field obtained from F_{mod} by adjoining the fields of definition of the 2-torsion points of any model of $E_F \times_F \overline{F}$ over F_{mod} [cf. Proposition 1.8, (ii), (iii)]. Moreover, we assume that the $(3 \cdot 5)$-torsion points of E_F are defined over F, and that*

$$F \quad = \quad F_{\mathrm{mod}}(\ \sqrt{-1},\ E_{F_{\mathrm{mod}}}[2 \cdot 3 \cdot 5]\ ) \quad \stackrel{\mathrm{def}}{=} \quad F_{\mathrm{tpd}}(\ \sqrt{-1},\ E_{F_{\mathrm{tpd}}}[3 \cdot 5]\ )$$

— i.e., that F is obtained from F_{tpd} by adjoining $\sqrt{-1}$, together with the fields of definition of the $(3 \cdot 5)$-torsion points of a model $E_{F_{\mathrm{tpd}}}$ of the elliptic curve $E_F \times_F \overline{F}$ over F_{tpd} determined by the **Legendre form** *of the Weierstrass equation [cf., e.g., the statement of Corollary 2.2, below; Proposition 1.8, (vi)]. [Thus, it follows from Proposition 1.8, (iv), that $E_F \cong E_{F_{\mathrm{tpd}}} \times_{F_{\mathrm{tpd}}} F$ over F, and from [IUTchI], Definition 3.1, (c), that $l \neq 5$.] If $F_{\mathrm{mod}} \subseteq F_{\Box} \subseteq K$ is any intermediate extension which is Galois over F_{mod}, then we shall write*

$$\mathfrak{d}^{F_{\Box}}_{\mathrm{ADiv}} \ \in \ \mathrm{ADiv}_{\mathbb{R}}(F_{\Box})$$

for the effective arithmetic divisor determined by the **different ideal** *of* $F_\square$ *over* $\mathbb{Q}$,

$$\mathfrak{q}^{F_\square}_{\mathrm{ADiv}} \in \mathrm{ADiv}_{\mathbb{R}}(F_\square)$$

for the effective arithmetic divisor determined by the ***q*-parameters** *of the elliptic curve* E_F *at the elements of* $\mathbb{V}(F_\square)^{\mathrm{bad}} \stackrel{\mathrm{def}}{=} \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \times_{\mathbb{V}_{\mathrm{mod}}} \mathbb{V}(F_\square)$ $(\neq \emptyset)$ *[cf. [GenEll], Remark 3.3.1],*

$$\mathfrak{f}^{F_\square}_{\mathrm{ADiv}} \in \mathrm{ADiv}_{\mathbb{R}}(F_\square)$$

for the effective arithmetic divisor whose support coincides with $\mathrm{Supp}(\mathfrak{q}^{F_\square}_{\mathrm{ADiv}})$, *but all of whose coefficients are equal to* 1 *— i.e., the* **conductor** *— and*

$$\log(\mathfrak{d}^{F_\square}_v) \stackrel{\mathrm{def}}{=} \underline{\deg}_{\mathbb{V}(F_\square)_v}(\mathfrak{d}^{F_\square}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{d}^{F_\square}_{v_{\mathbb{Q}}}) \stackrel{\mathrm{def}}{=} \underline{\deg}_{\mathbb{V}(F_\square)_{v_{\mathbb{Q}}}}(\mathfrak{d}^{F_\square}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{d}^{F_\square}) \stackrel{\mathrm{def}}{=} \underline{\deg}(\mathfrak{d}^{F_\square}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{q}_v) \stackrel{\mathrm{def}}{=} \underline{\deg}_{\mathbb{V}(F_\square)_v}(\mathfrak{q}^{F_\square}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{q}_{v_{\mathbb{Q}}}) \stackrel{\mathrm{def}}{=} \underline{\deg}_{\mathbb{V}(F_\square)_{v_{\mathbb{Q}}}}(\mathfrak{q}^{F_\square}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{q}) \stackrel{\mathrm{def}}{=} \underline{\deg}(\mathfrak{q}^{F_\square}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{f}^{F_\square}_v) \stackrel{\mathrm{def}}{=} \underline{\deg}_{\mathbb{V}(F_\square)_v}(\mathfrak{f}^{F_\square}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{f}^{F_\square}_{v_{\mathbb{Q}}}) \stackrel{\mathrm{def}}{=} \underline{\deg}_{\mathbb{V}(F_\square)_{v_{\mathbb{Q}}}}(\mathfrak{f}^{F_\square}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}$$

$$\log(\mathfrak{f}^{F_\square}) \stackrel{\mathrm{def}}{=} \underline{\deg}(\mathfrak{f}^{F_\square}_{\mathrm{ADiv}}) \in \mathbb{R}_{\geq 0}$$

— where $v \in \mathbb{V}_{\mathrm{mod}} = \mathbb{V}(F_{\mathrm{mod}})$, $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}} = \mathbb{V}(\mathbb{Q})$, $\mathbb{V}(F_\square)_v \stackrel{\mathrm{def}}{=} \mathbb{V}(F_\square) \times_{\mathbb{V}_{\mathrm{mod}}} \{v\}$, $\mathbb{V}(F_\square)_{v_{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} \mathbb{V}(F_\square) \times_{\mathbb{V}_{\mathbb{Q}}} \{v_{\mathbb{Q}}\}$. *Here, we observe that the various "*$\log(\mathfrak{q}_{(-)})$*'s" are independent of the choice of* $F_\square$, *and that the quantity "*$|\log(\underline{\underline{q}})| \in \mathbb{R}_{>0}$*" defined in [IUTchIII], Corollary 3.12, is equal to* $\frac{1}{2l} \cdot \log(\mathfrak{q}) \in \mathbb{R}$ *[cf. the definition of "*$\underline{\underline{q}}_{v}$*" in [IUTchI], Example 3.2, (iv)]. Then one may take the constant "*$C_\Theta \in \mathbb{R}$*" of [IUTchIII], Corollary 3.12, to be*

$$\frac{l+1}{4 \cdot |\log(\underline{\underline{q}})|} \cdot \Big\{ (1 + \tfrac{12 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 10 \cdot (e^*_{\mathrm{mod}} \cdot l + \eta_{\mathrm{prm}}) - \tfrac{1}{6} \cdot (1 - \tfrac{12}{l^2}) \cdot \log(\mathfrak{q}) \Big\} - 1$$

and hence, by applying the inequality "$C_\Theta \geq -1$*" of [IUTchIII], Corollary 3.12, conclude that*

$$\begin{aligned} \tfrac{1}{6} \cdot \log(\mathfrak{q}) &\leq (1 + \tfrac{20 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 20 \cdot (e^*_{\mathrm{mod}} \cdot l + \eta_{\mathrm{prm}}) \\ &\leq (1 + \tfrac{20 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F}) + \log(\mathfrak{f}^{F})) + 20 \cdot (e^*_{\mathrm{mod}} \cdot l + \eta_{\mathrm{prm}}) \end{aligned}$$

— where η_{prm} *is the positive real number of Proposition 1.6.*

Proof. For ease of reference, we divide our discussion into *steps*, as follows.

(i) We begin by recalling the following *elementary identities* for $n \in \mathbb{N}_{\geq 1}$:

(E1) $\frac{1}{n} \sum_{m=1}^{n} m = \frac{1}{2}(n+1)$;

(E2) $\frac{1}{n} \sum_{m=1}^{n} m^2 = \frac{1}{6}(2n+1)(n+1)$.

Also, we recall the following *elementary facts*:

(E3) For p a prime number, the cardinality $|GL_2(\mathbb{F}_p)|$ of $GL_2(\mathbb{F}_p)$ is given by $|GL_2(\mathbb{F}_p)| = p(p+1)(p-1)^2$.

(E4) For $p = 2, 3, 5$, the expression of (E3) may be computed as follows: $2(2+1)(2-1)^2 = 2 \cdot 3$; $3(3+1)(3-1)^2 = 3 \cdot 2^4$; $5(5+1)(5-1)^2 = 5 \cdot 2^5 \cdot 3$.

(E5) The degree of the extension $F_{\mathrm{mod}}(\sqrt{-1}\,)/F_{\mathrm{mod}}$ is ≤ 2.

(E6) We have: $0 \leq \log(2) \leq 1$, $1 \leq \log(3) \leq \log(\pi) \leq \log(5) \leq 2$.

(ii) Next, let us observe that the *inequality*

$$\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) \leq \log(\mathfrak{d}^F) + \log(\mathfrak{f}^F)$$

follows immediately from Proposition 1.3, (i), and the various definitions involved. On the other hand, the *inequality*

$$\begin{aligned}\log(\mathfrak{d}^F) + \log(\mathfrak{f}^F) &\leq \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + \log(2^{11} \cdot 3^3 \cdot 5^2) \\ &\leq \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + 21\end{aligned}$$

follows by applying Proposition 1.3, (i), at the primes that do *not* divide $2 \cdot 3 \cdot 5$ [where we recall that the extension F/F_{tpd} is *tamely ramified* over such primes — cf. Proposition 1.8, (vi), (vii)] and applying Proposition 1.3, (ii), together with (E3), (E4), (E5), (E6), and the fact that we have a natural outer inclusion $\mathrm{Gal}(F/F_{\mathrm{tpd}}) \hookrightarrow GL_2(\mathbb{F}_3) \times GL_2(\mathbb{F}_5) \times \mathbb{Z}/2\mathbb{Z}$, at the primes that *divide* $2 \cdot 3 \cdot 5$. In a similar vein, since the extension K/F is *tamely ramified* at the primes that do *not* divide l, and we have a natural outer inclusion $\mathrm{Gal}(K/F) \hookrightarrow GL_2(\mathbb{F}_l)$, the *inequality*

$$\begin{aligned}\log(\mathfrak{d}^K) \leq \log(\mathfrak{d}^K) + \log(\mathfrak{f}^K) &\leq \log(\mathfrak{d}^F) + \log(\mathfrak{f}^F) + 2 \cdot \log(l) \\ &\leq \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) + 2 \cdot \log(l) + 21\end{aligned}$$

follows immediately from Proposition 1.3, (i), (ii). Finally, for later reference, we observe that

$$(1 + \tfrac{4}{l}) \cdot \log(\mathfrak{d}^K) \leq (1 + \tfrac{4}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2 \cdot \log(l) + 46$$

— where we apply the estimates $\frac{\log(l)}{l} \leq \frac{1}{2}$ and $1 + \frac{4}{l} \leq 2$, both of which may be regarded as consequences of the fact that $l \geq 5$ [cf. also (E6)].

(iii) If $F_{\mathrm{tpd}} \subseteq F_\Box \subseteq K$ is any intermediate extension which is *Galois* over F_{mod}, then we shall write

$$\mathbb{V}(F_\Box)^{\mathrm{dst}} \subseteq \mathbb{V}(F_\Box)^{\mathrm{non}}$$

for the set of **"distinguished"** nonarchimedean valuations $v \in \mathbb{V}(F_\Box)^{\mathrm{non}}$, i.e., v that extend to a valuation $\in \mathbb{V}(K)^{\mathrm{non}}$ that *ramifies* over $\mathbb{Q}$. Now observe that it follows immediately from Proposition 1.8, (vi), (vii), together with our *assumption* on $\mathbb{V}(F)^{\mathrm{good}} \bigcap \mathbb{V}(F)^{\mathrm{non}}$, that

(D0) if $v \in \mathbb{V}(F_{\mathrm{tpd}})^{\mathrm{non}}$ does *not divide* $2 \cdot 3 \cdot 5 \cdot l$ and, moreover, is *not contained* in $\mathrm{Supp}(\mathfrak{q}^{F_{\mathrm{tpd}}}_{\mathrm{ADiv}})$, then the extension K/F_{tpd} is *unramified* over v.

Next, let us recall the well-known fact that the *determinant* of the Galois representation determined by the torsion points of an elliptic curve over a field of characteristic zero is the abelian Galois representation determined by the *cyclotomic character*. In particular, it follows [cf. the various definitions involved] that K contains a *primitive* $4 \cdot 3 \cdot 5 \cdot l$*-th root of unity*, hence is ramified over $\mathbb{Q}$ at any valuation $\in \mathbb{V}(K)^{\mathrm{non}}$ that divides $2 \cdot 3 \cdot 5 \cdot l$. Thus, one verifies immediately [i.e., by applying (D0); cf. also [IUTchI], Definition 3.1, (c)] that the following *conditions* on a valuation $v \in \mathbb{V}(F_{\square})^{\mathrm{non}}$ are *equivalent*:

(D1) $v \in \mathbb{V}(F_{\square})^{\mathrm{dst}}$.
(D2) The valuation v either divides $2 \cdot 3 \cdot 5 \cdot l$ or lies in $\mathrm{Supp}(\mathfrak{q}^{F_{\square}}_{\mathrm{ADiv}} + \mathfrak{d}^{F_{\square}}_{\mathrm{ADiv}})$.
(D3) The image of v in $\mathbb{V}(F_{\mathrm{tpd}})$ lies in $\mathbb{V}(F_{\mathrm{tpd}})^{\mathrm{dst}}$.

Let us write

$$\mathbb{V}^{\mathrm{dst}}_{\mathrm{mod}} \subseteq \mathbb{V}^{\mathrm{non}}_{\mathrm{mod}}; \quad \mathbb{V}^{\mathrm{dst}}_{\mathbb{Q}} \subseteq \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$$

for the respective *images* of $\mathbb{V}(F_{\mathrm{tpd}})^{\mathrm{dst}}$ in $\mathbb{V}_{\mathrm{mod}}$, $\mathbb{V}_{\mathbb{Q}}$ and, for $F_* \in \{F_{\mathrm{mod}}, \mathbb{Q}\}$ and $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$,

$$\mathfrak{s}^{F_*}_{\mathrm{ADiv}} \stackrel{\mathrm{def}}{=} \sum_{v \in \mathbb{V}(F_*)^{\mathrm{dst}}} e_v \cdot v \ \in \ \mathrm{ADiv}_{\mathbb{R}}(F_*)$$

$$\log(\mathfrak{s}^{F_*}_{v_{\mathbb{Q}}}) \ \stackrel{\mathrm{def}}{=} \ \underline{\deg}_{\mathbb{V}(F_*)_{v_{\mathbb{Q}}}}(\mathfrak{s}^{F_*}_{\mathrm{ADiv}}) \ \in \ \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{s}^{F_*}) \ \stackrel{\mathrm{def}}{=} \ \underline{\deg}(\mathfrak{s}^{F_*}_{\mathrm{ADiv}}) \ \in \ \mathbb{R}_{\geq 0}$$

$$\mathfrak{s}^{\leq}_{\mathrm{ADiv}} \stackrel{\mathrm{def}}{=} \sum_{w_{\mathbb{Q}} \in \mathbb{V}(\mathbb{Q})^{\mathrm{dst}}} \frac{\iota_{w_{\mathbb{Q}}}}{\log(p_{w_{\mathbb{Q}}})} \cdot w_{\mathbb{Q}} \ \in \ \mathrm{ADiv}_{\mathbb{R}}(\mathbb{Q})$$

$$\log(\mathfrak{s}^{\leq}_{v_{\mathbb{Q}}}) \ \stackrel{\mathrm{def}}{=} \ \underline{\deg}_{\mathbb{V}(\mathbb{Q})_{v_{\mathbb{Q}}}}(\mathfrak{s}^{\leq}_{\mathrm{ADiv}}) \ \in \ \mathbb{R}_{\geq 0}; \quad \log(\mathfrak{s}^{\leq}) \ \stackrel{\mathrm{def}}{=} \ \underline{\deg}(\mathfrak{s}^{\leq}_{\mathrm{ADiv}}) \ \in \ \mathbb{R}_{\geq 0}$$

— where we write $\mathbb{V}(F_*)_{v_{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} \mathbb{V}(F_*) \times_{\mathbb{V}_{\mathbb{Q}}} \{v_{\mathbb{Q}}\}$; we set $\iota_{w_{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} 1$ if $p_{w_{\mathbb{Q}}} \leq e^*_{\mathrm{mod}} \cdot l$, $\iota_{w_{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} 0$ if $p_{w_{\mathbb{Q}}} > e^*_{\mathrm{mod}} \cdot l$. Then one verifies immediately [again, by applying (D0); cf. also [IUTchI], Definition 3.1, (c)] that the following *conditions* on a valuation $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{non}}_{\mathbb{Q}}$ are *equivalent*:

(D4) $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{dst}}_{\mathbb{Q}}$.
(D5) The valuation $v_{\mathbb{Q}}$ *ramifies* in K.
(D6) Either $p_{v_{\mathbb{Q}}} \mid 2 \cdot 3 \cdot 5 \cdot l$ or $v_{\mathbb{Q}}$ lies in the image of $\mathrm{Supp}(\mathfrak{q}^{F_{\mathrm{tpd}}}_{\mathrm{ADiv}} + \mathfrak{d}^{F_{\mathrm{tpd}}}_{\mathrm{ADiv}})$.
(D7) Either $p_{v_{\mathbb{Q}}} \mid 2 \cdot 3 \cdot 5 \cdot l$ or $v_{\mathbb{Q}}$ lies in the image of $\mathrm{Supp}(\mathfrak{q}^{F}_{\mathrm{ADiv}} + \mathfrak{d}^{F}_{\mathrm{ADiv}})$.

Here, we observe in passing that, for $v \in \mathbb{V}(F_{\square})$,

(R1) $\log(e_v) \ \leq \ \log(2^{11} \cdot 3^3 \cdot 5 \cdot e_{\mathrm{mod}} \cdot l^4)$ if v *divides* l,
(R2) $\log(e_v) \ \leq \ \log(2^{11} \cdot 3^3 \cdot 5 \cdot e_{\mathrm{mod}} \cdot l)$ if v *divides* $2 \cdot 3 \cdot 5$ or lies in $\mathrm{Supp}(\mathfrak{q}^{F_{\square}}_{\mathrm{ADiv}})$ [hence does *not divide* l],
(R3) $\log(e_v) \ \leq \ \log(2^{11} \cdot 3^3 \cdot 5 \cdot e_{\mathrm{mod}})$ if v does *not divide* $2 \cdot 3 \cdot 5 \cdot l$ and, moreover, is *not contained* in $\mathrm{Supp}(\mathfrak{q}^{F_{\square}}_{\mathrm{ADiv}})$,

and hence that

(R4) if $e_v \geq p_v - 1 > p_v - 2$, then $p_v \;\leq\; 2^{12} \cdot 3^3 \cdot 5 \cdot e_{\mathrm{mod}} \cdot l \;=\; e^*_{\mathrm{mod}} \cdot l$, and $\log(e_v) \;\leq\; -3 + 4 \cdot \log(e^*_{\mathrm{mod}} \cdot l)$

— cf. (E3), (E4), (E5), (E6); (D0); Proposition 1.8, (v), (vii); [IUTchI], Definition 3.1, (c). Next, for later reference, we observe that the *inequality*

$$\frac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \;\leq\; \frac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(p_{v_{\mathbb{Q}}})$$

holds for *any* $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$; in particular, when $p_{v_{\mathbb{Q}}} = l$ (≥ 5), it holds that

$$\frac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \;\leq\; \frac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(p_{v_{\mathbb{Q}}}) \;\leq\; \frac{1}{2}$$

— cf. (E6). On the other hand, it follows immediately from Proposition 1.3, (i), by considering the *various possibilities* for elements $\in \mathrm{Supp}(\mathfrak{s}^{F_{\mathrm{mod}}}_{\mathrm{ADiv}})$, that

$$\log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \;\leq\; 2 \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}))$$

— and hence that

$$\frac{1}{p_{v_{\mathbb{Q}}}} \cdot \log(\mathfrak{s}^{F_{\mathrm{mod}}}_{v_{\mathbb{Q}}}) \;\leq\; \frac{2}{p_{v_{\mathbb{Q}}}} \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}_{v_{\mathbb{Q}}}))$$

— for any $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ such that $p_{v_{\mathbb{Q}}} \notin \{2, 3, 5, l\}$. In a similar vein, we conclude that

$$\begin{aligned} \log(\mathfrak{s}^{\mathbb{Q}}) \;&\leq\; 2 \cdot d_{\mathrm{mod}} \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + \log(2 \cdot 3 \cdot 5 \cdot l) \\ &\leq\; 2 \cdot d_{\mathrm{mod}} \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 5 + \log(l) \end{aligned}$$

and hence that

$$\frac{4}{l} \cdot \log(\mathfrak{s}^{\mathbb{Q}}) \;\leq\; \frac{8 \cdot d_{\mathrm{mod}}}{l} \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 6$$

— cf. (E6); the fact that $l \geq 5$. Combining this last inequality with the inequality of the final display of Step (ii) yields the *inequality*

$$(1 + \tfrac{4}{l}) \cdot \log(\mathfrak{d}^K) + \tfrac{4}{l} \cdot \log(\mathfrak{s}^{\mathbb{Q}}) \;\leq\; (1 + \tfrac{12 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2 \cdot \log(l) + 52$$

— where we apply the estimate $d_{\mathrm{mod}} \geq 1$.

(iv) In order to estimate the constant "C_Θ" of [IUTchIII], Corollary 3.12, we must, according to the various definitions given in the statement of [IUTchIII], Corollary 3.12, compute an *upper bound* for the

> *procession-normalized mono-analytic log-volume* of the *holomorphic hull* of the *union* of the *possible images* of a Θ*-pilot object*, relative to the relevant *Kummer isomorphisms* [cf. [IUTchIII], Theorem 3.11, (ii)], in the *multiradial representation* of [IUTchIII], Theorem 3.11, (i), which we regard as *subject* to the *indeterminacies* (Ind1), (Ind2), (Ind3) described in [IUTchIII], Theorem 3.11, (i), (ii).

Thus, we proceed to estimate this log-volume at *each* $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. Once one fixes $v_{\mathbb{Q}}$, this amounts to estimating the component of this log-volume in

$$\text{“}\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})\text{”}$$

[cf. the notation of [IUTchIII], Theorem 3.11, (i), (a)], for *each* $j \in \{1, \ldots, l^*\}$, which we shall also regard as an element of $\mathbb{F}_l^*$, and then computing the *average*, over $j \in \{1, \ldots, l^*\}$, of these estimates. Here, we recall [cf. [IUTchI], Proposition 6.9, (i); [IUTchIII], Proposition 3.4, (ii)] that $\mathbb{S}^{\pm}_{j+1} = \{0, 1, \ldots, j\}$. Also, we recall from [IUTchIII], Proposition 3.2, that “$\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}^{\pm}_{j+1};n,\circ}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}})$” is, by definition, a *tensor product* of $j+1$ copies, indexed by the elements of $\mathbb{S}^{\pm}_{j+1}$, of the *direct sum* of the $\mathbb{Q}$-spans of the *log-shells* associated to each of the elements of $\mathbb{V}(F_{\mathrm{mod}})_{v_{\mathbb{Q}}}$ [cf., especially, the second and third displays of [IUTchIII], Proposition 3.2]. In particular, for *each collection*

$$\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$$

of [not necessarily distinct!] elements of $\mathbb{V}(F_{\mathrm{mod}})_{v_{\mathbb{Q}}}$, we must estimate the component of the log-volume in question corresponding to the *tensor product* of the $\mathbb{Q}$-spans of the *log-shells* associated to this collection $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$ and then compute the *weighted average* [cf. the discussion of Remark 1.7.1], over possible collections $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$, of these estimates.

(v) Let $v_{\mathbb{Q}} \in \mathbb{V}^{\mathrm{dst}}_{\mathbb{Q}}$. Fix j, $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$ as in Step (iv). Write $\underline{v}_i \in \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}} = \mathbb{V}(F_{\mathrm{mod}})$ for the element corresponding to v_i. We would like to apply Proposition 1.4, (iii), to the present situation, by taking

- “I” to be $\mathbb{S}^{\pm}_{j+1}$;
- “$I^* \subseteq I$” to be the set of $i \in I$ such that $e_{\underline{v}_i} > p_{v_{\mathbb{Q}}} - 2$;
- “k_i” to be $K_{\underline{v}_i}$ [so “R_i” will be the ring of integers $\mathcal{O}_{K_{\underline{v}_i}}$ of $K_{\underline{v}_i}$];
- “$i^\dagger$” to be $j \in \mathbb{S}^{\pm}_{j+1}$;
- “λ” to be 0 if $\underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{good}}$;
- “λ” to be “ord$(-)$” of the element $\underline{\underline{q}}_{\underline{v}_j}^{j^2}$ [cf. the definition of “$\underline{\underline{q}}_{\underline{v}}$” in [IUTchI], Example 3.2, (iv)] if $\underline{v}_j \in \underline{\mathbb{V}}^{\mathrm{bad}}$.

Thus, the *inclusion* “$\phi(p^{\lambda} \cdot (R_I)^{\sim}) \subseteq p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^{\times})$” of Proposition 1.4, (iii), implies that the result of multiplying “$p^{\lfloor \lambda \rfloor - |I|} \cdot 2^{-|I|} \cdot \log_p(R_I^{\times})$” by a suitable *nonpositive* [cf. the inequalities concerning “$\mathfrak{d}_I + a_I$” that constitute the final portion of Proposition 1.4, (iii)] integer power of $p_{v_{\mathbb{Q}}}$ *contains* the “union of possible images of a Θ-pilot object” discussed in Step (iv). That is to say, the *indeterminacies* (Ind1) and (Ind2) are taken into account by the *arbitrary* nature of the automorphism “ϕ” [cf. Proposition 1.2], while the *indeterminacy* (Ind3) is taken into account by the fact that we are considering *upper bounds* [cf. the discussion of Step (x) of the proof of [IUTchIII], Corollary 3.12], together with the fact that the above-mentioned integer power of $p_{v_{\mathbb{Q}}}$ is *nonpositive*, which implies that the module obtained by multiplying by this power of $p_{v_{\mathbb{Q}}}$ *contains* “$p^{\lfloor \lambda \rfloor - |I|} \cdot 2^{-|I|} \cdot \log_p(R_I^{\times})$”. Thus, an upper bound on the component of the log-volume of the *holomorphic hull* under consideration may be obtained by computing an upper bound for the log-volume of the right-hand side of the inclusion "$p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \cdot \log_p(R_I^{\times}) \subseteq p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor - b_I} \cdot (R_I)^{\sim}$" of Proposition 1.4, (iii). Such an *upper bound*

$$\text{“}\Big(-\lambda + \mathfrak{d}_I + 1 \Big) \cdot \log(p) \ + \ \sum_{i \in I^*} \{3 + \log(e_i)\}\text{”}$$

is given in the second displayed inequality of Proposition 1.4, (iii). Here, we note that if $e_{\underline{v}_i} \leq p_{v_{\mathbb{Q}}} - 2$ for all $i \in I$, then this *upper bound* assumes the form

$$\text{“}\Big(-\lambda + \mathfrak{d}_I + 1 \Big) \cdot \log(p)\text{”}.$$

On the other hand, by (R4), if $e_{\underline{v}_i} > p_{v_{\mathbb{Q}}} - 2$ for some $i \in I$, then it follows that $p_{v_{\mathbb{Q}}} \leq e^*_{\mathrm{mod}} \cdot l$, and $\log(e_{\underline{v}_i}) \leq -3 + 4 \cdot \log(e^*_{\mathrm{mod}} \cdot l)$, so the *upper bound* in question may be taken to be

$$\text{“}\Big(-\lambda + \mathfrak{d}_I + 1 \Big) \cdot \log(p) \ + \ 4(j+1) \cdot l^*_{\mathrm{mod}}\text{”}$$

— where we write $l^*_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \log(e^*_{\mathrm{mod}} \cdot l)$. Also, we note that, unlike the other terms that appear in these *upper bounds*, "λ" is *asymmetric* with respect to the choice of "$i^\dagger \in I$" in $\mathbb{S}^{\pm}_{j+1}$. Since we would like to compute *weighted averages* [cf. the discussion of Remark 1.7.1], we thus observe that, after *symmetrizing* with respect to the choice of "$i^\dagger \in I$" in $\mathbb{S}^{\pm}_{j+1}$, this *upper bound* may be written in the form

$$\text{“}\beta_{\vec{e}}\text{”}$$

[cf. the notation of Proposition 1.7] if, in the situation of Proposition 1.7, one takes

- "E" to be $\mathbb{V}(F_{\mathrm{mod}})_{v_{\mathbb{Q}}}$;
- "n" to be $j+1$, so an element "$\vec{e} \in E^n$" corresponds precisely to a collection $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$;
- "λ_e", for an element $e \in E$ corresponding to $v \in \mathbb{V}(F_{\mathrm{mod}}) = \mathbb{V}_{\mathrm{mod}}$, to be $[(F_{\mathrm{mod}})_v : \mathbb{Q}_{v_{\mathbb{Q}}}] \in \mathbb{R}_{>0}$;
- "β_e", for an element $e \in E$ corresponding to $v \in \mathbb{V}(F_{\mathrm{mod}}) = \mathbb{V}_{\mathrm{mod}}$, to be

$$\log(\mathfrak{d}^K_v) - \tfrac{j^2}{2l(j+1)} \cdot \log(\mathfrak{q}_v) + \tfrac{1}{j+1} \cdot \log(p_{v_{\mathbb{Q}}}) + 4 \cdot \iota_{v_{\mathbb{Q}}} \cdot l^*_{\mathrm{mod}}$$

— where we recall that $\iota_{v_{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} 1$ if $p_{v_{\mathbb{Q}}} \leq e^*_{\mathrm{mod}} \cdot l$, $\quad \iota_{v_{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} 0$ if $p_{v_{\mathbb{Q}}} > e^*_{\mathrm{mod}} \cdot l$.

