Gravity as Temporal Geometry:
A Quantizable Reformulation of General Relativity

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Abstract

We reformulate gravity as the *geometry of time*: a single scalar field Φ controls the lapse $N = e^{\Phi}$, while the spatial geometry (γ_{ij}) follows from the ADM constraints and evolution equations with shift $\boldsymbol{\omega}$. Starting from the Einstein–Hilbert action we reconstruct $g_{\mu\nu}$ from $(\Phi, \boldsymbol{\omega}, \gamma_{ij})$ and derive the full constraint/evolution system, establishing classical equivalence with GR.

In the static, spherically symmetric sector (vacuum enforces $\partial_t \Phi = 0$ in this gauge), one ODE,

$$\partial_r \Phi = \frac{1 - e^{2\Phi}}{2r\, e^{2\Phi}},$$

integrates immediately to $e^{2\Phi} = 1 - \frac{r_s}{r}$, i.e. Schwarzschild, reproducing the standard tests. Horizons are coordinate artifacts of the diagonal foliation and are regular in Painlevé–Gullstrand and Eddington–Finkelstein charts. Rotation resides in the shift: solving the momentum constraint outside a compact source yields the azimuthal component ω_φ and the Lense–Thirring rate with the Kerr normalization. In spherical dynamics, $\partial_t \Phi$ is sourced by radial energy flux, reproducing the Vaidya/Bondi mass law. Linearized vacuum contains only the two transverse–traceless tensor modes, propagating at c with the standard GR energy flux.

Cosmology maps via $d\tau = e^{\Phi} dt$ and $a = e^{-\Phi}$, exactly reproducing the Friedmann equations (including k and Λ). Quantization is posed as a constrained QFT: Φ and $\boldsymbol{\omega}$ enforce the Hamiltonian/momentum constraints (instantaneous/Coulomb-like), while the propagating quanta are the TT tensors with the same low-energy EFT status as GR. We make no claims here about improved renormalizability or extra polarizations; such extensions are deferred to Part II.

Conventions: $G = c = 1$ unless explicitly restored.

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1 Introduction and Main Claim

Thesis. Gravity is the dynamics of a temporal scalar field Φ (with $N = e^{\Phi}$). Spatial geometry γ_{ij} and shift $\boldsymbol{\omega}$ follow via constraints and evolution (definitions in Sec. 2; one-page overview in Fig. 1). We claim classical equivalence to GR and a clean, conservative route to quantization of the physical radiative sector.

Advantages of this formulation. (i) Spherical problems reduce to a single ODE for Φ rather than coupled PDEs; (ii) The temporal nature of gravity becomes manifest—all gravitational effects stem from time dilation; (iii) Horizons are naturally regular in this foliation; (iv) The constraint structure cleanly separates physical (TT) from gauge degrees of freedom; (v) Practical calculations in atom interferometry and gravitational wave detection map directly to Φ.

Caution (no extra degree of freedom).
Φ is the logarithm of the lapse ($N = e^{\Phi}$). In the full 3+1 system, the lapse and shift are Lagrange multipliers enforcing the Hamiltonian/momentum constraints. None of the scalar relations in this paper introduce an independent propagating "scalar graviton"; radiative content remains the two TT tensor modes. *Rotation and frame dragging reside in the shift* $\boldsymbol{\omega}$ *(the gravitomagnetic potential); the lapse* Φ *remains gravito-electric.*

Acceptance check

Equivalence statement. Given fields $(\Phi, \boldsymbol{\omega}, \gamma_{ij})$ satisfying the constraint/evolution system of Sec. 3, the reconstructed $g_{\mu\nu}$ solves Einstein's equations with the same $T_{\mu\nu}$. (Proof sketch in Sec. 3 and App. C.)

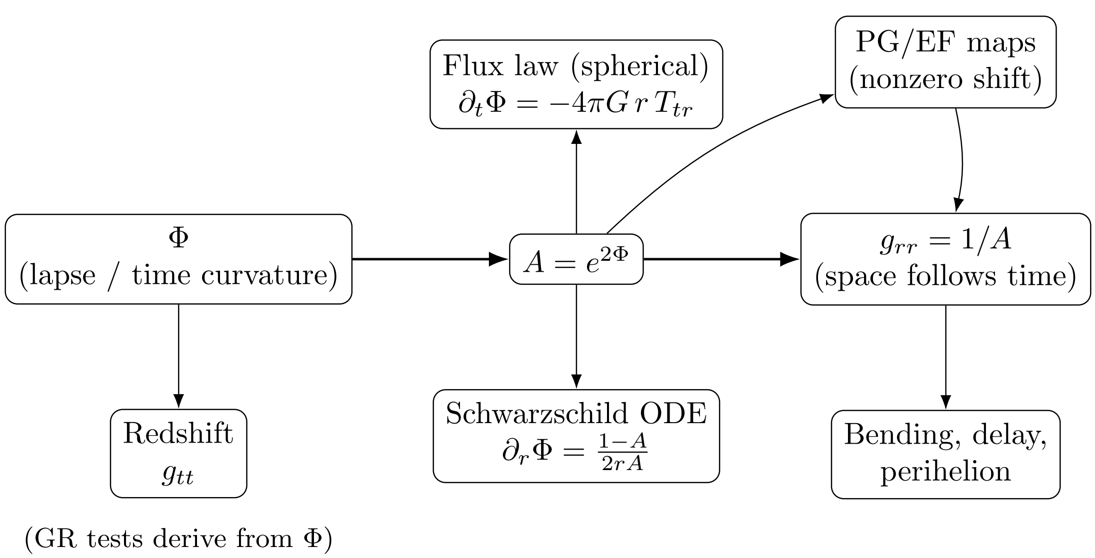


Figure 1: Lapse-first 3+1 split: $N = e^{\Phi}$, shift $\boldsymbol{\omega}$, spatial metric γ_{ij}. The central variable is $A \equiv e^{2\Phi}$. In diagonal spherical gauge, $g_{rr} = 1/A$ ("space follows time"). Horizon-regular Painlevé–Gullstrand / Eddington–Finkelstein maps correspond to *nonzero* shift. (Most textbook GR tests derive directly from Φ.)

Conventions & Notation (read this first)

Units & signature. $G = c = 1$ unless shown; metric signature $(-,+,+,+)$; Greek indices = spacetime, Latin = space.

Lapse-first variables. $N = e^{\Phi}$, $N_i \equiv \omega_i$, γ_{ij}, with

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \qquad A \equiv e^{2\Phi} = N^2.$$

In diagonal spherical gauge: $ds^2 = -A\,dt^2 + A^{-1}dr^2 + r^2 d\Omega^2$.

Times and coordinates. $d\tau = N\,dt = e^{\Phi}dt$. Spherical (r, θ, φ).

Background vs. perturbations. $\Phi = \Phi_0 + \delta\Phi$, $\gamma_{ij} = \bar{\gamma}_{ij} + h_{ij}$, $N = e^{\Phi_0}(1 + \delta\Phi)$. We *do not* use ϕ for $\delta\Phi$ (reserve ϕ for interferometer phase; φ is the azimuthal angle).

Derivatives. Overdot $\dot{} = d/d\tau$. Use ∂_t, ∂_r; avoid primes for time. Cosmology: $a(\tau) = e^{-\Phi}$, $H = \dot{a}/a = -\dot{\Phi}$, $1 + z = e^{\Phi}$ (with $\Phi_0 = 0$).

Gauges. "Lapse-first, shift-allowed." Zero-shift (diagonal) is convenient for static spherical intuition; PG/EF maps (nonzero shift) are regular at horizons and for flux. Physics is gauge-invariant.

Quick identities. $r_s = 2GM/c^2$; weak field $g_{tt} \simeq -(1 + 2\Phi) \Rightarrow \Phi \simeq \Phi_{\rm Newt}/c^2$. Sourced spherical evolution: $G_{tr} = -2\,\partial_t\Phi/r \Rightarrow \partial_t\Phi = -\dfrac{4\pi G}{c^4}\,r\,T_{tr}$. Frame dragging lives in ω_i (gravitomagnetism), not in $\delta\Phi$.

2 Variables and Dictionary (Rosetta Map)

We use a 3+1 split with lapse $N = e^{\Phi}$, shift $\boldsymbol{\omega}$ and spatial metric γ_{ij}. The reconstructed spacetime metric is

$$g_{tt} = -N^2 + \omega_i\omega^i, \qquad g_{ti} = \omega_i, \qquad g_{ij} = \gamma_{ij}. \tag{1}$$

Here $N^i \equiv \gamma^{ij}N_j = \gamma^{ij}\omega_j$ and $\omega^i = \gamma^{ij}\omega_j$.

