SOLUTIONS MANUAL

Communication Systems Engineering

Second Edition

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Contents

Chapter 2

Problem 2.1
1)

$$
\begin{aligned}
\epsilon^2 &= \int_{-\infty}^{\infty} \left| x(t) - \sum_{i=1}^{N} \alpha_i \phi_i(t) \right|^2 dt \\
&= \int_{-\infty}^{\infty} \left(x(t) - \sum_{i=1}^{N} \alpha_i \phi_i(t) \right) \left(x^*(t) - \sum_{j=1}^{N} \alpha_j^* \phi_j^*(t) \right) dt \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} \phi_i(t) x^*(t) dt - \sum_{j=1}^{N} \alpha_j^* \int_{-\infty}^{\infty} \phi_j^*(t) x(t) dt \\
&\quad + \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j^* \int_{-\infty}^{\infty} \phi_i(t) \phi_j^* dt \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt + \sum_{i=1}^{N} |\alpha_i|^2 - \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} \phi_i(t) x^*(t) dt - \sum_{j=1}^{N} \alpha_j^* \int_{-\infty}^{\infty} \phi_j^*(t) x(t) dt
\end{aligned}
$$

Completing the square in terms of α_i we obtain

$$
\epsilon^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \left| \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2 + \sum_{i=1}^{N} \left| \alpha_i - \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2
$$

The first two terms are independent of α's and the last term is always positive. Therefore the minimum is achieved for

$$
\alpha_i = \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt
$$

which causes the last term to vanish.

2) With this choice of α_i's

$$
\begin{aligned}
\epsilon^2 &= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \left| \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2 \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} |\alpha_i|^2
\end{aligned}
$$

Problem 2.2
1) The signal $x_1(t)$ is periodic with period $T_0 = 2$. Thus

$$
\begin{aligned}
x_{1,n} &= \frac{1}{2} \int_{-1}^{1} \Lambda(t) e^{-j2\pi \frac{n}{2} t} dt = \frac{1}{2} \int_{-1}^{1} \Lambda(t) e^{-j\pi n t} dt \\
&= \frac{1}{2} \int_{-1}^{0} (t+1) e^{-j\pi n t} dt + \frac{1}{2} \int_{0}^{1} (-t+1) e^{-j\pi n t} dt \\
&= \frac{1}{2} \left(\frac{j}{\pi n} t e^{-j\pi n t} + \frac{1}{\pi^2 n^2} e^{-j\pi n t} \right) \Bigg|_{-1}^{0} + \frac{j}{2\pi n} e^{-j\pi n t} \Bigg|_{-1}^{0} \\
&\quad - \frac{1}{2} \left(\frac{j}{\pi n} t e^{-j\pi n t} + \frac{1}{\pi^2 n^2} e^{-j\pi n t} \right) \Bigg|_{0}^{1} + \frac{j}{2\pi n} e^{-j\pi n t} \Bigg|_{0}^{1} \\
&\quad \frac{1}{\pi^2 n^2} - \frac{1}{2\pi^2 n^2} (e^{j\pi n} + e^{-j\pi n}) = \frac{1}{\pi^2 n^2} (1 - \cos(\pi n))
\end{aligned}
$$

When $n = 0$ then

$$x_{1,0} = \frac{1}{2}\int_{-1}^{1} \Lambda(t)dt = \frac{1}{2}$$

Thus

$$x_1(t) = \frac{1}{2} + 2\sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2}(1 - \cos(\pi n))\cos(\pi n t)$$

2) $x_2(t) = 1$. It follows then that $x_{2,0} = 1$ and $x_{2,n} = 0$, $\forall n \neq 0$.

3) The signal is periodic with period $T_0 = 1$. Thus

$$\begin{aligned}
x_{3,n} &= \frac{1}{T_0}\int_0^{T_0} e^t e^{-j2\pi n t}dt = \int_0^1 e^{(-j2\pi n+1)t}dt \\
&= \frac{1}{-j2\pi n + 1}e^{(-j2\pi n+1)t}\Big|_0^1 = \frac{e^{(-j2\pi n+1)} - 1}{-j2\pi n + 1} \\
&= \frac{e-1}{1 - j2\pi n} = \frac{e-1}{\sqrt{1 + 4\pi^2 n^2}}(1 + j2\pi n)
\end{aligned}$$

4) The signal $\cos(t)$ is periodic with period $T_1 = 2\pi$ whereas $\cos(2.5t)$ is periodic with period $T_2 = 0.8\pi$. It follows then that $\cos(t) + \cos(2.5t)$ is periodic with period $T = 4\pi$. The trigonometric Fourier series of the even signal $\cos(t) + \cos(2.5t)$ is

$$\begin{aligned}
\cos(t) + \cos(2.5t) &= \sum_{n=1}^{\infty} \alpha_n \cos(2\pi\frac{n}{T_0}t) \\
&= \sum_{n=1}^{\infty} \alpha_n \cos(\frac{n}{2}t)
\end{aligned}$$

By equating the coefficients of $\cos(\frac{n}{2}t)$ of both sides we observe that $a_n = 0$ for all n unless $n = 2, 5$ in which case $a_2 = a_5 = 1$. Hence $x_{4,2} = x_{4,5} = \frac{1}{2}$ and $x_{4,n} = 0$ for all other values of n.

5) The signal $x_5(t)$ is periodic with period $T_0 = 1$. For $n = 0$

$$x_{5,0} = \int_0^1 (-t+1)dt = (-\frac{1}{2}t^2 + t)\Big|_0^1 = \frac{1}{2}$$

For $n \neq 0$

$$\begin{aligned}
x_{5,n} &= \int_0^1 (-t+1)e^{-j2\pi n t}dt \\
&= -\left(\frac{j}{2\pi n}te^{-j2\pi n t} + \frac{1}{4\pi^2 n^2}e^{-j2\pi n t}\right)\Big|_0^1 + \frac{j}{2\pi n}e^{-j2\pi n t}\Big|_0^1 \\
&= -\frac{j}{2\pi n}
\end{aligned}$$

Thus,

$$x_5(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\sin 2\pi n t$$

6) The signal $x_6(t)$ is periodic with period $T_0 = 2T$. We can write $x_6(t)$ as

$$x_6(t) = \sum_{n=-\infty}^{\infty} \delta(t - n2T) - \sum_{n=-\infty}^{\infty} \delta(t - T - n2T)$$

$$
\begin{aligned}
&= \frac{1}{2T}\sum_{n=-\infty}^{\infty} e^{j\pi\frac{n}{T}t} - \frac{1}{2T}\sum_{n=-\infty}^{\infty} e^{j\pi\frac{n}{T}(t-T)} \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{2T}(1-e^{-j\pi n})e^{j2\pi\frac{n}{2T}t}
\end{aligned}
$$

However, this is the Fourier series expansion of $x_6(t)$ and we identify $x_{6,n}$ as

$$
x_{6,n} = \frac{1}{2T}(1-e^{-j\pi n}) = \frac{1}{2T}(1-(-1)^n) = \begin{cases} 0 & n \text{ even} \\ \frac{1}{T} & n \text{ odd} \end{cases}
$$

7) The signal is periodic with period T. Thus,

$$
\begin{aligned}
x_{7,n} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta'(t)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{1}{T}(-1)\frac{d}{dt}e^{-j2\pi\frac{n}{T}t}\Big|_{t=0} = \frac{j2\pi n}{T^2}
\end{aligned}
$$

8) The signal $x_8(t)$ is real even and periodic with period $T_0 = \frac{1}{2f_0}$. Hence, $x_{8,n} = a_{8,n}/2$ or

$$
\begin{aligned}
x_{8,n} &= 2f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t)\cos(2\pi n 2f_0 t)dt \\
&= f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+2n)t)dt + f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-2n)t)dt \\
&= \frac{1}{2\pi(1+2n)}\sin(2\pi f_0(1+2n)t)\Big|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-2n)}\sin(2\pi f_0(1-2n)t)\Big|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\
&= \frac{(-1)^n}{\pi}\left[\frac{1}{(1+2n)} + \frac{1}{(1-2n)}\right]
\end{aligned}
$$

9) The signal $x_9(t) = \cos(2\pi f_0 t) + |\cos(2\pi f_0 t)|$ is even and periodic with period $T_0 = 1/f_0$. It is equal to $2\cos(2\pi f_0 t)$ in the interval $[-\frac{1}{4f_0}, \frac{1}{4f_0}]$ and zero in the interval $[\frac{1}{4f_0}, \frac{3}{4f_0}]$. Thus

$$
\begin{aligned}
x_{9,n} &= 2f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t)\cos(2\pi n f_0 t)dt \\
&= f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+n)t)dt + f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-n)t)dt \\
&= \frac{1}{2\pi(1+n)}\sin(2\pi f_0(1+n)t)\Big|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-n)}\sin(2\pi f_0(1-n)t)\Big|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\
&= \frac{1}{\pi(1+n)}\sin(\frac{\pi}{2}(1+n)) + \frac{1}{\pi(1-n)}\sin(\frac{\pi}{2}(1-n))
\end{aligned}
$$

Thus $x_{9,n}$ is zero for odd values of n unless $n = \pm 1$ in which case $x_{9,\pm 1} = \frac{1}{2}$. When n is even $(n = 2l)$ then

$$
x_{9,2l} = \frac{(-1)^l}{\pi}\left[\frac{1}{1+2l} + \frac{1}{1-2l}\right]
$$

Problem 2.3

It follows directly from the uniqueness of the decomposition of a real signal in an even and odd part. Nevertheless for a real periodic signal

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(2\pi\frac{n}{T_0}t) + b_n \sin(2\pi\frac{n}{T_0}t)\right]$$

The even part of $x(t)$ is

$$\begin{aligned}
x_e(t) &= \frac{x(t)+x(-t)}{2} \\
&= \frac{1}{2}\left(a_0 + \sum_{n=1}^{\infty} a_n(\cos(2\pi\frac{n}{T_0}t) + \cos(-2\pi\frac{n}{T_0}t))\right. \\
&\qquad \left. + b_n(\sin(2\pi\frac{n}{T_0}t) + \sin(-2\pi\frac{n}{T_0}t))\right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi\frac{n}{T_0}t)
\end{aligned}$$

The last is true since $\cos(\theta)$ is even so that $\cos(\theta)+\cos(-\theta) = 2\cos\theta$ whereas the oddness of $\sin(\theta)$ provides $\sin(\theta)+\sin(-\theta) = \sin(\theta) - \sin(\theta) = 0$.

The odd part of $x(t)$ is

$$\begin{aligned}
x_o(t) &= \frac{x(t)-x(-t)}{2} \\
&- \sum_{n=1}^{\infty} b_n \sin(2\pi\frac{n}{T_0}t)
\end{aligned}$$

Problem 2.4

a) The signal is periodic with period T. Thus

$$\begin{aligned}
x_n &= \frac{1}{T}\int_0^T e^{-t}e^{-j2\pi\frac{n}{T}t}dt = \frac{1}{T}\int_0^T e^{-(j2\pi\frac{n}{T}+1)t}dt \\
&= -\frac{1}{T\left(j2\pi\frac{n}{T}+1\right)}e^{-(j2\pi\frac{n}{T}+1)t}\Big|_0^T = -\frac{1}{j2\pi n + T}\left[e^{-(j2\pi n+T)} - 1\right] \\
&= \frac{1}{j2\pi n+T}[1-e^{-T}] = \frac{T-j2\pi n}{T^2+4\pi^2n^2}[1-e^{-T}]
\end{aligned}$$

If we write $x_n = \frac{a_n - jb_n}{2}$ we obtain the trigonometric Fourier series expansion coefficients as

$$a_n = \frac{2T}{T^2+4\pi^2n^2}[1-e^{-T}], \qquad b_n = \frac{4\pi n}{T^2+4\pi^2n^2}[1-e^{-T}]$$

b) The signal is periodic with period $2T$. Since the signal is odd we obtain $x_0 = 0$. For $n \neq 0$

$$\begin{aligned}
x_n &= \frac{1}{2T}\int_{-T}^{T} x(t)e^{-j2\pi\frac{n}{2T}t}dt = \frac{1}{2T}\int_{-T}^{T}\frac{t}{T}e^{-j2\pi\frac{n}{2T}t}dt \\
&= \frac{1}{2T^2}\int_{-T}^{T} te^{-j\pi\frac{n}{T}t}dt \\
&= \frac{1}{2T^2}\left(\frac{jT}{\pi n}te^{-j\pi\frac{n}{T}t} + \frac{T^2}{\pi^2n^2}e^{-j\pi\frac{n}{T}t}\right)\Big|_{-T}^{T} \\
&= \frac{1}{2T^2}\left[\frac{jT^2}{\pi n}e^{-j\pi n} + \frac{T^2}{\pi^2n^2}e^{-j\pi n} + \frac{jT^2}{\pi n}e^{j\pi n} - \frac{T^2}{\pi^2n^2}e^{j\pi n}\right] \\
&= \frac{j}{\pi n}(-1)^n
\end{aligned}$$

The trigonometric Fourier series expansion coefficients are:

$$a_n = 0, \qquad b_n = (-1)^{n+1}\frac{2}{\pi n}$$

c) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{3}{2}$$

If $n \neq 0$ then

$$\begin{aligned}
x_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{-\frac{T}{2}}^{\frac{T}{2}} + \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{-\frac{T}{4}}^{\frac{T}{4}} \\
&= \frac{j}{2\pi n}\left[e^{-j\pi n} - e^{j\pi n} + e^{-j\pi\frac{n}{2}} - e^{-j\pi\frac{n}{2}}\right] \\
&= \frac{1}{\pi n}\sin(\pi\frac{n}{2}) = \frac{1}{2}\mathrm{sinc}(\frac{n}{2})
\end{aligned}$$

Note that $x_n = 0$ for n even and $x_{2l+1} = \frac{1}{\pi(2l+1)}(-1)^l$. The trigonometric Fourier series expansion coefficients are:

$$a_0 = 3, \quad , a_{2l} = 0, \quad , a_{2l+1} = \frac{2}{\pi(2l+1)}(-1)^l, \quad , b_n = 0, \ \forall n$$

d) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{2}{3}$$

If $n \neq 0$ then

$$\begin{aligned}
x_n &= \frac{1}{T}\int_0^T x(t)e^{-j2\pi\frac{n}{T}t}dt = \frac{1}{T}\int_0^{\frac{T}{3}} \frac{3}{T}te^{-j2\pi\frac{n}{T}t}dt \\
&\quad + \frac{1}{T}\int_{\frac{T}{3}}^{\frac{2T}{3}} e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{\frac{2T}{3}}^{T}(-\frac{3}{T}t + 3)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{3}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2n^2}e^{-j2\pi\frac{n}{T}t}\right)\Big|_0^{\frac{T}{3}} \\
&\quad - \frac{3}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2n^2}e^{-j2\pi\frac{n}{T}t}\right)\Big|_{\frac{2T}{3}}^{T} \\
&\quad + \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{\frac{T}{3}}^{\frac{2T}{3}} + \frac{3}{T}\frac{jT}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{\frac{2T}{3}}^{T} \\
&= \frac{3}{2\pi^2n^2}[\cos(\frac{2\pi n}{3}) - 1]
\end{aligned}$$

The trigonometric Fourier series expansion coefficients are:

$$a_0 = \frac{4}{3}, \quad a_n = \frac{3}{\pi^2n^2}[\cos(\frac{2\pi n}{3}) - 1], \quad b_n = 0, \ \forall n$$

e) The signal is periodic with period T. Since the signal is odd $x_0 = a_0 = 0$. For $n \neq 0$

$$\begin{aligned}
x_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{4}} -e^{-j2\pi\frac{n}{T}t}dt \\
&+\frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \frac{4}{T}te^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{4}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2n^2}e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{-\frac{T}{4}}^{\frac{T}{4}} \\
&-\frac{1}{T}\left(\frac{jT}{2\pi n}e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{-\frac{T}{2}}^{-\frac{T}{4}} + \frac{1}{T}\left(\frac{jT}{2\pi n}e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{\frac{T}{4}}^{\frac{T}{2}} \\
&= \frac{j}{\pi n}\left[(-1)^n - \frac{2\sin(\frac{\pi n}{2})}{\pi n}\right] = \frac{j}{\pi n}\left[(-1)^n - \text{sinc}(\frac{n}{2})\right]
\end{aligned}$$

For n even, $\text{sinc}(\frac{n}{2}) = 0$ and $x_n = \frac{j}{\pi n}$. The trigonometric Fourier series expansion coefficients are:

$$a_n = 0, \ \forall n, \quad b_n = \begin{cases} -\frac{1}{\pi l} & n = 2l \\ \frac{2}{\pi(2l+1)}[1 + \frac{2(-1)^l}{\pi(2l+1)}] & n = 2l+1 \end{cases}$$

f) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_{-\frac{T}{3}}^{\frac{T}{3}} x(t)dt = 1$$

For $n \neq 0$

$$\begin{aligned}
x_n &= \frac{1}{T}\int_{-\frac{T}{3}}^{0}(\frac{3}{T}t+2)e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{0}^{\frac{T}{3}}(-\frac{3}{T}t+2)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{3}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2n^2}e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{-\frac{T}{3}}^{0} \\
&-\frac{3}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2n^2}e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{0}^{\frac{T}{3}} \\
&+\frac{2}{T}\frac{jT}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Bigg|_{-\frac{T}{3}}^{0} + \frac{2}{T}\frac{jT}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Bigg|_{0}^{\frac{T}{3}} \\
&= \frac{3}{\pi^2n^2}\left[\frac{1}{2} - \cos(\frac{2\pi n}{3})\right] + \frac{1}{\pi n}\sin(\frac{2\pi n}{3})
\end{aligned}$$

The trigonometric Fourier series expansion coefficients are:

$$a_0 = 2, \quad a_n = 2\left[\frac{3}{\pi^2n^2}\left(\frac{1}{2} - \cos(\frac{2\pi n}{3})\right) + \frac{1}{\pi n}\sin(\frac{2\pi n}{3})\right], \quad b_n = 0, \ \forall n$$

Problem 2.5

1) The signal $y(t) = x(t - t_0)$ is periodic with period $T = T_0$.

$$\begin{aligned}
y_n &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t-t_0)e^{-j2\pi\frac{n}{T_0}t}dt \\
&= \frac{1}{T_0}\int_{\alpha-t_0}^{\alpha-t_0+T_0} x(v)e^{-j2\pi\frac{n}{T_0}}(v+t_0)dv \\
&= e^{-j2\pi\frac{n}{T_0}t_0}\frac{1}{T_0}\int_{\alpha-t_0}^{\alpha-t_0+T_0} x(v)e^{-j2\pi\frac{n}{T_0}v}dv \\
&= x_ne^{-j2\pi\frac{n}{T_0}t_0}
\end{aligned}$$

where we used the change of variables $v = t - t_0$

2) For $y(t)$ to be periodic there must exist T such that $y(t + mT) = y(t)$. But $y(t + T) = x(t + T)e^{j2\pi f_0 t}e^{j2\pi f_0 T}$ so that $y(t)$ is periodic if $T = T_0$ (the period of $x(t)$) and $f_0 T = k$ for some k in $\mathcal{Z}$. In this case

$$\begin{aligned} y_n &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}e^{j2\pi f_0 t}dt \\ &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{(n-k)}{T_0}t}dt = x_{n-k} \end{aligned}$$

3) The signal $y(t)$ is periodic with period $T = T_0/\alpha$.

$$\begin{aligned} y_n &= \frac{1}{T}\int_{\beta}^{\beta+T} y(t)e^{-j2\pi\frac{n}{T}t}dt = \frac{\alpha}{T_0}\int_{\beta}^{\beta+\frac{T_0}{\alpha}} x(\alpha t)e^{-j2\pi\frac{n\alpha}{T_0}t}dt \\ &= \frac{1}{T_0}\int_{\beta\alpha}^{\beta\alpha+T_0} x(v)e^{-j2\pi\frac{n}{T_0}v}dv = x_n \end{aligned}$$

where we used the change of variables $v = \alpha t$.

4)

$$\begin{aligned} y_n &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x'(t)e^{-j2\pi\frac{n}{T_0}t}dt \\ &= \frac{1}{T_0}x(t)e^{-j2\pi\frac{n}{T_0}t}\Big|_{\alpha}^{\alpha+T_0} - \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0}(-j2\pi\frac{n}{T_0})e^{-j2\pi\frac{n}{T_0}t}dt \\ &= j2\pi\frac{n}{T_0}\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}dt = j2\pi\frac{n}{T_0}x_n \end{aligned}$$

Problem 2.6

$$\begin{aligned} \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)y^*(t)dt &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0}\sum_{n=-\infty}^{\infty} x_n e^{\frac{j2\pi n}{T_0}t}\sum_{m=-\infty}^{\infty} y_m^* e^{-\frac{j2\pi m}{T_0}t}dt \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^* \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} e^{\frac{j2\pi(n-m)}{T_0}t}dt \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^* \delta_{mn} = \sum_{n=-\infty}^{\infty} x_n y_n^* \end{aligned}$$

Problem 2.7

Using the results of Problem 2.6 we obtain

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)x^*(t)dt = \sum_{n=-\infty}^{\infty}|x_n|^2$$

Since the signal has finite power

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0}|x(t)|^2 dt = K < \infty$$

Thus, $\sum_{n=-\infty}^{\infty}|x_n|^2 = K < \infty$. The last implies that $|x_n| \to 0$ as $n \to \infty$. To see this write

$$\sum_{n=-\infty}^{\infty}|x_n|^2 = \sum_{n=-\infty}^{-M}|x_n|^2 + \sum_{n=-M}^{M}|x_n|^2 + \sum_{n=M}^{\infty}|x_n|^2$$

Each of the previous terms is positive and bounded by K. Assume that $|x_n|^2$ does not converge to zero as n goes to infinity and choose $\epsilon = 1$. Then there exists a subsequence of x_n, x_{n_k}, such that

$$|x_{n_k}| > \epsilon = 1, \qquad \text{for } n_k > N \geq M$$

Then

$$\sum_{n=M}^{\infty} |x_n|^2 \geq \sum_{n=N}^{\infty} |x_n|^2 \geq \sum_{n_k} |x_{n_k}|^2 = \infty$$

This contradicts our assumption that $\sum_{n=M}^{\infty} |x_n|^2$ is finite. Thus $|x_n|$, and consequently x_n, should converge to zero as $n \to \infty$.

Problem 2.8

The power content of $x(t)$ is

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt$$

But $|x(t)|^2$ is periodic with period $T_0/2 = 1$ so that

$$P_x = \frac{2}{T_0} \int_0^{T_0/2} |x(t)|^2 dt = \frac{2}{3T_0} t^3 \Big|_0^{T_0/2} = \frac{1}{3}$$

From Parseval's theorem

$$P_x = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |x_n|^2 = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

For the signal under consideration

$$a_n = \begin{cases} -\frac{4}{\pi^2 n^2} & \text{n odd} \\ 0 & \text{n even} \end{cases} \qquad b_n = \begin{cases} -\frac{2}{\pi n} & \text{n odd} \\ 0 & \text{n even} \end{cases}$$

Thus,

$$\begin{aligned} \frac{1}{3} &= \frac{1}{2} \sum_{n=1}^{\infty} a^2 + \frac{1}{2} \sum_{n=1}^{\infty} b^2 \\ &= \frac{8}{\pi^4} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} + \frac{2}{\pi^2} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} \end{aligned}$$

But,

$$\sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} = \frac{\pi^2}{8}$$

and by substituting this in the previous formula we obtain

$$\sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} = \frac{\pi^4}{96}$$

Problem 2.9

1) Since $(a-b)^2 \geq 0$ we have that

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

with equality if $a = b$. Let

$$A = \left[\sum_{i=1}^{n} \alpha_i^2\right]^{\frac{1}{2}}, \qquad B = \left[\sum_{i=1}^{n} \beta_i^2\right]^{\frac{1}{2}}$$

Then substituting α_i/A for a and β_i/B for b in the previous inequality we obtain

$$\frac{\alpha_i}{A}\frac{\beta_i}{B} \leq \frac{1}{2}\frac{\alpha_i^2}{A^2} + \frac{1}{2}\frac{\beta_i^2}{B^2}$$

with equality if $\frac{\alpha_i}{\beta_i} = \frac{A}{B} = k$ or $\alpha_i = k\beta_i$ for all i. Summing both sides from $i = 1$ to n we obtain

$$\begin{aligned}
\sum_{i=1}^{n} \frac{\alpha_i \beta_i}{AB} &\leq \frac{1}{2}\sum_{i=1}^{n}\frac{\alpha_i^2}{A^2} + \frac{1}{2}\sum_{i=1}^{n}\frac{\beta_i^2}{B^2} \\
&= \frac{1}{2A^2}\sum_{i=1}^{n}\alpha_i^2 + \frac{1}{2B^2}\sum_{i=1}^{n}\beta_i^2 = \frac{1}{2A^2}A^2 + \frac{1}{2B^2}B^2 = 1
\end{aligned}$$

Thus,

$$\frac{1}{AB}\sum_{i=1}^{n}\alpha_i\beta_i \leq 1 \Rightarrow \sum_{i=1}^{n}\alpha_i\beta_i \leq \left[\sum_{i=1}^{n}\alpha_i^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}\beta_i^2\right]^{\frac{1}{2}}$$

Equality holds if $\alpha_i = k\beta_i$, for $i = 1, \ldots, n$.

2) The second equation is trivial since $|x_i y_i^*| = |x_i||y_i^*|$. To see this write x_i and y_i in polar coordinates as $x_i = \rho_{x_i}e^{j\theta_{x_i}}$ and $y_i = \rho_{y_i}e^{j\theta_{y_i}}$. Then, $|x_i y_i^*| = |\rho_{x_i}\rho_{y_i}e^{j(\theta_{x_i}-\theta_{y_i})}| = \rho_{x_i}\rho_{y_i} = |x_i||y_i| = |x_i||y_i^*|$. We turn now to prove the first inequality. Let z_i be any complex with real and imaginary components $z_{i,R}$ and $z_{i,I}$ respectively. Then,

$$\begin{aligned}
\left|\sum_{i=1}^{n} z_i\right|^2 &= \left|\sum_{i=1}^{n} z_{i,R} + j\sum_{i=1}^{n} z_{i,I}\right|^2 = \left(\sum_{i=1}^{n} z_{i,R}\right)^2 + \left(\sum_{i=1}^{n} z_{i,I}\right)^2 \\
&= \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}z_{m,R} + z_{i,I}z_{m,I})
\end{aligned}$$

Since $(z_{i,R}z_{m,I} - z_{m,R}z_{i,I})^2 \geq 0$ we obtain

$$(z_{i,R}z_{m,R} + z_{i,I}z_{m,I})^2 \leq (z_{i,R}^2 + z_{i,I}^2)(z_{m,R}^2 + z_{m,I}^2)$$

Using this inequality in the previous equation we get

$$\begin{aligned}
\left|\sum_{i=1}^{n} z_i\right|^2 &= \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}z_{m,R} + z_{i,I}z_{m,I}) \\
&\leq \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}(z_{m,R}^2 + z_{m,I}^2)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)\left(\sum_{m=1}^{n}(z_{m,R}^2 + z_{m,I}^2)^{\frac{1}{2}}\right) = \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)^2
\end{aligned}$$

Thus

$$\left|\sum_{i=1}^{n} z_i\right|^2 \leq \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)^2 \quad \text{or} \quad \left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n}|z_i|$$

The inequality now follows if we substitute $z_i = x_i y_i^*$. Equality is obtained if $\frac{z_{i,R}}{z_{i,I}} = \frac{z_{m,R}}{z_{m,I}} = k_1$ or $\angle z_i = \angle z_m = \theta$.

3) From 2) we obtain

$$\left|\sum_{i=1}^{n} x_i y_i^*\right|^2 \leq \sum_{i=1}^{n}|x_i||y_i|$$

But $|x_i|$, $|y_i|$ are real positive numbers so from 1)

$$\sum_{i=1}^{n} |x_i||y_i| \leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} |y_i|^2\right]^{\frac{1}{2}}$$

Combining the two inequalities we get

$$\left|\sum_{i=1}^{n} x_i y_i^*\right|^2 \leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} |y_i|^2\right]^{\frac{1}{2}}$$

From part 1) equality holds if $\alpha_i = k\beta_i$ or $|x_i| = k|y_i|$ and from part 2) $x_i y_i^* = |x_i y_i^*| e^{j\theta}$. Therefore, the two conditions are

$$\begin{cases} |x_i| = k|y_i| \\ \angle x_i - \angle y_i = \theta \end{cases}$$

which imply that for all i, $x_i = K y_i$ for some complex constant K.

3) The same procedure can be used to prove the Cauchy-Schwartz inequality for integrals. An easier approach is obtained if one considers the inequality

$$|x(t) + \alpha y(t)| \geq 0, \qquad \text{for all } \alpha$$

Then

$$\begin{aligned} 0 &\leq \int_{-\infty}^{\infty} |x(t) + \alpha y(t)|^2 dt = \int_{-\infty}^{\infty} (x(t) + \alpha y(t))(x^*(t) + \alpha^* y^*(t)) dt \\ &= \int_{-\infty}^{\infty} |x(t)|^2 dt + \alpha \int_{-\infty}^{\infty} x^*(t) y(t) dt + \alpha^* \int_{-\infty}^{\infty} x(t) y^*(t) dt + |a|^2 \int_{-\infty}^{\infty} |y(t)|^2 dt \end{aligned}$$

The inequality is true for $\int_{-\infty}^{\infty} x^*(t) y(t) dt = 0$. Suppose that $\int_{-\infty}^{\infty} x^*(t) y(t) dt \neq 0$ and set

$$\alpha = -\frac{\int_{-\infty}^{\infty} |x(t)|^2 dt}{\int_{-\infty}^{\infty} x^*(t) y(t) dt}$$

Then,

$$0 \leq -\int_{-\infty}^{\infty} |x(t)|^2 dt + \frac{[\int_{-\infty}^{\infty} |x(t)|^2 dt]^2 \int_{-\infty}^{\infty} |y(t)|^2 dt}{|\int_{-\infty}^{\infty} x(t) y^*(t) dt|^2}$$

and

$$\left|\int_{-\infty}^{\infty} x(t) y^*(t) dt\right| \leq \left[\int_{-\infty}^{\infty} |x(t)|^2 dt\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} |y(t)|^2 dt\right]^{\frac{1}{2}}$$

Equality holds if $x(t) = -\alpha y(t)$ a.e. for some complex α.

Problem 2.10

1) Using the Fourier transform pair

$$e^{-\alpha|t|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2 + (2\pi f)^2} = \frac{2\alpha}{4\pi^2} \frac{1}{\frac{\alpha^2}{4\pi^2} + f^2}$$

and the duality property of the Fourier transform: $X(f) = \mathcal{F}[x(t)] \Rightarrow x(-f) = \mathcal{F}[X(t)]$ we obtain

$$\left(\frac{2\alpha}{4\pi^2}\right) \mathcal{F}\left[\frac{1}{\frac{\alpha^2}{4\pi^2} + t^2}\right] = e^{-\alpha|f|}$$

With $\alpha = 2\pi$ we get the desired result

$$\mathcal{F}\left[\frac{1}{1+t^2}\right] = \pi e^{-2\pi|f|}$$

2)

$$\begin{aligned}
\mathcal{F}[x(t)] &= \mathcal{F}[\Pi(t-3)+\Pi(t+3)] \\
&= \text{sinc}(f)e^{-j2\pi f3}+\text{sinc}(f)e^{j2\pi f3} \\
&= 2\text{sinc}(f)\cos(2\pi 3f)
\end{aligned}$$

3)

$$\begin{aligned}
\mathcal{F}[x(t)] &= \mathcal{F}[\Lambda(2t+3)+\Lambda(3t-2)] \\
&= \mathcal{F}[\Lambda(2(t+\frac{3}{2}))+\Lambda(3(t-\frac{2}{3})] \\
&= \frac{1}{2}\text{sinc}^2(\frac{f}{2})e^{j\pi f3}+\frac{1}{3}\text{sinc}^2(\frac{f}{3})e^{-j2\pi f\frac{2}{3}}
\end{aligned}$$

4) $T(f)=\mathcal{F}[\text{sinc}^3(t)]=\mathcal{F}[\text{sinc}^2(t)\text{sinc}(t)]=\Lambda(f)\star\Pi(f)$. But

$$\Pi(f)\star\Lambda(f)=\int_{-\infty}^{\infty}\Pi(\theta)\Lambda(f-\theta)d\theta=\int_{-\frac{1}{2}}^{\frac{1}{2}}\Lambda(f-\theta)d\theta=\int_{f-\frac{1}{2}}^{f+\frac{1}{2}}\Lambda(v)dv$$

$$\begin{aligned}
\text{For}\quad & f\leq-\frac{3}{2}\Longrightarrow T(f)=0 \\
\text{For}\quad & -\frac{3}{2}<f\leq-\frac{1}{2}\Longrightarrow T(f)=\int_{-1}^{f+\frac{1}{2}}(v+1)dv=(\frac{1}{2}v^2+v)\Big|_{-1}^{f+\frac{1}{2}}=\frac{1}{2}f^2+\frac{3}{2}f+\frac{9}{8} \\
\text{For}\quad & -\frac{1}{2}<f\leq\frac{1}{2}\Longrightarrow T(f)=\int_{f-\frac{1}{2}}^{0}(v+1)dv+\int_{0}^{f+\frac{1}{2}}(-v+1)dv \\
& \qquad =(\frac{1}{2}v^2+v)\Big|_{f-\frac{1}{2}}^{0}+(-\frac{1}{2}v^2+v)\Big|_{0}^{f+\frac{1}{2}}=-f^2+\frac{3}{4} \\
\text{For}\quad & \frac{1}{2}<f\leq\frac{3}{2}\Longrightarrow T(f)=\int_{f-\frac{1}{2}}^{1}(-v+1)dv=(-\frac{1}{2}v^2+v)\Big|_{f-\frac{1}{2}}^{1}=\frac{1}{2}f^2-\frac{3}{2}f+\frac{9}{8} \\
\text{For}\quad & \frac{3}{2}<f\Longrightarrow T(f)=0
\end{aligned}$$

Thus,

$$T(f)=\begin{cases}
0 & f\leq-\frac{3}{2} \\
\frac{1}{2}f^2+\frac{3}{2}f+\frac{9}{8} & -\frac{3}{2}<f\leq-\frac{1}{2} \\
-f^2+\frac{3}{4} & -\frac{1}{2}<f\leq\frac{1}{2} \\
\frac{1}{2}f^2-\frac{3}{2}f+\frac{9}{8} & \frac{1}{2}<f\leq\frac{3}{2} \\
0 & \frac{3}{2}<f
\end{cases}$$

5)

$$\mathcal{F}[t\text{sinc}(t)]=\frac{1}{\pi}\mathcal{F}[\sin(\pi t)]=\frac{j}{2\pi}\left[\delta(f+\frac{1}{2})-\delta(f-\frac{1}{2})\right]$$

The same result is obtain if we recognize that multiplication by t results in differentiation in the frequency domain. Thus

$$\mathcal{F}[t\text{sinc}]=\frac{j}{2\pi}\frac{d}{df}\Pi(f)=\frac{j}{2\pi}\left[\delta(f+\frac{1}{2})-\delta(f-\frac{1}{2})\right]$$

6)

$$\begin{aligned}\mathcal{F}[t\cos(2\pi f_0 t)] &= \frac{j}{2\pi}\frac{d}{df}\left(\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)\right)\\ &= \frac{j}{4\pi}\left(\delta'(f-f_0)+\delta'(f+f_0)\right)\end{aligned}$$

7)

$$\mathcal{F}[e^{-\alpha|t|}\cos(\beta t)] = \frac{1}{2}\left[\frac{2\alpha}{\alpha^2+(2\pi(f-\frac{\beta}{2\pi}))^2}+\frac{2\alpha}{\alpha^2+(2\pi(f+\frac{\beta}{2\pi}))^2}\right]$$

8)

$$\begin{aligned}\mathcal{F}[te^{-\alpha|t|}\cos(\beta t)] &= \frac{j}{2\pi}\frac{d}{df}\left(\frac{\alpha}{\alpha^2+(2\pi(f-\frac{\beta}{2\pi}))^2}+\frac{\alpha}{\alpha^2+(2\pi(f+\frac{\beta}{2\pi}))^2}\right)\\ &= -j\left[\frac{2\alpha\pi(f-\frac{\beta}{2\pi})}{\left(\alpha^2+(2\pi(f-\frac{\beta}{2\pi}))^2\right)^2}+\frac{2\alpha\pi(f+\frac{\beta}{2\pi})}{\left(\alpha^2+(2\pi(f+\frac{\beta}{2\pi}))^2\right)^2}\right]\end{aligned}$$

Problem 2.11

$$\begin{aligned}\mathcal{F}[\frac{1}{2}(\delta(t+\frac{1}{2})+\delta(t-\frac{1}{2}))] &= \int_{-\infty}^{\infty}\frac{1}{2}(\delta(t+\frac{1}{2})+\delta(t-\frac{1}{2}))e^{-j2\pi ft}dt\\ &= \frac{1}{2}(e^{-j\pi f}+e^{-j\pi f}) = \cos(\pi f)\end{aligned}$$

Using the duality property of the Fourier transform:

$$X(f) = \mathcal{F}[x(t)] \Longrightarrow x(f) = \mathcal{F}[X(-t)]$$

we obtain

$$\mathcal{F}[\cos(-\pi t)] = \mathcal{F}[\cos(\pi t)] = \frac{1}{2}(\delta(f+\frac{1}{2})+\delta(f-\frac{1}{2}))$$

Note that $\sin(\pi t) = \cos(\pi t + \frac{\pi}{2})$. Thus

$$\begin{aligned}\mathcal{F}[\sin(\pi t)] &= \mathcal{F}[\cos(\pi(t+\frac{1}{2}))] = \frac{1}{2}(\delta(f+\frac{1}{2})+\delta(f-\frac{1}{2}))e^{j\pi f}\\ &= \frac{1}{2}e^{j\pi\frac{1}{2}}\delta(f+\frac{1}{2})+\frac{1}{2}e^{-j\pi\frac{1}{2}}\delta(f-\frac{1}{2})\\ &= \frac{j}{2}\delta(f+\frac{1}{2})-\frac{j}{2}\delta(f-\frac{1}{2})\end{aligned}$$

Problem 2.12

a) We can write $x(t)$ as $x(t) = 2\Pi(\frac{t}{4}) - 2\Lambda(\frac{t}{2})$. Then

$$\mathcal{F}[x(t)] = \mathcal{F}[2\Pi(\frac{t}{4})] - \mathcal{F}[2\Lambda(\frac{t}{2})] = 8\mathrm{sinc}(4f) - 4\mathrm{sinc}^2(2f)$$

b)

$$x(t) = 2\Pi(\frac{t}{4}) - \Lambda(t) \Longrightarrow \mathcal{F}[x(t)] = 8\mathrm{sinc}(4f) - \mathrm{sinc}^2(f)$$

c)

$$\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \int_{-1}^{0}(t+1)e^{-j2\pi ft}dt + \int_{0}^{1}(t-1)e^{-j2\pi ft}dt \\
&= \left(\frac{j}{2\pi f}t + \frac{1}{4\pi^2 f^2}\right)e^{-j2\pi ft}\Big|_{-1}^{0} + \frac{j}{2\pi f}e^{-j2\pi ft}\Big|_{-1}^{0} \\
&\quad + \left(\frac{j}{2\pi f}t + \frac{1}{4\pi^2 f^2}\right)e^{-j2\pi ft}\Big|_{0}^{1} - \frac{j}{2\pi f}e^{-j2\pi ft}\Big|_{0}^{1} \\
&= \frac{j}{\pi f}(1-\sin(\pi f))
\end{aligned}$$

d) We can write $x(t)$ as $x(t) = \Lambda(t+1) - \Lambda(t-1)$. Thus

$$X(f) = \mathrm{sinc}^2(f)e^{j2\pi f} - \mathrm{sinc}^2(f)e^{-j2\pi f} = 2j\mathrm{sinc}^2(f)\sin(2\pi f)$$

e) We can write $x(t)$ as $x(t) = \Lambda(t+1) + \Lambda(t) + \Lambda(t-1)$. Hence,

$$X(f) = \mathrm{sinc}^2(f)(1 + e^{j2\pi f} + e^{-j2\pi f}) = \mathrm{sinc}^2(f)(1 + 2\cos(2\pi f))$$

f) We can write $x(t)$ as

$$x(t) = \left[\Pi\left(2f_0(t - \frac{1}{4f_0})\right) - \Pi\left(2f_0(t - \frac{1}{4f_0})\right)\right]\sin(2\pi f_0 t)$$

Then

$$\begin{aligned}
X(f) &= \left[\frac{1}{2f_0}\mathrm{sinc}\left(\frac{f}{2f_0}\right)e^{-j2\pi\frac{1}{4f_0}f} - \frac{1}{2f_0}\mathrm{sinc}\left(\frac{f}{2f_0})\right)e^{j2\pi\frac{1}{4f_0}f}\right] \\
&\quad \star\frac{j}{2}(\delta(f+f_0) - \delta(f+f_0)) \\
&= \frac{1}{2f_0}\mathrm{sinc}\left(\frac{f+f_0}{2f_0}\right)\sin\left(\pi\frac{f+f_0}{2f_0}\right) - \frac{1}{2f_0}\mathrm{sinc}\left(\frac{f-f_0}{2f_0}\right)\sin\left(\pi\frac{f-f_0}{2f_0}\right)
\end{aligned}$$

Problem 2.13

We start with

$$\mathcal{F}[x(at)] = \int_{-\infty}\infty x(at)e^{-j2\pi ft}dt$$

and make the change in variable $u = at$, then,

$$\begin{aligned}
\mathcal{F}[x(at)] &= \frac{1}{|a|}\int_{-\infty}\infty x(u)e^{-j2\pi fu/a}du \\
&= \frac{1}{|a|}X\left(\frac{f}{a}\right)
\end{aligned}$$

where we have treated the cases $a > 0$ and $a < 0$ separately.