Here, we note that it follows immediately from the first equality of the first display of Proposition 1.7 that, after passing to *weighted averages*, the operation of *symmetrizing* with respect to the choice of "$i^\dagger \in I$" in $\mathbb{S}^{\pm}_{j+1}$ does *not affect* the computation of the *upper bound* under consideration. Thus, by applying Proposition 1.7, we obtain that the resulting **"weighted average upper bound"** is given by

$$(j+1) \cdot \log(\mathfrak{d}^K_{v_{\mathbb{Q}}}) - \tfrac{j^2}{2l} \cdot \log(\mathfrak{q}_{v_{\mathbb{Q}}}) + \log(\mathfrak{s}^{\mathbb{Q}}_{v_{\mathbb{Q}}}) + 4(j+1) \cdot l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}_{v_{\mathbb{Q}}})$$

— where we recall the notational conventions introduced in Step (iii). Thus, it remains to compute the *average over* $j \in \mathbb{F}_l^{\circledast}$. By *averaging* over $j \in \{1, \ldots, l^{\circledast} = \frac{l-1}{2}\}$ and applying (E1), (E2), we obtain the **"procession-normalized upper bound"**

$$\begin{aligned}
&\tfrac{(l^{\circledast}+3)}{2} \cdot \log(\mathfrak{d}^K_{v_\mathbb{Q}}) - \tfrac{(2l^{\circledast}+1)(l^{\circledast}+1)}{12l} \cdot \log(\mathfrak{q}_{v_\mathbb{Q}}) + \log(\mathfrak{s}^{\mathbb{Q}}_{v_\mathbb{Q}}) + 2(l^{\circledast}+3) \cdot l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}_{v_\mathbb{Q}}) \\
&\quad = \tfrac{l+5}{4} \cdot \log(\mathfrak{d}^K_{v_\mathbb{Q}}) - \tfrac{l+1}{24} \cdot \log(\mathfrak{q}_{v_\mathbb{Q}}) + \log(\mathfrak{s}^{\mathbb{Q}}_{v_\mathbb{Q}}) + (l+5) \cdot l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}_{v_\mathbb{Q}}) \\
&\quad \leq \tfrac{l+1}{4} \cdot \left\{ (1 + \tfrac{4}{l}) \cdot \log(\mathfrak{d}^K_{v_\mathbb{Q}}) - \tfrac{1}{6} \cdot \log(\mathfrak{q}_{v_\mathbb{Q}}) + \tfrac{4}{l} \cdot \log(\mathfrak{s}^{\mathbb{Q}}_{v_\mathbb{Q}}) + \tfrac{20}{3} \cdot l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}_{v_\mathbb{Q}}) \right\}
\end{aligned}$$

— where, in the passage to the final displayed inequality, we apply the estimates $\frac{1}{l+1} \leq \frac{1}{l}$ and $\frac{4(l+5)}{l+1} \leq \frac{20}{3}$, both of which may be regarded as consequences of the fact that $l \geq 5$.

(vi) Next, let $v_\mathbb{Q} \in \mathbb{V}^{\mathrm{non}}_\mathbb{Q} \setminus \mathbb{V}^{\mathrm{dst}}_\mathbb{Q}$. Fix j, $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$ as in Step (iv). Write $\underline{v}_i \in \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}} = \mathbb{V}(F_{\mathrm{mod}})$ for the element corresponding to v_i. We would like to apply Proposition 1.4, (iv), to the present situation, by taking

- "I" to be $\mathbb{S}^{\pm}_{j+1}$;
- "k_i" to be $K_{\underline{v}_i}$ [so "R_i" will be the ring of integers $\mathcal{O}_{K_{\underline{v}_i}}$ of $K_{\underline{v}_i}$].

Here, we note that our assumption that $v_\mathbb{Q} \in \mathbb{V}^{\mathrm{non}}_\mathbb{Q} \setminus \mathbb{V}^{\mathrm{dst}}_\mathbb{Q}$ implies that the hypotheses of Proposition 1.4, (iv), are satisfied. Thus, the *inclusion* "$\phi((R_I)^\sim) \subseteq (R_I)^\sim$" of Proposition 1.4, (iv), implies that the tensor product of log-shells under consideration *contains* the "union of possible images of a Θ-pilot object" discussed in Step (iv). That is to say, the *indeterminacies* (Ind1) and (Ind2) are taken into account by the *arbitrary* nature of the automorphism "ϕ" [cf. Proposition 1.2], while the *indeterminacy* (Ind3) is taken into account by the fact that we are considering *upper bounds* [cf. the discussion of Step (x) of the proof of [IUTchIII], Corollary 3.12], together with the fact that the "container of possible images" is *precisely equal* to the tensor product of log-shells under consideration. Thus, an upper bound on the component of the log-volume under consideration may be obtained by computing an upper bound for the log-volume of the right-hand side "$(R_I)^\sim$" of the above inclusion. Such an *upper bound*

$$\text{"}0\text{"}$$

is given in the final equality of Proposition 1.4, (iv). One may then compute a **"weighted average upper bound"** and then a *"procession-normalized upper bound"*, as was done in Step (v). The resulting **"procession-normalized upper bound"** is clearly equal to 0.

(vii) Next, let $v_\mathbb{Q} \in \mathbb{V}^{\mathrm{arc}}_\mathbb{Q}$. Fix j, $\{v_i\}_{i \in \mathbb{S}^{\pm}_{j+1}}$ as in Step (iv). Write $\underline{v}_i \in \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}} = \mathbb{V}(F_{\mathrm{mod}})$ for the element corresponding to v_i. We would like to apply Proposition 1.5, (iii), (iv), to the present situation, by taking

- "I" to be $\mathbb{S}^{\pm}_{j+1}$ [so $|I| = j+1$];
- "V" to be $\mathbb{V}(F_{\mathrm{mod}})_{v_\mathbb{Q}}$;

· "$\mathbb{C}_v$" to be $K_{\underline{v}}$, where we write $\underline{v} \in \underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$ for the element determined by $v \in V$.

Then it follows from Proposition 1.5, (iii), (iv), that

$$\pi^{j+1} \cdot B_I$$

serves as a *container* for the "union of possible images of a Θ-pilot object" discussed in Step (iv). That is to say, the *indeterminacies* (Ind1) and (Ind2) are taken into account by the fact that $B_I \subseteq M_I$ is *preserved* by *arbitrary automorphisms* of the type discussed in Proposition 1.5, (iii), while the *indeterminacy* (Ind3) is taken into account by the fact that we are considering *upper bounds* [cf. the discussion of Step (x) of the proof of [IUTchIII], Corollary 3.12], together with the fact that, by Proposition 1.5, (iv), together with our choice of the factor π^{j+1}, this "container of possible images" contains the elements of M_I obtained by forming the tensor product of elements of the log-shells under consideration. Thus, an upper bound on the component of the log-volume under consideration may be obtained by computing an upper bound for the log-volume of this container. Such an *upper bound*

$$(j+1) \cdot \log(\pi)$$

follows immediately from the fact that [in order to ensure compatibility with *arithmetic degrees* of arithmetic line bundles — cf. [IUTchIII], Proposition 3.9, (iii) — one is obliged to adopt normalizations which imply that] *the log-volume of B_I is equal to* 0. One may then compute a **"weighted average upper bound"** and then a *"procession-normalized upper bound"*, as was done in Step (v). The resulting **"procession-normalized upper bound"** is given by

$$\tfrac{l+5}{4} \cdot \log(\pi) \;\leq\; \tfrac{l+1}{4} \cdot 4$$

— cf. (E1), (E6); the fact that $l \geq 5$.

(viii) Now we return to the discussion of Step (iv). In order to compute the desired *upper bound* for "C_Θ", it suffices to **sum over** $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$ the various local **"procession-normalized upper bounds"** obtained in Steps (v), (vi), (vii) for $v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}$. By applying the *inequality* of the final display of Step (iii), we thus obtain the following *upper bound* for "$C_\Theta \cdot |\log(\underline{\underline{q}})|$", i.e., the product of "C_Θ" and $\frac{1}{2l} \cdot \log(\mathfrak{q})$:

$$\tfrac{l+1}{4} \cdot \left\{ (1 + \tfrac{12 \cdot d_{\mathrm{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) + 2 \cdot \log(l) + 56 - \tfrac{1}{6} \cdot (1 - \tfrac{12}{l^2}) \cdot \log(\mathfrak{q}) \right.$$
$$\left. + \tfrac{20}{3} \cdot l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}) \right\} - \tfrac{1}{2l} \cdot \log(\mathfrak{q})$$

— where we apply the estimate $\frac{l+1}{4} \cdot \frac{1}{6} \cdot \frac{12}{l^2} \geq \frac{1}{2l}$ [cf. the fact that $l \geq 1$].

Now let us recall the constant "η_{prm}" of Proposition 1.6. By applying Proposition 1.6, we compute:

$$l^*_{\mathrm{mod}} \cdot \log(\mathfrak{s}^{\leq}) \;\leq\; \log(e^*_{\mathrm{mod}} \cdot l) \cdot \sum_{p \,\leq\, e^*_{\mathrm{mod}} \cdot l} 1 \;\leq\; \tfrac{4}{3} \cdot \log(e^*_{\mathrm{mod}} \cdot l) \cdot \tfrac{e^*_{\mathrm{mod}} \cdot l}{\log(e^*_{\mathrm{mod}} \cdot l)}$$
$$= \tfrac{4}{3} \cdot e^*_{\mathrm{mod}} \cdot l$$

— where the sum ranges over the primes $p \leq e^*_{\text{mod}} \cdot l$ — if $e^*_{\text{mod}} \cdot l \geq \eta_{\text{prm}}$;

$$l^*_{\text{mod}} \cdot \log(\mathfrak{s}^{\leq}) \;\leq\; \log(e^*_{\text{mod}} \cdot l) \cdot \sum_{p \,\leq\, e^*_{\text{mod}} \cdot l} 1 \;\leq\; \tfrac{4}{3} \cdot \log(\eta_{\text{prm}}) \cdot \tfrac{\eta_{\text{prm}}}{\log(\eta_{\text{prm}})} \;=\; \tfrac{4}{3} \cdot \eta_{\text{prm}}$$

— where the sum ranges over the primes $p \leq e^*_{\text{mod}} \cdot l$ — if $e^*_{\text{mod}} \cdot l < \eta_{\text{prm}}$. Thus, we conclude that

$$l^*_{\text{mod}} \cdot \log(\mathfrak{s}^{\leq}) \;\leq\; \tfrac{4}{3} \cdot (e^*_{\text{mod}} \cdot l + \eta_{\text{prm}})$$

[i.e., regardless of the size of $e^*_{\text{mod}} \cdot l$]. Also, let us observe that

$$\tfrac{1}{3} \cdot \tfrac{4}{3} \cdot (e^*_{\text{mod}} \cdot l + \eta_{\text{prm}}) \;\geq\; \tfrac{1}{3} \cdot \tfrac{4}{3} \cdot e^*_{\text{mod}} \cdot l \;\geq\; 2 \cdot 2 \cdot 2^{12} \cdot 3 \cdot 5 \cdot l \;\geq\; 2 \cdot \log(l) + 56$$

— where we apply the estimates $e_{\text{mod}} \geq 1$, $\;2^{12} \cdot 3 \cdot 5 \geq 56$, $\;l \geq 5 \geq 1$, $\;l \geq \log(l)$ [cf. the fact that $l \geq 5$]. Thus, substituting back into our *original upper bound* for "$C_\Theta \cdot |\log(\underline{\underline{q}})|$", we obtain the following *upper bound* for "C_Θ":

$$\tfrac{l+1}{4 \cdot |\log(\underline{\underline{q}})|} \cdot \Big\{(1 + \tfrac{12 \cdot d_{\text{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\text{tpd}}}) + \log(\mathfrak{f}^{F_{\text{tpd}}})) + 10 \cdot (e^*_{\text{mod}} \cdot l + \eta_{\text{prm}}) - \tfrac{1}{6} \cdot (1 - \tfrac{12}{l^2}) \cdot \log(\mathfrak{q})\Big\} - 1$$

— where we apply the estimate $\frac{20+1}{3} \cdot \frac{4}{3} = \frac{7 \cdot 4}{3} \leq 10$ — i.e., as asserted in the statement of Theorem 1.10. The final portion of Theorem 1.10 follows immediately from [IUTchIII], Corollary 3.12, by applying the inequality of the first display of Step (ii), together with the estimates

$$(1 - \tfrac{12}{l^2})^{-1} \;\leq\; 2; \quad (1 - \tfrac{12}{l^2})^{-1} \cdot (1 + \tfrac{12 \cdot d_{\text{mod}}}{l}) \;\leq\; 1 + \tfrac{20 \cdot d_{\text{mod}}}{l}$$

[cf. the fact that $l \geq 7$, $d_{\text{mod}} \geq 1$]. $\bigcirc$

Remark 1.10.1. One of the main original motivations for the development of the theory discussed in the present series of papers was to create a *framework*, or *geometry*, within which a suitable analogue of the *scheme-theoretic Hodge-Arakelov theory* of [HASurI], [HASurII] could be realized in such a way that the *obstructions to diophantine applications* that arose in the scheme-theoretic formulation of [HASurI], [HASurII] [cf. the discussion of [HASurI], §1.5.1; [HASurII], Remark 3.7] could be *avoided*. From this point of view, it is of interest to observe that the computation of the *"leading term"* of the inequality of the final display of the statement of Theorem 1.10 — i.e., of the term

$$\tfrac{(l^{\circledast}+3)}{2} \cdot \log(\mathfrak{d}^K_{v_{\mathbb{Q}}}) - \tfrac{(2l^{\circledast}+1)(l^{\circledast}+1)}{12l} \cdot \log(\mathfrak{q}_{v_{\mathbb{Q}}})$$

that occurs in the final display of Step (v) of the proof of Theorem 1.10 — via the identities (E1), (E2) is *essentially identical* to the computation of the leading term that occurs in the proof of [HASurI], Theorem A [cf. the discussion following the statement of Theorem A in [HASurI], §1.1]. That is to say, in some sense,

> the **computations** performed in the proof of Theorem 1.10 were already essentially known to the author around the year 2000; the problem then was to construct an appropriate **framework**, or **geometry**, in which these computations could be performed!

This sort of situation may be compared to the computations underlying the **Weil Conjectures** priori to the construction of a "Weil cohomology" in which those computations could be performed, or, alternatively, to various computations of invariants in topology or differential geometry that were motivated by computations in **physics**, again prior to the construction of a suitable mathematical framework in which those computations could be performed.

Remark 1.10.2. The computation performed in the proof of Theorem 1.10 may be thought of as the computation of a sort of **derivative** in the $\mathbb{F}_l^{\circledast}$**-direction**, which, relative to the analogy between the theory of the present series of papers and the p-adic Teichmüller theory of [pOrd], [pTeich], corresponds to the *derivative* of the *canonical Frobenius lifting* — cf. the discussion of [IUTchIII], Remark 3.12.4, (iii). In this context, it is useful to recall the *arithmetic Kodaira-Spencer morphism* that occurs in scheme-theoretic Hodge-Arakelov theory [cf. [HASurII], §3]. In particular, in [HASurII], Corollary 3.6, it is shown that, when suitably formulated, a *"certain portion"* of this arithmetic Kodaira-Spencer morphism *coincides* with the usual *geometric Kodaira-Spencer morphism.* From the point of view of the *action* of $GL_2(\mathbb{F}_l)$ on the l-torsion points involved, this "certain portion" consists of the *unipotent matrices*

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

of $GL_2(\mathbb{F}_l)$. By contrast, the $\mathbb{F}_l^{\circledast}$*-symmetries* that occur in the present series of papers correspond to the *toral matrices*

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

of $GL_2(\mathbb{F}_l)$ — cf. the discussion of [IUTchI], Example 4.3, (i). As we shall see in §2 below, in the present series of papers, we shall ultimately take l to be *"large"*. When l is "sufficiently large", $GL_2(\mathbb{F}_l)$ may be thought of as a *"good approximation"* for $GL_2(\mathbb{Z})$ or $GL_2(\mathbb{R})$ — cf. the discussion of [IUTchI], Remark 6.12.3, (i), (iii). In the case of $GL_2(\mathbb{R})$, "toral subgroups" may be thought of as corresponding to the *isotropy subgroups* [isomorphic to $\mathbb{S}^1$] of points that arise from the action of $GL_2(\mathbb{R})$ on the *upper half-plane*, i.e., subgroups which may be thought of as a sort of *geometric, group-theoretic representation* of **tangent vectors** at a point.

Remark 1.10.3. The *"terms involving l"* that occur in the inequality of the final display of Theorem 1.10 may be thought of as an *inevitable consequence* of the *fundamental role* played in the theory of the present series of papers by the *l-torsion points* of the elliptic curve under consideration. Here, we note that it is of crucial importance to work over the *field of rationality* of the l-torsion points [i.e., "K" as opposed to "F"] *not only* when considering the global portions of the various ΘNF-

and $\Theta^{\pm \mathrm{ell}}$*-Hodge theaters* involved, but also when considering the local portions — i.e., the *prime-strips* — of these ΘNF- and $\Theta^{\pm \mathrm{ell}}$-Hodge theaters. That is to say, these local portions are necessary, for instance, in order to *glue together* the ΘNF- and $\Theta^{\pm \mathrm{ell}}$-Hodge theaters that appear so as to form a $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater* [cf. the discussion of [IUTchI], Remark 6.12.2]. In particular, to allow, within these local portions, any sort of *"Galois indeterminacy"* with respect to the l-torsion points — even, for instance, at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$, which, at first glance, might appear *irrelevant* to the theory of *Hodge-Arakelov-theoretic evaluation at l-torsion points* developed in [IUTchII] — would have the effect of *invalidating* the various *delicate manipulations involving l-torsion points* discussed in [IUTchI], §4, §6 [cf., e.g., [IUTchI], Propositions 4.7, 6.5].

Remark 1.10.4. The various *fluctuations in log-volume* — i.e., whose computation is the subject of Theorem 1.10! — that arise from the *multiradial representation* of [IUTchIII], Theorem 3.11, (i), may be thought of as a sort of **"inter-universal analytic torsion"**. Indeed,

> in general, **"analytic torsion"** may be understood as a sort of *measure* — in **"metrized"** [e.g., log-volume!] terms — of the degree of **deviation** of the *"holomorphic functions"* [such as sections of a line bundle] on a variety — i.e., which depend, in an essential way, on the **holomorphic moduli** of the variety! — from the *"real analytic functions"* — i.e., which are **invariant** with respect to **deformations of the holomorphic moduli** of the variety.

For instance:

(a) In *"classical" Arakelov theory*, analytic torsion typically arises as [the logarithm of] a sort of **normalized determinant** of the **Laplacian** acting on some space of real analytic [or L^2-] sections of a line bundle on a complex variety equipped with a real analytic Kähler metric [cf., e.g., [Arak], Chapters V, VI]. Here, we recall that in this sort of situation, the space of *holomorphic* sections of the line bundle is given by the kernel of the Laplacian; the definition of the *Laplacian* depends, in an essential way, on the *Kähler metric*, hence, in particular, on the *holomorphic moduli* of the variety under consideration [cf., e.g., the case of the Poincaré metric on a hyperbolic Riemann surface!].

(b) In the *scheme-theoretic Hodge-Arakelov theory* discussed in [HASurI], [HASurII], the main theorem consists of a sort of *comparison isomorphism* [cf. [HASurI], Theorem A] between a certain subspace of the space of *global sections* of the pullback of an ample line bundle on an elliptic curve to the *universal vectorial extension of the elliptic curve* and the space of *set-theoretic functions on the torsion points* of the elliptic curve. That is to say, the former space of sections contains, in a natural way, the space of *holomorphic* sections of the ample line bundle on the elliptic curve, while the latter space of functions may be thought of as a sort of "discrete approximation" of the space of *real analytic* functions on the elliptic curve [cf. the discussion of [HASurI], §1.3.2, §1.3.4]. In this context, the **"Gaussian poles"** [cf. the discussion of [HASurI], §1.1] arise as a measure of the *discrepancy of integral structures* between these two spaces in a neighborhood of the divisor at infinity of the moduli stack of elliptic curves, hence may be thought of as a sort of *"analytic torsion at the divisor at infinity"* [cf. the discussion of [HASurI], §1.2].

(c) In the case of the *multiradial representation* of [IUTchIII], Theorem 3.11, (i), the fluctuations of log-volume computed in Theorem 1.10 arise precisely as a result of the execution of a *comparison of an* **"alien" arithmetic holomorphic structure** to this **multiradial representation**, which is compatible with the *permutation symmetries of the étale-picture*, i.e., which is "invariant with respect to deformations of the arithmetic holomorphic moduli of the number field under consideration" in the sense that it makes sense *simultaneously with respect to distinct arithmetic holomorphic structures* [cf. [IUTchIII], Remark 3.11.1; [IUTchIII], Remark 3.12.3, (ii)]. Here, it is of interest to observe that the object of this comparison consists of the *values of the theta function*, i.e., in essence, a "holomorphic section of an ample line bundle". In particular, the resulting fluctuations of log-volume may be thought as a sort of *"analytic torsion"*. By analogy to the terminology *"Gaussian poles"* discussed in (b) above, it is natural to think of the terms involving the *different* $\mathfrak{d}^K_{(-)}$ that appear in the computation underlying Theorem 1.10 [cf., e.g., the final display of Step (v) of the proof of Theorem 1.10] as **"differential poles"** [cf. the discussion of Remarks 1.10.1, 1.10.2]. Finally, in the context of the *normalized determinants* that appear in (a), it is interesting to note the role played by the **prime number theorem** — i.e., in essence, the **Riemann zeta function** [cf. Proposition 1.6 and its proof] — in the computation of **"inter-universal analytic torsion"** given in the proof of Theorem 1.10.

Remark 1.10.5. The above remarks focused on the *conceptual* aspects of the theory surrounding Theorem 1.10. Before proceeding, however, we pause to discuss briefly certain aspects of Theorem 1.10 that are of interest from a **computational** point of view, i.e., in the spirit of conventional *analytic number theory*.

(i) First, we begin by observing that, unlike the inequalities that appear in the various results [cf. Corollaries 2.2, (ii); 2.3] obtained in §2 below, the inequalities obtained in Theorem 1.10 involve only **essentially explicit constants** and, moreover, *do not require one to* **exclude** *some non-explicit* **finite set** of "isomorphism classes of exceptional elliptic curves". From this point of view,

> the inequalities obtained in Theorem 1.10 are suited to application to **computations** concerning various **explicit diophantine equations**, such as, for instance, the equations that appear in "Fermat's Last Theorem".

Such explicit computations in the case of specific diophantine equations are, however, beyond the scope of the present paper.

(ii) One topic of interest in the context of computational aspects of Theorem 1.10 is the **asymptotic behavior** of the bound that appears in, say, the first inequality of the final display of Theorem 1.10. Let us assume, for simplicity, that $F_{\mathrm{tpd}} = \mathbb{Q}$ [so $d_{\mathrm{mod}} = 1$]. Also, to simplify the notation, let us write $\delta \stackrel{\mathrm{def}}{=} \log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}}) = \log(\mathfrak{f}^{F_{\mathrm{tpd}}})$. Then the bound under consideration assumes the form

$$\delta + * \cdot \tfrac{\delta}{l} + * \cdot l + *$$

— where, in the present discussion, the "$*$'s" are to be understood as denoting *fixed positive real numbers*. Thus, the *leading term* [cf. the discussion of Remark 1.10.1] is equal to δ. The remaining terms give rise to the *"ϵ terms"* [and bounded discrepancy] of the inequalities of Corollaries 2.2, (ii); 2.3, obtained in §2 below. Thus, if one ignores "bounded discrepancies", it is of interest to consider the behavior of the *"ϵ terms"*

$$* \cdot \tfrac{\delta}{l} \ + \ * \cdot l$$

as one allows the *initial* Θ*-data* under consideration to *vary* [i.e., subject to the condition "$F_{\text{tpd}} = \mathbb{Q}$"]. In this context, one **fundamental observation** is the following: although l is subject to various other conditions, no matter how "skillfully" one chooses l, the resulting "ϵ terms" are always

$$\geq \ * \cdot \delta^{1/2}$$

— an estimate that may be obtained by thinking of l as $\approx \delta^{\alpha}$, for some real number α, and comparing δ^{α} and $\delta^{1-\alpha}$. This estimate is of particular interest in the context of various explicit examples constructed by Masser and others [cf. [Mss]; the discussion of [vFr], §2] in which explicit "*abc* sums" are constructed for which the quantity on the left-hand side of the inequality of Theorem 1.10 under consideration *exceeds* the order of δ +

$$* \cdot \tfrac{\delta^{1/2}}{\log(\delta)}$$

— cf. [vFr], Equation (6). In particular, the asymptotic estimates given by Theorem 1.10 are **consistent** with the known asymptotic behavior of these explicit *abc* sums. Indeed, the exponent "$\frac{1}{2}$" that appears in the *fundamental observation* discussed above *coincides precisely* with the "expectation" expressed by van Frankenhuijsen in the final portion of the discussion of [vFr], §2! In the present paper, although we are unable to in fact achieve bounds on the "ϵ terms" of the order $* \cdot \delta^{1/2}$, we do succeed in obtaining bounds on the "ϵ terms" of the order

$$* \cdot \delta^{1/2} \cdot \log(\delta)$$

— albeit under the assumption that the *abc* sums under consideration are **compactly bounded** away from **infinity** at the prime 2, as well as at the archimedean prime [cf. Corollary 2.2, (ii); Remark 2.2.1 below for more details].

(iii) In the context of the discussion of (ii), it is of interest to observe that the "$* \cdot l$" portion of the "ϵ terms" that appear arises from the estimates given in Step (viii) of the proof of Theorem 1.10 for the quantity "$\log(\mathfrak{s}^{\leq})$". From the point of view of the discussion of [vFr], §3, this quantity corresponds essentially to a "certain portion" of the quantity "$\omega(abc)$" associated to an *abc* sum. That is to say, whereas "$\omega(abc)$" denotes the *total number of prime factors* that occur in the product *abc*, the quantity "$\log(\mathfrak{s}^{\leq})$" corresponds, roughly speaking, to the number of these prime factors that are $\leq e^{*}_{\text{mod}} \cdot l$. The appearance [i.e., in the proof of Theorem 1.10] of such a term which is closely related to "$\omega(abc)$" is of interest from the point of view of the discussion of [vFr], §3, partly since it is [*not precisely identical* to, but nonetheless] **reminiscent** of the various *refinements of the ABC Conjecture* proposed by Baker [i.e., which are the main topic of the discussion of

[vFr], §3]. The appearance [i.e., in the proof of Theorem 1.10] of such a term which is closely related to "$\omega(abc)$" is also of interest from the point of view of the explicit *abc* sums discussed in (ii) that give rise to asymptotic behavior $\geq * \cdot \frac{\delta^{1/2}}{\log(\delta)}$. That is to say, according to the discussion of [vFr], §3, Remark 1, this sort of *abc* sum tends to give rise to a

> **relatively large** value for $\omega(abc)$ — i.e., a state of affairs that is consistent with the *crucial role* played by the "ϵ term" related to $\omega(abc)$ in the computation of the lower bound "$\geq \ * \cdot \delta^{1/2}$" that appears in the *fundamental observation* of (ii).