Rosetta Map / Dictionary

Time-first	**GR / Cosmology**
$N = e^{\Phi}$	Lapse
$\boldsymbol{\omega}$	Shift, carries rotation/frame dragging
γ_{ij}	Spatial 3-metric
$\mathrm{d}\tau = e^{\Phi}\,\mathrm{d}t$	Proper (cosmic) time increment
$a = e^{-\Phi}$	FRW scale factor (Sec. 9)
$H = -\dot{\Phi}(\tau)$	Hubble rate in cosmic time

Lapse-first, shift-allowed. Throughout we take the lapse $N = e^{\Phi}$ as the primary scalar while allowing a nonzero shift $\boldsymbol{\omega}$ when dynamics or nonsphericity demand it; the diagonal, zero-shift form is a convenient gauge for static cases, not a physical restriction.

Table 1: Core symbols: lapse-first variables $\leftrightarrow$ metric components.

Forward (variables $\rightarrow$ metric)	**Reverse (metric + time function t $\rightarrow$ variables)**
$N = e^{\Phi}, \quad N_i \equiv \omega_i, \quad \gamma_{ij}$	$N = \big(-g^{\mu\nu}\partial_\mu t\,\partial_\nu t\big)^{-1/2}, \quad N_i = g_{0i}, \quad \gamma_{ij} = g_{ij}$
$g_{00} = -N^2 + \gamma_{ij}N^iN^j, \quad g_{0i} = N_i, \quad g_{ij} = \gamma_{ij}$	$\Phi = \ln N, \quad \omega_i = N_i$
$K_{ij} = \dfrac{1}{2N}\Big(-\partial_t\gamma_{ij} + D_iN_j + D_jN_i\Big), \quad D_i$ is the Levi–Civita connection of γ_{ij}.	

Note

Gauges. We use diagonal (zero-shift) in static spherical cases; for horizons or flows, we switch to Painlevé–Gullstrand (PG) or Eddington–Finkelstein (EF). Full maps in App. A.

3 Constraints and Evolution from the Action

We work in a lapse-first 3+1 split [1] with lapse $N \equiv e^{\Phi}$, shift covector ω_i (shift vector $N^i = \gamma^{ij}\omega_j$), and spatial metric γ_{ij}. We use signature $(-,+,+,+)$ and set $G = c = 1$ unless shown.

$$\mathrm{d}s^2 = -N^2\,\mathrm{d}t^2 + \gamma_{ij}\,(\,\mathrm{d}x^i + N^i\mathrm{d}t)(\,\mathrm{d}x^j + N^j\mathrm{d}t). \tag{2}$$

Here $N_i \equiv \omega_i$ and $N^i = \gamma^{ij}N_j = \gamma^{ij}\omega_j$. The extrinsic curvature is

$$K_{ij} \;=\; \frac{1}{2N}\left(-\partial_t\gamma_{ij} + \nabla_iN_j + \nabla_jN_i\right), \qquad N_j \equiv \gamma_{jk}N^k = \omega_j, \tag{3}$$

where D_i (synonymous with ∇_i below) is the Levi–Civita connection of γ_{ij}.

3.1 3+1 decomposition of R and the action

Boundary terms. We include the Gibbons–Hawking–York term so the Dirichlet variational problem is well-posed. The field redefinition $N = e^{\Phi} > 0$ leaves the boundary structure unchanged: $\delta N = N\,\delta\Phi$ so $N\sqrt{\gamma}$ multiplies the same total-derivative pieces, and no additional boundary contributions are induced. Up to the Gibbons–Hawking–York boundary term, the Einstein–Hilbert action reduces to

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int dt\,d^3x\,N\sqrt{\gamma}\,\big({}^{(3)}R + K_{ij}K^{ij} - K^2\big), \tag{4}$$

with $K \equiv \gamma^{ij}K_{ij}$ and ${}^{(3)}R$ the Ricci scalar of γ_{ij}. The only canonical variables are (γ_{ij}, π^{ij}), with

$$\pi^{ij} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_t\gamma_{ij})} = \frac{\sqrt{\gamma}}{16\pi G}\big(K^{ij} - \gamma^{ij}K\big). \tag{5}$$

There are no ∂_tN or ∂_tN^i terms, so their conjugate momenta vanish (primary constraints).

Table 2: Variations and their corresponding equations (ADM form).

Variation	Equation type	Representative equation (ADM form)
δN (or $\delta\Phi$ with $\delta N = N\,\delta\Phi$)	Hamiltonian constraint	${}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi G\,\rho$
δN_i (or $\delta\omega_i$)	Momentum constraints	$\nabla_j\big(K^{ij} - \gamma^{ij}K\big) = 8\pi G\, j^i$
$\delta\gamma_{ij}$	Evolution (γ_{ij})	$\partial_t\gamma_{ij} = -2NK_{ij} + \nabla_i N_j + \nabla_j N_i$
$\delta\gamma_{ij}$	Evolution (K_{ij})	$\partial_t K_{ij} = -\nabla_i\nabla_j N + N\big({}^{(3)}R_{ij} + KK_{ij} - 2K_i{}^k K_{kj}\big) + \mathcal{L}_{\vec{N}}K_{ij} - 8\pi GN\big(S_{ij} - \tfrac{1}{2}\gamma_{ij}(S-\rho)\big)$

Matter projections: $\rho \equiv n_\mu n_\nu T^{\mu\nu}$, $\quad j^i \equiv -\gamma^i{}_\mu\, n_\nu\, T^{\mu\nu}$, $\quad S_{ij} \equiv \gamma_{i\mu}\gamma_{j\nu}T^{\mu\nu}$, $\quad S \equiv \gamma^{ij}S_{ij}$.

3.2 Hamiltonian and momentum constraints

Varying w.r.t. the lapse $N = e^\Phi$ gives the Hamiltonian constraint

$$\boxed{\mathcal{H}_\perp \;\equiv\; \frac{16\pi G}{\sqrt{\gamma}}\big(\pi_{ij}\pi^{ij} - \tfrac{1}{2}\pi^2\big) - \frac{\sqrt{\gamma}}{16\pi G}\,{}^{(3)}R \;=\; 0}, \tag{6}$$

and varying w.r.t. the shift $N^i = \gamma^{ij}\omega_j$ gives the momentum (diffeomorphism) constraints. We use $\nabla_i \equiv D_i$ for the Levi–Civita connection of γ_{ij}.

$$\boxed{\mathcal{H}_i \;\equiv\; -2\,\nabla_j\pi^j{}_i \;=\; 0}. \tag{7}$$

The canonical Hamiltonian reads

$$\mathcal{H}_{\rm can} \;=\; N\,\mathcal{H}_\perp + N^i\mathcal{H}_i, \qquad (N = e^\Phi,\ N^i = \gamma^{ij}\omega_j). \tag{8}$$

3.3 Evolution for γ_{ij} and K_{ij}

The first evolution equation is simply the definition of K_{ij} rewritten:

$$\partial_t\gamma_{ij} \;=\; -2NK_{ij} + \nabla_i N_j + \nabla_j N_i \;=\; -2e^\Phi K_{ij} + \nabla_i\omega_j + \nabla_j\omega_i. \tag{9}$$

The second evolution equation follows from $\delta S\,/\,\delta\gamma_{ij}$ (or Hamilton's equations):

$$(\partial_t - \mathcal{L}_{\vec{N}})K_{ij} = -\nabla_i\nabla_j N + N\Big({}^{(3)}R_{ij} + KK_{ij} - 2K_i{}^k K_{kj}\Big) - 8\pi GN\big(S_{ij} - \tfrac{1}{2}\gamma_{ij}(S-\rho)\big)\,. \tag{10}$$

Here $\mathcal{L}_{\vec{N}}$ is the Lie derivative along N^i, and ∇_i is the Levi–Civita connection of γ_{ij} (synonymous with D_i used elsewhere). In vacuum ($T_{\mu\nu} = 0$) the last term vanishes.

3.4 Acceptance check: non-propagating lapse/shift

Because $S_{\rm EH}$ contains no $\partial_t N$ or $\partial_t N^i$, their conjugate momenta vanish:

$$p_N \equiv \frac{\partial\mathcal{L}}{\partial(\partial_t N)} = 0, \qquad p_i \equiv \frac{\partial\mathcal{L}}{\partial(\partial_t N^i)} = 0, \tag{11}$$

which are *primary* constraints. Time preservation generates the *secondary* constraints $\mathcal{H}_\perp = 0$ and $\mathcal{H}_i = 0$. All are first-class; $N = e^{\Phi}$ and N^i enter only as Lagrange multipliers in (8). Hence in vacuum they carry no propagating degrees of freedom. Linearizing about Minkowski and imposing the constraints/gauge leaves only the two transverse–traceless tensor modes h_{ij}^{TT} as propagating DOF.

DOF ledger (vacuum). On each slice: $(\gamma_{ij}, \pi^{ij}) = 12$ phase-space DOF. Four first-class constraints $(\mathcal{H}_\perp, \mathcal{H}_i)$ remove 8, leaving 4 phase-space = 2 configuration DOF (the two TT graviton polarizations). The lapse $N = e^{\Phi}$ and shift ω_i are non-dynamical multipliers.

