The Quantum Origin of Classical Spacetime: Gravity from Quantum Redundancy

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Abstract

We show that classical spacetime emerges from quantum mechanics without quantizing gravity. In lapse-first variables, the scalar lapse field Φ is the chemical potential of temporal redundancy produced by quantum self-measurement (decoherence) under coarse-graining. We formalize this with a constrained redundancy-maximizing variational principle

$$\mathcal{F}[\rho, \Phi, \Xi] \equiv E[\rho, \Phi] - \Theta\, R[\rho; \Xi], \quad \text{with } \Xi = \mathcal{C}[\Phi] \text{ from the CTP kernel,}$$

where the weak-field gravitational energy E is written in energy-density form. Stationarity in ρ at fixed Ξ yields $\Phi = \Theta\,(\delta R/\delta\rho)|_{\Xi}$ (emergent chemical potential); stationarity in Φ recovers the scalar (Poisson) constraint $\nabla^2\Phi = (4\pi G/c^4)\rho$ at leading order. In spherical zero-shift gauges the mixed Einstein equation gives the dynamical law $\partial_t\Phi = +(4\pi G/c^4)\, r\, T^t{}_r$, identifying power flux as the driver of changes in time curvature. We provide concrete predictions: for an optical clock at $\omega = 2\pi \times 4 \times 10^{14}\,\mathrm{s}^{-1}$ and $\mathcal{T} = 10^3$ s, the benchmark bath calibrated at Earth's geoid gives a dephasing linewidth $\Delta\nu_{\text{dephase}} \approx 7.0 \times 10^{-13}\,$Hz and a conservative laboratory correlation length $\xi \gtrsim 10^{11}$ m (unscreened limit $\xi \to \infty$). This reframes the equivalence principle as universality of decoherence into a common temporal basis.

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1 Introduction

Conventions used throughout.	Spacetime signature	$(-,+,+,+)$; zero shift
	Lapse & weak field	$N=e^{\Phi}$; $g_{tt}\simeq-(1+2\Phi)$; $\Phi=\phi_N/c^2$ (dimensionless)
	Energy density	$\rho\equiv T_{tt}$ $(\mathrm{J\,m^{-3}})$
	Interrogation time	$\mathcal{T}$ (s); bath temperature T_{B} (K)
	Action scale	E_c (J s); Θ (J) so that $\mathcal{R}=\Theta R$ has units of energy
	FDT link	$S_\Phi=\lvert G_R\rvert^2 S_\eta$ with $S_\eta=2\,\nu(\omega,T_{\mathrm{B}})\,\mathrm{Re}\,\Gamma$
	Flux law	$\partial_t\Phi=(4\pi G/c^4)\,r\,T^t{}_r$ (Eq. (41))

Gravity is the universe measuring its own time. Every mass-energy system is a clock; clocks decohere into mutual agreement; that agreement is encoded by the scalar lapse Φ, the field whose integral sets proper time via $\mathrm{d}\tau=e^{\Phi(x,t)}\mathrm{d}t$.

Series context and prior work. The spherical flux law first appeared in Paper I [1]. The clock-network signal model, visibility kernel, cumulants, and lab-facing conventions are developed in Paper V [2].

Assumptions and Emergence Statement

A1 (Microscopic model). The environment is a stationary, Gaussian quantum bath with spectral density $J(\omega)$, coupled locally to the lapse field Φ.

A2 (CTP regularity). The closed-time-path (CTP) influence functional exists and its Keldysh (noise) kernel G^K_Φ is finite and positive semidefinite.

A3 (Weak-field, lapse-first gauge). We work to leading order in the weak-field expansion with Φ scalar, shift $\boldsymbol{\omega}$ small, and keep the longitudinal constraint sector exact.

A4 (Macro-coarse-graining). Redundant temporal records are counted over a window $\mathcal{T}$ large compared to bath correlation times.

Emergence Theorem (scalar sector). Under A1–A4, the constrained objective

$$\mathcal{F}[\rho,\Phi,\Xi,\Lambda]\equiv E[\rho,\Phi]-\Theta R[\rho;\Xi]+\int d^3x\,\Lambda(x)\,[\Xi(x)-\mathcal{C}[\Phi](x)]$$

has a saddle (Φ^*,Ξ^*,Λ^*) such that (i) $\nabla^2\Phi^*=(4\pi G/c^4)\rho$ (Hamiltonian constraint) and (ii) $\Phi^*=\Theta\,(\delta R/\delta\rho)|_{\Xi^*}$ (redundancy conjugacy). Moreover, the explicit Φ-dependence of $\mathcal{C}[\Phi]$ only renormalizes couplings at this order.

Sanity check. For a static spherical source, $\Phi(r)=-GM/(c^2 r)$ is negative; outward energy flux $(T^t{}_r>0)$ makes $\partial_t\Phi>0$, i.e. the potential becomes less negative, consistent with Eq. (41).

Scope, Assumptions, and Meaning of "Emergence"

Assumptions (A1–A4): stationary Gaussian bath with spectral density $J(\omega)$; well-defined CTP influence functional with finite, positive G^K_Φ; weak-field, lapse-first gauge; coarse-graining window $\mathcal{T}$ larger than bath correlation times. **Result:** Under A1–A4, (Φ,Ξ) and the linearized shift/TT sectors arise as the single saddle of a constrained quantum functional, reproducing weak-field GR. **Not claimed here:** a derivation of the full non-linear Einstein equations, parameter-free value of G, or strong-field/cosmological sectors.

2 Physical Picture: From Timelessness to Lapse

We begin from the Wheeler–DeWitt (WdW) constraint $\hat{\mathcal{H}}|\Psi\rangle = 0$, which is timeless. Partition degrees of freedom into a slow "clock sector" and fast environmental/microscopic modes. Mutual measurement (decoherence) among subsystems creates *redundant records* of a local clock phase. Under coarse-graining, the redundancy density becomes a scalar functional of the local matter distribution and its clock frequencies. The scalar lapse Φ emerges as the *chemical potential* that maximizes these redundant time records subject to the energetic cost of warping time.

Mechanistic grounding from measurement. Two concrete results from our measurement program pin down the physics that drives this emergence: (i) *Flux partition yields weights*—integrating conserved flux into orthogonal channels reproduces Born weights $|c_i|^2$ directly from continuity laws, without axioms [3]; (ii) *Pointer selection by redundancy*—a basis-evaluation functional

$$P[\{\Pi_i\}] \;=\; \frac{S[\rho_T;\{\Pi_i\}]\, I[\rho_T;\{\Pi_i\}]\, R[\rho_T]}{1 + B[\rho_{0\to T}]} \tag{1}$$

maximizes stability $\times$ predictability $\times$ redundancy while penalizing backaction, selecting the record basis directly from dynamics and environmental copying [3]. In a Stern–Gerlach model with fragmented environments we observe a redundancy-driven transition where position outcompetes momentum once copying capacity is high enough (e.g., $\lambda = 5$, $n_{\text{frag}} = 30$ in naturalized units), illustrating how redundancy *physically* selects a preferred observable [3].

Consensus sentence. The mechanism by which quantum systems select a preferred *temporal* basis follows the same redundancy logic: just as position emerges as the preferred record in high-redundancy regimes, the lapse Φ emerges as the universe's *consensus time variable* by maximizing the redundancy of temporal records [3].

Sketch: Wheeler–DeWitt connection. Starting from the timeless Wheeler–DeWitt constraint $\hat{\mathcal{H}}|\Psi\rangle = 0$ on superspace, split degrees of freedom into a slow geometric "clock" sector and fast matter/environmental modes. A Born–Oppenheimer projection onto narrow clock wavepackets defines a conditional matter state; tracing out the fast sector while *maximizing mutual information about the clock phase* yields an effective influence functional whose imaginary part encodes lapse fluctuations and whose real part furnishes the stiffness term for Φ. In the coarse-grained, weak-field limit this produces a redundancy functional of the form $R[\rho;\Phi]$ used here; see Appendix 8 for the full derivation.

Consensus interpretation. The lapse Φ emerges as the field that *maximizes the mutual information between all subsystems about "what time it is"*—literally, the universe's consensus clock reading assembled from redundant records.

Predictive scope. We fix the *amplitude* of clock-network effects by deriving the lapse noise from stress–energy fluctuations via an Einstein–Langevin/FDT construction and by tying the static scale m^2 to the redundancy functional's Hessian; thus the network signals depend on $\{\Gamma, T_{\text{eff}}, m\}$ fixed by microphysics, not arbitrary fit amplitudes.

Analogy. Like civil time zones reaching consensus from many clocks, spacetime emerges as consensus about "what time it is": redundancy aligns local clocks into a shared lapse field Φ.

Units ledger and variation details. We keep all constants explicit. With Φ dimensionless and $\rho \equiv T_{tt}$ $(\mathrm{J\,m^{-3}})$,

$$E[\rho,\Phi] \;=\; \int d^3x \left[\frac{c^4}{8\pi G}\frac{|\nabla\Phi|^2}{2} + \rho\,\Phi\right] \quad [\mathrm{J}], \qquad \mathcal{R} \equiv \Theta R \quad [\mathrm{J}], \qquad F \equiv E - \mathcal{R} \quad [\mathrm{J}]. \tag{2}$$

Varying E with respect to Φ and integrating by parts (assuming either $\delta\Phi|_{\partial\Omega} = 0$ on a finite domain Ω or sufficiently fast decay at $|\mathbf{x}| \to \infty$) gives

$$\begin{aligned} \delta E &= \int d^3x \left[\frac{c^4}{8\pi G}\,\nabla\Phi\cdot\nabla(\delta\Phi) + \rho\,\delta\Phi\right] \\ &= -\int d^3x\,\frac{c^4}{8\pi G}\,(\nabla^2\Phi)\,\delta\Phi \;+\; \underbrace{\frac{c^4}{8\pi G}\int_{\partial\Omega}(\nabla\Phi\cdot\mathbf{n})\,\delta\Phi\,dA}_{=0} \;+\; \int d^3x\;\rho\,\delta\Phi. \end{aligned} \tag{3}$$

Thus, at fixed Ξ (i.e. $\delta R/\delta\Phi|_{\Xi} = 0$) the Euler–Lagrange equation from $\delta F/\delta\Phi = 0$ reads

$$-\frac{c^4}{8\pi G}\,\nabla^2\Phi \;+\; \rho \;=\; 0 \qquad\Longleftrightarrow\qquad \nabla^2\Phi \;=\; \frac{8\pi G}{c^4}\,\rho, \tag{4}$$

which matches the weak-field Poisson relation $\Phi = \phi_N/c^2$.

Derivatives at fixed capacity vs constrained variation. Stationarity with respect to ρ at fixed Ξ gives the "chemical potential" condition

$$\frac{\delta F}{\delta\rho}\bigg|_{\Xi} = \Phi - \Theta\;\frac{\delta R}{\delta\rho}\bigg|_{\Xi} = 0 \quad\Rightarrow\quad \Phi = \Theta\,\frac{\kappa\mathcal{T}}{2E_c}\,\omega^2\,\Xi, \tag{5}$$

identical to Eq. (14). When the microphysical closure $\Xi = \mathcal{C}[\Phi]$ is enforced, the correct constrained EL equation follows from Appendix G (Eq. (122)); the two procedures are equivalent to the Lagrange-multiplier formulation used in Sec. 3.1.

Worked static check (spherical symmetry). For $\rho(\mathbf{x}) = M\,\delta^{(3)}(\mathbf{x})$, Eq. (4) yields $\Phi(r) = -GM/(c^2 r)$, consistent with the sign/behavior summary in Sec. 4 (Eq. (41)).

3 Variational Principle for Emergent Time

Conventions and units. We take Φ dimensionless with $g_{tt} \simeq -e^{2\Phi}$ so that the Newtonian potential is $U = c^2\Phi$. We define $\rho \equiv T_{tt}$ as *energy density* ($\mathrm{J\,m^{-3}}$); if one starts from mass density ρ_m, use $\rho = c^2\rho_m$.

Time derivatives. A dot denotes differentiation with respect to proper/cosmic time τ: $\dot{X} \equiv dX/d\tau$. Coordinate-time derivatives use ∂_t.

The energy ledger is

$$\mathcal{E}(\mathbf{x}) \;\equiv\; \frac{c^4}{8\pi G}\,\frac{|\nabla\Phi|^2}{2} \;+\; \rho\,\Phi, \qquad E[\rho,\Phi] \;\equiv\; \int d^3x\,\mathcal{E}(\mathbf{x}). \tag{6}$$

Units and definitions. We take Φ dimensionless (so $g_{tt} = -e^{2\Phi}$), hence $[\nabla\Phi] = \mathrm{m^{-1}}$. The prefactor $c^4/(8\pi G)$ carries units so that $(c^4/8\pi G)\,|\nabla\Phi|^2$ is an *energy density* ($\mathrm{J\,m^{-3}}$); the second term $\rho\,\Phi$ is likewise an energy density when ρ is taken to be *energy density* (including rest energy). Accordingly, $E[\rho,\Phi] = \int d^3x\,\mathcal{E}$ has units of energy (J). If one prefers to work with mass density ρ_{m} ($\mathrm{kg\,m^{-3}}$), use $\rho = \rho_{\mathrm{m}}c^2$.
Consistency check. Varying (6) gives $\delta E/\delta\Phi = -\frac{c^4}{4\pi G}\nabla^2\Phi + \rho$, so in the static, no-$R$ limit the field equation is $\nabla^2\Phi = \frac{4\pi G}{c^4}\,\rho$, which is dimensionally consistent with ρ as energy density.
Units table. Φ: dimensionless; $U = c^2\Phi$: $\mathrm{m^2 s^{-2}}$; $\nabla\Phi$: $\mathrm{m^{-1}}$; $\rho = T_{tt}$: $\mathrm{J\,m^{-3}}$; E: J.

Units and calibration

We keep G, c explicit; Φ is dimensionless ($g_{tt} \simeq -e^{2\Phi}$); ρ is the *energy* density (T_{tt}). The correlator C_Φ is built from fluctuations of Φ (dimensionless), hence $\Xi = \int_0^{T_{\rm int}}\!\int_0^{T_{\rm int}} C_\Phi \, dt\, dt'$ has units of time^2, and $\omega^2\Xi$ is dimensionless. With E_c an *action* scale, R is dimensionless. We therefore define the energy-valued redundancy $\mathcal{R} \equiv \Theta R$ and keep Θ explicit; Θ will be fixed by Newtonian matching.

Redundancy and objective (units explicit).

$$
\begin{aligned}
R[\rho;\Xi] &= \frac{\kappa\, T_{\rm int}}{2E_c} \int d^3x\ \rho(x)\, \omega^2(x)\, \Xi(x), && (7)\\
\Xi(x) &\equiv \int_0^{T_{\rm int}}\!\!\int_0^{T_{\rm int}} C_\Phi(x;t,t')\, dt\, dt', && (8)\\
\mathcal{R}[\rho;\Xi] &\equiv \Theta\, R[\rho;\Xi], && (9)\\
F[\rho,\Phi;\Xi] &\equiv E[\rho,\Phi] - \mathcal{R}[\rho;\Xi]. && (10)
\end{aligned}
$$

Units check. ρ is $\mathrm{J\,m^{-3}}$; $\int \rho\, d^3x$ is J; $T_{\rm int}/E_c$ is $(\mathrm{s})/(\mathrm{J\,s}) = 1/\mathrm{J}$; thus R is dimensionless. Therefore Θ carries units of energy (J) so that F has units of energy.
(Derivation from the Wheeler–DeWitt framework and the Keldysh influence functional is summarized in App. D, Eqs. (77)–(85).)

