Breaking Bones

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1 Introduction

What are Bones? Bones are living tissues that have their own blood vessel and are made up of living cells which help them to grow. Bones provide important functions that organisms rely on, such as protection of key vital organs. Initially, humans at birth are born with 300 soft bones [1]. As humans develop, soft bones are eventually replaced with hard bone. The human skeleton itself is consist of 206 bones. Everyday, the human body goes through environmental strains. As a result, cells normally die out but are reproduced. Bones in the same way are no exception to this case. In this paper, we will be observing how certain factors can influence a bones structural integrity. Based on our observation so far, bones fractures typically occur as a result of old age. However, we are more interested in seeing how a bones behave given some sudden amount of force that can either be relatively small or even significantly large. Depending on the specific degree of severity, we want to specifically investigate points before (some elasticity) and after (plasticity) a bone fracture has occurred. Ultimately, we hope that this paper will help people gain a better understanding of bone structure and how it behaves under a certain amount of force.

2 Problem Statement

A human bone (specifically a femur) is placed on two support points. The distance between the two end supports is defined by L (mm). Another point that is situated above the bone will apply the load (F (N)) on the bone itself. Initially, the bone is positioned at an undeformed state (zero displacement) and the loading point will make contact with the bone using a small pre-load (less than 1 N) value in order to keep the bone in place. Eventually, the loading point will then move progressively downward with increasing applied load and displacement (d).

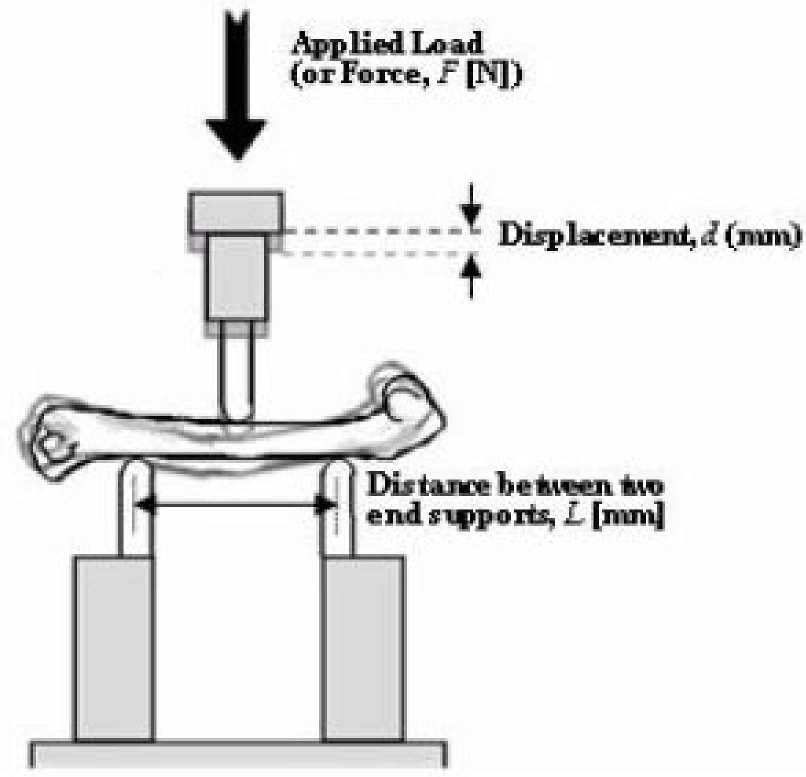


Figure: Three-Point Bending Test [2]