4 Static spherical ODE ⇒ Schwarzschild and classic tests

For static, spherically symmetric vacuum with zero shift,

$$ds^2 = -A(r)\,dt^2 + A(r)^{-1}\,dr^2 + r^2 d\Omega^2, \qquad A(r) \equiv e^{2\Phi(r)}. \tag{12}$$

4.1 Vacuum ODE for $\Phi(r)$

For the metric (12), the vacuum Einstein equations (Birkhoff's theorem) reduce to a single independent ODE, which we take as

$$r\,A'(r) = 1 - A(r). \tag{13}$$

Using $A' = 2A\,\partial_r\Phi$, (13) is equivalent to the lapse-first ODE

$$\partial_r\Phi = \frac{1 - e^{2\Phi(r)}}{2r\,e^{2\Phi(r)}}. \tag{14}$$

4.2 Integration to Schwarzschild

Integrating (13) gives

$$A(r) = 1 - \frac{C}{r}, \qquad C = 2GM \equiv r_s, \tag{15}$$

so

$$e^{2\Phi(r)} = A(r) = 1 - \frac{2GM}{r}, \qquad ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 + r^2 d\Omega^2. \tag{16}$$

4.3 Classic tests

Gauge-invariant outputs. All quoted observables (gravitational redshift, light bending, Shapiro delay, perihelion precession, horizon invariants, GW luminosity) are computed in forms independent of slicing/coordinates.

Gravitational redshift. Static observers at r_{e} (emit) and r_{o} (observe) measure

$$\frac{\nu_o}{\nu_e} = \sqrt{\frac{A(r_o)}{A(r_e)}} = e^{\Phi(r_o) - \Phi(r_e)}, \qquad z \equiv \frac{\nu_e}{\nu_o} - 1 \simeq \Phi(r_e) - \Phi(r_o) \simeq \frac{GM}{r_e} - \frac{GM}{r_o}. \tag{17}$$

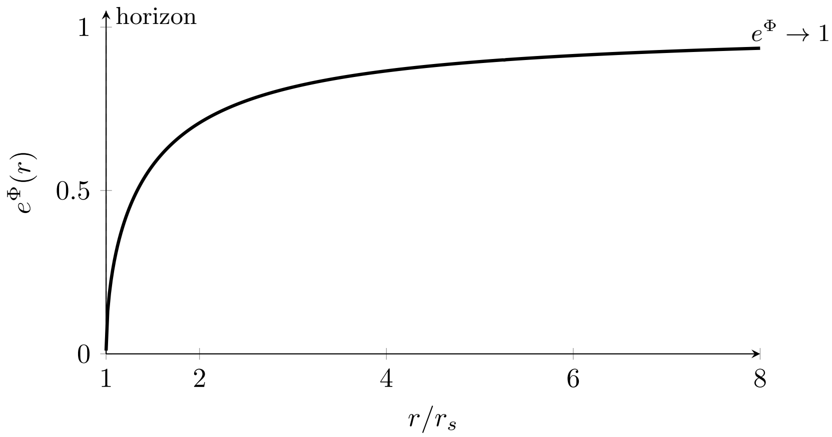


Figure 2: Clock rate $e^{\Phi(r)} = \sqrt{A(r)} = \sqrt{1 - r_s/r}$ vs radius ($r_s \equiv 2GM$); weak-field $g_{tt} \simeq -(1 + 2\Phi)$.

Light bending. A null geodesic with impact parameter b is deflected by

$$\boxed{\Delta\varphi \;=\; \frac{4GM}{b} \;+\; \mathcal{O}\!\left(\frac{(GM)^2}{b^2}\right)} \tag{18}$$

Shapiro time delay.

$$\Delta t^{(1\mathrm{w})}_{\mathrm{Shapiro}} = 2GM\,\ln\!\left(\frac{r_1 + r_2 + R_{12}}{r_1 + r_2 - R_{12}}\right) \;+\; \mathcal{O}(G^2),$$

where $r_{1,2}$ are the radial distances of the endpoints from the lens and R_{12} is their Euclidean separation. For superior conjunction (Sun between Earth at $r_\oplus$ and target at r_T, impact parameter b),

$$\Delta t^{(1\mathrm{w})}_{\mathrm{Shapiro}} \;\simeq\; 2GM\,\ln\!\left(\frac{4\,r_\oplus r_T}{b^2}\right), \qquad \Delta t^{(2\mathrm{w})}_{\mathrm{Shapiro}} \;\simeq\; 4GM\,\ln\!\left(\frac{4\,r_\oplus r_T}{b^2}\right).$$

Perihelion precession. For a bound orbit (semi-major axis a, eccentricity e), the per-orbit advance of perihelion is

$$\boxed{\Delta\varpi \;=\; \frac{6\pi GM}{a(1-e^2)} \;+\; \mathcal{O}(G^2)} \tag{19}$$

Acceptance check. Equations (17)–(19) (i.e., (17)–(21) in this section) are the standard GR results for the Schwarzschild geometry to leading post-Newtonian order, with Eddington parameter $\gamma = 1$. Thus the lapse-first ODE (14) reproduces all four classic tests exactly in vacuum.

5 Horizons, Regular Foliations, and Invariants

We exhibit explicit horizon-regular coordinates (PG, EF), the curvature invariant $\mathcal{K} \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, and the proper time for a radially infalling observer to cross $r = 2GM$.

5.1 Finite proper time across the horizon

For the diagonal Schwarzschild chart

$$ds^2 = -A(r)\,dt^2 + A(r)^{-1}\,dr^2 + r^2 d\Omega^2, \qquad A(r) = 1 - \frac{r_s}{r}, \quad r_s \equiv 2GM,$$

a radial timelike geodesic with specific energy

$$E \;\equiv\; -u_t \;=\; A(r)\,\frac{dt}{d\tau}$$

obeys (from $u^\mu u_\mu = -1$ and the conserved E)

$$\left(\frac{dr}{d\tau}\right)^2 \;=\; E^2 - A(r), \qquad \frac{dt}{d\tau} \;=\; \frac{E}{A(r)}. \tag{20}$$

Hence the proper-time element is

$$d\tau \;=\; \frac{dr}{\sqrt{E^2 - A(r)}} \;=\; \frac{dr}{\sqrt{E^2 - 1 + \dfrac{r_s}{r}}}. \tag{21}$$

At the horizon $r = r_s$ the denominator is $\sqrt{E^2}$, so the integrand is finite and the crossing time is finite. For the common case $E = 1$ (fall from rest at infinity),

$$\tau(r) \;=\; \frac{2}{3\sqrt{2GM}}\Big(r_0^{3/2} - r^{3/2}\Big), \tag{22}$$

so the proper time from r_0 to the horizon is

$$\tau_{r_0\to r_s} \;=\; \frac{2}{3\sqrt{2GM}}\Big(r_0^{3/2} - r_s^{3/2}\Big) \;<\; \infty. \tag{23}$$

Thus an infaller crosses $r = 2GM$ smoothly in finite τ.

5.2 Acceptance check

In EF and PG coordinates the metric components are finite at $r = r_s$; the Kretschmann invariant $\mathcal{K} \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ (see Eq. (77) below) is finite at $r = r_s$ and diverges only at $r = 0$; and a radial timelike geodesic crosses $r = r_s$ in finite proper time, Eqs. (21)–(23). Therefore the horizon is not a physical singularity, only a coordinate artifact of the diagonal chart.

6 Rotation as shift: momentum constraint ⇒ slow–Kerr calibration

We consider a stationary, weakly rotating vacuum exterior to a compact source of mass M and angular momentum vector $\mathbf{J}$. Take the background spatial metric to be asymptotically flat and, to leading (dipole) order, treat $\gamma_{ij} \simeq \delta_{ij}$; the lapse is the static Schwarzschild lapse $N^2 = 1 - 2GM/r + \mathcal{O}(J^2)$, while the *rotation* lives entirely in the shift N^i (covector $\omega_i = \gamma_{ij}N^j$).

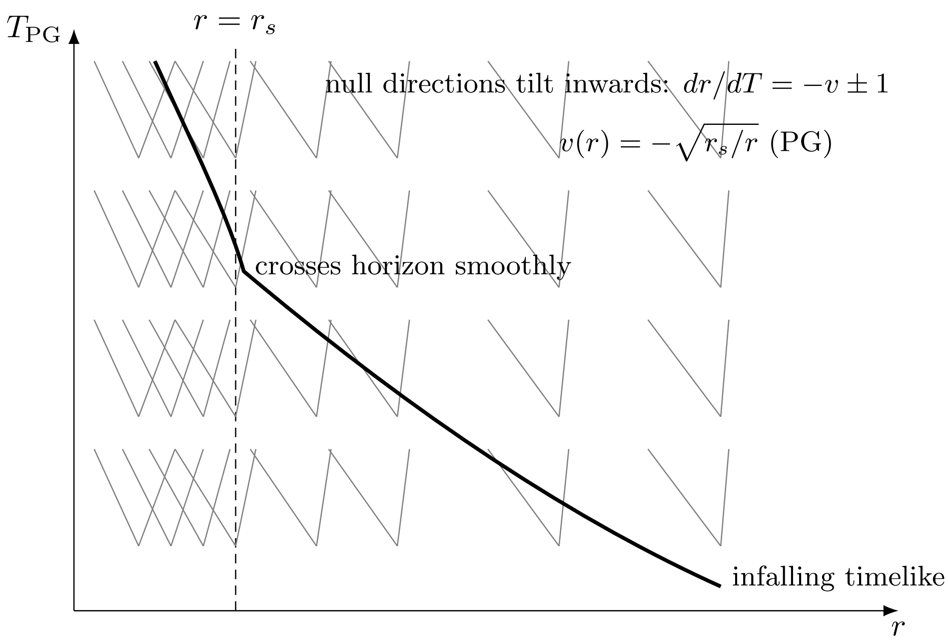


Figure 3: Painlevé–Gullstrand coordinates: an infaller crosses $r = r_s$ smoothly in finite proper time; null directions tilt inward across the horizon; the flow speed is $v(r) = -\sqrt{r_s/r}$.