By contrast, the *abc* sums of the form "$2^n = p + qr$" [where p, q, and r are prime numbers] considered in [vFr], §3, Remark 1, give rise to a

> **relatively small** value for $\omega(abc)$ [indeed, $\omega(abc) \leq 4$] — i.e., a situation that suggests *relatively small/essentially negligible* "ϵ terms" in the bound of Theorem 1.10 under consideration.

Such essentially negligible "ϵ terms" are, however, *consistent* with the fact [cf. [vFr], §3, Remark 1] that, for such *abc* sums, the left-hand side of the inequality of Theorem 1.10 under consideration is roughly $\approx \frac{1}{2}\cdot$ *the leading term* of the bound on the right-hand side, hence, in particular, is *amply bounded by the leading term* on the right-hand side, *without any "help" from the "ϵ terms"*.

Remark 1.10.6.

(i) In the context of the discussion of Remark 1.10.5, it is important to remember that

> the bound on "$\frac{1}{6} \cdot \log(\mathfrak{q})$" given in Theorem 1.10 only concerns the *q-parameters* at the nonarchimedean valuations contained in $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$, all of which are necessarily of **odd residue characteristic**

— cf. [IUTchI], Definition 3.1, (b). This observation is of relevance to the examples of *abc* sums constructed in [Mss] [cf. the discussion of Remark 1.10.5, (ii)], since it does not appear, at first glance, that there is any way to effectively control the *contributions at the prime* 2 in these examples, that is to say, in the notation of the Proposition of [Mss], to control the power of 2 that divides the integer "c" of the Proposition of [Mss], or, alternatively, in the notation of the proof of this Proposition on [Mss], p. 22, to control the power of 2 that divides the difference "$x_i - x_{i-1}$". On the other hand, it was pointed out to the author by *A. Venkatesh* that in fact it is not difficult to modify the construction of these examples of *abc* sums given in [Mss] so as to obtain *similar asymptotic estimates* to those obtained in [Mss] [cf. the discussion of Remark 1.10.5, (ii)], *even without taking into account the contributions at the prime* 2.

(ii) In the context of the discussion of (i), it is of interest to recall **why** nonarchimedean primes of **even** *residue characteristic* where the elliptic curve under consideration has bad multiplicative reduction are *excluded* from $\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}$ in the theory of the present series of papers. In a word, the reason that the theory encounters difficulties at primes over 2 is that it depends, in a quite essential way, on the theory of the **étale theta function** developed in [EtTh], which *fails* at primes over 2 [cf. the assumption that "*p is odd*" in [EtTh], Theorem 1.10, (iii); [EtTh], Definition 2.5; [EtTh], Corollary 2.18]. From the point of view of the theory of [IUTchI], [IUTchII], and [IUTchIII] [cf., especially, the theory of [IUTchII], §1, §2: [IUTchII], Corollary 1.12; [IUTchII], Corollary 2.4, (ii), (iii); [IUTchII], Corollary 2.6], one of the *key consequences* of the theory of [EtTh] is the **simultaneous multiradiality** of the algorithms that give rise to

(1) **constant multiple rigidity** and
(2) **cyclotomic rigidity**.

At a more concrete level, (1) is obtained by **evaluating** the usual series for the theta function [cf. [EtTh], Proposition 1.4] at the 2-torsion point in the "irreducible component labeled zero". One computes easily that the resulting "special value" is a **unit** for *odd* p, but is equal to a [nonzero] **non-unit** when $p = 2$. In particular, since (1) is established by *dividing* the series of [EtTh], Proposition 1.4 [i.e., the usual series for the theta function], by this special value, it follows that

(a) the *"integral structure"* on the theta function determined by this special value

coincides with

(b) the *"integral structure"* on the theta function determined by the natural integral structure on the pole at the origin

for **odd** p [cf. [EtTh], Theorem 1.10, (iii)], but **not** when $p = 2$. That is to say, when $p = 2$, a *nontrivial denominator* arises. Here, we recall that it is crucial to evaluate at 2-torsion points, i.e., as opposed to, say, more general points in the irreducible component labeled zero for reasons discussed in [IUTchII], Remark 2.5.1, (ii) [cf. also the discussion of [IUTchII], Remark 1.12.2, (i), (ii), (iii), (iv)]. This nontrivial denominator is fundamentally incompatible with the *multiradiality* of the algorithms of (1), (2) in that it is incompatible with the *fundamental splitting*, or *"decoupling"*, into *"purely radial"* [i.e., roughly speaking, "value group"] and *"purely coric"* [i.e., roughly speaking, "unit"] components discussed in [IUTchII], Remarks 1.11.4, (i); 1.12.2, (vi) [cf. also the discussion of [IUTchII], Remark 1.11.5]. That is to say, on the one hand,

> the **multiradiality of (1)** may only be established if the possible values at the evaluation points in the irreducible component labeled zero are known, a priori, to be **units**, i.e., if one works relative to the **integral structure (a)**

— cf. the discussion of [IUTchII], Remark 1.12.2, (i), (ii), (iii), (iv). On the other hand, if one tries to work

> **simultaneously** with the **integral structure (b)**, hence with the **nontrivial denominator** discussed above, then the **multiradiality of (2)** is **violated**.

Here, we recall that the *integral structure (b)*, which is referred to as the *"canonical integral structure"* in [EtTh], Proposition 1.4, (iii); [EtTh], Theorem 1.10, (iii), is in some sense the "integral structure of common sense".

(iii) It is not entirely clear to the author at the time of writing to what extent the *integral structure (b)* is *necessary* in order to carry out the theory developed in the present series of papers. Indeed, [EtTh], as well as the present series of papers, was written in a way that [unlike the discussion of (ii)!] "takes for granted" the fact that the two integral structures (a), (b) discussed above *coincide* for *odd* p, i.e., in a way which *identifies* these two integral structures and hence does not *specify*, at various key points in the discussion, whether one is in fact working with integral structure (a) or with integral structure (b). On the other hand, if it is indeed the case that *not only* the integral structure (a), *but also* the integral structure (b) plays an *essential role* in the present series of papers, then it follows [cf. the discussion of (ii)!] that the theory of the present series of papers is **fundamentally incompatible** with the inclusion in $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$ of nonarchimedean primes of **even** *residue characteristic* where the elliptic curve under consideration has bad multiplicative reduction.

(iv) In the context of the discussion of (ii), (iii), it is perhaps useful to recall that the classical theory of theta functions also tends to [depending on your point of view!] *"break down"* or *"assume a completely different form"* at the prime 2. For instance, this phenomenon can be seen throughout Mumford's theory of *algebraic theta functions*, which may be thought of as a sort of predecessor to the *scheme-theoretic Hodge-Arakelov theory* of [HASurI], [HASurII], which, in turn, may be thought of as a sort of predecessor to the theory of the present series of papers. In a similar vein, it is of interest to recall that the prime 2 is also *excluded* in the *p-adic Teichmüller theory* of [pOrd], [pTeich]. This is done in order to avoid the complications that occur in the theory of the Lie algebra sl_2 over fields of characteristic 2.

Remark 1.10.7.

(i) Since $e^*_{\mathrm{mod}} \leq d^*_{\mathrm{mod}}$, one may *replace* "$e^*_{\mathrm{mod}}$" by "$d^*_{\mathrm{mod}}$" in the final two displays of the statement of Theorem 1.10.

(ii) By contrast, at least if one adheres to the framework of the theory of the present series of papers,

> it is **not** possible to **replace** "d_{mod}" by "e_{mod}" in the final two displays of the statement of Theorem 1.10.

The fundamental reason for this is that, in the construction of the **multiradial representation** of [IUTchIII], Theorem 3.11, (i), it is necessary to consider tensor products of copies, labeled by $j \in \mathbb{F}^{\circledast}_l$, of F_{mod} over $\mathbb{Q}$ [cf. [IUTchIII], Proposition

3.3!]. That is to say, it is *fundamentally impossible* [i.e., relative to the framework of the theory of the present series of papers] to *identify* the F_{mod}-linear structures for distinct labels j, since the various tensor packets that appear in the multiradial representation must be constructed in such a way as to depend only on the **additive structure** [i.e., not the module structure over some sort of *ring* such as F_{mod}!] of the **[mono-analytic!] log-shells** involved. Working with tensor powers of copies of F_{mod} over $\mathbb{Q}$ means that there is *no way to avoid*, when one localizes at a prime number p, working with tensor products between localizations of F_{mod} at *distinct primes* of F_{mod} that divide p. Moreover, whenever *even one* of these primes of F_{mod} is lies under a prime of K that *ramifies* over $\mathbb{Q}$ [cf. condition (D5) of Step (iii) of the proof of Theorem 1.10], the computation of Step (v) of the proof of Theorem 1.10 necessarily gives rise to a "$\log(p)$" term — i.e., that appears in "$\log(\mathfrak{s}^{\mathbb{Q}})$" — that arises from *"rounding up" non-integral powers of* p [i.e., as in the inclusions of Proposition 1.4, (iii)], since only *integral powers* of p make sense in the *multiradial representation*. That is to say, whereas integral powers of p only require the use of the *additive structure* of the **[mono-analytic!] log-shells** involved, *non-integral powers* only make sense if one is equipped with the module structure over some sort of *ring* such as F_{mod}!

Section 2: Diophantine Inequalities

In the present §2, we combine Theorem 1.10 with the theory of [GenEll] to give a proof of the **ABC Conjecture**, or, equivalently, **Vojta's Conjecture for hyperbolic curves** [cf. Corollary 2.3 below].

We begin by reviewing some well-known estimates.

Proposition 2.1. (Well-known Estimates)

(i) **(Linearization of Logarithms)** *We have* $\log(x) \leq x$ *for all* $(\mathbb{R} \ni)$ $x \geq 1$.

(ii) **(The Prime Number Theorem)** *There exists a real number* $\xi_{\mathrm{prm}} \geq 5$ *such that*

$$\frac{2}{3} \cdot x \quad \leq \quad \theta(x) \stackrel{\mathrm{def}}{=} \sum_{p \leq x} \log(p) \quad \leq \quad \frac{4}{3} \cdot x$$

— where the sum ranges over the prime numbers p *such that* $p \leq x$ *— for all* $(\mathbb{R} \ni)$ $x \geq \xi_{\mathrm{prm}}$. *In particular, if* $\mathcal{A}$ *is a finite set of prime numbers, and we write*

$$\theta_{\mathcal{A}} \stackrel{\mathrm{def}}{=} \sum_{p \in \mathcal{A}} \log(p)$$

[where we take the sum to be 0 *if* $\mathcal{A} = \emptyset$*], then there exists a prime number* $p \notin \mathcal{A}$ *such that* $p \leq 2(\theta_{\mathcal{A}} + \xi_{\mathrm{prm}})$.

Proof. Assertion (i) is well-known and entirely elementary. Assertion (ii) is a well-known consequence of the *Prime Number Theorem* [cf., e.g., [Edw], p. 76; [GenEll], Lemma 4.1; [GenEll], Remark 4.1.1]. ○

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. In the following discussion, we shall apply the *notation* and *terminology* of [GenEll]. Let X be a smooth, proper, geometrically connected curve over a number field; $D \subseteq X$ a reduced divisor; $U_X \stackrel{\mathrm{def}}{=} X \backslash D$; d a positive integer. Write ω_X for the canonical sheaf on X. Suppose that U_X is a *hyperbolic curve*, i.e., that the degree of the line bundle $\omega_X(D)$ is *positive*. Then we recall the following notation:

- $U_X(\overline{\mathbb{Q}})^{\leq d} \subseteq U_X(\overline{\mathbb{Q}})$ denotes the subset of $\overline{\mathbb{Q}}$-rational points *defined over a finite extension field* of $\mathbb{Q}$ of *degree* $\leq d$ [cf. [GenEll], Example 1.3, (i)].

- log-diff_X denotes the *[normalized] log-different* function on $U_X(\overline{\mathbb{Q}})$ [cf. [GenEll], Definition 1.5, (iii)].

- log-cond_D denotes the *[normalized] log-conductor* function on $U_X(\overline{\mathbb{Q}})$ [cf. [GenEll], Definition 1.5, (iv)].

- $\mathrm{ht}_{\omega_X(D)}$ denotes the *[normalized] height* function on $U_X(\overline{\mathbb{Q}})$ associated to $\omega_X(D)$, which is well-defined up to a *"bounded discrepancy"* [cf. [GenEll], Proposition 1.4, (iii)].

In order to apply the theory of the present series of papers, it is neceesary to *construct suitable initial* Θ*-data*, as follows.

Corollary 2.2. **(Construction of Suitable Initial Θ-Data)** *Suppose that $X = \mathbb{P}^1_{\mathbb{Q}}$ is the* **projective line** *over $\mathbb{Q}$, and that $D \subseteq X$ is the divisor consisting of the* **three points** *"0", "1", and "∞". We shall regard X as the* **"λ-line"** *— i.e., we shall regard the standard coordinate on $X = \mathbb{P}^1_{\mathbb{Q}}$ as the "λ" in the* **Legendre form** *"$y^2 = x(x-1)(x-\lambda)$" of the Weierstrass equation defining an elliptic curve — and hence as being equipped with a natural* **classifying morphism** *$U_X \to (\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}}$ [cf. the discussion preceding Proposition 1.8]. Let*

$$\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$$

be a **compactly bounded subset** *[i.e., regarded as a subset of $X(\overline{\mathbb{Q}})$ — cf. Remark 2.3.1, (vi), below; [GenEll], Example 1.3, (ii)] whose* **support** *contains the nonarchimedean prime "2". Suppose further that $\mathcal{K}_V$ satisfies the following condition:*

($*_{j\text{-inv}}$) *If $v \in \mathbb{V}(\mathbb{Q})$ denotes the nonarchimedean prime "2", then the image of the subset $\mathcal{K}_v \subseteq U_X(\overline{\mathbb{Q}}_v)$ associated to $\mathcal{K}_V$ [cf. the notational conventions of [GenEll], Example 1.3, (ii)] via the* ***j*-invariant** *$U_X \to (\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}} \to \mathbb{A}^1_{\mathbb{Q}}$ is a* **bounded** *subset of $\mathbb{A}^1_{\mathbb{Q}}(\overline{\mathbb{Q}}_v) = \overline{\mathbb{Q}}_v$, i.e., is contained in a subset of the form $2^{N_{j\text{-inv}}} \cdot \mathcal{O}_{\overline{\mathbb{Q}}_v} \subseteq \overline{\mathbb{Q}}_v$, where $N_{j\text{-inv}} \in \mathbb{Z}$, and $\mathcal{O}_{\overline{\mathbb{Q}}_v} \subseteq \overline{\mathbb{Q}}_v$ denotes the ring of integers.*

Then:

(i) Write "$\log(\mathfrak{q}^{\forall}_{(-)})$" (respectively, "$\log(\mathfrak{q}^{\dagger 2}_{(-)})$") for the $\mathbb{R}$-valued function on $\mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$, hence also on $U_X(\overline{\mathbb{Q}})$, obtained by forming the normalized degree "$\underline{\deg}(-)$" of the effective arithmetic divisor determined by the ***q*-parameters** *of an elliptic curve over a number field at* **arbitrary** *nonarchimedean primes (respectively, at the nonarchimedean primes that do* **not divide** *2) [cf. the invariant "$\log(\mathfrak{q})$" associated, in the statement of Theorem 1.10, to the elliptic curve E_F]. Also, we shall write ht_∞ for the* **[normalized] height** *function on $U_X(\overline{\mathbb{Q}})$ — a function which is well-defined up to a* **"bounded discrepancy"** *[cf. the discussion preceding [GenEll], Proposition 3.4] — determined by the pull-back to X of the divisor at infinity of the natural compactification $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Q}}$ of $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}}$. Then we have an* **equality of "bounded discrepancy classes"** *[cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), below]*

$$\tfrac{1}{6} \cdot \log(\mathfrak{q}^{\dagger 2}_{(-)}) \;\approx\; \tfrac{1}{6} \cdot \log(\mathfrak{q}^{\forall}_{(-)}) \;\approx\; \tfrac{1}{6} \cdot \mathrm{ht}_\infty \;\approx\; \mathrm{ht}_{\omega_X(D)}$$

of functions on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$.

(ii) There exist

- *a positive real number H_{unif} which is* **independent** *of $\mathcal{K}_V$ and*
- *positive real numbers $C_{\mathcal{K}}$ and $H_{\mathcal{K}}$ which* **depend only** *on the choice of the* **compactly bounded subset** *$\mathcal{K}_V$*

such that the following property is satisfied: Let d be a positive integer, ϵ_d a positive real number ≤ 1. Set $\delta \stackrel{\mathrm{def}}{=} 2^{12} \cdot 3^3 \cdot 5 \cdot d$. Then there exists a **finite** *subset $\mathfrak{Exc}_d \subseteq U_X(\overline{\mathbb{Q}})^{\leq d}$ which depends only on $\mathcal{K}_V$, d, and ϵ_d, contains all points corresponding to elliptic curves that admit automorphisms of order > 2, and satisfies the following property:*

The function "$\log(\mathfrak{q}^{\forall}_{(-)})$" of (i) is

$$\leq \ H_{\mathrm{unif}} \cdot \epsilon_d^{-3} \cdot d^{4+\epsilon_d} + H_{\mathcal{K}}$$

on $\mathfrak{Exc}_d$. Let E_F be an **elliptic curve** *over a number field $F \subseteq \overline{\mathbb{Q}}$ that determines a $\overline{\mathbb{Q}}$-valued point of $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}}$ which lifts [not necessarily uniquely!] to a point $x_E \in U_X(F) \bigcap U_X(\overline{\mathbb{Q}})^{\leq d}$ such that*

$$x_E \in \mathcal{K}_V, \quad x_E \notin \mathfrak{Exc}_d.$$

Write F_{mod} for the **minimal field of definition** *of the corresponding point $\in \mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$ and*

$$F_{\mathrm{mod}} \quad \subseteq \quad F_{\mathrm{tpd}} \stackrel{\mathrm{def}}{=} F_{\mathrm{mod}}(\ E_{F_{\mathrm{mod}}}[2]\ ) \quad \subseteq \quad F$$

for the **"tripodal"** *intermediate field obtained from F_{mod} by adjoining the fields of definition of the 2-torsion points of any model of $E_F \times_F \overline{\mathbb{Q}}$ over F_{mod} [cf. Proposition 1.8, (ii), (iii)]. Moreover, we assume that the $(3 \cdot 5)$-torsion points of E_F are defined over F, and that*

$$F \quad = \quad F_{\mathrm{mod}}(\ \sqrt{-1},\ E_{F_{\mathrm{mod}}}[2 \cdot 3 \cdot 5]\ ) \quad \stackrel{\mathrm{def}}{=} \quad F_{\mathrm{tpd}}(\ \sqrt{-1},\ E_{F_{\mathrm{tpd}}}[3 \cdot 5]\ )$$

— i.e., that F is obtained from F_{tpd} by adjoining $\sqrt{-1}$, together with the fields of definition of the $(3 \cdot 5)$-torsion points of a model $E_{F_{\mathrm{tpd}}}$ of the elliptic curve $E_F \times_F \overline{\mathbb{Q}}$ over F_{tpd} determined by the **Legendre form** *of the Weierstrass equation discussed above [cf. Proposition 1.8, (vi)]. [Thus, it follows from Proposition 1.8, (iv), that $E_F \cong E_{F_{\mathrm{tpd}}} \times_{F_{\mathrm{tpd}}} F$ over F, so $x_E \in U_X(F_{\mathrm{tpd}}) \subseteq U_X(F)$; it follows from Proposition 1.8, (v), that E_F has* **stable reduction** *at every element of $\mathbb{V}(F)^{\mathrm{non}}$.] Write $\log(\mathfrak{q}^{\forall})$ (respectively, $\log(\mathfrak{q}^{\dagger 2})$) for the result of applying the function "$\log(\mathfrak{q}^{\forall}_{(-)})$" (respectively, "$\log(\mathfrak{q}^{\dagger 2}_{(-)})$") of (i) to x_E. Then E_F and F_{mod} arise as the "E_F" and "F_{mod}" for a collection of* **initial Θ-data** *as in Theorem 1.10 that, in the notation of Theorem 1.10, satisfies the following conditions:*

(C1) $(\log(\mathfrak{q}^{\forall}))^{1/2} \ \leq \ l \ \leq \ 10\delta \cdot (\log(\mathfrak{q}^{\forall}))^{1/2} \cdot \log(2\delta \cdot \log(\mathfrak{q}^{\forall}))$*;*
(C2) we have inequalities

$$\begin{aligned} \tfrac{1}{6} \cdot \log(\mathfrak{q}) \ \leq \ \tfrac{1}{6} \cdot \log(\mathfrak{q}^{\dagger 2}) \ &\leq \ \tfrac{1}{6} \cdot \log(\mathfrak{q}^{\forall}) \\ &\leq \ (1 + \epsilon_E) \cdot (\text{log-diff}_X(x_E) + \text{log-cond}_D(x_E)) + C_{\mathcal{K}} \end{aligned}$$

— where we write

$$\epsilon_E \ \stackrel{\mathrm{def}}{=} \ (60\delta)^2 \cdot \frac{\log(2\delta \cdot (\log(\mathfrak{q}^{\forall})))}{(\log(\mathfrak{q}^{\forall}))^{1/2}}$$

[i.e., so ϵ_E **depends** *on the integer* d*, as well as on the elliptic curve* E_F*!], and we observe, relative to the notation of Theorem 1.10, that [it follows tautologically from the definitions that] we have an equality* $\text{log-diff}_X(x_E) = \log(\mathfrak{d}^{F_{\text{tpd}}})$*, as well as inequalities*

$$\log(\mathfrak{f}^{F_{\text{tpd}}}) \leq \text{log-cond}_D(x_E) \leq \log(\mathfrak{f}^{F_{\text{tpd}}}) + \log(2l).$$

(iii) The positive real number H_{unif} *of (ii) [which is* **independent** *of* $\mathcal{K}_V$*!] may be chosen in such a way that the following property is satisfied: Let* d *be a positive integer,* ϵ_d *and* ϵ *positive real numbers* ≤ 1*. Then there exists a* **finite** *subset* $\mathfrak{Exc}_{\epsilon,d} \subseteq U_X(\overline{\mathbb{Q}})^{\leq d}$ *which depends only on* $\mathcal{K}_V$*,* ϵ*,* d*, and* ϵ_d *such that, in the notation of (ii), the function "*$\log(\mathfrak{q}^{\forall}_{(-)})$*" of (i) is*

$$\leq \; H_{\text{unif}} \cdot \epsilon^{-3} \cdot \epsilon_d^{-3} \cdot d^{4+\epsilon_d} + H_{\mathcal{K}}$$

on $\mathfrak{Exc}_{\epsilon,d}$*, and, moreover, the invariant* ϵ_E *associated to an elliptic curve* E_F *as in (ii) [i.e., that satisfies certain conditions which* **depend** *on* $\mathcal{K}_V$ *and* d*] satisfies the inequality* $\epsilon_E \leq \epsilon$ *whenever the point* $x_E \in U_X(F)$ *satisfies the condition* $x_E \notin \mathfrak{Exc}_{\epsilon,d}$*.*

Proof. First, we consider assertion (i). We begin by observing that, in light of the *condition* $(*_{j\text{-inv}})$ that was imposed on $\mathcal{K}_V$, it follows immediately from the various definitions involved that

$$\log(\mathfrak{q}^{\dagger 2}_{(-)}) \;\approx\; \log(\mathfrak{q}^{\forall}_{(-)})$$

— where we observe that the function "$\log(\mathfrak{q}^{\forall}_{(-)})$" may be *identified* with the function "$\underline{\deg}_{\infty}$" of the discussion preceding [GenEll], Proposition 3.4 — on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$. In a similar vein, since the *support* of $\mathcal{K}_V$ contains the unique archimedean prime of $\mathbb{Q}$, it follows immediately from the various definitions involved [cf. also Remark 2.3.1, (vi), below] that

$$\log(\mathfrak{q}^{\forall}_{(-)}) \;\approx\; \text{ht}_{\infty}$$

on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$ [cf. the argument of the final paragraph of the proof of [GenEll], Lemma 3.7]. Thus, we conclude that $\log(\mathfrak{q}^{\dagger 2}_{(-)}) \approx \log(\mathfrak{q}^{\forall}_{(-)}) \approx \text{ht}_{\infty}$ on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$. Finally, since [as is well-known] the pull-back to X of the divisor at infinity of the natural compactification $(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Q}}$ of $(\mathcal{M}_{\text{ell}})_{\mathbb{Q}}$ is of *degree* 6, while the line bundle $\omega_X(D)$ is of *degree* 1, the equality of BD-classes $\frac{1}{6} \cdot \text{ht}_{\infty} \approx \text{ht}_{\omega_X(D)}$ on $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$ follows immediately from [GenEll], Proposition 1.4, (i), (iii). This completes the proof of assertion (i).

Next, we consider assertion (ii). First, let us recall that if the once-punctured elliptic curve associated to E_F *fails to admit an* F*-core*, then there are only *four possibilities* for the j-invariant of E_F [cf. [CanLift], Proposition 2.7]. Thus, if we take the set $\mathfrak{Exc}_d$ to be the [*finite*!] collection of points corresponding to these four j-invariants, then we may assume that the once-punctured elliptic curve associated to E_F *admits an* F*-core*, hence, in particular, does not have any automorphisms of order > 2 over $\overline{\mathbb{Q}}$. In the discussion to follow, it will be necessary to *enlarge* the finite set $\mathfrak{Exc}_d$ several times, always in a fashion that *depends only on* $\mathcal{K}_V$, d, and ϵ_d [i.e., but not on x_E!] and in such a way that the function "$\log(\mathfrak{q}^{\forall}_{(-)})$" of (i) is $\leq H_{\text{unif}} \cdot \epsilon_d^{-3} \cdot d^{4+\epsilon_d} + H_{\mathcal{K}}$ on $\mathfrak{Exc}_d$ for some positive real number H_{unif} that is *independent* of $\mathcal{K}_V$ and some positive real number $H_{\mathcal{K}}$ that *depends only on* $\mathcal{K}_V$ [i.e., but not on d or ϵ_d!].

Next, let us write

$$h \stackrel{\text{def}}{=} \log(\mathfrak{q}^{\forall}) \ = \ \tfrac{1}{[F:\mathbb{Q}]} \cdot \sum_{v \in \mathbb{V}(F)^{\text{non}}} h_v \cdot f_v \cdot \log(p_v)$$

— that is to say, $h_v = 0$ for those v at which E_F has *good reduction*; $h_v \in \mathbb{N}_{\geq 1}$ is the *local height* of E_F [cf. [GenEll], Definition 3.3] for those v at which E_F has *bad multiplicative reduction*. Now it follows [by assertion (i); [GenEll], Proposition 1.4, (iv)] that the inequality $h^{1/2} \ < \ \xi_{\text{prm}}$ [cf. the notation of Proposition 2.1, (ii)] implies that there is only a *finite number of possibilities* for the j-invariant of E_F. Thus, by possibly *enlarging* the finite set $\mathfrak{Exc}_d$ [in a fashion that *depends only on* $\mathcal{K}_V$, d, and ϵ_d and in such a way that $h \leq H_{\text{unif}}$ on $\mathfrak{Exc}_d$ for some positive real number H_{unif} that is *independent* of $\mathcal{K}_V$], we may assume without loss of generality that the inequality

$$h^{1/2} \ \geq \ \xi_{\text{prm}} \ \geq \ 5$$

holds. Thus, since $[F : \mathbb{Q}] \leq \delta$ [cf. the properties (E3), (E4), (E5) in the proof of Theorem 1.10], it follows that

$$\begin{aligned}\delta \cdot h^{1/2} \ &\geq \ [F:\mathbb{Q}] \cdot h^{1/2} \ = \ \sum_{v} h^{-1/2} \cdot h_v \cdot f_v \cdot \log(p_v) \ \geq \ \sum_{v} h^{-1/2} \cdot h_v \cdot \log(p_v) \\ &\geq \ \sum_{h_v \ \geq \ h^{1/2}} h^{-1/2} \cdot h_v \cdot \log(p_v) \ \geq \ \sum_{h_v \ \geq \ h^{1/2}} \log(p_v)\end{aligned}$$

and

$$\begin{aligned}2\delta \cdot h^{1/2} \cdot \log(2\delta \cdot h) \ &\geq \ 2 \cdot [F:\mathbb{Q}] \cdot h^{1/2} \cdot \log(2 \cdot [F:\mathbb{Q}] \cdot h) \\ &\geq \ \sum_{h_v \neq 0} 2 \cdot h^{-1/2} \cdot \log(2 \cdot h_v \cdot f_v \cdot \log(p_v)) \cdot h_v \cdot f_v \cdot \log(p_v) \\ &\geq \ \sum_{h_v \neq 0} h^{-1/2} \cdot \log(h_v) \cdot h_v \ \geq \ \sum_{h_v \ \geq \ h^{1/2}} h^{-1/2} \cdot \log(h_v) \cdot h_v \\ &\geq \ \sum_{h_v \ \geq \ h^{1/2}} \log(h_v)\end{aligned}$$

— where the sums are all over $v \in \mathbb{V}(F)^{\text{non}}$ [possibly subject to various conditions, as indicated], and we apply the elementary estimate $2 \cdot \log(p_v) \ \geq \ 2 \cdot \log(2) \ = \ \log(4) \ \geq \ 1$ [cf. the property (E6) in the proof of Theorem 1.10].