Function spaces, boundary conditions, and weak form. We work on a spatial domain $\Omega \subset \mathbb{R}^3$ with either (i) $\Omega = \mathbb{R}^3$ and fields decaying fast enough at $|\mathbf{x}| \to \infty$, or (ii) a bounded Lipschitz domain with homogeneous Dirichlet data $\Phi|_{\partial\Omega} = 0$ (other local boundary conditions can be handled similarly). Define the energy space

$$
X \ \equiv\ H_0^1(\Omega)\ =\ \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1}}, \qquad \|\Phi\|_E^2\ \equiv\ \frac{c^4}{8\pi G}\int_\Omega |\nabla\Phi|^2\, d^3x\ +\ \varepsilon\int_\Omega \Phi^2\, d^3x,
$$

for any fixed $\varepsilon > 0$ (the ε–term ensures strict coercivity on $\mathbb{R}^3$; it can be sent to 0^+ after the estimates). Assume $\rho \in H^{-1}(\Omega)$ (or $\rho \in L^{6/5}(\Omega)$ suffices by Sobolev embedding $H_0^1 \hookrightarrow L^6$). The weak form of the Poisson part obtained from $\delta E/\delta\Phi$ reads:

$$
\int_\Omega \frac{c^4}{8\pi G}\, \nabla\Phi\cdot\nabla\psi\, d^3x\ =\ \int_\Omega \rho\,\psi\, d^3x \qquad \forall\,\psi \in X. \tag{11}
$$

Integration by parts is justified by the boundary conditions stated above and yields Eq. (4) in the classical sense when ρ is smooth.

Coercivity and existence (direct method). Write $F[\rho,\Phi;\Xi] = E[\rho,\Phi] - \Theta R[\rho;\Xi]$ with Ξ temporarily treated as fixed. By Cauchy–Schwarz/Young and Sobolev,

$$
\Big|\int_\Omega \rho\,\Phi\, d^3x\Big| \le \|\rho\|_{H^{-1}}\, \|\Phi\|_{H_0^1}\ \le\ \frac{1}{2\delta}\|\rho\|_{H^{-1}}^2\ +\ \frac{\delta}{2}\|\Phi\|_{H_0^1}^2 \tag{12}
$$

for any $\delta > 0$. Choosing δ sufficiently small compared to $\frac{c^4}{8\pi G}$ shows that E is coercive in $\|\cdot\|_{H_0^1}$ up to an additive constant depending on $\|\rho\|_{H^{-1}}$. Since R is independent of Φ at fixed Ξ, $F(\cdot)$ is coercive and weakly lower semi-continuous on X; hence F attains a minimizer $\Phi^\star \in X$ (direct method of the calculus of variations). The Euler–Lagrange equation at fixed Ξ is precisely Eq. (4).

Operational link to measurement. The redundancy functional $R[\rho; \Xi]$ here plays the *same role* as the $R[\rho_T]$ factor in the pointer functional (1) from the measurement paper [3]: it counts the fraction (or effective density) of environment fragments that carry *consistent temporal records*. Maximizing $F = E - \mathcal{R}$ therefore operationalizes the statement that the preferred scalar field is the one whose records are *most redundantly copyable* across space. In this picture, $\Phi = \Theta\ (\delta R/\delta\rho)|_{\Xi}$ (Eq. (14)) is precisely the *chemical potential of temporal redundancy*.
Intuition. $\overline{\omega^2}(\mathbf{x})$ counts how rapidly local systems can discriminate time intervals (clock sharpness), while $\Xi(\mathbf{x})$ measures the coherence time over which phase information persists under lapse fluctuations. Their product sets the *local rate of temporal information production*, and the integral aggregates the universe's redundancy capacity. Here $\overline{\omega^2}$ is a coarse-grained clock-frequency squared (set by local microscopic spectra) and Ξ encodes low-frequency lapse fluctuations (e.g., derived from the zero-frequency weight of the Φ noise correlator via the closure relation).

Constrained variation with closure $\Xi = \mathcal{C}[\Phi]$. When the microphysical closure $\Xi = \mathcal{C}[\Phi]$ is imposed, the correct stationarity condition is the constrained Euler–Lagrange equation

$$\left.\frac{\delta F}{\delta \Phi}\right|_{\Xi} + \Big(D\mathcal{C}[\Phi]\Big)^{*} \left.\frac{\delta F}{\delta \Xi}\right|_{\Phi} = 0, \qquad \Xi = \mathcal{C}[\Phi], \tag{13}$$

which is Eq. (122) derived in Appendix G [4]. Equivalently, introducing the multiplier field Λ in the augmented functional and eliminating it reproduces (13) (see App. G for details).

Notation for constrained variations. We denote functional derivatives at fixed capacity by a vertical bar: $\delta/\delta\rho|_{\Xi}$. When we later enforce the closure $\Xi = \mathcal{C}[\Phi]$ with a Lagrange multiplier Λ (Eqs. (14)–(35)), we drop the vertical bar because Ξ then appears explicitly as an independent variable. This keeps the two equivalent formalisms notationally consistent.
Derivative conventions. $F_{\Phi} \equiv \delta F/\delta\Phi\big|_{\Xi}$ and $F_{\Xi} \equiv \delta F/\delta\Xi\big|_{\Phi}$ are *partial* functional derivatives. The total derivative of the reduced functional $\tilde{F}[\Phi] = F[\rho, \Phi, C[\Phi]]$ includes the chain term $(DC)^{*} F_{\Xi}$ as in Eq. (122).

Stationarity of the constrained objective $\mathcal{F}$ at fixed Ξ yields $\Phi = \Theta\,(\delta R/\delta\rho)|_{\Xi}$; together with $\Xi = \mathcal{C}[\Phi]$ from the CTP constraint this identifies Φ as the redundancy conjugate:

$$\left.\frac{\delta F}{\delta \rho}\right|_{\Xi} = \Phi - \Theta \left.\frac{\delta R}{\delta \rho}\right|_{\Xi} = 0 \quad \Rightarrow \quad \Phi = \Theta \left.\frac{\delta R}{\delta \rho}\right|_{\Xi} = \Theta \frac{\kappa T_{\rm int}}{2E_c}\,\omega^2\,\Xi. \tag{14}$$

Fixed-point map and practical solver. Combining the chemical-potential relation $\Phi = C\,\omega^2\,\Xi$ with the closure $\Xi = \mathcal{C}[\Phi]$ gives the fixed-point problem

$$\Phi = \mathcal{T}[\Phi] \equiv C\,\omega^2\,\mathcal{C}[\Phi], \qquad C \equiv \Theta\,\frac{\kappa\mathcal{T}}{2E_c}. \tag{15}$$

Assume $\mathcal{C}$ is Fréchet-differentiable on a convex set $\mathcal{D} \subset X$ and Lipschitz in the energy norm: $\|\mathcal{C}[\Phi_1] - \mathcal{C}[\Phi_2]\|_E \le L_{\mathcal{C}}\,\|\Phi_1 - \Phi_2\|_E$. If

$$\|\,\omega^2\,\|_{L^\infty(\Omega)}\,C\,L_{\mathcal{C}} < 1, \tag{16}$$

then $\mathcal{T}$ is a contraction on $\mathcal{D}$ and the Picard iteration converges geometrically to the unique fixed point:

$$\Phi^{(n+1)} = (1-\alpha)\,\Phi^{(n)} + \alpha\,\mathcal{T}[\Phi^{(n)}], \qquad 0 < \alpha \le 1 \quad \text{with} \quad \alpha\,\|\,\omega^2\,\|_\infty\,C\,L_{\mathcal{C}} < 1. \tag{17}$$

Notes. (i) In the weak-field/Newtonian matching regime $C \ll 1$, so (16) is naturally satisfied. (ii) In equilibrium linear response with FDT (App. E), bounded noise spectrum and a dissipative kernel ensure $L_{\mathcal{C}} < \infty$.

Calibration of Θ (explicit Newtonian matching)

Define $C \equiv \Theta\, \kappa T_{\rm int}/(2E_c)$. Choose a reference situation $x_\star$ where $\Phi_\star$ is known from Poisson (e.g., outside a spherical mass M: $\Phi_\star(r) = -GM/(c^2 r)$). Pick a probe clock with angular frequency $\omega_\star$ and compute the capacity $\Xi_\star$ from the microphysical model (Appendix E). The stationarity condition $\Phi_\star = C\,\omega_\star^2\,\Xi_\star$ fixes

$$\Theta \;=\; \frac{2E_c}{\kappa\, T_{\rm int}}\,\frac{\Phi_\star}{\omega_\star^2\,\Xi_\star}\,. \tag{18}$$

Units check. $\Phi_\star$ is dimensionless; $\omega_\star^2 \Xi_\star$ is dimensionless; E_c is action (J s) and $T_{\rm int}$ is time (s), so $2E_c/(\kappa T_{\rm int})$ has units of energy (J). Hence Θ has units of energy, as required for $\mathcal{R} = \Theta R$ and $F = E - \mathcal{R}$.
Units sanity: R is dimensionless, Θ carries J, hence $\mathcal{R} = \Theta R$ and $F = E - \mathcal{R}$ both have units of energy.

Example (Earth, geoid): $\Phi_\oplus \simeq -GM_\oplus/(c^2 R_\oplus) \approx -6.96\times 10^{-10}$. For an optical clock ($\omega_\star \sim 2\pi\times 4\times 10^{14}\,{\rm s}^{-1}$) and an Ohmic bath (App. E) with $\Xi_\star = 2T_{\rm int}\, k_B T_B\, \eta\, |G^R_\Phi(0)|^2$, Eq. (18) yields Θ in terms of the bath parameters $(T_B, \eta, |G^R_\Phi(0)|)$.

Redundancy–to–curvature efficiency. We quantify how effectively redundancy sources drive curvature by

$$\mathcal{E}_{\rm R}(x) \;\equiv\; \frac{\Phi(x)}{C\,\omega^2(x)\,\Xi(x)}\,, \qquad C \equiv \Theta\,\frac{\kappa T_{\rm int}}{2E_c}. \tag{19}$$

In a stationary, homogeneous bath of identical clocks one expects $\mathcal{E}_{\rm R} \simeq 1$; departures encode microphysical inefficiencies or modelling mismatch. This quantity lives purely in the gravitational (Φ) sector and will *not* be used for dephasing/noise coupling in Sec. 5.1.
Physical meaning. $\mathcal{E}_{\rm R}(x)$ is the dimensionless ratio between the curvature Φ demanded by the Hamiltonian constraint and the curvature supplied by the local environment–clock pair via the redundancy ledger. When the bath is stationary and the same clock is used (so ω and Ξ are intensive and source–independent), $\mathcal{E}_{\rm R} \simeq 1$ expresses *equilibrium redundancy production.* In that case the proportionality $\Phi = C\,\omega^2 \Xi$ is fixed everywhere by the calibration point, so outside a localized source Poisson's equation enforces the usual Green–function scaling $\Phi(r) \propto M/r$—i.e., linear mass scaling. Deviations $\mathcal{E}_{\rm R} \neq 1$ quantify departures from that equilibrium (e.g., bath inhomogeneity or a different clock), effectively acting as a small local renormalization of the redundancy–curvature conversion.

At the calibration point $x_\star$, $\mathcal{E}_{\rm R}(x_\star) = 1$. In a stationary bath with fixed clock $\omega(x) = \omega_\star$ and $\Xi(x) = \Xi_\star$, one has $\mathcal{E}_{\rm R}(x) = \Phi(x)/\Phi_\star$.

Stationarity w.r.t. Φ gives the Poisson equation

$$\frac{\delta F}{\delta \Phi} = \frac{c^4}{8\pi G}(-\nabla^2\Phi) + \rho = 0 \quad\Rightarrow\quad \boxed{\nabla^2\Phi = \frac{4\pi G}{c^4}\,\rho}. \tag{20}$$

Remark. If one substitutes the closure $\Xi = \Xi[\Phi]$ *before* varying, a small term $\delta R/\delta\Phi$ appears that renormalizes couplings at leading order (absorbed into G); we therefore hold Ξ fixed in the variation and impose (8) afterwards (Appendix D).
No circularity: in the variational step we treat Ξ as an auxiliary field and enforce the microphysical closure $\Xi = \mathcal{K}[\Phi]$ with a multiplier Λ (see Section 3.1); the resulting Euler–Lagrange system (34)–(36) is equivalent to the fixed-point problem (38), which is a contraction under Eq. (39), ensuring a unique solution.

Constraints vs dynamics. Equation (20) is the *Hamiltonian constraint* for the lapse $N = e^{\Phi}$ on each time slice: given ρ it *instantaneously* fixes Φ on that slice. It is *not* an evolution equation. Evolution of Φ *between* slices comes from the mixed Einstein equations and, in general, also involves the shift (gravitomagnetic) sector and the transverse–traceless modes. In spherical, zero-shift settings the mixed $t-r$ component yields the *flux law* (Eq. (41), derived in Sec. 4).
Sign conventions and sanity check. We use signature $(-,+,+,+)$, zero shift, and $N = e^{\Phi}$ so that in the weak field $g_{tt} \simeq -(1+2\Phi)$ with $\Phi = \phi_N/c^2$ (negative near a gravitating source). Here $T^t{}_r$ is the outward energy–flux density; for outward radiation $T^t{}_r > 0$ one has $\partial_t\Phi > 0$ (mass inside r decreases $\Rightarrow$ potential becomes less negative), while for inward accretion $T^t{}_r < 0$ one finds $\partial_t\Phi < 0$ (mass increases $\Rightarrow$ potential deepens). This matches the expected behavior.

No scalar graviton

Why Φ satisfies a constraint (not a wave equation). The scalar Φ is the *lapse function* (time-reparameterization gauge) in ADM formalism. It enforces the Hamiltonian constraint $\mathcal{H}_\perp \approx 0$ on each spatial slice, fixing the geometry *instantaneously* given the matter distribution $\rho(x)$.
There is **no** scalar graviton: Φ has no kinetic term $\sim (\partial_t\Phi)^2$ and satisfies $\nabla^2\Phi = (4\pi G/c^4)\rho$ slice-by-slice. Evolution of Φ *between* slices comes from the mixed Einstein tensor (flux law), not from Φ dynamics.
Only the two transverse-traceless (TT) modes h_{ij}^{TT} satisfy $\Box h_{ij}^{\mathrm{TT}} = 0$ and propagate at light speed. This matches the standard GR degree-of-freedom count and explains why scalar–tensor theories (which *do* have $\Box\phi = 4\pi G\rho/c^4$) are phenomenologically distinct from pure GR.

We write the radial energy flux as $T^t{}_r$ (mixed indices). This is the minimal dynamical complement to Eq. (20). (ADM gauge/DOF bookkeeping: lapse and shift enforce constraints; only two TT modes propagate.)

Vector/tensor redundancy-ledgers and EL equations (linearized)

Write the weak-field energies (standard GEM + TT):

$$E_V[\boldsymbol{\omega}] = \frac{c^4}{16\pi G}\int d^3x\, |\nabla\times\boldsymbol{\omega}|^2, \tag{21}$$

$$E_T[h^{\mathrm{TT}}] = \frac{c^4}{32\pi G}\int d^3x\, \Big(|\dot h_{ij}^{\mathrm{TT}}|^2 + c^2|\nabla h_{ij}^{\mathrm{TT}}|^2\Big). \tag{22}$$

The extended constrained objective becomes

$$\begin{aligned}\mathcal{F}[\rho,\Phi,\mathbf{J},\boldsymbol{\omega},\dot h^{\mathrm{TT}},\Xi,\boldsymbol{\Omega},\Xi_T,\Lambda,\boldsymbol{\Lambda},\Lambda_T]&\\ = E[\rho,\Phi] + E_V[\boldsymbol{\omega}] + E_T[h^{\mathrm{TT}}]&\\ - \Theta R[\rho;\Xi] - \Theta R_V[\mathbf{J};\boldsymbol{\Omega}] - \Theta R_T[\dot h^{\mathrm{TT}};\Xi_T]&\\ + \int d^3x\,\Lambda[\Xi - \mathcal{C}[\Phi]] + \int d^3x\,\boldsymbol{\Lambda}\cdot[\boldsymbol{\Omega} - \boldsymbol{\mathcal{C}}_V[\boldsymbol{\omega}]]&\\ + \int d^3x\,\Lambda_T[\Xi_T - \mathcal{C}_T[\dot h^{\mathrm{TT}}]].&\end{aligned} \tag{23}$$

Extremizing yields the full linearized Einstein equations in our lapse-first gauge:

$$\nabla^2\Phi = \frac{4\pi G}{c^4}\rho, \tag{24}$$

$$\nabla^2\boldsymbol{\omega} = -\frac{16\pi G}{c^4}\,\mathbf{J}, \tag{25}$$

$$\Box\, h_{ij}^{\rm TT} = 0. \tag{26}$$

The first two are Hamiltonian constraints sourced by $(\rho, \mathbf{J})$; only the two TT modes in (26) propagate. This matches GR's weak-field DOF count: the redundancy framework pins constraints to the scalar/vector sectors and dynamics to the transverse-traceless tensor sector.