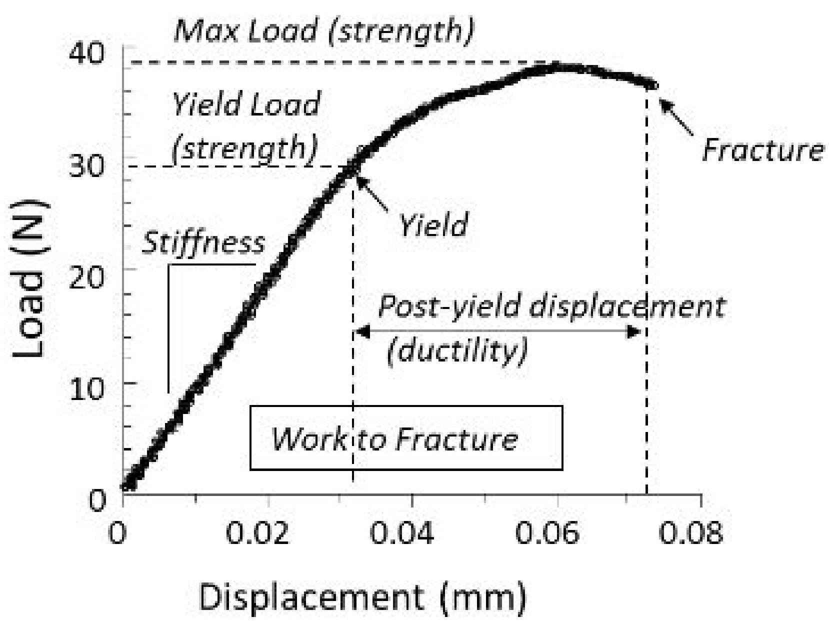
Assuming that the age range of the femur from is 22 - 40.

Unfortunately, there was not a significant amount of data found for this system. Data sets are in short supply, as ideal bones are not always available. Experiments cannot use dry or salvaged bones, because they are more brittle and do not represent a living bone. Therefore, we took to research online to determine what occurs in a system like this and what factors will contribute to bone strength.

2.1 Literature Review

2.1.1 Three-Point Bending

Even though, we did not have the actual resources to conduct the experiment we had to think of a way where we can mimic a bone fracture in a controlled setting. We felt that it was really important to go through the process and to get a better understanding of the significance of how we apply the topics that we learned in class to other situations. [2] Silva's paper on three-point testing, really appealed to us the most when it came to this project. Even though Silva uses mice in the research, we felt that three point testing experiment is still a good way to mimic bone fractures. Silva's paper goes into detail regarding a bones structural (mechanical) properties as it undergoes some gradual amount of force. Silva talks about certain properties that influences bone fractures such as stiffness, yield load, maximum load, post-yield displacement, and work-to-fracture.Stiffness according to Silva, is a measure of the resistance offered by the whole bone to the applied displacement during the elastic region and is analogous to a simple spring constant (K) from physics. Yield load, measures how much load a bone can sustain before it suffers permanent damage. Essentially, it is one measure of a bones strength. Maximum load, is a simple measure of whole bones strength. Post-yield displacement (PYD) measures ductility of a bone. Work-to-fracture represents the work that needs to be to in order to fracture the bone. Besides properties, the paper provides test data from Silva's experiment with different mice of specific age ranges.

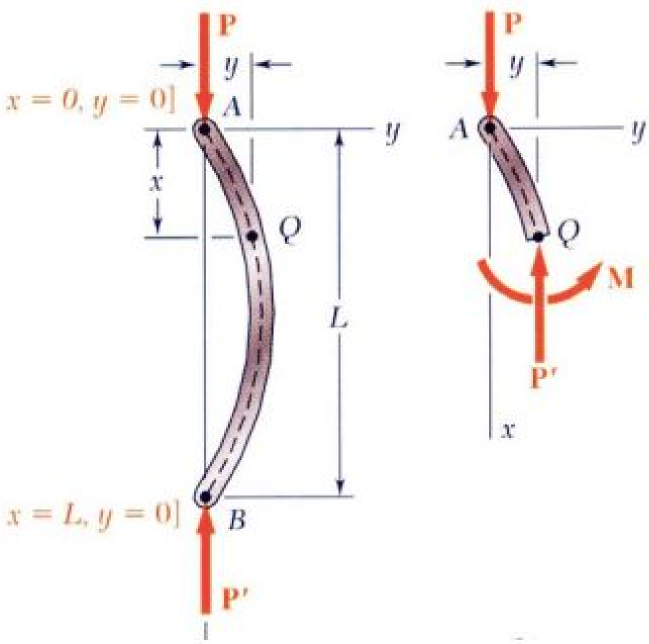


<u>Figure: Displacement vs Load</u> [1]