Thus, in summary, we conclude from the estimates made above that if we take

$$\mathcal{A}$$

to be the [*finite*!] set of prime numbers p such that p either

(S1) is $\leq h^{1/2}$,
(S2) *divides a nonzero* h_v for some $v \in \mathbb{V}(F)^{\text{non}}$, or
(S3) is *equal to* p_v for some $v \in \mathbb{V}(F)^{\text{non}}$ for which $h_v \geq h^{1/2}$,

then it follows from Proposition 2.1, (ii), together with our assumption that $h^{1/2} \geq \xi_{\text{prm}}$, that, in the notation of Proposition 2.1, (ii),

$$\begin{aligned}\theta_{\mathcal{A}} &\leq 2 \cdot h^{1/2} + \delta \cdot h^{1/2} + 2\delta \cdot h^{1/2} \cdot \log(2\delta \cdot h) \\ &\leq 4\delta \cdot h^{1/2} \cdot \log(2\delta \cdot h) \\ &\leq -\xi_{\text{prm}} + 5\delta \cdot h^{1/2} \cdot \log(2\delta \cdot h)\end{aligned}$$

— where we apply the estimates $\delta \geq 2$ and $\log(2\delta \cdot h) \geq \log(4) \geq 1$ [cf. the property (E6) in the proof of Theorem 1.10]. In particular, it follows from Proposition 2.1, (i), (ii), together with our assumption that $h^{1/2} \geq 5 \geq 1$, that there exists a *prime number* l such that

(P1) $(5 \leq)$ $h^{1/2} \leq l \leq 10\delta \cdot h^{1/2} \cdot \log(2\delta \cdot h)$ $(\leq 20 \cdot \delta^2 \cdot h^2)$ [cf. the condition (C1) in the statement of Corollary 2.2];
(P2) l does *not divide* any nonzero h_v for $v \in \mathbb{V}(F)^{\text{non}}$;
(P3) if $l = p_v$ for some $v \in \mathbb{V}(F)^{\text{non}}$, then $h_v < h^{1/2}$.

Next, let us *observe* that, again by possibly *enlarging* the finite set $\mathfrak{Exc}_d$ [in a fashion that *depends only on* $\mathcal{K}_V$, d, and ϵ_d and in such a way that $h \leq H_{\mathcal{K}}$ on $\mathfrak{Exc}_d$ for some positive real number $H_{\mathcal{K}}$ that *depends only on* $\mathcal{K}_V$], we may assume without loss of generality that, in the terminology of [GenEll], Lemma 3.5,

(P4) E_F does *not* admit an *l-cyclic* subgroup scheme.

Indeed, the existence of an l-cyclic subgroup scheme of E_F would imply that

$$\tfrac{l-2}{24} \cdot \log(\mathfrak{q}^{\forall}) \leq 2 \cdot \log(l) + T_{\mathcal{K}}$$

— where we apply assertion (i), (P2), the displayed inequality of [GenEll], Lemma 3.5, and the final inequality of the display of [GenEll], Proposition 3.4; we take the "ϵ" of [GenEll], Lemma 3.5, to be 1; we write $T_{\mathcal{K}}$ for the positive real number [which depends only on the choice of the *compactly bounded subset* $\mathcal{K}_V$] that results from the various *"bounded discrepancies"* implicit in these inequalities. Since $l \geq 5$ [cf. (P1)], it follows that $1 \leq 2 \cdot \log(l) \leq 48 \cdot \frac{l-2}{24}$ [cf. the property (E6) in the proof of Theorem 1.10], and hence that the inequality of the preceding display implies that $\log(\mathfrak{q}^{\forall})$ is *bounded*. On the other hand, [by assertion (i); [GenEll], Proposition 1.4, (iv)] this implies that there is only a *finite number of possibilities* for the j-invariant of E_F. This completes the proof of the above *observation*.

Next, let us note that it follows immediately from (P1), together with Proposition 2.1, (i), that

$$\begin{aligned}h^{1/2} \cdot \log(l) &\leq h^{1/2} \cdot \log(20 \cdot \delta^2 \cdot h^2) \leq 2 \cdot h^{1/2} \cdot \log(5\delta \cdot h) \\ &\leq 8 \cdot h^{1/2} \cdot \log(2 \cdot \delta^{1/4} \cdot h^{1/4}) \leq 8 \cdot h^{1/2} \cdot 2 \cdot \delta^{1/4} \cdot h^{1/4} \\ &= 16 \cdot \delta^{1/4} \cdot h^{3/4}\end{aligned}$$

— where we apply the estimates $20 \leq 5^2$ and $5 \leq 2^4$. In particular, we *observe* that, again by possibly *enlarging* the finite set $\mathfrak{Exc}_d$ [in a fashion that *depends only on* $\mathcal{K}_V$, d, and ϵ_d and in such a way that $h \leq H_{\text{unif}} \cdot d + H_{\mathcal{K}}$ on $\mathfrak{Exc}_d$ for some positive real number H_{unif} that is *independent* of $\mathcal{K}_V$ and some positive real number $H_{\mathcal{K}}$ that *depends only on* $\mathcal{K}_V$], we may assume without loss of generality that

(P5) if we write $\mathbb{V}_{\text{mod}}^{\text{bad}}$ for the set of nonarchimedean valuations $\in \mathbb{V}_{\text{mod}} \overset{\text{def}}{=} \mathbb{V}(F_{\text{mod}})$ that do *not divide* $2l$ and at which E_F has *bad multiplicative reduction*, then $\mathbb{V}_{\text{mod}}^{\text{bad}} \neq \emptyset$.

Indeed, if $\mathbb{V}_{\text{mod}}^{\text{bad}} = \emptyset$, then it follows, in light of the definition of h, from (P3), assertion (i), and the computation performed above, that

$$h \;\approx\; \log(\mathfrak{q}^{\dagger 2}) \;\leq\; h^{1/2} \cdot \log(l) \;\leq\; 16 \cdot \delta^{1/4} \cdot h^{3/4}$$

— an inequality which implies that $h^{1/4}$, hence h itself, is *bounded*. On the other hand, [by assertion (i); [GenEll], Proposition 1.4, (iv)] this implies that there is only a *finite number of possibilities* for the j-invariant of E_F. This completes the proof of the above *observation*. This property (P5) implies that

(P6) the image of the outer homomorphism $\text{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(\mathbb{F}_l)$ determined by the *l-torsion points* of E_F *contains the subgroup* $SL_2(\mathbb{F}_l) \subseteq GL_2(\mathbb{F}_l)$.

Indeed, since, by (P5), E_F has *bad multiplicative reduction* at some valuation $\in \mathbb{V}_{\text{mod}}^{\text{bad}} \neq \emptyset$, (P6) follows formally from (P2), (P4), and [GenEll], Lemma 3.1, (iii) [cf. the proof of the final portion of [GenEll], Theorem 3.8].

Now it follows *formally* from (P1), (P2), (P5), and (P6) that, if one takes "$\overline{F}$" to be $\overline{\mathbb{Q}}$, "F" to be the number field F of the above discussion, "X_F" to be the once-punctured elliptic curve associated to E_F, "l" to be the prime number l of the above discussion, and "$\mathbb{V}_{\text{mod}}^{\text{bad}}$" to be the set $\mathbb{V}_{\text{mod}}^{\text{bad}}$ of (P5), then there exist data "$\underline{C}_K$", "$\underline{\mathbb{V}}$", and "$\underline{\epsilon}$" such that *all of the conditions of [IUTchI], Definition 3.1, (a), (b), (c), (d), (e), (f), are satisfied*, and, moreover, that

(P7) *the resulting* **initial Θ-data**

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \mathbb{V}_{\text{mod}}^{\text{bad}},\ \underline{\epsilon})$$

satisfies the various conditions in the statement of Theorem 1.10.

Here, we note in passing that the crucial *existence* of data "$\underline{\mathbb{V}}$" and "$\underline{\epsilon}$" satisfying the requisite conditions follows, in essence, as a consequence of the fact [i.e., (P6)] that the Galois action on l-torsion points contains the *full special linear group* $SL_2(\mathbb{F}_l)$.

In light of (P7), we may apply Theorem 1.10 [cf. also Remark 1.10.7, (i)] to conclude that

$$\begin{aligned}
\frac{1}{6} \cdot \log(\mathfrak{q}) \;&\leq\; (1 + \tfrac{20 \cdot d_{\text{mod}}}{l}) \cdot (\log(\mathfrak{d}^{F_{\text{tpd}}}) + \log(\mathfrak{f}^{F_{\text{tpd}}})) \;+\; 20 \cdot (d^*_{\text{mod}} \cdot l + \eta_{\text{prm}}) \\
&\leq\; (1 + \delta \cdot h^{-1/2}) \cdot (\log(\mathfrak{d}^{F_{\text{tpd}}}) + \log(\mathfrak{f}^{F_{\text{tpd}}})) \\
&\qquad\qquad + \; 200 \cdot \delta^2 \cdot h^{1/2} \cdot \log(2\delta \cdot h) \;+\; 20\eta_{\text{prm}}
\end{aligned}$$

— where we apply (P1), as well as the estimates $20 \cdot d_{\mathrm{mod}} \ \leq \ d^*_{\mathrm{mod}} \ \leq \ \delta$.

Next, let us *observe* that it follows from (P3), together with the computation of the discussion preceding (P5), that

$$\begin{aligned} \tfrac{1}{6} \cdot \log(\mathfrak{q}^{\dagger 2}) \ - \ \tfrac{1}{6} \cdot \log(\mathfrak{q}) \ &\leq \ \tfrac{1}{6} \cdot h^{1/2} \cdot \log(l) \ \leq \ \tfrac{1}{3} \cdot h^{1/2} \cdot \log(5\delta \cdot h) \\ &\leq \ h^{1/2} \cdot \log(2\delta \cdot h) \end{aligned}$$

— where we apply the estimates $1 \leq h$ and $5 \leq 2^3$. Thus, since, by assertion (i), the difference $\frac{1}{6} \cdot \log(\mathfrak{q}^{\forall}) - \frac{1}{6} \cdot \log(\mathfrak{q}^{\dagger 2})$ is *bounded* by some positive real number $B_{\mathcal{K}}$ [which depends only on the choice of the *compactly bounded subset* $\mathcal{K}_V$], we conclude that

$$\begin{aligned} \tfrac{1}{6} \cdot h \ = \ \tfrac{1}{6} \cdot \log(\mathfrak{q}^{\forall}) \ &\leq \ (1 + \delta \cdot h^{-1/2}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) \\ &\qquad + \ (15\delta)^2 \cdot h^{1/2} \cdot \log(2\delta \cdot h) \ + \ \tfrac{1}{2} \cdot C_{\mathcal{K}} \\ &\leq \ (1 + \delta \cdot h^{-1/2}) \cdot (\log(\mathfrak{d}^{F_{\mathrm{tpd}}}) + \log(\mathfrak{f}^{F_{\mathrm{tpd}}})) \\ &\qquad + \ \tfrac{1}{6} \cdot h \cdot \tfrac{2}{5} \cdot (60\delta)^2 \cdot h^{-1/2} \cdot \log(2\delta \cdot h) \ + \ \tfrac{1}{2} \cdot C_{\mathcal{K}} \end{aligned}$$

— where we write $C_{\mathcal{K}} \ \stackrel{\mathrm{def}}{=} \ 40\eta_{\mathrm{prm}} \ + \ 2B_{\mathcal{K}}$, and we apply the estimate $6 \cdot 5 \leq 2 \cdot 4^2$.

Now let us set

$$\begin{aligned} \epsilon_E \ &\stackrel{\mathrm{def}}{=} \ (60\delta)^2 \cdot h^{-1/2} \cdot \log(2\delta \cdot h) \ \ (\geq \ 5 \cdot \delta \cdot h^{-1/2}); \\ \epsilon^*_d \ &\stackrel{\mathrm{def}}{=} \ \tfrac{1}{16} \cdot \epsilon_d \ \ (< \tfrac{1}{2} \ \leq \ 1) \end{aligned}$$

— where we apply the estimates $h \geq 1$, $\log(2\delta \cdot h) \ \geq \ \log(2\delta) \ \geq \ \log(4) \ \geq \ 1$ [cf. the property (E6) in the proof of Theorem 1.10], and $\epsilon_d \leq 1$. Note that the inequality

$$\begin{aligned} 1 \ < \epsilon_E \ &= \ (60\delta)^2 \cdot h^{-1/2} \cdot \log(2\delta \cdot h) \\ &= \ (\epsilon^*_d)^{-1} \cdot (60\delta)^2 \cdot h^{-1/2} \cdot \log(2^{\epsilon^*_d} \cdot \delta^{\epsilon^*_d} \cdot h^{\epsilon^*_d}) \\ &\leq \ (\epsilon^*_d)^{-1} \cdot (60\delta)^{2+\epsilon^*_d} \cdot h^{-(1/2-\epsilon^*_d)} \\ &\leq \ \left\{(\epsilon^*_d)^{-3} \cdot (60\delta)^{4+\epsilon_d} \cdot h^{-1}\right\}^{(1/2-\epsilon^*_d)} \end{aligned}$$

— where we apply Proposition 2.1, (i), together with the estimates

$$\frac{1}{\frac{1}{2} - \epsilon^*_d} \ = \ \frac{16}{8 - \epsilon_d} \ \leq \ 3; \qquad \frac{2 + \epsilon^*_d}{\frac{1}{2} - \epsilon^*_d} \ = \ \frac{32 + \epsilon_d}{8 - \epsilon_d} \ \leq \ 4 + \epsilon_d \ \leq \ 5$$

[both of which are consequences of the fact that $0 < \epsilon_d \leq 1 \leq 3$], as well as the estimates $0 < \epsilon^*_d \leq 1$, $60\delta \geq 2\delta \geq 1$, and $h \geq 1$ — implies a *bound* on h, hence, [by assertion (i); [GenEll], Proposition 1.4, (iv)] that there is only a *finite number of possibilities* for the j-invariant of E_F. Thus, by possibly *enlarging* the finite set $\mathfrak{Exc}_d$ [in a fashion that *depends only on* $\mathcal{K}_V$, d, and ϵ_d and in such a way that $h \leq H_{\mathrm{unif}} \cdot \epsilon_d^{-3} \cdot d^{4+\epsilon_d} + H_{\mathcal{K}}$ on $\mathfrak{Exc}_d$ for some positive real number H_{unif} that is *independent* of $\mathcal{K}_V$ and some positive real number $H_{\mathcal{K}}$ that *depends only on* $\mathcal{K}_V$], we may assume without loss of generality that $\epsilon_E \leq 1$.

Thus, in summary, we obtain inequalities

$$\begin{aligned}\tfrac{1}{6}\cdot h \;&\leq\; (1-\tfrac{2}{5}\cdot\epsilon_E)^{-1}(1+\tfrac{1}{5}\cdot\epsilon_E)\cdot(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}}))+(1-\tfrac{2}{5}\cdot\epsilon_E)^{-1}\cdot\tfrac{1}{2}\cdot C_{\mathcal{K}}\\ &\leq\; (1+\epsilon_E)\cdot(\log(\mathfrak{d}^{F_{\mathrm{tpd}}})+\log(\mathfrak{f}^{F_{\mathrm{tpd}}}))+C_{\mathcal{K}}\end{aligned}$$

by applying the estimates

$$\frac{1+\frac{1}{5}\cdot\epsilon_E}{1-\frac{2}{5}\cdot\epsilon_E} \quad\leq\quad 1+\epsilon_E; \qquad 1-\tfrac{2}{5}\cdot\epsilon_E \quad\geq\quad \tfrac{1}{2}$$

— both of which are consequences of the fact that $0<\epsilon_E\leq 1$. Thus, in light of (P1), together with the observation that it follows immediately from the definitions [cf. also Proposition 1.8, (vi)] that we have an equality $\text{log-diff}_X(x_E)=\log(\mathfrak{d}^{F_{\mathrm{tpd}}})$, as well as inequalities $\log(\mathfrak{f}^{F_{\mathrm{tpd}}})\leq\text{log-cond}_D(x_E)\leq\log(\mathfrak{f}^{F_{\mathrm{tpd}}})+\log(2l)$, we conclude that both of the conditions (C1), (C2) in the statement of assertion (ii) hold for $C_{\mathcal{K}}$ as defined above. This completes the proof of assertion (ii). Finally, assertion (iii) follows immediately by applying the argument applied above in the proof of assertion (ii) in the case of the inequality "$1<\epsilon_E$" to the inequality "$\epsilon<\epsilon_E$". ◯

Remark 2.2.1.

(i) Before proceeding, we pause to examine the **asymptotic behavior** of the **bound** obtained in Corollary 2.2, (ii), in the spirit of the discussion of Remark 1.10.5, (ii). For simplicity, we assume that $F_{\mathrm{tpd}}=\mathbb{Q}$ [so $d_{\mathrm{mod}}=1$]; we write $h\stackrel{\mathrm{def}}{=}\log(\mathfrak{q}^{\forall})$ [cf. the proof of Corollary 2.2, (ii)] and $\delta\stackrel{\mathrm{def}}{=}\text{log-diff}_X(x_E)+\text{log-cond}_D(x_E)=\text{log-cond}_D(x_E)$ [i.e., notation that is closely related to the notation of Remark 1.10.5, (ii), but *differs* substantially from the notation of Corollary 2.2, (ii)]. Thus, it follows immediately from the definitions that $1<\log(3)\leq\delta$ and $1<\log(3)\leq h$. In particular, the bound under consideration may be written in the form

$$\tfrac{1}{6}\cdot h \;\leq\; \delta \;+\; *\cdot\delta^{1/2}\cdot\log(\delta)$$

— where "$*$" is to be understood as denoting a *fixed positive real number*; we observe that the *ratio* h/δ is always a positive real number which is *bounded below* by the definition of h and δ and *bounded above* precisely as a consequence of the bound under consideration. In this context, it is of interest to observe that the form of the "ϵ term" $\delta^{1/2}\cdot\log(\delta)$ is **strongly reminiscent** of well-known intepretations of the **Riemann hypothesis** in terms of the asymptotic behavior of the function defined by considering the number of prime numbers less than a given natural number. Indeed, from the point of view of **weights** [cf. also the discussion of Remark 2.2.2 below], it is natural to regard the [logarithmic] height of a line bundle as an object that has the *same weight* as a **single Tate twist**, or, from a more classical point of view, "$2\pi i$" raised to the *power* 1. On the other hand, again from the point of view of weights, the variable "s" of the *Riemann zeta function* $\zeta(s)$ may be thought of as corresponding precisely to the *number of Tate twists* under consideration, so a *single Tate twist* corresponds to "$s=1$". Thus, from this point of view, "$s=\frac{1}{2}$", i.e., the *critical line* that appears in the *Riemann hypothesis*, corresponds precisely to the *square roots* of the [logarithmic] heights under consideration, i.e., to $h^{1/2}$, $\delta^{1/2}$. Moreover, from the point of view of the computations that underlie Theorem

1.10 and Corollary 2.2, (ii) [cf., especially, the proof of Corollary 2.2, (ii); Steps (v), (viii) of the proof of Theorem 1.10; the contribution of "b_i" in the *second displayed inequality* of Proposition 1.4, (iii)], this $\delta^{1/2}$ arises as a result of a sort of **"balance"**, or **"duality"** — i.e., that occurs as one *increases* the size of the auxiliary prime l [cf. the discussion of Remark 1.10.5, (ii)] — between the **archimedean decrease** in the "ϵ term" $\frac{\delta}{l}$ and the **nonarchimedean increase** in the "ϵ term" l [i.e., that arises from a certain estimate, in the proof of Proposition 1.2, (i), (ii), of the *radius of convergence* of the *p-adic logarithm*]. That is to say, such a **global arithmetic duality** is reminiscent of the **functional equation** of the Riemann zeta function [cf. the discussion of (iii) below].

(ii) In [vFr], §2, it is conjectured that, in the notation of the discussion of (i),

$$\limsup \frac{\log\left(\frac{1}{6}\cdot h - \delta\right)}{\log(h)} \;=\; \frac{1}{2}$$

and observed that the "$\frac{1}{2}$" that appears here is **strongly reminiscent** of the "$\frac{1}{2}$" that appears in the **Riemann hypothesis**. In the situation of Corollary 2.2, (ii), bounds are only obtained on *abc* sums that belong to the **compactly bounded subset** $\mathcal{K}_V$ under consideration; such bounds, i.e., as discussed in (i), thus imply that this $\limsup$ is $\leq \frac{1}{2}$. On the other hand, it is shown in [vFr], §2 [cf. also the references quoted in [vFr]], that, if one allows *arbitrary abc* sums [i.e., which are not necessarily assumed to be contained in a single compactly bounded subset $\mathcal{K}_V$], then this $\limsup$ is $\geq \frac{1}{2}$. It is not clear to the author at the time of writing whether or not such estimates [i.e., to the effect that the $\limsup$ under consideration is $\geq \frac{1}{2}$] hold even if one imposes the restriction that the *abc* sums under consideration be contained in a single compactly bounded subset $\mathcal{K}_V$.

(iii) In the well-known classical theory of the **Riemann zeta function**, the Riemann zeta function is closely related to the **theta function**, i.e., by means of the **Mellin transform**. In light of the *central role* played by theta functions in the theory of the present series of papers, it is tempting to hope, especially in the context of the observations of (i), (ii), that perhaps some extension of the theory of the present series of papers — i.e., some sort of **"inter-universal Mellin transform"** — may be obtained that allows one to relate the theory of the present series of papers to the Riemann zeta function.

(iv) In the context of the discussion of (iii), it is of interest to recall that, relative to the analogy between *number fields* and *one-dimensional function fields over finite fields*, the theory of the present series of papers may be thought of as being analogous to the theory surrounding the **derivative** of a lifting of the **Frobenius morphism** [cf. the discussion of [IUTchI], §I4; [IUTchIII], Remark 3.12.4]. On the other hand, the analogue of the **Riemann hypothesis** for one-dimensional function fields over finite fields may be proven by considering the elementary geometry of the [graph of the] **Frobenius morphism**. This state of affairs suggests that perhaps some sort of **"integral"** of the theory of the present series of papers could shed light on the Riemann hypothesis in the case of number fields.

(v) One way to summarize the point of view discussed in (i), (ii), and (iii) is as follows: The *asymptotic behavior* discussed in (i) suggests that perhaps one should expect that the *inequality* constituted by well-known intepretations of the **Riemann hypothesis** in terms of the asymptotic behavior of the function defined by considering the number of prime numbers less than a given natural number may be obtained as some sort of *"restriction"*

$$\textbf{(ABC inequality)}|_{\textbf{canonical number}}$$

of some sort of *"ABC inequality"* [i.e., some sort of bound of the sort obtained in Corollary 2.2, (ii)] to some sort of *"canonical number"* [i.e., where the term "number" is to be understood as referring to an *abc* sum]. Here, the descriptive "canonical" is to be understood as expressing the idea that one is not so much interested in considering a *fixed explicit "number/abc sum"*, but rather some sort of *suitable abstraction* of the sort of sequence of numbers/*abc* sums that gives rise to the lim sup value of "$\frac{1}{2}$" discussed in (ii). Of course, it is by no means clear precisely how such an "abstraction" should be formulated, but the idea is that it should represent

some sort of **average** over **all possible addition operations**

in the number field [in this case, $\mathbb{Q}$] under consideration or [perhaps equivalently]

> some sort of **"arithmetic measure or distribution"** constituted by such a collection of **all possible addition operations** that somehow amounts to a sort of arithmetic analogue of the measure that gives rise to the classical **Mellin transform**

[i.e., that appears in the discussion of (iii)].

Remark 2.2.2. In the context of the discussion of **weights** in Remark 2.2.1, (i), it is of interest to recall the significance of the **Gaussian integral**

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx \;=\; \sqrt{\pi}$$

in the theory of the present series of papers [cf. [IUTchII], Introduction; [IUTchII], Remark 1.12.5, as well as Remark 1.10.1 of the present paper]. Indeed, typically discussions of the *Riemann zeta function* $\zeta(s)$, or more general *L-functions*, in the context of conventional arithmetic geometry are concerned principally with the behavior of such functions at *integral values* [i.e., $\in \mathbb{Z}$] of the variable s. Such integral values of the variable s correspond to *integral Tate twists*, i.e., at a more concrete level, to *integral powers* of the quantity $2\pi i$. If one neglects nonzero factors $\in \mathbb{Q}(i)$, then such integral powers may be regarded as *integral powers* of π [or 2π]. At the level of *classical integrals*, the notion of a single Tate twist may be thought of as corresponding to the integral

$$\int_{\mathbb{S}^1} d\theta \;=\; 2\pi$$

over the unit circle $\mathbb{S}^1$; at the level of *schemes*, the notion of a single Tate twist may be thought of as corresponding to the scheme $\mathbb{G}_{\mathrm{m}}$. On the other hand, whereas the conventional theory of Tate twists in arithmetic geometry only involves *integral powers* of a single Tate twist, i.e., corresponding, in essence, to integral powers of π, the Gaussian integral may be thought of as a sort of *fundamental integral representation* of the notion of a **"Tate semi-twist"**. From this point of view, *scheme-theoretic Hodge-Arakelov theory* may be thought of as a sort of *fundamental scheme-theoretic represention* of the notion of a **"Tate semi-twist"** [cf. the discussion of [IUTchII], Remark 1.12.5]. Thus, in summary,

(a) the **Gaussian integral**,
(b) **scheme-theoretic Hodge-Arakelov theory**,
(c) the **inter-universal Teichmüller theory** developed in the present series of papers, and
(d) the **Riemann hypothesis**,

may all be thought of as **"phenomena of weight $\frac{1}{2}$"**, i.e., at a concrete level, phenomena that revolve around *arithmetic versions of "$\sqrt{\pi}$"*. Moreover, we observe that in the first three of these four examples, the essential nature of the notion of "weight $\frac{1}{2}$" may be thought of as being reflected in some sort of **exponential** of a **quadratic form**. This state of affairs is strongly reminiscent of

(1) the **Griffiths semi-transversality** of the **crystalline theta object** that occurs in scheme-theoretic Hodge-Arakelov theory [cf. [HASurII], Theorem 2.8; [IUTchII], Remark 1.12.5, (i)], which corresponds essentially [cf. the discussion of the proof of [HASurII], Theorem 2.10] to the *quadratic form* that appears in the exponents of the well-known series expansion of the *theta function*;

(2) the *quadratic* nature of the **commutator** of the **theta group**, which is applied, in [EtTh] [cf. the discussion of [IUTchIII], Remark 2.1.1], to derive the various *rigidity properties* which are interpreted, in [IUTchII], §1, as *multiradiality* properties — an interpretation that is *strongly reminiscent*, if one interprets "multiradiality" in terms of *"connections"* and *"parallel transport"* [cf. [IUTchII], Remark 1.7.1], of the quadratic form discussed in (1);

(3) the essentially *quadratic* nature of the **"ϵ term"** $* \cdot \frac{\delta}{l} + * \cdot l$ [which, we recall, occurs at the level of addition of *heights*, i.e., *log-volumes*!] in the discussion of Remark 1.10.5, (ii).