6.1 Momentum constraint outside the source

In the stationary, linearized limit ($\partial_t \gamma_{ij} = 0$, $K_{ij} = \frac{1}{2}(\partial_i N_j + \partial_j N_i)$), the momentum constraint

$$\mathcal{H}_i \equiv -2\,\nabla_j\big(K^j{}_i - \delta^j{}_i K\big) = 8\pi G\, S_i$$

reduces (with $\nabla \to \partial$ and gauge $\partial_i N^i = 0$) to the vector Poisson equation

$$\nabla^2 N_i = -\,16\pi G\, S_i, \qquad \partial_i N^i = 0. \tag{24}$$

where S_i is the matter momentum density ($S_i \simeq T_{0i}$ for slow sources). Outside the compact body, $S_i = 0$ and

$$\nabla \times (\nabla \times \vec{N}) = \mathbf{0}, \qquad \vec{N} \equiv (N^1, N^2, N^3), \qquad \vec{N} \to \mathbf{0} \text{ as } r \to \infty. \tag{25}$$

Matching to the total angular momentum $\mathbf{J} = \int \mathrm{d}^3x\, \mathbf{x} \times \mathbf{S}$ fixes the unique decaying solution (up to a pure gradient):

$$\vec{N}(\mathbf{r}) \;=\; -\,\frac{2G}{c^3}\,\frac{\mathbf{J} \times \mathbf{r}}{r^3}, \qquad \nabla \times \vec{N} \;=\; \frac{2G}{c^3 r^3}\big[\,3\,\mathbf{n}\,(\mathbf{J}\cdot\mathbf{n}) \;-\; \mathbf{J}\,\big], \qquad \mathbf{n} \equiv \frac{\mathbf{r}}{r}.$$

(We keep c explicit here for later comparison; set $c = 1$ elsewhere.)

6.2 From shift to $g_{t\varphi}$ and $\Omega(r)$

In spherical coordinates, the only nonzero component is $N^\varphi(r) = -\dfrac{2GJ}{c^3 r^3}$ (for $\mathbf{J} \parallel \hat{\mathbf{z}}$). Since $g_{ti} = \gamma_{ij} N^j \equiv N_i$ and $\gamma_{\varphi\varphi} = r^2 \sin^2\theta$, we obtain

$$\boxed{g_{t\varphi} \;=\; N_\varphi \;=\; \gamma_{\varphi\varphi}\, N^\varphi \;=\; -\,\frac{2GJ}{c^3 r}\,\sin^2\theta} \tag{26}$$

The local inertial frame (ZAMO) angular velocity is

$$\Omega_{\rm ZAMO} \;=\; -\,N^{\varphi} \;=\; \frac{2GJ}{c^3 r^3} \qquad (\text{for } \mathbf{J} \parallel \hat{\mathbf{z}}).$$

Equivalently, the slow–rotation line element can be written as

$$ds^2 \simeq -\Big(1-\frac{2GM}{r}\Big)dt^2 \;-\; 2\,\Omega_{\rm ZAMO}(r)\, r^2 \sin^2\!\theta \; dt\, d\varphi \;+\; \Big(1-\frac{2GM}{r}\Big)^{-1} dr^2 + r^2 d\Omega^2, \tag{27}$$

so that $2g_{t\varphi} dt\, d\varphi$ equals the cross term and

$$\Omega(r) \;=\; \Omega_{\rm ZAMO}(r) \;=\; -\,N^{\varphi} \;=\; \frac{2GJ}{c^3 r^3}. \tag{28}$$

6.3 Exact factor checks: Lense–Thirring and Kerr

Match to Lense–Thirring (frame dragging). The metric cross term is controlled by the ZAMO angular velocity $\Omega_{\rm ZAMO}(r)$ in (28). The *gyroscope* (spin) Lense–Thirring precession rate has magnitude

$$\Omega_{\rm LT}(r) \;=\; \frac{2GJ}{c^2 r^3}, \tag{29}$$

which differs from $\Omega_{\rm ZAMO}$ by one power of c because it is a physical spin-precession frequency rather than a coordinate angular velocity. Both share the same J normalization.

Match to linearized Kerr $g_{t\varphi}$. Expanding the Kerr metric to first order in $a \equiv J/(Mc)$ and at large r gives

$$g_{t\varphi}^{\rm Kerr} \;=\; -\,\frac{2GJ}{c^3 r}\,\sin^2\theta \;+\; \mathcal{O}\!\left(\frac{a^2}{r^2}\right), \tag{30}$$

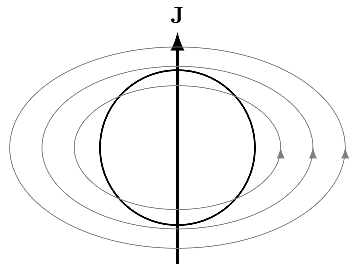
which exactly matches (26). Therefore the shift field derived from the momentum constraint reproduces the slow–Kerr cross term with the correct normalization and angular dependence.

Vector (dipole) gravitomagnetic field. From the Poisson equation (24) and the far–field solution for $\vec{N}$ in Sec. 6.1, the curl of the shift (the gravitomagnetic field) has the standard dipole form

$$\nabla\times\vec{N} \;=\; \frac{2G}{c^3 r^3}\Big[\,3\,\mathbf{n}\,(\mathbf{J}\!\cdot\!\mathbf{n}) - \mathbf{J}\,\Big], \tag{31}$$

agreeing with the Lense–Thirring dipole structure and fixing all numerical factors by the $g_{t\varphi}$ calibration above.

Callout. Frame dragging is carried by the shift (gravitomagnetic potential): $\vec{B}_g = \nabla\times\vec{N}$ with $\vec{N} = \gamma^{ij}\omega_j$. Time variation of Φ alone does *not* generate $\vec{B}_g$ in the absence of mass currents.



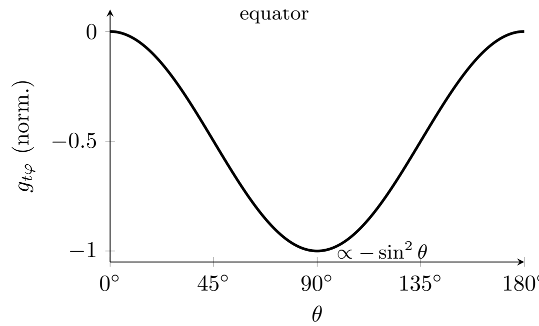


Figure 4: Rotation lives in ω_i: gravitomagnetic flow lines around a spinning mass.

Gravito-EM conventions (linearized, lapse-first)

Metric perturbations (Minkowski background, Cartesian indices):

$$g_{00} = -(1 + 2\Phi), \qquad g_{0i} = \omega_i, \qquad g_{ij} = (1 - 2\Phi)\,\delta_{ij}.$$

Fields and sources (definitions): $\mathbf{E}_g \equiv -\nabla\Phi - \frac{1}{2}\,\partial_t\boldsymbol{\omega}, \quad \mathbf{B}_g \equiv \nabla \times \boldsymbol{\omega}.$

$$\nabla^2\Phi = 4\pi G\,\rho, \qquad \nabla^2\boldsymbol{\omega} = -\,16\pi G\,\mathbf{J} \quad \text{(quasi-static)}.$$

Ampère-like relation (linearized Einstein eqs):

$$\nabla \times \mathbf{B}_g = -\,16\pi G\,\mathbf{J} + 2\,\partial_t\mathbf{E}_g.$$

Lense–Thirring calibration (slow rotation, far field):

$$g_{t\varphi} = -\,\frac{2GJ}{c^3 r}\,\sin^2\theta, \qquad \Omega_{\mathrm{LT}}(r) = \frac{2GJ}{c^2 r^3}.$$

6.4 Acceptance check

Equations (26) and (29) give $g_{t\varphi} = -(2GJ/c^3 r)\sin^2\theta$ and $\Omega_{\mathrm{LT}}(r) = 2GJ/(c^2 r^3)$, exactly matching the linearized Kerr result and the slow-rotation frame-dragging rate. Thus, in the lapse-first picture, *rotation is the shift*: the momentum constraint outside the source determines a unique (dipolar) $\vec{N}$ with the correct normalization.