Note that in the above expression if $a > 1$, then $x(at)$ is a contracted form of $x(t)$ whereas if $a < 1$, $x(at)$ is an expanded version of $x(t)$. This means that if we expand a signal in the time domain its frequency domain representation (Fourier transform) contracts and if we contract a signal in the time domain its frequency domain representation expands. This is exactly what one expects since contracting a signal in the time domain makes the changes in the signal more abrupt, thus, increasing its frequency content.

Problem 2.14
We have

$$\begin{aligned}
\mathcal{F}[x(t) \star y(t)] &= \int_{-\infty} \infty \left[\int_{-\infty} \infty x(\tau) y(t-\tau)\, d\tau \right] e^{-j2\pi f t} dt \\
&= \int_{-\infty} \infty x(\tau) \left[\int_{-\infty} \infty y(t-\tau) e^{-j2\pi f (t-\tau)}\, dt \right] e^{-j2\pi f \tau} d\tau
\end{aligned}$$

Now with the change of variable $u = t - \tau$, we have

$$\begin{aligned}
\int_{-\infty} \infty y(t-\tau) e^{-j2\pi f (t-\tau)} dt &= \int_{-\infty} \infty f y(u) e^{-j2\pi f u} du \\
&= \mathcal{F}[y(t)] \\
&= Y(f)
\end{aligned}$$

and, therefore,

$$\begin{aligned}
\mathcal{F}[x(t) \star y(t)] &= \int_{-\infty} \infty x(\tau) Y(f) e^{-j2\pi f \tau} d\tau \\
&= X(f) \cdot Y(f)
\end{aligned}$$

Problem 2.15
We start with the Fourier transform of $x(t - t_0)$,

$$\mathcal{F}[x(t-t_0)] = \int_{-\infty} \infty x(t-t_0) e^{-j2\pi f t} dt$$

With a change of variable of $u = t - t_0$, we obtain

$$\begin{aligned}
\mathcal{F}[x(t-t_0)] &= \int_{-\infty} \infty x(u) e^{-j2\pi f t_0} e^{-j2\pi f u} du \\
&= e^{-j2\pi f t_0} \int_{-\infty} \infty x(u) e^{-j2\pi f u} du \\
&= e^{-j2\pi f t_0} \mathcal{F}[x(t)]
\end{aligned}$$

Problem 2.16

$$\begin{aligned}
\int_{-\infty} \infty x(t) y^*(t)\, dt &= \int_{-\infty} \infty \left[\int_{-\infty} \infty X(f) e^{j2\pi f t} df \right] \left[\int_{-\infty} \infty Y(f') e^{j2\pi f' t} df' \right]^* dt \\
&= \int_{-\infty} \infty \left[\int_{-\infty} \infty X(f) e^{j2\pi f t} df \right] \left[\int_{-\infty} \infty Y^*(f') e^{-j2\pi f' t} df' \right] dt \\
&= \int_{-\infty} \infty X(f) \left[\int_{-\infty} \infty Y^*(f') \left[\int_{-\infty} \infty e^{j2\pi t (f-f')}\, dt \right] df' \right] df
\end{aligned}$$

Now using properties of the impulse function.

$$\int_{-\infty} \infty e^{j2\pi t (f-f')} dt = \delta(f - f')$$

and therefore

$$\begin{aligned}
\int_{-\infty} \infty x(t) y^*(t)\, dt &= \int_{-\infty} \infty X(f) \left[\int_{-\infty} \infty Y^*(f') \delta(f-f')\, df' \right] df \\
&= \int_{-\infty} \infty X(f) Y^*(f)\, df
\end{aligned}$$

where we have employed the sifting property of the impulse signal in the last step.

Problem 2.17
(Convolution theorem:)

$$\mathcal{F}[x(t) \star y(t)] = \mathcal{F}[x(t)]\mathcal{F}[y(t)] = X(f)Y(f)$$

Thus

$$\begin{aligned}
\text{sinc}(t) \star \text{sinc}(t) &= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t) \star \text{sinc}(t)]] \\
&= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t)] \cdot \mathcal{F}[\text{sinc}(t)]] \\
&= \mathcal{F}^{-1}[\Pi(f)\Pi(f)] = \mathcal{F}^{-1}[\Pi(f)] \\
&= \text{sinc}(t)
\end{aligned}$$

Problem 2.18

$$\begin{aligned}
\mathcal{F}[x(t)y(t)] &= \int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X(\theta)e^{j2\pi\theta t}d\theta\right) y(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} X(\theta) \left(\int_{-\infty}^{\infty} y(t)e^{-j2\pi(f-\theta)t}dt\right) d\theta \\
&= \int_{-\infty}^{\infty} X(\theta)Y(f-\theta)d\theta = X(f) \star Y(f)
\end{aligned}$$

Problem 2.19
1) Clearly

$$\begin{aligned}
x_1(t+kT_0) &= \sum_{n=-\infty}^{\infty} x(t+kT_0-nT_0) = \sum_{n=-\infty}^{\infty} x(t-(n-k)T_0) \\
&= \sum_{m=-\infty}^{\infty} x(t-mT_0) = x_1(t)
\end{aligned}$$

where we used the change of variable $m = n - k$.

2)

$$x_1(t) = x(t) \star \sum_{n=-\infty}^{\infty} \delta(t-nT_0)$$

This is because

$$\int_{-\infty}^{\infty} x(\tau) \sum_{n=-\infty}^{\infty} \delta(t-\tau-nT_0)d\tau = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-nT_0)d\tau = \sum_{n=-\infty}^{\infty} x(t-nT_0)$$

3)

$$\begin{aligned}
\mathcal{F}[x_1(t)] &= \mathcal{F}[x(t) \star \sum_{n=-\infty}^{\infty} \delta(t-nT_0)] = \mathcal{F}[x(t)]\mathcal{F}[\sum_{n=-\infty}^{\infty} \delta(t-nT_0)] \\
&= X(f)\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f-\frac{n}{T_0}) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_0})\delta(f-\frac{n}{T_0})
\end{aligned}$$

Problem 2.20

1) By Parseval's theorem

$$\int_{-\infty}^{\infty} \text{sinc}^5(t)dt = \int_{-\infty}^{\infty} \text{sinc}^3(t)\text{sinc}^2(t)dt = \int_{-\infty}^{\infty} \Lambda(f)T(f)df$$

where

$$T(f) = \mathcal{F}[\text{sinc}^3(t)] = \mathcal{F}[\text{sinc}^2(t)\text{sinc}(t)] = \Pi(f) \star \Lambda(f)$$

But

$$\Pi(f) \star \Lambda(f) = \int_{-\infty}^{\infty} \Pi(\theta)\Lambda(f-\theta)d\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Lambda(f-\theta)d\theta = \int_{f-\frac{1}{2}}^{f+\frac{1}{2}} \Lambda(v)dv$$

$$\begin{aligned}
&\text{For} \quad f \leq -\frac{3}{2} \Longrightarrow T(f) = 0\\
&\text{For} \quad -\frac{3}{2} < f \leq -\frac{1}{2} \Longrightarrow T(f) = \int_{-1}^{f+\frac{1}{2}} (v+1)dv = (\frac{1}{2}v^2+v)\Big|_{-1}^{f+\frac{1}{2}} = \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8}\\
&\text{For} \quad -\frac{1}{2} < f \leq \frac{1}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{0} (v+1)dv + \int_{0}^{f+\frac{1}{2}} (-v+1)dv\\
&\qquad\qquad = (\frac{1}{2}v^2+v)\Big|_{f-\frac{1}{2}}^{0} + (-\frac{1}{2}v^2+v)\Big|_{0}^{f+\frac{1}{2}} = -f^2 + \frac{3}{4}\\
&\text{For} \quad \frac{1}{2} < f \leq \frac{3}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{1} (-v+1)dv = (-\frac{1}{2}v^2+v)\Big|_{f-\frac{1}{2}}^{1} = \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8}\\
&\text{For} \quad \frac{3}{2} < f \Longrightarrow T(f) = 0
\end{aligned}$$

Thus,

$$T(f) = \begin{cases} 0 & f \leq -\frac{3}{2} \\ \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8} & -\frac{3}{2} < f \leq -\frac{1}{2} \\ -f^2 + \frac{3}{4} & -\frac{1}{2} < f \leq \frac{1}{2} \\ \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8} & \frac{1}{2} < f \leq \frac{3}{2} \\ 0 & \frac{3}{2} < f \end{cases}$$

Hence,

$$\begin{aligned}
\int_{-\infty}^{\infty} \Lambda(f)T(f)df &= \int_{-1}^{-\frac{1}{2}} (\frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8})(f+1)df + \int_{-\frac{1}{2}}^{0} (-f^2 + \frac{3}{4})(f+1)df\\
&\quad + \int_{0}^{\frac{1}{2}} (-f^2 + \frac{3}{4})(-f+1)df + \int_{\frac{1}{2}}^{1} (\frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8})(-f+1)df\\
&= \frac{41}{64}
\end{aligned}$$

2)

$$\begin{aligned}
\int_{0}^{\infty} e^{-\alpha t}\text{sinc}(t)dt &= \int_{-\infty}^{\infty} e^{-\alpha t}u_{-1}(t)\text{sinc}(t)dt\\
&= \int_{-\infty}^{\infty} \frac{1}{\alpha + j2\pi f}\Pi(f)df = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\alpha + j2\pi f}df\\
&= \frac{1}{j2\pi}\ln(\alpha + j2\pi f)|_{-1/2}^{1/2} = \frac{1}{j2\pi}\ln(\frac{\alpha + j\pi}{\alpha - j\pi}) = \frac{1}{\pi}\tan^{-1}\frac{\pi}{\alpha}
\end{aligned}$$

3)

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}\mathrm{sinc}^2(t)dt &= \int_{-\infty}^\infty e^{-\alpha t}u_{-1}(t)\mathrm{sinc}^2(t)dt \\
&= \int_{-\infty}^\infty \frac{1}{\alpha+j2\pi f}\Lambda(f)df df \\
&= \int_{-1}^0 \frac{f+1}{\alpha+j\pi f}df + \int_0^1 \frac{-f+1}{\alpha+j\pi f}df
\end{aligned}$$

But $\int \frac{x}{a+bx}dx = \frac{x}{b} - \frac{a}{b^2}\ln(a+bx)$ so that

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}\mathrm{sinc}^2(t)dt &= (\frac{f}{j2\pi} + \frac{\alpha}{4\pi^2}\ln(\alpha+j2\pi f))\Big|_{-1}^0 \\
&\quad -(\frac{f}{j2\pi} + \frac{\alpha}{4\pi^2}\ln(\alpha+j2\pi f))\Big|_0^1 + \frac{1}{j2\pi}\ln(\alpha+j2\pi f)\Big|_{-1}^1 \\
&= \frac{1}{\pi}\tan^{-1}(\frac{2\pi}{\alpha}) + \frac{\alpha}{2\pi^2}\ln(\frac{\alpha}{\sqrt{\alpha^2+4\pi^2}})
\end{aligned}$$

4)

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}\cos(\beta t)dt &= \int_{-\infty}^\infty e^{-\alpha t}u_{-1}(t)\cos(\beta t)dt \\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{1}{\alpha+j2\pi f}(\delta(f-\frac{\beta}{2\pi}) + \delta(f+\frac{\beta}{2\pi}))dt \\
&= \frac{1}{2}[\frac{1}{\alpha+j\beta} + \frac{1}{\alpha-j\beta}] = \frac{\alpha}{\alpha^2+\beta^2}
\end{aligned}$$

Problem 2.21

Using the convolution theorem we obtain

$$\begin{aligned}
Y(f) &= X(f)H(f) = (\frac{1}{\alpha+j2\pi f})(\frac{1}{\beta+j2\pi f}) \\
&= \frac{1}{(\beta-\alpha)}\frac{1}{\alpha+j2\pi f} - \frac{1}{(\beta-\alpha)}\frac{1}{\beta+j2\pi f}
\end{aligned}$$

Thus

$$y(t) = \mathcal{F}^{-1}[Y(f)] = \frac{1}{(\beta-\alpha)}[e^{-\alpha t} - e^{-\beta t}]u_{-1}(t)$$

If $\alpha = \beta$ then $X(f) = H(f) = \frac{1}{\alpha+j2\pi f}$. In this case

$$y(t) = \mathcal{F}^{-1}[Y(f)] = \mathcal{F}^{-1}[(\frac{1}{\alpha+j2\pi f})^2] = te^{-\alpha t}u_{-1}(t)$$

The signal is of the energy-type with energy content

$$\begin{aligned}
E_y &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}|y(t)|^2dt = \lim_{T\to\infty}\int_0^{\frac{T}{2}}\frac{1}{(\beta-\alpha)^2}(e^{-\alpha t}-e^{-\beta t})^2dt \\
&= \lim_{T\to\infty}\frac{1}{(\beta-\alpha)^2}\left[-\frac{1}{2\alpha}e^{-2\alpha t}\Big|_0^{T/2} - \frac{1}{2\beta}e^{-2\beta t}\Big|_0^{T/2} + \frac{2}{(\alpha+\beta)}e^{-(\alpha+\beta)t}\Big|_0^{T/2}\right] \\
&= \frac{1}{(\beta-\alpha)^2}[\frac{1}{2\alpha} + \frac{1}{2\beta} - \frac{2}{\alpha+\beta}] = \frac{1}{2\alpha\beta(\alpha+\beta)}
\end{aligned}$$

Problem 2.22

$$x_\alpha(t) = \begin{cases} x(t) & \alpha \le t < \alpha + T_0 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$X_\alpha(f) = \int_{-\infty}^{\infty} x_\alpha(t) e^{-j2\pi f t} dt = \int_{\alpha}^{\alpha+T_0} x(t) e^{-j2\pi f t} dt$$

Evaluating $X_\alpha(f)$ for $f = \frac{n}{T_0}$ we obtain

$$X_\alpha(\frac{n}{T_0}) = \int_{\alpha}^{\alpha+T_0} x(t) e^{-j2\pi \frac{n}{T_0} t} dt = T_0 x_n$$

where x_n are the coefficients in the Fourier series expansion of $x(t)$. Thus $X_\alpha(\frac{n}{T_0})$ is independent of the choice of α.

Problem 2.23

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x(t - nT_s) &= x(t) \star \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \frac{1}{T_s} x(t) \star \sum_{n=-\infty}^{\infty} e^{j2\pi \frac{n}{T_s} t} \\
&= \frac{1}{T_s} \mathcal{F}^{-1} \left[X(f) \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_s}) \right] \\
&= \frac{1}{T_s} \mathcal{F}^{-1} \left[\sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right) \delta(f - \frac{n}{T_s}) \right] \\
&= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right) e^{j2\pi \frac{n}{T_s} t}
\end{aligned}$$

If we set $t = 0$ in the previous relation we obtain Poisson's sum formula

$$\sum_{n=-\infty}^{\infty} x(-nT_s) = \sum_{m=-\infty}^{\infty} x(mT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right)$$

Problem 2.24

1) We know that

$$e^{-\alpha|t|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$

Applying Poisson's sum formula with $T_s = 1$ we obtain

$$\sum_{n=-\infty}^{\infty} e^{-\alpha|n|} = \sum_{n=-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + 4\pi^2 n^2}$$

2) Use the Fourier transform pair $\Pi(t) \rightarrow \text{sinc}(f)$ in the Poisson's sum formula with $T_s = K$. Then

$$\sum_{n=-\infty}^{\infty} \Pi(nK) = \frac{1}{K} \sum_{n=-\infty}^{\infty} \text{sinc}(\frac{n}{K})$$

But $\Pi(nK) = 1$ for $n = 0$ and $\Pi(nK) = 0$ for $|n| \ge 1$ and $K \in \{1, 2, \ldots\}$. Thus the left side of the previous relation reduces to 1 and

$$K = \sum_{n=-\infty}^{\infty} \text{sinc}(\frac{n}{K})$$

3) Use the Fourier transform pair $\Lambda(t) \rightarrow \text{sinc}^2(f)$ in the Poisson's sum formula with $T_s = K$. Then

$$\sum_{n=-\infty}^{\infty} \Lambda(nK) = \frac{1}{K} \sum_{n=-\infty}^{\infty} \text{sinc}^2(\frac{n}{K})$$

Reasoning as before we see that $\sum_{n=-\infty}^{\infty} \Lambda(nK) = 1$ since for $K \in \{1, 2, \ldots\}$

$$\Lambda(nK) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, $K = \sum_{n=-\infty}^{\infty} \text{sinc}^2(\frac{n}{K})$

Problem 2.25

Let $H(f)$ be the Fourier transform of $h(t)$. Then

$$H(f)\mathcal{F}[e^{-\alpha t}u_{-1}(t)] = \mathcal{F}[\delta(t)] \Longrightarrow H(f)\frac{1}{\alpha + j2\pi f} = 1 \Longrightarrow H(f) = \alpha + j2\pi f$$

The response of the system to $e^{-\alpha t}\cos(\beta t)u_{-1}(t)$ is

$$y(t) = \mathcal{F}^{-1}\left[H(f)\mathcal{F}[e^{-\alpha t}\cos(\beta t)u_{-1}(t)]\right]$$

But

$$\begin{aligned} \mathcal{F}[e^{-\alpha t}\cos(\beta t)u_{-1}(t)] &= \mathcal{F}[\frac{1}{2}e^{-\alpha t}u_{-1}(t)e^{j\beta t} + \frac{1}{2}e^{-\alpha t}u_{-1}(t)e^{-j\beta t}] \\ &= \frac{1}{2}\left[\frac{1}{\alpha + j2\pi(f - \frac{\beta}{2\pi})} + \frac{1}{\alpha + j2\pi(f + \frac{\beta}{2\pi})}\right] \end{aligned}$$

so that

$$Y(f) = \mathcal{F}[y(t)] = \frac{\alpha + j2\pi f}{2}\left[\frac{1}{\alpha + j2\pi(f - \frac{\beta}{2\pi})} + \frac{1}{\alpha + j2\pi(f + \frac{\beta}{2\pi})}\right]$$

Using the linearity property of the Fourier transform, the Convolution theorem and the fact that $\delta'(t) \xrightarrow{\mathcal{F}} j2\pi f$ we obtain

$$\begin{aligned} y(t) &= \alpha e^{-\alpha t}\cos(\beta t)u_{-1}(t) + (e^{-\alpha t}\cos(\beta t)u_{-1}(t)) \star \delta'(t) \\ &= e^{-\alpha t}\cos(\beta t)\delta(t) - \beta e^{-\alpha t}\sin(\beta t)u_{-1}(t) \\ &= \delta(t) - \beta e^{-\alpha t}\sin(\beta t)u_{-1}(t) \end{aligned}$$

Problem 2.26

1)

$$\begin{aligned} y(t) &= x(t) \star h(t) = x(t) \star (\delta(t) + \delta'(t)) \\ &= x(t) + \frac{d}{dt}x(t) \end{aligned}$$

With $x(t) = e^{-\alpha|t|}$ we obtain $y(t) = e^{-\alpha|t|} - \alpha e^{-\alpha|t|}\text{sgn}(t)$.

2)

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau = e^{-\beta t}\int_0^t e^{-(\alpha-\beta)\tau}d\tau \end{aligned}$$

$$\begin{aligned}
\text{If} \qquad & \alpha = \beta \Rightarrow y(t) = te^{-\alpha t}u_{-1}(t) \\
& \alpha \neq \beta \Rightarrow y(t) = e^{-\beta t}\frac{1}{\beta-\alpha}e^{-(\alpha-\beta)t}\Big|_0^t u_{-1}(t) = \frac{1}{\beta-\alpha}\left[e^{-\alpha t} - e^{-\beta t}\right]u_{-1}(t)
\end{aligned}$$

3)

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} e^{-\alpha\tau}\cos(\gamma\tau)u_{-1}(\tau)e^{-\beta(t-\tau)}u_{-1}(t-\tau)d\tau \\
&= \int_0^t e^{-\alpha\tau}\cos(\gamma\tau)e^{-\beta(t-\tau)}d\tau = e^{-\beta t}\int_0^t e^{(\beta-\alpha)\tau}\cos(\gamma\tau)d\tau
\end{aligned}$$

$$\begin{aligned}
\text{If } \alpha = \beta \Rightarrow y(t) &= e^{-\beta t}\int_0^t \cos(\gamma\tau)d\tau u_{-1}(t) = \frac{e^{-\beta t}}{\gamma}\sin(\gamma t)u_{-1}(t) \\
\text{If } \alpha = \beta \Rightarrow y(t) &= e^{-\beta t}\int_0^t e^{(\beta-\alpha)\tau}\cos(\gamma\tau)d\tau u_{-1}(t) \\
&= \frac{e^{-\beta t}}{(\beta-\alpha)^2+\gamma^2}\left((\beta-\alpha)\cos(\gamma\tau)+\gamma\sin(\gamma\tau)\right)e^{(\beta-\alpha)\tau}\Big|_0^t u_{-1}(t) \\
&= \frac{e^{-\alpha t}}{(\beta-\alpha)^2+\gamma^2}\left((\beta-\alpha)\cos(\gamma t)+\gamma\sin(\gamma t)\right)u_{-1}(t) \\
&\quad -\frac{e^{-\beta t}(\beta-\alpha)}{(\beta-\alpha)^2+\gamma^2}u_{-1}(t)
\end{aligned}$$

4)

$$y(t) = \int_{-\infty}^{\infty} e^{-\alpha|\tau|}e^{-\beta(t-\tau)}u_{-1}(t-\tau)d\tau = \int_{-\infty}^{t} e^{-\alpha|\tau|}e^{-\beta(t-\tau)}d\tau$$

Consider first the case that $\alpha \neq \beta$. Then

$$\begin{aligned}
\text{If } t < 0 \Rightarrow y(t) &= e^{-\beta t}\int_{-\infty}^t e^{(\beta+\alpha)\tau}d\tau = \frac{1}{\alpha+\beta}e^{\alpha t} \\
\text{If } t < 0 \Rightarrow y(t) &= \int_{-\infty}^0 e^{\alpha\tau}e^{-\beta(t-\tau)}d\tau + \int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau \\
&= \frac{e^{-\beta t}}{\alpha+\beta}e^{(\alpha+\beta)\tau}\Big|_{-\infty}^0 + \frac{e^{-\beta t}}{\beta-\alpha}e^{(\beta-\alpha)\tau}\Big|_0^t \\
&= -\frac{2\alpha e^{-\beta t}}{\beta^2-\alpha^2} + \frac{e^{-\alpha t}}{\beta-\alpha}
\end{aligned}$$

Thus

$$y(t) = \begin{cases} \frac{1}{\alpha+\beta}e^{\alpha t} & t \leq 0 \\ -\frac{2\alpha e^{-\beta t}}{\beta^2-\alpha^2} + \frac{e^{-\alpha t}}{\beta-\alpha} & t > 0 \end{cases}$$

In the case of $\alpha = \beta$

$$\begin{aligned}
\text{If } t < 0 \Rightarrow y(t) &= e^{-\alpha t}\int_{-\infty}^t e^{2\alpha\tau}d\tau = \frac{1}{2\alpha}e^{\alpha t} \\
\text{If } t < 0 \Rightarrow y(t) &= \int_{-\infty}^0 e^{-\alpha t}e^{2\alpha\tau}d\tau + \int_0^t e^{-\alpha t}d\tau \\
&= \frac{e^{-\alpha t}}{2\alpha}e^{2\alpha\tau}\Big|_{-\infty}^0 + te^{-\alpha t} \\
&= [\frac{1}{2\alpha}+t]e^{-\alpha t}
\end{aligned}$$

5) Using the convolution theorem we obtain

$$Y(f) = \Pi(f)\Lambda(f) = \begin{cases} 0 & \frac{1}{2} < |f| \\ f+1 & -\frac{1}{2} < f \leq 0 \\ -f+1 & 0 \leq f < \frac{1}{2} \end{cases}$$

Thus

$$\begin{aligned}
y(t) &= \mathcal{F}^{-1}[Y(f)] = \int_{-\frac{1}{2}}^{\frac{1}{2}} Y(f) e^{j2\pi f t} df \\
&= \int_{-\frac{1}{2}}^{0} (f+1) e^{j2\pi f t} df + \int_{0}^{\frac{1}{2}} (-f+1) e^{j2\pi f t} df \\
&= \left(\frac{1}{j2\pi t} f e^{j2\pi f t} + \frac{1}{4\pi^2 t^2} e^{j2\pi f t} \right) \Big|_{-\frac{1}{2}}^{0} + \frac{1}{j2\pi t} e^{j2\pi f t} \Big|_{-\frac{1}{2}}^{0} \\
&\quad - \left(\frac{1}{j2\pi t} f e^{j2\pi f t} + \frac{1}{4\pi^2 t^2} e^{j2\pi f t} \right) \Big|_{0}^{\frac{1}{2}} + \frac{1}{j2\pi t} e^{j2\pi f t} \Big|_{0}^{\frac{1}{2}} \\
&= \frac{1}{2\pi^2 t^2} [1 - \cos(\pi t)] + \frac{1}{2\pi t} \sin(\pi t)
\end{aligned}$$

Problem 2.27

Let the response of the LTI system be $h(t)$ with Fourier transform $H(f)$. Then, from the convolution theorem we obtain

$$Y(f) = H(f)X(f) \Longrightarrow \Lambda(f) = \Pi(f)H(f)$$

However, this relation cannot hold since $\Pi(f) = 0$ for $\frac{1}{2} < |f|$ whereas $\Lambda(f) \neq 0$ for $1 < |f| \leq 1/2$.

Problem 2.28

1) No. The input $\Pi(t)$ has a spectrum with zeros at frequencies $f = k$, $(k \neq 0,\ k \in \mathcal{Z})$ and the information about the spectrum of the system at those frequencies will not be present at the output. The spectrum of the signal $\cos(2\pi t)$ consists of two impulses at $f = \pm 1$ but we do not know the response of the system at these frequencies.

2)

$$\begin{aligned}
h_1(t) \star \Pi(t) &= \Pi(t) \star \Pi(t) = \Lambda(t) \\
h_2(t) \star \Pi(t) &= (\Pi(t) + \cos(2\pi t)) \star \Pi(t) \\
&= \Lambda(t) + \frac{1}{2}\mathcal{F}^{-1}\left[\delta(f-1)\mathrm{sinc}^2(f) + \delta(f+1)\mathrm{sinc}^2(f)\right] \\
&= \Lambda(t) + \frac{1}{2}\mathcal{F}^{-1}\left[\delta(f-1)\mathrm{sinc}^2(1) + \delta(f+1)\mathrm{sinc}^2(-1)\right] \\
&= \Lambda(t)
\end{aligned}$$

Thus both signals are candidates for the impulse response of the system.

3) $\mathcal{F}[u_{-1}(t)] = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$. Thus the system has a nonzero spectrum for every f and all the frequencies of the system will be excited by this input. $\mathcal{F}[e^{-at}u_{-1}(t)] = \frac{1}{a+j2\pi f}$. Again the spectrum is nonzero for all f and the response to this signal uniquely determines the system. In general the spectrum of the input must not vanish at any frequency. In this case the influence of the system will be present at the output for every frequency.

Problem 2.29

1)

$$
\begin{aligned}
E_{x_1} &= \int_{-\infty}^{\infty} x_1^2(t)\,dt \\
&= \int_0^{\infty} e^{-2t}\cos^2 t\,dt \\
&< \int_0^{\infty} e^{-2t}\,dt = \frac{1}{2}
\end{aligned}
$$

where we have used $\cos^2 t \le 1$. Therefore, $x_1(t)$ is energy type. To find the energy we have

$$
\begin{aligned}
\int_0^{\infty} e^{-2t}\cos^2 t\,dt &= \frac{1}{2}\int_0^{\infty} e^{-2t}\,dt + \frac{1}{2}\int_0^{\infty} e^{-2t}\cos 2t\,dt \\
&= \frac{1}{4} + \frac{1}{2}\left[-\frac{1}{4}e^{-2t}\cos(2t) + \frac{1}{4}e^{-2*t}\sin(2t)\right]_0^{\infty} \\
&= \frac{3}{8}
\end{aligned}
$$

2)

$$
\begin{aligned}
E_{x_2} = \int_{-\infty}^{\infty} x_2^2(t)\,dt &= \int_{-\infty}^{0} e^{-2t}\cos^2(t)\,dt + \int_0^{\infty} e^{-2t}\cos^2(t)\,dt \\
&= \int_0^{\infty} e^{2t}\cos^2(t)\,dt + \frac{3}{8} \\
&= \frac{3}{8} - \frac{3}{8} + \lim_{t\to\infty}\frac{e^{2t}}{8}\left(2\cos^2(t) + \sin(2t) + 1\right) \\
&= \lim_{t\to\infty}\frac{e^{2t}}{8} f(t)
\end{aligned}
$$

where $f(t) = 2\cos^2(t) + \sin(2t) + 1$. By taking the derivative and setting it equal to zero we can find the minimum of $f(t)$ and show that $f(t) > 0.5$. This shows that $\lim_{t\to\infty}\frac{e^{2t}}{8}f(t) \ge \lim_{t\to\infty}\frac{e^{2t}}{16} = \infty$. This shows that the signal is not energy-type.

To check if the signal is power type, we obviously have $\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-2t}\cos^2 t\,dt = 0$. Therefore

$$
\begin{aligned}
P &= \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{2t}\cos^2(t)\,dt \\
&= \lim_{T\to\infty}\frac{1/4\,e^{2\,T}\left(\cos(T)\right)^2 + 1/4\,e^{2\,T}\cos(T)\sin(T) + 1/8\,\left(e^{T}\right)^2 - 3/8}{T} \\
&= \infty
\end{aligned}
$$

Therefore $x_2(t)$ is neither power- nor energy-type.

3)

$$
\begin{aligned}
E_{x_3} = \int_{-\infty}^{\infty} (\mathrm{sgn}(t))^2\,dt &= \int_{-\infty}^{\infty} 1\,dt \\
&= \infty
\end{aligned}
$$

and hence the signal is not energy-type. To find the power

$$
\begin{aligned}
P_{x_3} &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} (\mathrm{sgn}(t)))^2\,dt \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} 1^2\,dt \\
&= \lim_{T\to\infty}\frac{1}{2T}2T = 1
\end{aligned}
$$

4) Since $x_4(t)$ is periodic (or almost periodic when f_1/f_2 is not rational) the signal is not energy type. To see whether it is power type, we have

$$\begin{aligned}
P_{x_4} &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (A\cos 2\pi f_1 t + B\cos 2\pi f_2 t)^2\, dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left(A^2\cos^2 2\pi f_1 t + B^2\cos^2 2\pi f_2 t + 2AB\cos 2\pi f_1 t\cos 2\pi f_2 t\right) dt \\
&= \frac{A^2+B^2}{2}
\end{aligned}$$

Problem 2.30

1)

$$\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|Ae^{j(2\pi f_0 t+\theta)}\right|^2 dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2\, dt \\
&= A^2
\end{aligned}$$

2)

$$\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T}\int_{0}^{T} 1^2\, dt \\
&= \frac{1}{2}
\end{aligned}$$

3)

$$\begin{aligned}
E &= \lim_{T\to\infty} \int_{0}^{T} K^2/\sqrt{t}\, dt \\
&= \lim_{T\to\infty} \left[2K^2\sqrt{t}\right]_0^T \\
&= \lim_{T\to\infty} 2K^2\sqrt{T} \\
&= \infty
\end{aligned}$$

therefore, it is not energy-type. To find the power

$$\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} K^2/\sqrt{t}\, dt \\
&= \lim_{T\to\infty} \frac{1}{2T} 2K^2\sqrt{T} \\
&= 0
\end{aligned}$$

and hence it is not power-type either.

Problem 2.31

1) $x(t) = e^{-\alpha t}u_{-1}(t)$. The spectrum of the signal is $X(f) = \frac{1}{\alpha + j2\pi f}$ and the energy spectral density

$$\mathcal{G}_X(f) = |X(f)|^2 = \frac{1}{\alpha^2 + 4\pi^2 f^2}$$

Thus,

$$R_X(\tau) = \mathcal{F}^{-1}[\mathcal{G}_X(f)] = \frac{1}{2\alpha}e^{-\alpha|\tau|}$$

The energy content of the signal is

$$E_X = R_X(0) = \frac{1}{2\alpha}$$

2) $x(t) = \mathrm{sinc}(t)$. Clearly $X(f) = \Pi(f)$ so that $\mathcal{G}_X(f) = |X(f)|^2 = \Pi^2(f) = \Pi(f)$. The energy content of the signal is

$$E_X = \int_{-\infty}^{\infty} \Pi(f)df = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Pi(f)df = 1$$

3) $x(t) = \sum_{n=-\infty}^{\infty} \Lambda(t-2n)$. The signal is periodic and thus it is not of the energy type. The power content of the signal is

$$\begin{aligned} P_x &= \frac{1}{2}\int_{-1}^{1} |x(t)|^2 dt = \frac{1}{2}\int_{-1}^{0}(t+1)^2 dt + \int_{0}^{1}(-t+1)^2 dt \\ &= \frac{1}{2}\left(\frac{1}{3}t^3 + t^2 + t\right)\Big|_{-1}^{0} + \frac{1}{2}\left(\frac{1}{3}t^3 - t^2 + t\right)\Big|_{0}^{1} \\ &= \frac{1}{3} \end{aligned}$$

The same result is obtain if we let

$$\mathcal{S}_X(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta(f - \frac{n}{2})$$

with $x_0 = \frac{1}{2}$, $x_{2l} = 0$ and $x_{2l+1} = \frac{2}{\pi(2l+1)}$ (see Problem 2.2). Then

$$\begin{aligned} P_X &= \sum_{n=-\infty}^{\infty} |x_n|^2 \\ &= \frac{1}{4} + \frac{8}{\pi^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^4} = \frac{1}{4} + \frac{8}{\pi^2}\frac{\pi^2}{96} = \frac{1}{3} \end{aligned}$$

4)

$$E_X = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} |u_{-1}(t)|^2 dt = \lim_{T\to\infty}\int_{0}^{\frac{T}{2}} dt = \lim_{T\to\infty}\frac{T}{2} = \infty$$

Thus, the signal is not of the energy type.

$$P_X = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |u_{-1}(t)|^2 dt = \lim_{T\to\infty}\frac{1}{T}\frac{T}{2} = \frac{1}{2}$$

Hence, the signal is of the power type and its power content is $\frac{1}{2}$. To find the power spectral density we find first the autocorrelation $R_X(\tau)$.

$$\begin{aligned} R_X(\tau) &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u_{-1}(t)u_{-1}(t-\tau)dt \\ &= \lim_{T\to\infty}\frac{1}{T}\int_{\tau}^{\frac{T}{2}} dt \\ &= \lim_{T\to\infty}\frac{1}{T}(\frac{T}{2} - \tau) = \frac{1}{2} \end{aligned}$$

Thus, $\mathcal{S}_X(f) = \mathcal{F}[R_X(\tau)] = \frac{1}{2}\delta(f)$.

5) Clearly $|X(f)|^2 = \pi^2 \mathrm{sgn}^2(f) = \pi^2$ and $E_X = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \pi^2 dt = \infty$. The signal is not of the energy type for the energy content is not bounded. Consider now the signal

$$x_T(t) = \frac{1}{t}\Pi(\frac{t}{T})$$

Then,

$$X_T(f) = -j\pi \mathrm{sgn}(f) \star T\mathrm{sinc}(fT)$$

and

$$\mathcal{S}_X(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} = \lim_{T\to\infty} \pi^2 T \left| \int_{-\infty}^{f} \mathrm{sinc}(vT)dv - \int_{f}^{\infty} \mathrm{sinc}(vT)dv \right|^2$$

However, the squared term on the right side is bounded away from zero so that $\mathcal{S}_X(f)$ is ∞. The signal is not of the power type either.