Remark 2.2.3. The discussion of Remark 2.2.1 centers around the content of Corollary 2.2, (ii), in the case of *elliptic curves defined over* $\mathbb{Q}$. On the other hand, if, in the context of Corollary 2.2, (ii), (iii), one considers the case where d is an *arbitrary positive integer* [i.e., which is *not necessarily bounded*, as in the situation of Corollary 2.3 below!], then the inequalities obtained in (C2) of Corollary 2.2, (ii), may be regarded, by applying Corollary 2.2, (iii), as a sort of **"weak version"** of the so-called **"uniform ABC Conjecture"**. That is to say, these inequalities constitute only a "weak version" in the sense that they are restricted to rational points that lie in the **compactly bounded subset** $\mathcal{K}_V$, and, moreover, the bounds given for the function "$\log(\mathfrak{q}^{\forall}_{(-)})$" [i.e., in essence, the *"height"*] on $\mathfrak{Exc}_d$ and $\mathfrak{Exc}_{\epsilon,d}$ **depend** on the positive integer d [cf. also Remark 2.3.2, (i), below].

Remark 2.2.4. Before proceeding, it is perhaps of interest to consider the ideas discussed in Remarks 2.2.1, 2.2.3 above in the context of the analogy between the theory of the present series of papers and the p-adic Teichmüller theory of [pOrd], [pTeich] [cf. also [InpTch]].

(i) The analogy between the theory of the present series of papers and the p-adic Teichmüller theory of [pOrd], [pTeich] [cf. also [InpTch]] is discussed in detail in [IUTchIII], Remark 1.4.1, (iii); [IUTchIII], Remark 3.12.4. In a word, this discussion concerns similarities between the *log-theta-lattice* considered in the present series of papers and the *canonical Frobenius lifting* on the ordinary locus of a *canonical curve* of the sort that appears in the theory of [pOrd]. Such a canonical curve is associated, in the theory of [pOrd], to a hyperbolic curve equipped with a nilpotent ordinary indigenous bundle over a **perfect field** of positive characteristic p. On the other hand, the theory of [pOrd] also addresses the *universal case*, i.e., of the tautological hyperbolic curve equipped with a nilpotent ordinary indigenous bundle over the *moduli stack* of such data in positive characteristic. In particular, one constructs, in the theory of [pOrd], a *canonical Frobenius lifting* over a canonical p-adic lifting of this moduli stack. This moduli stack is smooth of dimension $3g-3+r$ [i.e., in the case of hyperbolic curves of type (g, r)] over $\mathbb{F}_p$, hence, in particular, is **far from perfect** [i.e., as an algebraic stack in positive characteristic]. Thus, in some sense,

> the *gap* between the theory of the present series of papers, on the one hand, and the notion discussed in Remark 2.2.1, (v), of a **"canonical number/arithmetic measure/distribution"**, on the other, may be understood, in the context of the analogy with p-adic Teichmüller theory, as corresponding to the *gap* between the theory of [pOrd] specialized to the case of *"canonical curves"*, i.e., over **perfect** *base fields*, and the full, non-specialized version of the theory of [pOrd], i.e., which concerns *canonical Frobenius liftings* over the **non-perfect** *moduli stack* of hyperbolic curves equipped with a nilpotent ordinary indigenous bundle.

That is to say, in a word, one has a *correspondence*

$$\textbf{"canonical number"} \quad \longleftrightarrow \quad \textbf{modular Frobenius liftings}.$$

(ii) In general, the *gap* between *perfect* and *non-perfect* schemes in positive characteristic is reflected precisely in the extent to which the *Frobenius morphism* on the scheme under consideration *fails to be an isomorphism*. Put another way, the "phenomenon" of non-perfect schemes in positive characteristic may be thought of as a reflection of the *distortion* arising from the *Frobenius morphism* in positive characteristic. In the context of the theory of the present series of papers [cf. [IUTchIII], Remark 1.4.1, (iii)], the Frobenius morphism in positive characteristic corresponds to the **$\mathfrak{log}$-link**. Moreover, in the context of the inequalities obtained in Theorem 1.10, the term "$* \cdot l$" [cf. the discussion of Remark 1.10.5, (ii)] arises, in the computations that underlie the proof of Theorem 1.10, precisely by applying the *prime number theorem* [i.e., Proposition 1.6] to sum up the *log-volumes* of the *log-shells* [cf. Propositions 1.2, (ii); 1.4, (iii)] at various nonarchimedean primes of the number field. In this context, we make the following *observations*:

· These log-volumes of log-shells may be thought of as numerical measures of the **distortions** *of the integral structure* [i.e., relative to the "arithmetic holomorphic" integral structures determined by the various local rings of integers "$\mathcal{O}$"] that arise from the $\mathfrak{log}$**-link**.

· Estimates arising from the *prime number theorem* are closely related to the aspects of the **Riemann zeta function** that are discussed in Remark 2.2.1.

· The *prime number* l is, ultimately, in the computations of Corollary 2.2, (ii) [cf., especially, condition "(C1)"], taken to be roughly of the order of the square root of the *height* of the elliptic curve under consideration. That is to say, since the height of an elliptic curve *"roughly controls"* [i.e., up to finitely many possibilities] the *moduli* of the elliptic curve, the prime number l may be thought of as a sort of **rough numerical representation** of the **moduli** of the elliptic curve under consideration.

Thus, in summary, these observations strongly support the point of view that

> the *computations that underlie the proof of Theorem 1.10* may be thought of as constituting one *convincing piece of evidence* for the point of view discussed in (i) above.

(iii) In the context of the discussion of (i), (ii), it is of interest to recall that the *modular Frobenius liftings* of [pOrd] are *not defined* over the *algebraic* moduli stack of hyperbolic curves over $\mathbb{Z}_p$, but *rather* over the *p-adic formal algebraic stack* [which is *formally étale* over the corresponding algebraic moduli stack of hyperbolic curves over $\mathbb{Z}_p$] constituted by the canonical lifting to $\mathbb{Z}_p$ of the moduli stack of hyperbolic curves equipped with a nilpotent ordinary indigenous bundle. That is to say,

> the *gap* between this [*"p-adically analytic"*] p-adic formal algebraic stack parametrizing **"ordinary"** data and the corresponding algebraic moduli stack of hyperbolic curves over $\mathbb{Z}_p$ is highly reminiscent, in the context of Corollary 2.2, (ii) [cf. also Remark 2.2.3], of the *gap* between the [*"arithmetically analytic"*] **compactly bounded subsets** "$\mathcal{K}_V$" [i.e., consisting of elliptic curves that satisfy the condition of being in *"sufficiently general position"* — a condition that may be thought of as a sort of *"global arithmetic version of ordinariness"*] and the entire set of algebraic points "$\mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$".

Ultimately, this gap between "$\mathcal{K}_V$" and "$\mathcal{M}_{\mathrm{ell}}(\overline{\mathbb{Q}})$" will be bridged, in Corollary 2.3 below, by applying [GenEll], Theorem 2.1, which may be thought of as a sort of **arithmetic analytic continuation** by means of [noncritical] **Belyi maps** [cf. the discussion of Belyi maps in the Introduction to [GenEll]]. This state of affairs is reminiscent of the *"arithmetic analytic continuation via Belyi maps"* that occurs in the theory of [AbsTopIII] [i.e., in essence, the theory of *Belyi cuspidalizations*] that is applied in [IUTchI], §5 [cf. [IUTchI], Remark 5.1.4]. Finally, in this context, we recall that the *open immersion* "$\widehat{\kappa}$" that appears in the discussion towards the end of [InpTch], §2.6 — i.e., which embeds a sort of *perfection* of the p-adic formal algebraic stack discussed above into an essentially *algebraic* stack given by a certain pro-finite covering of the corresponding algebraic moduli stack of hyperbolic curves over $\mathbb{Z}_p$ determined by considering *representations of the geometric fundamental group into* $PGL_2(\mathbb{Z}_p)$ — may be thought of as a sort of *p-adic analytic continuation* to this corresponding algebraic moduli stack of the essentially "p-adically analytic" theory of modular Frobenius liftings developed in [pOrd].

(iv) Finally, in the context of the discussion of (i), (ii), (iii), we observe that the issue discussed in Remark 2.2.1 of considering the *asymptotic behavior* of the theory of the present series of papers when $l \to \infty$ may be thought of as the problem of understanding how the theory of the present series of papers behaves

> as one passes from the **discrete approximation** of the elliptic curve under consideration constituted by the *l-torsion points* of the elliptic curve to the **"full continuous theory"**

[cf. the discussion of [IUTchI], Remark 6.12.3, (i) ; [HASurI], §1.3.4]. This point of view is of interest in light of the theory of **Bernoulli numbers**, i.e., which, on the one hand, is, as is well-known, closely related to the **values** [at positive even integers] of the **Riemann zeta function** [cf. the discussion of Remark 2.2.1], and, on the other hand, is closely related to the passage from the

$$\textbf{discrete difference operator } f(x) \mapsto f(x+1) - f(x)$$

— for, say, real-valued real analytic functions $f(-)$ on the real line — to the

$$\textbf{continuous derivative operator } f(x) \mapsto \frac{d}{dx} f(x)$$

— where we recall that the operator $f(x) \mapsto f(x+1)$ may be thought of as the operator "$e^{\frac{d}{dx}}$" obtained by *exponentiating* this continuous derivative operator.

We are now ready to state and prove the *main theorem* of the present §2, which may also be regarded as the *main application* of the theory developed in the present series of papers.

Corollary 2.3. **(Diophantine Inequalities)** *Let X be a smooth, proper, geometrically connected curve over a number field; $D \subseteq X$ a reduced divisor; $U_X \stackrel{\text{def}}{=} X \setminus D$; d a positive integer; $\epsilon \in \mathbb{R}_{>0}$ a positive real number. Write ω_X for the canonical sheaf on X. Suppose that U_X is a* **hyperbolic curve**, *i.e., that the degree of the line bundle $\omega_X(D)$ is* **positive**. *Then, relative to the notation reviewed above, one has an* **inequality of "bounded discrepancy classes"**

$$\mathrm{ht}_{\omega_X(D)} \ \lesssim \ (1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D)$$

of functions on $U_X(\overline{\mathbb{Q}})^{\leq d}$ — i.e., the function $(1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D) - \mathrm{ht}_{\omega_X(D)}$ is bounded below by a **constant** *on $U_X(\overline{\mathbb{Q}})^{\leq d}$ [cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), below].*

Proof. One verifies immediately that the content of the statement of Corollary 2.3 coincides precisely with the content of [GenEll], Theorem 2.1, (i). Thus, it follows from the *equivalence* of [GenEll], Theorem 2.1, that, in order to complete the proof of Corollary 2.3, it suffices to verify that [GenEll], Theorem 2.1, (ii), holds. That is to say, we may assume without loss of generality that:

- $X = \mathbb{P}^1_{\mathbb{Q}}$ is the *projective line* over $\mathbb{Q}$;
- $D \subseteq X$ is the divisor consisting of the *three points* "0", "1", and "∞";
- $\mathcal{K}_V \subseteq U_X(\overline{\mathbb{Q}})$ is a *compactly bounded subset* [cf. Remark 2.3.1, (vi), below] whose *support* contains the nonarchimedean prime "2";
- $\mathcal{K}_V$ satisfies the condition "$(*_{j\text{-inv}})$" of Corollary 2.2.

[Here, we note, with regard to the condition "$(*_{j\text{-inv}})$" of Corollary 2.2, that this condition only concerns the behavior of $\mathcal{K}_V \bigcap U_X(\overline{\mathbb{Q}})^{\leq d}$ *as d varies*; that is to say, this condition is *entirely vacuous* in situations, i.e., such as the situation considered in [GenEll], Theorem 2.1, (ii), in which one is only concerned with $\mathcal{K}_V \bigcap U_X(\overline{\mathbb{Q}})^{\leq d}$ *for a fixed d*.] Then it suffices to show that the *inequality of BD-classes of functions* [cf. [GenEll], Definition 1.2, (ii), as well as Remark 2.3.1, (ii), below]

$$\mathrm{ht}_{\omega_X(D)} \lesssim (1+\epsilon)(\text{log-diff}_X + \text{log-cond}_D)$$

holds on $\mathcal{K}_V \bigcap U_X(\overline{\mathbb{Q}})^{\leq d}$. But such an inequality follows immediately, in light of the [relevant] *equality of BD-classes* of Corollary 2.2, (i), from Corollary 2.2, (ii) [cf. condition (C2)], (iii) [where we note that it follows immediately from the various definitions involved that $d_{\mathrm{mod}} \leq d$]. This completes the proof of Corollary 2.3. ◯

Remark 2.3.1. We take this opportunity to correct some *unfortunate misprints* in [GenEll].

(i) The notation "$\mathrm{ord}_v(-) : F_v \to \mathbb{Z}$" in the final sentence of the first paragraph following [GenEll], Definition 1.1, should read "$\mathrm{ord}_v(-) : F_v^\times \to \mathbb{Z}$".

(ii) In [GenEll], Definition 1.2, (ii), the *non-resp'd* and *first resp'd* items in the display should be *reversed*! That is to say, the notation "$\alpha \lesssim_{\mathcal{F}} \beta$" corresponds to "$\alpha(x) - \beta(x) \leq C$"; the notation "$\alpha \gtrsim_{\mathcal{F}} \beta$" corresponds to "$\beta(x) - \alpha(x) \leq C$".

(iii) The first portion of the first sentence of the statement of [GenEll], Corollary 4.4, should read: "Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$; ...".

(iv) The "$\text{log-diff}_{\overline{\mathcal{M}}_{\mathrm{ell}}}([E_L]))$" in the second inequality of the final display of the statement of [GenEll], Corollary 4.4, should read "$\text{log-diff}_{\overline{\mathcal{M}}_{\mathrm{ell}}}([E_L])$".

(v) The *equality*

$$\mathrm{ht}_E \approx (\deg(E)/\deg(\omega_X)) \cdot \mathrm{ht}_{\omega_X}$$

implicit in the final "$\approx$" of the final display of the proof of [GenEll], Theorem 2.1, should be replaced by an *inequality*

$$\mathrm{ht}_E \lesssim 2 \cdot (\deg(E)/\deg(\omega_X)) \cdot \mathrm{ht}_{\omega_X}$$

[which follows immediately from [GenEll], Proposition 1.4, (ii)], and the expression "$\deg(E)/\deg(\omega_X)$" in the inequality imposed on the *choice* of ϵ' should be replaced by the expression "$2 \cdot (\deg(E)/\deg(\omega_X))$".

(vi) Suppose that we are in the situation of [GenEll], Example 1.3, (ii). Let $U \subseteq X$ be an open subscheme. Then a *"compactly bounded subset"*

$$\mathcal{K}_V \ \subseteq \ U(\overline{\mathbb{Q}}) \ \ (\subseteq X(\overline{\mathbb{Q}}))$$

of $U(\overline{\mathbb{Q}})$ is to be understood as a subset which forms a compactly bounded subset of $X(\overline{\mathbb{Q}})$ [i.e., in the sense discussed in [GenEll], Example 1.3, (ii)] and, moreover, satisfies the property that for each $v \in V^{\mathrm{arc}} \stackrel{\mathrm{def}}{=} V \bigcap \mathbb{V}(\mathbb{Q})^{\mathrm{arc}}$ (respectively, $v \in V^{\mathrm{non}} \stackrel{\mathrm{def}}{=} V \bigcap \mathbb{V}(\mathbb{Q})^{\mathrm{non}}$), the compact domain $\mathcal{K}_v \subseteq X^{\mathrm{arc}}$ (respectively, $\mathcal{K}_v \subseteq X(\overline{\mathbb{Q}}_v)$) is, in fact, *contained in*

$$U(\mathbb{C}) \subseteq X(\mathbb{C}) = X^{\mathrm{arc}} \text{ (respectively, } U(\overline{\mathbb{Q}}_v) \subseteq X(\overline{\mathbb{Q}}_v)).$$

In particular, this convention should be applied to the use of the term "compactly bounded subset" in the statements of [GenEll], Theorem 2.1; [GenEll], Lemma 3.7; [GenEll], Theorem 3.8; [GenEll], Corollary 4.4, as well as in the present paper [cf. the statement of Corollary 2.2; the proof of Corollary 2.3]. Although this convention was not discussed explicitly in [GenEll], Example 1.3, (ii), it is, in effect, discussed *explicitly* in the discussion of "compactly bounded subsets" at the beginning of the Introduction to [GenEll]. Moreover, this convention is *implicit* in the arguments involving compactly bounded subsets in the proof of [GenEll], Theorem 2.1.

(vii) In the discussion following the second display of [GenEll], Example 1.3, (ii), the phrase "(respectively, $X(\mathbb{Q}_v)$)" should read "(respectively, $X(\overline{\mathbb{Q}}_v)$)".

(viii) The first display of the paragraph immediately following [GenEll], Remark 3.3.1, should read as follows:

$$|\alpha|^2 \ \stackrel{\mathrm{def}}{=} \ \left| \int_{E_v} \alpha \wedge \overline{\alpha} \ \right|$$

[i.e., the integral should be replaced by the absolute value of the integral].

Remark 2.3.2.

(i) The reader will note that, by arguing with a "bit more care", it is not difficult to give **stronger** versions of the various **estimates** that occur in Theorem 1.10; Corollaries 2.2, 2.3 and their proofs. Such stronger estimates are, however, beyond the scope of the present series of papers, so we shall not pursue this topic further in the present paper.

(ii) On the other hand, we observe that the constant "1" in the inequality of the display of Corollary 2.3 **cannot be improved** — cf. the examples constructed in [Mss]; the discussion of Remark 1.10.5, (ii), (iii). This observation is closely related to discussions of how the theory of the present series of papers *breaks down* if one attempts to replace the **first power** of the **étale theta function** by its ***N*-th power** for some integer $N \geq 2$ [cf. the discussion in the final portion of Step (xi) of the proof of [IUTchIII], Corollary 3.12; the discussion of [IUTchIII], Remark 3.12.1, (ii)]. Such an "N-th power operation" may also be thought of as corresponding to the operation of replacing each Tate curve that occurs at an element $\in \underline{\mathbb{V}}^{\mathrm{bad}}$ by the Tate curve whose q-parameter is given by the N*-th power* of the q-parameter of the original Tate curve. This sort of operation on Tate curves may, in turn, be thought of as an **isogeny** of the sort that occurs in [GenEll], Lemma 3.5. On the other hand, the content of the proof of [GenEll], Lemma 3.5, consists essentially of a computation to the effect that even if one attempts to consider such "N-th power isogenies" at *certain* elements $\in \underline{\mathbb{V}}^{\mathrm{bad}}$, the *global height* of the elliptic curve over a number field that arises from such an isogeny will typically remain, up to a relatively small discrepancy, *unchanged*. In this context, we recall that this sort of **invariance**, up to a relatively small discrepancy, of the **global height** under **isogeny** is one of the essential observations that underlies the theory of [Falt] — a state of affairs that is also of interest in light of the observations of Remark 2.3.3 below.

Remark 2.3.3. Corollary 2.3 may be thought of as an **effective** version of the **Mordell Conjecture**. From this point of view, it is perhaps of interest to compare the **"essential ingredients"** that are applied in the proof of Corollary 2.3 [i.e., in effect, that are applied in the present series of papers!] with the "essential ingredients" applied in [Falt]. The following discussion benefited substantially from numerous e-mail and skype exchanges with *Ivan Fesenko* during the summer of 2015.

(i) Although the author does not wish to make any pretensions to completeness in any rigorous sense, perhaps a rough, informal list of "essential ingredients" in the case of [Falt] may be given as follows:

(a) results in elementary algebraic number theory related to the *"geometry of numbers"*, such as the theory of *heights* and the *Hermite-Minkowski theorem*;
(b) the *global class field theory* of *number fields*;
(c) the *p-adic* theory of *Hodge-Tate decompositions*;
(d) the *p-adic* theory of *finite flat group schemes*;
(e) generalities in algebraic geometry concerning *isogenies* and *Tate modules* of *abelian varieties*;
(f) generalities in algebraic geometry concerning *polarizations* of *abelian varieties*;
(g) the *logarithmic geometry* of toroidal compactifications of the moduli stack of abelian varieties.

With regard to the global class field theory of (b), we observe that there are *numerous different approaches* to *"dissecting"* the proofs of the main results of global class field theory into more primitive components. To some extent, these different approaches correspond to different points of view arising from subsequent research on topics related to global class field theory. Here, we wish to consider the approach taken in [Lang1], Chapters VIII, IX, X, XI, which is attibuted [cf. the Introduction to [Lang1], Part Two] to Weber. It is of interest, in the context of the discussion of (vii) below, that this is apparently the *oldest approach* to proving certain portions of global class field theory. It is also of interest that this approach motivates the approach to global class field theory via consideration of *density of primes in arithmetic progressions* and *splitting laws*. This aspect of this approach of [Lang1] is closely related to various issues that appear in [Falt] [cf. [Lang1], Chapter VIII,

§5]. Moreover, as we shall see in the following discussion, this approach of [Lang1] to global class field theory is well-suited to discussions of *comparisons* between the theory of [Falt] and the *inter-universal Teichmüller theory* developed in the present series of papers. At a technical level, the dissection of the global class field theory of (b), as developed in [Lang1], into more primitive components may be summarized as follows:

(b-1) the *local class field theory* of *p-adic local fields* [cf. [Lang1], Chapter IX, §3; [Lang1], Chapter XI, §4];
(b-2) the theory of *global density of primes* [cf. the discussion surrounding the *Universal Norm Index Inequality* in [Lang1], Chapter VIII, §3];
(b-3) results in elementary algebraic number theory related to the *"geometry of numbers"* that give rise to the *Unit Theorem* [cf. [Lang1], Chapter V, §1; [Lang1], Chapter IX, §4];
(b-4) the *global reciprocity law*, i.e., in effect, the *existence of a conductor for the Artin symbol* [cf. [Lang1], Chapter X, §2];
(b-5) *Kummer theory* [cf. [Lang1], Chapter XI, §1].

Here, we recall that (b-1), (b-2), and (b-3) are applied in [Lang1], Chapter IX, §5, to verify the *Universal Norm Index Equality* for cyclic extensions. This Universal Norm Index Equality is then applied in [Lang1], Chapter X, §1, and combined with the theory of *cyclotomic extensions* in [Lang1], Chapter X, §2, to verify (b-4). Finally, (b-4) is combined with (b-5) in [Lang1], Chapter XI, §2, to complete the proof of the *Existence Theorem* for class fields.

(ii) From the point of view of the theory of the present series of papers, (a), together with (b-3), is reminiscent of the elementary algebraic number theory characterization of **nonzero global integers** as **roots of unity**, which plays an important role in the theory of the present series of papers [cf. [IUTchIII], the proof of Proposition 3.10]. Moreover, (a) is also reminiscent of the **arithmetic degrees** of line bundles that appear, for instance, in the form of **global realified Frobenioids**, throughout the theory of the present series of papers. Next, we observe that (b-1) is reminiscent of the ***p*-adic absolute anabelian geometry** of [AbsTopIII] [cf., e.g., [AbsTopIII], Corollary 1.10, (i)]. On the other hand, (b-2) is reminiscent of repeated applications of the **Prime Number Theorem** in the present paper [cf. Propositions 1.6; 2.1, (ii)]; this comparison between (b-2) and the Prime Number Theorem will be discussed in more detail in (iv) below. Next, we observe [cf. the discussion of the latter portion of [IUTchIII], Remark 3.12.1, (iii)] that (b-4) is reminiscent of the application of the elementary fact "$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^\times = \{1\}$" in the multiradial algorithms for **cyclotomic rigidity isomorphisms** in the **number field** case [cf. [IUTchI], Example 5.1, (v), as well as the discussion of [IUTchIII], Remarks 2.3.2, 2.3.3], that is to say, not only in the sense that

> both are closely related to the various **cyclotomes** that appear in global class field theory theory or inter-universal Teichmüller theory,

but also in the sense that

> both may be regarded as analogues of the *usual* **product formula** [i.e., which appears at the level of *Frobenius-like* monoids isomorphic to the multiplicative group of nonzero elements of a number field] at the level of

certain [*étale-like*!] **profinite Galois groups** related to global number fields.

On the other hand, (b-5) is reminiscent of the central role played through inter-universal Teichmüller theory by constructions modeled on classical **Kummer theory**. In fact, these comparisons involving (b-4) and (b-5) are closely related to one another and will be discussed in more detail in (v), (vi), and (vii) below. Next, we recall that **Hodge-Tate decompositions** as in (c) play a central role in the proofs of the main results of [pGC], which, in turn, underlie the theory of [AbsTopIII]. The ramification computations concerning finite flat group schemes as in (d) are reminiscent of various p-adic ramification computations concerning **log-shells** in [AbsTopIII], as well as in Propositions 1.1, 1.2, 1.3, 1.4 of the present paper. Whereas [Falt] revolves around the *abelian/linear* theory of abelian varieties [cf. (e)], the theory of the present series of papers depends, in an essential way, on various intricate manipulations involving *finite étale coverings* of *hyperbolic curves*, such as the use of **Belyi maps** in [GenEll], as well as in the **Belyi cuspidalizations** applied in [AbsTopIII]. The theory of polarizations of abelian varieties applied in [Falt] [cf. (f)] is reminiscent of the essential role played by *commutators* of **theta groups** in the theory of [EtTh], which, in turn, plays a central role in the theory of the present series of papers. Finally, the *logarithmic geometry* of (g) is reminiscent of the **combinatorial anabelian geometry** of [SemiAnbd], which is applied, in [IUTchI], §2, to the logarithmic geometry of coverings of stable curves.

(iii) One way to summarize the discussion of (ii) is as follows:

many aspects of the theory of [Falt] may be regarded as **"distant abelian ancestors"** *of certain aspects of the* **"anabelian-based theory"** *of inter-universal Teichmüller theory.*

Alternatively, one may observe that the overwhelmingly *scheme-theoretic* nature of the theory applied in [Falt] lies in stark contrast to the highly *non-scheme-theoretic* nature of the *absolute anabelian geometry* and theory of *monoids/Frobenioids* applied in the present series of papers: that is to say,

many aspects of the theory of [Falt] may be regarded as **"distant arithmetically holomorphic ancestors"** *of certain aspects of the* **multiradial** *and* **mono-analytic** *[i.e.,* **"arithmetically real analytic"***] theory developed in inter-universal Teichmüller theory.*

One way to understand this **fundamental difference** between the theory of [Falt] and inter-universal Teichmüller theory is by considering the naive goal of constructing some sort of **"Frobenius morphism"** on a **number field** [cf. the discussion of [FrdI], §I3], i.e., which has the effect of **multiplying arithmetic degrees** by a positive factor > 1: whereas the theory of [Falt] [cf., e.g., the argument of the proof of [GenEll], Lemma 3.5, as discussed in Remark 2.3.2, (ii)] may regarded as a reflection of the point of view that,

so long as one respects the **arithmetic holomorphic structure** *of scheme theory, such a "Frobenius morphism" on a number field* **cannot exist**,

the essential content of inter-universal Teichmüller theory may be summarized in a word as the assertion that,

> *if one dismantles this arithmetic holomorphic structure in a suitably canonical fashion and allows oneself to work with* **multiradial/mono-analytic** *[i.e.,* **"arithmetically quasi-conformal"***] structures, then one* **can indeed construct***, in a very* **canonical** *fashion, such a "Frobenius morphism" on a number field.*

(iv) In the context of the comparison discussed in (ii) concerning (b-2), it is of interest to note that the *fundamental difference* discussed in (iii) between the theory of [Falt] and inter-universal Teichmüller theory is, in some sense, reflected in the *difference* between the theory of *global density of primes* [i.e., (b-2)] and the *Prime Number Theorem*. That is to say, the **coherence** of the sorts of collections of primes that appear in the theory of global density of primes may be thought of as a sort of representation, in the context of *analytic number theory*, of the **arithmetic holomorphic structure** of conventional scheme theory. By contrast, in the context of the Prime Number Theorem, primes of a number field appear, so to speak, *one by one*, i.e., in a fashion that is only possible if one **deactivates**, in the context of analytic number theory, the *coherence* that underlies the **aggregrations of primes** that appear in the theory of global density of primes. That is to say, this approach to treating primes "one by one" may be thought of as corresponding to the *dismantling of arithmetic holomorphic structures* that occurs in the context of the **multiradial/mono-analytic** structures that appear in inter-universal Teichmüller theory. Here, it is also of interest to note that the way in which one "deactivates aggregations of primes" in the context of the Prime Number Theorem may be thought of [cf. the discussion of [IUTchIII], Remark 3.12.2, (i), (c)] as a sort of **dismantling** of the ring structure of a number field into its underlying **additive** [i.e., counting primes *"one by one"*!] and **multiplicative** structures [i.e., the very notion of a *prime*!].