For the vector sector, the standard GEM calibration gives

$$\boldsymbol{\omega}(\mathbf{r}) = -\frac{2G}{c^3}\,\frac{\mathbf{J}\times\mathbf{r}}{r^3} \quad\Rightarrow\quad g_{t\varphi} = -\frac{2GJ}{c^3 r}\,\sin^2\theta,$$

which matches the linearized Kerr normalization.

Deriving the vector/tensor equations (sketch)

Vector (gravitomagnetic) sector. Work in the standard weak-field gauge with $\nabla\cdot\boldsymbol{\omega} = 0$. Use the magnetostatic GEM energy and a redundancy ledger linear in the source current:

$$E_V[\boldsymbol{\omega}] = \frac{c^4}{16\pi G}\int d^3x\,|\nabla\times\boldsymbol{\omega}|^2, \tag{27}$$

$$R_V[\mathbf{J};\,\boldsymbol{\omega},\Xi_V] = \frac{\kappa_V T_{\rm int}}{2E_c}\int d^3x\,\mathbf{J}\cdot\boldsymbol{\omega}\,\Xi_V(x), \tag{28}$$

with Ξ_V constrained by the CTP kernel via a multiplier field $\boldsymbol{\Lambda}_V$ (not shown explicitly). Extremize $F_V = E_V - \Theta_V R_V$ at fixed Ξ_V (the constraint is imposed after variation, as in Sec. 3.1). The variations are

$$\frac{\delta E_V}{\delta\boldsymbol{\omega}} = \frac{c^4}{8\pi G}\,\nabla\times(\nabla\times\boldsymbol{\omega}) = -\frac{c^4}{8\pi G}\,\nabla^2\boldsymbol{\omega} \quad (\nabla\cdot\boldsymbol{\omega}=0), \qquad \frac{\delta R_V}{\delta\boldsymbol{\omega}} = \frac{\kappa_V T_{\rm int}}{2E_c}\,\mathbf{J}\,\Xi_V.$$

Stationarity $\delta F_V/\delta\boldsymbol{\omega} = 0$ gives

$$-\frac{c^4}{8\pi G}\,\nabla^2\boldsymbol{\omega} \;-\; \Theta_V\,\frac{\kappa_V T_{\rm int}}{2E_c}\,\mathbf{J}\,\Xi_V \;=\; 0.$$

Choosing the (vector) Newtonian matching $\Theta_V = \frac{4E_c}{\kappa_V T_{\rm int}\,\Xi_V}$ (the exact analog of Eq. (18)) yields

$$\nabla^2\boldsymbol{\omega} \;=\; -\frac{16\pi G}{c^4}\,\mathbf{J}, \tag{29}$$

i.e. the standard GEM relation for the shift in the Coulomb-like gauge.

Remark (ledger in $\boldsymbol{\Omega}$ *vs.* $\boldsymbol{\omega}$*).* One can equivalently write $R_V \propto \int(\nabla\times\boldsymbol{\omega})\cdot\mathbf{K}\,\Xi_V$ and integrate by parts to a $J\cdot\boldsymbol{\omega}$ form using $\nabla\cdot\mathbf{J} = 0$ and vanishing boundary terms. Thus the $J\cdot\boldsymbol{\omega}$ choice is merely the variationally convenient representative of the same gauge-invariant content.

Tensor (TT) sector. Propagating modes require a time-integrated action (not an instantaneous energy minimum). Use the linearized TT action and a redundancy action that weights $\dot{h}^{\mathrm{TT}}$ with its slow capacity Ξ_T:

$$S_T[h^{\mathrm{TT}}] = \int dt\, \frac{c^4}{64\pi G} \int d^3x \left(\frac{1}{c^2}\, \dot{h}^{\mathrm{TT}}_{ij} \dot{h}^{\mathrm{TT}}_{ij} - |\nabla h^{\mathrm{TT}}_{ij}|^2 \right), \tag{30}$$

$$S_{R_T}[h^{\mathrm{TT}}; \Xi_T] = \Theta_T\, \frac{\kappa_T}{2E_c} \int dt \int d^3x\; \Xi_T(x)\, \dot{h}^{\mathrm{TT}}_{ij} \dot{h}^{\mathrm{TT}}_{ij}. \tag{31}$$

The constrained action is $S_{\mathrm{tot}} = S_T - S_{R_T} + \int d^3x\, \Lambda_T(x) \big[\Xi_T(x) - \mathcal{C}_T[h^{\mathrm{TT}}](x) \big]$. Varying w.r.t. h^{TT}_{ij} and integrating by parts in t gives (keeping Ξ_T fixed during variation)

$$\frac{c^2}{32\pi G}\, \ddot{h}^{\mathrm{TT}}_{ij} - \frac{c^4}{32\pi G}\, \nabla^2 h^{\mathrm{TT}}_{ij} \;-\; \Theta_T\, \frac{\kappa_T}{E_c}\, \Xi_T\, \ddot{h}^{\mathrm{TT}}_{ij} \;=\; 0 \quad \text{(TT gauge)}.$$

The Ξ_T term *renormalizes* the kinetic coefficient. Writing $Z_T \equiv 1 - \frac{32\pi G}{c^2}\, \Theta_T\, \frac{\kappa_T}{E_c}\, \Xi_T$ and dividing through by $c^2/32\pi G$,

$$Z_T\, \ddot{h}^{\mathrm{TT}}_{ij} - c^2 \nabla^2 h^{\mathrm{TT}}_{ij} = 0.$$

For weak bath dressing (Appendix E) one has $Z_T \simeq 1$; more generally, a finite Z_T is absorbed into a field renormalization $h^{\mathrm{TT}} \to h^{\mathrm{TT}}/\sqrt{Z_T}$, leaving the dispersion unchanged. Thus, to leading order,

$$\Box\, h^{\mathrm{TT}}_{ij} \;=\; 0, \tag{32}$$

which is Eq. (26). Any residual $Z_T \neq 1$ would show up as a universal wave-speed renormalization; solar-system and GW constraints bound this to be negligible, fixing $\Theta_T \Xi_T$ accordingly.
CTP closure (no circularity). As in the scalar sector, $\Xi_{V,T}$ are slow capacities fixed by their CTP kernels $\mathcal{C}_{V,T}[\boldsymbol{\omega}]$ and $\mathcal{C}_T[h^{\mathrm{TT}}]$ via multipliers; their explicit field-dependence only renormalizes coefficients (Z-factors) at this order.

Self-consistency without circularity

To avoid any appearance of circular dependence, we treat (Φ, Ξ) as independent fields and impose the microphysical closure $\Xi = \mathcal{K}[\Phi]$ with a Lagrange multiplier $\Lambda(x)$. Define the augmented objective

$$\mathcal{F}_{\mathrm{aug}}[\rho, \Phi, \Xi, \Lambda] \;=\; E[\rho, \Phi] \;-\; \Theta\, R[\rho; \Xi] \;+\; \int d^3x\, \Lambda(x)\, \big[\Xi(x) - \mathcal{K}[\Phi](x) \big]. \tag{33}$$

Here $\mathcal{K}[\Phi]$ is the capacity functional defined in Appendix E (e.g. $\Xi = \int_0^{T_{\mathrm{int}}} \int_0^{T_{\mathrm{int}}} C_\Phi\, dt\, dt'$), and R is the redundancy functional from Eq. (7). Stationarity yields the coupled Euler–Lagrange equations

$$\frac{\delta \mathcal{F}_{\mathrm{aug}}}{\delta \Phi(x)} = -\frac{c^4}{8\pi G}\, \nabla^2 \Phi(x) \;+\; \rho(x) \;-\! \int d^3x'\, \Lambda(x')\, \mathcal{K}^{(1)}(x', x; \Phi) \;=\; 0, \tag{34}$$

$$\frac{\delta \mathcal{F}_{\mathrm{aug}}}{\delta \Xi(x)} = -\Theta\, \frac{\kappa T_{\mathrm{int}}}{2E_c}\, \rho(x)\, \omega^2(x) \;+\; \Lambda(x) \;=\; 0, \tag{35}$$

$$\frac{\delta \mathcal{F}_{\mathrm{aug}}}{\delta \Lambda(x)} = \Xi(x) - \mathcal{K}[\Phi](x) \;=\; 0, \tag{36}$$

where $\mathcal{K}^{(1)}(x', x; \Phi) \equiv \delta \mathcal{K}[\Phi](x') / \delta \Phi(x)$. Eliminating Λ with (35) and using (36) gives the closed field equation

$$-\frac{c^4}{8\pi G}\, \nabla^2 \Phi(x) \;+\; \rho(x) \;-\; \Theta\, \frac{\kappa T_{\mathrm{int}}}{2E_c} \int d^3x'\, \rho(x')\, \omega^2(x')\, \mathcal{K}^{(1)}(x', x; \Phi) \;=\; 0. \tag{37}$$

Equivalently, combining (35) and (36) with the chemical-potential condition (14) yields the pointwise fixed-point form

$$\Phi(x) \;=\; C\,\omega^2(x)\,\mathcal{K}[\Phi](x), \qquad C \;\equiv\; \Theta\,\frac{\kappa T_{\rm int}}{2E_c}. \tag{38}$$

Well-posedness (no circularity). Let $(X, \|\cdot\|_E)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ under the energy norm $\|\Phi\|_E^2 = \frac{c^4}{8\pi G}\int |\nabla\Phi|^2 d^3x + \varepsilon\int \Phi^2 d^3x$ for any $\varepsilon > 0$. Assume the capacity map is Fréchet-differentiable and Lipschitz on a convex set $\mathcal{D} \subset X$:

$$\|\mathcal{K}[\Phi_1] - \mathcal{K}[\Phi_2]\|_E \;\le\; L_{\mathcal{K}}\,\|\Phi_1 - \Phi_2\|_E \qquad \forall\,\Phi_1, \Phi_2 \in \mathcal{D}. \tag{39}$$

If $C\,\|\omega^2\|_{L^\infty}\,L_{\mathcal{K}} < 1$ then the map $\mathcal{T}[\Phi] \equiv C\,\omega^2\,\mathcal{K}[\Phi]$ is a contraction on $\mathcal{D}$ and the fixed-point equation (38) admits a unique solution $\Phi^\star \in \mathcal{D}$ (Banach fixed-point theorem). Moreover, the Picard iteration

$$\Phi^{(n+1)} \;=\; C\,\omega^2\,\mathcal{K}[\Phi^{(n)}], \qquad n = 0, 1, 2, \ldots, \tag{40}$$

converges to $\Phi^\star$ for any $\Phi^{(0)} \in \mathcal{D}$, and the error contracts geometrically. In the weak-field Newtonian limit $\kappa \to 0$ one has $C \to 0$, so the contraction condition is automatically satisfied and $\Phi^\star$ reduces to the Poisson solution.

Remarks. (i) In equilibrium linear response, $\mathcal{K}[\Phi]$ is obtained from the Φ-spectrum via FDT (Appendix E); bounded noise spectrum and dissipative kernel imply a finite $L_{\mathcal{K}}$. (ii) For numerics we use a relaxed iteration $\Phi^{(n+1)} \leftarrow (1-\alpha)\Phi^{(n)} + \alpha\,\mathcal{T}[\Phi^{(n)}]$ with $\alpha \in (0, 1]$; choose α so that $\alpha\,C\,\|\omega^2\|_{L^\infty}\,L_{\mathcal{K}} < 1$ on the grid.

Weights are already supplied by flux. On the quantum side, the flux-partition identity yields the Born weights $|c_i|^2$ directly from a continuity law [3]. Thus the measurement program supplies both (i) *weights* (via flux partition) and (ii) *basis* (via redundancy), furnishing the concrete bridge needed for the emergent-time variational picture.

Together, Eq. (14) (chemical potential at fixed Ξ) and the *constraint* Eq. (20) fix the scalar geometry on each slice, setting the spatial coherence scale $c^4/8\pi G$. Dynamics between slices comes from the flux law (Eq. (41), spherical case) or, more generally, from the shift/tensor sectors.

> **No scalar loophole for frame dragging.** In the linearized lapse-first gauge, $(\partial_t^2 - \nabla^2)\,\boldsymbol{\omega}_T = 16\pi G\,\mathbf{J}_T/c^4$. Thus $\partial_t\Phi$ (scalar sector) cannot produce $\mathbf{B}_g = \nabla\times\boldsymbol{\omega}$; physical frame dragging requires transverse mass currents.

> **Summary of Sec. 3.** (i) $F = E - \Theta R$ with E coercive on H_0^1, R dimensionless, Θ [J]. (ii) At fixed Ξ, $\delta F/\delta\Phi = 0$ gives the Poisson-type equation (4); $\delta F/\delta\rho|_\Xi = 0$ gives the chemical potential relation (14). (iii) With closure $\Xi = \mathcal{C}[\Phi]$, the correct EL is (13) (App. G). (iv) The self-consistent solution solves the fixed-point map (15) and converges by (17) when (16) holds.

Key Insight. We do not *quantize* gravity because gravity *is* what emerges when quantum systems create redundant time records. The Planck scale is not where gravity "becomes quantum"; it is where individual quantum systems become sufficiently heavy and coherent to act as reliable clocks—feeding the redundancy that defines classical time.

4 Consistency & Known Limits

Cosmology. In homogeneous/isotropic settings one may map the scale factor to the lapse via $a = e^{-\Phi}$ so that $H \equiv \dot{a}/a = -\dot{\Phi}$. Standard Friedmann equations are recovered in this variable, making the scalar time potential explicit.

Dynamics (spherical sector). In spherical, zero-shift gauges (EF/PG), the mixed Einstein equation gives

$$\partial_t \Phi = +\frac{4\pi G}{c^4}\, r\, T^t{}_r,\text{[1]} \tag{41}$$

Notation. We write $T^t{}_r \equiv T^t{}_i n^i$ on S_r; indices are placed to avoid metric-dependent sign slips, and n^i is the outward unit normal. *Provenance.* This operational EF/PG flux law is derived carefully in App. F.1 of Ref. [2] and was first noted in Paper I [1].

Derivation (energy balance on a sphere). Let $E_{\rm enc}(t,r) \equiv \int_{|\mathbf{x}|<r} T^{tt}(t,\mathbf{x})\, d^3x$ and $M_{\rm enc} \equiv E_{\rm enc}/c^2$. Energy conservation gives

$$\dot{E}_{\rm enc}(t,r) \;=\; -\int_{S_r} T^t{}_i(t,\mathbf{x})\, n^i\, dA \;=\; -\,4\pi r^2\, \langle T^t{}_i n^i\rangle_\Omega, \tag{42}$$

with n^i the outward unit normal and $\langle\cdot\rangle_\Omega$ the solid-angle average [5, 6]. In the weak field, $\Phi(t,r) = -\dfrac{GM_{\rm enc}(t,r)}{c^2\, r}$, hence

$$\partial_t \Phi(t,r) = -\frac{G}{c^2 r}\, \dot{M}_{\rm enc}(t,r) = \frac{G}{c^4 r}\int_{S_r} T^t{}_i n^i\, dA = \frac{G}{c^4}\, r\, 4\pi\, \langle T^t{}_i n^i\rangle_\Omega. \tag{43}$$

For spherical symmetry $T^t{}_i n^i \equiv T^t{}_r$ on S_r, giving Eq. (41).

Equivalent forms (local vs. averaged).

$$\text{(spherical)}\quad \partial_t\Phi(t,r) \;=\; \frac{4\pi G}{c^4}\, r\, T^t{}_r(t,r), \tag{44}$$

$$\text{(solid-angle average)}\quad \partial_t\Phi(t,r) \;=\; \frac{G}{c^4}\,\frac{r}{4\pi}\int d\Omega\; T^t{}_i(t,r,\Omega)\, n^i. \tag{45}$$

Eq. (45) is the general statement; Eq. (44) follows if the flux is angle-independent on S_r.
Sanity checks. (i) **Signs:** With $(-,+,+,+)$ and $N = e^\Phi$, $\Phi = \phi_N/c^2 < 0$ near a source. Outward energy flow has $T^t{}_r > 0$, so $\partial_t\Phi > 0$ (shallower potential); inward accretion $T^t{}_r < 0$ gives $\partial_t\Phi < 0$ (deeper potential). (ii) **Units:** $[T^t{}_r]$ = energy flux density = $\mathrm{J\,m^{-2}\,s^{-1}}$; multiplying by r (m) and G/c^4 ($\mathrm{m\,J^{-1}}$) yields a pure 1/s, matching $\partial_t\Phi$ since Φ is dimensionless.
GR cross-check. In the null-radiation (Vaidya) solution one has $\dot{M} = -4\pi r^2 T^r{}_t$ along outgoing null slices [7]. Linearizing and converting to our t-slicing yields Eq. (45); the spherical case reduces to Eq. (44).

so time curvature changes only when power flows through space; this is the *dynamical complement* to the constraint Eq. (20).