Problem 2.32
1)
a) If $\alpha \neq \gamma$,

$$\begin{aligned}
|Y(f)|^2 &= |X(f)|^2 |H(f)|^2 \\
&= \frac{1}{(\alpha^2+4\pi^2 f^2)(\beta^2+4\pi^2 f^2)} \\
&= \frac{1}{\beta^2-\alpha^2}\left[\frac{1}{\alpha^2+4\pi^2 f^2} - \frac{1}{\beta^2+4\pi^2 f^2}\right]
\end{aligned}$$

From this, $R_Y(\tau) = \frac{1}{\beta^2-\alpha^2}\left[\frac{1}{2\alpha}e^{-\alpha|\tau|} - \frac{1}{2\beta}e^{-\beta|\tau|}\right]$ and $E_y = R_y(0) = \frac{1}{2\alpha\beta(\alpha+\beta)}$.
If $\alpha = \gamma$ then

$$\mathcal{G}_Y(f) = |Y(f)|^2 = |X(f)|^2|H(f)|^2 = \frac{1}{(\alpha^2+4\pi^2 f^2)^2}$$

The energy content of the signal is

$$\begin{aligned}
E_Y &= \int_{-\infty}^{\infty} \frac{1}{(\alpha^2+4\pi^2 f^2)^2} \\
&= \frac{1}{4\alpha^2}\int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2+4\pi^2 f^2}\frac{2\alpha}{\alpha^2+4\pi^2 f^2} df \\
&= \frac{1}{4\alpha^2}\int_{-\infty}^{\infty} e^{-2\alpha|t|}dt = \frac{1}{4\alpha^2}2\int_{0}^{\infty} e^{-2\alpha t}dt \\
&= \frac{1}{2\alpha^2} - \frac{1}{2\alpha}e^{-2\alpha t}\Big|_0^{\infty} = \frac{1}{4\alpha^3}
\end{aligned}$$

b) $H(f) = \frac{1}{\gamma+j2\pi f} \Longrightarrow |H(f)|^2 = \frac{1}{\gamma^2+4\pi^2 f^2}$. The energy spectral density of the output is

$$\mathcal{G}_Y(f) = \mathcal{G}_X(f)|H(f)|^2 = \frac{1}{\gamma^2+4\pi^2 f^2}\Pi(f)$$

The energy content of the signal is

$$\begin{aligned}
E_Y &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\gamma^2+4\pi^2 f^2} df = \frac{1}{2\pi\gamma}\arctan\frac{f\gamma}{2\pi}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= \frac{1}{\pi\gamma}\arctan\frac{f\gamma}{4\pi}
\end{aligned}$$

c) The power spectral density of the output is

$$\begin{aligned}
\mathcal{S}_Y(f) &= \sum_{n=-\infty}^{\infty} |x_n|^2 |H(\frac{n}{2})|^2 \delta(f - \frac{n}{2}) \\
&= \frac{1}{4\gamma^2}\delta(f) + 2\sum_{l=0}^{\infty} \frac{|x_{2l+1}|^2}{\gamma^2 + \pi^2(2l+1)^2}\delta(f - \frac{2l+1}{2}) \\
&= \frac{1}{4\gamma^2}\delta(f) + \frac{8}{\pi^2}\sum_{l=0}^{\infty} \frac{1}{(2l+1)^4(\gamma^2 + \pi^2(2l+1)^2)}\delta(f - \frac{2l+1}{2})
\end{aligned}$$

The power content of the output signal is

$$\begin{aligned}
P_Y &= \sum_{n=-\infty}^{\infty} |x_n|^2 |H(\frac{n}{2})|^2 \\
&= \frac{1}{4\gamma^2} + \frac{8}{\pi^2}\sum_{l=0}^{\infty}\left[\frac{1}{\gamma^2(2l+1)^4} + \frac{\pi^4}{\gamma^4(\gamma^2+\pi^2(2l+1)^2)} - \frac{\pi^2}{\gamma^4(2l+1)^2}\right] \\
&= \frac{1}{4\gamma^2} + \frac{8}{\pi^2}\left(\frac{\pi^2}{\gamma^2 96} - \frac{\pi^4}{8\gamma^4} + \frac{\pi^2}{\gamma^4}\sum_{l=0}^{\infty}\frac{1}{\frac{\gamma^2}{\pi^2} + (2l+1)^2}\right) \\
&= \frac{1}{3\gamma^2} - \frac{\pi^2}{\gamma^4} + \frac{2\pi^2}{\gamma^5}\tanh(\frac{\gamma}{2})
\end{aligned}$$

where we have used the fact

$$\tanh(\frac{\pi x}{2}) = \frac{4x}{\pi}\sum_{l=0}^{\infty}\frac{1}{x^2+(2l+1)^2}, \qquad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

d) The power spectral density of the output signal is

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \frac{1}{2}\frac{1}{\gamma^2 + 4\pi^2 f^2}\delta(f) = \frac{1}{2\gamma^2}\delta(f)$$

The power content of the signal is

$$P_Y = \int_{-\infty}^{\infty} \mathcal{S}_Y(f)df = \frac{1}{2\gamma^2}$$

e) $X(f) = -j\pi\text{sgn}(f)$ so that $|X(f)|^2 = \pi^2$ for all f except $f = 0$ for which $|X(f)|^2 = 0$. Thus, the energy spectral density of the output is

$$\mathcal{G}_Y(f) = |X(f)|^2|H(f)|^2 = \frac{\pi^2}{\gamma^2 + 4\pi^2 f^2}$$

and the energy content of the signal

$$E_Y = \pi^2\int_{-\infty}^{\infty}\frac{1}{\gamma^2+4\pi^2 f^2}df = \pi^2\frac{1}{2\pi\gamma}\arctan(\frac{f2\pi}{\gamma})\Big|_{-\infty}^{\infty} = \frac{\pi^2}{2\gamma}$$

2)
a) $h(t) = \text{sinc}(6t) \Longrightarrow H(f) = \frac{1}{6}\Pi(\frac{f}{6})$ The energy spectral density of the output signal is $\mathcal{G}_Y(f) = \mathcal{G}_X(f)|H(f)|^2$ and with $\mathcal{G}_X(f) = \frac{1}{\alpha^2+4\pi^2 f^2}$ we obtain

$$\mathcal{G}_Y(f) = \frac{1}{\alpha^2+4\pi^2 f^2}\frac{1}{36}\Pi^2(\frac{f}{6}) = \frac{1}{36(\alpha^2+4\pi^2 f^2)}\Pi(\frac{f}{6})$$

The energy content of the signal is

$$\begin{aligned}
E_Y &= \int_{-\infty}^{\infty} \mathcal{G}_Y(f)df = \frac{1}{36}\int_{-3}^{3} \frac{1}{\alpha^2+4\pi^2 f^2}df \\
&= \frac{1}{36(2\alpha\pi)}\arctan(f\frac{2\pi}{\alpha})\bigg|_{-3}^{3} \\
&= \frac{1}{36\alpha\pi}\arctan(\frac{6\pi}{\alpha})
\end{aligned}$$

b) The energy spectral density is $\mathcal{G}_Y(f) = \frac{1}{36}\Pi(\frac{f}{6})\Pi(f) = \frac{1}{36}\Pi(f)$ and the energy content of the output

$$E_Y(f) = \frac{1}{36}\int_{-\frac{1}{2}}^{\frac{1}{2}} df = \frac{1}{36}$$

c)

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \sum_{n=-\infty}^{\infty} |x_n|^2 \frac{1}{36}\Pi(\frac{n}{12})\delta(f-\frac{n}{2})$$

Since $\Pi(\frac{n}{12})$ is nonzero only for n such that $\frac{n}{12} \leq \frac{1}{2}$ and $x_0 = \frac{1}{2}$, $x_{2l} = 0$ and $x_{2l+1} = \frac{2}{\pi(2l+1)^2}$ (see Problem 2.2), we obtain

$$\begin{aligned}
\mathcal{S}_Y(f) &= \frac{1}{4\cdot 36}\delta(f) + \sum_{l=-3}^{2} \frac{1}{36}|x_{2l+1}|^2 \delta(f-\frac{2l+1}{2}) \\
&= \frac{1}{144}\delta(f) + \frac{1}{9\pi^2}\sum_{l=-3}^{2}\frac{1}{(2l+1)^4}\delta(f-\frac{2l+1}{2})
\end{aligned}$$

The power content of the signal is

$$P_Y = \frac{1}{144} + \frac{2}{9\pi^2}(1+\frac{1}{81}+\frac{1}{625}) == \frac{1}{144} + \frac{.2253}{\pi^2}$$

d) $\mathcal{S}_X(f) = \frac{1}{2}\delta(f)$, $|H(f)|^2 = \frac{1}{36}\Pi(\frac{f}{6})$. Hence, $\mathcal{S}_Y(f) = \frac{1}{72}\Pi(\frac{f}{6})\delta(f) = \frac{1}{72}\delta(f)$. The power content of the signal is $P_Y = \int_{-\infty}^{\infty}\frac{1}{72}\delta(f)df = \frac{1}{72}$.
e) $y(t) = \text{sinc}(6t) \star \frac{1}{t} = \pi\text{sinc}(6t)\star\frac{1}{\pi t}$. However, convolution with $\frac{1}{\pi t}$ is the Hilbert transform which is known to conserve the energy of the signal provided that there are no impulses at the origin in the frequency domain ($f = 0$). This is the case of $\pi\text{sinc}(6t)$, so that

$$E_Y = \int_{-\infty}^{\infty}\pi^2\text{sinc}^2(6t)dt = \pi^2\int_{-\infty}^{\infty}\frac{1}{36}\Pi^2(\frac{f}{36})df = \frac{\pi^2}{36}\int_{-3}^{3}df = \frac{\pi^2}{6}$$

The energy spectral density is

$$\mathcal{G}_Y(f) = \frac{1}{36}\Pi^2(\frac{f}{6})\pi^2\text{sgn}^2(f)$$

3) $\frac{1}{\pi t}$ is the impulse response of the Hilbert transform filter, which is known to preserve the energy of the input signal. $|H(f)|^2 = \text{sgn}^2(f)$
a) The energy spectral density of the output signal is

$$\mathcal{G}_Y(f) = \mathcal{G}_X(f)\text{sgn}^2(f) = \begin{cases} \mathcal{G}_X(f) & f \neq 0 \\ 0 & f = 0 \end{cases}$$

Since $\mathcal{G}_X(f)$ does not contain any impulses at the origin

$$E_Y = E_X = \frac{1}{2\alpha}$$

b) Arguing as in the previous question

$$\mathcal{G}_Y(f) = \mathcal{G}_X(f)\text{sgn}^2(f) = \begin{cases} \Pi(f) & f \neq 0 \\ 0 & f = 0 \end{cases}$$

Since $\Pi(f)$ does not contain any impulses at the origin

$$E_Y = E_X = 1$$

c)

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)\text{sgn}^2(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta(f - \frac{n}{2}), \quad n \neq 0$$

But, $x_{2l} = 0$, $x_{2l+1} = \frac{1}{\pi(2l+1)}$ so that

$$\mathcal{S}_Y(f) = 2\sum_{l=0}^{\infty} |x_{2l+1}|^2 \delta(f - \frac{n}{2}) = \frac{8}{\pi^2} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} \delta(f - \frac{n}{2})$$

The power content of the output signal is

$$P_Y = \frac{8}{\pi^2} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} = \frac{8}{\pi^2} \frac{\pi^2}{96} = \frac{1}{12}$$

d) $\mathcal{S}_X(f) = \frac{1}{2}\delta(f)$ and $|H(f)|^2 = \text{sgn}^2(f)$. Thus $\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = 0$, and the power content of the signal is zero.
e) The signal $\frac{1}{t}$ has infinite energy and power content, and since $\mathcal{G}_Y(f) = \mathcal{G}_X(f)\text{sgn}^2(f)$, $\mathcal{S}_Y(f) = \mathcal{S}_X(f)\text{sgn}^2(f)$ the same will be true for $y(t) = \frac{1}{t} \star \frac{1}{\pi t}$.

Problem 2.33
Note that

$$P_x = \int_{-\infty}^{\infty} \mathcal{S}_x(f)df = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$

But in the interval $[-\frac{T}{2}, \frac{T}{2}]$, $|x(t)|^2 = |x_T(t)|^2$ so that

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x_T(t)|^2 dt$$

Using Rayleigh's theorem

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x_T(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |X_T(f)|^2 df \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \mathcal{G}_{x_T}(f)df = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{T} \mathcal{G}_{x_T}(f)df \end{aligned}$$

Comparing the last with $P_x = \int_{-\infty}^{\infty} \mathcal{S}_x(f)df$ we see that

$$\mathcal{S}_x(f) = \lim_{T\to\infty} \frac{1}{T} \mathcal{G}_{x_T}(f)$$

Problem 2.34
Let $y(t)$ be the output signal, which is the convolution of $x(t)$, and $h(t)$, $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$. Using Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
|y(t)| &= \left|\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right| \\
&\leq \left[\int_{-\infty}^{\infty} |h(\tau)|^2 d\tau\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} |x(t-\tau)|^2 d\tau\right]^{\frac{1}{2}} \\
&\leq E_h^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} |x(t-\tau)|^2 d\tau\right]^{\frac{1}{2}}
\end{aligned}$$

Squaring the previous inequality and integrating from $-\infty$ to ∞ we obtain

$$\int_{-\infty}^{\infty} |y(t)|^2 dt \leq E_h \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x(t-\tau)|^2 d\tau dt$$

But by assumption $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x(t-\tau)|^2 d\tau dt$, E_h are finite, so that the energy of the output signal is finite.

Consider the LTI system with impulse response $h(t) = \sum_{n=-\infty}^{\infty} \Pi(t-2n)$. The signal is periodic with period $T = 2$, and the power content of the signal is $P_H = \frac{1}{2}$. If the input to this system is the energy type signal $x(t) = \Pi(t)$, then

$$y(t) = \sum_{n=-\infty}^{\infty} \Lambda(t-2n)$$

which is a power type signal with power content $P_Y = \frac{1}{2}$.

Problem 2.35
For no aliasing to occur we must sample at the Nyquist rate

$$f_s = 2 \cdot 6000 \text{ samples/sec } = 12000 \text{ samples/sec}$$

With a guard band of 2000

$$f_s - 2W = 2000 \Longrightarrow f_s = 14000$$

The reconstruction filter should not pick-up frequencies of the images of the spectrum $X(f)$. The nearest image spectrum is centered at f_s and occupies the frequency band $[f_s - W, f_s + W]$. Thus the highest frequency of the reconstruction filter ($= 10000$) should satisfy

$$10000 \leq f_s - W \Longrightarrow f_s \geq 16000$$

For the value $f_s = 16000$, K should be such that

$$K \cdot f_s = 1 \Longrightarrow K = (16000)^{-1}$$

Problem 2.36

$$x(t) = A\text{sinc}(1000\pi t) \Longrightarrow X(f) = \frac{A}{1000}\Pi(\frac{f}{1000})$$

Thus the bandwidth W of $x(t)$ is $1000/2 = 500$. Since we sample at $f_s = 2000$ there is a gap between the image spectra equal to

$$2000 - 500 - W = 1000$$

The reconstruction filter should have a bandwidth W' such that $500 < W' < 1500$. A filter that satisfy these conditions is

$$H(f) = T_s \Pi\left(\frac{f}{2W'}\right) = \frac{1}{2000}\Pi\left(\frac{f}{2W'}\right)$$

and the more general reconstruction filters have the form

$$H(f) = \begin{cases} \frac{1}{2000} & |f| < 500 \\ \text{arbitrary} & 500 < |f| < 1500 \\ 0 & |f| > 1500 \end{cases}$$

Problem 2.37

1)

$$\begin{aligned} x_p(t) &= \sum_{n=-\infty}^{\infty} x(nT_s)p(t-nT_s) \\ &= p(t) \star \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s) \\ &= p(t) \star x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT_s) \end{aligned}$$

Thus

$$\begin{aligned} X_p(f) &= P(f) \cdot \mathcal{F}\left[x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT_s)\right] \\ &= P(f)X(f) \star \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \delta(t-nT_s)\right] \\ &= P(f)X(f) \star \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_s}) \\ &= \frac{1}{T_s} P(f) \sum_{n=-\infty}^{\infty} X(f - \frac{n}{T_s}) \end{aligned}$$

2) In order to avoid aliasing $\frac{1}{T_s} > 2W$. Furthermore the spectrum $P(f)$ should be invertible for $|f| < W$.

3) $X(f)$ can be recovered using the reconstruction filter $\Pi(\frac{f}{2W_\Pi})$ with $W < W_\Pi < \frac{1}{T_s} - W$. In this case

$$X(f) = X_p(f)T_s P^{-1}(f)\Pi(\frac{f}{2W_\Pi})$$

Problem 2.38

1)

$$\begin{aligned} x_1(t) &= \sum_{n=-\infty}^{\infty} (-1)^n x(nT_s)\delta(t-nT_s) = x(t) \sum_{n=-\infty}^{\infty} (-1)^n \delta(t-nT_s) \\ &= x(t)\left[\sum_{l=-\infty}^{\infty} \delta(t-2lT_s) - \sum_{l=-\infty}^{\infty} \delta(t - T_s - 2lT_s)\right] \end{aligned}$$

Thus

$$
\begin{aligned}
X_1(f) &= X(f) \star \left[\frac{1}{2T_s} \sum_{l=-\infty}^{\infty} \delta(f - \frac{l}{2T_s}) - \frac{1}{2T_s} \sum_{l=-\infty}^{\infty} \delta(f - \frac{l}{2T_s}) e^{-j2\pi f T_s} \right] \\
&= \frac{1}{2T_s} \sum_{l=-\infty}^{\infty} X(f - \frac{l}{2T_s}) - \frac{1}{2T_s} \sum_{l=-\infty}^{\infty} X(f - \frac{l}{2T_s}) e^{-j2\pi \frac{l}{2T_s} T_s} \\
&= \frac{1}{2T_s} \sum_{l=-\infty}^{\infty} X(f - \frac{l}{2T_s}) - \frac{1}{2T_s} \sum_{l=-\infty}^{\infty} X(f - \frac{l}{2T_s}) (-1)^l \\
&= \frac{1}{T_s} \sum_{l=-\infty}^{\infty} X(f - \frac{1}{2T_s} - \frac{l}{T_s})
\end{aligned}
$$

2) The spectrum of $x(t)$ occupies the frequency band $[-W, W]$. Suppose that from the periodic spectrum $X_1(f)$ we isolate $X_k(f) = \frac{1}{T_s} X(f - \frac{1}{2T_s} - \frac{k}{T_s})$, with a bandpass filter, and we use it to reconstruct $x(t)$. Since $X_k(f)$ occupies the frequency band $[2kW, 2(k+1)W]$, then for all k, $X_k(f)$ cannot cover the whole interval $[-W, W]$. Thus at the output of the reconstruction filter there will exist frequency components which are not present in the input spectrum. Hence, the reconstruction filter has to be a time-varying filter. To see this in the time domain, note that the original spectrum has been shifted by $f' = \frac{1}{2T_s}$. In order to bring the spectrum back to the origin and reconstruct $x(t)$ the sampled signal $x_1(t)$ has to be multiplied by $e^{-j2\pi \frac{1}{2T_s} t} = e^{-j2\pi W t}$. However the system described by

$$y(t) = e^{j2\pi W t} x(t)$$

is a time-varying system.

3) Using a time-varying system we can reconstruct $x(t)$ as follows. Use the bandpass filter $T_s \Pi(\frac{f-W}{2W})$ to extract the component $X(f - \frac{1}{2T_s})$. Invert $X(f - \frac{1}{2T_s})$ and multiply the resultant signal by $e^{-j2\pi W t}$. Thus

$$x(t) = e^{-j2\pi W t} \mathcal{F}^{-1} \left[T_s \Pi(\frac{f-W}{2W}) X_1(f) \right]$$

Problem 2.39

1) The linear interpolation system can be viewed as a linear filter where the sampled signal $x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$ is passed through the filter with impulse response

$$h(t) = \begin{cases} 1 + \frac{t}{T_s} & -T_s \leq f \leq 0 \\ 1 - \frac{t}{T_s} & 0 \leq f \leq T_s \\ 0 & \text{otherwise} \end{cases}$$

To see this write

$$x_1(t) = \left[x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right] \star h(t) = \sum_{n=-\infty}^{\infty} x(nT_s) h(t - nT_s)$$

Comparing this with the interpolation formula in the interval $[nT_s, (n+1)T_s]$

$$
\begin{aligned}
x_1(t) &= x(nT_s) + \frac{t - nT_s}{T_s} \left(x((n+1)T_s) - x(nT_s) \right) \\
&= x(nT_s) \left[1 - \frac{t - nT_s}{T_s} \right] + x((n+1)T_s) \left[1 + \frac{t - (n+1)T_s}{T_s} \right] \\
&= x(nT_s) h(t - nT_s) + x((n+1)T_s) h(t - (n+1)T_s)
\end{aligned}
$$

we observe that $h(t)$ does not extend beyond $[-T_s, T_s]$ and in this interval its form should be the one described above. The power spectrum of $x_1(t)$ is $S_{X_1}(f) = |X_1(f)|^2$ where

$$\begin{aligned}
X_1(f) &= \mathcal{F}[x_1(t)] = \mathcal{F}\left[h(t) \star x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right] \\
&= H(f)\left[X(f) \star \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_s})\right] \\
&= \text{sinc}^2(fT_s) \sum_{n=-\infty}^{\infty} X(f - \frac{n}{T_s})
\end{aligned}$$

2) The system function $\text{sinc}^2(fT_s)$ has zeros at the frequencies f such that

$$fT_s = k, \quad k \in \mathcal{Z} - \{0\}$$

In order to recover $X(f)$, the bandwidth W of $x(t)$ should be smaller than $1/T_s$, so that the whole $X(f)$ lies inside the main lobe of $\text{sinc}^2(fT_s)$. This condition is automatically satisfied if we choose T_s such that to avoid aliasing ($2W < 1/T_s$). In this case we can recover $X(f)$ from $X_1(f)$ using the lowpass filter $\Pi(\frac{f}{2W})$.

$$\Pi(\frac{f}{2W})X_1(f) = \text{sinc}^2(fT_s)X(f)$$

or

$$X(f) = (\text{sinc}^2(fT_s))^{-1}\Pi(\frac{f}{2W})X_1(f)$$

If $T_s \ll 1/W$, then $\text{sinc}^2(fT_s) \approx 1$ for $|f| < W$ and $X(f)$ is available using $X(f) = \Pi(\frac{f}{2W})X_1(f)$.

Problem 2.40

1) $W = 50$Hz so that $T_s = 1/2W = 10^{-2}$sec. The reconstructed signal is

$$\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} x(nT_s)\text{sinc}(\frac{t}{T_s} - n) \\
&= -\sum_{n=-4}^{-1} \text{sinc}(\frac{t}{T_s} - n) + \sum_{n=1}^{4} \text{sinc}(\frac{t}{T_s} - n)
\end{aligned}$$

With $T_s = 10^{-2}$ and $t = 5 \cdot 10^{-3}$ we obtain

$$\begin{aligned}
x(.005) &= -\sum_{n=1}^{4} \text{sinc}(\frac{1}{2} + n) + \sum_{n=1}^{4} \text{sinc}(\frac{1}{2} - n) \\
&= -[\text{sinc}(\frac{3}{2}) + \text{sinc}(\frac{5}{2}) + \text{sinc}(\frac{7}{2}) + \text{sinc}(\frac{9}{2})] \\
&\quad +[\text{sinc}(-\frac{1}{2}) + \text{sinc}(-\frac{3}{2}) + \text{sinc}(-\frac{5}{2}) + \text{sinc}(-\frac{7}{2})] \\
&= \text{sinc}(\frac{1}{2}) - \text{sinc}(\frac{9}{2}) = \frac{2}{\pi}\sin(\frac{\pi}{2}) - \frac{2}{9\pi}\sin(\frac{9\pi}{2}) \\
&= \frac{16}{9\pi}
\end{aligned}$$

where we have used the fact that $\text{sinc}(t)$ is an even function.

2) Note that (see Problem 2.41)

$$\int_{-\infty}^{\infty} \text{sinc}(2Wt - m)\text{sinc}^*(2Wt - n)dt = \frac{1}{2W}\delta_{mn}$$

with δ_{mn} the Kronecker delta. Thus,

$$\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)x^*(t)dt \\
&= \sum_{n=-\infty}^{\infty} x(nT_s)x^*(mT_s) \int_{-\infty}^{\infty} \text{sinc}(2Wt-m)\text{sinc}^*(2Wt-n)dt \\
&= \sum_{n=-\infty}^{\infty} |x(nT_s)|^2 \frac{1}{2W}
\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2W}\left[\sum_{n=-4}^{-1} 1 + \sum_{n=1}^{4} 1\right] = \frac{4}{W} = 8 \cdot 10^{-2}$$

Problem 2.41

1) Using Parseval's theorem we obtain

$$\begin{aligned}
A &= \int_{-\infty}^{\infty} \text{sinc}(2Wt-m)\text{sinc}(2Wt-n)dt \\
&= \int_{-\infty}^{\infty} \mathcal{F}[\text{sinc}(2Wt-m)]\mathcal{F}[\text{sinc}(2Wt-n)]dt \\
&= \int_{-\infty}^{\infty} (\frac{1}{2W})^2 \Pi^2(\frac{f}{2W}) e^{-j2\pi f \frac{m-n}{2W}} df \\
&= \frac{1}{4W^2}\int_{-W}^{W} e^{-j2\pi f \frac{m-n}{2W}} df = \frac{1}{2W}\delta_{mn}
\end{aligned}$$

where δ_{mn} is the Kronecker's delta. The latter implies that $\{\text{sinc}(2Wt-m)\}$ form an orthogonal set of signals. In order to generate an orthonormal set of signals we have to weight each function by $1/\sqrt{2W}$.

2) The bandlimited signal $x(t)$ can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\text{sinc}(2Wt-n)$$

where $x(nT_s)$ are the samples taken at the Nyquist rate. This is an orthogonal expansion relation where the basis functions $\{\text{sinc}(2Wt-m)\}$ are weighted by $x(mT_s)$.

3)

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)\text{sinc}(2Wt-n)dt &= \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(mT_s)\text{sinc}(2Wt-m)\text{sinc}(2Wt-n)dt \\
&= \sum_{m=-\infty}^{\infty} x(mT_s) \int_{-\infty}^{\infty} \text{sinc}(2Wt-m)\text{sinc}(2Wt-n)dt \\
&= \sum_{m=-\infty}^{\infty} x(mT_s)\frac{1}{2W}\delta_{mn} = \frac{1}{2W}x(nT_s)
\end{aligned}$$

Problem 2.42
We define a new signal $y(t) = x(t+t_0)$. Then $y(t)$ is bandlimited with $Y(f) = e^{j2\pi f t_0}X(f)$ and the samples of $y(t)$ at $\{kT_s\}_{k=-\infty}^{\infty}$ are equal to the samples of $x(t)$ at $\{t_0 + kT_s\}_{k=-\infty}^{\infty}$. Applying the sampling theorem to the reconstruction of $y(t)$, we have

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} y(kT_s)\mathrm{sinc}\,(2W(t-kT_s)) \quad (1)\\ &= \sum_{k=-\infty}^{\infty} x(t_0+kT_s)\mathrm{sinc}\,(2W(t-kT_s)) \quad (2)\end{aligned}$$

and, hence,

$$x(t+t_0) = \sum_{k=-\infty}^{\infty} x(t_0+kT_s)\mathrm{sinc}\,(2W(t-kT_s))$$

Substituting $t = -t_0$ we obtain the following important interpolation relation.

$$x(0) = \sum_{k=-\infty}^{\infty} x(t_0+kT_s)\mathrm{sinc}\,(2W(t_0+kT_s))$$

Problem 2.43
We know that

$$\begin{aligned} x(t) &= x_c(t)\cos(2\pi f_0 t) - x_s(t)\sin(2\pi f_0 t)\\ \hat{x}(t) &= x_c(t)\sin(2\pi f_0 t) + x_s(t)\cos(2\pi f_0 t)\end{aligned}$$

We can write these relations in matrix notation as

$$\begin{pmatrix} x(t)\\ \hat{x}(t)\end{pmatrix} = \begin{pmatrix} \cos(2\pi f_0 t) & -\sin(2\pi f_0 t)\\ \sin(2\pi f_0 t) & \cos(2\pi f_0 t)\end{pmatrix}\begin{pmatrix} x_c(t)\\ x_s(t)\end{pmatrix} = R\begin{pmatrix} x_c(t)\\ x_s(t)\end{pmatrix}$$

The rotation matrix R is nonsingular ($\det(R) = 1$) and its inverse is

$$R^{-1} = \begin{pmatrix} \cos(2\pi f_0 t) & \sin(2\pi f_0 t)\\ -\sin(2\pi f_0 t) & \cos(2\pi f_0 t)\end{pmatrix}$$

Thus

$$\begin{pmatrix} x_c(t)\\ x_s(t)\end{pmatrix} = R^{-1}\begin{pmatrix} x(t)\\ \hat{x}(t)\end{pmatrix} = \begin{pmatrix} \cos(2\pi f_0 t) & \sin(2\pi f_0 t)\\ -\sin(2\pi f_0 t) & \cos(2\pi f_0 t)\end{pmatrix}\begin{pmatrix} x(t)\\ \hat{x}(t)\end{pmatrix}$$

and the result follows.

Problem 2.44
$x_c(t) = \mathrm{Re}[x_l(t)]$. Thus

$$x_c(t) = \frac{1}{2}[x_l(t) + x^*(t)]$$

Taking the Fourier transform of the previous relation we obtain

$$X_c(f) = \frac{1}{2}[X_l(f) + X_l^*(-f)]$$

Problem 2.45

$$\begin{aligned}
x_1(t) &= x(t)\sin(2\pi f_0 t)\\
X_1(f) &= -\frac{1}{2j}X(f+f_0) + \frac{1}{2j}X(f-f_0)
\end{aligned}$$

$$\begin{aligned}
x_2(t) &= \hat{x}(t)\\
X_2(f) &= -j\mathrm{sgn}(f)X(f)
\end{aligned}$$

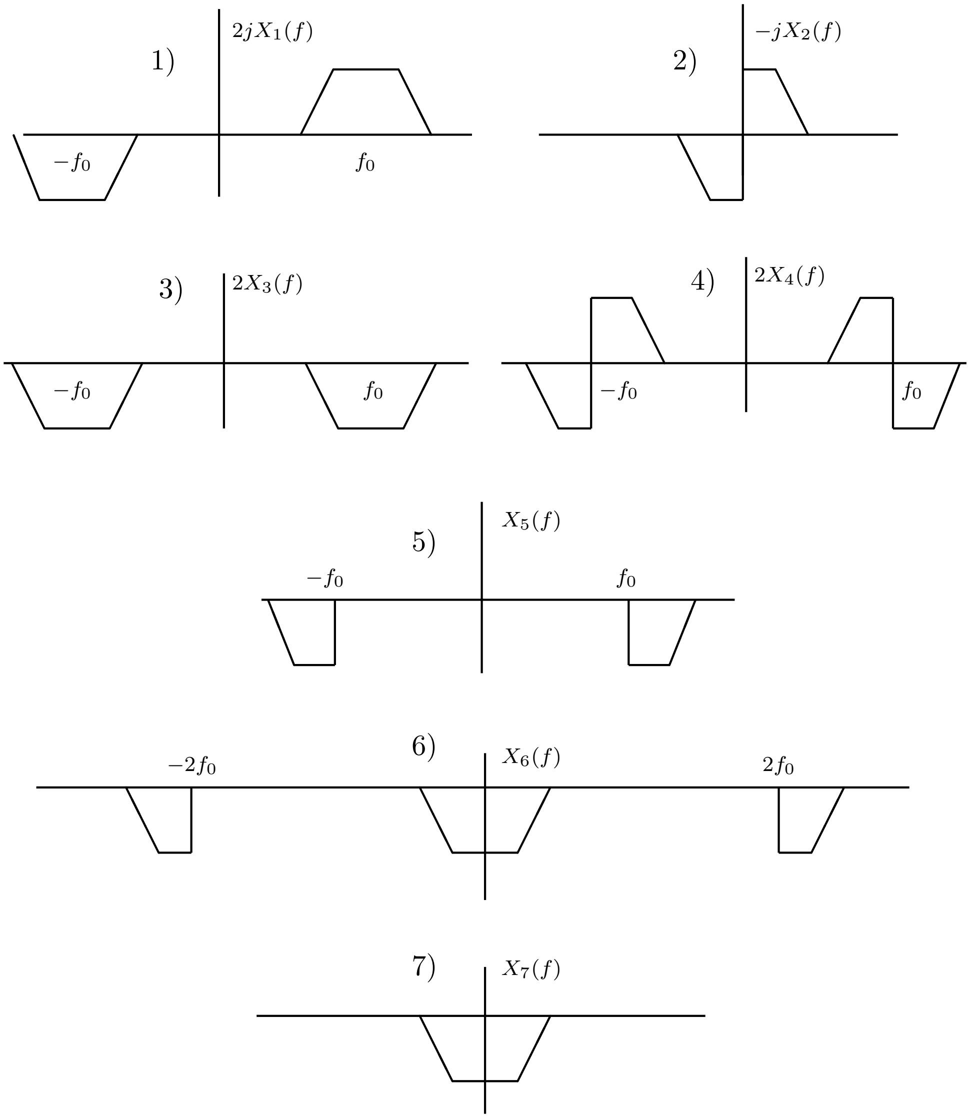
$$\begin{aligned}
x_3(t) &= \hat{x_1}(t) = x(t)\widehat{\sin(2\pi f_0 t)} = -x(t)\cos(2\pi f_0 t)\\
X_3(f) &= -\frac{1}{2}X(f+f_0) - \frac{1}{2}X(f-f_0)
\end{aligned}$$

$$\begin{aligned}
x_4(t) &= x_2(t)\sin(2\pi f_0 t) = \hat{x}(t)\sin(2\pi f_0 t)\\
X_4(f) &= -\frac{1}{2j}\hat{X}(f+f_0) + \frac{1}{2j}\hat{X}(f-f_0)\\
&= -\frac{1}{2j}\left[-j\mathrm{sgn}(f+f_0)X(f+f_0)\right] + \frac{1}{2j}\left[-j\mathrm{sgn}(f-f_0)X(f-f_0)\right]\\
&= \frac{1}{2}\mathrm{sgn}(f+f_0)X(f+f_0) - \frac{1}{2}\mathrm{sgn}(f-f_0)X(f-f_0)
\end{aligned}$$

$$\begin{aligned}
x_5(t) &= \hat{x}(t)\sin(2\pi f_0 t) + x(t)\cos(2\pi f_0 t)\\
X_5(f) &= X_4(f) - X_3(f) = \frac{1}{2}X(f+f_0)(\mathrm{sgn}(f+f_0)-1) - \frac{1}{2}X(f-f_0)(\mathrm{sgn}(f-f_0)+1)
\end{aligned}$$

$$\begin{aligned}
x_6(t) &= [\hat{x}(t)\sin(2\pi f_0 t) + x(t)\cos(2\pi f_0 t)]2\cos(2\pi f_0 t)\\
X_6(f) &= X_5(f+f_0) + X_5(f-f_0)\\
&= \frac{1}{2}X(f+2f_0)(\mathrm{sgn}(f+2f_0)-1) - \frac{1}{2}X(f)(\mathrm{sgn}(f)+1)\\
&\quad +\frac{1}{2}X(f)(\mathrm{sgn}(f)-1) - \frac{1}{2}X(f-2f_0)(\mathrm{sgn}(f-2f_0)+1)\\
&= -X(f) + \frac{1}{2}X(f+2f_0)(\mathrm{sgn}(f+2f_0)-1) - \frac{1}{2}X(f-2f_0)(\mathrm{sgn}(f-2f_0)+1)
\end{aligned}$$

$$\begin{aligned}
x_7(t) &= x_6(t) \star 2W\mathrm{sinc}(2Wt) = -x(t)\\
X_7(f) &= X_6(f)\Pi(\frac{f}{2W}) = -X(f)
\end{aligned}$$



Problem 2.46
If $x(t)$ is even then $X(f)$ is a real and even function and therefore $-j\,\mathrm{sgn}(f)X(f)$ is an imaginary and odd function. Hence, its inverse Fourier transform $\hat{x}(t)$ will be odd. If $x(t)$ is odd then $X(f)$ is imaginary and odd and $-j\,\mathrm{sgn}(f)X(f)$ is real and even and, therefore, $\hat{x}(t)$ is even.

Problem 2.47
Using Rayleigh's theorem of the Fourier transform we have

$$E_x = \int_{-\infty} \infty |x(t)|^2 dt = \int_{-\infty} \infty |X(f)|^2 df$$

and

$$E_{\hat{x}} = \int_{-\infty} \infty |\hat{x}(t)|^2 dt = \int_{-\infty} \infty |-j\mathrm{sgn}(f)X(f)|^2 df$$

Noting the fact that $|-j\mathrm{sgn}(f)|^2 = 1$ except for $f = 0$, and the fact that $X(f)$ does not contain any impulses at the origin we conclude that $E_x = E_{\hat{x}}$.

Problem 2.48
Here we use Parseval's Theorem of the Fourier Transform to obtain

$$\int_{-\infty} \infty x(t)\hat{x}(t)\,dt \quad = \quad \int_{-\infty} \infty X(f)[-j\mathrm{sgn}(f)X(f)]^* df$$

$$
\begin{aligned}
&= -j\int_{-\infty}^{0} |X(f)|^2 df + j\int_{0}^{+\infty} |X(f)|^2 df \\
&= 0
\end{aligned}
$$

where in the last step we have used the fact that $X(f)$ is Hermitian and therefore $|X(f)|^2$ is even.

Problem 2.49

We note that $C(f) = M(f) \star X(f)$. From the assumption on the bandwidth of $m(t)$ and $x(t)$ we see that $C(f)$ consists of two separate positive frequency and negative frequency regions that do not overlap. Let us denote these regions by $C_+(f)$ and $C_-(f)$ respectively. A moment's thought shows that

$$C_+(f) = M(f) \star X_+(f)$$

and

$$C_-(f) = M(f) \star X_-(f)$$

To find the Hilbert Transform of $c(t)$ we note that

$$
\begin{aligned}
\mathcal{F}[\hat{c}(t)] &= -j\mathrm{sgn}(f)C(f) \\
&= -jC_+(f) + jC_-(f) \\
&= -jM(f) \star X_+(f) + jM(f) \star X_-(f) \\
&= M(f) \star [-jX_+(f) + jX_-(f)] \\
&= M(f) \star [-j\mathrm{sgn}(f)X(f)] \\
&= M(f) \star \mathcal{F}[\hat{x}(t)]
\end{aligned}
$$

Returning to the time domain we obtain

$$\hat{c}(t) = m(t)\hat{x}(t)$$

Problem 2.50

It is enough to note that

$$\mathcal{F}[\hat{\hat{x}}(t)] = (-j\mathrm{sgn}(f))^2 X(f)$$

and hence

$$\mathcal{F}[\hat{\hat{x}}(t)] = -X(f)$$

where we have used the fact that $X(f)$ does not contain any impulses at the origin.