(v) The *fundamental difference* discussed in (iii) between the theory of [Falt] and inter-universal Teichmüller theory may also be seen in the context of the comparison discussed in (ii) concerning (b-4). Indeed, the **global reciprocity law** of (b-4), which plays a *central role* in global class field theory, depends, in an essential way, on *nontrivial relationships* between **local units** [such as the unit determined by a prime number l at a nonarchimedean prime of a number field of residue characteristic $\neq l$] at one prime of a number field and elements of **local value groups** [such as the element determined by l at a nonarchimedean prime of a number field of residue characteristic l] at another prime of the number field. Such nontrivial relationships are

> **fundamentally incompatible** with the **splittings/decouplings** of *local units* and *local value groups* that play a *central role* in the **dismantling** of **arithmetic holomorphic structures** that occurs in inter-universal Teichmüller theory [cf. the discussion of [IUTchIII], Remark 2.3.3, (i); [IUTchIII], Remark 3.12.2, (i), (a)].

This incompatibility [i.e., with *nontrivial relationships* between *local units* and *local value groups* at nonarchimedean primes with distinct residue characteristics] may also be seen quite *explicitly* in the structure of the various types of *prime-strips* that appear in inter-universal Teichmüller theory [cf. [IUTchI], Fig. I1.2]. That is to say, such nontrivial relationships, which form the content of the global reciprocity law of global class field theory, may be thought of as a sort of **global Galois-theoretic representation** of the **constraints** that constitute the **arithmetic holomorphic structure** of conventional scheme theory.

(vi) Another fundamental aspect of the comparison discussed in (ii) concerning (b-4) may be seen in the fact that whereas the global reciprocity law of global class field theory concerns the **global reciprocity map**, the cyclotomic rigidity algorithms of inter-universal Teichmüller theory to which (b-4) was compared appear in the context of **Kummer-theoretic isomorphisms**. That is to say, although both the global reciprocity map and Kummer-theoretic isomorphisms involve correspondences between multiplicative monoids associated to number fields and multiplicative monoids that arise from global Galois groups, one *fundamental difference* between these two types of correspondence lies in the fact that whereas

> **Kummer-theoretic isomorphisms** satisfy very **strong covariant** [with respect to *functions*] **functoriality** properties,

the *reciprocity maps* that appear in various versions of class field theory *tend not to satisfy such strong functoriality properties.* This presence or absence of strong functoriality properties is, to a substantial extent, a reflection of the fact that whereas *Kummer theory* may be performed in a *very straightforward, tautological, "general nonsense"* fashion in a *wide variety of situations*,

> **class field theory** may only be conducted in **very special arithmetic situations**.

This presence of strong functoriality properties [i.e., in the case of Kummer theory] is the essential reason for the *central role* played by *Kummer theory* [cf. (b-5)] in *inter-universal Teichmüller theory*, as well as in many situations that arise in *anabelian geometry* in general [cf., e.g., the theory of [Cusp]]. Indeed, the very **tautological/ubiquitous/strongly functorial nature of Kummer theory** makes it well-suited to the sort of **dismantling** of ring structures that occurs in inter-universal Teichmüller theory, as well as to the various **evaluation** operations of **functions** at **special points** that play a *central role*, in the context of **Galois evaluation**, in inter-universal Teichmüller theory [cf. the discussion of [IUTchIII], Remark 2.3.3]. By contrast, although there exist various higher-dimensional versions of class field theory involving higher algebraic K-groups, these versions of class field theory are *fundamentally incompatible* with the crucial *evaluation of function* operations of the sort that occur in inter-universal Teichmüller theory. Indeed, more generally, except for very *exceptional classical cases* involving *exponential functions* in the case of $\mathbb{Q}$ or *modular and elliptic functions* in the case of imaginary quadratic fields,

> **class field theory** tends to be very **ill-suited** to situations that involve the **evaluation of special functions at special points**.

Moreover, even if one restricts one's attention, for instance, to functoriality with respect to passing to a finite extension field, the functoriality of the reciprocity maps that occur in class field theory are [unlike *Kummer-theoretic isomorphisms*!] **contravariant** [with respect to functions] and can only be made *covariant* if one applies some sort of nontrivial **duality** result to reverse the direction of the maps —

a state of affairs that makes class field theory very difficult to apply not only in inter-universal Teichmüller theory, but also in many situations that arise in anabelian geometry. On the other hand, in the context of inter-universal Teichmüller theory, the *price*, so to speak, that one pays for the very *convenient, "general nonsense"* nature of Kummer theory lies in

> the **highly nontrivial nature** — which may be seen, for instance, in the establishment of various **multiradiality** properties — of the **cyclotomic rigidity algorithms** that appear in inter-universal Teichmüller theory [cf. the discussion of [IUTchIII], Remark 2.3.3].

Here, we recall that such cyclotomic rigidity algorithms — which *never appear* in discussions of conventional arithmetic geometry in which the *arithmetic holomorphic structure* is *held fixed* — play a *central role* in inter-universal Teichmüller theory precisely because of the *indeterminacies* that arise as a consequence of the *dismantling* of the arithmetic holomorphic structure. Finally, in this context, it is of interest to recall that, although *local class field theory* is, in a certain limited sense, applied in inter-universal Teichmüller theory, i.e., in order to obtain *cyclotomic rigidity algorithms* for *MLF-Galois pairs* [cf. [IUTchII], Proposition 1.3, (ii)], it is only *"of limited use"* in the sense that the resulting cyclotomic rigidity algorithms are **uniradial** [i.e., fail to be *multiradial* — cf. [IUTchIII], Figs. 2.1, 3.7, and the surrounding discussions].

(vii) The *fundamental incompatibility* — i.e., except in very *exceptional classical cases* involving *exponential functions* in the case of $\mathbb{Q}$ or *modular and elliptic functions* in the case of imaginary quadratic fields — discussed in (vi) of *class field theory* with situations that involve the *evaluation of special functions at special points* is highly reminiscent of the *original point of view* of class field theory in the early twentieth century [cf. **Kronecker's Jugendtraum**, **Hilbert's twelfth problem**], i.e., to the effect that further development of class field theory should proceed precisely by *extending* the theory involving *evaluation of special functions at special points* that exists in these "exceptional classical cases" to the case of *arbitrary number fields.* This state of affairs is, in turn, highly reminiscent of the fact that the approach taken in the above discussion to "dissecting global class field theory" is the **oldest/original approach** to global class field theory, as well as of the fact that this original approach is the most well-suited to discussions of *comparisons* between the theory of [Falt] and inter-universal Teichmüller theory. This state of affairs is also highly reminiscent of the discussion in [Pano], §3, §4, of the *numerous analogies* between inter-universal Teichmüller theory and the classical [i.e., dating back to the nineteenth century!] theory surrounding **Jacobi's identity for the theta function** on the upper half-plane and **Gaussian distributions/integrals**. Finally, this collection of observations, taken as a whole, may be summarized as follows:

> *Many of the ideas that appear in* **inter-universal Teichmüller theory** *bear a* **much closer resemblance** *to the mathematics of the* **late nineteenth** *and* **early twentieth centuries** *— i.e., to the mathematics of* **Gauss***,* **Jacobi***,* **Kummer***,* **Kronecker***,* **Weber***,* **Frobenius***,* **Hilbert***, and* **Teichmüller** *— than to the mathematics of the mid- to late twentieth century. This close resemblance suggests strongly that, relative to*

> *the mathematics of the late nineteenth and early twentieth centuries, the course of development of a substantial portion of the mathematics of the mid- to late twentieth century should* **not** *be regarded as* **"unique"** *or* **"inevitable"***, but rather as being merely* **one possible choice** *among many viable and fruitful alternatives that existed* **a priori**.

Here, we note that although the use, in inter-universal Teichmüller theory, of **Belyi** maps, as well as of the ***p*-adic anabelian geometry** of the 1990's [i.e., [pGC]], may at first glance look like an incidence of "exceptions" to the "rule" constituted by this point of view, these "exceptions" may be thought of as "proving the rule" in the sense that they are *far from typical* of the mathematics of the late twentieth century.

Remark 2.3.4. Various aspects of the theory of the present series of papers are *substantially reminiscent* of the theory surrounding **Bogomolov's proof** of the **geometric** version of the **Szpiro Conjecture**, as discussed in [ABKP], [Zh]. Put another way, these aspects of the theory of the present series of papers may be thought of as **arithmetic analogues** of the geometric theory surrounding Bogomolov's proof. Alternatively, Bogomolov's proof may be thought of as a sort of **useful elementary guide**, or **blueprint** [perhaps even a sort of **Rosetta stone**!], for understanding substantial portions of the theory of the present series of papers. The author would like to express his gratitude to *Ivan Fesenko* for bringing to his attention, via numerous discussions in person, e-mails, and skype conversations between December 2014 and January 2015, the possibility of the existence of such fascinating connections between Bogomolov's proof and the theory of the present series of papers. We discuss these analogies in more detail in [BogIUT].

Remark 2.3.5. In [Par], a proof is given of the **Mordell Conjecture** *for function fields over the* **complex numbers**. Like the proof of Bogomolov discussed in Remark 2.3.4, Parshin's proof involves **metric estimates** of **"displacements"** that arise from actions of elements of the [usual topological] **fundamental group** of the complex hyperbolic curve that serves as the *base scheme of the given family of curves*. In particular, we observe that one may pose the following *question*:

> Is it possible to apply *some portion of the ideas of the* **inter-universal Teichmüller theory** developed in the present series of papers to obtain a *proof of the* **Mordell Conjecture** *over number fields* **without** making use of **Belyi maps** as in the proof of Corollary 2.3 [i.e., the proof of [GenEll], Theorem 2.1]?

This question was posed to the author by *Felipe Voloch* in an e-mail in September 2015. The answer to this question is, as far as the author can see at the time of writing, *"no"*. On the other hand, this question is interesting in the context of the discussion of Remarks 2.3.3 and 2.3.4 in that it serves to highlight various interesting aspects of inter-universal Teichmüller theory, as we explain in the following discussion.

(i) First, we recall [cf., e.g., [Lang2], Chapter I, §1, §2, for more details] that the starting point of the theory of the *Kobayashi distance* on a [Kobayashi] hyperbolic complex manifold is the well-known *Schwarz lemma* of elementary complex analysis and its consequences for the geometry of holomorphic maps from the open unit disc D in the complex plane to an arbitrary complex manifold. In the following discussion, we shall refer to this geometry as the *Schwarz-theoretic geometry* of D. Perhaps the most *fundamental difference* between the proofs of Parshin and Bogomolov lies in the fact that

(PB1) Whereas Parshin's proof revolves around *estimates of displacements* arising from actions of elements of the fundamental group on a certain *two-dimensional complete [Kobayashi] hyperbolic complex manifold* by means of the **holomorphic** geometry of the **Kobayashi distance**, i.e., in effect, the **Schwarz-theoretic geometry** of D, Bogomolov's proof [cf. the review of Bogomolov's proof given in [BogIUT]] revolves around *estimates of displacements* arising from actions of elements of the fundamental group on a *one-dimensional real analytic manifold* [i.e., a universal covering of a copy of the unit circle $\mathbb{S}^1$] by means of the **real analytic symplectic geometry** of the upper half-plane.

Here, it is already interesting to note that this fundamental gap, in the case of results over *complex function fields*, between the *holomorphic* geometry applied in Parshin's proof of the *Mordell Conjecture* and the *real analytic symplectic* geometry applied in Bogomolov's proof of the *Szpiro Conjecture* is highly reminiscent of the fundamental gap discussed in Remark 2.3.3, (iii), in the case of results over *number fields*, between the **arithmetically holomorphic** nature of the proof of the *Mordell Conjecture* given in [Falt] and the **"arithmetically quasi-conformal"** nature of the proof of the *Szpiro Conjecture* [cf. Corollary 2.3] via inter-universal Teichmüller theory given in the present series of papers. That is to say,

> Parshin's proof is best understood **not** as a *"weaker, or simplified, version of Bogomolov's proof obtained by extracting certain portions of Bogomolov's proof"*, but rather as a proof that reflects a *fundamentally qualitatively different geometry* — i.e., **holomorphic**, as opposed to **real analytic** — from Bogomolov's proof.

This point of view already suggests rather strongly, relative to the analogy between Bogomolov's proof and inter-universal Teichmüller theory [cf. [BogIUT]] that it is *unnatural/unrealistic* to expect to obtain a new proof of the Mordell Conjecture over number fields by applying *some portion of the ideas of the inter-universal Teichmüller theory.*

(ii) At a more technical level, the fundamental difference (PB1) discussed in (i) may be seen in the fact that

(PB2) whereas Parshin's proof involves **numerous holomorphic maps** from the open unit disc D into one- and two-dimensional complex manifolds [i.e., in essence, the universal coverings of the *base space* and *total space* of the family of curves under consideration], Bogomolov's proof revolves around the real analytic symplectic geometry of a **fixed copy** of the open unit disc D [or, equivalently, the upper half-plane], i.e., in Bogomolov's proof, *one never considers holomorphic maps from D to itself which are not biholomorphic.*

The essentially arbitrary nature of these numerous holomorphic maps that appear in Parshin's proof is reflected in the fact that

(PB3) Parshin's proof is well-suited to proving a **rough qualitative** [i.e., **"finiteness"**] result for families of curves of **arbitrary genus** ≥ 2, whereas Bogomolov's proof is well-suited to proving a much finer **explicit inequality**, but only in the case of families of **elliptic curves**.

Another technical aspect of the proofs of Parshin and Bogomolov that is closely related to both (PB2) and (PB3) is the fact that

(PB4) whereas the estimation apparatus of Bogomolov's proof depends in an essential way on special properties of particular types of elements — such as **unipotent** elements or **commutators** — of the fundamental group under consideration, the estimation apparatus of Parshin's proof is uniform for **arbitrary [“sufficiently small”] elements** of the fundamental group under consideration.

(iii) Although, as discussed in (ii), it is difficult to see how Parshin's proof could be *"embedded"* into [i.e., obtained as a *"suitable portion of"*] Bogomolov's proof, the **Schwarz-theoretic geometry** of D admits a *"natural embedding"* into [i.e., admits a natural analogy to a suitable portion of] inter-universal Teichmüller theory, namely, in the form of the theory of *categories of localizations* of the sort that appear in [GeoAnbd], §2; [AbsTopI], §4; [AbsTopII], §3. This theory of categories of localizations culminates in the theory of **Belyi cuspidalizations**, which is discussed in [AbsTopII], §3, and applied to obtained the **mono-anabelian reconstruction algorithms** of [AbsTopIII], §1. Moreover, the analogy between such categories of localizations and the classical *Schwarz-theoretic geometry* of D [or, equivalently, the upper half-plane] is discussed in the Introduction to [GeoAnbd], as well as in [IUTchI], Remark 5.1.4. This theory of categories of localizations may be summarized roughly as follows:

> In the context of **absolute anabelian geometry** over number fields and their nonarchimedean localizations, **Belyi maps** play the role of the **Schwarz-theoretic geometry** of the open unit disc D, i.e., the role of realizing a sort of **arithmetic** version of **analytic continuation**.

This point of view is also interesting from the point of view of the discussion of Remark 2.2.4, (iii), i.e., to the effect that [noncritical] **Belyi maps** play the role of realizing a sort of **arithmetic** version of **analytic continuation** in the proof of [GenEll], Theorem 2.1. That is to say, from the point of view of the *question* posed at the beginning of the present Remark 2.3.5:

> Even if, in the context of inter-universal Teichmüller theory, one attempts to search for an analogue of Parshin's proof in the form of a "suitable portion" of the inter-universal Teichmüller theory developed in [IUTchI], [IUTchII], [IUTchIII] [i.e., even if one *avoids* consideration of the application of [noncritical] Belyi maps in the proof of Corollary 2.3 via [GenEll], Theorem 2.1], one is ultimately led — i.e., from the point of view of considering arithmetic analogues of the classical complex theory of **analytic continuation** and the **Schwarz-theoretic geometry** of the open unit disc D — to the Belyi maps that appear in the **Belyi cuspidalizations** of [AbsTopII], §3; [AbsTopIII], §1.

Put another way, it appears that any *search* in the realm of inter-universal Teichmüller theory either for *some* proof of the Mordell Conjecture [over number fields] or for *some* analogue of Parshin's proof [of the Mordell Conjecture over complex function fields] appears to lead inevitably to *some* application of **Belyi maps** to realize *some* sort of arithmetic analogue of the classical complex theory of **analytic continuation** and the **Schwarz-theoretic geometry** of the open unit disc D.

Section 3: Inter-universal Formalism: the Language of Species

In the present §3, we develop — albeit from an extremely *naive/non-expert* point of view, relative to the theory of foundations! — the language of **species**. Roughly speaking, a "species" is a **"type of mathematical object"**, such as a "group", a "ring", a "scheme", etc. In some sense, this language may be thought of as an *explicit description* of certain tasks typically executed at an *implicit, intuitive level* by mathematicians [i.e., mathematicians who are not equipped with a detailed knowledge of the theory of foundations!] via a sort of "mental arithmetic" in the course of interpreting various mathematical arguments. In the context of the theory developed in the present series of papers, however, it is useful to describe these intuitive operations explicitly.

In the following discussion, we shall work with various **models** — consisting of *"sets"* and a *relation* "$\in$" — of the standard *ZFC axioms* of axiomatic set theory [i.e., the nine axioms of *Zermelo-Fraenkel*, together with the *axiom of choice* — cf., e.g., [Drk], Chapter 1, §3]. We shall refer to such models as **ZFC-models**. Recall that a *(Grothendieck) universe* V is a set satisfying the following axioms [cf. [McLn], p. 194]:

(i) V is *transitive*, i.e., if $y \in x$, $x \in V$, then $y \in V$.
(ii) The set of *natural numbers* $\mathbb{N} \in V$.
(iii) If $x \in V$, then the *power set of* x also belongs to V.
(iv) If $x \in V$, then the *union of all members of* x also belongs to V.
(v) If $x \in V$, $y \subseteq V$, and $f : x \to y$ is a *surjection*, then $y \in V$.

We shall say that a set E is a *V-set* if $E \in V$.

The various ZFC-models that we work with may be thought of as [but are *not restricted* to be!] the ZFC-models determined by various *universes* that are sets relative to some *ambient ZFC-model* which, in addition to the standard axioms of ZFC set theory, satisfies the following **existence axiom** [attributed to the "Grothendieck school" — cf. the discussion of [McLn], p. 193]:

($\dagger^{\mathrm{G}}$) *Given any set x, there exists a universe V such that $x \in V$.*

We shall refer to a ZFC-model that also satisfies this additional axiom of the Grothendieck school as a *ZFCG-model*. This existence axiom ($\dagger^{\mathrm{G}}$) implies, in particular, that:

> *Given a set I and a collection of universes V_i, where $i \in I$, indexed by I [i.e., a 'function' $I \ni i \mapsto V_i$], there exists a [larger] universe V such that $V_i \in V$, for $i \in I$.*

Indeed, since the graph of the function $I \ni i \mapsto V_i$ is a *set*, it follows that $\{V_i\}_{i \in I}$ is a *set*. Thus, it follows from the *existence axiom* ($\dagger^{\mathrm{G}}$) that there exists a universe V such that $\{V_i\}_{i \in I} \in V$. Hence, by condition (i), we conclude that $V_i \in V$, for all $i \in I$, as desired. Note that this means, in particular, that there exist *infinite ascending chains of universes*

$$V_0 \in V_1 \in V_2 \in V_3 \in \ldots \in V_n \in \ldots \in V$$

— where n ranges over the natural numbers. On the other hand, by the *axiom of foundation*, there do not exist *infinite descending chains of universes*

$$V_0 \ni V_1 \ni V_2 \ni V_3 \ni \ldots \ni V_n \ni \ldots$$

— where n ranges over the natural numbers.

Although we shall not discuss in detail here the quite difficult issue of *whether or not there actually exist ZFCG-models*, we remark in passing that it may be possible to justify the stance of ignoring such issues in the context of the present series of papers — at least from the point of view of establishing the validity of various *"final results"* that may be formulated in ZFC-models — by invoking the work of *Feferman* [cf. [Ffmn]]. Precise statements concerning such issues, however, lie beyond the scope of the present paper [as well as of the level of expertise of the author!].

In the following discussion, we use the phrase *"set-theoretic formula"* as it is conventionally used in discussions of *axiomatic set theory* [cf., e.g., [Drk], Chapter 1, §2], with the following *proviso*: In the following discussion, it should be understood that every set-theoretic formula that appears is *"absolute"* in the sense that its validity for a collection of sets contained in some universe V relative to the model of set theory determined by V is *equivalent*, for any universe W such that $V \in W$, to its validity for the same collection of sets relative to the model of set theory determined by W [cf., e.g., [Drk], Chapter 3, Definition 4.2].

Definition 3.1.

(i) A 0-*species* $\mathfrak{S}_0$ is a collection of conditions given by a *set-theoretic formula*

$$\Phi_0(\mathfrak{E})$$

involving an ordered collection $\mathfrak{E} = (\mathfrak{E}_1, \ldots, \mathfrak{E}_{n_0})$ of sets $\mathfrak{E}_1, \ldots, \mathfrak{E}_{n_0}$ [which we think of as "indeterminates"], for some integer $n_0 \geq 1$; in this situation, we shall refer to $\mathfrak{E}$ as a *collection of species-data* for $\mathfrak{S}_0$. If $\mathfrak{S}_0$ is a 0-species given by a set-theoretic formula $\Phi_0(\mathfrak{E})$, then a 0-*specimen* of $\mathfrak{S}_0$ is a *specific* ordered collection of n_0 sets $E = (E_1, \ldots, E_{n_0})$ in some *specific* ZFC-model that satisfies $\Phi_0(E)$. If E is a 0-specimen of a 0-species $\mathfrak{S}_0$, then we shall write $E \in \mathfrak{S}_0$. If, moreover, it holds, in any ZFC-model, that the 0-specimens of $\mathfrak{S}_0$ form a *set*, then we shall refer to $\mathfrak{S}_0$ as 0-*small*.

(ii) Let $\mathfrak{S}_0$ be a 0-species. Then a 1-*species* $\mathfrak{S}_1$ *acting on* $\mathfrak{S}_0$ is a collection of *set-theoretic formulas* Φ_1, $\Phi_{1\circ 1}$ satisfying the following conditions:

(a) Φ_1 is a set-theoretic formula

$$\Phi_1(\mathfrak{E}, \mathfrak{E}', \mathfrak{F})$$

involving two collections of species-data $\mathfrak{E}$, $\mathfrak{E}'$ for $\mathfrak{S}_0$ [i.e., the conditions $\Phi_0(\mathfrak{E})$, $\Phi_0(\mathfrak{E}')$ hold] and an ordered collection $\mathfrak{F} = (\mathfrak{F}_1, \ldots, \mathfrak{F}_{n_1})$ of ["in-determinate"] sets $\mathfrak{F}_1, \ldots, \mathfrak{F}_{n_1}$, for some integer $n_1 \geq 1$; in this situation, we shall refer to $(\mathfrak{E}, \mathfrak{E}', \mathfrak{F})$ as a *collection of species-data* for $\mathfrak{S}_1$ and write

$\mathfrak{F} : \mathfrak{E} \to \mathfrak{E}'$. If, in some ZFC-model, $E, E' \in \mathfrak{S}_0$, and F is a *specific* ordered collection of n_1 sets that satisfies the condition $\Phi_1(E, E', F)$, then we shall refer to the data (E, E', F) as a 1-*specimen* of $\mathfrak{S}_1$ and write $(E, E', F) \in \mathfrak{S}_1$; alternatively, we shall denote a 1-specimen (E, E', F) via the notation $F : E \to E'$ and refer to E (respectively, E') as the *domain* (respectively, *codomain*) of $F : E \to E'$.

(b) $\Phi_{1\circ 1}$ is a set-theoretic formula

$$\Phi_{1\circ 1}(\mathfrak{E}, \mathfrak{E}', \mathfrak{E}'', \mathfrak{F}, \mathfrak{F}', \mathfrak{F}'')$$

involving three collections of species-data $\mathfrak{F} : \mathfrak{E} \to \mathfrak{E}'$, $\mathfrak{F}' : \mathfrak{E}' \to \mathfrak{E}''$, $\mathfrak{F}'' : \mathfrak{E} \to \mathfrak{E}''$ for $\mathfrak{S}_1$ [i.e., the conditions $\Phi_0(\mathfrak{E})$; $\Phi_0(\mathfrak{E}')$; $\Phi_0(\mathfrak{E}'')$; $\Phi_1(\mathfrak{E}, \mathfrak{E}', \mathfrak{F})$; $\Phi_1(\mathfrak{E}', \mathfrak{E}'', \mathfrak{F}')$; $\Phi_1(\mathfrak{E}, \mathfrak{E}'', \mathfrak{F}'')$ hold]; in this situation, we shall refer to $\mathfrak{F}''$ as a *composite of* $\mathfrak{F}$ *with* $\mathfrak{F}'$ and write $\mathfrak{F}'' = \mathfrak{F}' \circ \mathfrak{F}$ [which is, *a priori*, an abuse of notation, since there may exist *many* composites of $\mathfrak{F}$ with $\mathfrak{F}'$ — cf. (c) below]; we shall use similar terminology and notation for 1-specimens in specific ZFC-models.

(c) Given a pair of 1-specimens $F : E \to E'$, $F' : E' \to E''$ of $\mathfrak{S}_1$ in some ZFC-model, there *exists a unique composite* $F'' : E \to E''$ of F with F' in the given ZFC-model.

(d) Composition of 1-specimens $F : E \to E'$, $F' : E' \to E''$, $F'' : E'' \to E'''$ of $\mathfrak{S}_1$ in a ZFC-model is *associative*.

(e) For any 0-specimen E of $\mathfrak{S}_0$ in a ZFC-model, there exists a [necessarily unique] 1-specimen $F : E \to E$ of $\mathfrak{S}_1$ [in the given ZFC-model] — which we shall refer to as the *identity* 1-*specimen* id_E of E — such that for any 1-specimens $F' : E' \to E$, $F'' : E \to E''$ of $\mathfrak{S}_1$ [in the given ZFC-model] we have $F \circ F' = F'$, $F'' \circ F = F''$.

If, moreover, it holds, in any ZFC-model, that for any two 0-specimens E, E' of $\mathfrak{S}_0$, the 1-specimens $F : E \to E'$ of $\mathfrak{S}_1$ [i.e., the 1-specimens of $\mathfrak{S}_1$ with domain E and codomain E'] form a *set*, then we shall refer to $\mathfrak{S}_1$ as 1-*small*.

(iii) A *species* $\mathfrak{S}$ is defined to be a pair consisting of a 0-species $\mathfrak{S}_0$ and a 1-species $\mathfrak{S}_1$ acting on $\mathfrak{S}_0$. Fix a species $\mathfrak{S} = (\mathfrak{S}_0, \mathfrak{S}_1)$. Let $i \in \{0, 1\}$. Then we shall refer to an i-specimen of $\mathfrak{S}_i$ as an *i-specimen of* $\mathfrak{S}$. We shall refer to a 0-specimen (respectively, 1-specimen) of $\mathfrak{S}$ as a *species-object* (respectively, a *species-morphism*) of $\mathfrak{S}$. We shall say that $\mathfrak{S}$ is *i-small* if $\mathfrak{S}_i$ is i-small. We shall refer to a species-morphism $F : E \to E'$ as a *species-isomorphism* if there exists a species-morphism $F' : E' \to E$ such that the composites $F \circ F'$, $F' \circ F$ are *identity* species-morphisms; in this situation, we shall say that E, E' are *species-isomorphic*. [Thus, one verifies immediately that *composites* of *species-isomorphisms* are species-isomorphisms.] We shall refer to a species-isomorphism whose domain and codomain are equal as a *species-automorphism*. We shall refer to as *model-free* [cf. Remark 3.1.1 below] an i-specimen of $\mathfrak{S}$ equipped with a description via a *set-theoretic formula* that is "*independent* of the ZFC-model in which it is given" in the sense that for any pair of universes V_1, V_2 of some ZFC-model such that $V_1 \in V_2$, the set-theoretic formula determines the *same* i-specimen of $\mathfrak{S}$, whether interpreted relative to the ZFC-model determined by V_1 or the ZFC-model determined by V_2.