Gauge & DOF ledger. In ADM form, the lapse $N = e^\Phi$ and shift ω_i enforce constraints; only transverse-traceless modes propagate. Our variational picture is a coarse-grained scalar sector consistent with that DOF counting; vector (frame-drag) effects live in the shift and are not sourced by Φ alone.

Background derivations. For explicit FRW mapping, flux-law details, and gauge bookkeeping in lapse-first variables, see [1, 8].

[1]With our conventions (signature $(-,+,+,+)$, $N = e^\Phi$, $\Phi < 0$ near masses), outward flux $T^t{}_r > 0$ raises Φ (mass inside r decreases), consistent with the sign in Eq. (41).

Beyond spherical: For rotation, non-spherical flows, and waves one must include the *shift* ω_i (gravitomagnetic potential) and the TT modes. In the weak-field, lapse-first gauge this reproduces the standard Maxwell-like GEM structure, with $\nabla^2\Phi = (4\pi G/c^4)\,\rho$ as a constraint and $\nabla^2\boldsymbol{\omega} \sim -16\pi G\,\mathbf{J}$ sourcing frame dragging. We defer that general dynamical treatment to the companion strong-field paper.

5 Decoherence: Distinctive Scaling

Fourier & window conventions. We use $f(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} f(\omega)$ and $f(\omega) = \int dt\, e^{i\omega t} f(t)$. For a rectangular interrogation window of length $\mathcal{T}$,

$$W_{\mathcal{T}}(\omega) \;=\; \int_0^{\mathcal{T}} dt\, e^{i\omega t} \;=\; e^{i\omega\mathcal{T}/2}\,\frac{2\sin(\omega\mathcal{T}/2)}{\omega}, \qquad |W_{\mathcal{T}}(\omega)|^2 = \frac{4\sin^2(\omega\mathcal{T}/2)}{\omega^2}.$$

The capacity

$$\Xi = \int_0^{\mathcal{T}}\!\!\int_0^{\mathcal{T}} C_\Phi(t-t')\,dt\,dt' = \int \frac{d\omega}{2\pi}\,|W_{\mathcal{T}}(\omega)|^2\,S_\Phi(\omega),$$

so for long $\mathcal{T}$: $\Xi \simeq \mathcal{T}\,S_\Phi(0)$ (Wiener–Khinchin) [9–12].

For a two-path clock of angular frequency ω, differential lapse noise with correlator $C_\Phi(t,t')$ produces Gaussian dephasing

$$\mathcal{V} \simeq \exp\Big[-\tfrac{1}{2}\,\omega^2\!\int_0^{\mathcal{T}}\!\!\int_0^{\mathcal{T}} C_\Phi(t,t')\,\mathrm{d}t\,\mathrm{d}t'\Big], \tag{46}$$

Visibility and variance relation. Let $S_\Phi(f)$ be the one-sided PSD of Φ noise pulled back to the worldline. The spectral visibility kernel is $K_{\mathcal{T}}(f) = |\Omega_{\mathcal{T}}(f)|^2$ and

$$\mathrm{Var}[\varphi] \;=\; \omega^2\int_0^\infty \frac{df}{2}\,K_{\mathcal{T}}(f)\,S_\Phi(f) \;=\; \omega^2\,\Xi,$$

so that $\mathcal{V} = \exp[-\frac{1}{2}\,\mathrm{Var}[\varphi]]$.
Clock-network derivation and conventions. A full derivation for worldline pullbacks, interrogation transfer functions $\Omega_{A,B}(f)$, and the spectral visibility kernel is given in Ref. [2, Sec. 7.2, Eq. (26)]. We follow the same two-sided Fourier/PSD conventions there for all visibility integrals.
Worldline pullback. For stationary clocks and slow motion, the stochastic phase increment is linear in the lapse fluctuation along the measured paths; see Ref. [2, Eq. (19)–(23)].
equivalently a rate $\Gamma_\phi = \frac{\omega^2}{2} S_\Phi(0)$ for long $\mathcal{T}$, where $S_\Phi(0)$ is the zero-frequency power. As Φ is sourced *linearly* by ρ (Eq. (20)), we predict

$$\boxed{\Gamma_\phi \;\propto\; \omega^2\,M^\alpha, \quad \alpha \approx 1} \tag{47}$$

Redundancy-driven basis selection. The same redundancy mechanism that favors position over momentum in collisional SG models [3] implies that temporal records are favored when environmental channels transmit proper-time differences efficiently, producing the universal ω^2 dependence while maintaining a *linear* (not quadratic) mass scaling at fixed geometry.
Origin of $\alpha \approx 1$. Since Φ is sourced linearly by energy density via Eq. (20), the zero-frequency lapse noise scales linearly with source mass at fixed geometry, yielding $\Gamma_\phi \propto \omega^2 M$ rather than the M^2-type growth found in CSL/GRW models. at fixed geometry, in contrast to CSL/GRW models that typically yield M^2-like scaling. Geometry and network connectivity set the proportionality.

5.1 Predictive stochastic lapse from stress–energy fluctuations

We consider small departures $\delta\Phi$ about the stationary solution $\Phi_\star$ of our variational principle $F[\rho,\Phi]$. Linearizing the Euler–Lagrange equation yields an effective Langevin dynamics

$$(-\nabla^2 + m^2)\,\delta\Phi(t,\mathbf{x}) \;+\; \int d^3x' \int_{-\infty}^{t} dt'\;\Gamma(\mathbf{x}-\mathbf{x}',\,t-t')\,\partial_{t'}\big[\delta\Phi(t',\mathbf{x}')\big] \;=\; \eta(t,\mathbf{x}). \tag{48}$$

Notation: $\partial_{t'}$ acts only on the bracketed $\delta\Phi(t',\mathbf{x}')$; the kernel $\Gamma(\mathbf{x}-\mathbf{x}',t-t')$ is held fixed, and the causal limit $t' \le t$ is made explicit.
The equation features a retarded damping kernel Γ and a zero-mean source η encoding stress–energy fluctuations of the environment [13–15].

Mass scale from redundancy. The curvature of the redundancy functional fixes the static "mass" of the scalar sector:

$$m^2 \equiv \frac{8\pi G}{c^4}\Big[\frac{\delta^2 R}{\delta\Phi^2}\Big]_{\Phi_\star}, \qquad \xi \equiv m^{-1}. \tag{49}$$

Stochastic coupling in the clock sector. Dephasing depends on the stochastic coupling between the probe clock and lapse fluctuations. We denote this by

$$\zeta_\phi \in [0,\infty)\,, \tag{50}$$

which parametrizes the strength with which clock phase responds to Φ-noise in our Ohmic/FDT model.

> **Notation.** ω is the carrier angular frequency of the probe; $\Omega_{\mathcal{T}}(f)$ is the window transfer function (its magnitude squared defines $K_{\mathcal{T}}$). We write ζ_ϕ for the stochastic clock–lapse coupling (§5.1) and $\mathcal{E}_{\rm R}$ for the redundancy→curvature efficiency (Sec. 3); these live in different sectors and are not interchangeable. Interrogation time is denoted $\mathcal{T}$ (to avoid confusion with the bath temperature $T_{\rm B}$ or a global time function $T(x)$).

Notation warning: ζ_ϕ is entirely distinct from the redundancy–to–curvature efficiency $\mathcal{E}_{\rm R}$ of Sec. 3; the former belongs to the *dephasing (clock)* sector, the latter to the *gravitational (Φ)* sector.

Response and spectrum (with FDT). Fourier transforming $(\omega,\mathbf{k})$, the retarded Green function for Eq. (48) is

$$G_R(\omega,k) \;=\; \big[k^2 + m^2 - i\,\omega\,\Gamma(\omega,k)\big]^{-1}, \qquad \delta\Phi = G_R\,\eta. \tag{51}$$

The lapse spectrum follows from the linear response:

$$S_\Phi(\omega,k) \;=\; \big|G_R(\omega,k)\big|^2\, S_\eta(\omega,k). \tag{52}$$

Near equilibrium, the fluctuation–dissipation theorem (FDT) relates the noise spectrum to the damping kernel as [16, 17]

$$S_\eta(\omega,k) \;=\; 2\,\nu(\omega,T_{\rm B})\,\mathrm{Re}\,\Gamma(\omega,k), \tag{53}$$

with

$$\nu(\omega,T_{\rm B}) \;=\; \begin{cases} k_{\rm B}T_{\rm B}, & \text{classical (high-}T\text{) limit,} \\ \dfrac{\hbar\omega}{2}\coth\!\big(\dfrac{\hbar\omega}{2k_{\rm B}T_{\rm B}}\big), & \text{quantum (symmetrized) FDT.} \end{cases} \tag{54}$$

Substituting (53) into (52) yields the explicit form

$$S_\Phi(\omega,k) \;=\; 2\,\nu(\omega,T_\mathrm{B})\,\left|G_R(\omega,k)\right|^2\,\mathrm{Re}\,\Gamma(\omega,k). \tag{55}$$

Notes. (i) If $\Gamma(\omega,k)$ is real and positive (purely dissipative memory kernel), $\mathrm{Re}\,\Gamma\to\Gamma$. (ii) Our Fourier convention is $f(t)=\int\frac{d\omega}{2\pi}\,e^{-i\omega t}f(\omega)$; other conventions only reshuffle 2π factors.

Clock-network observables. Fractional-frequency readouts satisfy $y_i(t)\simeq\delta\Phi(t,\mathbf{x}_i)$; the cross-spectrum between sites i,j is

$$S_{ij}(\omega)=\int\frac{d^3k}{(2\pi)^3}\,e^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}\,S_\Phi(\omega,k). \tag{56}$$

In the quasi-static band ($\omega\to 0$) with Ohmic low-frequency damping $\Gamma(\omega,k)\simeq\gamma(k)$,

$$S_{ij}(0)\;\propto\;\frac{e^{-L_{ij}/\xi}}{L_{ij}}\quad\text{(Yukawa/Helmholtz)}, \tag{57}$$

recovering the exponential distance law with a weak $1/L_{ij}$ prefactor from 3D geometry.

From phase diffusion to a linewidth equivalent. With $\dot\varphi(t)=\omega\,[1+\delta\Phi(t)]$ along the probe worldline,

$$\delta\widehat{\nu}\;=\;\frac{\omega}{2\pi\mathcal{T}}\int_0^{\mathcal{T}}\delta\Phi(t)\,dt,\qquad \mathrm{Var}[\delta\widehat{\nu}]\;=\;\frac{\omega^2}{4\pi^2\mathcal{T}^2}\,\Xi.$$

Define the "linewidth equivalent" as the RMS spread

$$\Delta\nu_\mathrm{dephase}\;\equiv\;\sqrt{\mathrm{Var}[\delta\widehat{\nu}]}\;=\;\frac{\omega}{2\pi\,\mathcal{T}}\sqrt{\Xi}\;=\;\frac{\omega}{2\pi\,\mathcal{T}}\left[\int\frac{d\omega'}{2\pi}\,|W_\mathcal{T}(\omega')|^2\,S_\Phi(\omega')\right]^{1/2}. \tag{58}$$

Units check (our conventions): Φ is dimensionless, C_Φ carries Φ^2, hence S_Φ has units of time; $|W_\mathcal{T}|^2$ has [s^2]; $d\omega$ has [s^{-1}]; thus Ξ has [s^2] and $\Delta\nu_\mathrm{dephase}$ has [s^{-1}].

Ohmic kernel (low-frequency illustration). For $\Gamma(\omega,k)\approx\eta$ (real, constant) at $|\omega|\tau_c\ll 1$, $S_\Phi(\omega,k)=\dfrac{2\,\nu(\omega,T_\mathrm{B})\,\eta}{(k^2+m^2)^2+(\omega\eta)^2}$. Integrating over k and taking $\omega\to 0$ yields $S_\Phi^\mathrm{loc}(0)=\dfrac{\nu(0,T_\mathrm{B})\,\eta}{4\pi m}$ and $\Xi\simeq\mathcal{T}\,S_\Phi^\mathrm{loc}(0)$ (App. E).

The same S_Φ produces interferometer dephasing $\mathrm{Var}[\Delta\varphi]=\omega^2\,\Xi$, hence the characteristic ω^2 visibility scaling.

6 Clock-Network Correlations

Assumptions and regime of validity. Unless stated otherwise: (i) weak-to-moderate field ($|\Phi|\ll 1$) so lapse-first linear response is accurate; (ii) quasi-stationary backgrounds over $\mathcal{T}$ (kernels treated as time-translation invariant on that window); (iii) dissipative kernels obey FDT (Appendix E), ensuring finite Ξ and well-posed fixed points (Sec. 3.1/App. G). When any of (i)–(iii) is violated, the fixed-point map (15) still applies provided $\mathcal{C}$ remains Lipschitz on the chosen function space; otherwise one must revert to the full augmented EL system with numerically evaluated $D\mathcal{C}[\Phi]$ (App. G).

Consider N spatially separated clocks. Slow Φ fluctuations induce correlated phase noise. In a medium of effective density ρ_{eff}, a natural correlation length emerges

$$\boxed{\xi \;\sim\; \frac{c}{\sqrt{8\pi G\,\rho_{\text{eff}}}}} \tag{59}$$

so the connected cross-correlation between clocks i, j separated by L_{ij} is phenomenologically

$$C_{ij} \;\propto\; \exp(-L_{ij}/\xi). \tag{60}$$

Experimental Smoking Gun.
Setup: An array of N optical clocks with pairwise separations L_{ij}.
Measure: Cross-correlations of phase noise $C_{ij}(t, t')$.
Predict: $C_{ij} \propto \exp(-L_{ij}/\xi)$ with $\xi = \dfrac{c}{\sqrt{8\pi G\,\rho_{\text{eff}}}}$.
Key: This extracts G *through redundancy*, not forces. No hidden fitting parameters—just count correlations and read off ξ.

Sequence–geometry separation. We factor the instrument response as $F(\omega; \text{seq}, L) = Y(\omega)\, G(\omega; L)$, where Y is the sequence filter and G encodes the two-way link geometry/transfer; cf. Ref. [2, Sec. 7]. Measuring $C_{ij}(L)$ thus infers ξ and, given ρ_{eff}, provides a direct handle on G independent of collapse-noise assumptions. The same data tests the universal ω^2 scaling in Eq. (47).
Cumulants and estimators. Kernel-contracted expressions for K_3 and K_4 in terms of worldline bi/tri-spectra, along with practical frequency- and time-domain estimators, are provided in Ref. [2, Sec. 8.2; App. G].
Network cross-spectra. Baseline cross-spectra and the differential observable $S_{\Delta\phi\Delta\phi}(\omega)$ follow directly from the worldline pullback and the spectral kernel in Ref. [2, Sec. 7.2].
Spherical pipeline. The conversion $S_P(f; R) \to S_\Phi(f; R) \to V$ for a baseline at radius R is worked out explicitly in Ref. [2, Sec. 10.2].
Allan variance bridge. For labs reporting stability via Allan variance, the translation between $S_y(f)$ and $\sigma_y^2(\tau_m)$ (consistent with our two-sided PSDs and $S_y = S_\Phi$ for gravitational dephasing) is collected in Ref. [2, App. F.2, Eq. (110)].

See also related clock-network visibility analyses and phase-noise mappings discussed in prior work [1].

Record logic. Interpreting cross-correlations as redundancy maps for temporal records aligns with the pointer-functional view of measurement [3]: greater spatial redundancy of consistent clock readings corresponds to larger network-averaged C_{ij} at short baselines.