Problem 2.51

Using the result of Problem 2.49 and noting that the Hilbert transform of cos is sin we have

$$x(t)\widehat{\cos(2\pi f_0 t)} = x(t)\sin(2\pi f_0 t)$$

Problem 2.52

$$
\begin{aligned}
\mathcal{F}[A\widehat{\sin(2\pi f_0 t + \theta)}] &= -j\mathrm{sgn}(f)A\left[-\frac{1}{2j}\delta(f+f_0)e^{j2\pi f\frac{\theta}{2f_0}} + \frac{1}{2j}\delta(f-f_0)e^{-j2\pi f\frac{\theta}{2f_0}}\right] \\
&= \frac{A}{2}\left[\mathrm{sgn}(-f_0)\delta(f+f_0)e^{j2\pi f\frac{\theta}{2f_0}} - \mathrm{sgn}(-f_0)\delta(f-f_0)e^{-j2\pi f\frac{\theta}{2f_0}}\right] \\
&= -\frac{A}{2}\left[\delta(f+f_0)e^{j2\pi f\frac{\theta}{2f_0}} + \delta(f-f_0)e^{-j2\pi f\frac{\theta}{2f_0}}\right] \\
&= -A\mathcal{F}[\cos(2\pi f_0 t + \theta)]
\end{aligned}
$$

Thus, $\widehat{A\sin(2\pi f_0 t+\theta)} = -A\cos(2\pi f_0 t+\theta)$

Problem 2.53

Taking the Fourier transform of $\widehat{e^{j2\pi f_0 t}}$ we obtain

$$\mathcal{F}[\widehat{e^{j2\pi f_0 t}}] = -j\text{sgn}(f)\delta(f-f_0) = -j\text{sgn}(f_0)\delta(f-f_0)$$

Thus,

$$\widehat{e^{j2\pi f_0 t}} = \mathcal{F}^{-1}[-j\text{sgn}(f_0)\delta(f-f_0)] = -j\text{sgn}(f_0)e^{j2\pi f_0 t}$$

Problem 2.54

$$\begin{aligned}
\mathcal{F}\left[\widehat{\frac{d}{dt}x(t)}\right] &= \mathcal{F}[\widehat{x(t)\star\delta'(t)}] = -j\text{sgn}(f)\mathcal{F}[x(t)\star\delta'(t)] \\
&= -j\text{sgn}(f)j2\pi f X(f) = 2\pi f\text{sgn}(f)X(f) \\
&= 2\pi|f|X(f)
\end{aligned}$$

Problem 2.55

We need to prove that $\widehat{x'(t)} = (\hat{x}(t))'$.

$$\begin{aligned}
\mathcal{F}[\widehat{x'(t)}] &= \mathcal{F}[\widehat{x(t)\star\delta'(t)}] = -j\text{sgn}(f)\mathcal{F}[x(t)\star\delta'(t)] = -j\text{sgn}(f)X(f)j2\pi f \\
&= \mathcal{F}[\hat{x}(t)]j2\pi f = \mathcal{F}[(\hat{x}(t))']
\end{aligned}$$

Taking the inverse Fourier transform of both sides of the previous relation we obtain, $\widehat{x'(t)} = (\hat{x}(t))'$

Problem 2.56

$$\begin{aligned}
x(t) = \text{sinc}t\cos 2\pi f_0 t &\implies X(f) = \frac{1}{2}\Pi(f+f_0)) + \frac{1}{2}\Pi(f-f_0)) \\
h(t) = \text{sinc}^2 t\sin 2\pi f_0 t &\implies H(f) = -\frac{1}{2j}\Lambda(f+f_0)) + \frac{1}{2j}\Lambda(f-f_0))
\end{aligned}$$

The lowpass equivalents are

$$\begin{aligned}
X_l(f) &= 2u(f+f_0)X(f+f_0) = \Pi(f) \\
H_l(f) &= 2u(f+f_0)H(f+f_0) = \frac{1}{j}\Lambda(f) \\
Y_l(f) &= \frac{1}{2}X_l(f)H_l(f) = \begin{cases} \frac{1}{2j}(f+1) & -\frac{1}{2} < f \le 0 \\ \frac{1}{2j}(-f+1) & 0 \le f < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Taking the inverse Fourier transform of $Y_l(f)$ we can find the lowpass equivalent response of the system. Thus,

$$\begin{aligned}
y_l(t) &= \mathcal{F}^{-1}[Y_l(f)] \\
&= \frac{1}{2j}\int_{-\frac{1}{2}}^{0}(f+1)e^{j2\pi ft}df + \frac{1}{2j}\int_{0}^{\frac{1}{2}}(-f+1)e^{j2\pi ft}df \\
&= \frac{1}{2j}\left[\frac{1}{j2\pi t}fe^{j2\pi ft} + \frac{1}{4\pi^2t^2}e^{j2\pi ft}\right]\Bigg|_{-\frac{1}{2}}^{0} + \frac{1}{2j}\frac{1}{j2\pi t}e^{j2\pi ft}\Bigg|_{-\frac{1}{2}}^{0} \\
&\quad -\frac{1}{2j}\left[\frac{1}{j2\pi t}fe^{j2\pi ft} + \frac{1}{4\pi^2t^2}e^{j2\pi ft}\right]\Bigg|_{0}^{\frac{1}{2}} + \frac{1}{2j}\frac{1}{j2\pi t}e^{j2\pi ft}\Bigg|_{0}^{\frac{1}{2}} \\
&= j\left[-\frac{1}{4\pi t}\sin\pi t + \frac{1}{4\pi^2t^2}(\cos\pi t - 1)\right]
\end{aligned}$$

The output of the system $y(t)$ can now be found from $y(t) = \mathrm{Re}[y_l(t)e^{j2\pi f_0 t}]$. Thus

$$\begin{aligned} y(t) &= \mathrm{Re}\left[(j[-\frac{1}{4\pi t}\sin \pi t + \frac{1}{4\pi^2 t^2}(\cos \pi t - 1)])(\cos 2\pi f_0 t + j \sin 2\pi f_0 t)\right] \\ &= [\frac{1}{4\pi^2 t^2}(1 - \cos \pi t) + \frac{1}{4\pi t}\sin \pi t] \sin 2\pi f_0 t \end{aligned}$$

Problem 2.57

1) The spectrum of the output signal $y(t)$ is the product of $X(f)$ and $H(f)$. Thus,

$$Y(f) = H(f)X(f) = X(f)A(f_0)e^{j(\theta(f_0)+(f-f_0)\theta'(f)|_{f=f_0})}$$

$y(t)$ is a narrowband signal centered at frequencies $f = \pm f_0$. To obtain the lowpass equivalent signal we have to shift the spectrum (positive band) of $y(t)$ to the right by f_0. Hence,

$$Y_l(f) = u(f+f_0)X(f+f_0)A(f_0)e^{j(\theta(f_0)+f\theta'(f)|_{f=f_0})} = X_l(f)A(f_0)e^{j(\theta(f_0)+f\theta'(f)|_{f=f_0})}$$

2) Taking the inverse Fourier transform of the previous relation, we obtain

$$\begin{aligned} y_l(t) &= \mathcal{F}^{-1}\left[X_l(f)A(f_0)e^{j\theta(f_0)}e^{jf\theta'(f)|_{f=f_0}}\right] \\ &= A(f_0)x_l(t + \frac{1}{2\pi}\theta'(f)|_{f=f_0}) \end{aligned}$$

With $y(t) = \mathrm{Re}[y_l(t)e^{j2\pi f_0 t}]$ and $x_l(t) = V_x(t)e^{j\Theta_x(t)}$ we get

$$\begin{aligned} y(t) &= \mathrm{Re}[y_l(t)e^{j2\pi f_0 t}] \\ &= \mathrm{Re}\left[A(f_0)x_l(t + \frac{1}{2\pi}\theta'(f)|_{f=f_0})e^{j\theta(f_0)}e^{j2\pi f_0 t}\right] \\ &= \mathrm{Re}\left[A(f_0)V_x(t + \frac{1}{2\pi}\theta'(f)|_{f=f_0})e^{j2\pi f_0 t}e^{j\Theta_x(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0})}\right] \\ &= A(f_0)V_x(t - t_g)\cos(2\pi f_0 t + \theta(f_0) + \Theta_x(t + \frac{1}{2\pi}\theta'(f)|_{f=f_0})) \\ &= A(f_0)V_x(t - t_g)\cos(2\pi f_0(t + \frac{\theta(f_0)}{2\pi f_0}) + \Theta_x(t + \frac{1}{2\pi}\theta'(f)|_{f=f_0})) \\ &= A(f_0)V_x(t - t_g)\cos(2\pi f_0(t - t_p) + \Theta_x(t + \frac{1}{2\pi}\theta'(f)|_{f=f_0})) \end{aligned}$$

where

$$t_g = -\frac{1}{2\pi}\theta'(f)|_{f=f_0}, \qquad t_p = -\frac{1}{2\pi}\frac{\theta(f_0)}{f_0} = -\frac{1}{2\pi}\frac{\theta(f)}{f}\bigg|_{f=f_0}$$

3) t_g can be considered as a time lag of the envelope of the signal, whereas t_p is the time corresponding to a phase delay of $\frac{1}{2\pi}\frac{\theta(f_0)}{f_0}$.

Problem 2.58

1) We can write $H_\theta(f)$ as follows

$$H_\theta(f) = \begin{cases} \cos\theta - j\sin\theta & f > 0 \\ 0 & f = 0 \\ \cos\theta + j\sin\theta & f < 0 \end{cases} = \cos\theta - j\mathrm{sgn}(f)\sin\theta$$

Thus,

$$h_\theta(t) = \mathcal{F}^{-1}[H_\theta(f)] = \cos\theta\delta(t) + \frac{1}{\pi t}\sin\theta$$

2)

$$\begin{aligned}
x_\theta(t) &= x(t) \star h_\theta(t) = x(t) \star (\cos\theta\delta(t) + \frac{1}{\pi t}\sin\theta) \\
&= \cos\theta x(t) \star \delta(t) + \sin\theta \frac{1}{\pi t} \star x(t) \\
&= \cos\theta x(t) + \sin\theta \hat{x}(t)
\end{aligned}$$

3)

$$\begin{aligned}
\int_{-\infty}^{\infty} |x_\theta(t)|^2 dt &= \int_{-\infty}^{\infty} |\cos\theta x(t) + \sin\theta\hat{x}(t)|^2 dt \\
&= \cos^2\theta \int_{-\infty}^{\infty} |x(t)|^2 dt + \sin^2\theta \int_{-\infty}^{\infty} |\hat{x}(t)|^2 dt \\
&\quad + \cos\theta\sin\theta \int_{-\infty}^{\infty} x(t)\hat{x}^*(t)dt + \cos\theta\sin\theta \int_{-\infty}^{\infty} x^*(t)\hat{x}(t)dt
\end{aligned}$$

But $\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{x}(t)|^2 dt = E_x$ and $\int_{-\infty}^{\infty} x(t)\hat{x}^*(t)dt = 0$ since $x(t)$ and $\hat{x}(t)$ are orthogonal. Thus,

$$E_{x_\theta} = E_x(\cos^2\theta + \sin^2\theta) = E_x$$

Problem 2.59

1)

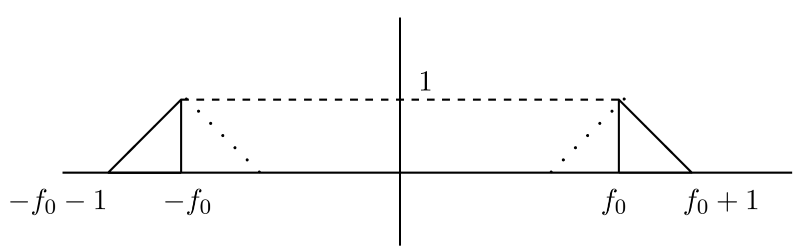
$$\begin{aligned}
z(t) &= x(t) + j\hat{x}(t) = m(t)\cos(2\pi f_0 t) - \hat{m}(t)\sin(2\pi f_0 t) \\
&\quad + j[m(t)\widehat{\cos(2\pi f_0 t)} - \hat{m}(t)\widehat{\sin(2\pi f_0 t)}) \\
&= m(t)\cos(2\pi f_0 t) - \hat{m}(t)\sin(2\pi f_0 t) \\
&\quad + jm(t)\sin(2\pi f_0 t) + j\hat{m}(t)\cos(2\pi f_0 t) \\
&= (m(t) + j\hat{m}(t))e^{j2\pi f_0 t}
\end{aligned}$$

The lowpass equivalent signal is given by

$$x_l(t) = z(t)e^{-j2\pi f_0 t} = m(t) + j\hat{m}(t)$$

2) The Fourier transform of $m(t)$ is $\Lambda(f)$. Thus

$$\begin{aligned}
X(f) &= \frac{\Lambda(f+f_0) + \Lambda(f-f_0)}{2} - (-j\text{sgn}(f)\Lambda(f)) \star \\
&\quad \left[-\frac{1}{2j}\delta(f+f_0) + \frac{1}{2j}\delta(f-f_0)\right] \\
&= \frac{1}{2}\Lambda(f+f_0)\left[1 - \text{sgn}(f+f_0)\right] + \frac{1}{2}\Lambda(f-f_0)\left[1 + \text{sgn}(f-f_0)\right]
\end{aligned}$$



The bandwidth of $x(t)$ is $W = 1$.

3)

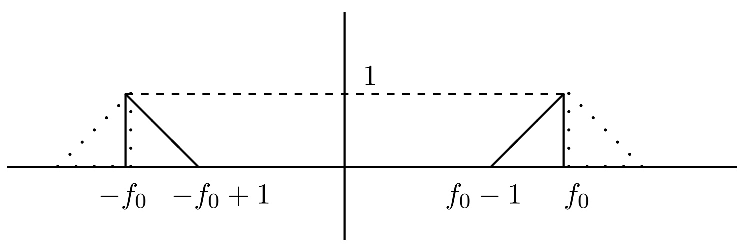
$$
\begin{aligned}
z(t) &= x(t) + j\hat{x}(t) = m(t)\cos(2\pi f_0 t) + \hat{m}(t)\sin(2\pi f_0 t) \\
&\quad + j[\widehat{m(t)\cos(2\pi f_0 t)} + \widehat{\hat{m}(t)\sin(2\pi f_0 t)}] \\
&= m(t)\cos(2\pi f_0 t) + \hat{m}(t)\sin(2\pi f_0 t) \\
&\quad + jm(t)\sin(2\pi f_0 t) - j\hat{m}(t)\cos(2\pi f_0 t) \\
&= (m(t) - j\hat{m}(t))e^{j2\pi f_0 t}
\end{aligned}
$$

The lowpass equivalent signal is given by

$$x_l(t) = z(t)e^{-j2\pi f_0 t} = m(t) - j\hat{m}(t)$$

The Fourier transform of $x(t)$ is

$$
\begin{aligned}
X(f) &= \frac{\Lambda(f+f_0) + \Lambda(f-f_0)}{2} - (j\mathrm{sgn}(f)\Lambda(f)) \star \\
&\qquad \left[-\frac{1}{2j}\delta(f+f_0) + \frac{1}{2j}\delta(f-f_0)\right] \\
&= \frac{1}{2}\Lambda(f+f_0)\left[1 + \mathrm{sgn}(f+f_0)\right] + \frac{1}{2}\Lambda(f-f_0)\left[1 - \mathrm{sgn}(f-f_0)\right]
\end{aligned}
$$



Chapter 3

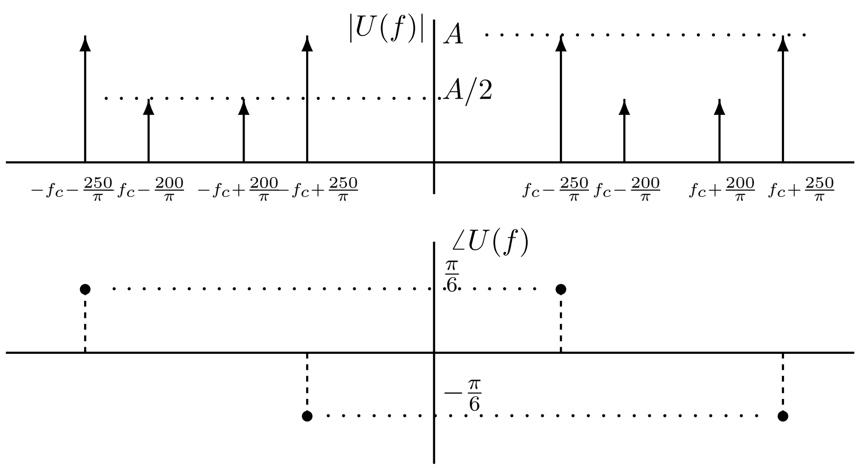
Problem 3.1
The modulated signal is

$$\begin{aligned}
u(t) &= m(t)c(t) = Am(t)\cos(2\pi 4\times 10^3 t) \\
&= A\left[2\cos(2\pi\frac{200}{\pi}t) + 4\sin(2\pi\frac{250}{\pi}t + \frac{\pi}{3})\right]\cos(2\pi 4\times 10^3 t) \\
&= A\cos(2\pi(4\times 10^3 + \frac{200}{\pi})t) + A\cos(2\pi(4\times 10^3 - \frac{200}{\pi})t) \\
&\quad +2A\sin(2\pi(4\times 10^3 + \frac{250}{\pi})t + \frac{\pi}{3}) - 2A\sin(2\pi(4\times 10^3 - \frac{250}{\pi})t - \frac{\pi}{3})
\end{aligned}$$

Taking the Fourier transform of the previous relation, we obtain

$$\begin{aligned}
U(f) &= A\left[\delta(f-\frac{200}{\pi}) + \delta(f+\frac{200}{\pi}) + \frac{2}{j}e^{j\frac{\pi}{3}}\delta(f-\frac{250}{\pi}) - \frac{2}{j}e^{-j\frac{\pi}{3}}\delta(f+\frac{250}{\pi})\right] \\
&\quad \star\frac{1}{2}[\delta(f-4\times 10^3) + \delta(f+4\times 10^3)] \\
&= \frac{A}{2}\left[\delta(f-4\times 10^3 - \frac{200}{\pi}) + \delta(f-4\times 10^3 + \frac{200}{\pi})\right. \\
&\quad +2e^{-j\frac{\pi}{6}}\delta(f-4\times 10^3 - \frac{250}{\pi}) + 2e^{j\frac{\pi}{6}}\delta(f-4\times 10^3 + \frac{250}{\pi}) \\
&\quad +\delta(f+4\times 10^3 - \frac{200}{\pi}) + \delta(f+4\times 10^3 + \frac{200}{\pi}) \\
&\quad \left.+2e^{-j\frac{\pi}{6}}\delta(f+4\times 10^3 - \frac{250}{\pi}) + 2e^{j\frac{\pi}{6}}\delta(f+4\times 10^3 + \frac{250}{\pi})\right]
\end{aligned}$$

The next figure depicts the magnitude and the phase of the spectrum $U(f)$.



To find the power content of the modulated signal we write $u^2(t)$ as

$$\begin{aligned}
u^2(t) &= A^2\cos^2(2\pi(4\times 10^3 + \frac{200}{\pi})t) + A^2\cos^2(2\pi(4\times 10^3 - \frac{200}{\pi})t) \\
&\quad +4A^2\sin^2(2\pi(4\times 10^3 + \frac{250}{\pi})t + \frac{\pi}{3}) + 4A^2\sin^2(2\pi(4\times 10^3 - \frac{250}{\pi})t - \frac{\pi}{3}) \\
&\quad +\text{terms of cosine and sine functions in the first power}
\end{aligned}$$

Hence,

$$P = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} u^2(t)dt = \frac{A^2}{2} + \frac{A^2}{2} + \frac{4A^2}{2} + \frac{4A^2}{2} = 5A^2$$

Problem 3.2

$$u(t) = m(t)c(t) = A(\text{sinc}(t) + \text{sinc}^2(t))\cos(2\pi f_c t)$$

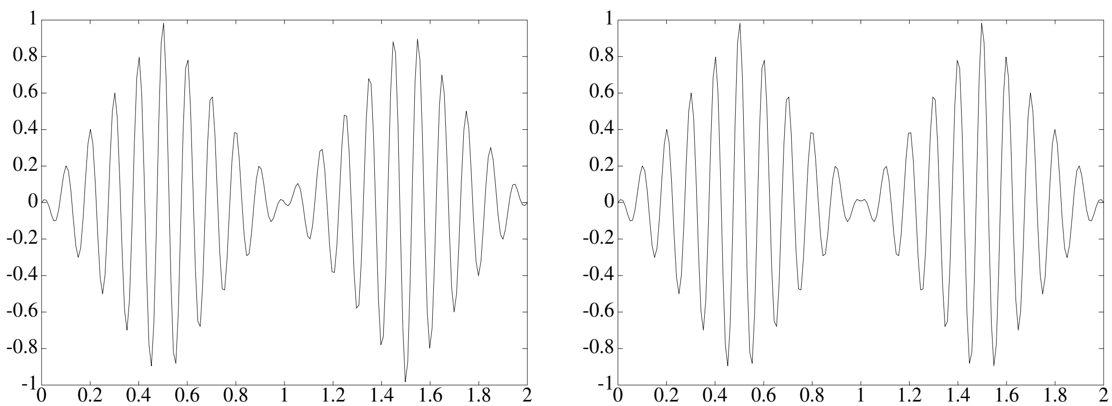
Taking the Fourier transform of both sides, we obtain

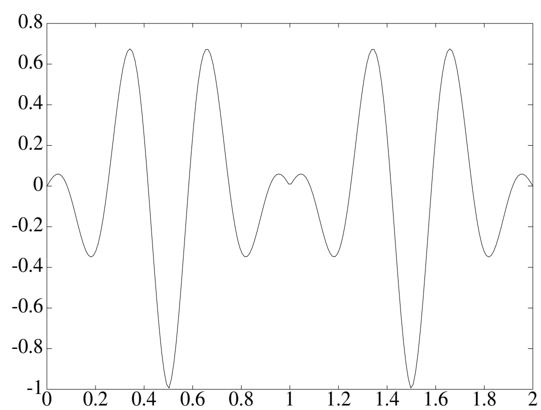
$$\begin{aligned} U(f) &= \frac{A}{2}\left[\Pi(f) + \Lambda(f)\right] \star (\delta(f - f_c) + \delta(f + f_c)) \\ &= \frac{A}{2}\left[\Pi(f - f_c) + \Lambda(f - f_c) + \Pi(f + f_c) + \Lambda(f + f_c)\right] \end{aligned}$$

$\Pi(f - f_c) \neq 0$ for $|f - f_c| < \frac{1}{2}$, whereas $\Lambda(f - f_c) \neq 0$ for $|f - f_c| < 1$. Hence, the bandwidth of the bandpass filter is 2.

Problem 3.3

The following figure shows the modulated signals for $A = 1$ and $f_0 = 10$. As it is observed both signals have the same envelope but there is a phase reversal at $t = 1$ for the second signal $Am_2(t)\cos(2\pi f_0 t)$ (right plot). This discontinuity is shown clearly in the next figure where we plotted $Am_2(t)\cos(2\pi f_0 t)$ with $f_0 = 3$.





Problem 3.4

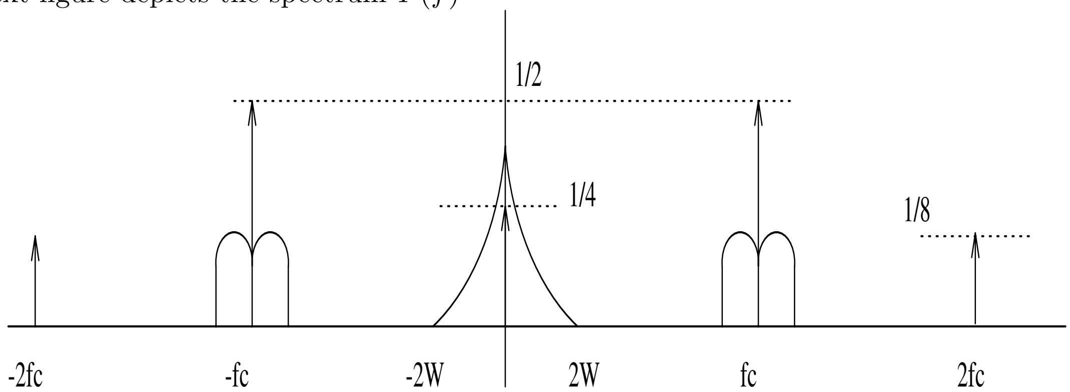
$$y(t) = x(t) + \frac{1}{2}x^2(t)$$

$$
\begin{aligned}
&= m(t) + \cos(2\pi f_c t) + \frac{1}{2}\left(m^2(t) + \cos^2(2\pi f_c t) + 2m(t)\cos(2\pi f_c t)\right) \\
&= m(t) + \cos(2\pi f_c t) + \frac{1}{2}m^2(t) + \frac{1}{4} + \frac{1}{4}\cos(2\pi 2 f_c t) + m(t)\cos(2\pi f_c t)
\end{aligned}
$$

Taking the Fourier transform of the previous, we obtain

$$
\begin{aligned}
Y(f) &= M(f) + \frac{1}{2}M(f) \star M(f) + \frac{1}{2}\left(M(f-f_c) + M(f+f_c)\right) \\
&\quad + \frac{1}{4}\delta(f) + \frac{1}{2}\left(\delta(f-f_c) + \delta(f+f_c)\right) + \frac{1}{8}\left(\delta(f-2f_c) + \delta(f+2f_c)\right)
\end{aligned}
$$

The next figure depicts the spectrum $Y(f)$



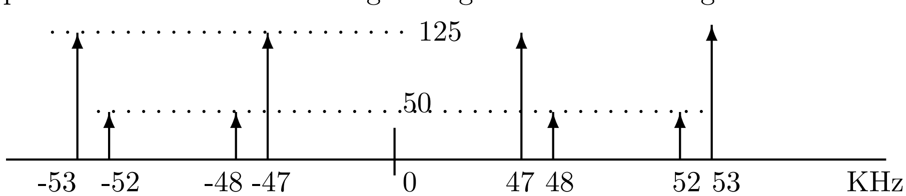
Problem 3.5

$$
\begin{aligned}
u(t) &= m(t) \cdot c(t) \\
&= 100(2\cos(2\pi 2000 t) + 5\cos(2\pi 3000 t))\cos(2\pi f_c t)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
U(f) &= \frac{100}{2}\left[\delta(f-2000) + \delta(f+2000) + \frac{5}{2}(\delta(f-3000) + \delta(f+3000))\right] \\
&\quad \star \left[\delta(f-50000) + \delta(f+50000)\right] \\
&= 50\left[\delta(f-52000) + \delta(f-48000) + \frac{5}{2}\delta(f-53000) + \frac{5}{2}\delta(f-47000)\right. \\
&\quad \left. + \delta(f+52000) + \delta(f+48000) + \frac{5}{2}\delta(f+53000) + \frac{5}{2}\delta(f+47000)\right]
\end{aligned}
$$

A plot of the spectrum of the modulated signal is given in the next figure



Problem 3.6

The mixed signal $y(t)$ is given by

$$
\begin{aligned}
y(t) &= u(t) \cdot x_L(t) = Am(t)\cos(2\pi f_c t)\cos(2\pi f_c t + \theta) \\
&= \frac{A}{2}m(t)\left[\cos(2\pi 2 f_c t + \theta) + \cos(\theta)\right]
\end{aligned}
$$

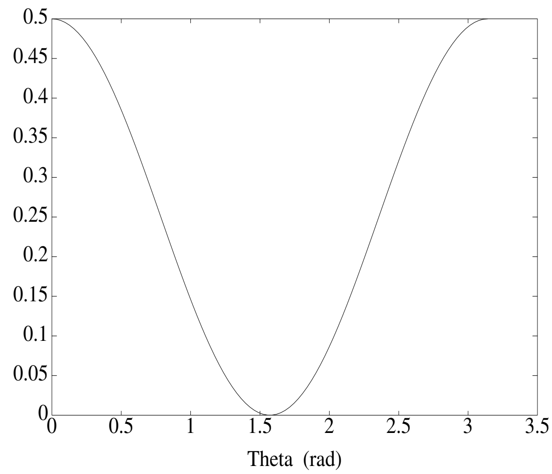
The lowpass filter will cut-off the frequencies above W, where W is the bandwidth of the message signal $m(t)$. Thus, the output of the lowpass filter is

$$z(t) = \frac{A}{2} m(t) \cos(\theta)$$

If the power of $m(t)$ is P_M, then the power of the output signal $z(t)$ is $P_{\text{out}} = P_M \frac{A^2}{4} \cos^2(\theta)$. The power of the modulated signal $u(t) = Am(t)\cos(2\pi f_c t)$ is $P_U = \frac{A^2}{2} P_M$. Hence,

$$\frac{P_{\text{out}}}{P_U} = \frac{1}{2}\cos^2(\theta)$$

A plot of $\frac{P_{\text{out}}}{P_U}$ for $0 \leq \theta \leq \pi$ is given in the next figure.

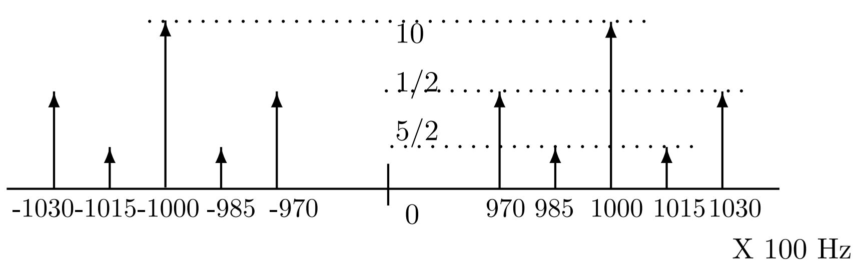


Problem 3.7

1) The spectrum of $u(t)$ is

$$\begin{aligned}
U(f) &= \frac{20}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right] \\
&+ \frac{2}{4}\left[\delta(f - f_c - 1500) + \delta(f - f_c + 1500)\right. \\
&\left. + \delta(f + f_c - 1500) + \delta(f + f_c + 1500)\right] \\
&+ \frac{10}{4}\left[\delta(f - f_c - 3000) + \delta(f - f_c + 3000)\right. \\
&\left. + \delta(f + f_c - 3000) + \delta(f + f_c + 3000)\right]
\end{aligned}$$

The next figure depicts the spectrum of $u(t)$.



2) The square of the modulated signal is

$$\begin{aligned}
u^2(t) &= 400\cos^2(2\pi f_c t) + \cos^2(2\pi(f_c - 1500)t) + \cos^2(2\pi(f_c + 1500)t) \\
&+ 25\cos^2(2\pi(f_c - 3000)t) + 25\cos^2(2\pi(f_c + 3000)t) \\
&+ \text{terms that are multiples of cosines}
\end{aligned}$$

If we integrate $u^2(t)$ from $-\frac{T}{2}$ to $\frac{T}{2}$, normalize the integral by $\frac{1}{T}$ and take the limit as $T \to \infty$, then all the terms involving cosines tend to zero, whereas the squares of the cosines give a value of $\frac{1}{2}$. Hence, the power content at the frequency $f_c = 10^5$ Hz is $P_{f_c} = \frac{400}{2} = 200$, the power content at the frequency P_{f_c+1500} is the same as the power content at the frequency P_{f_c-1500} and equal to $\frac{1}{2}$, whereas $P_{f_c+3000} = P_{f_c-3000} = \frac{25}{2}$.

3)

$$\begin{aligned} u(t) &= (20 + 2\cos(2\pi 1500t) + 10\cos(2\pi 3000t))\cos(2\pi f_c t) \\ &= 20(1 + \frac{1}{10}\cos(2\pi 1500t) + \frac{1}{2}\cos(2\pi 3000t))\cos(2\pi f_c t) \end{aligned}$$

This is the form of a conventional AM signal with message signal

$$\begin{aligned} m(t) &= \frac{1}{10}\cos(2\pi 1500t) + \frac{1}{2}\cos(2\pi 3000t) \\ &= \cos^2(2\pi 1500t) + \frac{1}{10}\cos(2\pi 1500t) - \frac{1}{2} \end{aligned}$$

The minimum of $g(z) = z^2 + \frac{1}{10}z - \frac{1}{2}$ is achieved for $z = -\frac{1}{20}$ and it is $\min(g(z)) = -\frac{201}{400}$. Since $z = -\frac{1}{20}$ is in the range of $\cos(2\pi 1500t)$, we conclude that the minimum value of $m(t)$ is $-\frac{201}{400}$. Hence, the modulation index is

$$\alpha = -\frac{201}{400}$$

4)

$$\begin{aligned} u(t) &= 20\cos(2\pi f_c t) + \cos(2\pi(f_c - 1500)t) + \cos(2\pi(f_c - 1500)t) \\ &= 5\cos(2\pi(f_c - 3000)t) + 5\cos(2\pi(f_c + 3000)t) \end{aligned}$$

The power in the sidebands is

$$P_{\text{sidebands}} = \frac{1}{2} + \frac{1}{2} + \frac{25}{2} + \frac{25}{2} = 26$$

The total power is $P_{\text{total}} = P_{\text{carrier}} + P_{\text{sidebands}} = 200 + 26 = 226$. The ratio of the sidebands power to the total power is

$$\frac{P_{\text{sidebands}}}{P_{\text{total}}} = \frac{26}{226}$$

Problem 3.8

1)

$$\begin{aligned} u(t) &= m(t)c(t) \\ &= 100(\cos(2\pi 1000t) + 2\cos(2\pi 2000t))\cos(2\pi f_c t) \\ &= 100\cos(2\pi 1000t)\cos(2\pi f_c t) + 200\cos(2\pi 2000t)\cos(2\pi f_c t) \\ &= \frac{100}{2}\left[\cos(2\pi(f_c + 1000)t) + \cos(2\pi(f_c - 1000)t)\right] \\ &\quad \frac{200}{2}\left[\cos(2\pi(f_c + 2000)t) + \cos(2\pi(f_c - 2000)t)\right] \end{aligned}$$

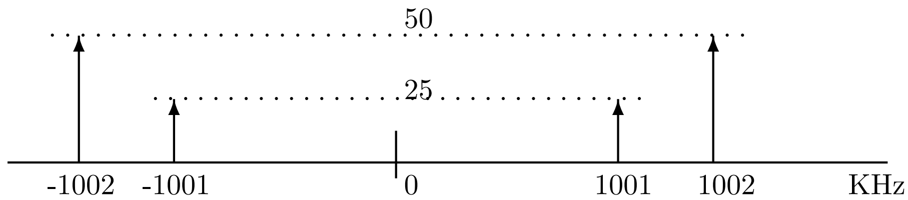
Thus, the upper sideband (USB) signal is

$$u_u(t) = 50\cos(2\pi(f_c + 1000)t) + 100\cos(2\pi(f_c + 2000)t)$$

2) Taking the Fourier transform of both sides, we obtain

$$\begin{aligned} U_u(f) &= 25\left(\delta(f-(f_c+1000))+\delta(f+(f_c+1000))\right) \\ &\quad +50\left(\delta(f-(f_c+2000))+\delta(f+(f_c+2000))\right) \end{aligned}$$

A plot of $U_u(f)$ is given in the next figure.



Problem 3.9

If we let

$$x(t) = -\Pi\left(\frac{t+\frac{T_p}{4}}{\frac{T_p}{2}}\right) + \Pi\left(\frac{t-\frac{T_p}{4}}{\frac{T_p}{2}}\right)$$

then using the results of Problem 2.23, we obtain

$$\begin{aligned} v(t) &= m(t)s(t) = m(t)\sum_{n=-\infty}^{\infty} x(t-nT_p) \\ &= m(t)\frac{1}{T_p}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})e^{j2\pi\frac{n}{T_p}t} \end{aligned}$$

where

$$\begin{aligned} X(\frac{n}{T_p}) &= \mathcal{F}\left[-\Pi\left(\frac{t+\frac{T_p}{4}}{\frac{T_p}{2}}\right) + \Pi\left(\frac{t-\frac{T_p}{4}}{\frac{T_p}{2}}\right)\right]\Bigg|_{f=\frac{n}{T_p}} \\ &= \frac{T_p}{2}\text{sinc}(f\frac{T_p}{2})\left(e^{-j2\pi f\frac{T_p}{4}} - e^{j2\pi f\frac{T_p}{4}}\right)\Bigg|_{f=\frac{n}{T_p}} \\ &= \frac{T_p}{2}\text{sinc}(\frac{n}{2})(-2j)\sin(n\frac{\pi}{2}) \end{aligned}$$

Hence, the Fourier transform of $v(t)$ is

$$V(f) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\text{sinc}(\frac{n}{2})(-2j)\sin(n\frac{\pi}{2})M(f-\frac{n}{T_p})$$

The bandpass filter will cut-off all the frequencies except the ones centered at $\frac{1}{T_p}$, that is for $n=\pm 1$. Thus, the output spectrum is

$$\begin{aligned} U(f) &= \text{sinc}(\frac{1}{2})(-j)M(f-\frac{1}{T_p}) + \text{sinc}(\frac{1}{2})jM(f+\frac{1}{T_p}) \\ &= -\frac{2}{\pi}jM(f-\frac{1}{T_p}) + \frac{2}{\pi}jM(f+\frac{1}{T_p}) \\ &= \frac{4}{\pi}M(f)\star\left[\frac{1}{2j}\delta(f-\frac{1}{T_p}) - \frac{1}{2j}\delta(f+\frac{1}{T_p})\right] \end{aligned}$$

Taking the inverse Fourier transform of the previous expression, we obtain

$$u(t) = \frac{4}{\pi}m(t)\sin(2\pi\frac{1}{T_p}t)$$

which has the form of a DSB-SC AM signal, with $c(t) = \frac{4}{\pi}\sin(2\pi\frac{1}{T_p}t)$ being the carrier signal.

Problem 3.10
Assume that $s(t)$ is a periodic signal with period T_p, i.e. $s(t) = \sum_n x(t - nT_p)$. Then

$$
\begin{aligned}
v(t) &= m(t)s(t) = m(t)\sum_{n=-\infty}^{\infty} x(t - nT_p) \\
&= m(t)\frac{1}{T_p}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})e^{j2\pi\frac{n}{T_p}t} \\
&= \frac{1}{T_p}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})m(t)e^{j2\pi\frac{n}{T_p}t}
\end{aligned}
$$

where $X(\frac{n}{T_p}) = \mathcal{F}[x(t)]|_{f=\frac{n}{T_p}}$. The Fourier transform of $v(t)$ is

$$
\begin{aligned}
V(f) &= \frac{1}{T_p}\mathcal{F}\left[\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})m(t)e^{j2\pi\frac{n}{T_p}t}\right] \\
&= \frac{1}{T_p}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})M(f - \frac{n}{T_p})
\end{aligned}
$$

The bandpass filter will cut-off all the frequency components except the ones centered at $f_c = \pm\frac{1}{T_p}$. Hence, the spectrum at the output of the BPF is

$$
U(f) = \frac{1}{T_p}X(\frac{1}{T_p})M(f - \frac{1}{T_p}) + \frac{1}{T_p}X(-\frac{1}{T_p})M(f + \frac{1}{T_p})
$$

In the time domain the output of the BPF is given by

$$
\begin{aligned}
u(t) &= \frac{1}{T_p}X(\frac{1}{T_p})m(t)e^{j2\pi\frac{1}{T_p}t} + \frac{1}{T_p}X^*(\frac{1}{T_p})m(t)e^{-j2\pi\frac{1}{T_p}t} \\
&= \frac{1}{T_p}m(t)\left[X(\frac{1}{T_p})e^{j2\pi\frac{1}{T_p}t} + X^*(\frac{1}{T_p})e^{-j2\pi\frac{1}{T_p}t}\right] \\
&= \frac{1}{T_p}2\mathrm{Re}(\mathrm{X}(\frac{1}{\mathrm{T_p}}))\mathrm{m(t)}\cos(2\pi\frac{1}{\mathrm{T_p}}\mathrm{t})
\end{aligned}
$$

As it is observed $u(t)$ has the form a modulated DSB-SC signal. The amplitude of the modulating signal is $A_c = \frac{1}{T_p}2\mathrm{Re}(\mathrm{X}(\frac{1}{\mathrm{T_p}}))$ and the carrier frequency $f_c = \frac{1}{T_p}$.