(iv) We shall refer to as the *category determined by* $\mathfrak{S}$ in a ZFC-model the *category* whose objects are the *species-objects* of $\mathfrak{S}$ in the given ZFC-model and whose arrows are the *species-morphisms* of $\mathfrak{S}$ in the given ZFC-model. [One verifies immediately that this description does indeed determine a category.]

Remark 3.1.1. We observe that any of the familiar descriptions of $\mathbb{N}$ [cf., e.g., [Drk], Chapter 2, Definitions 2.3, 2.9], $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}_p$, or $\mathbb{R}$, for instance, yield *species* [all of whose species-morphisms are identity species-morphisms] each of which has a *unique* species-object in any given ZFC-model. Such species are *not to be confused* with such species as the species of "monoids isomorphic to $\mathbb{N}$ and monoid isomorphisms", which admits *many species-objects* [all of which are species-isomorphic] in any ZFC-model. On the other hand, the set-theoretic formula used, for instance, to define the former "species $\mathbb{N}$" may be applied to define a *"model-free species-object* $\mathbb{N}$*"* of the latter "species of monoids isomorphic to $\mathbb{N}$".

Remark 3.1.2.

(i) It is important to remember when working with species that

> the **essence** *of a species* lies *not in the specific sets* that occur as species-objects or species-morphisms of the species in various ZFC-models, but rather in the **collection of rules**, i.e., *set-theoretic formulas*, that govern the construction of such sets in an *unspecified, "indeterminate" ZFC-model.*

Put another way, the emphasis in the theory of species lies in the *programs* — i.e., **"software"** — that yield the desired output data, *not on the output data itself.* From this point of view, one way to describe the various set-theoretic formulas that constitute a species is as a *"deterministic algorithm"* [a term suggested to the author by Minhyong Kim] for constructing the sets to be considered.

(ii) One interesting point of view that arose in discussions between the author and F. Kato is the following. The relationship between the classical approach to discussing mathematics relative to a *fixed model of set theory* — an approach in which *specific sets* play a central role — and the *"species-theoretic"* approach considered here — in which the **rules**, given by set-theoretic formulas for constructing the sets of interest [i.e., not specific sets themselves!], play a central role — may be regarded as *analogous* to the relationship between *classical approaches to algebraic varieties* — in which specific sets of solutions of polynomial equations in an algebraically closed field play a central role — and *scheme theory* — in which the functor determined by a scheme, i.e., the polynomial equations, or "rules", that determine solutions, as opposed to specific sets of solutions themselves, play a central role. That is to say, in summary:

$$[\text{fixed model of set theory approach} : \text{species-theoretic approach}] \quad \longleftrightarrow \quad [\text{varieties} : \text{schemes}]$$

A similar analogy — i.e., of the form

$$[\text{fixed model of set theory approach} : \text{species-theoretic approach}] \quad \longleftrightarrow \quad [\text{groups of specific matrices} : \text{abstract groups}]$$

— may be made to the notion of an "abstract group", as opposed to a "group of specific matrices". That is to say, just as a "group of specific matrices may be thought of as a *specific representation* of an "abstract group", the category of objects determined by a species in a specific ZFC-model may be thought of as a *specific representation* of an "abstract species".

(iii) If, in the context of the discussion of (i), (ii), one tries to form a sort of *quotient*, in which "programs" that yield the same sets as "output data" are *identified*, then one must contend with the resulting *indeterminacy*, i.e., working with programs is only well-defined up to internal modifications of the programs in question that does not affect the final output. This leads to somewhat *intractable problems* concerning the internal structure of such programs — a topic that lies well beyond the scope of the present work.

Remark 3.1.3.

(i) Typically, in the discussion to follow, we shall not write out explicitly the various set-theoretic formulas involved in the definition of a species. Rather, it is to be understood that the set-theoretic formulas to be used are those arising from the *conventional descriptions* of the mathematical objects involved. When applying such conventional descriptions, however, it is important to check that they are *well-defined* and *do* **not** *depend* upon the use of **arbitrary choices** that are not describable via well-defined set-theoretic formulas.

(ii) The fact that the data involved in a species is given by abstract *set-theoretic formulas* imparts a certain **canonicality** to the mathematical notion constituted by the species, a canonicality that is **not shared**, for instance, by mathematical objects whose construction depends on an **invocation of the axiom of choice** in some particular ZFC-model [cf. the discussion of (i) above]. Moreover, by furnishing a stock of such "canonical notions", the theory of species allows one, in effect, to *compute the* **extent of deviation** of various *"non-canonical objects"* [i.e., whose construction depends upon the invocation of the axiom of choice!] from a sort of *"canonical norm"*.

Remark 3.1.4. Note that because the data involved in a species is given by abstract *set-theoretic formulas*, the mathematical notion constituted by the species is **immune** to, i.e., unaffected by, **extensions of the universe** — i.e., such as the ascending chain $V_0 \in V_1 \in V_2 \in V_3 \in \ldots \in V_n \in \ldots \in V$ that appears in the discussion preceding Definition 3.1 — in which one works. This is the sense in which we apply the term **"inter-universal"**. That is to say, *"inter-universal geometry"* allows one to relate the "geometries" that occur in distinct universes.

Remark 3.1.5. Similar remarks to the remarks made in Remarks 3.1.2, 3.1.3, and 3.1.4 concerning the significance of working with *set-theoretic formulas* may be made with regard to the notions of *mutations*, *morphisms of mutations*, *mutation-histories*, *observables*, and *cores* to be introduced in Definition 3.3 below.

One fundamental example of a species is the following.

Example 3.2. **Categories.** The notions of a [small] category and an isomorphism class of [covariant] functors between two given [small] categories yield an example of a *species*. That is to say, at a set-theoretic level, one may think of a [small] *category* as, for instance, a set of arrows, together with a set of composition relations, that satisfies certain properties; one may think of a [covariant] *functor* between [small] categories as the set given by the graph of the map on arrows determined by the functor [which satisfies certain properties]; one may think of an *isomorphism class of functors* as a collection of such graphs, i.e., the graphs determined by the functors in the isomorphism class, which satisfies certain properties. Then one has *"dictionaries"*

$$\text{0-species} \quad \longleftrightarrow \quad \text{the notion of a category}$$

$$\text{1-species} \quad \longleftrightarrow \quad \text{the notion of an isomorphism class of functors}$$

at the level of *notions* and

$$\text{a 0-specimen} \quad \longleftrightarrow \quad \text{a particular [small] category}$$

$$\text{a 1-specimen} \quad \longleftrightarrow \quad \text{a particular isomorphism class of functors}$$

at the level of *specific mathematical objects* in a specific ZFC-model. Moreover, one verifies easily that species-isomorphisms between 0-species correspond to isomorphism classes of equivalences of categories in the usual sense.

Remark 3.2.1. Note that in the case of Example 3.2, one could also define a notion of *"2-species"*, *"2-specimens"*, etc., via the notion of an "isomorphism of functors", and then take the 1-species under consideration to be the notion of a functor [i.e., not an isomorphism class of functors]. Indeed, more generally, one could define a notion of "n-species" for arbitrary integers $n \geq 1$. Since, however, this approach would only serve to add an *unnecessary level of complexity* to the theory, we choose here to take the approach of working with "functors considered up to isomorphism".

Definition 3.3. Let $\mathfrak{S} = (\mathfrak{S}_0, \mathfrak{S}_1)$; $\underline{\mathfrak{S}} = (\underline{\mathfrak{S}}_0, \underline{\mathfrak{S}}_1)$ be *species*.

(i) A *mutation* $\mathfrak{M} : \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}}$ is defined to be a collection of *set-theoretic formulas* Ψ_0, Ψ_1 satisfying the following properties:

(a) Ψ_0 is a set-theoretic formula

$$\Psi_0(\mathfrak{E}, \underline{\mathfrak{E}})$$

involving a collection of species-data $\mathfrak{E}$ for $\mathfrak{S}_0$ and a collection of species-data $\underline{\mathfrak{E}}$ for $\underline{\mathfrak{S}}_0$; in this situation, we shall write $\mathfrak{M}(\mathfrak{E})$ for $\underline{\mathfrak{E}}$. Moreover, if, in some ZFC-model, $E \in \mathfrak{S}_0$, then we require that there *exist a unique* $\underline{E} \in \underline{\mathfrak{S}}_0$ such that $\Psi_0(E, \underline{E})$ holds; in this situation, we shall write $\mathfrak{M}(E)$ for $\underline{E}$.

(b) Ψ_1 is a set-theoretic formula

$$\Psi_1(\mathfrak{E}, \mathfrak{E}', \mathfrak{F}, \underline{\mathfrak{F}})$$

involving a collection of species-data $\mathfrak{F} : \mathfrak{E} \to \mathfrak{E}'$ for $\mathfrak{S}_1$ and a collection of species-data $\underline{\mathfrak{F}} : \underline{\mathfrak{E}} \to \underline{\mathfrak{E}}'$ for $\underline{\mathfrak{S}}_1$, where $\underline{\mathfrak{E}} = \mathfrak{M}(\mathfrak{E})$, $\underline{\mathfrak{E}}' = \mathfrak{M}(\mathfrak{E}')$; in this situation, we shall write $\mathfrak{M}(\mathfrak{F})$ for $\underline{\mathfrak{F}}$. Moreover, if, in some ZFC-model, $(F : E \to E') \in \mathfrak{S}_1$, then we require that there *exist a unique* $(\underline{F} : \underline{E} \to \underline{E}') \in \underline{\mathfrak{S}}_1$ such that $\Psi_0(E, E', F, \underline{F})$ holds; in this situation, we shall write $\mathfrak{M}(F)$ for $\underline{F}$. Finally, we require that the assignment $F \mapsto \mathfrak{M}(F)$ be compatible with *composites* and map *identity* species-morphisms of $\mathfrak{S}$ to identity species-morphisms of $\underline{\mathfrak{S}}$. In particular, if one fixes a ZFC-model, then $\mathfrak{M}$ determines a *functor* from the category determined by $\mathfrak{S}$ in the given ZFC-model to the category determined by $\underline{\mathfrak{S}}$ in the given ZFC-model.

There are evident notions of *"composition of mutations"* and *"identity mutations"*.

(ii) Let $\mathfrak{M}, \mathfrak{M}' : \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}}$ be *mutations.* Then a *morphism of mutations* $\mathfrak{Z} : \mathfrak{M} \to \mathfrak{M}'$ is defined to be a *set-theoretic formula* Ξ satisfying the following properties:

(a) Ξ is a set-theoretic formula

$$\Xi(\mathfrak{E}, \underline{\mathfrak{F}})$$

involving a collection of species-data $\mathfrak{E}$ for $\mathfrak{S}_0$ and a collection of species-data $\underline{\mathfrak{F}} : \mathfrak{M}(\mathfrak{E}) \to \mathfrak{M}'(\mathfrak{E})$ for $\mathfrak{S}_1$; in this situation, we shall write $\mathfrak{Z}(\mathfrak{E})$ for $\underline{\mathfrak{F}}$. Moreover, if, in some ZFC-model, $E \in \mathfrak{S}_0$, then we require that there *exist a unique* $\underline{F} \in \underline{\mathfrak{S}}_1$ such that $\Xi(E, \underline{F})$ holds; in this situation, we shall write $\mathfrak{Z}(E)$ for $\underline{F}$.

(b) Suppose, in some ZFC-model, that $F : E_1 \to E_2$ is a species-morphism of $\mathfrak{S}$. Then one has an equality of composite species-morphisms $\mathfrak{M}'(F) \circ \mathfrak{Z}(E_1) = \mathfrak{Z}(E_2) \circ \mathfrak{M}(F) : \mathfrak{M}(E_1) \to \mathfrak{M}'(E_2)$. In particular, if one fixes a ZFC-model, then a morphism of mutations $\mathfrak{M} \to \mathfrak{M}'$ determines a *natural transformation* between the functors determined by $\mathfrak{M}$, $\mathfrak{M}'$ in the ZFC-model — cf. (i).

There are evident notions of *"composition of morphisms of mutations"* and *"identity morphisms of mutations"*. If it holds that for every species-object E of $\mathfrak{S}$, $\mathfrak{Z}(E)$ is a *species-isomorphism*, then we shall refer to $\mathfrak{Z}$ as an *isomorphism of mutations.* In particular, one verifies immediately that $\mathfrak{Z}$ is an isomorphism of mutations if and only if there exists a morphism of mutations $\mathfrak{Z}' : \mathfrak{M}' \to \mathfrak{M}$ such that the composite morphisms of mutations $\mathfrak{Z}' \circ \mathfrak{Z} : \mathfrak{M} \to \mathfrak{M}$, $\mathfrak{Z} \circ \mathfrak{Z}' : \mathfrak{M}' \to \mathfrak{M}'$ are the respective *identity* morphisms of the mutations $\mathfrak{M}$, $\mathfrak{M}'$.

(iii) Let $\mathfrak{M} : \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}}$ be a *mutation.* Then we shall say that $\mathfrak{M}$ is a *mutation-equivalence* if there exists a mutation $\mathfrak{M}' : \underline{\mathfrak{S}} \rightsquigarrow \mathfrak{S}$, together with isomorphisms of mutations between the composites $\mathfrak{M} \circ \mathfrak{M}'$, $\mathfrak{M}' \circ \mathfrak{M}$ and the respective identity mutations. In this situation, we shall say that $\mathfrak{M}$, $\mathfrak{M}'$ are *mutation-quasi-inverses* to one another. Finally, we observe that, if we suppose further that $\mathfrak{S}$, $\underline{\mathfrak{S}}$ are 1-*small*, then for any two given species-objects in the domain species of a mutation-equivalence, the *mutation-equivalence* induces a *bijection* between the set of *species-morphisms* (respectively, *species-isomorphisms*) between the two given species-objects [of the domain species] and the set of species-morphisms (respectively, species-isomorphisms) between the two species-objects [of the codomain species] obtained by applying the mutation-equivalence to the two given species-objects.

(iv) Let $\vec{\Gamma}$ be an *oriented graph*, i.e., a graph Γ, which we shall refer to as the *underlying graph* of $\vec{\Gamma}$, equipped with the additional data of a total ordering, for each edge e of Γ, on the set [of cardinality 2] of *branches* of e [cf., e.g., [AbsTopIII], §0]. Then we define a *mutation-history* $\mathfrak{H} = (\vec{\Gamma}, \mathfrak{S}^*, \mathfrak{M}^*)$ *[indexed by* $\vec{\Gamma}$*]* to be a collection of data as follows:

(a) for each vertex v of $\vec{\Gamma}$, a *species* $\mathfrak{S}^v$;

(b) for each edge e of $\vec{\Gamma}$, running from a vertex v_1 to a vertex v_2, a *mutation* $\mathfrak{M}^e : \mathfrak{S}^{v_1} \rightsquigarrow \mathfrak{S}^{v_2}$.

In this situation, we shall refer to the vertices, edges, and branches of $\vec{\Gamma}$ as vertices, edges, and branches of $\mathfrak{H}$. Thus, the notion of a "mutation-history" may be thought of as a *species-theoretic* version of the notion of a "diagram of categories" given in [AbsTopIII], Definition 3.5, (i).

(v) Let $\mathfrak{H} = (\vec{\Gamma}, \mathfrak{S}^*, \mathfrak{M}^*)$ be a *mutation-history*; $\underline{\mathfrak{S}}$ a *species*. For simplicity, we assume that the underlying graph of $\vec{\Gamma}$ is *simply connected*. Then we shall refer to as a(n) *[*$\underline{\mathfrak{S}}$*-valued] covariant* (respectively, *contravariant*) *observable* $\mathfrak{V}$ of the mutation-history $\mathfrak{H}$ a collection of data as follows:

(a) for each vertex v of $\vec{\Gamma}$, a mutation $\mathfrak{V}^v : \mathfrak{S}^v \to \underline{\mathfrak{S}}$, which we shall refer to as the *observation mutation* at v;

(b) for each edge e of $\vec{\Gamma}$, running from a vertex v_1 to a vertex v_2, a *morphism of mutations* $\mathfrak{V}^e : \mathfrak{V}^{v_1} \to \mathfrak{V}^{v_2} \circ \mathfrak{M}^e$ (respectively, $\mathfrak{V}^e : \mathfrak{V}^{v_2} \circ \mathfrak{M}^e \to \mathfrak{V}^{v_1}$).

If $\mathfrak{V}$ is a covariant observable such that all of the morphisms of mutations "$\mathfrak{V}^e$" are *isomorphisms of mutations*, then we shall refer to the covariant observable $\mathfrak{V}$ as a *core*. Thus, one may think of a core $\mathfrak{C}$ of a mutation-history as lying *"under"* the entire mutation-history in a *"uniform fashion"*. Also, we shall refer to the "property [of an observable] of being a core" as the *"coricity"* of the observable. Finally, we note that the notions of an "observable" and a "core" given here may be thought of as simplified, *species-theoretic* versions of the notions of "observable" and "core" given in [AbsTopIII], Definition 3.5, (iii).

Remark 3.3.1.

(i) One well-known consequence of the *axiom of foundation* of axiomatic set theory is the assertion that **"$\in$-loops"**

$$a \in b \in c \in \ldots \in a$$

can *never occur* in the set theory in which one works. On the other hand, there are many situations in mathematics in which one wishes to somehow **"identify"** mathematical objects that arise at *higher levels* of the $\in$-structure of the set theory under consideration with mathematical objects that arise at *lower levels* of this $\in$-structure. In some sense, the notions of a *"set"* and of a *"bijection of sets"* allow one to achieve such "identifications". That is to say, the mathematical objects at both higher and lower levels of the $\in$-structure constitute examples of the *same mathematical notion of a "set"*, so that one may consider "bijections of sets" between those sets without violating the axiom of foundation. In some sense, the notion of a **species** may be thought of as a natural *extension* of this observation. That is to say,

> the notion of a "species" allows one to consider, for instance, *species-isomorphisms* between species-objects that occur at *different levels* of the $\in$-structure of the set theory under consideration — i.e., roughly speaking, to **"simulate $\in$-loops"** — *without violating the axiom of foundation.*

Moreover, typically the sorts of species-objects at different levels of the $\in$-structure that one wishes to somehow have "identified" with one another occur as the result of executing the *mutations* that arise in some sort of **mutation-history**

$$\dots \rightsquigarrow \mathfrak{S} \rightsquigarrow \underline{\mathfrak{S}} \rightsquigarrow \underline{\underline{\mathfrak{S}}} \rightsquigarrow \dots \rightsquigarrow \mathfrak{S} \rightsquigarrow \dots$$

[where $\mathfrak{S} = (\mathfrak{S}_0, \mathfrak{S}_1)$; $\underline{\mathfrak{S}} = (\underline{\mathfrak{S}}_0, \underline{\mathfrak{S}}_1)$; $\underline{\underline{\mathfrak{S}}} = (\underline{\underline{\mathfrak{S}}}_0, \underline{\underline{\mathfrak{S}}}_1)$ are *species*] — e.g., the *"output species-objects"* of the "$\mathfrak{S}$" on the *right* that arise from applying various mutations to the *"input species-objects"* of the "$\mathfrak{S}$" on the *left*.

(ii) In the context of constructing *"loops"* in a mutation-history as in the final display of (i), we observe that

> the **simpler** the structure of the **species** involved, the **easier** it is to construct **"loops"**.

It is for this reason that species such as the species determined by the notion of a **category** [cf. Example 3.2] are easier to work with, from the point of view of constructing "loops", than more complicated species such as the species determined by the notion of a *scheme*. This is one of the *principal motivations* for the *"geometry of categories"* — of which *"absolute anabelian geometry"* is the special case that arises when the categories involved are Galois categories — i.e., for the theory of *representing scheme-theoretic geometries via categories* [cf., e.g., the Introductions of [MnLg], [SemiAnbd], [Cusp], [FrdI]]. At a more concrete level, the utility of working with categories to reconstruct objects that occurred at earlier stages of some sort of "series of constructions" [cf. the mutation-history of the final display of (i)!] may be seen in the *"reconstruction of the underlying scheme"* in various situations throughout [MnLg] by applying the *natural equivalence of categories* of the final display of [MnLg], Definition 1.1, (iv), from a certain category constructed from a log scheme, as well as in the theory of *"slim exponentiation"* discussed in the Appendix to [FrdI].

(iii) Again in the context of mutation-histories such as the one given in the final display of (i), although one may, on certain occasions, wish to apply various mutations that *fundamentally alter* the structure of the mathematical objects involved and hence give rise to *"output species-objects"* of the "$\mathfrak{S}$" on the *right* that are related in a *highly nontrivial fashion* to the *"input species-objects"* of the "$\mathfrak{S}$" on the *left*, it is also of interest to consider

> *"portions"* of the various mathematical objects that occur that are left **unaltered** by the various mutations that one applies.

This is precisely the reason for the introduction of the notion of a *core* of a mutation-history. One important consequence of the construction of various cores associated to a mutation-history is that often

> one may apply various cores associated to a mutation-history to **describe**, by means of **non-coric observables**, the portions of the various mathematical objects that occur which *are* **altered** by the various mutations that one applies *in terms of* the **unaltered** portions, i.e., **cores**.

Indeed, this point of view plays a *central role* in the theory of the present series of papers — cf. the discussion of Remark 3.6.1, (ii), below.

Remark 3.3.2. One somewhat *naive* point of view that constituted one of the original motivations for the author in the development of theory of the present series of papers is the following. In the classical theory of schemes, when considering **local systems** on a scheme, there is no reason to restrict oneself to considering local systems valued in, say, modules over a finite ring. If, moreover, there is no reason to make such a restriction, then one is naturally led to consider, for instance, local systems of *schemes* [cf., e.g., the theory of the "Galois mantle" in [pTeich]], or, indeed, local systems of more general **collections of mathematical objects**. One may then ask what happens if one tries to consider local systems on the schemes that occur as fibers of a local system of schemes. [More concretely, if X is, for instance, a connected scheme, then one may consider local systems $\mathcal{X}$ over X whose *fibers* are isomorphic to X; then one may repeat this process, by considering such local systems *over each fiber* of the local system $\mathcal{X}$ on X, etc.] In this way, one is eventually led to the consideration of **"systems of nested local systems"** — i.e., a local system over a local system over a local system, etc. It is precisely this point of view that underlies the notion of *"successive iteration of a given mutation-history"*, relative to the terminology formulated in the present §3. If, moreover, one thinks of such "successive iterates of a given mutation-history" as being a sort of abstraction of the naive idea of a "system of nested local systems", then the notion of a **core** may be thought of as a sort of mathematical object that is *invariant* with respect to the application of the operations that gave rise to the "system of nested local systems".

Example 3.4. Topological Spaces and Fundamental Groups.

(i) One verifies easily that the notions of a *topological space* and a *continuous map* between topological spaces determine an example of a *species* $\mathfrak{S}^{\mathrm{top}}$. In a similar vein, the notions of a *universal covering* $\widetilde{X} \to X$ of a pathwise connected topological space X and a *continuous map* between such universal coverings $\widetilde{X} \to X$, $\widetilde{Y} \to Y$ [i.e., a pair of compatible continuous maps $\widetilde{X} \to \widetilde{Y}$, $X \to Y$], considered up to composition with a *deck transformation* of the universal covering $\widetilde{Y} \to Y$, determine an example of a *species* $\mathfrak{S}^{\text{u-top}}$. We leave to the reader the routine task of writing out the various *set-theoretic formulas* that define the species structures of $\mathfrak{S}^{\mathrm{top}}$, $\mathfrak{S}^{\text{u-top}}$. Here, we note that at a set-theoretic level, the species-morphisms of $\mathfrak{S}^{\text{u-top}}$

are *collections* of continuous maps [between two given universal coverings], any two of which differ from one another by composition with a deck transformation.

(ii) One verifies easily that the notions of a *group* and an *outer homomorphism* between groups [i.e., a homomorphism considered up to composition with an inner automorphism of the codomain group] determine an example of a *species* $\mathfrak{S}^{\mathrm{gp}}$. We leave to the reader the routine task of writing out the various *set-theoretic formulas* that define the species structure of $\mathfrak{S}^{\mathrm{gp}}$. Here, we note that at a set-theoretic level, the species-morphisms of $\mathfrak{S}^{\mathrm{gp}}$ are *collections* of homomorphisms [between two given groups], any two of which differ from one another by composition with an inner automorphism.

(iii) Now one verifies easily that the assignment

$$(\widetilde{X} \to X) \quad \mapsto \quad \mathrm{Aut}(\widetilde{X}/X)$$

— where $(\widetilde{X} \to X)$ is a species-object of $\mathfrak{S}^{\text{u-top}}$, and $\mathrm{Aut}(\widetilde{X}/X)$ denotes the group of deck transformations of the universal covering $\widetilde{X} \to X$ — determines a *mutation* $\mathfrak{S}^{\text{u-top}} \rightsquigarrow \mathfrak{S}^{\mathrm{gp}}$. That is to say, the *"fundamental group"* may be thought of as a sort of mutation.

Example 3.5. Absolute Anabelian Geometry.

(i) Let $\mathcal{S}$ be a class of connected normal schemes that is closed under isomorphism [of schemes]. Suppose that there exists a set $E_{\mathcal{S}}$ of schemes describable by a *set-theoretic formula* with the property that every scheme of $\mathcal{S}$ is isomorphic to some scheme belonging to $E_{\mathcal{S}}$. Then just as in the case of universal coverings of topological spaces discussed in Example 3.4, (i), one verifies easily, by applying the set-theoretic formula describing $E_{\mathcal{S}}$, that the *universal pro-finite étale coverings* $\widetilde{X} \to X$ of schemes X belonging to $\mathcal{S}$ and *isomorphisms* of such coverings considered up to composition with a *deck transformation* give rise to a *species* $\mathfrak{S}^{\mathcal{S}}$.

(ii) Let $\mathcal{G}$ be a class of topological groups that is closed under isomorphism [of topological groups]. Suppose that there exists a set $E_{\mathcal{G}}$ of topological groups describable by a *set-theoretic formula* with the property that every topological group of $\mathcal{G}$ is isomorphic to some topological group belonging to $E_{\mathcal{G}}$. Then just as in the case of abstract groups discussed in Example 3.4, (ii), one verifies easily, by applying the set-theoretic formula describing $E_{\mathcal{G}}$, that *topological groups* belonging to $\mathcal{G}$ and [bi-continuous] *outer isomorphisms* between such topological groups give rise to a *species* $\mathfrak{S}^{\mathcal{G}}$.

(iii) Let $\mathcal{S}$ be as in (i). Then for an appropriate choice of $\mathcal{G}$, by associating to a universal pro-finite étale covering the resulting group of deck transformations, one obtains a *mutation*

$$\Pi : \quad \mathfrak{S}^{\mathcal{S}} \quad \rightsquigarrow \quad \mathfrak{S}^{\mathcal{G}}$$

[cf. Example 3.4, (iii)]. Then one way to define the notion that the schemes belonging to the class $\mathcal{S}$ are **"[absolute] anabelian"** is to require the specification of a *mutation*

$$\mathbb{A} : \quad \mathfrak{S}^{\mathcal{G}} \quad \rightsquigarrow \quad \mathfrak{S}^{\mathcal{S}}$$

which forms a *mutation-quasi-inverse* to Π. Here, we note that the existence of the bijections [i.e., *"fully faithfulness"*] discussed in Definition 3.3, (iii), is, in essence, the condition that is usually taken as the definition of "anabelian". By contrast, the *species-theoretic* approach of the present discussion may be thought of as an explicit mathematical formulation of the **algorithmic** *approach to [absolute] anabelian geometry* discussed in the Introduction to [AbsTopI].