7 Spherical dynamics ⇒ Vaidya/Bondi

We derive the spherical flux law for $\Phi(t, r)$, map it to ingoing EF (v), and recover the Bondi mass balance at $\mathscr{I}^{\pm}$, including a worked null–shell example.

7.1 Flux law $\partial_t \Phi = -4\pi G\, r\, T_{tr}$

In the spherical, zero-shift ansatz

$$\mathrm{d}s^2 = -e^{2\Phi(t,r)}\,\mathrm{d}t^2 + e^{-2\Phi(t,r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2, \qquad A \equiv e^{2\Phi}, \tag{32}$$

the mixed Einstein component is

$$G_{tr} \;=\; -\frac{2}{r}\,\partial_t\Phi. \tag{33}$$

With $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ this yields the spherical flux evolution law

$$\partial_t\Phi = -\,4\pi G\, r\, T_{tr}, \tag{34}$$

$$\Longleftrightarrow \qquad \partial_t A = -\,8\pi G\, r\, A\, T_{tr}, \qquad (A \equiv e^{2\Phi}). \tag{35}$$

7.2 EF map and Vaidya mass function

Define the tortoise coordinate by $dr_* = dr/A$ and the *advanced* EF time $v \equiv t + r_*$. Then

$$ds^2 = -A\, dv^2 + 2\, dv\, dr + r^2 d\Omega^2. \tag{36}$$

Allowing ingoing null dust gives the Vaidya metric

$$ds^2 = -\left(1 - \frac{2G\, m(v)}{r}\right) dv^2 + 2\, dv\, dr + r^2 d\Omega^2, \qquad T_{vv} = \frac{1}{4\pi r^2}\,\frac{dm}{dv}. \tag{37}$$

The diagonal/EF components relate via $v = t + r_*$:

$$\left.\frac{\partial v}{\partial t}\right|_r = 1, \qquad \left.\frac{\partial v}{\partial r}\right|_t = \frac{1}{A},$$

so for ingoing null dust

$$T_{tr} \;=\; T_{vv}\,\frac{\partial v}{\partial t}\frac{\partial v}{\partial r} \;=\; \frac{T_{vv}}{A}. \tag{38}$$

Using (35) with (38) gives

$$\partial_t A \;=\; -\,8\pi G\, r\, T_{vv}. \tag{39}$$

Identifying $A = 1 - \dfrac{2G\, m(v)}{r}$ implies

$$\partial_v A \;=\; -\,\frac{2G}{r}\,\frac{dm}{dv} \;=\; -\,8\pi G\, r\, T_{vv}, \tag{40}$$

which matches the standard Vaidya/Bondi balance.

7.3 Bondi balance at $\mathscr{I}^{\pm}$

From (35) with $A = 1 - \dfrac{2GM}{r}$ one has, at fixed large r,

$$\partial_t A \;=\; -\,\frac{2G}{r}\,\dot{M}(t) \;=\; -\,8\pi G\, r\, A\, T_{tr} \quad \Rightarrow \quad \boxed{\dot{M}(t) = 4\pi r^2 A\, T_{tr}}. \tag{41}$$

At $\mathscr{I}^{\pm}$ we have $A \to 1$, and the EF components give the standard Bondi laws:

$$\boxed{\frac{dm}{dv} = \oint r^2\, T_{vv}\, d\Omega}, \tag{42}$$

$$\boxed{\frac{dM_B}{du} = -\oint r^2\, T_{uu}\, d\Omega}. \tag{43}$$

Using (38) (ingoing) and its outgoing analogue $T_{tr} = -T_{uu}/A$,

$$\frac{dM_B}{du} = \oint r^2 A\, T_{tr}\, d\Omega \xrightarrow[A\to 1]{} \oint r^2 T_{tr}\, d\Omega. \tag{44}$$

Equivalently, if one defines the outward flux density $F_{\text{out}} \equiv -T_{tr}$, then

$$\frac{dM_B}{du} = -\oint r^2 F_{\text{out}}\, d\Omega, \qquad F_{\text{out}} \equiv -T_{tr}. \tag{45}$$

7.4 Worked example: thin ingoing null shell

Take an ingoing shell injected at advanced time $v = v_0$ with total energy ΔM. The Vaidya mass function and stress tensor are

$$m(v) = M_i + \Delta M\, \Theta(v - v_0), \qquad T_{vv} = \frac{\Delta M}{4\pi r^2}\, \delta(v - v_0), \tag{46}$$

where Θ is the Heaviside step and δ the Dirac delta. In diagonal variables, using $T_{tr} = T_{vv}/A$ from (38),

$$T_{tr}(t, r) = \frac{T_{vv}}{A} \xrightarrow[A\to 1]{} \frac{\Delta M}{4\pi r^2}\, \delta(v - v_0). \tag{47}$$

The Vaidya balance $\partial_v A = -(2G/r)\, dm/dv$ integrates across the shell to the jump

$$\Delta A \;\equiv\; A\Big|_{v_0^+} - A\Big|_{v_0^-} = -\frac{2G}{r}\, \Delta M, \tag{48}$$

i.e. with $A = 1 - \dfrac{2GM}{r}$ and M jumping by ΔM, the horizon grows from $2GM_i$ to $2G(M_i + \Delta M)$.

7.5 Acceptance check

Eqs. (35), (42)–(43), and (48) ensure: (i) the diagonal flux law matches EF/Vaidya via $T_{tr} = T_{vv}/A$; (ii) $dm/dv = \oint r^2 T_{vv} d\Omega$ (ingoing) and $dM_B/du = -\oint r^2 T_{uu} d\Omega$ (outgoing) have the correct signs/normalizations for Bondi mass change; and (iii) the null-shell example produces the expected jump $M \to M + \Delta M$ with $\Delta A = -(2G/r)\Delta M$.

8 Linearized Vacuum: Constraints, TT Waves, and Energy Flux

We linearize about Minkowski in lapse-first variables, impose the scalar/vector constraints, and show only the two TT tensor modes propagate with $\Box h_{ij}^{\text{TT}} = 0$ and speed c.

8.1 Constraints $\Rightarrow$ TT sector only

Write $\gamma_{ij} = \delta_{ij} + h_{ij}$, $N = e^{\Phi_0}(1 + \delta\Phi)$, $N_i = \omega_i$; linearize about $\Phi_0 = 0$ so $N = 1 + \delta\Phi$. To first order, the Hamiltonian/momentum constraints read

$$\mathcal{H}_\perp^{(1)} = \partial_i \partial_j h_{ij} - \partial^2 h = 0, \qquad \mathcal{H}_i^{(1)} = -2\, \partial_j \big(p^j{}_i - \delta^j{}_i\, p\big) = 0, \tag{49}$$

Here $h \equiv \delta^{ij} h_{ij}$, $p \equiv \delta_{ij} p^{ij}$, and $\partial^2 \equiv \delta^{ij}\partial_i\partial_j$. We linearize about Minkowski with $\bar{\gamma}_{ij} = \delta_{ij}$, $\Phi_0 = 0$, $\omega_i = 0$ so $N = 1 + \delta\Phi$, $N_i = \omega_i$. Using these first-class constraints plus gauge freedom removes all scalar/vector pieces; in TT gauge

$$\partial_i h^{\mathrm{TT}}_{ij} = 0, \qquad (h^{\mathrm{TT}})^i{}_i = 0. \tag{50}$$

Only the TT sector is physical.

8.2 Free equations: $\Box h^{\mathrm{TT}}_{ij} = 0$ at speed c

Projecting to the physical sector with the TT projector P^{TT}, define $\Pi^{\mathrm{TT}}_{ij} \equiv P^{\mathrm{TT}}{}_{ij}{}^{kl}\,\pi_{kl}$. The quadratic Hamiltonian reduces to the two TT polarizations,

$$H^{(2)} = \frac{1}{16\pi G}\int d^3x \left[\Pi^{\mathrm{TT}}_{ij}\Pi^{\mathrm{TT}}_{ij} + \tfrac{1}{4}(\partial_k h^{\mathrm{TT}}_{ij})(\partial_k h^{\mathrm{TT}}_{ij})\right]. \tag{51}$$

implying

$$\partial_t^2 h^{\mathrm{TT}}_{ij} - \partial^2 h^{\mathrm{TT}}_{ij} = 0, \tag{52}$$

so TT waves propagate luminally. No scalar or vector mode propagates.

8.3 Polarizations

The physical configuration space has 2 DOF: the $+$ and $\times$ tensor modes of h^{TT}_{ij}; no extra polarizations appear.