Problem 3.11
1) The spectrum of the modulated signal $Am(t)\cos(2\pi f_c t)$ is

$$
V(f) = \frac{A}{2}[M(f - f_c) + M(f + f_c)]
$$

The spectrum of the signal at the output of the highpass filter is

$$
U(f) = \frac{A}{2}[M(f + f_c)u_{-1}(-f - f_c) + M(f - f_c)u_{-1}(f - f_c)]
$$

Multiplying the output of the HPF with $A\cos(2\pi(f_c + W)t)$ results in the signal $z(t)$ with spectrum

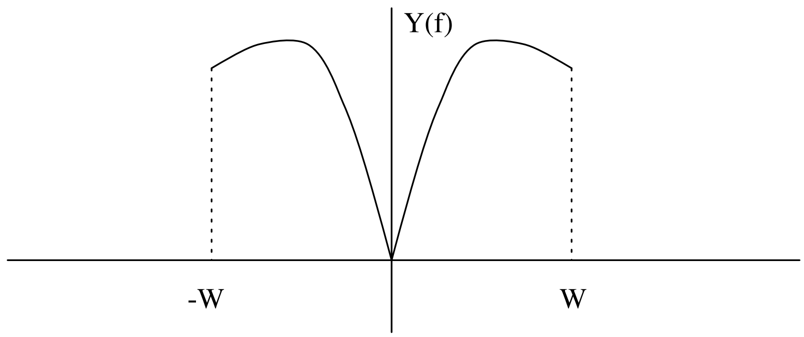
$$
\begin{aligned}
Z(f) &= \frac{A}{2}[M(f + f_c)u_{-1}(-f - f_c) + M(f - f_c)u_{-1}(f - f_c)] \\
&\quad \star\frac{A}{2}[\delta(f - (f_c + W)) + \delta(f + f_c + W)]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{A^2}{4}(M(f+f_c-f_c-W)u_{-1}(-f+f_c+W-f_c) \\
&\quad +M(f+f_c-f_c+W)u_{-1}(f+f_c+W-f_c) \\
&\quad +M(f-2f_c-W)u_{-1}(f-2f_c-W) \\
&\quad +M(f+2f_c+W)u_{-1}(-f-2f_c-W)) \\
&= \frac{A^2}{4}(M(f-W)u_{-1}(-f+W)+M(f+W)u_{-1}(f+W) \\
&\quad +M(f-2f_c-W)u_{-1}(f-2f_c-W)+M(f+2f_c+W)u_{-1}(-f-2f_c-W))
\end{aligned}
$$

The LPF will cut-off the double frequency components, leaving the spectrum

$$
Y(f) = \frac{A^2}{4}[M(f-W)u_{-1}(-f+W)+M(f+W)u_{-1}(f+W)]
$$

The next figure depicts $Y(f)$ for $M(f)$ as shown in Fig. P-5.12.



2) As it is observed from the spectrum $Y(f)$, the system shifts the positive frequency components to the negative frequency axis and the negative frequency components to the positive frequency axis. If we transmit the signal $y(t)$ through the system, then we will get a scaled version of the original spectrum $M(f)$.

Problem 3.12
The modulated signal can be written as

$$
\begin{aligned}
u(t) &= m(t)\cos(2\pi f_c t+\phi) \\
&= m(t)\cos(2\pi f_c t)\cos(\phi) - m(t)\sin(2\pi f_c t)\sin(\phi) \\
&= u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t)
\end{aligned}
$$

where we identify $u_c(t) = m(t)\cos(\phi)$ as the in-phase component and $u_s(t) = m(t)\sin(\phi)$ as the quadrature component. The envelope of the bandpass signal is

$$
\begin{aligned}
V_u(t) &= \sqrt{u_c^2(t)+u_s^2(t)} = \sqrt{m^2(t)\cos^2(\phi)+m^2(t)\sin^2(\phi)} \\
&= \sqrt{m^2(t)} = |m(t)|
\end{aligned}
$$

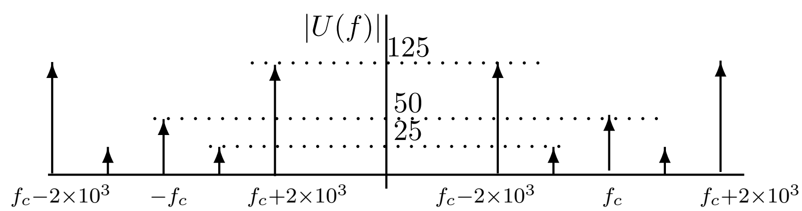
Hence, the envelope is proportional to the absolute value of the message signal.

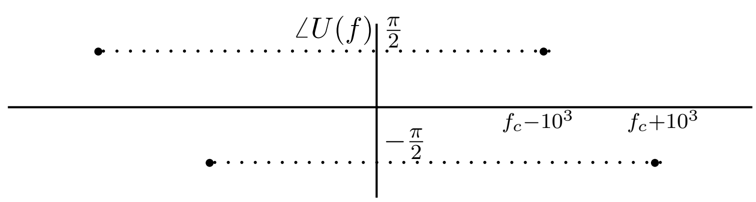
Problem 3.13
1) The modulated signal is

$$
\begin{aligned}
u(t) &= 100[1+m(t)]\cos(2\pi 8\times 10^5 t) \\
&= 100\cos(2\pi 8\times 10^5 t) + 100\sin(2\pi 10^3 t)\cos(2\pi 8\times 10^5 t) \\
&\quad +500\cos(2\pi 2\times 10^3 t)\cos(2\pi 8\times 10^5 t) \\
&= 100\cos(2\pi 8\times 10^5 t) + 50[\sin(2\pi(10^3+8\times 10^5)t) - \sin(2\pi(8\times 10^5-10^3)t)] \\
&\quad +250[\cos(2\pi(2\times 10^3+8\times 10^5)t) + \cos(2\pi(8\times 10^5-2\times 10^3)t)]
\end{aligned}
$$

Taking the Fourier transform of the previous expression, we obtain

$$
\begin{aligned}
U(f) &= 50[\delta(f-8\times 10^5)+\delta(f+8\times 10^5)] \\
&\quad +25\left[\frac{1}{j}\delta(f-8\times 10^5-10^3)-\frac{1}{j}\delta(f+8\times 10^5+10^3)\right] \\
&\quad -25\left[\frac{1}{j}\delta(f-8\times 10^5+10^3)-\frac{1}{j}\delta(f+8\times 10^5-10^3)\right] \\
&\quad +125\left[\delta(f-8\times 10^5-2\times 10^3)+\delta(f+8\times 10^5+2\times 10^3)\right] \\
&\quad +125\left[\delta(f-8\times 10^5-2\times 10^3)+\delta(f+8\times 10^5+2\times 10^3)\right] \\
&= 50[\delta(f-8\times 10^5)+\delta(f+8\times 10^5)] \\
&\quad +25\left[\delta(f-8\times 10^5-10^3)e^{-j\frac{\pi}{2}}+\delta(f+8\times 10^5+10^3)e^{j\frac{\pi}{2}}\right] \\
&\quad +25\left[\delta(f-8\times 10^5+10^3)e^{j\frac{\pi}{2}}+\delta(f+8\times 10^5-10^3)e^{-j\frac{\pi}{2}}\right] \\
&\quad +125\left[\delta(f-8\times 10^5-2\times 10^3)+\delta(f+8\times 10^5+2\times 10^3)\right] \\
&\quad +125\left[\delta(f-8\times 10^5-2\times 10^3)+\delta(f+8\times 10^5+2\times 10^3)\right]
\end{aligned}
$$





2) The average power in the carrier is

$$P_{\text{carrier}} = \frac{A_c^2}{2} = \frac{100^2}{2} = 5000$$

The power in the sidebands is

$$P_{\text{sidebands}} = \frac{50^2}{2}+\frac{50^2}{2}+\frac{250^2}{2}+\frac{250^2}{2} = 65000$$

3) The message signal can be written as

$$
\begin{aligned}
m(t) &= \sin(2\pi 10^3 t)+5\cos(2\pi 2\times 10^3 t) \\
&= -10\sin(2\pi 10^3 t)+\sin(2\pi 10^3 t)+5
\end{aligned}
$$

As it is seen the minimum value of $m(t)$ is -6 and is achieved for $\sin(2\pi 10^3 t) = -1$ or $t = \frac{3}{4\times 10^3}+\frac{1}{10^3}k$, with $k\in Z$. Hence, the modulation index is $\alpha = 6$.

4) The power delivered to the load is

$$P_{\text{load}} = \frac{|u(t)|^2}{50} = \frac{100^2(1+m(t))^2\cos^2(2\pi f_c t)}{50}$$

The maximum absolute value of $1 + m(t)$ is 6.025 and is achieved for $\sin(2\pi 10^3 t) = \frac{1}{20}$ or $t = \frac{\arcsin(\frac{1}{20})}{2\pi 10^3} + \frac{k}{10^3}$. Since $2 \times 10^3 \ll f_c$ the peak power delivered to the load is approximately equal to

$$\max(P_{\text{load}}) = \frac{(100 \times 6.025)^2}{50} = 72.6012$$

Problem 3.14
1)

$$\begin{aligned} u(t) &= 5\cos(1800\pi t) + 20\cos(2000\pi t) + 5\cos(2200\pi t) \\ &= 20(1 + \frac{1}{2}\cos(200\pi t))\cos(2000\pi t) \end{aligned}$$

The modulating signal is $m(t) = \cos(2\pi 100 t)$ whereas the carrier signal is $c(t) = 20\cos(2\pi 1000 t)$.

2) Since $-1 \leq \cos(2\pi 100 t) \leq 1$, we immediately have that the modulation index is $\alpha = \frac{1}{2}$.

3) The power of the carrier component is $P_{\text{carrier}} = \frac{400}{2} = 200$, whereas the power in the sidebands is $P_{\text{sidebands}} = \frac{400\alpha^2}{2} = 50$. Hence,

$$\frac{P_{\text{sidebands}}}{P_{\text{carrier}}} = \frac{50}{200} = \frac{1}{4}$$

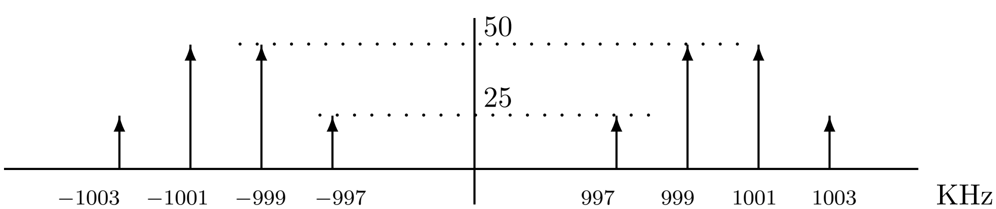
Problem 3.15
1) The modulated signal is written as

$$\begin{aligned} u(t) &= 100(2\cos(2\pi 10^3 t) + \cos(2\pi 3 \times 10^3 t))\cos(2\pi f_c t) \\ &= 200\cos(2\pi 10^3 t)\cos(2\pi f_c t) + 100\cos(2\pi 3 \times 10^3 t)\cos(2\pi f_c t) \\ &= 100\left[\cos(2\pi(f_c + 10^3)t) + \cos(2\pi(f_c - 10^3)t)\right] \\ &\quad +50\left[\cos(2\pi(f_c + 3 \times 10^3)t) + \cos(2\pi(f_c - 3 \times 10^3)t)\right] \end{aligned}$$

Taking the Fourier transform of the previous expression, we obtain

$$\begin{aligned} U(f) &= 50\left[\delta(f - (f_c + 10^3)) + \delta(f + f_c + 10^3)\right. \\ &+ \left.\delta(f - (f_c - 10^3)) + \delta(f + f_c - 10^3)\right] \\ &+ 25\left[\delta(f - (f_c + 3 \times 10^3)) + \delta(f + f_c + 3 \times 10^3)\right. \\ &+ \left.\delta(f - (f_c - 3 \times 10^3)) + \delta(f + f_c - 3 \times 10^3)\right] \end{aligned}$$

The spectrum of the signal is depicted in the next figure



2) The average power in the frequencies $f_c + 1000$ and $f_c - 1000$ is

$$P_{f_c+1000} = P_{f_c-1000} = \frac{100^2}{2} = 5000$$

The average power in the frequencies $f_c + 3000$ and $f_c - 3000$ is

$$P_{f_c+3000} = P_{f_c-3000} = \frac{50^2}{2} = 1250$$

Problem 3.16

1) The Hilbert transform of $\cos(2\pi 1000t)$ is $\sin(2\pi 1000t)$, whereas the Hilbert transform of$\sin(2\pi 1000t)$ is $-\cos(2\pi 1000t)$. Thus

$$\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$$

2) The expression for the LSSB AM signal is

$$u_l(t) = A_c m(t)\cos(2\pi f_c t) + A_c \hat{m}(t)\sin(2\pi f_c t)$$

Substituting $A_c = 100$, $m(t) = \cos(2\pi 1000t) + 2\sin(2\pi 1000t)$ and $\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$ in the previous, we obtain

$$\begin{aligned}
u_l(t) &= 100\left[\cos(2\pi 1000t) + 2\sin(2\pi 1000t)\right]\cos(2\pi f_c t) \\
&+ 100\left[\sin(2\pi 1000t) - 2\cos(2\pi 1000t)\right]\sin(2\pi f_c t) \\
&= 100\left[\cos(2\pi 1000t)\cos(2\pi f_c t) + \sin(2\pi 1000t)\sin(2\pi f_c t)\right] \\
&+ 200\left[\cos(2\pi f_c t)\sin(2\pi 1000t) - \sin(2\pi f_c t)\cos(2\pi 1000t)\right] \\
&= 100\cos(2\pi(f_c - 1000)t) - 200\sin(2\pi(f_c - 1000)t)
\end{aligned}$$

3) Taking the Fourier transform of the previous expression we obtain

$$\begin{aligned}
U_l(f) &= 50\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right) \\
&+ 100j\left(\delta(f - f_c + 1000) - \delta(f + f_c - 1000)\right) \\
&= (50 + 100j)\delta(f - f_c + 1000) + (50 - 100j)\delta(f + f_c - 1000)
\end{aligned}$$

Hence, the magnitude spectrum is given by

$$\begin{aligned}
|U_l(f)| &= \sqrt{50^2 + 100^2}\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right) \\
&= 10\sqrt{125}\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right)
\end{aligned}$$

Problem 3.17

The input to the upper LPF is

$$\begin{aligned}
u_u(t) &= \cos(2\pi f_m t)\cos(2\pi f_1 t) \\
&= \frac{1}{2}\left[\cos(2\pi(f_1 - f_m)t) + \cos(2\pi(f_1 + f_m)t)\right]
\end{aligned}$$

whereas the input to the lower LPF is

$$\begin{aligned}
u_l(t) &= \cos(2\pi f_m t)\sin(2\pi f_1 t) \\
&= \frac{1}{2}\left[\sin(2\pi(f_1 - f_m)t) + \sin(2\pi(f_1 + f_m)t)\right]
\end{aligned}$$

If we select f_1 such that $|f_1 - f_m| < W$ and $f_1 + f_m > W$, then the two lowpass filters will cut-off the frequency components outside the interval $[-W, W]$, so that the output of the upper and lower LPF is

$$\begin{aligned}
y_u(t) &= \cos(2\pi(f_1 - f_m)t) \\
y_l(t) &= \sin(2\pi(f_1 - f_m)t)
\end{aligned}$$

The output of the Weaver's modulator is

$$u(t) = \cos(2\pi(f_1 - f_m)t)\cos(2\pi f_2 t) - \sin(2\pi(f_1 - f_m)t)\sin(2\pi f_2 t)$$

which has the form of a SSB signal since $\sin(2\pi(f_1 - f_m)t)$ is the Hilbert transform of $\cos(2\pi(f_1 - f_m)t)$. If we write $u(t)$ as

$$u(t) = \cos(2\pi(f_1 + f_2 - f_m)t)$$

then with $f_1+f_2-f_m = f_c+f_m$ we obtain an USSB signal centered at f_c, whereas with $f_1+f_2-f_m = f_c - f_m$ we obtain the LSSB signal. In both cases the choice of f_c and f_1 uniquely determine f_2.

Problem 3.18

The signal $x(t)$ is $m(t) + \cos(2\pi f_0 t)$. The spectrum of this signal is $X(f) = M(f) + \frac{1}{2}(\delta(f - f_0) + \delta(f + f_0))$ and its bandwidth equals to $W_x = f_0$. The signal $y_1(t)$ after the Square Law Device is

$$\begin{aligned} y_1(t) &= x^2(t) = (m(t) + \cos(2\pi f_0 t))^2 \\ &= m^2(t) + \cos^2(2\pi f_0 t) + 2m(t)\cos(2\pi f_0 t) \\ &= m^2(t) + \frac{1}{2} + \frac{1}{2}\cos(2\pi 2 f_0 t) + 2m(t)\cos(2\pi f_0 t) \end{aligned}$$

The spectrum of this signal is given by

$$Y_1(f) = M(f) \star M(f) + \frac{1}{2}\delta(f) + \frac{1}{4}(\delta(f - 2f_0) + \delta(f + 2f_0)) + M(f - f_0) + M(f + f_0)$$

and its bandwidth is $W_1 = 2f_0$. The bandpass filter will cut-off the low-frequency components $M(f)\star M(f)+\frac{1}{2}\delta(f)$ and the terms with the double frequency components $\frac{1}{4}(\delta(f-2f_0)+\delta(f+2f_0))$. Thus the spectrum $Y_2(f)$ is given by

$$Y_2(f) = M(f - f_0) + M(f + f_0)$$

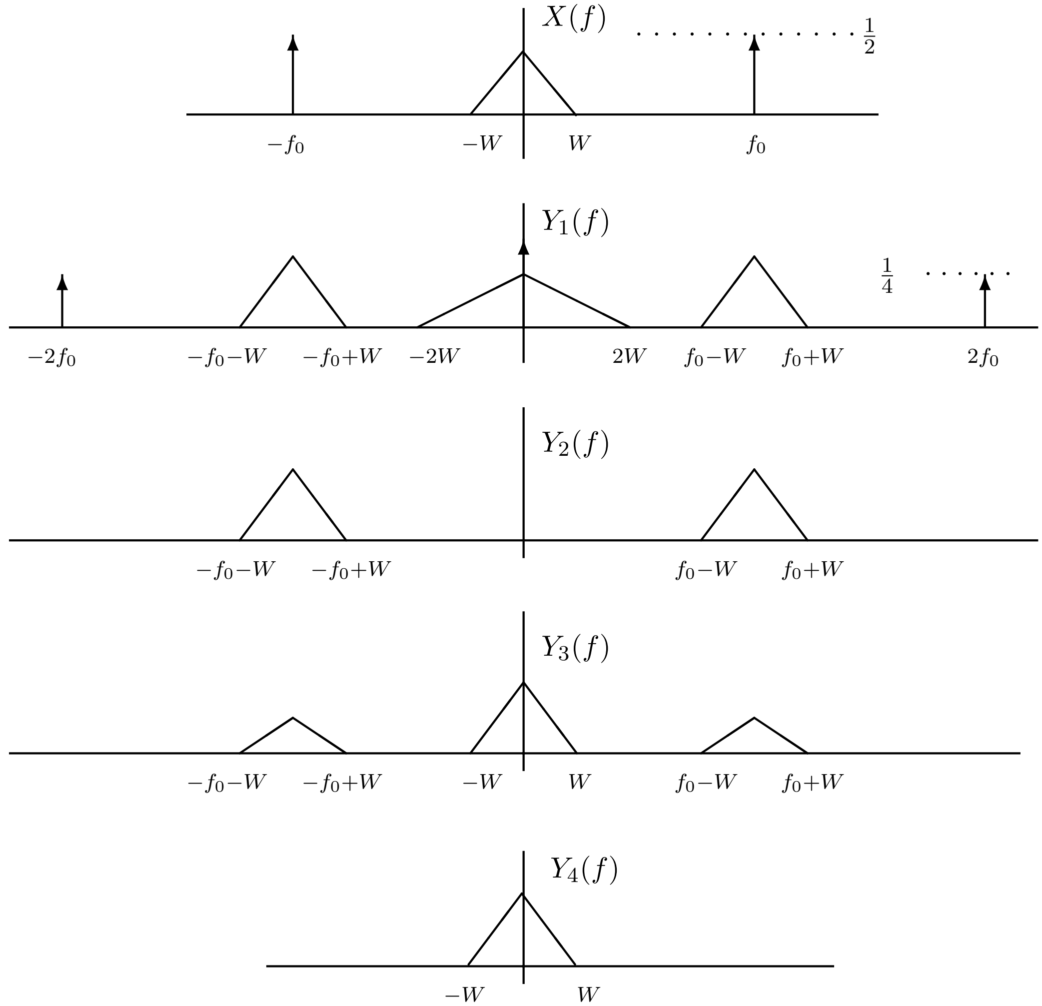
and the bandwidth of $y_2(t)$ is $W_2 = 2W$. The signal $y_3(t)$ is

$$y_3(t) = 2m(t)\cos^2(2\pi f_0 t) = m(t) + m(t)\cos(2\pi f_0 t)$$

with spectrum

$$Y_3(t) = M(f) + \frac{1}{2}(M(f - f_0) + M(f + f_0))$$

and bandwidth $W_3 = f_0 + W$. The lowpass filter will eliminate the spectral components $\frac{1}{2}(M(f - f_0) + M(f + f_0))$, so that $y_4(t) = m(t)$ with spectrum $Y_4 = M(f)$ and bandwidth $W_4 = W$. The next figure depicts the spectra of the signals $x(t)$, $y_1(t)$, $y_2(t)$, $y_3(t)$ and $y_4(t)$.



Problem 3.19

1)

$$
\begin{aligned}
y(t) &= ax(t) + bx^2(t) \\
&= a(m(t) + \cos(2\pi f_0 t)) + b(m(t) + \cos(2\pi f_0 t))^2 \\
&= am(t) + bm^2(t) + a\cos(2\pi f_0 t) \\
&\quad + b\cos^2(2\pi f_0 t) + 2bm(t)\cos(2\pi f_0 t)
\end{aligned}
$$

2) The filter should reject the low frequency components, the terms of double frequency and pass only the signal with spectrum centered at f_0. Thus the filter should be a BPF with center frequency f_0 and bandwidth W such that $f_0 - W_M > f_0 - \frac{W}{2} > 2W_M$ where W_M is the bandwidth of the message signal $m(t)$.

3) The AM output signal can be written as

$$u(t) = a(1 + \frac{2b}{a}m(t))\cos(2\pi f_0 t)$$

Since $A_m = \max[|m(t)|]$ we conclude that the modulation index is

$$\alpha = \frac{2bA_m}{a}$$

Problem 3.20

1) When USSB is employed the bandwidth of the modulated signal is the same with the bandwidth of the message signal. Hence,

$$W_{\text{USSB}} = W = 10^4 \text{ Hz}$$

2) When DSB is used, then the bandwidth of the transmitted signal is twice the bandwidth of the message signal. Thus,

$$W_{\text{DSB}} = 2W = 2 \times 10^4 \text{ Hz}$$

3) If conventional AM is employed, then

$$W_{\text{AM}} = 2W = 2 \times 10^4 \text{ Hz}$$

4) Using Carson's rule, the effective bandwidth of the FM modulated signal is

$$B_c = (2\beta + 1)W = 2\left(\frac{k_f \max[|m(t)|]}{W} + 1\right)W = 2(k_f + W) = 140000 \text{ Hz}$$

Problem 3.21

1) The lowpass equivalent transfer function of the system is

$$H_l(f) = 2u_{-1}(f + f_c)H(f + f_c) = 2\begin{cases} \frac{1}{W}f + \frac{1}{2} & |f| \leq \frac{W}{2} \\ 1 & \frac{W}{2} < f \leq W \end{cases}$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned}
h_l(t) &= \mathcal{F}^{-1}[H_l(f)] = \int_{-\frac{W}{2}}^{W} H_l(f)e^{j2\pi ft}df \\
&= 2\int_{-\frac{W}{2}}^{\frac{W}{2}} (\frac{1}{W}f + \frac{1}{2})e^{j2\pi ft}df + 2\int_{\frac{W}{2}}^{W} e^{j2\pi ft}df \\
&= \frac{2}{W}\left(\frac{1}{j2\pi t}fe^{j2\pi ft} + \frac{1}{4\pi^2 t^2}e^{j2\pi ft}\right)\Bigg|_{-\frac{W}{2}}^{\frac{W}{2}} + \frac{1}{j2\pi t}e^{j2\pi ft}\Bigg|_{-\frac{W}{2}}^{\frac{W}{2}} + \frac{2}{j2\pi t}e^{j2\pi ft}\Bigg|_{\frac{W}{2}}^{W} \\
&= \frac{1}{j\pi t}e^{j2\pi Wt} + \frac{j}{\pi^2 t^2 W}\sin(\pi Wt) \\
&= \frac{j}{\pi t}\left[\text{sinc}(Wt) - e^{j2\pi Wt}\right]
\end{aligned}$$

2) An expression for the modulated signal is obtained as follows

$$\begin{aligned}
u(t) &= \text{Re}[(m(t) \star h_l(t))e^{j2\pi f_c t}] \\
&= \text{Re}\left[(m(t) \star \frac{j}{\pi t}(\text{sinc}(Wt) - e^{j2\pi Wt}))e^{j2\pi f_c t}\right] \\
&= \text{Re}\left[(m(t) \star (\frac{j}{\pi t}\text{sinc}(Wt)))e^{j2\pi f_c t} + (m(t) \star \frac{1}{j\pi t}e^{j2\pi Wt})e^{j2\pi f_c t}\right]
\end{aligned}$$

Note that

$$\mathcal{F}[m(t) \star \frac{1}{j\pi t}e^{j2\pi Wt}] = -M(f)\text{sgn}(f - W) = M(f)$$

since $\text{sgn}(f-W) = -1$ for $f < W$. Thus,

$$\begin{aligned}
u(t) &= \text{Re}\left[(m(t) \star (\frac{j}{\pi t}\text{sinc}(Wt)))e^{j2\pi f_c t} + m(t)e^{j2\pi f_c t}\right] \\
&= m(t)\cos(2\pi f_c t) - m(t) \star (\frac{1}{\pi t}\text{sinc}(Wt))\sin(2\pi f_c t)
\end{aligned}$$

Problem 3.22

a) A DSB modulated signal is written as

$$\begin{aligned}
u(t) &= Am(t)\cos(2\pi f_0 t + \phi) \\
&= Am(t)\cos(\phi)\cos(2\pi f_0 t) - Am(t)\sin(\phi)\sin(2\pi f_0 t)
\end{aligned}$$

Hence,

$$\begin{aligned}
x_c(t) &= Am(t)\cos(\phi) \\
x_s(t) &= Am(t)\sin(\phi) \\
V(t) &= \sqrt{A^2 m^2(t)(\cos^2(\phi) + \sin^2(\phi))} = |Am(t)| \\
\Theta(t) &= \arctan\left(\frac{Am(t)\cos(\phi)}{Am(t)\sin(\phi)}\right) = \arctan(\tan(\phi)) = \phi
\end{aligned}$$

b) A SSB signal has the form

$$u_{\text{SSB}}(t) = Am(t)\cos(2\pi f_0 t) \mp A\hat{m}(t)\sin(2\pi f_0 t)$$

Thus, for the USSB signal (minus sign)

$$\begin{aligned}
x_c(t) &= Am(t) \\
x_s(t) &= A\hat{m}(t) \\
V(t) &= \sqrt{A^2(m^2(t) + \hat{m}^2(t))} = A\sqrt{m^2(t) + \hat{m}^2(t)} \\
\Theta(t) &= \arctan\left(\frac{\hat{m}(t)}{m(t)}\right)
\end{aligned}$$

For the LSSB signal (plus sign)

$$\begin{aligned}
x_c(t) &= Am(t) \\
x_s(t) &= -A\hat{m}(t) \\
V(t) &= \sqrt{A^2(m^2(t) + \hat{m}^2(t))} = A\sqrt{m^2(t) + \hat{m}^2(t)} \\
\Theta(t) &= \arctan\left(-\frac{\hat{m}(t)}{m(t)}\right)
\end{aligned}$$

c) If conventional AM is employed, then

$$\begin{aligned}
u(t) &= A(1 + m(t))\cos(2\pi f_0 t + \phi) \\
&= A(1 + m(t))\cos(\phi)\cos(2\pi f_0 t) - A(1 + m(t))\sin(\phi)\sin(2\pi f_0 t)
\end{aligned}$$

Hence,

$$\begin{aligned}
x_c(t) &= A(1 + m(t))\cos(\phi) \\
x_s(t) &= A(1 + m(t))\sin(\phi) \\
V(t) &= \sqrt{A^2(1 + m(t))^2(\cos^2(\phi) + \sin^2(\phi))} = A|(1 + m(t))| \\
\Theta(t) &= \arctan\left(\frac{A(1 + m(t))\cos(\phi)}{A(1 + m(t))\sin(\phi)}\right) = \arctan(\tan(\phi)) = \phi
\end{aligned}$$

d) A PM modulated signal has the form

$$\begin{aligned} u(t) &= A\cos(2\pi f_c t + k_p m(t)) \\ &= \text{Re}\left[Ae^{j2\pi f_c t}e^{jk_p m(t)}\right] \end{aligned}$$

From the latter expression we identify the lowpass equivalent signal as

$$u_l(t) = Ae^{jk_p m(t)} = x_c(t) + jx_s(t)$$

Thus,

$$\begin{aligned} x_c(t) &= A\cos(k_p m(t)) \\ x_s(t) &= A\sin(k_p m(t)) \\ V(t) &= \sqrt{A^2(\cos^2(k_p m(t)) + \sin^2(k_p m(t)))} = A \\ \Theta(t) &= \arctan\left(\frac{A\cos(k_p m(t))}{A\sin(k_p m(t))}\right) = k_p m(t) \end{aligned}$$

e) To get the expressions for an FM signal we replace $k_p m(t)$ by $2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau$ in the previous relations. Hence,

$$\begin{aligned} x_c(t) &= A\cos(2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau) \\ x_s(t) &= A\sin(2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau) \\ V(t) &= A \\ \Theta(t) &= 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau \end{aligned}$$

Problem 3.23

1) If SSB is employed, the transmitted signal is

$$u(t) = Am(t)\cos(2\pi f_0 t) \mp A\hat{m}(t)\sin(2\pi f_0 t)$$

Provided that the spectrum of $m(t)$ does not contain any impulses at the origin $P_M = P_{\hat{M}} = \frac{1}{2}$ and

$$P_{\text{SSB}} = \frac{A^2 P_M}{2} + \frac{A^2 P_{\hat{M}}}{2} = A^2 P_M = 400\frac{1}{2} = 200$$

The bandwidth of the modulated signal $u(t)$ is the same with that of the message signal. Hence,

$$W_{\text{SSB}} = 10000 \text{ Hz}$$

2) In the case of DSB-SC modulation $u(t) = Am(t)\cos(2\pi f_0 t)$. The power content of the modulated signal is

$$P_{\text{DSB}} = \frac{A^2 P_M}{2} = 200\frac{1}{2} = 100$$

and the bandwidth $W_{\text{DSB}} = 2W = 20000$ Hz.

3) If conventional AM is employed with modulation index $\alpha = 0.6$, the transmitted signal is

$$u(t) = A[1 + \alpha m(t)]\cos(2\pi f_0 t)$$

The power content is

$$P_{\text{AM}} = \frac{A^2}{2} + \frac{A^2\alpha^2 P_M}{2} = 200 + 200 \cdot 0.6^2 \cdot 0.5 = 236$$

The bandwidth of the signal is $W_{\text{AM}} = 2W = 20000$ Hz.

4) If the modulation is FM with $k_f = 50000$, then

$$P_{\text{FM}} = \frac{A^2}{2} = 200$$

and the effective bandwidth is approximated by Carson's rule as

$$B_c = 2(\beta + 1)W = 2\left(\frac{50000}{W} + 1\right) W = 120000 \text{ Hz}$$

Problem 3.24

1) Since $\mathcal{F}[\text{sinc}(400t)] = \frac{1}{400}\Pi(\frac{f}{400})$, the bandwidth of the message signal is $W = 200$ and the resulting modulation index

$$\beta_f = \frac{k_f \max[|m(t)|]}{W} = \frac{k_f 10}{W} = 6 \Longrightarrow k_f = 120$$

Hence, the modulated signal is

$$\begin{aligned} u(t) &= A\cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau) \\ &= 100\cos(2\pi f_c t + +2\pi 1200 \int_{-\infty}^{t} \text{sinc}(400\tau)d\tau) \end{aligned}$$

2) The maximum frequency deviation of the modulated signal is

$$\Delta f_{\max} = \beta_f W = 6 \times 200 = 1200$$

3) Since the modulated signal is essentially a sinusoidal signal with amplitude $A = 100$, we have

$$P = \frac{A^2}{2} = 5000$$

4) Using Carson's rule, the effective bandwidth of the modulated signal can be approximated by

$$B_c = 2(\beta_f + 1)W = 2(6+1)200 = 2800 \text{ Hz}$$

Problem 3.25

1) The maximum phase deviation of the PM signal is

$$\Delta\phi_{\max} = k_p \max[|m(t)|] = k_p$$

The phase of the FM modulated signal is

$$\begin{aligned} \phi(t) &= 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau = 2\pi k_f \int_{0}^{t} m(\tau)d\tau \\ &= \begin{cases} 2\pi k_f \int_0^t \tau d\tau = \pi k_f t^2 & 0 \le t < 1 \\ \pi k_f + 2\pi k_f \int_1^t d\tau = \pi k_f + 2\pi k_f (t-1) & 1 \le t < 2 \\ \pi k_f + 2\pi k_f - 2\pi k_f \int_2^t d\tau = 3\pi k_f - 2\pi k_f (t-2) & 2 \le t < 3 \\ \pi k_f & 3 \le t \end{cases} \end{aligned}$$

The maximum value of $\phi(t)$ is achieved for $t = 2$ and is equal to $3\pi k_f$. Thus, the desired relation between k_p and k_f is

$$k_p = 3\pi k_f$$

2) The instantaneous frequency for the PM modulated signal is

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_c + \frac{1}{2\pi}k_p\frac{d}{dt}m(t)$$

For the $m(t)$ given in Fig. P-3.25, the maximum value of $\frac{d}{dt}m(t)$ is achieved for t in $[0, 1]$ and it is equal to one. Hence,

$$\max(f_i(t)) = f_c + \frac{1}{2\pi}$$

For the FM signal $f_i(t) = f_c + k_f m(t)$. Thus, the maximum instantaneous frequency is

$$\max(f_i(t)) = f_c + k_f = f_c + 1$$

Problem 3.26

1) Since an angle modulated signal is essentially a sinusoidal signal with constant amplitude, we have

$$P = \frac{A_c^2}{2} \Longrightarrow P = \frac{100^2}{2} = 5000$$

The same result is obtained if we use the expansion

$$u(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t)$$

along with the identity

$$J_0^2(\beta) + 2\sum_{n=1}^{\infty} J_n^2(\beta) = 1$$

2) The maximum phase deviation is

$$\Delta\phi_{\max} = \max|4\sin(2000\pi t)| = 4$$

3) The instantaneous frequency is

$$\begin{aligned} f_i &= f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) \\ &= f_c + \frac{4}{2\pi}\cos(2000\pi t)2000\pi = f_c + 4000\cos(2000\pi t) \end{aligned}$$

Hence, the maximum frequency deviation is

$$\Delta f_{\max} = \max|f_i - f_c| = 4000$$

4) The angle modulated signal can be interpreted both as a PM and an FM signal. It is a PM signal with phase deviation constant $k_p = 4$ and message signal $m(t) = \sin(2000\pi t)$ and it is an FM signal with frequency deviation constant $k_f = 4000$ and message signal $m(t) = \cos(2000\pi t)$.

Problem 3.27

The modulated signal can be written as

$$u(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t)$$

The power in the frequency component $f = f_c + kf_m$ is $P_k = \frac{A_c^2}{2}J_n^2(\beta)$. Hence, the power in the carrier is $P_{\text{carrier}} = \frac{A_c^2}{2}J_0^2(\beta)$ and in order to be zero the modulation index β should be one of the roots of $J_0(x)$. The smallest root of $J_0(x)$ is found from tables to be equal 2.404. Thus,

$$\beta_{\min} = 2.404$$

Problem 3.28

1) If the output of the narrowband FM modulator is,

$$u(t) = A\cos(2\pi f_0 t + \phi(t))$$

then the output of the upper frequency multiplier ($\times n_1$) is

$$u_1(t) = A\cos(2\pi n_1 f_0 t + n_1\phi(t))$$

After mixing with the output of the second frequency multiplier $u_2(t) = A\cos(2\pi n_2 f_0 t)$ we obtain the signal

$$\begin{aligned} y(t) &= A^2\cos(2\pi n_1 f_0 t + n_1\phi(t))\cos(2\pi n_2 f_0 t) \\ &= \frac{A^2}{2}\left(\cos(2\pi(n_1+n_2)f_0 + n_1\phi(t)) + \cos(2\pi(n_1-n_2)f_0 + n_1\phi(t))\right) \end{aligned}$$

The bandwidth of the signal is $W = 15$ KHz, so the maximum frequency deviation is $\Delta f = \beta_f W = 0.1 \times 15 = 1.5$ KHz. In order to achieve a frequency deviation of $f = 75$ KHz at the output of the wideband modulator, the frequency multiplier n_1 should be equal to

$$n_1 = \frac{f}{\Delta f} = \frac{75}{1.5} = 50$$

Using an up-converter the frequency modulated signal is given by

$$y(t) = \frac{A^2}{2}\cos(2\pi(n_1+n_2)f_0 + n_1\phi(t))$$

Since the carrier frequency $f_c = (n_1+n_2)f_0$ is 104 MHz, n_2 should be such that

$$(n_1+n_2)100 = 104 \times 10^3 \Longrightarrow n_1 + n_2 = 1040 \text{ or } n_2 = 990$$

2) The maximum allowable drift (d_f) of the 100 kHz oscillator should be such that

$$(n_1+n_2)d_f = 2 \Longrightarrow d_f = \frac{2}{1040} = .0019 \text{ Hz}$$

Problem 3.29

The modulated PM signal is given by

$$\begin{aligned} u(t) &= A_c\cos(2\pi f_c t + k_p m(t)) = A_c \mathrm{Re}\left[e^{j2\pi f_c t}e^{jk_p m(t)}\right] \\ &= A_c \mathrm{Re}\left[e^{j2\pi f_c t}e^{jm(t)}\right] \end{aligned}$$

The signal $e^{jm(t)}$ is periodic with period $T_m = \frac{1}{f_m}$ and Fourier series expansion

$$\begin{aligned}
c_n &= \frac{1}{T_m}\int_0^{T_m} e^{jm(t)}e^{-j2\pi n f_m t}dt \\
&= \frac{1}{T_m}\int_0^{\frac{T_m}{2}} e^{j}e^{-j2\pi n f_m t}dt + \frac{1}{T_m}\int_{\frac{T_m}{2}}^{T_m} e^{-j}e^{-j2\pi n f_m t}dt \\
&= -\frac{e^{j}}{T_m j2\pi n f_m}e^{-j2\pi n f_m t}\Big|_0^{\frac{T_m}{2}} - \frac{e^{-j}}{T_m j2\pi n f_m}e^{-j2\pi n f_m t}\Big|_{\frac{T_m}{2}}^{T_m} \\
&= \frac{(-1)^n - 1}{2\pi n}j(e^{j} - e^{-j}) = \begin{cases} 0 & n = 2l \\ \frac{2}{\pi(2l+1)}\sin(1) & n = 2l+1 \end{cases}
\end{aligned}$$

Hence,

$$e^{jm(t)} = \sum_{l=-\infty}^{\infty} \frac{2}{\pi(2l+1)}\sin(1)e^{j2\pi l f_m t}$$

and

$$\begin{aligned}
u(t) &= A_c\text{Re}\left[e^{j2\pi f_c t}e^{jm(t)}\right] = A_c\text{Re}\left[e^{j2\pi f_c t}\sum_{l=-\infty}^{\infty}\frac{2}{\pi(2l+1)}\sin(1)e^{j2\pi l f_m t}\right] \\
&= A_c \sum_{l=-\infty}^{\infty}\left|\frac{2\sin(1)}{\pi(2l+1)}\right|\cos(2\pi(f_c + l f_m)t + \phi_l)
\end{aligned}$$

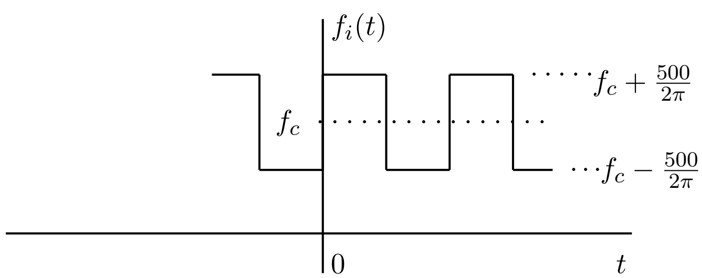
where $\phi_l = 0$ for $l \geq 0$ and $\phi_l = \pi$ for negative values of l.