The above figure gives measures of important, but not necessarily independent, bone properties. Since the figure shows a specific case, it would be beneficial to study how the behavior of this graph changes by simply looking at changes in density of the bone. By studying change in density we have an idea how age itself affects the bone. Since bone density is a parameter we can control via nutrition and exercise, we can potentially maximize the work to fracture (area under the curve above). It is also worth checking how ductility may change if the strength and stiffness are increased.

2.1.2 Euler Buckling

Buckling is the failure of a structure to remain static under a load [4]. Euler buckling finds the critical load for something to buckle. When it comes to bones buckling, they break when they are under a load equal to the critical buckling load. Buckling is also a bifurcation of the system of a bone under load in a state of static equilibrium. The following diagram shows the system and governing differential equation.



$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}y$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

Figure: Euler Buckling with two hinge points.

Where P = load the bone is under, E = modulus, I = minimum area moment of inertia.

The solution to this second order differential equation will give the force which is equal to the critical load which will break the bone, this is Euler's critical load which is described as,

$$P_{critical} = \frac{EI\pi^2}{L^2}$$, L = length of the bone

P critical is a force and it can be compared to the force derived from our dimensional analysis.

One important concept in the Euler buckling system is the second moment of inertia, in which describes the smallest area of the cross section of the object. At the smallest point of the bone, it will have the smallest area, and the smallest “disk” around the centroid of the bone will have the most stress applied to it.

$$\mathcal{J} = \begin{matrix} \frac{1}{4}\pi a^4 & 0 \\ 0 & \frac{1}{4}\pi a^4 \end{matrix}$$

Figure: Matrix defining the calculations for second moment of inertia

2.1.3 Hooke’s Law

During the research process, we considered a spring as a model with its constant and its elasticity. The bone breaking model acts in many ways like a spring, but with a high elastic modulus. To simplify many of the parameters we can assume that the bone acts like a single mass with no hollow points. We will use Hooke’s Law on the bone to being a single linear spring to develop a model for elasticity, length of change, tensile strength(yield strength), force, and cross-sectional area.[10]

We will define the rod that has length *L* and cross-sectional area *A*. Its tensile stress *σ* is linearly proportional to its fractional extension or strain *ε* by the modulus of elasticity *E*:

$$\sigma = E\varepsilon$$

The modulus of elasticity may often be considered constant. In turn,

$$\varepsilon = \frac{\Delta L}{L}$$

(that is, the fractional change in length), and since

$$\sigma = \frac{F}{A}$$

it follows that:

$$\frac{\sigma}{E} = \frac{F}{AE}$$

The change in length may be expressed as

$$\sigma = \frac{F}{A}$$

This work was done by individuals and scientist throughout the years and thus will be used to derive specific equations later on in this paper. [11]

Yield point(strength) is something very difficult to find without using actual data as the bases for reference. Thus, we attempted to find a relationship between the tensile stress and elasticity of Isotropic material (uniform properties in all directions). To determine the yield point we used the total strain energy theory, credited to Beltrami-Haigh, as a way to calculate the total energy required to pass the yield point. [11]

The total strain energy theory (Beltrami-Haigh) allows us to create a simple equation for energy with Elasticity and tensile strength as parameters. [12]

$$U_t = \frac{1}{2}E[\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\ ]$$

U_t = total strain energy

We can use this power law equation to form a log-log plot of the tensile strength to elasticity modulus.

$$\sigma_y = C(\varepsilon')^m$$

$$log(\sigma_y) \ = \ log(C) + mlog(\varepsilon')$$

However, due to time restraints for completion of the project and the late information found on a new path, we were unable to model a proper equation and graph.