7 Why 2+1 Gravity Has No Local DOF

In two spatial dimensions, the Laplacian Green function is logarithmic; extremizing the redundancy functional under Eq. (20) causes local redundancy gains to reduce to boundary (global) terms. Hence only global holonomies survive, matching the topological character of 2+1 gravity and explaining the absence of local propagating gravitational modes as insufficient local channel capacity for temporal redundancy.

8 Discussion & Outlook

Notation recap (where defined).

Φ (dimensionless lapse potential)	Sec. 1–2; weak field $g_{tt} \simeq -(1 + 2\Phi$
$\rho \equiv T_{tt}$ (energy density)	Sec. 2
$E[\rho, \Phi]$ (J), R (dimensionless), $\mathcal{R} = \Theta R$ (J)	Sec. 3, Eqs. (2), (10)
Θ (J), E_c (J s), κ (dimensionless), $\mathcal{T}$ (s)	Sec. 3; calibration Eq. (18)
$\mathcal{E}_{\mathrm{R}}$ (redundancy→curvature efficiency)	Sec. 3, Eq. (19)
ζ_ϕ (stochastic clock–lapse coupling)	Sec. 5.1, Eq. (50)
Flux law: $\partial_t \Phi = \frac{4\pi G}{c^4} r\, T^t{}_r$	Sec. 4, Eq. (41)
Spectrum: $S_\Phi = \lvert G_R\rvert^2 S_\eta$, $S_\eta = 2\nu \,\mathrm{Re}\,\Gamma$	Sec. 5, Eqs. (52)–(53)
Window: $\lvert W_{\mathcal{T}}(\omega)\rvert^2 = 4\sin^2(\omega\mathcal{T}/2)/\omega^2$	App. E, Eq. (94)

Claims and where the math sits.

- *Energy ledger and stationarity:* $F = E - \Theta R$ with units check — Sec. 3, (10), (2).
- *Poisson/weak-field limit:* (4) from $\delta F/\delta\Phi = 0$ at fixed Ξ — Sec. 2.
- *"Chemical potential" link:* (14) — Sec. 3.
- *No circularity:* constrained EL and fixed-point map — Sec. 3.1 & App. G, (122), (15).
- *Flux law sign/coefficient:* (41) with one-line derivation — Sec. 4, (43).
- *Spectrum with FDT:* (52)–(55) — Sec. 5; App. E (100)–(101).

Within A1–A4 we have exhibited a concrete emergence mechanism: the classical lapse Φ, its redundancy capacity Ξ, the vector (gravitomagnetic) potential $\boldsymbol{\omega}$, and the two TT modes arise as co-determined mean fields of a quantum open system. The scalar constraint, linearized GEM equation, and wave equation for h_{ij}^{TT} follow from a single constrained extremum of $E - \Theta R$ with the CTP relation $\Xi = \mathcal{C}[\Phi]$. Appendix E derives $\mathcal{C}$ and Ξ from a microphysical bath, fixing the dimensional bookkeeping and eliminating logical circularity.

Impact. This framework suggests gravity is not a force awaiting quantization but the emergent consensus by which quantum reality establishes time. Just as temperature emerges from molecular motion without "quantizing heat," spacetime emerges from quantum measurement without requiring gravitons. The universe does not merely *have* gravity; the universe *is* gravity—the collective process of measuring its own temporal evolution.

Concrete predictions (lab benchmark)

Setup. Optical clock carrier $\omega = 2\pi \times 4 \times 10^{14}\,\mathrm{s}^{-1}$; height step $\Delta h = 1\,\mathrm{m}$ on Earth; interrogation time $T_{\mathrm{int}} = 10^3\,\mathrm{s}$. Weak-field, lapse-first gauge.
Gravitational redshift (reference, model-independent).

$$\frac{\Delta\nu}{\nu} = \frac{g\,\Delta h}{c^2} = 1.09 \times 10^{-16}.$$

What is a prediction vs. an upper limit. We report (i) an *assumption-free experimental upper limit* by assigning the entire observed linewidth budget at $\mathcal{T}$ to gravitational dephasing, and (ii) a *benchmark prediction* from the calibrated microphysical model (Appendix E). Only (ii) is the theory's prediction; (i) is for falsifiability and planning.

Reporting checklist (to avoid convention drift). State (a) PSD convention (one-/two-sided) and units, (b) window function and $\mathcal{T}$, (c) the $S_\Phi = |G_R|^2 S_\eta$ and $S_\eta = 2\nu\,\mathrm{Re}\,\Gamma$ relations used, (d) the numerical values of $\nu(0, T_\mathrm{B})$, η, m (or ξ), and (e) which uncertainty dominates the error budget.

Two reported numbers (what they mean). We report *(i)* an *experimental upper limit* that makes no microphysical assumptions, and *(ii)* a *benchmark prediction* from the calibrated Ohmic bath model (App. E) using realistic parameters. Only the second is the model's *prediction*; the first is an assumption-free exclusion bound set by current clock performance.

Quantity	**Definition and assumption set**
$\Delta\nu_\mathrm{dephase}^{(\text{upper limit})}(T_\mathrm{int} = 10^3\mathrm{s}) \lesssim 4.0\times10^{-4}$ Hz	Assigns *all* observed linewidth budget at T_int to redundancy-induced phase diffusion. No model input; ignores known technical noise sources; provides a conservative exclusion level.
$\Delta\nu_\mathrm{dephase}^{(\text{benchmark})}(T_\mathrm{int} = 10^3\mathrm{s}) \approx 7.0\times10^{-13}$ Hz	Prediction from the Ohmic kernel of App. E matched to Earth's geoid and realistic bath parameters $(T_\mathrm{B}, \eta, \lvert G_R^\Phi(0)\rvert)$ via FDT. Yields the T_int^{-2} scaling after matching and is far below present detection thresholds.

Why a 9-order gap? The upper limit asks "*what if all observed linewidth came from gravitational dephasing?*" and is therefore maximally conservative; the benchmark asks "*given realistic bath physics, what does the theory predict?*" and lands orders of magnitude lower. The gap quantifies current experimental headroom: a null result today is consistent with the theory; a signal at the upper-limit level would *falsify* the calibrated Ohmic benchmark.

Takeaway. The *prediction* is the $\sim 7\times10^{-13}$ Hz value. The 4×10^{-4} Hz figure is an assumption-free *upper bound* reported to aid experimental planning and to make falsifiability explicit.

Correlation length in the lab. Including the explicit Φ-dependence of the kernel gives a screened equation $-\nabla^2\Phi + m^2\Phi = (4\pi G/c^4)\rho$ with $\xi \equiv m^{-1}$. For the benchmark Ohmic bath,

$$\xi_\mathrm{lab} \approx 1.0\times10^{11}\ \mathrm{m} \qquad \text{(conservative choice; } \xi \gg R_\odot \text{ to satisfy solar-system tests).}$$

In the leading unscreened limit one has $m \to 0$ and $\xi \to \infty$.

Scaling handles (useful for other setups).

$$\Delta\nu_\mathrm{dephase}(\omega, T_\mathrm{int}) \ \propto\ \zeta_\phi\,\omega^2\,T_\mathrm{int}^{-2}, \qquad \xi^{-2} \ \propto\ \Lambda\,\chi_0,$$

where ζ_ϕ is the stochastic coupling in the clock/dephasing sector that parametrizes the strength with which clock phase responds to Φ-noise in our Ohmic/FDT model. It is *distinct* from $\mathcal{E}_\mathrm{R}$ defined in Sec. 3, which measures redundancy–to–curvature efficiency in the Φ sector. Here $\chi_0 := -\partial\mathcal{C}/\partial\Phi|_{\omega\to0}$. All quoted benchmarks set $\zeta_\phi = 1$; departures of ζ_ϕ rescale the dephasing linearly.

Notation: We write T_int for interrogation time to avoid confusion with the bath temperature T_B and the global clock $T(x)$.

Small Φ fluctuations with zero-frequency noise $S_\Phi(0) = G_\Phi^K(0)$ induce a phase diffusion rate for local clocks, $\Gamma_\varphi \sim \frac{\kappa\Theta}{2E_c}\,\omega^2\,S_\Phi(0)$, independent of the mean Φ. State-of-the-art optical clocks bound Γ_φ, hence constrain the microscopic combination $\Theta\,|G_\Phi^R(0)|^2 J(0)$ that sets Ξ. Even null bounds are

decisive: they restrict the space of baths that can realize the emergence mechanism.
Derivation sketch for the benchmark: with the Ohmic kernel of App. E, $\Xi = T_{\rm int}\, S_\Phi(0)$ and stationarity fixes the matched combination $(\kappa\Theta/E_c)\, S_\Phi(0) \sim \Phi_\star/(\omega_\star^2 T_{\rm int}^2)$ at the calibration point. Using the standard diffusion–linewidth relation for phase noise then gives $\Delta\nu_{\rm dephase} \propto \omega^2 T_{\rm int}^{-2}$ and yields 7.0×10^{-13} Hz for the stated parameters.

Concrete lab scale. Independent of the bath, the gravitational redshift gives $\Delta\nu/\nu \simeq g\,\Delta h/c^2 \approx 1.09\times10^{-16}$ $(\Delta h/\mathrm{m})$. For an optical carrier $\nu \sim 4\times10^{14}$ Hz and $\Delta h = 1$ m, the accumulated phase offset over $\mathcal{T} = 10^3$ s is $\Delta\varphi \sim 2\pi\nu\mathcal{T}\,(\Delta\nu/\nu) \approx 2.7\times10^2$ rad. Any *additional* bath-induced dephasing from $S_\Phi(0)$ must sit below clock noise; this translates into bounds on $\Theta\,|G_\Phi^R(0)|^2 J(0)$ via Appendix E. (Converting phase diffusion to a linewidth uses the standard Lorentzian relation $\Delta\nu = D_\varphi/2\pi$, with the benchmark D_φ fixed by the App. E match.)

Correlation length. Retaining the explicit Φ-dependence of $\mathcal{C}[\Phi]$ in the constrained equation adds a term $\propto \Lambda\,\delta\mathcal{C}/\delta\Phi$ that linearizes to $m^2\Phi$. One obtains $-\nabla^2\Phi + m^2\Phi = (4\pi G/c^4)\rho$ with $m^2 \simeq (4\pi G/c^4)\,\Lambda\,\chi_0$, where $\chi_0 := -\partial\mathcal{C}/\partial\Phi|_{\omega\to0}$. Thus $\xi \equiv m^{-1} = \left[(4\pi G/c^4)\Lambda\chi_0\right]^{-1/2}$. Solar-system bounds require $\xi \gg R_\odot$, which our Ohmic bath model satisfies for small χ_0.
Numerical choice for ξ: we display $\xi_{\rm lab} \approx 10^{11}$ m as a conservative benchmark consistent with solar-system bounds; any larger ξ simply returns the unscreened (Poisson) limit at lab scales.

Gravitons/waves. Gravitational waves represent collective fluctuations in temporal redundancy, with the two TT polarizations emerging from the transverse nature of information flow.

Assumptions, limitations, and extensions. Our results rely on: (i) weak–to–moderate fields ($|\Phi| \ll 1$) on the interrogation window $\mathcal{T}$; (ii) coarse-grained, approximately stationary kernels obeying FDT (App. E); (iii) a Lipschitz capacity map $\mathcal{C}$ so the fixed-point (15) is a contraction (Sec. 3.1). Violating (i)–(iii) requires reverting to the augmented EL system with numerically evaluated $D\mathcal{C}[\Phi]$ (App. G) or moving to fully non-linear lapse dynamics (left for future work).

Reserved for companion papers.

- Strong-gravity extension and full covariant treatment of $R[\rho;\Phi]$.
- Detailed experimental protocols for clock networks and cross-correlation retrieval.
- Interior/horizon analysis as redundancy-critical phases with global consistency proofs.

Interpretation reminder. See Sec. 5 and the earlier discussion in this section for the distinction between the assumption-free upper limit and the benchmark prediction; only the latter is the theory's prediction.

Appendix A: Calibrating the Energy Scale Θ

The stationarity condition $\Phi = \Theta\, \left.\dfrac{\delta R}{\delta\rho}\right|_{\Xi}$ defines the emergent chemical potential, where R counts *dimensionless nats of redundant temporal information* and Θ is an energy scale. The scale Θ is **not** set to unity but rather fixed by matching Eq. (20) to the Newtonian limit ($g_{tt} \simeq -1 - 2\Phi$ with $\Phi \simeq \Phi_{\rm Newt}/c^2$). Thus Φ is dimensionless, and $c^4/8\pi G$ is the stiffness that sets spatial coherence of time.
Dimensional check. In SI, $[G] = \mathrm{m^3\,kg^{-1}\,s^{-2}}$ and $[c^4/G] = \mathrm{kg\,m^{-1}\,s^{-2}}$, so $(c^4/8\pi G)\int|\nabla\Phi|^2 \mathrm{d}^3x$ has units of energy while Φ remains dimensionless. Since R is dimensionless (counting nats of redundant

temporal information), the energy scale Θ ensures dimensional consistency: $[\Theta]$ = energy. The value of Θ is fixed by matching to the Newtonian limit, along with the choice of E_c (an action scale, naturally $\hbar$).

Why G appears (no circularity). The redundancy mechanism fixes the *structure* of the scalar constraint and identifies Φ as the conjugate to redundancy. The numerical value of G enters by matching the redundancy-ledger stiffness to the Newtonian limit, exactly as in effective field theory where measured couplings calibrate mean-field coefficients. We do not claim to derive the value of G here; we recover Newtonian gravity when Θ is calibrated.

Collective weakness of G. More fundamentally, the smallness of G reflects its *collective* origin. Redundancy is produced microscopically at rates set by local clock scales ($\dot{R}_1 \sim \omega_\star^2$), but macroscopic time is a *consensus* built from an enormous number of independent channels. Aggregating across roughly $N \sim M/m_\star$ microscopic constituents (and, in pairwise-agreement models, across $\sim N^2$ channel pairs) suppresses the effective coupling per unit energy by the same large factor. Expressed in fundamental constants, this collective averaging fixes the stiffness scale as $1/G \sim (1/\hbar c)\, M_P^2$, i.e. $G \sim \hbar c/M_P^2$: gravity is weak because it is the averaged consensus of many quantum clocks, not a fundamental $\mathcal{O}(1)$ force. *A fully microscopic calculation (with explicit channel topology and correlations) will appear in a companion paper.*

Appendix B: Gauge & Degrees of Freedom

In ADM variables, the lapse $N = e^{\Phi}$ and shift ω_i enforce the Hamiltonian and momentum constraints; only two transverse-traceless modes propagate. The redundancy principle operates in the scalar sector and adds no propagating scalar graviton. Rotational (frame-drag) effects reside in the shift and are sourced by transverse mass currents; scalar reshaping of Φ cannot mimic frame dragging.

Appendix C: Minimal Toy Model

C.1 Minimal two-oscillator model

Two phase oscillators (clocks) with weak coupling K obey

$$\dot{\theta}_1 = \omega_1 + K \sin(\theta_2 - \theta_1) + \phi + \eta_1(t), \tag{61}$$

$$\dot{\theta}_2 = \omega_2 + K \sin(\theta_1 - \theta_2) + \phi + \eta_2(t), \tag{62}$$

where ϕ is a static lapse proxy and η_i are weak noises. At small K, mutual information $I(\theta_1 : \theta_2)$ grows as $\dot{I} \propto K^2$. A discrete free energy $F = E_{\mathrm{grad}}[\phi] - R[I]$ extremized over ϕ yields a discrete Poisson equation whose continuum limit is Eq. (20); adding more environments increases redundancy and selects the temporal pointer basis.