Problem 3.30

1) The instantaneous frequency is given by

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_c + \frac{1}{2\pi}100m(t)$$

A plot of $f_i(t)$ is given in the next figure



2) The peak frequency deviation is given by

$$\Delta f_{\max} = k_f \max[|m(t)|] = \frac{100}{2\pi}5 = \frac{250}{\pi}$$

Problem 3.31

1) The modulation index is

$$\beta = \frac{k_f \max[|m(t)|]}{f_m} = \frac{\Delta f_{\max}}{f_m} = \frac{20\times 10^3}{10^4} = 2$$

The modulated signal $u(t)$ has the form

$$
\begin{aligned}
u(t) &= \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c+nf_m)t+\phi_n) \\
&= \sum_{n=-\infty}^{\infty} 100 J_n(2)\cos(2\pi(10^8+n10^4)t+\phi_n)
\end{aligned}
$$

The power of the unmodulated carrier signal is $P = \frac{100^2}{2} = 5000$. The power in the frequency component $f = f_c + k10^4$ is

$$
P_{f_c+kf_m} = \frac{100^2 J_k^2(2)}{2}
$$

The next table shows the values of $J_k(2)$, the frequency $f_c + kf_m$, the amplitude $100J_k(2)$ and the power $P_{f_c+kf_m}$ for various values of k.

Index k	$J_k(2)$	Frequency Hz	Amplitude $100J_k(2)$	Power $P_{f_c+kf_m}$
0	.2239	10^8	22.39	250.63
1	.5767	10^8+10^4	57.67	1663.1
2	.3528	$10^8+2\times10^4$	35.28	622.46
3	.1289	$10^8+3\times10^4$	12.89	83.13
4	.0340	$10^8+4\times10^4$	3.40	5.7785

As it is observed from the table the signal components that have a power level greater than 500 (= 10% of the power of the unmodulated signal) are those with frequencies $10^8 + 10^4$ and $10^8 + 2\times10^4$. Since $J_n^2(\beta) = J_{-n}^2(\beta)$ it is conceivable that the signal components with frequency $10^8 - 10^4$ and $10^8 - 2\times10^4$ will satisfy the condition of minimum power level. Hence, there are four signal components that have a power of at least 10% of the power of the unmodulated signal. The components with frequencies $10^8 + 10^4$, $10^8 - 10^4$ have an amplitude equal to 57.67, whereas the signal components with frequencies $10^8 + 2\times10^4$, $10^8 - 2\times10^4$ have an amplitude equal to 35.28.

2) Using Carson's rule, the approximate bandwidth of the FM signal is

$$
B_c = 2(\beta+1)f_m = 2(2+1)10^4 = 6\times10^4 \text{ Hz}
$$

Problem 3.32
1)

$$
\begin{aligned}
\beta_p &= k_p \max[|m(t)|] = 1.5\times2 = 3 \\
\beta_f &= \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000\times2}{1000} = 6
\end{aligned}
$$

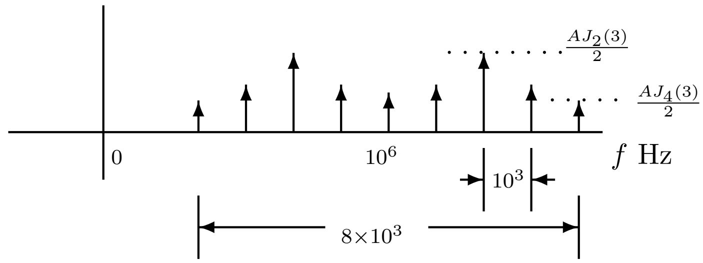
2) Using Carson's rule we obtain

$$
\begin{aligned}
B_{\text{PM}} &= 2(\beta_p+1)f_m = 8\times1000 = 8000 \\
B_{\text{FM}} &= 2(\beta_f+1)f_m = 14\times1000 = 14000
\end{aligned}
$$

3) The PM modulated signal can be written as

$$
u(t) = \sum_{n=-\infty}^{\infty} AJ_n(\beta_p)\cos(2\pi(10^6+n10^3)t)
$$

The next figure shows the amplitude of the spectrum for positive frequencies and for these components whose frequencies lie in the interval $[10^6 - 4\times 10^3, 10^6 + 4\times 10^3]$. Note that $J_0(3) = -.2601$, $J_1(3) = 0.3391$, $J_2(3) = 0.4861$, $J_3(3) = 0.3091$ and $J_4(3) = 0.1320$.

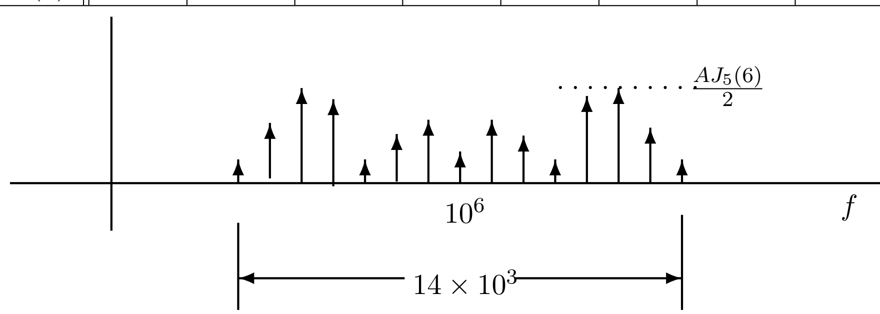


In the case of the FM modulated signal

$$\begin{aligned}
u(t) &= A\cos(2\pi f_c t + \beta_f \sin(2000\pi t)) \\
&= \sum_{n=-\infty}^{\infty} AJ_n(6)\cos(2\pi(10^6 + n10^3)t + \phi_n)
\end{aligned}$$

The next figure shows the amplitude of the spectrum for positive frequencies and for these components whose frequencies lie in the interval $[10^6 - 7\times 10^3, 10^6 - 7\times 10^3]$. The values of $J_n(6)$ for $n = 0,\ldots,7$ are given in the following table.

n	0	1	2	3	4	5	6	7
$J_n(6)$	.1506	-.2767	-.2429	.1148	.3578	.3621	.2458	.1296



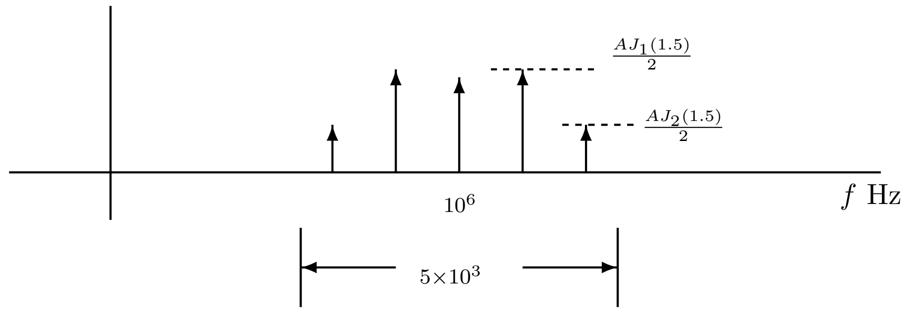
4) If the amplitude of $m(t)$ is decreased by a factor of two, then $m(t) = \cos(2\pi 10^3 t)$ and

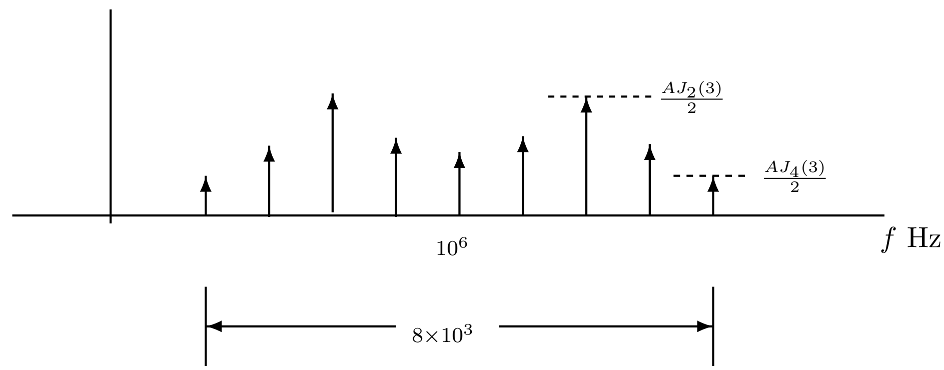
$$\begin{aligned}
\beta_p &= k_p \max[|m(t)|] = 1.5 \\
\beta_f &= \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000}{1000} = 3
\end{aligned}$$

The bandwidth is determined using Carson's rule as

$$\begin{aligned}
B_{\text{PM}} &= 2(\beta_p + 1)f_m = 5\times 1000 = 5000 \\
B_{\text{FM}} &= 2(\beta_f + 1)f_m = 8\times 1000 = 8000
\end{aligned}$$

The amplitude spectrum of the PM and FM modulated signals is plotted in the next figure for positive frequencies. Only those frequency components lying in the previous derived bandwidth are plotted. Note that $J_0(1.5) = .5118$, $J_1(1.5) = .5579$ and $J_2(1.5) = .2321$.





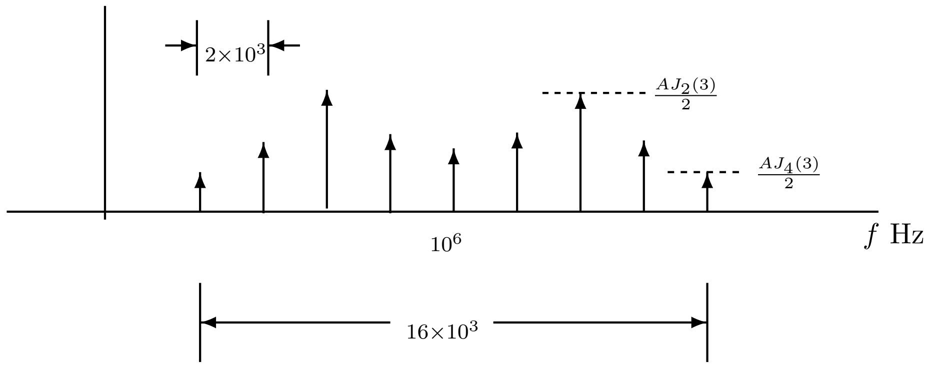
5) If the frequency of $m(t)$ is increased by a factor of two, then $m(t) = 2\cos(2\pi 2 \times 10^3 t)$ and

$$\begin{aligned}
\beta_p &= k_p \max[|m(t)|] = 1.5 \times 2 = 3 \\
\beta_f &= \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000 \times 2}{2000} = 3
\end{aligned}$$

The bandwidth is determined using Carson's rule as

$$\begin{aligned}
B_{\text{PM}} &= 2(\beta_p + 1)f_m = 8 \times 2000 = 16000 \\
B_{\text{FM}} &= 2(\beta_f + 1)f_m = 8 \times 2000 = 16000
\end{aligned}$$

The amplitude spectrum of the PM and FM modulated signals is plotted in the next figure for positive frequencies. Only those frequency components lying in the previous derived bandwidth are plotted. Note that doubling the frequency has no effect on the number of harmonics in the bandwidth of the PM signal, whereas it decreases the number of harmonics in the bandwidth of the FM signal from 14 to 8.



Problem 3.33

1) The PM modulated signal is

$$\begin{aligned} u(t) &= 100\cos(2\pi f_c t + \frac{\pi}{2}\cos(2\pi 1000 t)) \\ &= \sum_{n=-\infty}^{\infty} 100 J_n(\frac{\pi}{2})\cos(2\pi(10^8 + n10^3)t) \end{aligned}$$

The next table tabulates $J_n(\beta)$ for $\beta = \frac{\pi}{2}$ and $n = 0, \ldots, 4$.

n	0	1	2	3	4
$J_n(\beta)$	.4720	.5668	.2497	.0690	.0140

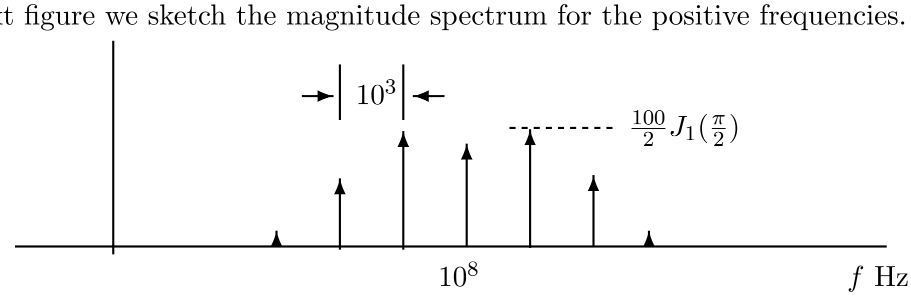
The total power of the modulated signal is $P_{\text{tot}} = \frac{100^2}{2} = 5000$. To find the effective bandwidth of the signal we calculate the index k such that

$$\sum_{n=-k}^{k} \frac{100^2}{2} J_n^2(\frac{\pi}{2}) \geq 0.99 \times 5000 \Longrightarrow \sum_{n=-k}^{k} J_n^2(\frac{\pi}{2}) \geq 0.99$$

By trial end error we find that the smallest index k is 2. Hence the effective bandwidth is

$$B_{\text{eff}} = 4 \times 10^3 = 4000$$

In the the next figure we sketch the magnitude spectrum for the positive frequencies.



2) Using Carson's rule, the approximate bandwidth of the PM signal is

$$B_{\text{PM}} = 2(\beta_p + 1)f_m = 2(\frac{\pi}{2} + 1)1000 = 5141.6$$

As it is observed, Carson's rule overestimates the effective bandwidth allowing in this way some margin for the missing harmonics.

Problem 3.34

1) Assuming that $u(t)$ is an FM signal it can be written as

$$\begin{aligned} u(t) &= 100\cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} \alpha\cos(2\pi f_m \tau)d\tau) \\ &= 100\cos(2\pi f_c t + \frac{k_f\alpha}{f_m}\sin(2\pi f_m t)) \end{aligned}$$

Thus, the modulation index is $\beta_f = \frac{k_f\alpha}{f_m} = 4$ and the bandwidth of the transmitted signal

$$B_{\text{FM}} = 2(\beta_f + 1)f_m = 10 \text{ KHz}$$

2) If we double the frequency, then

$$u(t) = 100\cos(2\pi f_c t + 4\sin(2\pi 2 f_m t))$$

Using the same argument as before we find that $\beta_f = 4$ and

$$B_{\text{FM}} = 2(\beta_f + 1)2f_m = 20 \text{ KHz}$$

3) If the signal $u(t)$ is PM modulated, then

$$\beta_p = \Delta\phi_{\max} = \max[4\sin(2\pi f_m t)] = 4$$

The bandwidth of the modulated signal is

$$B_{\text{PM}} = 2(\beta_p + 1)f_m = 10 \text{ KHz}$$

4) If f_m is doubled, then $\beta_p = \Delta\phi_{\max}$ remains unchanged whereas

$$B_{\text{PM}} = 2(\beta_p + 1)2f_m = 20 \text{ KHz}$$

Problem 3.35

1) If the signal $m(t) = m_1(t) + m_2(t)$ DSB modulates the carrier $A_c\cos(2\pi f_c t)$ the result is the signal

$$\begin{aligned} u(t) &= A_c m(t)\cos(2\pi f_c t) \\ &= A_c(m_1(t) + m_2(t))\cos(2\pi f_c t) \\ &= A_c m_1(t)\cos(2\pi f_c t) + A_c m_2(t)\cos(2\pi f_c t) \\ &= u_1(t) + u_2(t) \end{aligned}$$

where $u_1(t)$ and $u_2(t)$ are the DSB modulated signals corresponding to the message signals $m_1(t)$ and $m_2(t)$. Hence, AM modulation satisfies the superposition principle.

2) If $m(t)$ frequency modulates a carrier $A_c\cos(2\pi f_c t)$ the result is

$$\begin{aligned} u(t) &= A_c\cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty}(m_1(\tau) + m_2(\tau))d\tau) \\ &\neq A_c\cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} m_1(\tau)d\tau) \\ &\quad + A_c\cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} m_2(\tau)d\tau) \\ &= u_1(t) + u_2(t) \end{aligned}$$

where the inequality follows from the nonlinearity of the cosine function. Hence, angle modulation is not a linear modulation method.

Problem 3.36

The transfer function of the FM discriminator is

$$H(s) = \frac{R}{R + Ls + \frac{1}{Cs}} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Thus,

$$|H(f)|^2 = \frac{4\pi^2\left(\frac{R}{L}\right)^2 f^2}{\left(\frac{1}{LC} - 4\pi^2 f^2\right)^2 + 4\pi^2(\frac{R}{L})^2 f^2}$$

As it is observed $|H(f)|^2 \leq 1$ with equality if

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Since this filter is to be used as a slope detector, we require that the frequency content of the signal, which is $[80-6, 80+6]$ MHz, to fall inside the region over which $|H(f)|$ is almost linear. Such a region can be considered the interval $[f_{10}, f_{90}]$, where f_{10} is the frequency such that $|H(f_{10})| = 10\% \max[|H(f)|]$ and f_{90} is the frequency such that $|H(f_{10})| = 90\% \max[|H(f)|]$.

With $\max[|H(f)| = 1$, $f_{10} = 74 \times 10^6$ and $f_{90} = 86 \times 10^6$, we obtain the system of equations

$$4\pi^2 f_{10}^2 + \frac{50 \times 10^3}{L} 2\pi f_{10}[1 - 0.1^2]^{\frac{1}{2}} - \frac{1}{LC} = 0$$
$$4\pi^2 f_{90}^2 + \frac{50 \times 10^3}{L} 2\pi f_{90}[1 - 0.9^2]^{\frac{1}{2}} - \frac{1}{LC} = 0$$

Solving this system, we obtain

$$L = 14.98 \text{ mH} \quad C = 0.018013 \text{ pF}$$

Problem 3.37

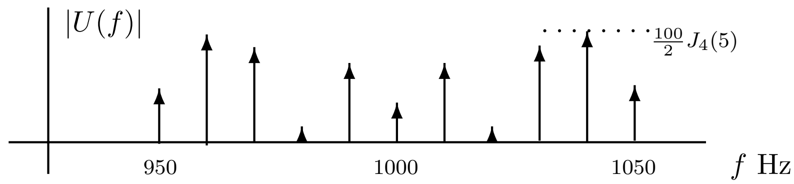
The case of $\phi(t) = \beta\cos(2\pi f_m t)$ has been treated in the text (see Section 3.3.2). the modulated signal is

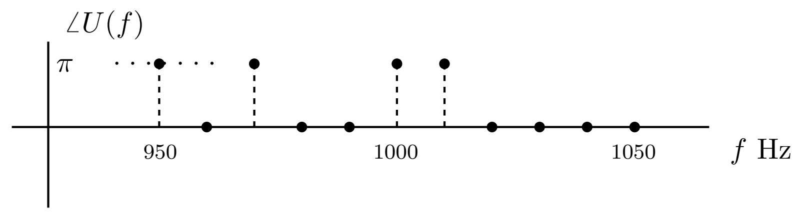
$$\begin{aligned} u(t) &= \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)) \\ &= \sum_{n=-\infty}^{\infty} 100 J_n(5)\cos(2\pi(10^3 + n10)) \end{aligned}$$

The following table shows the values of $J_n(5)$ for $n = 0, \ldots, 5$.

n	0	1	2	3	4	5
$J_n(5)$	-.178	-.328	.047	.365	.391	.261

In the next figure we plot the magnitude and the phase spectrum for frequencies in the range $[950, 1050]$ Hz. Note that $J_{-n}(\beta) = J_n(\beta)$ if n is even and $J_{-n}(\beta) = -J_n(\beta)$ if n is odd.





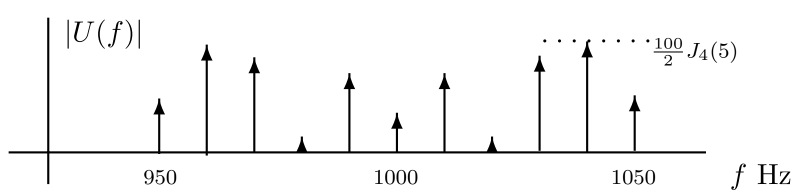
The Fourier Series expansion of $e^{j\beta\sin(2\pi f_m t)}$ is

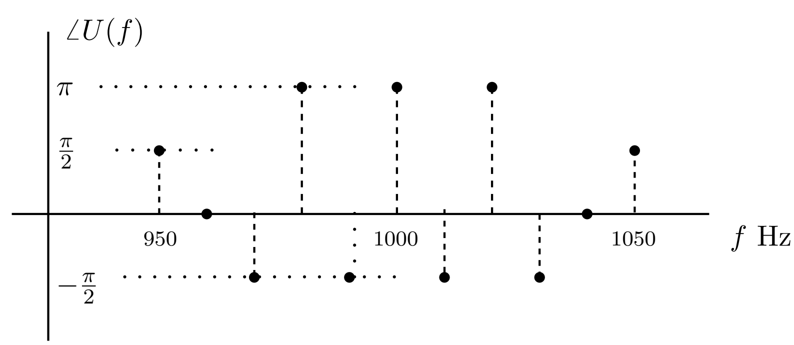
$$\begin{aligned} c_n &= f_m \int_{\frac{1}{4f_m}}^{\frac{5}{4f_m}} e^{j\beta\sin(2\pi f_m t)} e^{-j2\pi n f_m t} dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{j\beta\cos u - jnu} e^{j\frac{n\pi}{2}} du \\ &= e^{j\frac{n\pi}{2}} J_n(\beta) \end{aligned}$$

Hence,

$$\begin{aligned} u(t) &= A_c \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_c t} e^{j2\pi n f_m t}\right] \\ &= A_c \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} e^{j2\pi(f_c + n f_m)t + \frac{n\pi}{2}}\right] \end{aligned}$$

The magnitude and the phase spectra of $u(t)$ for $\beta = 5$ and frequencies in the interval $[950, 1000]$ Hz are shown in the next figure. Note that the phase spectrum has been plotted modulo 2π in the interval $(-\pi, \pi]$.





Problem 3.38

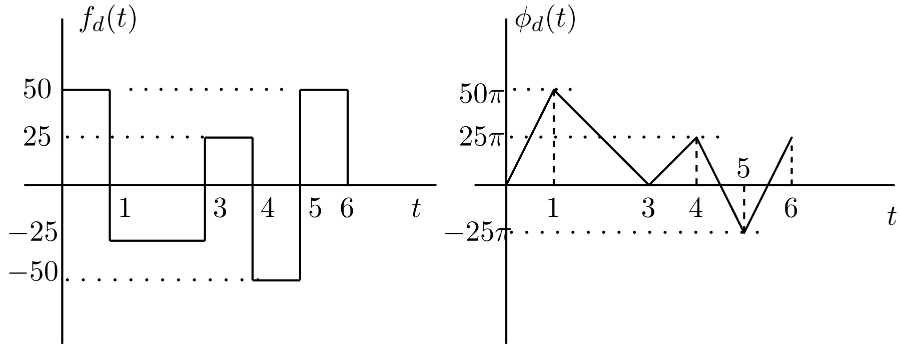
The frequency deviation is given by

$$f_d(t) = f_i(t) - f_c = k_f m(t)$$

whereas the phase deviation is obtained from

$$\phi_d(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau$$

In the next figure we plot the frequency and the phase deviation when $m(t)$ is as in Fig. P-3.38 with $k_f = 25$.



Problem 3.39
Using Carson's rule we obtain

$$B_c = 2(\beta+1)W = 2(\frac{k_f \max[|m(t)|]}{W} + 1)W = \begin{cases} 20020 & k_f = 10 \\ 20200 & k_f = 100 \\ 22000 & k_f = 1000 \end{cases}$$

Problem 3.40
The modulation index is

$$\beta = \frac{k_f \max[|m(t)|]}{f_m} = \frac{10 \times 10}{8} = 12.5$$

The output of the FM modulator can be written as

$$\begin{aligned} u(t) &= 10\cos(2\pi 2000t + 2\pi k_f \int_{-\infty}^{t} 10\cos(2\pi 8\tau)d\tau) \\ &= \sum_{n=-\infty}^{\infty} 10 J_n(12.5)\cos(2\pi(2000+n8)t + \phi_n) \end{aligned}$$

At the output of the BPF only the signal components with frequencies in the interval $[2000-32, 2000+32]$ will be present. These components are the terms of $u(t)$ for which $n = -4, \ldots, 4$. The power of the output signal is then

$$\frac{10^2}{2} J_0^2(12.5) + 2\sum_{n=1}^{4} \frac{10^2}{2} J_n^2(12.5) = 50 \times 0.2630 = 13.15$$

Since the total transmitted power is $P_{\text{tot}} = \frac{10^2}{2} = 50$, the power at the output of the bandpass filter is only 26.30% of the transmitted power.

Problem 3.41
1) The instantaneous frequency is

$$f_i(t) = f_c + k_f m_1(t)$$

The maximum of $f_i(t)$ is

$$\max[f_i(t)] = \max[f_c + k_f m_1(t)] = 10^6 + 5 \times 10^5 = 1.5 \text{ MHz}$$

2) The phase of the PM modulated signal is $\phi(t) = k_p m_1(t)$ and the instantaneous frequency

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_c + \frac{k_p}{2\pi}\frac{d}{dt}m_1(t)$$

The maximum of $f_i(t)$ is achieved for t in $[0,1]$ where $\frac{d}{dt}m_1(t) = 1$. Hence, $\max[f_i(t)] = 10^6 + \frac{3}{2\pi}$.

3) The maximum value of $m_2(t) = \text{sinc}(2\times 10^4 t)$ is 1 and it is achieved for $t = 0$. Hence,

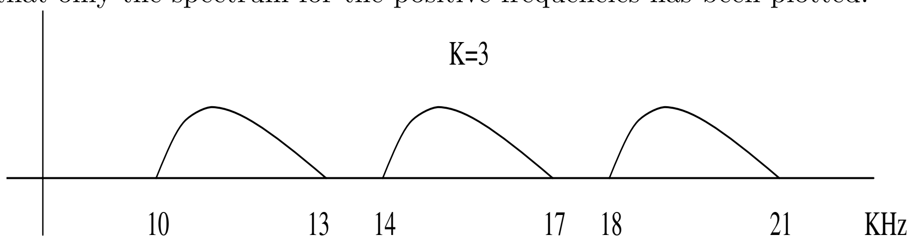
$$\max[f_i(t)] = \max[f_c + k_f m_2(t)] = 10^6 + 10^3 = 1.001 \text{ MHz}$$

Since, $\mathcal{F}[\text{sinc}(2\times 10^4 t)] = \frac{1}{2\times 10^4}\Pi(\frac{f}{2\times 10^4})$ the bandwidth of the message signal is $W = 10^4$. Thus, using Carson's rule, we obtain

$$B = 2(\frac{k_f \max[|m(t)|]}{W} + 1)W = 22 \text{ KHz}$$

Problem 3.42

1) The next figure illustrates the spectrum of the SSB signal assuming that USSB is employed and K=3. Note, that only the spectrum for the positive frequencies has been plotted.



2) With $LK = 60$ the possible values of the pair (L,K) (or (K,L)) are $\{(1,60),(2,30),(3,20),(4,15),(6,10)\}$. As it is seen the minimum value of $L+K$ is achieved for $L=6$, $K=10$ (or $L=10$, $K=6$).

3) Assuming that $L=6$ and $K=10$ we need 16 carriers with frequencies

$$\begin{array}{ll} f_{k_1} = 10 \text{ KHz} & f_{k_2} = 14 \text{ KHz} \\ f_{k_3} = 18 \text{ KHz} & f_{k_4} = 22 \text{ KHz} \\ f_{k_5} = 26 \text{ KHz} & f_{k_6} = 30 \text{ KHz} \\ f_{k_7} = 34 \text{ KHz} & f_{k_8} = 38 \text{ KHz} \\ f_{k_9} = 42 \text{ KHz} & f_{k_{10}} = 46 \text{ KHz} \end{array}$$

and

$$\begin{array}{ll} f_{l_1} = 290 \text{ KHz} & f_{l_2} = 330 \text{ KHz} \\ f_{l_3} = 370 \text{ KHz} & f_{l_4} = 410 \text{ KHz} \\ f_{l_5} = 450 \text{ KHz} & f_{l_6} = 490 \text{ KHz} \end{array}$$

Problem 3.43

Since 88 MHz $< f_c <$ 108 MHz and

$$|f_c - f_c'| = 2f_{\text{IF}} \quad \text{if } f_{\text{IF}} < f_{\text{LO}}$$

we conclude that in order for the image frequency f_c' to fall outside the interval $[88, 108]$ MHZ, the minimum frequency f_{IF} is such that

$$2f_{\text{IF}} = 108 - 88 \Longrightarrow f_{\text{IF}} = 10 \text{ MHz}$$

If $f_{\text{IF}} = 10$ MHz, then the range of f_{LO} is $[88+10, 108+10] = [98, 118]$ MHz.

Chapter 4

Problem 4.1
Let us denote by r_n (b_n) the event of drawing a red (black) ball with number n. Then

$$\begin{aligned}
1.\ E_1 &= \{r_2, r_4, b_2\} \\
2.\ E_2 &= \{r_2, r_3, r_4\} \\
3.\ E_3 &= \{r_1, r_2, b_1, b_2\} \\
4.\ E_4 &= \{r_1, r_2, r_4, b_1, b_2\} \\
5.\ E_5 &= \{r_2, r_4, b_2\} \cup [\{r_2, r_3, r_4\} \cap \{r_1, r_2, b_1, b_2\}] \\
&= \{r_2, r_4, b_2\} \cup \{r_2\} = \{r_2, r_4, b_2\}
\end{aligned}$$

Problem 4.2
Solution:
Since the seven balls equally likely to be drawn, the probability of each event E_i is proportional to its cardinality.

$$P(E_1) = \frac{3}{7}, \quad P(E_2) = \frac{3}{7}, \quad P(E_3) = \frac{4}{7}, \quad P(E_4) = \frac{5}{7}, \quad P(E_5) = \frac{3}{7}$$

Problem 4.3
Solution:
Let us denote by X the event that a car is of brand X, and by R the event that a car needs repair during its first year of purchase. Then
1)

$$\begin{aligned}
P(R) &= P(A, R) + P(B, R) + P(C, R) \\
&= P(R|A)P(A) + P(R|B)P(B) + P(R|C)P(C) \\
&= \frac{5}{100}\frac{20}{100} + \frac{10}{100}\frac{30}{100} + \frac{15}{100}\frac{50}{100} \\
&= \frac{11.5}{100}
\end{aligned}$$

2)

$$P(A|R) = \frac{P(A, R)}{P(R)} = \frac{P(R|A)P(A)}{P(R)} = \frac{.05.20}{.115} = .087$$

Problem 4.4
Solution:
If two events are mutually exclusive (disjoint) then $P(A \cup B) = P(A) \cup P(B)$ which implies that $P(A \cap B) = 0$. If the events are independent then $P(A \cap B) = P(A) \cap P(B)$. Combining these two conditions we obtain that two disjoint events are independent if

$$P(A \cap B) = P(A)P(B) = 0$$

Thus, at least on of the events should be of zero probability.

Problem 4.5

Let us denote by nS the event that n was produced by the source and sent over the channel, and by nC the event that n was observed at the output of the channel. Then

1)

$$\begin{aligned} P(1C) &= P(1C|1S)P(1S) + P(1C|0C)P(0C) \\ &= .8 \cdot .7 + .2 \cdot .3 = .62 \end{aligned}$$

where we have used the fact that $P(1S) = .7$, $P(0C) = .3$, $P(1C|0C) = .2$ and $P(1C|1S) = 1 - .2 = .8$

2)

$$P(1S|1C) = \frac{P(1C, 1S)}{P(1C)} = \frac{P(1C|1S)P(1S)}{P(1C)} = \frac{.8 \cdot .7}{.62} = .9032$$

Problem 4.6

1) X can take four different values. 0, if no head shows up, 1, if only one head shows up in the four flips of the coin, 2, for two heads and 3 if the outcome of each flip is head.

2) X follows the binomial distribution with $n = 3$. Thus

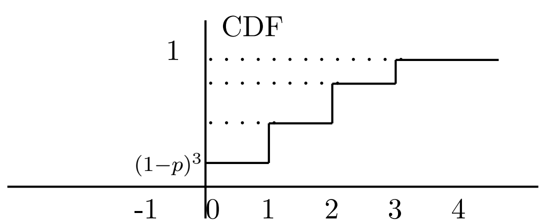
$$P(X = k) = \begin{cases} \binom{3}{k} p^k (1-p)^{3-k} & \text{for } 0 \leq k \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

3)

$$F_X(k) = \sum_{m=0}^{k} \binom{3}{m} p^m (1-p)^{3-m}$$

Hence

$$F_X(k) = \begin{cases} 0 & k < 0 \\ (1-p)^3 & k = 0 \\ (1-p)^3 + 3p(1-p)^2 & k = 1 \\ (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) & k = 2 \\ (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) + p^3 = 1 & k = 3 \\ 1 & k > 3 \end{cases}$$



4)

$$P(X > 1) = \sum_{k=2}^{3} \binom{3}{k} p^k (1-p)^{3-k} = 3p^2(1-p) + (1-p)^3$$

Problem 4.7

1) The random variables X and Y follow the binomial distribution with $n = 4$ and $p = 1/4$ and $1/2$ respectively. Thus

$$p(X = 0) = \binom{4}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^4 = \frac{3^4}{2^8} \qquad p(Y = 0) = \binom{4}{0} \left(\frac{1}{2}\right)^4 = \frac{1}{2^4}$$

$$p(X=1) = \left(\begin{array}{c}4\\1\end{array}\right)\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^3 = \frac{3^3 4}{2^8} \qquad p(Y=1) = \left(\begin{array}{c}4\\1\end{array}\right)\left(\frac{1}{2}\right)^4 = \frac{4}{2^4}$$

$$p(X=2) = \left(\begin{array}{c}4\\2\end{array}\right)\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 = \frac{3^3 2}{2^8} \qquad p(Y=2) = \left(\begin{array}{c}4\\2\end{array}\right)\left(\frac{1}{2}\right)^4 = \frac{6}{2^4}$$

$$p(X=3) = \left(\begin{array}{c}4\\3\end{array}\right)\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^1 = \frac{3 \cdot 4}{2^8} \qquad p(Y=3) = \left(\begin{array}{c}4\\3\end{array}\right)\left(\frac{1}{2}\right)^4 = \frac{4}{2^4}$$

$$p(X=4) = \left(\begin{array}{c}4\\4\end{array}\right)\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^0 = \frac{1}{2^8} \qquad p(Y=4) = \left(\begin{array}{c}4\\4\end{array}\right)\left(\frac{1}{2}\right)^4 = \frac{1}{2^4}$$

Since X and Y are independent we have

$$p(X=Y=2) = p(X=2)p(Y=2) = \frac{3^3 2}{2^8}\frac{6}{2^4} = \frac{81}{1024}$$

2)

$$\begin{aligned}
p(X=Y) &= p(X=0)p(Y=0) + p(X=1)p(Y=1) + p(X=2)p(Y=2) \\
&\quad + p(X=3)p(Y=3) + p(X=4)p(Y=4) \\
&= \frac{3^4}{2^12} + \frac{3^3 \cdot 4^2}{2^12} + \frac{3^4 \cdot 2^2}{2^12} + \frac{3 \cdot 4^2}{2^12} + \frac{1}{2^12} = \frac{886}{4096}
\end{aligned}$$

3)

$$\begin{aligned}
p(X>Y) &= p(Y=0)\left[p(X=1) + p(X=2) + p(X=3) + p(X=4)\right] + \\
&\quad p(Y=1)\left[p(X=2) + p(X=3) + p(X=4)\right] + \\
&\quad p(Y=2)\left[p(X=3) + p(X=4)\right] + \\
&\quad p(Y=3)\left[p(X=4)\right] \\
&= \frac{535}{4096}
\end{aligned}$$

4) In general $p(X+Y \leq 5) = \sum_{l=0}^{5}\sum_{m=0}^{l} p(X=l-m)p(Y=m)$. However it is easier to find $p(X+Y \leq 5)$ through $p(X+Y \leq 5) = 1 - p(X+Y > 5)$ because fewer terms are involved in the calculation of the probability $p(X+Y>5)$. Note also that $p(X+Y>5|X=0) = p(X+Y>5|X=1) = 0$.

$$\begin{aligned}
p(X+Y>5) &= p(X=2)p(Y=4) + p(X=3)[p(Y=3) + p(Y=4)] + \\
&\quad p(X=4)[p(Y=2) + p(Y=3) + p(Y=4)] \\
&= \frac{125}{4096}
\end{aligned}$$

Hence, $p(X+Y \leq 5) = 1 - p(X+Y>5) = 1 - \frac{125}{4096}$

Problem 4.8

1) Since $\lim_{x\to\infty} F_X(x) = 1$ and $F_X(x) = 1$ for all $x \geq 1$ we obtain $K = 1$.