3 Analysis

3.1 Dimensional Analysis

In order to get a better understanding of bone fractures, performed some dimensional analysis on quantities that we believe are important factors. Here we can, see how each of these quantities really do compare and how they would relate.

We performed dimensional analysis on both three-point bending and Euler buckling.

3.1.1a Three-Point Bending

Cross-Sectional Area: The cross-sectional area of where a force is applied to determines how much bone is within a certain point. The size of the bone determines how much force would need to be applied for fracture.

$A = [L^2]$

Bone Density: the density of a material plays a role in how flexible the object can be. The more dense an object is, the more force it will need to bend or break it.

$\rho = [ML^{-3}]$

Poisson Ratio (PR): A ratio, PR is great indicator in judging a bones structure. More specifically, a bones material property (brittle or ductile) By how much times does an object decontract (compress) perpendicular to y-direction, where in the x direction the bone is being pushed to compensate for the force.

$V = -(\varepsilon \text{ lateral} / \varepsilon \text{ longitudinal})$

$\varepsilon = \Delta L/L$

Where (ε) is the strain and is **Δ**L is the difference between the new length and the original length. One thing to note about Poisson Ratio is that it is already dimensionless as is.

mm = [L]

$V = -[L]/[L] = 1$

Young's Modulus: A measure of stiffness of a material. Young's Modulus was the work of Thomas Young, after Euler's initial concept several decades prior. The quantity describes the relationship between tensile stress (σ) and extensional strain (Ɛ). The resultant unit of the quantity is a measure of pressure.

$Y = \sigma / \varepsilon = [ML^{-1}T^{-2}]$

Length of Bone: Depending on the size, the amount of force required to fracture a bone can vary. Typically, longer bone lengths would require a lesser amount of force compared to a smaller bone length.

$\ell = [L]$

Force: Force applied to the bone is one of the principal quantities we must discuss. In our dimensional analysis, we try to find quantities that can directly calculate this value.

$F = ma = [MLT^{-2}]$

Dimensional Analysis:
Considering all the quantities above, this was our first DA that we worked on.

$$[\, A^a \rho^b V^c Y^d \ell^e F^f t^g \,] = L^{2a} * M^b L^{-3b} * M^d L^{-d} T^{-2d} * L^e * M^f L^f T^{-2f} * T^g$$

M: $b + d + f = 0$
L: $2a - 3b - d + e + f = 0$
T: $-2d - 2f + g = 0$

$b = -d - f$
$a = -d - 2f - e/2$
$g = 2d + 2f$

$$A^{a-d-2f-e/2} \rho^{-d-f} V^c Y^d \ell^e F^f t^{2d+2f}$$

π1 $= (Yt^2/A\rho)^d$, **π2** $= (Ft^2/A^2\rho)^f$, **π3** $= (\ell/A^{1/2})^d$, **π4** $= (V)^c$

By Buckingham Theorem
F(**π1 , π2 , π3 , π4)** = 0

So, π2 = G(π1 , π3 , π4)

$$(Ft^2/A^2\rho)^f = G(\, Yt^2/A\rho,\ \ell/A^{1/2},\ V)$$

$F = A^2\rho * G(Yt^2/A\rho, \ell /A^{1/2}, V) / t^2$

The results are as follows:

Force = $(A^2\rho / t^2) * \mathbf{G(Yt^2/A\rho, \ell /A^{1/2}, V)}$

This tells us some things:

- Force is dependent on area and density
- Over a longer time, it's possible the result from function G will outweigh the other quantities.
- Young's Modulus snuck its way in there because it describes the stiffness of the bone and its elasticity to the force. Bone is expected to have a higher value of Young's Modulus than, say, a rubber band, as it requires significantly more force to bend a bone than a rubber band.
- Work is related to force, as it is simply an applied force in a direction. This equation of force helps since work can be derived from it.
    - A possible equation for work might be derived if we include some displacement as a quantity