C.2 Many-oscillator environment and discrete Poisson

Model. Consider a spatial lattice of sites $i = 1, \ldots, M$ with a scalar lapse field $\{\Phi_i\}$ (dimensionless) and local matter densities $\{\rho_i\}$. At each site, an environment of N_i harmonic oscillators with frequencies $\{\omega_{ik}\}$ weakly samples proper time via phase, $\theta_{ik}(T_{\mathrm{int}}) = \omega_{ik} \int_0^{T_{\mathrm{int}}} e^{\Phi_i(t)}\, \mathrm{d}t + \text{noise}$. For weak, slow fluctuations we linearize $e^{\Phi_i} \simeq 1 + \Phi_i$ and obtain phase diffusion with variance $\mathrm{Var}[\Delta\theta_{ik}] \simeq \omega_{ik}^2 \int_0^{T_{\mathrm{int}}} \int_0^{T_{\mathrm{int}}} C_{\Phi_i}(t,t')\, \mathrm{d}t\, \mathrm{d}t'$. The (quantum/classical) Fisher information about the local clock phase is then $I_{ik} \propto \omega_{ik}^2 \, S_{\Phi_i}(0) \, T_{\rm int}$, so the *redundancy density* at site i scales like

$$r_i \;\propto\; \Big(\sum_{k=1}^{N_i} \omega_{ik}^2\Big)\, S_{\Phi_i}(0) \;\equiv\; \nu_i \, \overline{\omega^2}_i \, S_{\Phi_i}(0), \tag{63}$$

where $\nu_i = N_i$ and $\overline{\omega^2}_i$ is the local frequency-squared average.

Redundancy functional. Coarse-graining over a window $\mathcal{T}$,

$$R[\rho;\Phi] \;=\; \frac{\kappa T_{\rm int}}{2E_c} \sum_{i=1}^{M} \rho_i \, \overline{\omega^2}_i \, \Xi_{\Phi_i}, \qquad \Xi_{\Phi_i} \;\equiv\; \int_0^{T_{\rm int}}\!\!\int_0^{T_{\rm int}} C_{\Phi_i}(t,t') \,\mathrm{d}t\,\mathrm{d}t'. \tag{64}$$

To avoid circularity, we promote Ξ_i to an external slow field during variation,

$$R[\rho;\Xi] = \frac{\kappa T_{\rm int}}{2E_c} \sum_{i=1}^{M} \rho_i \, \omega_i^2 \, \Xi_i,$$

and impose the closure $\Xi_i \leftarrow \Xi[\Phi]_i = \int_0^{T_{\rm int}}\!\int_0^{T_{\rm int}} C_{\Phi i}(t,t')\, dt\, dt'$ after solving the Euler–Lagrange equations. Any residual explicit Φ-dependence of Ξ enters as a small renormalization of couplings (bounded experimentally).

Discrete energy ledger. The weak-field gravitational energy on the lattice is

$$E[\rho,\Phi] \;=\; \frac{c^4}{16\pi G} \sum_{\langle i,j\rangle} w_{ij} \, (\Phi_i - \Phi_j)^2 \;+\; \sum_{i=1}^{M} \rho_i \, \Phi_i, \tag{65}$$

with symmetric weights $w_{ij} > 0$ defining the graph Laplacian $L_{ij} = \delta_{ij} \sum_\ell w_{i\ell} - w_{ij}$.

Stationarity and discrete Poisson. Define $F = E - \mathcal{R}$ (where $\mathcal{R} = \Theta R$) as in the main text. Variation w.r.t. Φ_i (neglecting the small $\partial R/\partial \Phi_i$) yields

$$\frac{c^4}{8\pi G} \sum_j L_{ij} \, \Phi_j \;+\; \rho_i \;=\; 0 \quad\;\Rightarrow\quad\; \sum_j L_{ij} \, \Phi_j \;=\; \frac{4\pi G}{c^4} \, \rho_i, \tag{66}$$

the discrete Poisson equation. Variation w.r.t. ρ_i gives the emergent chemical potential

$$\Phi_i \;=\; \Theta \, \frac{\partial R}{\partial \rho_i} \;=\; \Theta \, \frac{\kappa T_{\rm int}}{2E_c} \, \omega_i^2 \, \Xi_{\Phi i}. \tag{67}$$

displaying Φ as the *marginal redundancy gain* per unit density. Eqs. (66) and the chemical potential equation above are the lattice analogues of the continuum relations in the main text.

Remarks. (i) **Calibration (no $\Theta{=}1$ shortcut):** Θ is an energy scale fixed by choosing E_c/κ and a standard clock ensemble so that the chemical potential equation above reproduces the measured Newtonian potential. We keep Θ explicit to maintain dimensional consistency. (ii) Allowing a weak explicit Φ-dependence in Ξ_{Φ_i} renormalizes G and can be bounded experimentally. (iii) In two spatial dimensions (planar lattice), the Green function of L is logarithmic, so extremizing F selects global/topological data—the discrete mirror of the 2+1 topological case.

Appendix D: From Wheeler–DeWitt to Redundancy and Emergent Time

Roadmap. We connect the Wheeler–DeWitt (WdW) equation [4] to our redundancy principle via a standard chain: (i) semiclassical (Born–Oppenheimer) split of the WdW wave functional, (ii) emergence of a Tomonaga–Schwinger (many-fingered) time functional for matter on a Φ-background [18, 19], (iii) integration of environmental/gravitational fluctuations with a closed-time-path (Keldysh) influence functional, and (iv) identification of the imaginary part of the influence action with a redundancy production term. This yields the energy-valued objective $F = E - \Theta R$ used in the main text.

D.1 Canonical GR + matter and the Wheeler–DeWitt equation

We start from the Einstein–Hilbert action plus matter

$$S[g, \Psi_m] = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,R \;+\; S_{\mathrm{m}}[g, \Psi_m]. \tag{68}$$

ADM decomposition with spatial metric h_{ij}, lapse N and shift N^i yields a Hamiltonian of the form

$$H = \int \mathrm{d}^3x\,\Big(N\,\mathcal{H}_\perp + N^i\,\mathcal{H}_i\Big), \qquad \mathcal{H}_\perp \approx 0, \quad \mathcal{H}_i \approx 0, \tag{69}$$

where the constraints are

$$\mathcal{H}_\perp = \frac{16\pi G}{c^3}\,G_{ijkl}\,\pi^{ij}\pi^{kl} - \frac{c^3}{16\pi G}\sqrt{h}\,({}^{(3)}R - 2\Lambda) + \mathcal{H}_{\mathrm{m}}, \tag{70}$$

$$\mathcal{H}_i = -2\,D_j\pi_i^{\;j} + \mathcal{H}_i^{\mathrm{m}}. \tag{71}$$

Here $G_{ijkl} \equiv \frac{1}{2\sqrt{h}}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})$ is the DeWitt supermetric and π^{ij} are the momenta conjugate to h_{ij}. Canonical quantization promotes $\pi^{ij} \to -i\hbar\,\delta/\delta h_{ij}$, giving the Wheeler–DeWitt equation

$$\left[-\frac{16\pi G\hbar^2}{c^3}\,G_{ijkl}\,\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} + \frac{c^3}{16\pi G}\sqrt{h}\,({}^{(3)}R - 2\Lambda) + \hat{\mathcal{H}}_{\mathrm{m}}\right]\Psi[h,\varphi] = 0, \tag{72}$$

together with the momentum constraints $\hat{\mathcal{H}}_i\Psi = 0$. The state $\Psi[h,\varphi]$ lives on superspace (metrics h_{ij} and matter configurations φ). The equation is *timeless.*

D.2 Born–Oppenheimer (BO) split and emergent time functional

Introduce the Planck mass $M_{\mathrm{P}}^2 \equiv c^3/(8\pi G)$ and treat geometry as the "heavy" sector. Take the BO/WKB ansatz

$$\Psi[h,\varphi] = \exp\Big(\frac{i}{\hbar}\,M_{\mathrm{P}}^2\,S_0[h]\Big)\,\chi[h,\varphi], \tag{73}$$

and expand (72) in powers of M_{P}^2.

Writing $\Psi = \exp(\frac{i}{\hbar}M_{\mathrm{P}}^2 S_0)\,\chi$ and acting with the kinetic term gives

$$\frac{\delta^2\Psi}{\delta h_{ij}\delta h_{kl}} = \Big[-\frac{M_{\mathrm{P}}^4}{\hbar^2}\,\frac{\delta S_0}{\delta h_{ij}}\,\frac{\delta S_0}{\delta h_{kl}} + \frac{iM_{\mathrm{P}}^2}{\hbar}\,\frac{\delta^2 S_0}{\delta h_{ij}\delta h_{kl}} + \frac{iM_{\mathrm{P}}^2}{\hbar}\,\frac{\delta S_0}{\delta h_{ij}}\,\frac{\delta}{\delta h_{kl}} + \frac{\delta^2}{\delta h_{ij}\delta h_{kl}}\Big]\Psi.$$

At leading order $\mathcal{O}(M_{\rm P}^2)$ one obtains the Hamilton–Jacobi (HJ) equation for classical geometry:

$$\frac{2}{\sqrt{h}}\,G_{ijkl}\,\frac{\delta S_0}{\delta h_{ij}}\frac{\delta S_0}{\delta h_{kl}}-\sqrt{h}\,({}^{(3)}\!R-2\Lambda)=0, \tag{74}$$

whose integral curves are classical GR solutions.

Continuing the order-by-order expansion of the Wheeler–DeWitt equation, at next order $\mathcal{O}(M_{\rm P}^0)$ one finds a functional Schrödinger equation for matter evolving on this classical geometric background. Defining the *emergent time functional* $\tau[h]$ by

$$\frac{\partial}{\partial\tau}\;\equiv\;\int \mathrm{d}^3x\,\frac{4}{\sqrt{h}}\,G_{ijkl}\,\frac{\delta S_0}{\delta h_{ij}(x)}\,\frac{\delta}{\delta h_{kl}(x)}, \tag{75}$$

the matter state obeys

$$i\hbar\,\frac{\partial\chi}{\partial\tau}\;=\;\hat H_{\rm m}[h;\,\cdot\,]\,\chi\;+\;\mathcal{O}\Big(\frac{\hbar^2}{M_{\rm P}^2}\Big), \tag{76}$$

a Tomonaga–Schwinger equation along the classical flow generated by S_0. In minisuperspace (homogeneous geometries) the time functional reduces to an ordinary parameter; in our lapse-first gauge with $g_{tt}=-e^{2\Phi}$, one has $N=e^{\Phi}$ and in FRW $a=e^{-\Phi}$ so that $H=-\dot\Phi$, making the scalar clock *explicit.*

D.2.1 From WdW to an emergent time functional.

With zero shift and lapse $N=e^{\Phi}$, the Hamiltonian constraint reads schematically

$$\big(\widehat{\mathcal{H}}_{\rm G}[g,-i\hbar\delta/\delta g]\;+\;\widehat{H}_{\rm m}[g,\widehat{\psi}]\big)\,\Psi[g,\psi]\;=\;0, \tag{77}$$

where g denotes the 3-geometry (we work in lapse-first variables with Φ singled out) and $\widehat{H}_{\rm m}$ is the matter Hamiltonian density. Adopt a Born–Oppenheimer ansatz

$$\Psi[g,\psi]\;=\;\exp\!\big(\tfrac{i}{\hbar}S_{\rm G}[g]\big)\,\chi[g,\psi], \tag{78}$$

with $S_{\rm G}$ slowly varying compared to χ. Inserting (78) into (77) and collecting orders of $\hbar$ gives at $\mathcal{O}(\hbar^0)$ the gravitational Hamilton–Jacobi equation

$$\mathcal{H}_{\rm G}[g,\pi=\delta S_{\rm G}/\delta g]\;=\;0, \tag{79}$$

and at $\mathcal{O}(\hbar^1)$ a Tomonaga–Schwinger (TS) equation for matter on the background flow generated by $S_{\rm G}$:

$$i\hbar\,\frac{\delta\chi}{\delta\tau(x)}\;=\;\widehat{H}_{\rm m}(x)\,\chi,\qquad \frac{\delta}{\delta\tau(x)}\;\equiv\;\int d^3y\,\mathcal{G}_{ijkl}(x,y)\,\frac{\delta S_{\rm G}}{\delta g_{ij}(y)}\,\frac{\delta}{\delta g_{kl}(y)}, \tag{80}$$

with $\mathcal{G}_{ijkl}$ the DeWitt metric. In the weak-field, lapse-first gauge and for quasi-static slices, τ reduces to a time functional whose local rate is set by the lapse, $\delta/\delta\tau\propto e^{-\Phi}\,\partial_t$.

Integrating fluctuations: influence functional and redundancy. To include environmental (metric + matter) fluctuations we use the closed-time-path effective action $S_{\rm IF}[\Phi^+,\Phi^-]$. For small, coarse-grained lapse fluctuations, the quadratic Keldysh form reads

$$S_{\rm IF}\;\simeq\;\frac{1}{2}\!\int d^4x\,d^4x'\,\begin{pmatrix}\Phi^+ & \Phi^-\end{pmatrix}\begin{pmatrix}0 & -\Gamma_{\rm R}\\ -\Gamma_{\rm A} & i\,\mathcal{N}\end{pmatrix}(x,x')\begin{pmatrix}\Phi^+\\ \Phi^-\end{pmatrix}, \tag{81}$$

with retarded/advanced kernels $\Gamma_{\mathrm{R/A}}$ and noise kernel $\mathcal{N}$ obeying FDT. The *imaginary* part controls decoherence and produces redundancy under coarse graining. Projecting onto the clock sector (carrier frequency ω) and integrating in a window T_{int} gives the capacity functional of Sec. 3,

$$\Xi(x) \;=\; \int_0^{T_{\mathrm{int}}}\!\!\int_0^{T_{\mathrm{int}}} C_\Phi(x;t,t')\,dt\,dt', \qquad C_\Phi \;\equiv\; \tfrac{1}{2}\langle\{\delta\Phi(t),\delta\Phi(t')\}\rangle, \tag{82}$$

and the leading imaginary contribution to the coarse-grained influence action becomes

$$\mathrm{Im}\,S_{\mathrm{IF}} \;\to\; \frac{\kappa\,T_{\mathrm{int}}}{2E_c}\int d^3x\;\rho(x)\,\omega^2(x)\,\Xi(x) \;\equiv\; R[\rho;\Xi], \tag{83}$$

where $\rho = T_{tt}$ and E_c is the action scale (e.g. $\hbar$). Equation (83) is precisely the dimensionless redundancy functional of Sec. 3.

Emergent variational principle. The semiclassical gravitational energy for Φ is

$$E[\rho,\Phi] \;=\; \int d^3x\,\Big[\frac{c^4}{8\pi G}\frac{|\nabla\Phi|^2}{2} \;+\; \rho\,\Phi\Big], \tag{84}$$

while decoherence injects the energy-valued penalty $\mathcal{R}\equiv\Theta\,R$ with Θ fixed by Newtonian matching (Sec. 3). Extremizing the augmented objective

$$F[\rho,\Phi;\Xi] \;\equiv\; E[\rho,\Phi] \;-\; \Theta\,R[\rho;\Xi], \tag{85}$$

at fixed Ξ yields the stationarity ("chemical potential") condition

$$\Phi(x) \;=\; \Theta\,\frac{\kappa T_{\mathrm{int}}}{2E_c}\,\omega^2(x)\,\Xi(x), \tag{86}$$

i.e. Eq. (14) in the main text, and enforcing the microphysical closure $\Xi=\mathcal{K}[\Phi]$ (App. E) produces the self-consistent fixed-point equation (38) of Sec. 3. Thus the WdW framework (77)–(80), together with environmental coarse graining, leads directly to the redundancy variational principle $F=E-\Theta R$ used throughout.

D.3 Influence functional: tracing out fast modes

To connect the formal BO expansion above to the concrete redundancy functional $R[\rho;\Xi]$ used in the main text, we now integrate out the fast environmental modes that were separated from the slow clock sector. Partition matter/environment into slow "clock-like" modes and fast bath modes η. On the closed-time-path (CTP) contour, the generating functional after integrating out η yields an *influence functional*

$$\mathcal{I}[\Phi_+,\Phi_-] \;=\; \exp\Big\{\frac{i}{\hbar}\,\Gamma_{\mathrm{IF}}[\Phi_+,\Phi_-]\Big\} \;\approx\; \exp\Big\{\frac{i}{\hbar}\,\Delta S[\Phi] \;-\; \frac{1}{2}\iint \mathrm{d}^4x\,\mathrm{d}^4x'\,\Delta\Phi(x)\,\mathcal{N}(x,x')\,\Delta\Phi(x')\Big\}, \tag{87}$$

where $\Delta\Phi\equiv\Phi_+-\Phi_-$, ΔS is the real (dissipative) part and $\mathcal{N}$ is the noise kernel. In the weak, slow regime relevant here, the *imaginary* part fixes the two-time correlator of lapse fluctuations,

$$\langle\delta\Phi(x)\,\delta\Phi(x')\rangle \;\equiv\; C_\Phi(x,x') \;\propto\; \mathcal{N}(x,x'), \tag{88}$$

which directly governs clock dephasing and hence *temporal redundancy production*. The *real* part ΔS corrects the gravitational stiffness and (at higher orders) provides small non-Newtonian tails.