2) The random variable is of the mixed-type since there is a discontinuity at $x=1$. $\lim_{\epsilon\to 0} F_X(1-\epsilon) = 1/2$ whereas $\lim_{\epsilon\to 0} F_X(1+\epsilon) = 1$

3)

$$P(\frac{1}{2} < X \leq 1) = F_X(1) - F_X(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4}$$

4)

$$P(\frac{1}{2} < X < 1) = F_X(1^-) - F_X(\frac{1}{2}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

5)

$$P(X > 2) = 1 - P(X \leq 2) = 1 - F_X(2) = 1 - 1 = 0$$

Problem 4.9

1)

$$\begin{aligned}
x < -1 \quad &\Rightarrow \quad F_X(x) = 0 \\
-1 \leq x \leq 0 \quad &\Rightarrow \quad F_X(x) = \int_{-1}^{x} (v+1)dv = (\frac{1}{2}v^2 + v)\Big|_{-1}^{x} = \frac{1}{2}x^2 + x + \frac{1}{2} \\
0 \leq x \leq 1 \quad &\Rightarrow \quad F_X(x) = \int_{-1}^{0} (v+1)dv + \int_{0}^{x} (-v+1)dv = -\frac{1}{2}x^2 + x + \frac{1}{2} \\
1 \leq x \quad &\Rightarrow \quad F_X(x) = 1
\end{aligned}$$

2)

$$p(X > \frac{1}{2}) = 1 - F_X(\frac{1}{2}) = 1 - \frac{7}{8} = \frac{1}{8}$$

3)

$$p(X > 0|X < \frac{1}{2}) = \frac{p(X > 0,\ X < \frac{1}{2})}{p(X < \frac{1}{2})} = \frac{F_X(\frac{1}{2}) - F_X(0)}{1 - p(X > \frac{1}{2})} = \frac{3}{7}$$

4) We find first the CDF

$$F_X(x|X > \frac{1}{2}) = p(X \leq x|X > \frac{1}{2}) = \frac{p(X \leq x,\ X > \frac{1}{2})}{p(X > \frac{1}{2})}$$

If $x \leq \frac{1}{2}$ then $p(X \leq x|X > \frac{1}{2}) = 0$ since the events $E_1 = \{X \leq \frac{1}{2}\}$ and $E_1 = \{X > \frac{1}{2}\}$ are disjoint. If $x > \frac{1}{2}$ then $p(X \leq x|X > \frac{1}{2}) = F_X(x) - F_X(\frac{1}{2})$ so that

$$F_X(x|X > \frac{1}{2}) = \frac{F_X(x) - F_X(\frac{1}{2})}{1 - F_X(\frac{1}{2})}$$

Differentiating this equation with respect to x we obtain

$$f_X(x|X > \frac{1}{2}) = \begin{cases} \frac{f_X(x)}{1-F_X(\frac{1}{2})} & x > \frac{1}{2} \\ 0 & x \leq \frac{1}{2} \end{cases}$$

5)

$$\begin{aligned}
E[X|X > 1/2] &= \int_{-\infty}^{\infty} x f_X(x|X > 1/2)dx \\
&= \frac{1}{1 - F_X(1/2)} \int_{\frac{1}{2}}^{\infty} x f_X(x)dx \\
&= 8 \int_{\frac{1}{2}}^{\infty} x(-x+1)dx = 8(-\frac{1}{3}x^3 + \frac{1}{2}x^2)\Big|_{\frac{1}{2}}^{1} \\
&= \frac{2}{3}
\end{aligned}$$

Problem 4.10
1) The random variable X is Gaussian with zero mean and variance $\sigma^2 = 10^{-8}$. Thus $p(X > x) = Q(\frac{x}{\sigma})$ and

$$\begin{aligned}
p(X > 10^{-4}) &= Q\left(\frac{10^{-4}}{10^{-4}}\right) = Q(1) = .159 \\
p(X > 4 \times 10^{-4}) &= Q\left(\frac{4 \times 10^{-4}}{10^{-4}}\right) = Q(4) = 3.17 \times 10^{-5} \\
p(-2 \times 10^{-4} < X \leq 10^{-4}) &= 1 - Q(1) - Q(2) = .8182
\end{aligned}$$

2)

$$p(X > 10^{-4} | X > 0) = \frac{p(X > 10^{-4},\ X > 0)}{p(X > 0)} = \frac{p(X > 10^{-4})}{p(X > 0)} = \frac{.159}{.5} = .318$$

3) $y = g(x) = xu(x)$. Clearly $f_Y(y) = 0$ and $F_Y(y) = 0$ for $y < 0$. If $y > 0$, then the equation $y = xu(x)$ has a unique solution $x_1 = y$. Hence, $F_Y(y) = F_X(y)$ and $f_Y(y) = f_X(y)$ for $y > 0$. $F_Y(y)$ is discontinuous at $y = 0$ and the jump of the discontinuity equals $F_X(0)$.

$$F_Y(0^+) - F_Y(0^-) = F_X(0) = \frac{1}{2}$$

In summary the PDF $f_Y(y)$ equals

$$f_Y(y) = f_X(y)u(y) + \frac{1}{2}\delta(y)$$

The general expression for finding $f_Y(y)$ can not be used because $g(x)$ is constant for some interval so that there is an uncountable number of solutions for x in this interval.

4)

$$\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty} y f_Y(y) dy \\
&= \int_{-\infty}^{\infty} y \left[f_X(y)u(y) + \frac{1}{2}\delta(y) \right] dy \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy = \frac{\sigma}{\sqrt{2\pi}}
\end{aligned}$$

5) $y = g(x) = |x|$. For a given $y > 0$ there are two solutions to the equation $y = g(x) = |x|$, that is $x_{1,2} = \pm y$. Hence for $y > 0$

$$\begin{aligned}
f_Y(y) &= \frac{f_X(x_1)}{|\mathrm{sgn}(x_1)|} + \frac{f_X(x_2)}{|\mathrm{sgn}(x_2)|} = f_X(y) + f_X(-y) \\
&= \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}
\end{aligned}$$

For $y < 0$ there are no solutions to the equation $y = |x|$ and $f_Y(y) = 0$.

$$E[Y] = \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy = \frac{2\sigma}{\sqrt{2\pi}}$$

Problem 4.11

1) $y = g(x) = ax^2$. Assume without loss of generality that $a > 0$. Then, if $y < 0$ the equation $y = ax^2$ has no real solutions and $f_Y(y) = 0$. If $y > 0$ there are two solutions to the system, namely $x_{1,2} = \sqrt{y/a}$. Hence,

$$\begin{aligned} f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} \\ &= \frac{f_X(\sqrt{y/a})}{2a\sqrt{y/a}} + \frac{f_X(-\sqrt{y/a})}{2a\sqrt{y/a}} \\ &= \frac{1}{\sqrt{ay}\sqrt{2\pi\sigma^2}} e^{-\frac{y}{2a\sigma^2}} \end{aligned}$$

2) The equation $y = g(x)$ has no solutions if $y < -b$. Thus $F_Y(y)$ and $f_Y(y)$ are zero for $y < -b$. If $-b \leq y \leq b$, then for a fixed y, $g(x) < y$ if $x < y$; hence $F_Y(y) = F_X(y)$. If $y > b$ then $g(x) \leq b < y$ for every x; hence $F_Y(y) = 1$. At the points $y = \pm b$, $F_Y(y)$ is discontinuous and the discontinuities equal to

$$F_Y(-b^+) - F_Y(-b^-) = F_X(-b)$$

and

$$F_Y(b^+) - F_Y(b^-) = 1 - F_X(b)$$

The PDF of $y = g(x)$ is

$$\begin{aligned} f_Y(y) &= F_X(-b)\delta(y+b) + (1 - F_X(b))\delta(y-b) + f_X(y)[u_{-1}(y+b) - u_{-1}(y-b)] \\ &= Q\left(\frac{b}{\sigma}\right)(\delta(y+b) + \delta(y-b)) + \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} [u_{-1}(y+b) - u_{-1}(y-b)] \end{aligned}$$

3) In the case of the hard limiter

$$\begin{aligned} p(Y = b) &= p(X < 0) = F_X(0) = \frac{1}{2} \\ p(Y = a) &= p(X > 0) = 1 - F_X(0) = \frac{1}{2} \end{aligned}$$

Thus $F_Y(y)$ is a staircase function and

$$f_Y(y) = F_X(0)\delta(y-b) + (1 - F_X(0))\delta(y-a)$$

4) The random variable $y = g(x)$ takes the values $y_n = x_n$ with probability

$$p(Y = y_n) = p(a_n \leq X \leq a_{n+1}) = F_X(a_{n+1}) - F_X(a_n)$$

Thus, $F_Y(y)$ is a staircase function with $F_Y(y) = 0$ if $y < x_1$ and $F_Y(y) = 1$ if $y > x_N$. The PDF is a sequence of impulse functions, that is

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^{N} [F_X(a_{i+1}) - F_X(a_i)]\,\delta(y - x_i) \\ &= \sum_{i=1}^{N} \left[Q\left(\frac{a_i}{\sigma}\right) - Q\left(\frac{a_{i+1}}{\sigma}\right)\right]\delta(y - x_i) \end{aligned}$$

Problem 4.12
The equation $x = \tan\phi$ has a unique solution in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, that is

$$\phi_1 = \arctan x$$

Furthermore

$$x'(\phi) = \left(\frac{\sin\phi}{\cos\phi}\right)' = \frac{1}{\cos^2\phi} = 1 + \frac{\sin^2\phi}{\cos^2\phi} = 1 + x^2$$

Thus,

$$f_X(x) = \frac{f_\Phi(\phi_1)}{|x'(\phi_1)|} = \frac{1}{\pi(1+x^2)}$$

We observe that $f_X(x)$ is the Cauchy density. Since $f_X(x)$ is even we immediately get $E[X] = 0$. However, the variance is

$$\begin{aligned} \sigma_X^2 &= E[X^2] - (E[X])^2 \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{1+x^2}dx = \infty \end{aligned}$$

Problem 4.13
1)

$$\begin{aligned} E[Y] &= \int_0^\infty y f_Y(y)dy \geq \int_\alpha^\infty y f_Y(y)dy \\ &\geq \alpha\int_\alpha^\infty y f_Y(y)dy = \alpha p(Y \geq \alpha) \end{aligned}$$

Thus $p(Y \geq \alpha) \leq E[Y]/\alpha$.

2) Clearly $p(|X - E[X]| > \epsilon) = p((X - E[X])^2 > \epsilon^2)$. Thus using the results of the previous question we obtain

$$p(|X - E[X]| > \epsilon) = p((X - E[X])^2 > \epsilon^2) \leq \frac{E[(X - E[X])^2]}{\epsilon^2} = \frac{\sigma^2}{\epsilon^2}$$

Problem 4.14
The characteristic function of the binomial distribution is

$$\begin{aligned} \psi_X(v) &= \sum_{k=0}^{n} e^{jvk}\left(\begin{array}{c} n \\ k \end{array}\right) p^k(1-p)^{n-k} \\ &= \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)(pe^{jv})^k(1-p)^{n-k} = (pe^{jv} + (1-p))^n \end{aligned}$$

Thus

$$\begin{aligned} E[X] &= m_X^{(1)} = \frac{1}{j}\frac{d}{dv}(pe^{jv} + (1-p))^n\Big|_{v=0} = \frac{1}{j}n(pe^{jv} + (1-p))^{n-1}pje^{jv}\Big|_{v=0} \\ &= n(p+1-p)^{n-1}p = np \\ E[X^2] &= m_X^{(2)} = (-1)\frac{d^2}{dv^2}(pe^{jv} + (1-p))^n\Big|_{v=0} \\ &= (-1)\frac{d}{dv}\left[n(pe^{jv} + (1-p)^{n-1}pje^{jv}\right]\Big|_{v=0} \\ &= \left[n(n-1)(pe^{jv} + (1-p))^{n-2}p^2e^{2jv} + n(pe^{jv} + (1-p))^{n-1}pe^{jv}\right]\Big|_{v=0} \\ &= n(n-1)(p+1-p)p^2 + n(p+1-p)p \\ &= n(n-1)p^2 + np \end{aligned}$$

Hence the variance of the binomial distribution is

$$\sigma^2 = E[X^2] - (E[X])^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)$$

Problem 4.15

The characteristic function of the Poisson distribution is

$$\psi_X(v) = \sum_{k=0}^{\infty} e^{jvk}\frac{\lambda^k}{k!}e^{-k} = \sum_{k=0}^{\infty}\frac{(e^{jv-1}\lambda)^k}{k!}$$

But $\sum_{k=0}^{\infty}\frac{a^k}{k!} = e^a$ so that $\psi_X(v) = e^{\lambda(e^{jv-1})}$. Hence

$$\begin{aligned}
E[X] &= m_X^{(1)} = \frac{1}{j}\frac{d}{dv}\psi_X(v)\Big|_{v=0} = \frac{1}{j}e^{\lambda(e^{jv-1})}j\lambda e^{jv}\Big|_{v=0} = \lambda \\
E[X^2] &= m_X^{(2)} = (-1)\frac{d^2}{dv^2}\psi_X(v)\Big|_{v=0} = (-1)\frac{d}{dv}\left[\lambda e^{\lambda(e^{jv-1})}e^{jv}j\right]\Big|_{v=0} \\
&= \left[\lambda^2 e^{\lambda(e^{jv-1})}e^{jv} + \lambda e^{\lambda(e^{jv-1})}e^{jv}\right]\Big|_{v=0} = \lambda^2 + \lambda
\end{aligned}$$

Hence the variance of the Poisson distribution is

$$\sigma^2 = E[X^2] - (E[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Problem 4.16

For n odd, x^n is odd and since the zero-mean Gaussian PDF is even their product is odd. Since the integral of an odd function over the interval $[-\infty, \infty]$ is zero, we obtain $E[X^n] = 0$ for n even. Let $I_n = \int_{-\infty}^{\infty} x^n \exp(-x^2/2\sigma^2)dx$ with n even. Then,

$$\begin{aligned}
\frac{d}{dx}I_n &= \int_{-\infty}^{\infty}\left[nx^{n-1}e^{-\frac{x^2}{2\sigma^2}} - \frac{1}{\sigma^2}x^{n+1}e^{-\frac{x^2}{2\sigma^2}}\right]dx = 0 \\
\frac{d^2}{dx^2}I_n &= \int_{-\infty}^{\infty}\left[n(n-1)x^{n-2}e^{-\frac{x^2}{2\sigma^2}} - \frac{2n+1}{\sigma^2}x^n e^{-\frac{x^2}{2\sigma^2}} + \frac{1}{\sigma^4}x^{n+2}e^{-\frac{x^2}{2\sigma^2}}\right]dx \\
&= n(n-1)I_{n-2} - \frac{2n+1}{\sigma^2}I_n + \frac{1}{\sigma^4}I_{n+2} = 0
\end{aligned}$$

Thus,

$$I_{n+2} = \sigma^2(2n+1)I_n - \sigma^4 n(n-1)I_{n-2}$$

with initial conditions $I_0 = \sqrt{2\pi\sigma^2}$, $I_2 = \sigma^2\sqrt{2\pi\sigma^2}$. We prove now that

$$I_n = 1 \times 3 \times 5 \times \cdots \times (n-1)\sigma^n\sqrt{2\pi\sigma^2}$$

The proof is by induction on n. For $n = 2$ it is certainly true since $I_2 = \sigma^2\sqrt{2\pi\sigma^2}$. We assume that the relation holds for n and we will show that it is true for I_{n+2}. Using the previous recursion we have

$$\begin{aligned}
I_{n+2} &= 1 \times 3 \times 5 \times \cdots \times (n-1)\sigma^{n+2}(2n+1)\sqrt{2\pi\sigma^2} \\
&\quad -1 \times 3 \times 5 \times \cdots \times (n-3)(n-1)n\sigma^{n-2}\sigma^4\sqrt{2\pi\sigma^2} \\
&= 1 \times 3 \times 5 \times \cdots \times (n-1)(n+1)\sigma^{n+2}\sqrt{2\pi\sigma^2}
\end{aligned}$$

Clearly $E[X^n] = \frac{1}{\sqrt{2\pi\sigma^2}}I_n$ and

$$E[X^n] = 1 \times 3 \times 5 \times \cdots \times (n-1)\sigma^n$$

Problem 4.17

1) $f_{X,Y}(x,y)$ is a PDF so that its integral over the support region of x, y should be one.

$$\begin{aligned}
\int_0^1\int_0^1 f_{X,Y}(x,y)dxdy &= K\int_0^1\int_0^1(x+y)dxdy \\
&= K\left[\int_0^1\int_0^1 xdxdy + \int_0^1\int_0^1 ydxdy\right] \\
&= K\left[\frac{1}{2}x^2\Big|_0^1 y|_0^1 + \frac{1}{2}y^2\Big|_0^1 x|_0^1\right] \\
&= K
\end{aligned}$$

Thus $K=1$.

2)

$$\begin{aligned}
p(X+Y>1) &= 1-P(X+Y\leq 1) \\
&= 1-\int_0^1\int_0^{1-x}(x+y)dxdy \\
&= 1-\int_0^1 x\int_0^{1-x}dydx - \int_0^1 dx\int_0^{1-x}ydy \\
&= 1-\int_0^1 x(1-x)dx - \int_0^1\frac{1}{2}(1-x)^2dx \\
&= \frac{2}{3}
\end{aligned}$$

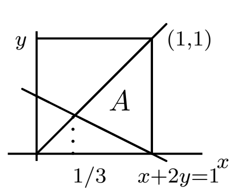
3) By exploiting the symmetry of $f_{X,Y}$ and the fact that it has to integrate to 1, one immediately sees that the answer to this question is 1/2. The "mechanical" solution is:

$$\begin{aligned}
p(X>Y) &= \int_0^1\int_y^1(x+y)dxdy \\
&= \int_0^1\int_y^1 xdxdy + \int_0^1\int_y^1 ydxdy \\
&= \int_0^1\frac{1}{2}x^2\Big|_y^1 dy + \int_0^1 yx\Big|_y^1 dy \\
&= \int_0^1\frac{1}{2}(1-y^2)dy + \int_0^1 y(1-y)dy \\
&= \frac{1}{2}
\end{aligned}$$

4)

$$p(X>Y|X+2Y>1) = p(X>Y, X+2Y>1)/p(X+2Y>1)$$

The region over which we integrate in order to find $p(X>Y, X+2Y>1)$ is marked with an A in the following figure.



Thus

$$\begin{aligned}
p(X>Y, X+2Y>1) &= \int_{\frac{1}{3}}^{1}\int_{\frac{1-x}{2}}^{x}(x+y)dxdy \\
&= \int_{\frac{1}{3}}^{1}\left[x(x-\frac{1-x}{2})+\frac{1}{2}(x^2-(\frac{1-x}{2})^2)\right]dx \\
&= \int_{\frac{1}{3}}^{1}\left(\frac{15}{8}x^2-\frac{1}{4}x-\frac{1}{8}\right)dx \\
&= \frac{49}{108} \\
p(X+2Y>1) &= \int_0^1\int_{\frac{1-x}{2}}^{1}(x+y)dxdy \\
&= \int_0^1\left[x(1-\frac{1-x}{2})+\frac{1}{2}(1-(\frac{1-x}{2})^2)\right]dx \\
&= \int_0^1\left(\frac{3}{8}x^2+\frac{3}{4}x+\frac{3}{8}\right)dx \\
&= \frac{3}{8}\times\frac{1}{3}x^3\Big|_0^1+\frac{3}{4}\times\frac{1}{2}x^2\Big|_0^1+\frac{3}{8}x\Big|_0^1 \\
&= \frac{7}{8}
\end{aligned}$$

Hence, $p(X>Y|X+2Y>1)=(49/108)/(7/8)=14/27$

5) When $X=Y$ the volume under integration has measure zero and thus

$$P(X=Y)=0$$

6) Conditioned on the fact that $X=Y$, the new p.d.f of X is

$$f_{X|X=Y}(x)=\frac{f_{X,Y}(x,x)}{\int_0^1 f_{X,Y}(x,x)dx}=2x.$$

In words, we re-normalize $f_{X,Y}(x,y)$ so that it integrates to 1 on the region characterized by $X=Y$. The result depends only on x. Then $p(X>\frac{1}{2}|X=Y)=\int_{1/2}^1 f_{X|X=Y}(x)dx=3/4$.

7)

$$\begin{aligned}
f_X(x) &= \int_0^1(x+y)dy=x+\int_0^1 ydy=x+\frac{1}{2} \\
f_Y(y) &= \int_0^1(x+y)dx=y+\int_0^1 xdx=y+\frac{1}{2}
\end{aligned}$$

8) $F_X(x|X+2Y>1)=p(X\leq x, X+2Y>1)/p(X+2Y>1)$

$$\begin{aligned}
p(X\leq x, X+2Y>1) &= \int_0^x\int_{\frac{1-v}{2}}^{1}(v+y)dvdy \\
&= \int_0^x\left[\frac{3}{8}v^2+\frac{3}{4}v+\frac{3}{8}\right]dv \\
&= \frac{1}{8}x^3+\frac{3}{8}x^2+\frac{3}{8}x
\end{aligned}$$

Hence,

$$f_X(x|X+2Y>1)=\frac{\frac{3}{8}x^2+\frac{6}{8}x+\frac{3}{8}}{p(X+2Y>1)}=\frac{3}{7}x^2+\frac{6}{7}x+\frac{3}{7}$$

$$
\begin{aligned}
E[X|X+2Y>1] &= \int_0^1 x f_X(x|X+2Y>1)dx \\
&= \int_0^1 \left(\frac{3}{7}x^3 + \frac{6}{7}x^2 + \frac{3}{7}x\right) \\
&= \frac{3}{7} \times \frac{1}{4}x^4\Big|_0^1 + \frac{6}{7} \times \frac{1}{3}x^3\Big|_0^1 + \frac{3}{7} \times \frac{1}{2}x^2\Big|_0^1 = \frac{17}{28}
\end{aligned}
$$

Problem 4.18

1)

$$F_Y(y) = p(Y \leq y) = p(X_1 \leq y \cup X_2 \leq y \cup \cdots \cup X_n \leq y)$$

Since the previous events are not necessarily disjoint, it is easier to work with the function $1 - [F_Y(y)] = 1 - p(Y \leq y)$ in order to take advantage of the independence of X_i's. Clearly

$$
\begin{aligned}
1 - p(Y \leq y) &= p(Y > y) = p(X_1 > y \cap X_2 > y \cap \cdots \cap X_n > y) \\
&= (1 - F_{X_1}(y))(1 - F_{X_2}(y)) \cdots (1 - F_{X_n}(y))
\end{aligned}
$$

Differentiating the previous with respect to y we obtain

$$f_Y(y) = f_{X_1}(y)\prod_{i\neq 1}^n (1 - F_{X_i}(y)) + f_{X_2}(y)\prod_{i\neq 2}^n (1 - F_{X_i}(y)) + \cdots + f_{X_n}(y)\prod_{i\neq n}^n (1 - F_{X_i}(y))$$

2)

$$
\begin{aligned}
F_Z(z) &= P(Z \leq z) = p(X_1 \leq z, X_2 \leq z, \cdots, X_n \leq z) \\
&= p(X_1 \leq z)p(X_2 \leq z) \cdots p(X_n \leq z)
\end{aligned}
$$

Differentiating the previous with respect to z we obtain

$$f_Z(z) = f_{X_1}(z)\prod_{i\neq 1}^n F_{X_i}(z) + f_{X_2}(z)\prod_{i\neq 2}^n F_{X_i}(z) + \cdots + f_{X_n}(z)\prod_{i\neq n}^n F_{X_i}(z)$$

Problem 4.19

$$E[X] = \int_0^\infty x\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx = \frac{1}{\sigma^2}\int_0^\infty x^2 e^{-\frac{x^2}{2\sigma^2}}dx$$

However for the Gaussian random variable of zero mean and variance σ^2

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^\infty x^2 e^{-\frac{x^2}{2\sigma^2}}dx = \sigma^2$$

Since the quantity under integration is even, we obtain that

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^\infty x^2 e^{-\frac{x^2}{2\sigma^2}}dx = \frac{1}{2}\sigma^2$$

Thus,

$$E[X] = \frac{1}{\sigma^2}\sqrt{2\pi\sigma^2}\frac{1}{2}\sigma^2 = \sigma\sqrt{\frac{\pi}{2}}$$

In order to find $VAR(X)$ we first calculate $E[X^2]$.

$$
\begin{aligned}
E[X^2] &= \frac{1}{\sigma^2}\int_0^\infty x^3 e^{-\frac{x^2}{2\sigma^2}}dx = -\int_0^\infty x d[e^{-\frac{x^2}{2\sigma^2}}] \\
&= -x^2 e^{-\frac{x^2}{2\sigma^2}}\Big|_0^\infty + \int_0^\infty 2xe^{-\frac{x^2}{2\sigma^2}}dx \\
&= 0 + 2\sigma^2\int_0^\infty \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx = 2\sigma^2
\end{aligned}
$$

Thus,

$$VAR(X) = E[X^2] - (E[X])^2 = 2\sigma^2 - \frac{\pi}{2}\sigma^2 = (2 - \frac{\pi}{2})\sigma^2$$

Problem 4.20

Let $Z = X + Y$. Then,

$$F_Z(z) = p(X + Y \leq z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x, y)dxdy$$

Differentiating with respect to z we obtain

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} \frac{d}{dz} \int_{-\infty}^{z-y} f_{X,Y}(x, y)dxdy \\ &= \int_{-\infty}^{\infty} f_{X,Y}(z - y, y)\frac{d}{dz}(z - y)dy \\ &= \int_{-\infty}^{\infty} f_{X,Y}(z - y, y)dy \\ &= \int_{-\infty}^{\infty} f_X(z - y)f_Y(y)dy \end{aligned}$$

where the last line follows from the independence of X and Y. Thus $f_Z(z)$ is the convolution of $f_X(x)$ and $f_Y(y)$. With $f_X(x) = \alpha e^{-\alpha x}u(x)$ and $f_Y(y) = \beta e^{-\beta x}u(x)$ we obtain

$$f_Z(z) = \int_0^z \alpha e^{-\alpha v}\beta e^{-\beta(z-v)}dv$$

If $\alpha = \beta$ then

$$f_Z(z) = \int_0^z \alpha^2 e^{-\alpha z}dv = \alpha^2 z e^{-\alpha z}u_{-1}(z)$$

If $\alpha \neq \beta$ then

$$f_Z(z) = \alpha\beta e^{-\beta z}\int_0^z e^{(\beta-\alpha)v}dv = \frac{\alpha\beta}{\beta - \alpha}\left[e^{-\alpha z} - e^{-\beta z}\right]u_{-1}(z)$$

Problem 4.21

1) $f_{X,Y}(x, y)$ is a PDF, hence its integral over the supporting region of x, and y is 1.

$$\begin{aligned} \int_0^{\infty}\int_y^{\infty} f_{X,Y}(x, y)dxdy &= \int_0^{\infty}\int_y^{\infty} Ke^{-x-y}dxdy \\ &= K\int_0^{\infty} e^{-y}\int_y^{\infty} e^{-x}dxdy \\ &= K\int_0^{\infty} e^{-2y}dy = K(-\frac{1}{2})e^{-2y}\Big|_0^{\infty} = K\frac{1}{2} \end{aligned}$$

Thus K should be equal to 2.

2)

$$\begin{aligned} f_X(x) &= \int_0^x 2e^{-x-y}dy = 2e^{-x}(-e^{-y})\Big|_0^x = 2e^{-x}(1 - e^{-x}) \\ f_Y(y) &= \int_y^{\infty} 2e^{-x-y}dy = 2e^{-y}(-e^{-x})\Big|_y^{\infty} = 2e^{-2y} \end{aligned}$$

3)

$$\begin{aligned}
f_X(x)f_Y(y) &= 2e^{-x}(1-e^{-x})2e^{-2y} = 2e^{-x-y}2e^{-y}(1-e^{-x}) \\
&\neq 2e^{-x-y} = f_{X,Y}(x,y)
\end{aligned}$$

Thus X and Y are not independent.

4) If $x < y$ then $f_{X|Y}(x|y) = 0$. If $x \geq y$, then with $u = x - y \geq 0$ we obtain

$$f_U(u) = f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2e^{-x-y}}{2e^{-2y}} = e^{-x+y} = e^{-u}$$

5)

$$\begin{aligned}
E[X|Y=y] &= \int_y^\infty xe^{-x+y}dx = e^y \int_y^\infty xe^{-x}dx \\
&= e^y \left[-xe^{-x}\Big|_y^\infty + \int_y^\infty e^{-x}dx \right] \\
&= e^y(ye^{-y} + e^{-y}) = y + 1
\end{aligned}$$

6) In this part of the problem we will use extensively the following definite integral

$$\int_0^\infty x^{\nu-1}e^{-\mu x}dx = \frac{1}{\mu^\nu}(\nu-1)!$$

$$\begin{aligned}
E[XY] &= \int_0^\infty \int_y^\infty xy2e^{-x-y}dxdy = \int_0^\infty 2ye^{-y}\int_y^\infty xe^{-x}dxdy \\
&= \int_0^\infty 2ye^{-y}(ye^{-y} + e^{-y})dy = 2\int_0^\infty y^2e^{-2y}dy + 2\int_0^\infty ye^{-2y}dy \\
&= 2\frac{1}{2^3}2! + 2\frac{1}{2^2}1! = 1
\end{aligned}$$

$$\begin{aligned}
E[X] &= 2\int_0^\infty xe^{-x}(1-e^{-x})dx = 2\int_0^\infty xe^{-x}dx - 2\int_0^\infty xe^{-2x}dx \\
&= 2 - 2\frac{1}{2^2} = \frac{3}{2} \\
E[Y] &= 2\int_0^\infty ye^{-2y}dy = 2\frac{1}{2^2} = \frac{1}{2} \\
E[X^2] &= 2\int_0^\infty x^2e^{-x}(1-e^{-x})dx = 2\int_0^\infty x^2e^{-x}dx - 2\int_0^\infty x^2e^{-2x}dx \\
&= 2\cdot 2! - 2\frac{1}{2^3}2! = \frac{7}{2} \\
E[Y^2] &= 2\int_0^\infty y^2e^{-2y}dy = 2\frac{1}{2^3}2! = \frac{1}{2}
\end{aligned}$$

Hence,

$$COV(X,Y) = E[XY] - E[X]E[Y] = 1 - \frac{3}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

and

$$\rho_{X,Y} = \frac{COV(X,Y)}{(E[X^2]-(E[X])^2)^{1/2}(E[Y^2]-(E[Y])^2)^{1/2}} = \frac{1}{\sqrt{5}}$$

Problem 4.22

$$
\begin{aligned}
E[X] &= \frac{1}{\pi}\int_0^{\pi} \cos\theta d\theta = \frac{1}{\pi}\sin\theta|_0^{\pi} = 0 \\
E[Y] &= \frac{1}{\pi}\int_0^{\pi} \sin\theta d\theta = \frac{1}{\pi}(-\cos\theta)|_0^{\pi} = \frac{2}{\pi} \\
E[XY] &= \int_0^{\pi} \cos\theta \ \sin\theta \frac{1}{\pi} d\theta \\
&= \frac{1}{2\pi}\int_0^{\pi} \sin 2\theta d\theta = \frac{1}{4\pi}\int_0^{2\pi} \sin x dx = 0 \\
COV(X,Y) &= E[XY] - E[X]E[Y] = 0
\end{aligned}
$$

Thus the random variables X and Y are uncorrelated. However they are not independent since $X^2 + Y^2 = 1$. To see this consider the probability $p(|X| < 1/2, Y \geq 1/2)$. Clearly $p(|X| < 1/2)p(Y \geq 1/2)$ is different than zero whereas $p(|X| < 1/2, Y \geq 1/2) = 0$. This is because $|X| < 1/2$ implies that $\pi/3 < \theta < 5\pi/3$ and for these values of θ, $Y = \sin\theta > \sqrt{3}/2 > 1/2$.

Problem 4.23

1) Clearly $X > r$, $Y > r$ implies that $X^2 > r^2$, $Y^2 > r^2$ so that $X^2+Y^2 > 2r^2$ or $\sqrt{X^2+Y^2} > \sqrt{2}r$. Thus the event $E_1(r) = \{X > r, Y > r\}$ is a subset of the event $E_2(r) = \{\sqrt{X^2+Y^2} > \sqrt{2}r|X, Y > 0\}$ and $p(E_1(r)) \leq p(E_2(r))$.

2) Since X and Y are independent

$$
p(E_1(r)) = p(X > r, Y > r) = p(X > r)p(Y > r) = Q^2(r)
$$

3) Using the rectangular to polar transformation $V = \sqrt{X^2+Y^2}$, $\Theta = \arctan\frac{Y}{X}$ it is proved (see text Eq. 4.1.22) that

$$
f_{V,\Theta}(v,\theta) = \frac{v}{2\pi\sigma^2}e^{-\frac{v^2}{2\sigma^2}}
$$

Hence, with $\sigma^2 = 1$ we obtain

$$
\begin{aligned}
p(\sqrt{X^2+Y^2} > \sqrt{2}r|X, Y > 0) &= \int_{\sqrt{2}r}^{\infty}\int_0^{\frac{\pi}{2}} \frac{v}{2\pi}e^{-\frac{v^2}{2}} dv d\theta \\
&= \frac{1}{4}\int_{\sqrt{2}r}^{\infty} v e^{-\frac{v^2}{2}} dv = \frac{1}{4}(-e^{-\frac{v^2}{2}})\Big|_{\sqrt{2}r}^{\infty} \\
&= \frac{1}{4}e^{-r^2}
\end{aligned}
$$

Combining the results of part 1), 2) and 3) we obtain

$$
Q^2(r) \leq \frac{1}{4}e^{-r^2} \quad \text{or} \quad Q(r) \leq \frac{1}{2}e^{-\frac{r^2}{2}}
$$

Problem 4.24

The following is a program written in Fortran to compute the Q function

```
REAL*8      x,t,a,q,pi,p,b1,b2,b3,b4,b5
PARAMETER        (p=.2316419d+00, b1=.31981530d+00,
```

```
     +       b2=-.356563782d+00, b3=1.781477937d+00,
     +       b4=-1.821255978d+00, b5=1.330274429d+00)
C-
             pi=4.*atan(1.)
C-INPUT
        PRINT*,  'Enter -x-'
        READ*,   x
C-
             t=1./(1.+p*x)
             a=b1*t + b2*t**2. + b3*t**3. + b4*t**4. + b5*t**5.
             q=(exp(-x**2./2.)/sqrt(2.*pi))*a
C-OUTPUT
        PRINT*,  q
C-
        STOP
        END
```

The results of this approximation along with the actual values of $Q(x)$ (taken from text Table 4.1) are tabulated in the following table. As it is observed a very good approximation is achieved.

x	$Q(x)$	Approximation
1.	1.59×10^{-1}	1.587×10^{-1}
1.5	6.68×10^{-2}	6.685×10^{-2}
2.	2.28×10^{-2}	2.276×10^{-2}
2.5	6.21×10^{-3}	6.214×10^{-3}
3.	1.35×10^{-3}	1.351×10^{-3}
3.5	2.33×10^{-4}	2.328×10^{-4}
4.	3.17×10^{-5}	3.171×10^{-5}
4.5	3.40×10^{-6}	3.404×10^{-6}
5.	2.87×10^{-7}	2.874×10^{-7}

Problem 4.25

The n-dimensional joint Gaussian distribution is

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(C)}} e^{-(\mathbf{x}-\mathbf{m})C^{-1}(\mathbf{x}-\mathbf{m})^t}$$

The Jacobian of the linear transformation $\mathbf{Y} = A\mathbf{X}^t + \mathbf{b}$ is $1/\det(A)$ and the solution to this equation is

$$\mathbf{x} = (\mathbf{y} - \mathbf{b})^t (A^{-1})^t$$

We may substitute for $\mathbf{x}$ in $f_{\mathbf{X}}(\mathbf{x})$ to obtain $f_{\mathbf{Y}}(\mathbf{y})$.

$$\begin{aligned}
f_{\mathbf{Y}}(\mathbf{y}) &= \frac{1}{(2\pi)^{n/2}(\det(C))^{1/2}|\det(A)|} \exp\Big(-[(\mathbf{y}-\mathbf{b})^t (A^{-1})^t - \mathbf{m}]C^{-1} \\
&\qquad\qquad [(\mathbf{y}-\mathbf{b})^t (A^{-1})^t - \mathbf{m}]^t\Big) \\
&= \frac{1}{(2\pi)^{n/2}(\det(C))^{1/2}|\det(A)|} \exp\Big(-[\mathbf{y}^t - b^t - \mathbf{m}A^t](A^t)^{-1}C^{-1}A^{-1} \\
&\qquad\qquad [\mathbf{y} - \mathbf{b} - A\mathbf{m}^t]\Big) \\
&= \frac{1}{(2\pi)^{n/2}(\det(C))^{1/2}|\det(A)|} \exp\Big(-[\mathbf{y}^t - b^t - \mathbf{m}A^t](ACA^t)^{-1} \\
&\qquad\qquad [\mathbf{y}^t - \mathbf{b}^t - \mathbf{m}A^t]^t\Big)
\end{aligned}$$

Thus $f_{\mathbf{Y}}(\mathbf{y})$ is a n-dimensional joint Gaussian distribution with mean and variance given by

$$\mathbf{m}_Y = \mathbf{b} + \mathbf{A}\mathbf{m}^t, \qquad C_Y = ACA^t$$

Problem 4.26

1) The joint distribution of X and Y is given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{1}{2}\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}\right\}$$

The linear transformations $Z = X + Y$ and $W = 2X - Y$ are written in matrix notation as

$$\begin{pmatrix} Z \\ W \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix}$$

Thus, (see Prob. 4.25)

$$f_{Z,W}(z,w) = \frac{1}{2\pi \det(M)^{1/2}}\exp\left\{-\frac{1}{2}\begin{pmatrix} Z & W \end{pmatrix} M^{-1}\begin{pmatrix} Z \\ W \end{pmatrix}\right\}$$

where

$$M = A\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}A^t = \begin{pmatrix} 2\sigma^2 & \sigma^2 \\ \sigma^2 & 5\sigma^2 \end{pmatrix} = \begin{pmatrix} \sigma_Z^2 & \rho_{Z,W}\sigma_Z\sigma_W \\ \rho_{Z,W}\sigma_Z\sigma_W & \sigma_W^2 \end{pmatrix}$$

From the last equality we identify $\sigma_Z^2 = 2\sigma^2$, $\sigma_W^2 = 5\sigma^2$ and $\rho_{Z,W} = 1/\sqrt{10}$

2)

$$\begin{aligned} F_R(r) &= p(R \le r) = p(\frac{X}{Y} \le r) \\ &= \int_0^\infty \int_{-\infty}^{yr} f_{X,Y}(x,y)dxdy + \int_{-\infty}^0 \int_{yr}^\infty f_{X,Y}(x,y)dxdy \end{aligned}$$

Differentiating $F_R(r)$ with respect to r we obtain the PDF $f_R(r)$. Note that

$$\begin{aligned} \frac{d}{da}\int_b^a f(x)dx &= f(a) \\ \frac{d}{db}\int_b^a f(x)dx &= -f(b) \end{aligned}$$

Thus,

$$\begin{aligned} F_R(r) &= \int_0^\infty \frac{d}{dr}\int_{-\infty}^{yr} f_{X,Y}(x,y)dxdy + \int_{-\infty}^0 \frac{d}{dr}\int_{yr}^\infty f_{X,Y}(x,y)dxdy \\ &= \int_0^\infty y f_{X,Y}(yr,y)dy - \int_{-\infty}^0 y f_{X,Y}(yr,y)dy \\ &= \int_{-\infty}^\infty |y| f_{X,Y}(yr,y)dy \end{aligned}$$

Hence,

$$\begin{aligned} f_R(r) &= \int_{-\infty}^\infty |y|\frac{1}{2\pi\sigma^2}e^{-\frac{y^2r^2+y^2}{2\sigma^2}}dy = 2\int_0^\infty y\frac{1}{2\pi\sigma^2}e^{-y^2(\frac{1+r^2}{2\sigma^2})}dy \\ &= 2\frac{1}{2\pi\sigma^2}\frac{2\sigma^2}{2(1+r^2)} = \frac{1}{\pi}\frac{1}{1+r^2} \end{aligned}$$

$f_R(r)$ is the Cauchy distribution; its mean is zero and the variance ∞.

Problem 4.27

The binormal joint density function is

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)} \times \right. \\ & \left. \left[\frac{(x-m_1)^2}{\sigma_1^2} + \frac{(y-m_2)^2}{\sigma_2^2} - \frac{2\rho(x-m_1)(y-m_2)}{\sigma_1\sigma_2}\right]\right\} \\ &= \frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp\left\{-(\mathbf{z}-\mathbf{m})C^{-1}(\mathbf{z}-\mathbf{m})^t\right\} \end{aligned}$$

where $\mathbf{z} = [x \;\; y]$, $\mathbf{m} = [m_1 \;\; m_2]$ and

$$C = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

1) With

$$C = \begin{pmatrix} 4 & -4 \\ -4 & 9 \end{pmatrix}$$

we obtain $\sigma_1^2 = 4$, $\sigma_2^2 = 9$ and $\rho\sigma_1\sigma_2 = -4$. Thus $\rho = -\frac{2}{3}$.

2) The transformation $Z = 2X + Y$, $W = X - 2Y$ is written in matrix notation as

$$\begin{pmatrix} Z \\ W \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$$

The ditribution $f_{Z,W}(z,w)$ is binormal with mean $\mathbf{m}' = \mathbf{m}A^t$, and covariance matrix $C' = ACA^t$. Hence

$$C' = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 4 & -4 \\ -4 & 9 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 9 & 2 \\ 2 & 56 \end{pmatrix}$$

The off-diagonal elements of C' are equal to $\rho\sigma_Z\sigma_W = COV(Z,W)$. Thus $COV(Z,W) = 2$.