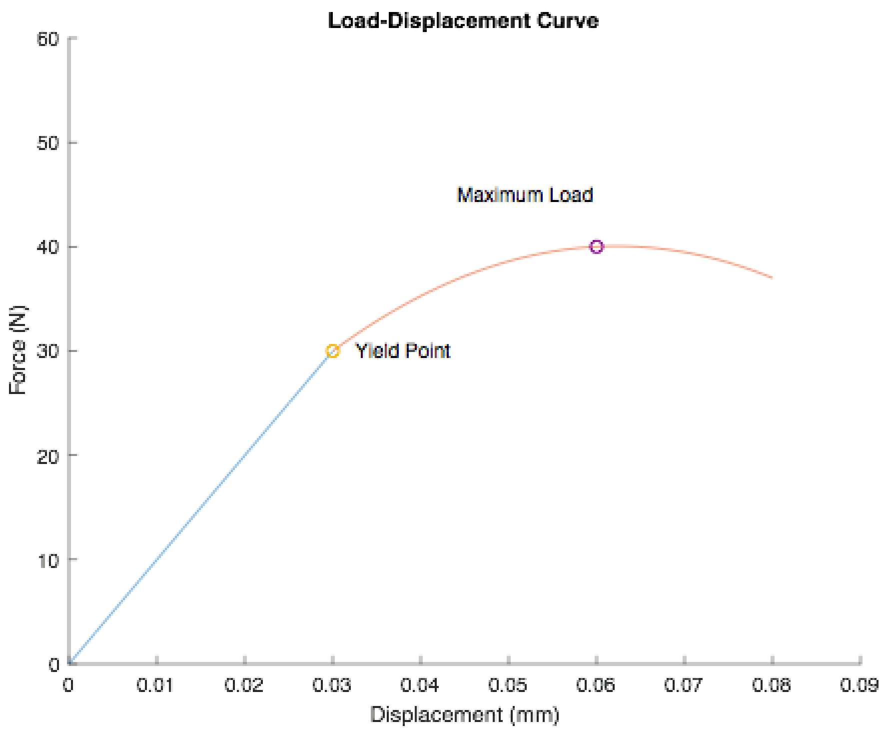


Figure: Work-To-Fracture graph

The above graph describes a few key moments in a three-point bending model. Firstly, the **yield point** describes the point at which permanent deformation will occur [9]. Before this point, a bone will return to its original form. After this point, the bone deforms plastically and non-linearly this is the point at which the material begins to break, causing microfractures. **Maximum Load** describes the point at which a bone will fail, and therefore fracture [9].