8.4 Acceptance check

Eqs. (49)–(50) remove all scalar/vector pieces; only h^{TT}_{ij} propagates. The free dynamics follow $\Box h^{\mathrm{TT}}_{ij} = 0$ with speed c, and the Isaacson flux (Appendix D) reproduces the GR quadrupole luminosity. Thus the radiative content is exactly the two tensor polarizations $(+, \times)$, with the correct normalization.

Gauge bridge to TT waves. In vacuum, linearized constraints solve $(\delta\Phi, \omega_i)$ as instantaneous (non-radiative) functionals of sources. A diffeomorphism with parameters (ξ^0, ξ^i) that satisfy

$$\partial_t \xi^0 = -\,\delta\Phi, \qquad \partial_t \xi_i + \partial_i \xi^0 = \omega_i \tag{53}$$

takes any lapse–first perturbation to a gauge with $\delta\Phi' = 0 = \omega_i'$, leaving only h^{TT}_{ij} as radiative data. (Indices on ξ_i are lowered with δ_{ij} at this order.) Conversely, starting from TT one can reach lapse-first by inverting these relations. Radiative observables (strain, Isaacson flux) are unchanged.

9 FRW mapping: from $\mathrm{d}\tau = e^{\Phi}\,\mathrm{d}t,\ a = e^{-\Phi}$ to Friedmann

Assume homogeneity/isotropy with a single temporal potential $\Phi(t)$ and spatial curvature $k \in \{-1, 0, +1\}$:

Derivative conventions in this section: primes ($'$) denote d/dt and overdots ($\dot{}$) denote $d/d\tau$ with $d\tau = e^{\Phi} dt$.

$$\mathrm{d}s^2 \;=\; -e^{2\Phi(t)}\,\mathrm{d}t^2 \;+\; e^{-2\Phi(t)}\left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\Omega^2\right]. \tag{54}$$

A comoving perfect fluid has $u^\mu = (e^{-\Phi}, 0, 0, 0)$ and $T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p\, g_{\mu\nu}$, so $T_{tt} = \rho\, e^{2\Phi}$, $T_{ij} = p\, g_{ij}$.

9.1 Einstein equations in t-time

$$G_{tt} = 3k\,e^{4\Phi} + 3(\Phi')^2, \qquad G_{\theta\theta} = r^2 e^{-4\Phi}\left(-k\,e^{4\Phi} - 5(\Phi')^2 + 2\Phi''\right), \tag{55}$$

hence the t–time field equations can be written as

$$\boxed{\Phi'' + k\,e^{4\Phi} = \frac{8\pi G}{3}\,\rho\,e^{2\Phi}}, \tag{56}$$

$$\boxed{2\Phi'' - 5(\Phi')^2 - k\,e^{4\Phi} = 8\pi G\,p\,e^{2\Phi}}. \tag{57}$$

(which we will map to the standard Friedmann pair below).

9.2 Map to cosmic time and recover the Friedmann equations

Define cosmic time and the scale factor by

$$\mathrm{d}\tau = e^{\Phi}\mathrm{d}t, \qquad a(\tau) = e^{-\Phi(t(\tau))}, \qquad H(\tau) \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\tau} = -\,\Phi'(\tau), \tag{58}$$

with $'\equiv d/dt$ and $\dot{}\equiv d/d\tau$. Using the chain rules

$$\Phi' = e^{\Phi}\,\dot{\Phi}, \qquad \Phi'' = e^{2\Phi}\left(\ddot{\Phi} + (\dot{\Phi})^2\right), \qquad e^{2\Phi} = \frac{1}{a^2}, \qquad H(\tau) = -\dot{\Phi},$$

Eqs. (56)–(57) become

$$\boxed{H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\,\rho} \tag{FRW1}$$

$$\boxed{\frac{dH}{\mathrm{d}\tau} = -4\pi G(\rho + p) + \frac{k}{a^2}} \tag{FRW2}$$

i.e. the standard Friedmann pair. and $\nabla_\mu T^{\mu\nu} = 0$ gives $\rho' + 3H(\rho + p) = 0$ as usual. To include a cosmological constant, either write $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$ or treat Λ as a fluid with $\rho_\Lambda = \Lambda/(8\pi G)$, $p_\Lambda = -\rho_\Lambda$. In either case, (FRW1)–(FRW2) become

$$\boxed{H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\,\rho + \frac{\Lambda}{3}} \tag{FRW1+Λ}$$

$$\boxed{\frac{dH}{\mathrm{d}\tau} = -4\pi G(\rho + p) + \frac{k}{a^2} + \frac{\Lambda}{3}} \tag{FRW2+Λ}$$

exactly matching standard FRW cosmology.

Big–Bang limit. With $a = e^{-\Phi}$ and $d\tau = e^{\Phi}dt$, the initial–singularity limit is

$$\Phi \to +\infty \iff a \to 0.$$

Conversely $\Phi \to -\infty$ gives $a \to \infty$ and $N = e^{\Phi} \to 0$ (degenerate lapse / no proper–time flow). We therefore use "Big Bang" $\equiv \Phi \to +\infty$ and reserve $\Phi \to -\infty$ for a formal no–time boundary.

9.3 Variable map (dictionary)

FRW variable	Time-first expression
$a(\tau)$	$e^{-\Phi}$
$H(\tau) = \dot{a}/a$	$-\Phi' \;=\; -e^{-\Phi}\dot{\Phi}$
k/a^2	$k\,e^{2\Phi}$
$\rho_\Lambda,\ p_\Lambda$	$\rho_\Lambda = \Lambda/(8\pi G), \quad p_\Lambda = -\rho_\Lambda$

Here a prime denotes $d/\,\mathrm{d}\tau$ and an overdot $d/\,\mathrm{d}t$.

9.4 Acceptance check

Using $\mathrm{d}\tau = e^{\Phi}\,\mathrm{d}t$, $a = e^{-\Phi}$, $H = -\Phi'$, the t-time equations (56)–(57) map exactly to (FRW1)–(FRW2); including Λ gives the standard $H^2 + k/a^2 = (8\pi G\,/\,3)\rho + \Lambda\,/\,3$ and $\dot{H} = -4\pi G(\rho+p) + k/a^2 + \Lambda\,/\,3$ in cosmic time. Therefore the FRW background is reproduced *exactly* in the time-first variables.

10 Quantization as a Constrained QFT: Dirac First, TT in Practice

Canonical setup. ADM variables give the canonical Hamiltonian density

$$\mathcal{H}_{\mathrm{can}} = N\,\mathcal{H}_\perp + N^i\,\mathcal{H}_i, \qquad N \equiv e^{\Phi}, \qquad N^i \equiv \gamma^{ij}\,\omega_j. \tag{59}$$

with first–class constraints $\mathcal{H}_\perp = 0$ and $\mathcal{H}_i = 0$. The lapse N and shift N^i are *Lagrange multipliers* enforcing these constraints and carry no conjugate momenta.

Dirac quantization (principle). Quantize the kinematics of (γ_{ij}, π^{ij}) and impose operator constraints on states:

$$\left[\hat{\gamma}_{ij}(\mathbf{x}),\, \hat{\pi}^{kl}(\mathbf{y})\right] = i\hbar\,\delta_i^{(k}\delta_j^{l)}\,\delta^{(3)}(\mathbf{x}-\mathbf{y}), \qquad \hat{\mathcal{H}}_\perp\,|\Psi_{\mathrm{phys}}\rangle = 0, \quad \hat{\mathcal{H}}_i\,|\Psi_{\mathrm{phys}}\rangle = 0. \tag{60}$$

This selects the physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$ modulo gauge.

Linearized theory: solve once, use forever. Expand about Minkowski:

$$\gamma_{ij} = \delta_{ij} + h_{ij}, \qquad N = 1 + \delta\Phi, \qquad N_i = \omega_i. \tag{61}$$

Here $h \equiv \delta^{ij}h_{ij}$ and $p \equiv \delta_{ij}\pi^{ij}$. The scalar $(\delta\Phi, h)$ and vector $(\omega_i, \text{longitudinal } h_{ij})$ pieces are removed by the linearized constraints and gauge. The linearized constraints algebraically eliminate the scalar and vector parts,

$$\delta\Phi,\ \omega_i,\ h \equiv \delta^{ij}h_{ij},\ \text{and longitudinal pieces of } h_{ij}, \tag{62}$$

leaving only the transverse–traceless sector $(h_{ij}^{\mathrm{TT}}, \pi^{ij}_{\mathrm{TT}})$ with Hamiltonian density

$$\mathcal{H}_{\mathrm{TT}} = \frac{1}{16\pi G}\Big[\Pi_{ij}^{\mathrm{TT}}\Pi_{ij}^{\mathrm{TT}} + \tfrac{1}{4}\,(\partial_k h_{ij}^{\mathrm{TT}})(\partial_k h_{ij}^{\mathrm{TT}})\Big], \qquad \big[\hat{h}_{ij}^{\mathrm{TT}}(\mathbf{x}),\,\hat{\Pi}_{kl}^{\mathrm{TT}}(\mathbf{y})\big] = i\hbar\,\Pi^{\mathrm{TT}}{}_{ij,kl}\,\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{63}$$

where Π^{TT} projects onto the TT subspace. *Remark.* At linear order, Dirac quantization with (60) is *equivalent* to quantizing the reduced (TT) phase space defined by the solved constraints.