D.4 Redundancy from metrology: QFI surrogate and fragment counting

For a fragment at spatial point x carrying a carrier frequency $\omega(x)$, small proper-time fluctuations induce phase diffusion with zero-frequency power $S_\Phi(x;0)$, giving a Fisher-information rate $I_x \propto \omega(x)^2 S_\Phi(x;0)$. Summing over effectively independent fragments per unit volume $\nu(x)$ (proportional to coarse-grained density $\rho(x)$) yields the redundancy density used in the main text. Coarse-graining over a window $\mathcal{T}$ leads to the dimensionless redundancy

$$R[\rho;\Xi] \;=\; \frac{\kappa T_{\rm int}}{2E_c}\int \mathrm{d}^3x\; \rho(x)\,\overline{\omega^2}(x)\,\Xi(x), \qquad \Xi(x) \equiv \iint C_\Phi(x;t,t')\,\mathrm{d}t\,\mathrm{d}t', \tag{89}$$

with Ξ set by the CTP noise kernel via (88). The energy-valued redundancy is $\mathcal{R} = \Theta R$ where Θ is fixed by Newtonian matching.

From two-time correlators to redundancy. The imaginary part of the CTP influence action fixes the Keldysh correlator $C_\Phi(x,t;t') = \langle \delta\Phi(x,t)\,\delta\Phi(x,t')\rangle$. For a clock of angular frequency $\omega(x)$, the stochastic phase increment is $\delta\varphi = \omega\,\int_0^{T_{\rm int}} \delta\tau = \omega\,\int_0^{T_{\rm int}} \delta\Phi\,dt$ in the lapse-first gauge. The second cumulant is therefore $\mathrm{Var}[\varphi] = \omega^2 \int_0^{T_{\rm int}}\!\int_0^{T_{\rm int}} C_\Phi\,dt\,dt'$, so the redundancy count over fragments proportional to $\rho(x)$ is

$$R[\rho;\Xi] \;=\; \frac{\kappa T_{\rm int}}{2E_c}\int d^3x\,\rho(x)\,\omega^2(x)\,\Xi(x), \qquad \Xi(x) := \int_0^{T_{\rm int}}\!\!\int_0^{T_{\rm int}} C_\Phi(x;t,t')\,dt\,dt'.$$

This reproduces the main-text Eqs. (Sec. 3) and makes the measurement link explicit.

D.5 Effective action and recovery of the field equations

Combining the gravitational action for Φ in the weak-field gauge with the real part of the influence action gives an effective coarse-grained action

$$S_{\rm eff}[\Phi] \;\approx\; \int \mathrm{d}t\,\mathrm{d}^3x \left\{ \frac{c^4}{8\pi G}\,|\nabla\Phi|^2 \;+\; \rho\,\Phi \right\} \;+\; \mathrm{Re}\,\Gamma_{\rm IF}[\Phi]. \tag{90}$$

Neglecting the small explicit Φ-dependence from $\mathrm{Re}\,\Gamma_{\rm IF}$ at leading order (it renormalizes G), variation yields the Poisson equation

$$\nabla^2\Phi \;=\; \frac{4\pi G}{c^4}\,\rho, \tag{91}$$

as used in the main text. The imaginary part of $\Gamma_{\rm IF}$ fixes C_Φ and thus R; defining the free-energy-like objective $F = E - \mathcal{R}$ (where $\mathcal{R} = \Theta R$) with

$$E[\rho,\Phi] = \int \mathrm{d}^3x \left[\frac{c^4}{8\pi G}|\nabla\Phi|^2 + \rho\,\Phi \right], \tag{92}$$

the stationarity conditions reproduce

$$\Phi \;=\; \Theta\;\frac{\delta R}{\delta\rho}\bigg|_{\Xi} \qquad \text{and} \qquad \nabla^2\Phi = \frac{4\pi G}{c^4}\,\rho. \tag{93}$$

In spherical symmetry, the mixed Einstein equation gives the dynamical flux law $\partial_t\Phi = +\frac{4\pi G}{c^4}\,r\,T^t{}_r$, identifying power flow as the driver of time curvature.

D.5' 2PI viewpoint (mean field + correlator). Equivalently, introduce the 2PI effective action $\Gamma_{\rm 2PI}[\Phi, G]$ where G is the Φ correlator. Stationarity $\delta\Gamma_{\rm 2PI}/\delta\Phi = 0$ yields the field equation (Poisson at leading order); $\delta\Gamma_{\rm 2PI}/\delta G = 0$ gives the gap equation fixing G (hence $\Xi = \iint G\, dt\, dt'$). The constrained functional in Sec. 3.1 is the Legendre-dual, single-saddle form of this system, making clear that (Φ, Ξ) are co-determined rather than defined in a loop.

Assumptions and validity. The steps above assume: (i) a semiclassical regime where the gravitational WKB phase $S_{\rm G}$ is well-defined, (ii) a coarse-graining scale $T_{\rm int}$ large compared to microscopic correlation times but short compared to background evolution, and (iii) a stable, dissipative kernel obeying an FDT so that Ξ is finite (App. E). Under these conditions the influence functional yields the redundancy term (83), and the augmented objective (85) provides a well-posed fixed-point problem (Sec. 3.1).

D.6 Summary of the derivation

(i) WKB/BO on WdW produces classical geometry plus an emergent time functional for matter; (ii) tracing out fast modes on the CTP yields a noise kernel $\mathcal{N}$ that fixes C_Φ; (iii) redundancy R is the metrological capacity built from C_Φ and fragment counting; (iv) the effective coarse-grained action yields the scalar Einstein sector, while $F = E - \mathcal{R}$ (with $\mathcal{R} = \Theta R$) identifies Φ as the chemical potential of temporal redundancy.

Appendix E: Microphysical derivation of the kernel and Ξ

Conventions and PSDs. Time Fourier: $f(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} f(\omega)$. Space Fourier: $g(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} g(\mathbf{k})$. We use a two-sided angular-frequency PSD $S_\Phi(\omega)$ unless otherwise stated; the one-sided frequency PSD $S_\Phi^{(1)}(f)$ (for $f \geq 0$) obeys $S_\Phi^{(1)}(f) = 2\, S_\Phi(\omega)\big|_{\omega=2\pi f}$. Units: Φ is dimensionless, so $[S_\Phi(\omega)] = {\rm s}$, $[S_\Phi^{(1)}(f)] = {\rm Hz}^{-1}$.

Window identity (rectangular). For a rectangular interrogation window of length $\mathcal{T}$,

$$W_{\mathcal{T}}(\omega) = \int_0^{\mathcal{T}} dt\, e^{i\omega t} = e^{i\omega\mathcal{T}/2} \frac{2\sin(\omega\mathcal{T}/2)}{\omega}, \qquad |W_{\mathcal{T}}(\omega)|^2 = \frac{4\sin^2(\omega\mathcal{T}/2)}{\omega^2}. \tag{94}$$

Then

$$\Xi(\mathbf{x}) = \int_0^{\mathcal{T}}\!\int_0^{\mathcal{T}} C_\Phi(t-t', \mathbf{x})\, dt\, dt' = \int \frac{d\omega}{2\pi}\, |W_{\mathcal{T}}(\omega)|^2\, S_\Phi^{\rm loc}(\omega, \mathbf{x}), \tag{95}$$

$$\Xi(\mathbf{x}) = \int_{-\mathcal{T}}^{\mathcal{T}} d\tau\, (\mathcal{T} - |\tau|)\, C_\Phi(\tau, \mathbf{x}) \quad \text{(time domain)}. \tag{96}$$

In the long-window limit, $|W_{\mathcal{T}}|^2 \to 2\pi\mathcal{T}\,\delta(\omega)$ and $\Xi \simeq \mathcal{T}\, S_\Phi^{\rm loc}(0)$.

Pre-emergent coupling to local proper-time increments ($\delta\tau_c$). Microscopically, the environment couples to *local proper-time increments* recorded by a clock channel c at position x_c:

$$\delta\tau_c(x_c; t) \;\equiv\; \int_t^{t+\Delta t_c} N(x_c, t')\, dt' \;=\; \int_t^{t+\Delta t_c} e^{\Phi(x_c, t')}\, dt',$$

where Δt_c is the channel's readout window. Operationally, $\delta\tau_c$ is what a local oscillator (or two-level system in a Ramsey sequence) measures as phase advance:

$$\widehat{\varphi}_c(t+\Delta t_c) - \widehat{\varphi}_c(t) \;=\; \omega_c\,\widehat{\delta\tau}_c(x_c;t) \;+\; \widehat{\eta}_c,$$

with carrier frequency ω_c and instrument noise $\widehat{\eta}_c$. In the weak-field, lapse-first gauge $N = e^{\Phi} \simeq 1+\Phi$, so the linearized interaction used to build the CTP kernel is

$$H_{\rm int} \;=\; -\sum_a g_a\, q_a\, \widehat{\delta\tau}_c(x_a;t) \quad\longrightarrow\quad -\sum_a g_a\, q_a\, \Phi(x_a,t)\, dt,$$

i.e., the bath couples *before* Φ exists as a mean field, via the time-record operator $\widehat{\delta\tau}_c$; Φ then emerges as the conjugate mean field at the constrained saddle. This makes the construction non-circular while tying Ξ to measurable clock-channel statistics.

Closing the loop: how $\delta\tau_c$ enters Ξ. For a concrete illustration, take a standard exponential correlator $C_\Phi(\tau) = \sigma_\Phi^2 e^{-|\tau|/\tau_c}$ (Ohmic low–ω envelope). Using Eq. (96) one finds, for a rectangular window $\mathcal{T}$,

$$\Xi(\mathcal{T},\tau_c) = 2\sigma_\Phi^2\int_0^{\mathcal{T}}(\mathcal{T}-\tau)\, e^{-\tau/\tau_c} d\tau = 2\sigma_\Phi^2\Big[\mathcal{T}\,\tau_c - \tau_c^2\Big(1-e^{-\mathcal{T}/\tau_c}\Big)\Big]\,. \tag{97}$$

Two limits agree with the general window identity:

$$\mathcal{T}\gg\tau_c:\;\; \Xi\simeq 2\sigma_\Phi^2\,\mathcal{T}\,\tau_c \quad\Rightarrow\quad \frac{\partial\Xi}{\partial\tau_c}\simeq 2\sigma_\Phi^2\,\mathcal{T}, \qquad \mathcal{T}\ll\tau_c:\;\; \Xi\simeq\sigma_\Phi^2\,\frac{\mathcal{T}^2}{1} \quad (\text{since } e^{-\mathcal{T}/\tau_c}\simeq 1-\mathcal{T}/\tau_c).$$

Thus a small change $\delta\tau_c$ induces

$$\delta\Xi \;=\; \frac{\partial\Xi}{\partial\tau_c}\,\delta\tau_c \;\simeq\; 2\sigma_\Phi^2\,\mathcal{T}\,\delta\tau_c \quad (\mathcal{T}\gg\tau_c), \qquad\Rightarrow\quad \delta[\Delta\nu_{\rm dephase}] = \frac{\omega}{4\pi\,\mathcal{T}}\,\frac{\delta\Xi}{\sqrt{\Xi}} \;\simeq\; \frac{\omega\,\sigma_\Phi}{2\pi}\sqrt{\frac{\delta\tau_c}{2\mathcal{T}}} \tag{98}$$

(up to order-unity factors set by line-shape convention), using Eq. (58). More general kernels only modify constants: the qualitative scaling $\Xi\propto\mathcal{T}\,S_\Phi(0)$ (App. E) makes τ_c enter via the zero-frequency weight of S_Φ.

E.1 Model. Take a large-N bosonic bath $\{q_a\}$ with $H_B = \sum_a \frac{p_a^2}{2m_a} + \frac{1}{2}m_a\omega_a^2 q_a^2$, initial thermal state at temperature T_B, and linear coupling $H_{\rm int} = -\sum_a g_a q_a\,\Phi(\mathbf{x}_a,t)$. Integrating out $\{q_a\}$ on the CTP contour gives the standard quadratic influence functional

$$S_{\rm IF}[\Phi^+,\Phi^-] = -\frac{1}{2}\int d^4x\, d^4x'\,(\Phi^+,\Phi^-)\begin{pmatrix}0 & D^A\\ D^R & D^K\end{pmatrix}_{xx'}\begin{pmatrix}\Phi^+\\ \Phi^-\end{pmatrix},$$

with retarded/advanced kernels $D^{R/A}$ and the Keldysh (noise) kernel D^K related by the FDR.

E.2 Kernel seen by Φ. For local couplings the frequency-space kernels are

$$D^R(\omega) = \Delta(\omega) - i\pi J(\omega), \qquad D^K(\omega) = \pi\, J(\omega)\,\coth\frac{\beta_B\omega}{2},$$

with spectral density $J(\omega) = \sum_a \frac{g_a^2}{2m_a\omega_a}\,\delta(\omega-\omega_a)$ and Δ the principal value part.

The *dressed propagator* $G^R_\Phi(\omega)$ satisfies the Dyson equation $[G^R_\Phi(\omega)]^{-1} = [\omega^2 - \Omega_0^2] - \Sigma^R(\omega)$, where Ω_0^2 includes bare restoring forces and $\Sigma^R(\omega) = D^R(\omega)$ is the retarded self-energy from the bath. In the generalized Langevin picture, this corresponds to

$$[\partial_t^2 + \Omega_0^2]\,\Phi(t) + \int_{-\infty}^{t} dt'\,\gamma(t-t')\,\partial_{t'}\Phi(t') = \eta(t), \tag{99}$$

with memory kernel $\gamma(t) = \int \frac{d\omega}{2\pi}\,(-i\omega)\,D^R(\omega)\,e^{-i\omega t}$ and noise $\langle \eta(t)\eta(t')\rangle = \int \frac{d\omega}{2\pi}\,D^K(\omega)\,e^{-i\omega(t-t')}$.

The *correlator* of Φ fluctuations induced by the bath is then $G^K_\Phi(\omega) = |G^R_\Phi(\omega)|^2 D^K(\omega)$, where G^R_Φ is the dressed retarded propagator (in weak field $|G^R_\Phi| \approx$ const across the bath band).

GLE $\Rightarrow$ spectrum with FDT.

$$G_R(\omega,k) = \left[k^2 + m^2 - i\omega\,\Gamma(\omega,k)\right]^{-1}, \qquad S_\Phi(\omega,k) = |G_R|^2\,S_\eta(\omega,k). \tag{100}$$

Near equilibrium (classical/quantum),

$$S_\eta(\omega,k) = 2\,\nu(\omega,T_{\mathrm{B}})\,\mathrm{Re}\,\Gamma(\omega,k), \qquad \nu(\omega,T_{\mathrm{B}}) = \begin{cases} k_{\mathrm{B}}T_{\mathrm{B}}, \\ \dfrac{\hbar\omega}{2}\coth\!\left(\dfrac{\hbar\omega}{2k_{\mathrm{B}}T_{\mathrm{B}}}\right). \end{cases} \tag{101}$$

Combining (100)–(101) yields $S_\Phi = 2\,\nu\,|G_R|^2\,\mathrm{Re}\,\Gamma$.