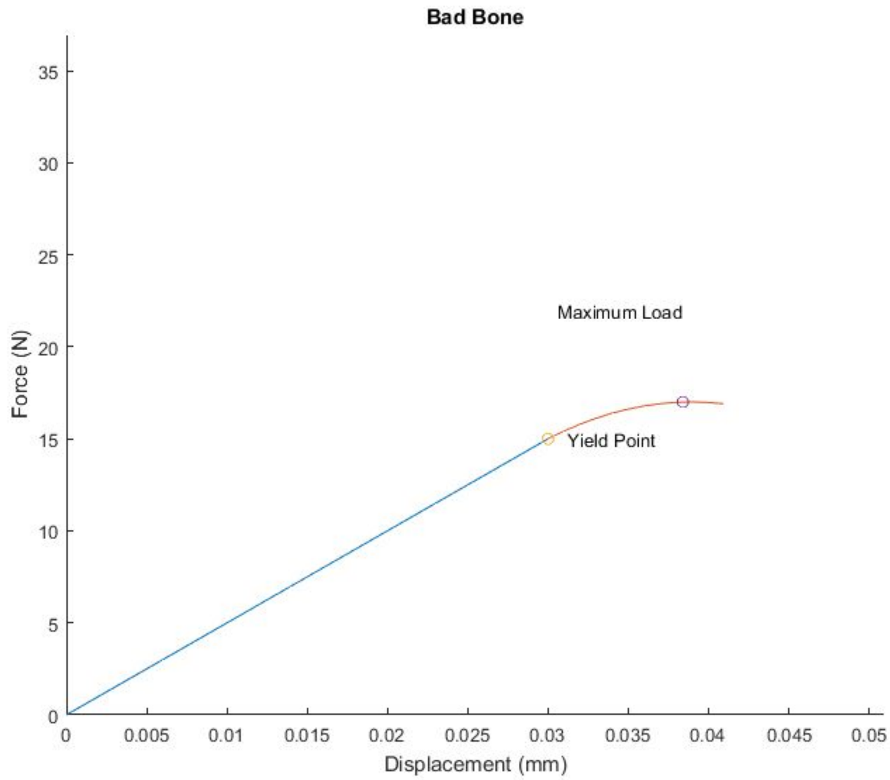


Figure: Bad Bone with a small PYD and is inelastic.

3.1.1b Euler Buckling

There are two major equations to consider during Euler Buckling. Firstly, there should be an equation that describes the amount of force to cause displacement, and secondly an equation to describe the amount of stress applied to the bone after the displacement occurs.

Important assumptions that will affect results:

- Bone is assumed to be a uniform, cylindrical shape. Its smallest cross-sectional area determines the radius.
- Bone is assumed to have a uniform density throughout the cylinder.
- The cortical bone solely determines its strength

Important Factors are:

Young's Modulus $Y = [ML^{-1}T^{-2}]$

Length of Bone $\ell = [L]$

Radius of Bone $r = [L]$

Force Applied $F = [MLT^{-2}]$

Minimum Second (Area) Moment of Inertia: Second moment of inertia describes the area

$I = [L^4]$

Dimensional Analysis

$$[Y^a \ell^b r^c F^d I^e] = M^a L^{-a} T^{-2a} * L^b * L^c * M^d L^d T^{-2d} * L^{4e} = 1$$

M: $a + d = 0$
L: $-a + b + c + d + 4e = 0$
T: $-2a - 2d = 0$

$a=-d$
$b=-2d-c-4e$
$c=c$
$d=d$
$e=e$

$$Y^{-d} l^{-2d-c-4e} r^c F^d I^e$$

$$\pi_1 = l^{-c} r^c = l^{-1} r \ , \pi_2 = Y^{-d} l^{-2d} D^d = Y^{-1} l^{-2} F \ , \pi_3 = l^{-4e} I^e = l^{-4} I$$

By Buckingham Theorem

$$F(\pi_1, \pi_2, \pi_3) \ = \ 0$$

Thus

$$\pi_2 = G(\pi_1, \pi_3) \ \rightarrow \ \frac{F}{Y l^2} = G(\frac{r}{l}, \frac{I}{l^4})$$

After performing dimensional analysis, we could not find correct equations for either of the following Euler buckling equations:

Critical Load

$$F = \frac{\pi^2 EI}{(KL)^2}$$

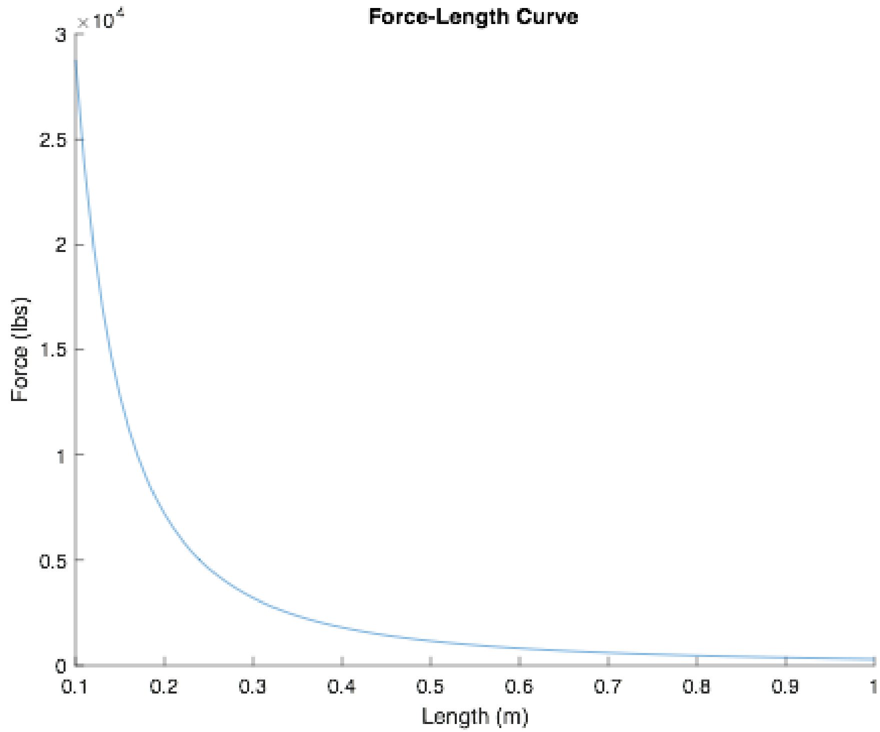
Quantity	Definition
I	Moment of Inertia
E	Young's Modulus
L	Length of Object
F	Force Applied
K	A constant that changes based on how the object is held.