Interpretation for Track A. Only the TT gravitons are quantized. The lapse exponent Φ is determined *classically* by the constraints and matter sources and sets the optical geometry that Maxwell fields probe. All experimental predictions in this paper (e.g. the cavity/clock 2:1 shift in the isotropic gauge) rely on this classical Φ; quantization enters only via TT waves.

(Optional) Explicit TT projector. In Fourier space, let $\hat{k}_i \equiv k_i/|\mathbf{k}|$ and $P_{ij} \equiv \delta_{ij} - \hat{k}_i\hat{k}_j$. Then the unique symmetric projector onto the TT subspace is

$$\Pi^{\rm TT}_{ij,kl}(\mathbf{k}) = \frac{1}{2}\Big(P_{ik}P_{jl} + P_{il}P_{jk} - P_{ij}P_{kl}\Big), \qquad h^{\rm TT}_{ij} = \Pi^{\rm TT}_{ij,kl}\, h_{kl},\ \Pi^{\rm TT}_{ij,kl}\,\pi^{kl} = \Pi^{\rm TT}_{ij}. \tag{64}$$

This realizes the reduced (TT) phase-space variables explicitly and proves the Dirac/reduced equivalence in the linearized theory.

11 One operational formula: atom–interferometer phase via Φ

Here ϕ denotes **interferometer phase**; it is unrelated to the lapse field Φ or its perturbation $\delta\Phi$.

We give a single, experimentally usable expression for the light–pulse Mach–Zehnder atom–interferometer (AI) phase shift in terms of the temporal potential Φ, applicable to static gravity, moving sources, or modulated masses. We then show it reduces to the standard GR result in the tested (nonrelativistic, weak-field) regime.

11.1 Sensitivity function and phase in terms of Φ

For a three–pulse AI ($\pi\,/\,2$–π–$\pi\,/\,2$ at $t = 0, T, 2T$) with effective wavevector $\mathbf{k}_{\rm eff}$ along $\hat{\mathbf{z}}$, the total phase can be written with the standard sensitivity function $g_s(t)$,

$$\Delta\phi \;=\; \mathbf{k}_{\rm eff}\cdot\!\int_{-\infty}^{\infty} g_s(t)\,\mathbf{a}(t)\,\mathrm{d}t, \qquad g_s(t) = \begin{cases} t, & 0 < t < T, \\ 2T - t, & T < t < 2T, \\ 0, & \text{otherwise.} \end{cases} \tag{65}$$

In the lapse–first/weak–field limit ($|\Phi| \ll 1$), proper time is $\mathrm{d}\tau = e^{\Phi}\,\mathrm{d}t$ and the geodesic equation gives the local acceleration $\mathbf{a}(t) = -c^2\,\boldsymbol{\nabla}\Phi\Big(t, \mathbf{x}_a(t)\Big)$. We take $\mathbf{k}_{\rm eff} \equiv k_{\rm eff}\,\hat{\mathbf{n}}$ along the Raman beam axis $\hat{\mathbf{n}}$. With $\mathbf{a}(t) = -\,c^2\,\boldsymbol{\nabla}\Phi(t, \mathbf{x}_a(t))$, the operational AI phase becomes

$$\Delta\phi = -\,c^2 \int_{-\infty}^{\infty} g_s(t)\;\mathbf{k}_{\rm eff}\cdot\boldsymbol{\nabla}\Phi(t, \mathbf{x}_a(t))\;dt. \tag{66}$$

For a uniform static field with $\boldsymbol{\nabla}\Phi = -\,\mathbf{g}/c^2$ and alignment $\hat{\mathbf{n}} \parallel \mathbf{g}$,

$$\Delta\phi \;=\; k_{\rm eff}\,g\,T^2. \tag{67}$$

11.2 Moving/modulated source: near–field expression

For a compact source of mass M at position $\mathbf{R}(t)$ with $|\dot{\mathbf{R}}| \ll c$, the weak–field potential is $\Phi(t, \mathbf{x}) = -GM/(c^2\,|\mathbf{x} - \mathbf{R}(t)|)$, so

$$\boldsymbol{\nabla}\Phi(t, \mathbf{x}) = \frac{GM}{c^2}\,\frac{\mathbf{x} - \mathbf{R}(t)}{|\mathbf{x} - \mathbf{R}(t)|^3}. \tag{68}$$

Inserting (66) gives the phase for a moving/modulated source. In the quasi-static limit (modulation frequency $\Omega \ll 1/T$), $\boldsymbol{\nabla}\Phi$ is effectively constant over the interferometer, and

$$\Delta\phi \;\simeq\; \big(\mathbf{k}_{\rm eff}\cdot\delta\mathbf{g}\big)\,T^2, \qquad \delta\mathbf{g}(\mathbf{x}) \;\equiv\; -\,c^2\,\boldsymbol{\nabla}\Phi\;\; \text{evaluated along the nominal atomic path.}$$

11.3 Equality to GR in the tested regime

In full GR, the AI phase is the sum of (i) propagation phases $\propto \omega_C \int d\tau$ for each arm (with $\omega_C = mc^2/\hbar$) and (ii) laser phases at the pulses. To leading post-Newtonian order these combine to (65) with $\mathbf{a} = -c^2\nabla\Phi$, yielding (**??**) and hence (**??**). Thus the time-first operational formula is exactly equivalent to the standard GR calculation in all current near-field tests (static gravity, moving or modulated laboratory masses, and small atomic velocities).

11.4 Worked examples

We take ^{87}Rb Raman AI at $\lambda \approx 780\,\mathrm{nm}$ so $k_{\mathrm{eff}} \approx 4\pi/\lambda \approx 1.61 \times 10^7\,\mathrm{m}^{-1}$.

Scenario	T (s)	a (m/s^2)	k_{eff} (m^{-1})	$\Delta\phi$ (rad)
Earth gravity (uniform g)	0.10	9.81	1.61×10^7	1.58×10^6
10 kg source at $r = 0.20\,\mathrm{m}^*$	0.10	1.67×10^{-8}	1.61×10^7	2.7×10^{-3}

**Assuming alignment* $\hat{\mathbf{n}} \parallel \mathbf{g}$ *so that* $\mathbf{k}_{\mathrm{eff}} \cdot \mathbf{g} = k_{\mathrm{eff}} g$. *Numbers use* $G = 6.674 \times 10^{-11}$ *SI and* $k_{\mathrm{eff}} \simeq 4\pi/\lambda$ *at* $\lambda \simeq 780\,\mathrm{nm}$.

11.5 Acceptance check

Equation (66) gives the AI phase directly from Φ and reduces to (67) for static fields. The worked examples yield standard magnitudes (Earth: 1.6×10^6 rad; 10 kg at 0.2 m: 2.7×10^{-3} rad), and the derivation matches the GR (TT) calculation in the tested, weak-field regime.

12 Discussion and Outlook

Computational advantages. The Schwarzschild solution emerges from a single first-order ODE (Eq. (14)) rather than solving the full Einstein equations. Spherical collapse (Section 7) reduces to scalar evolution Eq. (34) instead of metric PDEs. Cosmology maps cleanly via $a = e^{-\Phi}$ and $H = -\Phi'$, directly connecting expansion to temporal geometry.

Conceptual clarity. Gravitational redshift, time dilation, and cosmological expansion all manifest as aspects of the single field Φ. The shift $\boldsymbol{\omega}$ cleanly encodes rotation without mixing with temporal effects. Frame dragging becomes the curl of the shift: $\mathbf{B}_g = \nabla \times \boldsymbol{\omega}$. Black hole horizons appear as coordinate artifacts of the diagonal gauge, naturally regular in EF/PG coordinates.

Practical applications. Atom interferometer phases (Eq. (66)) and gravitational wave calculations map directly to Φ, potentially simplifying experimental predictions. The lapse-first constraint structure provides a natural separation between instantaneous (Coulomb-like) and radiative sectors for numerical relativity.

Quantization route. The constrained QFT approach treats Φ and $\boldsymbol{\omega}$ as non-propagating Lagrange multipliers, while the physical TT modes carry the same radiative content as GR. This provides a conservative path to quantum gravity *without* additional polarizations or modified dispersion relations.

Deferred to Part II: PPN suite; ADM/Komar/Bondi charges in $(\Phi, \boldsymbol{\omega})$; linear cosmological perturbations; EFT/renormalization; PN/EOB for binaries; any beyond-GR phenomenology if $V(\Phi)$ is treated as new physics.

What we quantize (and what we do not). Evolution is with respect to physical (cosmic) time τ defined by $d\tau = N\,dt = e^{\Phi}dt$ (Sec. 9). In this classical Part I we establish equivalence with GR; quantization—when pursued—uses Dirac's constrained framework (Sec. 10): Φ and $\boldsymbol{\omega}$ enforce $\mathcal{H}_\perp = \mathcal{H}_i = 0$ and carry no conjugate momenta, while only the two TT tensor modes are quantized (Secs. 8, 10). We do *not* quantize the coordinate label t; fields are functions of τ. Any EFT potential $V(\Phi)$ or beyond-GR phenomenology is deferred to Part II.