(iv) The framework of [absolute] anabelian geometry [cf., e.g., the framework discussed above in (iii)] gives a good example of the importance of specifying *precisely* **what species** *one is working with* in a given "series of constructions" [cf., e.g., the mutation-history of the final display of Remark 3.3.1, (i)]. That is to say, there is a quite substantial difference between working with a

profinite group in its sole capacity as a profinite group

and working with the same profinite group — which may happen to arise as the *étale fundamental group* of some scheme! —

> *regarded as being equipped with various data that arise from the construction of the profinite group as the étale fundamental group of some scheme.*

It is precisely this sort of issue that constituted one of the original motivations for the author in the development of the theory of species presented here.

Example 3.6. The Étale Site and Frobenius.

(i) Let p be a *prime number*. If S is a *reduced* scheme over $\mathbb{F}_p$, then denote by $S^{(p)}$ the scheme with the same topological space as S, but whose structure sheaf is given by the *subsheaf*

$$\mathcal{O}_{S^{(p)}} \stackrel{\text{def}}{=} (\mathcal{O}_S)^p \subseteq \mathcal{O}_S$$

of p-th powers of sections of S. Thus, the natural inclusion $\mathcal{O}_{S^{(p)}} \hookrightarrow \mathcal{O}_S$ induces a morphism $\Phi_S : S \to S^{(p)}$. Moreover, "raising to the p-th power" determines a *natural isomorphism* $\alpha_S : S^{(p)} \xrightarrow{\sim} S$ such that the resulting composite $\alpha_S \circ \Phi_S : S \to S$ is the *Frobenius morphism* of S. Write

$$\mathfrak{S}^{p\text{-sch}}$$

for the *species* of reduced quasi-compact schemes over $\mathbb{F}_p$ and quasi-compact morphisms of schemes. Then consider the *[small] category* $S_{\text{ét}}$ — i.e., "the *small étale site* of S" — defined as follows:

> An *object* of $S_{\text{ét}}$ is a(n) [necessarily quasi-affine, by *Zariski's Main Theorem*!]
>
> > *étale morphism of finite presentation* $T \to S$ equipped with a *finite open cover* $\{U_i\}_{i \in I}$ of S, together with *factorizations* $T|_{U_i} \subseteq \mathbb{A}^{N_i}_{U_i} \to U_i$ for each $i \in I$
>
> — where I is a finite subset of the set of open subschemes of S; $\mathbb{A}^{N_i}_{U_i}$ denotes a standard copy of affine N_i-space over U_i, for some integer $N_i \geq 1$; the

"$\subseteq$" exhibits $T|_{U_i}$ as a finitely presented subscheme of $\mathbb{A}^{N_i}_{U_i}$; we observe that *any étale morphism of finite presentation $T \to S$ necessarily admits such auxiliary data parametrized by some index set I.* A *morphism* of $S_{\text{ét}}$ from an object $T_1 \to S$ to an object $T_2 \to S$ [each of which is equipped with auxiliary data] is a(n) [necessarily étale of finite presentation] S-morphism $T_1 \to T_2$.

In particular, one may construct an *assignment*

$$S \mapsto S_{\text{ét}}$$

that maps a species-object S of $\mathfrak{S}^{p\text{-sch}}$ to the [small] category $S_{\text{ét}}$ in such a way that the assignment $S \mapsto S_{\text{ét}}$ is *contravariantly functorial* with respect to species-morphisms $S_1 \to S_2$ of $\mathfrak{S}^{p\text{-sch}}$, and, moreover, may be described via *set-theoretic formulas.* Thus, such an assignment determines an *"étale site mutation"*

$$\mathfrak{M}^{\text{ét}} : \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{\text{cat}}$$

— where we write $\mathfrak{S}^{\text{cat}}$ for the *species* of categories and isomorphism classes of contravariant functors [i.e., a slightly modified form of the species considered in Example 3.2]. Another natural assignment in the present context is the assignment

$$S \mapsto S^{\text{pf}}$$

which maps S to its *perfection* S^{pf}, i.e., the scheme determined by taking the inverse limit of the inverse system $\ldots \to S \to S \to S$ obtained by considering *iterates* of the *Frobenius morphism* of S. Thus, by considering the final copy of "S" in this inverse system, one obtains a natural morphism $\beta_S : S^{\text{pf}} \to S$. Finally, one obtains a *"perfection mutation"*

$$\mathfrak{M}^{\text{pf}} : \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}}$$

by considering the *set-theoretic formulas* underlying the assignment $S \mapsto S^{\text{pf}}$.

(ii) Write

$$\mathfrak{F}^{p\text{-sch}} : \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}}$$

for the **"Frobenius mutation"** obtained by considering the *set-theoretic formulas* underlying the assignment $S \mapsto S^{(p)}$. Thus, one may formulate the well-known *"invariance of the étale site under Frobenius"* [cf., e.g., [FK], Chapter I, Proposition 3.16] as the statement that the **"étale site mutation"** $\mathfrak{M}^{\text{ét}}$ exhibits $\mathfrak{S}^{\text{cat}}$ as a **core** — i.e., an *"invariant piece"* — of the **"Frobenius mutation-history"**

$$\ldots \rightsquigarrow \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \ldots$$

determined by the *"Frobenius mutation"* $\mathfrak{F}^{p\text{-sch}}$. In this context, we observe that the **"perfection mutation"** $\mathfrak{M}^{\text{pf}}$ also yields a **core** — i.e., another *"invariant piece"* — of the Frobenius mutation-history. On the other hand, the natural morphism $\Phi_S : S \to S^{(p)}$ may be interpreted as a *covariant* **observable** of this mutation-history whose *observation mutations* are the identity mutations $\text{id}_{\mathfrak{S}^{p\text{-sch}}} : \mathfrak{S}^{p\text{-sch}} \rightsquigarrow \mathfrak{S}^{p\text{-sch}}$. Since Φ_S is not, in general, an isomorphism, it follows that this observable constitutes an example of an **non-coric** observable. Nevertheless, the natural morphism $\beta_S : S^{\mathrm{pf}} \to S$ may be interpreted as a *morphism of mutations* $\mathfrak{M}^{\mathrm{pf}} \to \mathrm{id}_{\mathfrak{S}^{p\text{-sch}}}$ that serves to relate the *non-coric observable* just considered to the *coric observable* arising from $\mathfrak{M}^{\mathrm{pf}}$.

(iii) One may also develop a version of (i), (ii) for *log schemes*; we leave the routine details to the interested reader. Here, we pause to mention that the theory of log schemes motivates the following **"combinatorial monoid-theoretic"** version of the *non-coric observable* on the *Frobenius mutation-history* of (ii). Write

$$\mathfrak{S}^{\mathrm{mon}}$$

for the species of *torsion-free abelian monoids* and *morphisms of monoids*. If M is a species-object of $\mathfrak{S}^{\mathrm{mon}}$, then write $M^{(p)} \stackrel{\mathrm{def}}{=} p \cdot M \subseteq M$. Then the assignment $M \mapsto M^{(p)}$ determines a **"monoid-Frobenius mutation"**

$$\mathfrak{F}^{\mathrm{mon}} : \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \mathfrak{S}^{\mathrm{mon}}$$

and hence a **"monoid-Frobenius mutation-history"**

$$\ldots \rightsquigarrow \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \ldots$$

which is equipped with a **non-coric** *contravariant* **observable** determined by the natural inclusion morphism $M^{(p)} \hookrightarrow M$ and the observation mutations given by the identity mutations $\mathrm{id}_{\mathfrak{S}^{\mathrm{mon}}} : \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \mathfrak{S}^{\mathrm{mon}}$. On the other hand, the *p-perfection* M^{pf} of M, i.e., the inductive limit of the inductive system $M \hookrightarrow M \hookrightarrow M \hookrightarrow \ldots$ obtained by considering the inclusions given by *multiplying by p*, gives rise to a **"monoid-p-perfection mutation"**

$$\mathfrak{M}^{\text{pf-mon}} : \mathfrak{S}^{\mathrm{mon}} \rightsquigarrow \mathfrak{S}^{\mathrm{mon}}$$

— which may be interpreted as a **core** of the monoid-Frobenius mutation-history. Finally, the natural inclusion of monoids $M \hookrightarrow M^{\mathrm{pf}}$ may be interpreted as a *morphism of mutations* $\mathrm{id}_{\mathfrak{S}^{\mathrm{mon}}} \to \mathfrak{M}^{\text{pf-mon}}$ that serves to relate the *non-coric observable* just considered to the *coric observable* arising from $\mathfrak{M}^{\text{pf-mon}}$.

Remark 3.6.1.

(i) The various constructions of Example 3.6 may be thought of as providing, in the case of the phenomena of *"invariance of the étale site under Frobenius"* and *"invariance of the perfection under Frobenius"*, a *"species-theoretic intepretation"* — i.e., via consideration of

"coric" versus **"non-coric"** observables

— of the difference between **"étale-type"** and **"Frobenius-type"** structures [cf. the discussion of [FrdI], §I4]. This sort of approach via *"combinatorial patterns"* to expressing the difference between "étale-type" and "Frobenius-type" structures plays a *central role* in the theory of the present series of papers. Indeed, the mutation-histories and cores considered in Example 3.6, (ii), (iii), may be thought of as the **underlying motivating examples** for the theory of both

· the **vertical lines**, i.e., consisting of $\mathfrak{log}$**-links**, and

· the **horizontal lines**, i.e., consisting of $\Theta^{\times\boldsymbol{\mu}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\text{gau}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\text{LGP}}$-/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$**-links**,

of the **log-theta-lattice** [cf. [IUTchIII], Definitions 1.4, 3.8]. Finally, we recall that this approach to understanding the $\mathfrak{log}$-links may be seen in the introduction of the terminology of *"observables"* and *"cores"* in [AbsTopIII], Definition 3.5, (iii).

(ii) Example 3.6 also provides a good example of the *important theme* [cf. the discussion of Remark 3.3.1, (iii)] of

describing **non-coric** *data in terms of* **coric** *data*

— cf. the morphism $\beta_S : S^{\text{pf}} \to S$ of Example 3.6, (ii); the natural inclusion $M \hookrightarrow M^{\text{pf}}$ of Example 3.6, (iii). From the point of view of the *vertical* and *horizontal* lines of the **log-theta-lattice** [cf. the discussion of (i)], this theme may also be observed in the *vertically coric log-shells* that serve as a *common receptacle* for the various arrows of the $\mathfrak{log}$**-Kummer correspondences** of [IUTchIII], Theorem 3.11, (ii), as well as in the *multiradial representations* of [IUTchIII], Theorem 3.11, (i), which describe [certain aspects of] the **arithmetic holomorphic structure** on one vertical line of the log-theta-lattice in terms that may be understood relative to an **alien** *arithmetic holomorphic structure* on another vertical line — i.e., separated from the first vertical line by *horizontal arrows* — of the log-theta-lattice [cf. [IUTchIII], Remark 3.11.1; [IUTchIII], Remark 3.12.2, (ii)].

Remark 3.6.2.

(i) In the context of the theme of *"coric descriptions of non-coric data"* discussed in Remark 3.6.1, (ii), it is of interest to observe the significance of the use of **set-theoretic formulas** [cf. the discussion of Remarks 3.1.2, 3.1.3, 3.1.4, 3.1.5] to realize such descriptions. That is to say, descriptions in terms of *arbitrary choices* that depend on a particular model of set theory [cf. Remark 3.1.3] do not allow one to **calculate** *in terms that make sense in one universe* the operations performed in an **alien universe**! This is precisely the sort of situation that one encounters when one considers the vertical and horizontal arrows of the log-theta-lattice [cf. (ii) below], where *distinct universes* arise from the *distinct* **scheme-theoretic basepoints** on either side of such an arrow that correspond to *distinct* **ring theories**, i.e., ring theories that *cannot be related* to one another by means of a *ring homomorphism* — cf. the discussion of Remark 3.6.3 below. Indeed,

> it was precisely the need to understand this sort of situation that led the author to develop the **"inter-universal"** version of **Teichmüller theory** exposed in the present series of papers.

Finally, we observe that the *algorithmic approach* [i.e., as opposed to the *"fully faithfulness/Grothendieck Conjecture-style approach"* — cf. Example 3.5, (iii)] to reconstruction issues via *set-theoretic formulas* plays an essential role in this context. That is to say, although different algorithms, or *software*, may yield the *same output data*, it is only by working with **specific algorithms** that one may understand the *delicate inter-relations* that exist between various **components** of the structures that occur as one performs various operations [i.e., the mutations of a mutation-history]. In the case of the theory developed in the present series of papers, one central example of this phenomenon is the **cyclotomic rigidity isomorphisms** that underlie the theory of $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-link compatibility** discussed in [IUTchIII], Theorem 3.11, (iii), (c), (d) [cf. also [IUTchIII], Remarks 2.2.1, 2.3.2].

(ii) The **algorithmic** *approach to reconstruction* that is taken throughout the present series of papers, as well as, for instance, in [FrdI], [EtTh], and [AbsTopIII], was conceived by the author in the spirit of the **species-theoretic** *formulation* exposed in the present §3. Nevertheless, [cf. Remark 3.1.3, (i)] we shall not explicitly write out the various set-theoretic formulas involved in the various species, mutations, etc. that are *implicit* throughout the theory of the present series of papers. Rather, it is to be understood that the set-theoretic formulas to be used are those arising from the *conventional descriptions* that are given of the mathematical objects involved. When applying such conventional descriptions, however, the reader is obliged to check that they are *well-defined* and *do* **not** *depend* upon the use of **arbitrary choices** that are not describable via well-defined set-theoretic formulas.

(iii) The *sharp contrast* between

· the **canonicality** imparted by descriptions via *set-theoretic formulas* in the context of *extensions of the universe* in which one works

[cf. Remarks 3.1.3, 3.1.4] and

· the situation that arises if one allows, in one's descriptions, the various **arbitrary choices** arising from *invocations of the axiom of choice*

may be understood somewhat explicitly if one attempts to *"catalogue the various possibilities"* corresponding to various possible choices that may occur in one's description. That is to say, such a "cataloguing operation" typically obligates one to work with *"sets of very large cardinality"*, many of which must be constructed by means of **set-theoretic exponentiation** [i.e., such as the operation of passing from a set E to the power set "2^E" of all subsets of E]. Such a rapid outbreak of *"unwieldy large sets"* is reminiscent of the *rapid growth*, in the p-adic crystalline theory, of the p-adic valuations of the *denominators* that occur when one *formally integrates an* **arbitrary connection**, as opposed to a **"canonical connection"** of the sort that arises from a crystalline representation. In the p-adic theory, such "canonical connections" are typically related to **"canonical liftings"**, such as, for instance, those that occur in ***p*-adic Teichmüller theory** [cf. [pOrd], [pTeich]]. In this context, it is of interest to recall that the canonical liftings of p-adic Teichmüller theory may, under certain conditions, be thought of as liftings *"of minimal complexity"* in the sense that their Witt vector coordinates are given by *polynomials of* **minimal degree** — cf. the computations of [Finot].

Remark 3.6.3.

(i) In the context of Remark 3.6.2, it is useful to recall the *fundamental reason* for the need to pursue **"inter-universality"** in the present series of papers [cf. the discussion of [IUTchIII], Remark 1.2.4; [IUTchIII], Remark 1.4.2], namely,

> since *étale fundamental groups* — i.e., in essence, **Galois groups** — are defined as certain *automorphism groups of fields/rings*, the definition of such a Galois group as a certain automorphism group of some ring structure is **fundamentally incompatible** with the **vertical** and **horizontal** arrows of the **log-theta-lattice** [i.e., which do *not* arise from ring homomorphisms]!

In this respect, "transformations" such as the vertical and horizontal arrows of the log-theta-lattice *differ*, quite fundamentally, from "transformations" that are *compatible with the ring structures* on the domain and codomain, i.e., **morphisms of rings/schemes**, which **tautologically** give rise to **functorial morphisms** between the respective étale fundamental groups. Put another way, in the notation of [IUTchI], Definition 3.1, (e), (f) [which will be applied throughout the remainder of the present Remark 3.6.3], for, say, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$,

> the only *natural correspondence* that may be described by means of **set-theoretic formulas** between the isomorphs of the local base field Galois groups "$G_{\underline{v}}$" on either side of a vertical or horizontal arrow of the log-theta-lattice is the correspondence constituted by an **indeterminate isomorphism of topological groups**.

A similar statement may be made concerning the isomorphs of the *geometric fundamental group* $\Delta_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}})$ on either side of a *vertical* [but *not horizontal*! — cf. the discussion of (ii) below] arrow of the log-theta-lattice — that is to say,

> the only *natural correspondence* that may described by means of **set-theoretic formulas** between these isomorphs is the correspondence constituted by an **indeterminate isomorphism of topological groups** *equipped with some* **outer action** *by the respective isomorph of "$G_{\underline{v}}$"*

— cf. the discussion of [IUTchIII], Remark 1.2.4. Here, again we recall from the discussion of Remark 3.6.2, (i), (ii), that it is only by working with such correspondences that may be described by means of *set-theoretic formulas* that one may obtain descriptions that allow one to **calculate** the operations performed in *one universe* from the point of view of an **alien universe**.

(ii) One *fundamental difference* between the *vertical* and *horizontal* arrows of the log-theta-lattice is that whereas, for, say, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$,

(V1) one **identifies**, up to isomorphism, the isomorphs of the *full* arithmetic fundamental group "$\Pi_{\underline{v}}$" on either side of a **vertical** arrow,

(H1) one **distinguishes** the "$\Delta_{\underline{v}}$'s" on either side of a **horizontal** arrow, i.e., one *only identifies*, up to isomorphism, the local base field Galois groups "$G_{\underline{v}}$" on either side of a horizontal arrow.

— cf. the discussion of [IUTchIII], Remark 1.4.2. One way to understand the fundamental reason for this difference is as follows.

(V2) In order to construct the $\mathfrak{log}$**-link** — i.e., at a more concrete level, the power series that defines the $p_{\underline{v}}$-adic logarithm at $\underline{v}$ — it is necessary to avail oneself of the local **ring structures** at $\underline{v}$ [cf. the discussion of [IUTchIII], Definition 1.1, (i), (ii)], which may only be reconstructed from the *full* "$\Pi_{\underline{v}}$" [i.e., not from "$G_{\underline{v}}$ stripped of its structure as a *quotient* of $\Pi_{\underline{v}}$" — cf. the discussion of [IUTchIII], Remark 1.4.1, (i); [IUTchIII], Remark 2.1.1, (ii); [AbsTopIII], §I3].

(H2) In order to construct the $\Theta^{\times\boldsymbol{\mu}}_{\mathrm{gau}}$**-**/$\Theta^{\times\boldsymbol{\mu}}_{\mathrm{LGP}}$**-**/$\Theta^{\times\boldsymbol{\mu}}_{\mathfrak{lgp}}$**-links** — i.e., at a more concrete level, the correspondence

$$\underline{\underline{q}} \;\mapsto\; \left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\dots,l^*}$$

[cf. [IUTchII], Remark 4.11.1] — it is necessary, in effect, to construct an *"isomorphism"* between a mathematical object [i.e., the *theta values* "$\underline{\underline{q}}^{j^2}$"] that **depends**, in an essential way, on regarding the various "j" as **distinct labels** [which are constructed from "$\Delta_{\underline{v}}$"!] and a mathematical object [i.e., "$\underline{\underline{q}}$"] that is **independent** *of these labels*; it is then a **tautology** that such an "isomorphism" may only be achieved if the labels — i.e., in essence, "$\Delta_{\underline{v}}$" — on either side of the "isomorphism" are kept **distinct** from one another.

Here, we observe in passing that the "apparently *horizontal* arrow-related" issue discussed in (H2) of **simultaneous realization** of **"label-dependent"** and **"label-free"** mathematical objects is reminiscent of the *vertical* arrow portion of the **bicoricity** theory of [IUTchIII], Theorem 1.5 — cf. the discussion of [IUTchIII], Remark 1.5.1, (i), (ii); Step (vii) of the proof of [IUTchIII], Corollary 3.12.

Bibliography

[ABKP] J. Amorós, F. Bogomolov, L. Katzarkov, T. Pantev, Symplectic Lefshetz fibration with arbitrary fundamental groups, *J. Differential Geom.* **54** (2000), pp. 489-545.

[NerMod] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **21**, Springer-Verlag (1990).

[Drk] F. R. Drake, *Set Theory: an Introduction to Large Cardinals*, *Studies in Logic and the Foundations of Mathematics* **76**, North-Holland (1974).

[DmMn] H. Dym and H. P. McKean, *Fourier Series and Integrals*, Academic Press (1972).

[Edw] H. M. Edwards, *Riemann's Zeta Function*, Academic Press (1974).

[Falt] G. Faltings, Endlichkeitssätze für Abelschen Varietäten über Zahlkörpern, *Invent. Math.* **73** (1983), pp. 349-366.

[Ffmn] S. Feferman, Set-theoretical Foundations of Category Theory, *Reports of the Midwest Category Seminar III*, *Lecture Notes in Mathematics* **106**, Springer-Verlag (1969), pp. 201-247.

[Finot] L. R. A. Finotti, Minimal degree liftings of hyperelliptic curves, *J. Math. Sci. Univ. Tokyo* **11** (2004), pp. 1-47.

[FK] E. Freitag and R. Kiehl, *Étale Cohomology and the Weil Conjecture*, Springer Verlag (1988).

[Harts] R. Hartshorne, *Algebraic Geometry*, *Graduate Texts in Mathematics* **52**, Springer-Verlag (1977).

[Ih] Y. Ihara, Comparison of some quotients of fundamental groups of algebraic curves over p-adic fields, *Galois-Teichmüller Theory and Arithmetic Geometry*, *Adv. Stud. Pure Math.* **63**, Math. Soc. Japan (2012), pp. 221-250.

[Kobl] N. Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions*, *Graduate Texts in Mathematics* **58**, Springer-Verlag (1984).

[Lang1] S. Lang, *Algebraic number theory, Second Edition*, *Graduate Texts in Mathematics* **110**, Springer-Verlag (1986).

[Lang2] S. Lang, *Introduction to complex hyperbolic spaces*, Springer-Verlag (1987).

[McLn] S. MacLane, One Universe as a Foundation for Category Theory, *Reports of the Midwest Category Seminar III*, *Lecture Notes in Mathematics* **106**, Springer-Verlag (1969).

[Mss] D. W. Masser, *Note on a conjecture of Szpiro* in *Astérisque* **183** (1990), pp. 19-23.

[Milne] J. S. Milne, *Abelian Varieties* in *Arithmetic Geometry*, edited by G. Cornell and J. H. Silverman, Springer-Verlag (1986), pp. 103-150.

[pOrd] S. Mochizuki, A Theory of Ordinary p-adic Curves, *Publ. Res. Inst. Math. Sci.* **32** (1996), pp. 957-1151.

[pTeich] S. Mochizuki, *Foundations of p-adic Teichmüller Theory*, AMS/IP Studies in Advanced Mathematics **11**, American Mathematical Society/International Press (1999).

[InpTch] S. Mochizuki, An Introduction to p-adic Teichmüller Theory, *Cohomologies p-adiques et applications arithmétiques I*, *Astérisque* **278** (2002), pp. 1-49.

[pGC] S. Mochizuki, The Local Pro-p Anabelian Geometry of Curves, *Invent. Math.* **138** (1999), pp. 319-423.

[HASurI] S. Mochizuki, A Survey of the Hodge-Arakelov Theory of Elliptic Curves I, *Arithmetic Fundamental Groups and Noncommutative Algebra*, *Proceedings of Symposia in Pure Mathematics* **70**, American Mathematical Society (2002), pp. 533-569.

[HASurII] S. Mochizuki, A Survey of the Hodge-Arakelov Theory of Elliptic Curves II, *Algebraic Geometry 2000, Azumino*, *Adv. Stud. Pure Math.* **36**, Math. Soc. Japan (2002), pp. 81-114.

[CanLift] S. Mochizuki, The Absolute Anabelian Geometry of Canonical Curves, *Kazuya Kato's fiftieth birthday*, *Doc. Math. 2003, Extra Vol.*, pp. 609-640.

[GeoAnbd] S. Mochizuki, The Geometry of Anabelioids, *Publ. Res. Inst. Math. Sci.* **40** (2004), pp. 819-881.

[SemiAnbd] S. Mochizuki, Semi-graphs of Anabelioids, *Publ. Res. Inst. Math. Sci.* **42** (2006), pp. 221-322.

[Cusp] S. Mochizuki, Absolute anabelian cuspidalizations of proper hyperbolic curves, *J. Math. Kyoto Univ.* **47** (2007), pp. 451-539.

[FrdI] S. Mochizuki, The Geometry of Frobenioids I: The General Theory, *Kyushu J. Math.* **62** (2008), pp. 293-400.

[EtTh] S. Mochizuki, The Étale Theta Function and its Frobenioid-theoretic Manifestations, *Publ. Res. Inst. Math. Sci.* **45** (2009), pp. 227-349.

[GenEll] S. Mochizuki, Arithmetic Elliptic Curves in General Position, *Math. J. Okayama Univ.* **52** (2010), pp. 1-28.

[AbsTopI] S. Mochizuki, Topics in Absolute Anabelian Geometry I: Generalities, *J. Math. Sci. Univ. Tokyo* **19** (2012), pp. 139-242.

[AbsTopII] S. Mochizuki, Topics in Absolute Anabelian Geometry II: Decomposition Groups and Endomorphisms, *J. Math. Sci. Univ. Tokyo* **20** (2013), pp. 171-269.

[AbsTopIII] S. Mochizuki, Topics in Absolute Anabelian Geometry III: Global Reconstruction Algorithms, *J. Math. Sci. Univ. Tokyo* **22** (2015), pp. 939-1156.

[IUTchI] S. Mochizuki, *Inter-universal Teichmüller Theory I: Construction of Hodge Theaters*, RIMS Preprint **1756** (August 2012), to appear in *Publ. Res. Inst. Math. Sci.*

[IUTchII] S. Mochizuki, *Inter-universal Teichmüller Theory II: Hodge-Arakelov-theoretic Evaluation*, RIMS Preprint **1757** (August 2012), to appear in *Publ. Res. Inst. Math. Sci.*

[IUTchIII] S. Mochizuki, *Inter-universal Teichmüller Theory III: Canonical Splittings of the Log-theta-lattice*, RIMS Preprint **1758** (August 2012), to appear in *Publ. Res. Inst. Math. Sci.*

[Pano] S. Mochizuki, A Panoramic Overview of Inter-universal Teichmüller Theory, *Algebraic number theory and related topics 2012*, *RIMS Kōkyūroku Bessatsu* **B51**, Res. Inst. Math. Sci. (RIMS), Kyoto (2014), pp. 301-345.

[MnLg] S. Mochizuki, Monomorphisms in Categories of Log Schemes, *Kodai Math. J.* **38** (2015), pp. 365-429.

[BogIUT] S. Mochizuki, Bogomolov's proof of the geometric version of the Szpiro Conjecture from the point of view of inter-universal Teichmüller theory, *Res. Math. Sci.* **3** (2016), 3:6.

[Par] A. N. Parshin, Finiteness theorems and hyperbolic manifolds, *The Grothendieck Festschrift, Vol. III*, *Progress in Mathematics* **88**, Birkhäuser (1990).

[Silv] J. H. Silverman, *The Arithmetic of Elliptic Curves*, *Graduate Texts in Mathematics* **106**, Springer-Verlag (1986).

[Arak] C. Soulé, D. Abramovich, J.-F. Burnol, J. Kramer, *Lectures on Arakelov Geometry*, *Cambridge studies in advanced mathematics* **33**, Cambridge University Press (1994).

[vFr] M. van Frankenhuijsen, About the ABC conjecture and an alternative, *Number theory, analysis and geometry*, Springer-Verlag (2012), pp. 169-180.

[Vjt] P. Vojta, *Diophantine approximations and value distribution theory*, *Lecture Notes in Mathematics* **1239**, Springer-Verlag (1987).

[Zh] S. Zhang, Geometry of algebraic points, *First International Congress of Chinese Mathematicians (Beijing, 1998)*, *AMS/IP Stud. Adv. Math.* **20** (2001), pp. 185-198.

Updated versions of preprints are available at the following webpage:

`http://www.kurims.kyoto-u.ac.jp/~motizuki/papers-english.html`