Stress

$$\sigma = \frac{F}{A} = \frac{\pi^2 E}{\left(\frac{\ell}{r}\right)^2}$$

Quantity	Definition
σ	Stress
E	Young's Modulus
ℓ	Length of Object
r	Radius of Object
A	Area in which stress is applied

Figure:



Force-Length curve

In this graph, we can see that small objects require more force to buckle, and that large objects require increasingly smaller force to buckle. The line itself represents the maximum load, or the amount of force for fracture.

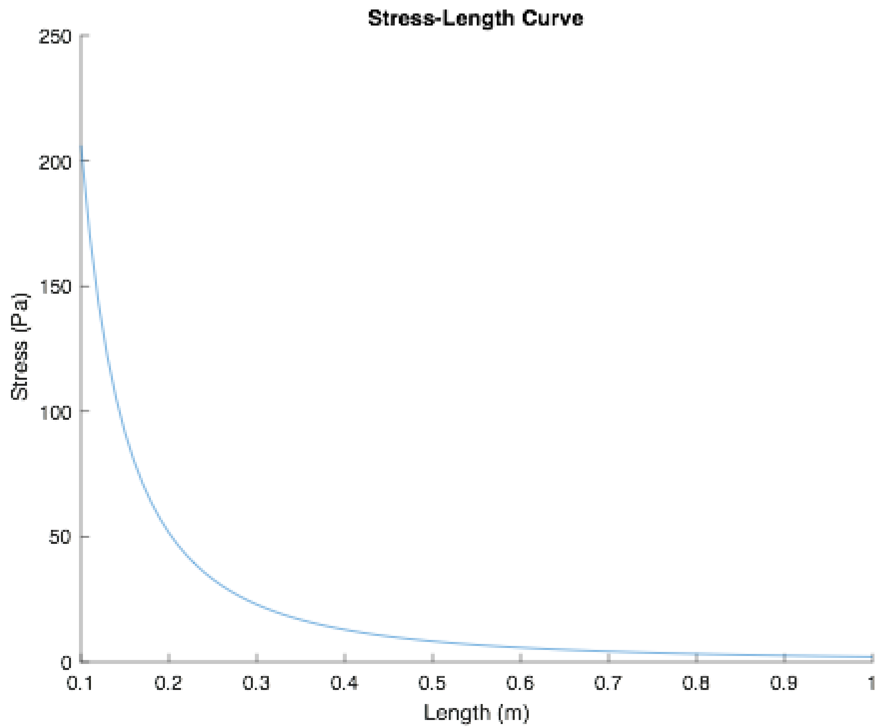


Figure: Stress-Length curve.
This figure describes the relationship between length and amount of stress applied to an object. As with the amount of force applied, the object will experience a different stress depending on length. Small objects experience more stress, bigger objects experience less. The line itself represents the maximum load, or the amount of force for fracture.