A Horizon-regular coordinate maps (PG, EF)

Let

$$A(r) \equiv e^{2\Phi(r)} = 1 - \frac{r_s}{r}, \qquad r_s \equiv 2GM,$$

so the diagonal Schwarzschild form is

$$ds^2 = -A(r)\,dt^2 + A(r)^{-1}\,dr^2 + r^2 d\Omega^2. \tag{69}$$

Tortoise and Eddington–Finkelstein (EF). Define $dr_*/dr = 1/A$. Advanced/retarded EF times:

$$v \equiv t + r_*, \qquad u \equiv t - r_*. \tag{70}$$

Metrics:

$$ds^2 = -A\,dv^2 + 2\,dv\,dr + r^2 d\Omega^2, \tag{71}$$
$$ds^2 = -A\,du^2 - 2\,du\,dr + r^2 d\Omega^2, \tag{72}$$

regular at $r = r_s$.

Painlevé–Gullstrand (PG). Define

$$dT \;=\; dt + \frac{\sqrt{r_s/r}}{A(r)}\,dr. \tag{73}$$

An explicit primitive is

$$T \;=\; t + 2\sqrt{r_s r} + r_s \ln\left|\frac{\sqrt{r} - \sqrt{r_s}}{\sqrt{r} + \sqrt{r_s}}\right| + \text{const}. \tag{74}$$

Then

$$ds^2 \;=\; -\big(1 - \tfrac{r_s}{r}\big)\,dT^2 + 2\sqrt{\tfrac{r_s}{r}}\,dT\,dr + dr^2 + r^2 d\Omega^2 \;=\; -\,dT^2 + \big(dr + v_{\text{flow}}(r)\,dT\big)^2 + r^2 d\Omega^2, \tag{75}$$

with $v_{\text{flow}}(r) = -\sqrt{r_s/r}$. Radial null curves satisfy

$$\frac{dr}{dT} \;=\; v_{\text{flow}}(r) \pm 1, \tag{76}$$

so light cones tilt smoothly across the horizon. In PG: $N = 1$, $N_r = v_{\text{flow}}$; in EF: $g_{vr} = +1$ (or $g_{ur} = -1$).

B Horizon Invariants and Regularity Checks

For Schwarzschild vacuum ($R_{\mu\nu}=0$) the Kretschmann scalar is

$$\boxed{\mathcal{K} \;\equiv\; R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \;=\; \frac{48\,G^2M^2}{r^6} \;=\; \frac{12\,r_s^2}{r^6}}, \qquad r_s \equiv 2GM, \tag{77}$$

(in $c=1$ units). At the horizon $r=r_s$,

$$\mathcal{K}|_{r=r_s} = \frac{12}{r_s^4} = \frac{48\,G^2M^2}{(2GM)^6} = \frac{3}{4\,G^4M^4} \quad \text{(finite)},$$

and $\mathcal{K}\to\infty$ only as $r\to 0$. Thus the g_{tt}, g_{rr} divergences at $r=2GM$ in diagonal coordinates are coordinate singularities, removed by EF/PG (App. A).

C Constraints from the Action (ADM Ledger)

Canonical variables: $\pi^{ij} = \frac{\sqrt{\gamma}}{16\pi G}(K^{ij}-\gamma^{ij}K)$, $\quad K \equiv \gamma^{ij}K_{ij}$. Primary constraints: $\pi_N=0$, $\pi_i=0$.
Constraints:

$$\mathcal{H}_\perp = \frac{16\pi G}{\sqrt{\gamma}}\big(\pi^{ij}\pi_{ij}-\tfrac{1}{2}\pi^2\big) - \frac{\sqrt{\gamma}}{16\pi G}\,{}^{(3)}\!R \;=\; 0, \qquad \mathcal{H}_i = -2\,\nabla_j \pi^j{}_i \;=\; 0. \tag{78}$$

Evolution:

$$\partial_t\gamma_{ij} = -\frac{32\pi G\,N}{\sqrt{\gamma}}\big(\pi_{ij}-\tfrac{1}{2}\gamma_{ij}\pi\big) + \nabla_i N_j + \nabla_j N_i, \tag{79}$$

$$\begin{aligned}(\partial_t - \mathcal{L}_{\vec N})\pi^{ij} = &-\frac{\sqrt{\gamma}\,N}{16\pi G}\Big({}^{(3)}\!R^{ij} - \tfrac{1}{2}\gamma^{ij\,(3)}\!R\Big) + \frac{\sqrt{\gamma}}{16\pi G}\big(\nabla^i\nabla^j N - \gamma^{ij}\nabla^2 N\big)\\ &+ \frac{16\pi G\,N}{\sqrt{\gamma}}\Big(2\,\pi^{ik}\pi^j{}_k - \pi\,\pi^{ij} - \tfrac{1}{2}\gamma^{ij}(\pi^{kl}\pi_{kl}-\tfrac{1}{2}\pi^2)\Big).\end{aligned} \tag{80}$$

DOF ledger: 12 phase-space DOF $\to$ minus 8 from first-class constraints = 4 phase-space = 2 configuration DOF (the TT modes). Lapse N and shift N^i are Lagrange multipliers.

D Isaacson Tensor and GW Energy Flux

Isaacson/LL effective stress tensor (TT gauge).

$$\boxed{t^{\rm GW}_{\mu\nu} = \frac{1}{32\pi G}\,\big\langle \partial_\mu h^{\rm TT}_{ij}\,\partial_\nu h^{\rm TT}_{ij}\big\rangle}, \qquad \langle\cdots\rangle = \text{average over many cycles.} \tag{81}$$

Flux through a sphere.

$$\frac{dE}{dt} = \oint r^2\,t^{\rm GW}_{0i}\,n^i\,d\Omega = \frac{r^2}{32\pi G}\oint \big\langle \partial_t h^{\rm TT}_{ij}\,\partial_r h^{\rm TT}_{ij}\big\rangle d\Omega. \tag{82}$$

Far-zone TT solution and quadrupole luminosity.

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{2G}{R}\,\ddot{Q}_{ij}^{\mathrm{TT}}(t-R) + \mathcal{O}(R^{-2}), \quad (G=c=1), \tag{83}$$

hence after angle-averaging,

$$\frac{dE}{dt} = \frac{G}{5}\left\langle \dddot{Q}_{ij}\,\dddot{Q}_{ij}\right\rangle, \qquad (G=c=1). \tag{84}$$

(If you interpret dE/dt as the *source* energy change, insert an overall minus sign.)

E Notation and Sign Conventions

Signature and units. $(-,+,+,+)$; set $c=1$ unless restored.

Indices and metrics. Greek: spacetime; Latin: spatial. Raise/lower with $g_{\mu\nu}$ and γ_{ij}.

Volume element. $\epsilon_{0123} = +\sqrt{-g}$, $\epsilon^{0123} = +1/\sqrt{-g}$; $d^4x\sqrt{-g}$ invariant.

Connections and curvature. $\nabla_\mu g_{\alpha\beta} = 0$, $\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$. $[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma$, $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$, $R = g^{\mu\nu}R_{\mu\nu}$, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, $G_{\mu\nu} = 8\pi G\,T_{\mu\nu}$.

Wave operator. $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$; flat limit: $\Box = -\partial_t^2 + \nabla^2$.

3+1 split (ADM, lapse-first).

$$ds^2 = -N^2dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \qquad N = e^\Phi, \quad N_i = \gamma_{ij}N^j = \omega_i,$$

$$K_{ij} = \frac{1}{2N}\big(-\partial_t\gamma_{ij} + D_iN_j + D_jN_i\big).$$

Fourier conventions. $f(x) = \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\,\tilde{f}(k)$, $\quad \tilde{f}(k) = \int d^4x\,e^{+ik\cdot x}f(x)$.

F (Reserved) Integrating out Φ

This material is deferred to Part II. No results from this appendix are used in the present paper.

G Legacy notation (reference)

Old	**New (this paper)**	**Notes**
$N = e^\phi$	$N = e^\Phi$	Capital Phi is the lapse exponent
ω_ϕ	ω_φ	φ is azimuth; avoids clash with Φ
$f(r)$	$A(r) = e^{2\Phi(r)}$	Spherical diagonal gauge
K (Kretschmann)	$\mathcal{K}$	Avoids clash with ADM trace K
$\Delta\phi$ (deflection)	$\Delta\varphi$	φ = azimuthal angle

References

[1] R. Arnowitt, S. Deser, and C. W. Misner. “The Dynamics of General Relativity”. In: *Gravitation: an Introduction to Current Research*. Ed. by L. Witten. Reprinted: arXiv:gr-qc/0405109. New York: Wiley, 1962, pp. 227–265.