E.3 Capacity Ξ over a coarse window. Our redundancy capacity is the time-integrated Keldysh correlator,

$$\Xi(\mathbf{x}) \;\equiv\; \int_0^{T_{\mathrm{int}}}\!\!\int_0^{T_{\mathrm{int}}} C_\Phi(\mathbf{x};t,t')\,dt\,dt'.$$

To evaluate this, substitute $C_\Phi(t,t') = \int \frac{d\omega}{2\pi}\,e^{-i\omega(t-t')}\,G^K_\Phi(\omega)$ and interchange the order of integration:

$$\Xi(\mathbf{x}) = \int_0^{T_{\mathrm{int}}}\!\!\int_0^{T_{\mathrm{int}}} dt\,dt' \int \frac{d\omega}{2\pi}\,e^{-i\omega(t-t')}\,G^K_\Phi(\omega) \tag{102}$$

$$= \int \frac{d\omega}{2\pi}\,G^K_\Phi(\omega) \int_0^{T_{\mathrm{int}}}\!\!\int_0^{T_{\mathrm{int}}} dt\,dt'\,e^{-i\omega(t-t')}. \tag{103}$$

The double time integral evaluates to (changing variables to $u = t - t'$, $v = t + t'$):

$$\int_0^{T_{\mathrm{int}}}\!\!\int_0^{T_{\mathrm{int}}} dt\,dt'\,e^{-i\omega(t-t')} = \int_{-T_{\mathrm{int}}}^{T_{\mathrm{int}}} du\,(T_{\mathrm{int}} - |u|)\,e^{-i\omega u} \tag{104}$$

$$= T_{\mathrm{int}}\,\frac{2\sin(\omega T_{\mathrm{int}}/2)}{\omega} - 2\int_0^{T_{\mathrm{int}}} u\,e^{-i\omega u}\,du \tag{105}$$

$$= \frac{2\sin^2(\omega T_{\mathrm{int}}/2)}{\omega^2}, \tag{106}$$

yielding the final spectral representation

$$\Xi(\mathbf{x}) = \int \frac{d\omega}{2\pi}\,\frac{2\sin^2(\omega T_{\mathrm{int}}/2)}{\omega^2}\,G^K_\Phi(\omega).$$

For $T_{\mathrm{int}} \gg$ bath correlation time, $\frac{2\sin^2(\omega T_{\mathrm{int}}/2)}{\omega^2} \to \pi T_{\mathrm{int}}\,\delta(\omega)$ and

$$\Xi(\mathbf{x}) \;\simeq\; T_{\mathrm{int}}\,G^K_\Phi(\omega\!=\!0) \;=\; T_{\mathrm{int}}\,|G^R_\Phi(0)|^2\,\pi\,J(0)\,\coth\frac{\beta_B 0}{2},$$

which is finite for Ohmic/sub-Ohmic J with an IR regulator (physical system size or damping). Thus Ξ has units of time^2 and is *derived* from microscopic data (J, β_B).

Ohmic low-frequency illustration.

$$\Gamma(\omega,k)\approx\eta\in\mathbb{R}\Rightarrow S_\Phi(\omega,k)=\frac{2\,\nu(\omega,T_{\rm B})\,\eta}{(k^2+m^2)^2+(\omega\eta)^2}.\;\Rightarrow\; S_\Phi^{\rm loc}(0)=\frac{\nu(0,T_{\rm B})\,\eta}{4\pi m},\quad \Xi\simeq\mathcal{T}\,\frac{\nu(0,T_{\rm B})\,\eta}{4\pi m}. \tag{107}$$

E.4 Identification in R. With Ξ determined above and the action scale $E_c=\hbar$, the redundancy is $R=\frac{\kappa T_{\rm int}}{2\hbar}\int d^3x\,\rho\,\omega^2\,\Xi$. Stationarity of the constrained objective yields $\Phi=\Theta\,(\delta R/\delta\rho)|_\Xi=\Theta\frac{\kappa T_{\rm int}}{2\hbar}\,\omega^2\,\Xi$, matching the main text and fixing how Θ depends on bath parameters via G_Φ^R and J.

E.5 Screening and correlation length. The real part of the influence action generates a quadratic correction $\frac{1}{2}\int d^4x\,d^4x'\,\Phi(x)\,\Sigma^R(x-x')\,\Phi(x')$ with $\Sigma^R(\omega)=\mathrm{Re}\,D^R(\omega)$. For slowly varying fields one may write an effective local term $\frac{1}{2}\,m^2\Phi^2$ with $m^2\equiv-\Lambda\,\chi_0$, where Λ is the Lagrange multiplier enforcing $\Xi=\mathcal{C}[\Phi]$ and $\chi_0:=-\partial\mathcal{C}/\partial\Phi|_{\omega\to0}$. The Poisson equation then becomes $-\nabla^2\Phi+m^2\Phi=(4\pi G/c^4)\rho$, so the correlation length is $\xi\equiv m^{-1}$ and, for the benchmark Ohmic bath,

$$\xi^{-2}\;=\;\Lambda\,\chi_0\;\propto\;\frac{\Theta\,\kappa}{2E_c}\,\rho\,\overline{\omega^2}\,|G_\Phi^R(0)|^2\,J(0)\,,$$

where the proportionality absorbs windowing constants. Using the same parameters as in the dephasing benchmark (App. E) yields $\xi_{\rm lab}\approx10^{11}$ m; smaller χ_0 (weaker bath response) pushes ξ larger.

E.6 Phase diffusion to frequency linewidth. Clock dephasing from lapse fluctuations follows the standard relation for phase noise. A clock with carrier frequency ω_0 subject to stochastic phase $\varphi(t)$ has instantaneous frequency $\omega(t)=\omega_0+d\varphi/dt$. For Gaussian phase diffusion with second cumulant

$$\langle[\varphi(t)]^2\rangle=2D_\varphi\cdot t\quad\text{(diffusion coefficient } D_\varphi\text{)},$$

the power spectral density of frequency fluctuations is $S_y(f)=2D_\varphi/(2\pi f)^2$ for $f\ll1/(2\pi\sqrt{D_\varphi})$. The corresponding linewidth (full width at half-maximum of the Lorentzian spectrum) is

$$\Delta\nu_{\rm FWHM}=\frac{D_\varphi}{2\pi}.$$

For our lapse-driven dephasing with $D_\varphi=\zeta_\phi\omega_0^2S_\Phi(0)$ (where ζ_ϕ is the stochastic coupling), the predicted linewidth broadening is

$$\Delta\nu_{\rm dephase}=\frac{\zeta_\phi\omega_0^2S_\Phi(0)}{2\pi}=\frac{\zeta_\phi\omega_0^2}{2\pi}\cdot G_\Phi^K(0).$$

Substituting the Ohmic result $G_\Phi^K(0)=2\pi^2\alpha\omega_{\rm IR}|G_\Phi^R(0)|^2$ yields the benchmark scaling $\Delta\nu_{\rm dephase}\propto\zeta_\phi\omega_0^2T_{\rm int}^{-2}$ after using the time-calibration relation.

Appendix F: Einstein–Langevin route to lapse fluctuations

Screened Poisson and Yukawa Green function.

$$-\nabla^2\Phi+m^2\Phi=\frac{8\pi G}{c^4}\rho\quad\Longleftrightarrow\quad\Phi(\mathbf{k})=\frac{8\pi G}{c^4}\frac{\rho(\mathbf{k})}{k^2+m^2}. \tag{108}$$

$$G_m(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2+m^2} = \frac{e^{-mr}}{4\pi r}, \quad \Phi(\mathbf{x}) = \frac{8\pi G}{c^4} \int d^3x' \, G_m(\mathbf{x}-\mathbf{x}') \, \rho(\mathbf{x}'). \tag{109}$$

The Yukawa Green's function is evaluated using standard methods [20]. Correlation length $\xi \equiv m^{-1}$ controls the decay $e^{-r/\xi}$.

Origin of m. A local-in-time piece $\Gamma_0 \, \delta(t-t')$ in the GLE kernel shifts the static self-energy, $k^2 \to k^2+m^2$ with $m^2 \propto \Gamma_0$ (cf. the by-parts form of the GLE in §5).

Starting from the mixed Einstein equation (flux law) $\partial_t \Phi = +(4\pi G/c^4) \, r \, T^t{}_r$, decompose $T^t{}_r = \langle T^t{}_r \rangle + \delta T^t{}_r$ and integrate out matter in the closed-time-path (CTP) formalism. The influence functional yields a nonlocal, causal self-energy for Φ and a Gaussian stochastic source with correlator given by the connected two-point function of the flux [21]:

$$\mathcal{D} \, \delta\Phi + \int \Gamma * \partial_t \delta\Phi = \eta, \tag{110}$$

$$\langle \eta(x)\eta(x') \rangle = \left(\frac{4\pi G}{c^4}\right)^2 \left\langle \delta\big[r T^t{}_r(x)\big] \, \delta\big[r' T^t{}_r(x')\big] \right\rangle_c, \tag{111}$$

where $\mathcal{D} = -\nabla^2 + m^2$ with m^2 fixed by Eq. (49). In Fourier space, the retarded propagator and spectrum are

$$G_R(\omega, k) = \big[k^2 + m^2 - i\omega \, \Gamma(\omega, k)\big]^{-1}, \tag{112}$$

$$S_\Phi(\omega, k) = |G_R(\omega, k)|^2 \, S_\eta(\omega, k), \qquad S_\eta(\omega, k) = 2 \coth\!\left(\frac{\omega}{2T_{\text{eff}}}\right) \mathrm{Im}\big[\omega \, \Gamma(\omega, k)\big]. \tag{113}$$

In the static limit with $\Gamma(\omega, k) \to \gamma(k)$, one obtains the Yukawa kernel and the distance law Eq. (57); different physically motivated Γ encode colored noise and finite correlation times without introducing free amplitudes beyond T_{eff}.

F.1 Screened Poisson solution step-by-step. The screened operator $\mathcal{D} = -\nabla^2 + m^2$ acts on $\delta\Phi(\mathbf{x})$ with source $\rho(\mathbf{x})$. In Fourier space, $\mathcal{D}$ becomes multiplication by $k^2 + m^2$, so the Green function is

$$\tilde{G}(\mathbf{k}) = \frac{1}{k^2 + m^2}.$$

The solution is $\tilde{\delta\Phi}(\mathbf{k}) = \frac{4\pi G}{c^4} \cdot \frac{\tilde{\rho}(\mathbf{k})}{k^2+m^2}$. Inverting the Fourier transform,

$$\delta\Phi(\mathbf{x}) = \frac{4\pi G}{c^4} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2+m^2} \int d^3x' \, e^{-i\mathbf{k}\cdot\mathbf{x}'} \rho(\mathbf{x}') \tag{114}$$

$$= \frac{4\pi G}{c^4} \int d^3x' \, \rho(\mathbf{x}') \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2+m^2}. \tag{115}$$

The remaining integral is the Fourier transform of the Yukawa potential:

$$\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2+m^2} = \frac{e^{-m|\mathbf{r}|}}{4\pi|\mathbf{r}|},$$

yielding the final solution

$$\delta\Phi(\mathbf{x}) = \frac{G}{c^4} \int d^3x' \, \rho(\mathbf{x}') \frac{e^{-m|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}.$$

F.2 Two-point correlator with screening. For the stochastic lapse correlator, the product of two Green functions gives

$$\langle\delta\Phi(\mathbf{x})\delta\Phi(\mathbf{x}')\rangle = \left(\frac{4\pi G}{c^4}\right)^2 \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{|k^2+m^2|^2} S_\eta(k) \tag{116}$$

$$= \left(\frac{4\pi G}{c^4}\right)^2 S_\eta(0) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{(k^2+m^2)^2}, \tag{117}$$

where we assumed white noise $S_\eta(k) \approx S_\eta(0)$ at long wavelengths. The integral becomes

$$\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{(k^2+m^2)^2} = \frac{me^{-mr}}{8\pi r},$$

giving the distance-dependent correlation

$$\langle\delta\Phi(\mathbf{x})\delta\Phi(\mathbf{x}')\rangle = \left(\frac{G}{c^4}\right)^2 S_\eta(0) \frac{me^{-m|\mathbf{x}-\mathbf{x}'|}}{2|\mathbf{x}-\mathbf{x}'|}.$$

F.3 Lab correlation length: explicit numbers. For the benchmark Ohmic bath with parameters $\alpha = 10^{-3}$, $\omega_{\rm IR} = 10^{-15}\,\mathrm{s}^{-1}$, and coupling scale $|G^R_\Phi(0)|^{-1} \sim M_{\rm Pl}c^2/\hbar$, the screening mass squared is

$$m^2 = \frac{4\pi G}{c^4}\Lambda\chi_0 \tag{118}$$

$$\sim \frac{4\pi G}{c^4} \cdot \frac{1}{\hbar} \cdot \alpha\omega_{\rm IR}|G^R_\Phi(0)|^2 \tag{119}$$

$$\sim \frac{4\pi G}{c^4\hbar} \cdot 10^{-3} \cdot 10^{-15} \cdot \left(\frac{\hbar}{M_{\rm Pl}c^2}\right)^2 \tag{120}$$

$$\sim 10^{-22}\,\mathrm{m}^{-2}. \tag{121}$$

This gives the correlation length $\xi = m^{-1} \sim 10^{11}$ m, much larger than the solar radius $R_\odot \sim 7\times10^8$ m, satisfying solar-system constraints.

F.4 Origin of screening mass. The mass parameter m^2 arises from the constraint $\Xi = \mathcal{C}[\Phi]$ via the Lagrange multiplier Λ. When $\mathcal{C}[\Phi]$ has Φ-dependence, variations yield an additional term $\Lambda\frac{\delta\mathcal{C}}{\delta\Phi}$ in the Euler-Lagrange equation. For small deviations $\delta\Phi$ about the stationary point,

$$\left.\frac{\delta^2\mathcal{C}}{\delta\Phi^2}\right|_{\Phi_\star} = -\chi_0 < 0$$

(negative for stability), so the linearized constraint contributes $+\Lambda\chi_0\delta\Phi$ to the field equation. This generates the screening mass $m^2 = \Lambda|\chi_0|$, with Λ determined by the global consistency requirement between the variational principle and the microphysical bath dynamics.

Appendix G: Constrained variation for $\Xi = \mathcal{C}[\Phi]$

Let $\Phi \in \mathcal{Y}$, $\Xi \in \mathcal{X}$ (Banach spaces) and $\mathcal{C} : \mathcal{Y} \to \mathcal{X}$ be Fréchet differentiable with bounded derivative $D\mathcal{C}[\Phi]$. Define the reduced functional

$$\tilde{F}[\Phi] \;\equiv\; F[\rho,\Phi,\mathcal{C}[\Phi]], \qquad F[\rho,\Phi,\Xi] \;=\; E[\rho,\Phi] - \Theta\, R[\rho;\Xi].$$

Writing partial functional derivatives at fixed arguments as $F_\Phi \equiv \delta F/\delta\Phi\big|_\Xi$ and $F_\Xi \equiv \delta F/\delta\Xi\big|_\Phi$, the first variation of $\tilde{F}$ is

$$\delta\tilde{F} = \langle F_\Phi,\, \delta\Phi\rangle + \langle F_\Xi,\, D\mathcal{C}[\Phi]\cdot\delta\Phi\rangle = \Big\langle F_\Phi + (D\mathcal{C}[\Phi])^* F_\Xi,\, \delta\Phi\Big\rangle.$$

Hence stationarity of $\tilde{F}$ yields the *constrained Euler–Lagrange* equation

$$F_\Phi(\rho,\Phi,\Xi) \;+\; (D\mathcal{C}[\Phi])^* F_\Xi(\rho,\Phi,\Xi) \;=\; 0 \qquad \text{with} \qquad \Xi = \mathcal{C}[\Phi]. \tag{122}$$

Equivalence with the multiplier formulation. Introduce the augmented functional

$$\mathcal{F}_{\text{aug}}[\rho,\Phi,\Xi,\Lambda] = E[\rho,\Phi] - \Theta\, R[\rho;\Xi] + \int d^3x\, \Lambda(x)\, \big[\Xi(x) - \mathcal{C}[\Phi](x)\big].$$

Stationarity with respect to (Φ,Ξ,Λ) gives

$$\begin{aligned} \frac{\delta\mathcal{F}_{\text{aug}}}{\delta\Phi} &:= F_\Phi - (D\mathcal{C}[\Phi])^*\Lambda = 0, \\ \frac{\delta\mathcal{F}_{\text{aug}}}{\delta\Xi} &:= F_\Xi + \Lambda = 0, \\ \frac{\delta\mathcal{F}_{\text{aug}}}{\delta\Lambda} &:= \Xi - \mathcal{C}[\Phi] = 0. \end{aligned}$$

Eliminating $\Lambda = -F_\Xi$ from the first line yields precisely Eq. (122), with the closure $\Xi = \mathcal{C}[\Phi]$. This proves the equivalence used in Sec. 3.1 and justifies treating (Φ,Ξ) as independent during variation without any circularity.

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