4 Conclusions

Overall, we took a mathematical approach to modeling bone fractures. Based on both the information we gathered and the calculations that were made, we at least have some better idea of what goes on in. Specifically, a better idea of fractures via modeling.

Based on what we gathered online, we learned that there is more of a likely chance for a bone to fracture considering different factors. Old age is a major factor in bone fragility [7]. This is likely due to low bone density as age progresses. This also explains why older individuals are more likely to suffer severe bone trauma from a fall. Another factor is bone density itself [8]. If the bone is less dense, the easier it is to bend the material in a direction. Some materials have a naturally-high elastic modulus (Young's Modulus) that resists bending and buckling, but density still plays a major role. Another factor for fractures is the angle of the applied force. Bone collagen are tiny fibers that exist within the bone, and face a certain direction [6]. On one hand, if force is applied to the bone perpendicular to the collagen fibers, it will require more force to fracture the bone. On the other hand, if the force is applied near-parallel to the fibers, the bone requires significantly less force to fracture.

Lastly, something that we need to keep in mind is that bone cells have to restructure themselves every year that is a result of general growth. As a result, the rate at which bones go through physical strain on top of restructuring increases the likelihood for bones to fracture even more [13].

4.1 Future Work To Consider

Since we only covered a small amount of the potential systems to describe in bone fractures, there are other systems that we should consider for later models. For example, there is 4-point bending that describes an area-force applied to a bone instead of a point. Another example is rotational torque applied to a bone. Previous trauma also affects bone strength and would provide information about why sports injuries can cause future ones.

The systems described here can easily apply to other materials, given the correct elastic modulus. This would be helpful to mechanical and civil engineers. Another step would be to investigate the relationship between stiffness, yield load, and maximum load. Small scale models will also be interesting, and finding solutions for engineers could reside here.

5 References

[1] Fun Facts About Bones and Joints, (2011), Retrieve from http://www.bidmc.org/YourHealth/Health-Notes/Bones-and-Joints/Bone-and-Joint-Basics/Bone-and-Joint-Facts.aspx

[2] M.J Silva. Bone Mechanical Testing by Three-Point Bending, Washington University Musculoskeletal Structure and Strength Core, (2016).

[3] Lakes, R. What is Poisson's Ratio?, (2011). Retrieved from http://silver.neep.wisc.edu/~lakes/PoissonIntro.html

[4] Euler Buckling. Wikipedia (accessed 2017). Retrieved from https://en.wikipedia.org/wiki/Buckling

[5] Area Moments of Inertia. WolframAlpha (accessed 2017). Retrieved from http://mathworld.wolfram.com/AreaMomentofInertia.html

[6] The Skeletal System. Powerpoint (accessed 2017) Retrieved from http://www2.sunysuffolk.edu/kennym/Ch7.pdf

[7] Aging and Bone. National Center for Biotechnology Information (2010). Retrieved from https://www.ncbi.nlm.nih.gov/pmc/articles/PMC2991386/

[8] Skeletal fragility definition. National Center for Biotechnology Information (2011). Retrived from https://www.ncbi.nlm.nih.gov/pmc/articles/PMC3279077/

[9] Strength of Materials Basics and Equations | Mechanics of Materials Equations. EngineersEdge.com (accessed 2017). Retrieved from https://www.engineersedge.com/strength_of_materials.htm

[10] Hooke's law. Wikipedia (accessed 2017). Retrieved from https://en.wikipedia.org/wiki/Hooke%27s_law

[11] Yield(engineering). Wikipedia (accessed 2017). Retrieved from https://en.wikipedia.org/wiki/Yield_(engineering)

[12] Total Strain Energy Theory (Beltrami-Haigh). Powerpoint (accessed 2017). Retrieved from http://uetmmmksk.weebly.com/uploads/3/6/0/0/3600114/theories_of_failure.pdf

[13] A Mathematical Model of Stress Fracture in Human Bone. Powerpoint (2003). Retrieve from https://www.csb.pitt.edu/BBSI/2003/presentation/sandvik.pdf