3) Z will be Gaussian with variance $\sigma_Z^2 = 9$ and mean

$$m_Z = [\; m_1 \quad m_2 \;] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4$$

Problem 4.28

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\sqrt{2\pi}\sigma_Y}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}} \exp[-A]$$

where

$$\begin{aligned} A &= \frac{(x-m_X)^2}{2(1-\rho_{X,Y}^2)\sigma_X^2} + \frac{(y-m_Y)^2}{2(1-\rho_{X,Y}^2)\sigma_Y^2} - 2\rho\frac{(x-m_X)(y-m_Y)}{2(1-\rho_{X,Y}^2)\sigma_X\sigma_Y} - \frac{(y-m_Y)^2}{2\sigma_Y^2} \\ &= \frac{1}{2(1-\rho_{X,Y}^2)\sigma_X^2}\left((x-m_X)^2 + \frac{(y-m_Y)^2\sigma_X^2\rho_{X,Y}^2}{\sigma_Y^2} - 2\rho\frac{(x-m_X)(y-m_Y)\sigma_X}{\sigma_Y}\right) \\ &= \frac{1}{2(1-\rho_{X,Y}^2)\sigma_X^2}\left[x - \left(m_X + (y-m_Y)\frac{\rho\sigma_X}{\sigma_Y}\right)\right]^2 \end{aligned}$$

Thus

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho_{X,Y}^2}} \exp\left\{-\frac{1}{2(1-\rho_{X,Y}^2)\sigma_X^2}\left[x - \left(m_X + (y-m_Y)\frac{\rho\sigma_X}{\sigma_Y}\right)\right]^2\right\}$$

which is a Gaussian PDF with mean $m_X + (y-m_Y)\rho\sigma_X/\sigma_Y$ and variance $(1-\rho_{X,Y}^2)\sigma_X^2$. If $\rho = 0$ then $f_{X|Y}(x|y) = f_X(x)$ which implies that Y does not provide any information about X or X, Y are independent. If $\rho = \pm 1$ then the variance of $f_{X|Y}(x|y)$ is zero which means that $X|Y$ is deterministic. This is to be expected since $\rho = \pm 1$ implies a linear relation $X = AY + b$ so that knowledge of Y provides all the information about X.

Problem 4.29

1) The random variables Z, W are a linear combination of the jointly Gaussian random variables X, Y. Thus they are jointly Gaussian with mean $\mathbf{m}' = \mathbf{m}A^t$ and covariance matrix $C' = ACA^t$, where $\mathbf{m}$, C is the mean and covariance matrix of the random variables X and Y and A is the transformation matrix. The binormal joint density function is

$$f_{Z,W}(z,w) = \frac{1}{\sqrt{(2\pi)^n \det(C)}|\det(A)|} \exp\left\{-([z\ \ w] - \mathbf{m}')C'^{-1}([z\ \ w] - \mathbf{m}')^t\right\}$$

If $\mathbf{m} = \mathbf{0}$, then $\mathbf{m}' = \mathbf{m}A^t = \mathbf{0}$. With

$$C = \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix} \qquad A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

we obtain $\det(A) = \cos^2\theta + \sin^2\theta = 1$ and

$$\begin{aligned} C' &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \sigma^2(1+\rho\sin 2\theta) & \rho\sigma^2(\cos^2\theta - \sin^2\theta) \\ \rho\sigma^2(\cos^2\theta - \sin^2\theta) & \sigma^2(1-\rho\sin 2\theta) \end{pmatrix} \end{aligned}$$

2) Since Z and W are jointly Gaussian with zero-mean, they are independent if they are uncorrelated. This implies that

$$\cos^2\theta - \sin^2\theta = 0 \Longrightarrow \theta = \frac{\pi}{4} + k\frac{\pi}{2}, \quad k \in \mathcal{Z}$$

Note also that if X and Y are independent, then $\rho = 0$ and any rotation will produce independent random variables again.

Problem 4.30

1) $f_{X,Y}(x,y)$ is a PDF and its integral over the supporting region of x and y should be one.

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy \\ &= \int_{-\infty}^{0}\int_{-\infty}^{0} \frac{K}{\pi}e^{-\frac{x^2+y^2}{2}}dxdy + \int_{0}^{\infty}\int_{0}^{\infty} \frac{K}{\pi}e^{-\frac{x^2+y^2}{2}}dxdy \\ &= \frac{K}{\pi}\int_{-\infty}^{0} e^{-\frac{x^2}{2}}dx \int_{-\infty}^{0} e^{-\frac{y^2}{2}}dx + \frac{K}{\pi}\int_{0}^{\infty} e^{-\frac{x^2}{2}}dx\int_{0}^{\infty} e^{-\frac{y^2}{2}}dx \\ &= \frac{K}{\pi}\left[2(\frac{1}{2}\sqrt{2\pi})^2\right] = K \end{aligned}$$

Thus $K = 1$

2) If $x < 0$ then

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{0} \frac{1}{\pi} e^{-\frac{x^2+y^2}{2}} dy = \frac{1}{\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{0} e^{-\frac{y^2}{2}} dy \\
&= \frac{1}{\pi} e^{-\frac{x^2}{2}} \frac{1}{2}\sqrt{2\pi} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}
\end{aligned}$$

If $x > 0$ then

$$\begin{aligned}
f_X(x) &= \int_{0}^{\infty} \frac{1}{\pi} e^{-\frac{x^2+y^2}{2}} dy = \frac{1}{\pi} e^{-\frac{x^2}{2}} \int_{0}^{\infty} e^{-\frac{y^2}{2}} dy \\
&= \frac{1}{\pi} e^{-\frac{x^2}{2}} \frac{1}{2}\sqrt{2\pi} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}
\end{aligned}$$

Thus for every x, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ which implies that $f_X(x)$ is a zero-mean Gaussian random variable with variance 1. Since $f_{X,Y}(x,y)$ is symmetric to its arguments and the same is true for the region of integration we conclude that $f_Y(y)$ is a zero-mean Gaussian random variable of variance 1.

3) $f_{X,Y}(x,y)$ has not the same form as a binormal distribution. For $xy < 0$, $f_{X,Y}(x,y) = 0$ but a binormal distribution is strictly positive for every x, y.

4) The random variables X and Y are not independent for if $xy < 0$ then $f_X(x)f_Y(y) \neq 0$ whereas $f_{X,Y}(x,y) = 0$.

5)

$$\begin{aligned}
E[XY] &= \frac{1}{\pi} \int_{-\infty}^{0} \int_{-\infty}^{0} XY e^{-\frac{x^2+y^2}{2}} dx dy + \frac{1}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\frac{x^2+y^2}{2}} dx dy \\
&= \frac{1}{\pi} \int_{-\infty}^{0} X e^{-\frac{x^2}{2}} dx \int_{-\infty}^{0} Y e^{-\frac{y^2}{2}} dy + \frac{1}{\pi} \int_{0}^{\infty} X e^{-\frac{x^2}{2}} dx \int_{0}^{\infty} Y e^{-\frac{y^2}{2}} dy \\
&= \frac{1}{\pi}(-1)(-1) + \frac{1}{\pi} = \frac{2}{\pi}
\end{aligned}$$

Thus the random variables X and Y are correlated since $E[XY] \neq 0$ and $E[X] = E[Y] = 0$, so that $E[XY] - E[X]E[Y] \neq 0$.

6) In general $f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$. If $y > 0$, then

$$f_{X|Y}(x,y) = \begin{cases} 0 & x < 0 \\ \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} & x \geq 0 \end{cases}$$

If $y \leq 0$, then

$$f_{X|Y}(x,y) = \begin{cases} 0 & x > 0 \\ \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} & x < 0 \end{cases}$$

Thus

$$f_{X|Y}(x,y) = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} u(xy)$$

which is not a Gaussian distribution.

Problem 4.31

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{(x-m)^2 + y^2}{2\sigma^2}\right\}$$

With the transformation

$$V = \sqrt{X^2+Y^2}, \qquad \Theta = \arctan\frac{Y}{X}$$

we obtain

$$\begin{aligned}
f_{V,\Theta}(v,\theta) &= v f_{X,Y}(v\cos\theta, v\sin\theta) \\
&= \frac{v}{2\pi\sigma^2}\exp\left\{-\frac{(v\cos\theta - m)^2 + v^2\sin\theta}{2\sigma^2}\right\} \\
&= \frac{v}{2\pi\sigma^2}\exp\left\{-\frac{v^2+m^2-2mv\cos\theta}{2\sigma^2}\right\}
\end{aligned}$$

To obtain the marginal probability density function for the magnitude, we integrate over θ so that

$$\begin{aligned}
f_V(v) &= \int_0^{2\pi} \frac{v}{2\pi\sigma^2} e^{-\frac{v^2+m^2}{2\sigma^2}} e^{\frac{mv\cos\theta}{\sigma^2}} d\theta \\
&= \frac{v}{\sigma^2} e^{-\frac{v^2+m^2}{2\sigma^2}} \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{mv\cos\theta}{\sigma^2}} d\theta \\
&= \frac{v}{\sigma^2} e^{-\frac{v^2+m^2}{2\sigma^2}} I_0(\frac{mv}{\sigma^2})
\end{aligned}$$

where

$$I_0(\frac{mv}{\sigma^2}) = \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{mv\cos\theta}{\sigma^2}} d\theta$$

With $m=0$ we obtain

$$f_V(v) = \begin{cases} \frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}} & v > 0 \\ 0 & v \le 0 \end{cases}$$

which is the Rayleigh distribution.

Problem 4.32

1) Let X_i be a random variable taking the values 1, 0, with probability $\frac{1}{4}$ and $\frac{3}{4}$ respectively. Then, $m_{X_i} = \frac{1}{4}\cdot 1 + \frac{3}{4}\cdot 0 = \frac{1}{4}$. The weak law of large numbers states that the random variable $Y = \frac{1}{n}\sum_{i=1}^n X_i$ has mean which converges to m_{X_i} with probability one. Using Chebychev's inequality (see Problem 4.13) we have $p(|Y - m_{X_i}| \ge \epsilon) \le \frac{\sigma_Y^2}{\epsilon^2}$ for every $\epsilon > 0$. Hence, with $n = 2000$, $Z = \sum_{i=1}^{2000} X_i$, $m_{X_i} = \frac{1}{4}$ we obtain

$$p(|Z - 500| \ge 2000\epsilon) \le \frac{\sigma_Y^2}{\epsilon^2} \Rightarrow p(500 - 2000\epsilon \le Z \le 500 + 2000\epsilon) \ge 1 - \frac{\sigma_Y^2}{\epsilon^2}$$

The variance σ_Y^2 of $Y = \frac{1}{n}\sum_{i=1}^n X_i$ is $\frac{1}{n}\sigma_{X_i}^2$, where $\sigma_{X_i}^2 = p(1-p) = \frac{3}{16}$ (see Problem 4.13). Thus, with $\epsilon = 0.001$ we obtain

$$p(480 \le Z \le 520) \ge 1 - \frac{3/16}{2\times 10^{-1}} = .063$$

2) Using the C.L.T. the CDF of the random variable $Y = \frac{1}{n}\sum_{i=1}^n X_i$ converges to the CDF of the random variable $N(m_{X_i}, \frac{\sigma}{\sqrt{n}})$. Hence

$$P = p\left(\frac{480}{n} \le Y \le \frac{520}{n}\right) = Q\left(\frac{\frac{480}{n} - m_{X_i}}{\sigma}\right) - Q\left(\frac{\frac{520}{n} - m_{X_i}}{\sigma}\right)$$

With $n = 2000$, $m_{X_i} = \frac{1}{4}$, $\sigma^2 = \frac{p(1-p)}{n}$ we obtain

$$\begin{aligned}
P &= Q\left(\frac{480-500}{\sqrt{2000p(1-p)}}\right) - Q\left(\frac{520-500}{\sqrt{2000p(1-p)}}\right) \\
&= 1 - 2Q\left(\frac{20}{\sqrt{375}}\right) = .682
\end{aligned}$$

Problem 4.33
Consider the random variable vector

$$\mathbf{x} = [\ \omega_1 \quad \omega_1+\omega_2 \quad \ldots \quad \omega_1+\omega_2+\cdots+\omega_n\ ]^t$$

where each ω_i is the outcome of a Gaussian random variable distributed according to $N(0,1)$. Since

$$m_{\mathbf{x},i} = E[\omega_1+\omega_2+\cdots+\omega_i)] = E[\omega_1]+E[\omega_2]+\cdots+E[\omega_i] = 0$$

we obtain

$$m_{\mathbf{x}} = \mathbf{0}$$

The covariance matrix is

$$C = E[(\mathbf{x}-m_{\mathbf{x}})(\mathbf{x}-m_{\mathbf{x}})^t] = E[\mathbf{x}\mathbf{x}^t]$$

The i,j element $(C_{i,j})$ of this matrix is

$$\begin{aligned}
C_{i,j} &= E[(\omega_1+\omega_2+\cdots+\omega_i)(\omega_1+\omega_2+\cdots+\omega_j)] \\
&= E[(\omega_1+\omega_2+\cdots+\omega_{\min(i,j)})(\omega_1+\omega_2+\cdots+\omega_{\min(i,j)})] \\
&\quad +E[(\omega_1+\omega_2+\cdots+\omega_{\min(i,j)})(\omega_{\min(i,j)+1}+\cdots+\omega_{\max(i,j)})]
\end{aligned}$$

The expectation in the last line of the previous equation is zero. This is true since all the random variables inside the first parenthesis are different from the random variables in the second parenthesis, and for uncorrelated random variables of zero mean $E[\omega_k\omega_l]$ when $k \neq l$. Hence,

$$\begin{aligned}
C_{i,j} &= E[(\omega_1+\omega_2+\cdots+\omega_{\min(i,j)})(\omega_1+\omega_2+\cdots+\omega_{\min(i,j)})] \\
&= \sum_{k=1}^{\min(i,j)}\sum_{l=1}^{\min(i,j)} E[\omega_k\omega_l] = \sum_{k=1}^{\min(i,j)} E[\omega_k\omega_k] + \sum_{k,l=1}\sum_{k\neq l} E[\omega_k\omega_l] \\
&= \sum_{k=1}^{\min(i,j)} 1 = \min(i,j)
\end{aligned}$$

Thus

$$C = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & & 2 \\ \vdots & & \ddots & \vdots \\ 1 & 2 & \cdots & n \end{pmatrix}$$

Problem 4.34
The random variable $X(t_0)$ is uniformly distributed over $[-1\ \ 1]$. Hence,

$$m_X(t_0) = E[X(t_0)] = E[X] = 0$$

As it is observed the mean $m_X(t_0)$ is independent of the time instant t_0.

Problem 4.35

$$m_X(t) = E[A+Bt] = E[A]+E[B]t = 0$$

where the last equality follows from the fact that A, B are uniformly distributed over $[-1\ \ 1]$ so that $E[A]=E[B]=0$.

$$\begin{aligned}
R_X(t_1,t_2) &= E[X(t_1)X(t_2)] = E[(A+Bt_1)(A+Bt_2)] \\
&= E[A^2]+E[AB]t_2+E[BA]t_1+E[B^2]t_1t_2
\end{aligned}$$

The random variables A, B are independent so that $E[AB] = E[A]E[B] = 0$. Furthermore

$$E[A^2] = E[B^2] = \int_{-1}^{1} x^2 \frac{1}{2} dx = \frac{1}{6} x^3 |_{-1}^{1} = \frac{1}{3}$$

Thus

$$R_X(t_1, t_2) = \frac{1}{3} + \frac{1}{3} t_1 t_2$$

Problem 4.36

Since the joint density function of $\{X(t_i\}_{i=1}^n$ is a jointly Gaussian density of zero-mean the autocorrelation matrix of the random vector process is simply its covariance matrix. The i, j element of the matrix is

$$\begin{aligned} R_X(t_i, t_j) &= COV(X(t_i)X(t_j)) + m_X(t_i)m_X(t_j) = COV(X(t_i)X(t_j)) \\ &= \sigma^2 \min(t_i, t_j) \end{aligned}$$

Problem 4.37

Since $X(t) = X$ with the random variable uniformly distributed over $[-1 \ \ 1]$ we obtain

$$f_{X(t_1),X(t_2),\cdots,X(t_n)}(x_1, x_2, \ldots, x_n) = f_{X,X,\cdots,X}(x_1, x_2, \ldots, x_n)$$

for all $t_1, \ldots, t_n$ and n. Hence, the statistical properties of the process are time independent and by definition we have a stationary process.

Problem 4.38

The process is not wide sense stationary for the autocorrelation function depends on the values of t_1, t_2 and not on their difference. To see this suppose that $t_1 = t_2 = t$. If the process was wide sense stationary, then $R_X(t,t) = R_X(0)$. However, $R_X(t,t) = \sigma^2 t$ and it depends on t as it is opposed to $R_X(0)$ which is independent of t.

Problem 4.39

If a process $X(t)$ is M^{th} order stationary, then for all $n \leq M$, and Δ

$$f_{X(t_1)X(t_2)\cdots X(t_n)}(x_1, x_2, \cdots, x_n) = f_{X(t_1+\Delta)\cdots X(t_n+\Delta)}(x_1, \cdots x_n)$$

If we let $n = 1$, then

$$m_X(0) = E[X(0)] = \int_{-\infty}^{\infty} x f_{X(0)}(x) dx = \int_{-\infty}^{\infty} x f_{X(0+t)}(x) dx = m_X(t)$$

for all t. Hence, $m_x(t)$ is constant. With $n = 2$ we obtain

$$f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(t_1+\Delta)X(t_2+\Delta)}(x_1, x_2), \qquad \forall\Delta$$

If we let $\Delta = -t_1$, then

$$f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(0)X(t_2-t_1)}(x_1, x_2)$$

which means that

$$R_x(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X(0)X(t_2-t_1)}(x_1, x_2) dx_1 dx_2$$

depends only on the difference $\tau = t_1 - t_2$ and not on the individual values of t_1, t_2. Thus the M^{th} order stationary process, has a constant mean and an autocorrelation function dependent on $\tau = t_1 - t_2$ only. Hence, it is a wide sense stationary process.

Problem 4.40

1) $f(\tau)$ cannot be the autocorrelation function of a random process for $f(0) = 0 < f(1/4f_0) = 1$. Thus the maximum absolute value of $f(\tau)$ is not achieved at the origin $\tau = 0$.

2) $f(\tau)$ cannot be the autocorrelation function of a random process for $f(0) = 0$ whereas $f(\tau) \neq 0$ for $\tau \neq 0$. The maximum absolute value of $f(\tau)$ is not achieved at the origin.

3) $f(0) = 1$ whereas $f(\tau) > f(0)$ for $|\tau| > 1$. Thus $f(\tau)$ cannot be the autocorrelation function of a random process.

4) $f(\tau)$ is even and the maximum is achieved at the origin ($\tau = 0$). We can write $f(\tau)$ as

$$f(\tau) = 1.2\Lambda(\tau) - \Lambda(\tau - 1) - \Lambda(\tau + 1)$$

Taking the Fourier transform of both sides we obtain

$$\mathcal{S}(f) = 1.2\mathrm{sinc}^2(f) - \mathrm{sinc}^2(f)\left(e^{-j2\pi f} + e^{j2\pi f}\right) = \mathrm{sinc}^2(f)(1.2 - 2\cos(2\pi f))$$

As we observe the power spectrum $S(f)$ can take negative values, i.e. for $f = 0$. Thus $f(\tau)$ can not be the autocorrelation function of a random process.

Problem 4.41

As we have seen in Problem 4.38 the process is not stationary and thus it is not ergodic. This in accordance to our definition of ergodicity as a property of stationary and ergodic processes.

Problem 4.42

The random variable ω_i takes the values $\{1, 2, \ldots, 6\}$ with probability $\frac{1}{6}$. Thus

$$\begin{aligned}
E_X &= E\left[\int_{-\infty}^{\infty} X^2(t)dt\right] \\
&= E\left[\int_{-\infty}^{\infty} \omega_i^2 e^{-2t} u_{-1}^2(t)dt\right] = E\left[\int_0^{\infty} \omega_i^2 e^{-2t} dt\right] \\
&= \int_0^{\infty} E[\omega_i^2] e^{-2t} dt = \int_0^{\infty} \frac{1}{6}\sum_{i=1}^{6} i^2 e^{-2t} dt \\
&= \frac{91}{6}\int_0^{\infty} e^{-2t} dt = \frac{91}{6}(-\frac{1}{2}e^{-2t})\Big|_0^{\infty} \\
&= \frac{91}{12}
\end{aligned}$$

Thus the process is an energy-type process. However, this process is not stationary for

$$m_X(t) = E[X(t) = E[\omega_i]e^{-t}u_{-1}(t) = \frac{21}{6}e^{-t}u_{-1}(t)$$

is not constant.

Problem 4.43

1) We find first the probability of an even number of transitions in the interval $(0, \tau]$.

$$\begin{aligned}
p_N(n = even) &= p_N(0) + p_N(2) + p_N(4) + \cdots \\
&= \frac{1}{1 + \alpha\tau}\sum_{l=0}^{\infty}\left(\frac{\alpha\tau}{1 + \alpha\tau}\right)^2 \\
&= \frac{1}{1 + \alpha\tau}\frac{1}{1 - \frac{(\alpha\tau)^2}{(1+\alpha\tau)^2}} \\
&= \frac{1 + \alpha\tau}{1 + 2\alpha\tau}
\end{aligned}$$

The probability $p_N(n = odd)$ is simply $1 - p_N(n = even) = \frac{\alpha\tau}{1+2\alpha\tau}$. The random process $Z(t)$ takes the value of 1 (at time instant t) if an even number of transitions occurred given that $Z(0) = 1$, or if an odd number of transitions occurred given that $Z(0) = 0$. Thus,

$$\begin{aligned}
m_Z(t) &= E[Z(t)] = 1 \cdot p(Z(t) = 1) + 0 \cdot p(Z(t) = 0) \\
&= p(Z(t) = 1|Z(0) = 1)p(Z(0) = 1) + p(Z(t) = 1|Z(0) = 0)p(Z(0) = 0) \\
&= p_N(n = even)\frac{1}{2} + p_N(n = odd)\frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

2) To determine $R_Z(t_1, t_2)$ note that $Z(t + \tau) = 1$ if $Z(t) = 1$ and an even number of transitions occurred in the interval $(t, t + \tau]$, or if $Z(t) = 0$ and an odd number of transitions have taken place in $(t, t + \tau]$. Hence,

$$\begin{aligned}
R_Z(t + \tau, t) &= E[Z(t + \tau)Z(t)] \\
&= 1 \cdot p(Z(t + \tau) = 1, Z(t) = 1) + 0 \cdot p(Z(t + \tau) = 1, Z(t) = 0) \\
&\quad +0 \cdot p(Z(t + \tau) = 0, Z(t) = 1) + 0 \cdot p(Z(t + \tau) = 0, Z(t) = 0) \\
&= p(Z(t + \tau) = 1, Z(t) = 1) = p(Z(t + \tau) = 1|Z(t) = 1)p(Z(t) = 1) \\
&= \frac{1}{2}\frac{1 + \alpha\tau}{1 + 2\alpha\tau}
\end{aligned}$$

As it is observed $R_Z(t + \tau, t)$ depends only on τ and thus the process is stationary. The process is not cyclostationary.

3) Since the process is stationary

$$P_Z = R_Z(0) = \frac{1}{2}$$

Problem 4.44

1)

$$\begin{aligned}
m_X(t) &= E[X(t)] = E[X \cos(2\pi f_0 t)] + E[Y \sin(2\pi f_0 t)] \\
&= E[X] \cos(2\pi f_0 t) + E[Y] \sin(2\pi f_0 t) \\
&= 0
\end{aligned}$$

where the last equality follows from the fact that $E[X] = E[Y] = 0$.

2)

$$\begin{aligned}
R_X(t + \tau, t) &= E[(X \cos(2\pi f_0(t + \tau)) + Y \sin(2\pi f_0(t + \tau))) \\
&\quad (X \cos(2\pi f_0 t) + Y \sin(2\pi f_0 t))] \\
&= E[X^2 \cos(2\pi f_0(t + \tau)) \cos(2\pi f_0 t)] + \\
&\quad E[XY \cos(2\pi f_0(t + \tau)) \sin(2\pi f_0 t)] + \\
&\quad E[YX \sin(2\pi f_0(t + \tau)) \cos(2\pi f_0 t)] + \\
&\quad E[Y^2 \sin(2\pi f_0(t + \tau)) \sin(2\pi f_0 t)] \\
&= \frac{\sigma^2}{2}[\cos(2\pi f_0(2t + \tau)) + \cos(2\pi f_0 \tau)] + \\
&\quad \frac{\sigma^2}{2}[\cos(2\pi f_0 \tau) - \cos(2\pi f_0(2t + \tau))] \\
&= \sigma^2 \cos(2\pi f_0 \tau)
\end{aligned}$$

where we have used the fact that $E[XY] = 0$. Thus the process is stationary for $R_X(t+\tau,t)$ depends only on τ.

3) Since the process is stationary $P_X = R_X(0) = \sigma^2$.

4) If $\sigma_X^2 \neq \sigma_Y^2$, then

$$m_X(t) = E[X]\cos(2\pi f_0 t) + E[Y]\sin(2\pi f_0 t) = 0$$

and

$$\begin{aligned}
R_X(t+\tau,t) &= E[X^2]\cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t) + \\
&\quad E[Y^2]\sin(2\pi f_0(t+\tau))\sin(2\pi f_0 t) \\
&= \frac{\sigma_X^2}{2}\left[\cos(2\pi f_0(2t+\tau)) - \cos(2\pi f_0\tau)\right] + \\
&\quad \frac{\sigma_Y^2}{2}\left[\cos(2\pi f_0\tau) - \cos(2\pi f_0(2t+\tau))\right] \\
&= \frac{\sigma_X^2 - \sigma_Y^2}{2}\cos(2\pi f_0(2t+\tau) + \\
&\quad \frac{\sigma_X^2 + \sigma_Y^2}{2}\cos(2\pi f_0\tau)
\end{aligned}$$

The process is not stationary for $R_X(t+\tau,t)$ does not depend only on τ but on t as well. However the process is cyclostationary with period $T_0 = \frac{1}{2f_0}$. Note that if X or Y is not of zero mean then the period of the cyclostationary process is $T_0 = \frac{1}{f_0}$. The power spectral density of $X(t)$ is

$$P_X = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(\frac{\sigma_X^2 - \sigma_Y^2}{2}\cos(2\pi f_0 2t) + \frac{\sigma_X^2 + \sigma_Y^2}{2}\right)dt = \infty$$

Problem 4.45
1)

$$\begin{aligned}
m_X(t) &= E\left[X(t)\right] = E\left[\sum_{k=-\infty}^{\infty} A_k p(t-kT)\right] \\
&= \sum_{k=-\infty}^{\infty} E[A_k]p(t-kT) \\
&= m\sum_{k=-\infty}^{\infty} p(t-kT)
\end{aligned}$$

2)

$$\begin{aligned}
R_X(t+\tau,t) &= E\left[X(t+\tau)X(t)\right] \\
&= E\left[\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} A_k A_l p(t+\tau-kT)p(t-lT)\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} E[A_k A_l]p(t+\tau-kT)p(t-lT) \\
&= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} R_A(k-l)p(t+\tau-kT)p(t-lT)
\end{aligned}$$

3)

$$\begin{aligned}
R_X(t+T+\tau,t+T) &= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} R_A(k-l)p(t+T+\tau-kT)p(t+T-lT) \\
&= \sum_{k'=-\infty}^{\infty}\sum_{l'=-\infty}^{\infty} R_A(k'+1-(l'+1))p(t+\tau-k'T)p(t-l'T) \\
&= \sum_{k'=-\infty}^{\infty}\sum_{l'=-\infty}^{\infty} R_A(k'-l')p(t+\tau-k'T)p(t-l'T) \\
&= R_X(t+\tau,t)
\end{aligned}$$

where we have used the change of variables $k' = k-1$, $l' = l-1$. Since $m_X(t)$ and $R_X(t+\tau,t)$ are periodic, the process is cyclostationary.

4)

$$\begin{aligned}
\bar{R}_X(\tau) &= \frac{1}{T}\int_0^T R_X(t+\tau,t)dt \\
&= \frac{1}{T}\int_0^T \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} R_A(k-l)p(t+\tau-kT)p(t-lT)dt \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} R_A(n)\sum_{l=-\infty}^{\infty}\int_0^T p(t+\tau-lT-nT)p(t-lT)dt \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} R_A(n)\sum_{l=-\infty}^{\infty}\int_{-lT}^{T-lT} p(t'+\tau-nT)p(t')dt' \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} R_A(n)\int_{-\infty}^{\infty} p(t'+\tau-nT)p(t')dt' \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} R_A(n)R_p(\tau-nT)
\end{aligned}$$

where $R_p(\tau-nT) = \int_{-\infty}^{\infty} p(t'+\tau-nT)p(t')dt' = p(t)\star p(-t)|_{t=\tau-nT}$

5)

$$\begin{aligned}
S_X(f) &= \mathcal{F}[\bar{R}_X(\tau)] = \mathcal{F}\left[\frac{1}{T}\sum_{n=-\infty}^{\infty} R_A(n)R_p(\tau-nT)\right] \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} R_A(n)\int_{-\infty}^{\infty} R_p(\tau-nT)e^{-j2\pi f\tau}d\tau \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} R_A(n)\int_{-\infty}^{\infty} R_p(\tau')e^{-j2\pi f(\tau'+nT)}d\tau' \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} R_A(n)e^{-j2\pi fnT}\int_{-\infty}^{\infty} R_p(\tau')e^{-j2\pi f\tau'}d\tau'
\end{aligned}$$

But, $R_p(\tau') = p(\tau')\star p(-\tau')$ so that

$$\begin{aligned}
\int_{-\infty}^{\infty} R_p(\tau')e^{-j2\pi f\tau'}d\tau' &= \int_{-\infty}^{\infty} p(\tau')e^{-j2\pi f\tau'}d\tau' \int_{-\infty}^{\infty} p(-\tau')e^{-j2\pi f\tau'}d\tau' \\
&= P(f)P^*(f) = |P(f)|^2
\end{aligned}$$

where we have used the fact that for real signals $P(-f) = P^*(f)$. Substituting the relation above to the expression for $\mathcal{S}_X(f)$ we obtain

$$\mathcal{S}_X(f) = \frac{|P(f)|^2}{T}\sum_{n=-\infty}^{\infty} R_A(n)e^{-j2\pi fnT}$$

$$= \frac{|P(f)|^2}{T}\left[R_A(0) + 2\sum_{n=1}^{\infty} R_A(n)\cos(2\pi f n T)\right]$$

where we have used the assumption $R_A(n) = R_A(-n)$ and the fact $e^{j2\pi f nT} + e^{-j2\pi f nT} = 2\cos(2\pi f nT)$

Problem 4.46

1) The autocorrelation function of A_n's is $R_A(k-l) = E[A_k A_l] = \delta_{kl}$ where δ_{kl} is the Kronecker's delta. Furthermore

$$P(f) = \mathcal{F}\left[\Pi(\frac{t-\frac{T}{2}}{T})\right] = T\text{sinc}(Tf)e^{-j2\pi f\frac{T}{2}}$$

Hence, using the results of Problem 4.45 we obtain

$$\mathcal{S}_X(f) = T\text{sinc}^2(Tf)$$

2) In this case $E[A_n] = \frac{1}{2}$ and $R_A(k-l) = E[A_k A_l]$. If $k = l$, then $R_A(0) = E[A_k^2] = \frac{1}{2}$. If $k \neq l$, then $R_A(k-l) = E[A_k A_l] = E[A_k]E[A_l] = \frac{1}{4}$. The power spectral density of the process is

$$\mathcal{S}_X(f) = T\text{sinc}^2(Tf)\left[\frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty}\cos(2\pi k f T)\right]$$

3) If $p(t) = \Pi(\frac{t-3T/2}{3T})$ and $A_n = \pm 1$ with equal probability, then

$$\begin{aligned}
\mathcal{S}_X(f) &= \frac{|P(f)|^2}{T}R_A(0) = \frac{1}{T}\left|3T\text{sinc}(3Tf)e^{-j2\pi f\frac{3T}{2}}\right|^2 \\
&= 9T\text{sinc}^2(3Tf)
\end{aligned}$$

For the second part the power spectral density is

$$\mathcal{S}_X(f) = 9T\text{sinc}^2(3Tf)\left[\frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty}\cos(2\pi k f T)\right]$$

Problem 4.47

1) $E[B_n] = E[A_n] + E[A_{n-1}] = 0$. To find the autocorrelation sequence of B_n's we write

$$\begin{aligned}
R_B(k-l) &= E[B_k B_l] = E[(A_k + A_{k-1})(A_l + A_{l-1})] \\
&= E[A_k A_l] + E[A_k A_{l-1}] + E[A_{k-1}A_l] + E[A_{k-1}A_{l-1}]
\end{aligned}$$

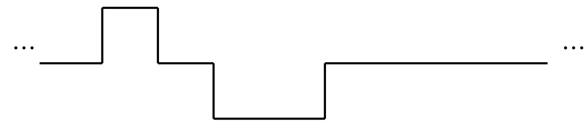
If $k = l$, then $R_B(0) = E[A_k^2] + E[A_{k-1}^2] = 2$. If $k = l-1$, then $R_B(1) = E[A_k A_{l-1}]] = 1$. Similarly, if $k = l+1$, $R_B(-1) = E[A_{k-1}A_l]] = 1$. Thus,

$$R_B(k-l) = \begin{cases} 2 & k-l=0 \\ 1 & k-l=\pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Using the results of Problem 4.45 we obtain

$$\begin{aligned}
\mathcal{S}_X(f) &= \frac{|P(f)|^2}{T}\left(R_B(0) + 2\sum_{k=1}^{\infty} R_B(k)\cos(2\pi k f T)\right) \\
&= \frac{|P(f)|^2}{T}(2 + 2\cos(2\pi f T))
\end{aligned}$$

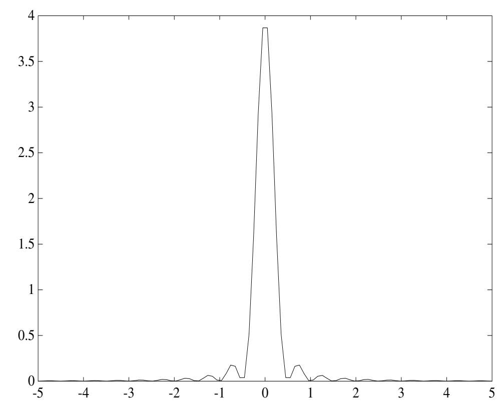
2) Consider the sample sequence of A_n's $\{\cdots, -1, 1, 1, -1, -1, -1, 1, -1, 1, -1, \cdots\}$. Then the corresponding sequence of B_n's is $\{\cdots, 0, 2, 0, -2, -2, 0, 0, 0, 0, \cdots\}$. The following figure depicts the corresponding sample function $X(t)$.



If $p(t) = \Pi(\frac{t-T/2}{T})$, then $|P(f)|^2 = T^2 \text{sinc}^2(Tf)$ and the power spectral density is

$$\mathcal{S}_X(f) = T\text{sinc}^2(Tf)(2 + 2\cos(2\pi fT))$$

In the next figure we plot the power spectral density for $T = 1$.



3) If $B_n = A_n + \alpha A_{n-1}$, then

$$R_B(k-l) = \begin{cases} 1+\alpha^2 & k-l=0 \\ \alpha & k-l=\pm 1 \\ 0 & \text{otherwise} \end{cases}$$

The power spectral density in this case is given by

$$\mathcal{S}_X(f) = \frac{|P(f)|^2}{T}(1 + \alpha^2 + 2\alpha\cos(2\pi fT))$$

Problem 4.48
In general the mean of a function of two random variables, $g(X, Y)$, can be found as

$$E[g(X,Y)] = E[E[g(X,Y)|X]]$$

where the outer expectation is with respect to the random variable X.
1)

$$m_Y(t) = E[X(t+\Theta)] = E[E[X(t+\Theta)|\Theta]]$$

where

$$\begin{aligned} E[X(t+\Theta)|\Theta] &= \int X(t+\theta) f_{X(t)|\Theta}(x|\theta)dx \\ &= \int X(t+\theta) f_{X(t)}(x)dx = m_X(t+\theta) \end{aligned}$$

where we have used the independence of $X(t)$ and Θ. Thus

$$m_Y(t) = E[m_X(t+\theta)] = \frac{1}{T}\int_0^T m_X(t+\theta)d\theta = m_Y$$

where the last equality follows from the periodicity of $m_X(t+\theta)$. Similarly for the autocorrelation function

$$\begin{aligned}
R_Y(t+\tau,t) &= E\left[E[X(t+\tau+\Theta)X(t+\Theta)|\Theta]\right] \\
&= E\left[R_X(t+\tau+\theta,t+\theta)\right] \\
&= \frac{1}{T}\int_0^T R_X(t+\tau+\theta,t+\theta)d\theta \\
&= \frac{1}{T}\int_0^T R_X(t'+\tau,t')dt'
\end{aligned}$$

where we have used the change of variables $t'=t+\theta$ and the periodicity of $R_X(t+\tau,t)$

2)

$$\begin{aligned}
\mathcal{S}_Y(f) &= E\left[\lim_{T\to\infty}\frac{|Y_T(f)|^2}{T}\right] = E\left[E\left[\lim_{T\to\infty}\frac{|Y_T(f)|^2}{T}\Big|\Theta\right]\right] \\
&= E\left[E\left[\lim_{T\to\infty}\frac{|X_T(f)e^{j2\pi f\theta}|^2}{T}\Big|\Theta\right]\right] = E\left[E\left[\lim_{T\to\infty}\frac{|X_T(f)|^2}{T}\right]\right] \\
&= E\left[\mathcal{S}_X(f)\right] = \mathcal{S}_X(f)
\end{aligned}$$

3) Since $\mathcal{S}_Y(f)=\mathcal{F}[\frac{1}{T}\int_0^T R_X(t+\tau,t)dt]$ and $\mathcal{S}_Y(f)=\mathcal{S}_X(f)$ we conclude that

$$\mathcal{S}_X(f)=\mathcal{F}\left[\frac{1}{T}\int_0^T R_X(t+\tau,t)dt\right]$$

Problem 4.49

Using Parseval's relation we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} f^2\mathcal{S}_X(f)df &= \int_{-\infty}^{\infty}\mathcal{F}^{-1}[f^2]\mathcal{F}^{-1}[\mathcal{S}_X(f)]d\tau \\
&= \int_{-\infty}^{\infty} -\frac{1}{4\pi^2}\delta^{(2)}(\tau)R_X(\tau)d\tau \\
&= -\frac{1}{4\pi^2}(-1)^2\frac{d^2}{d\tau^2}R_X(\tau)|_{\tau=0} \\
&= -\frac{1}{4\pi^2}\frac{d^2}{d\tau^2}R_X(\tau)|_{\tau=0}
\end{aligned}$$

Also,

$$\int_{-\infty}^{\infty}\mathcal{S}_X(f)df = R_X(0)$$

Combining the two relations we obtain

$$W_{RMS}=\frac{\int_{-\infty}^{\infty} f^2\mathcal{S}_X(f)df}{\int_{-\infty}^{\infty}\mathcal{S}_X(f)df} = -\frac{1}{4\pi^2 R_X(0)}\frac{d^2}{d\tau^2}R_X(\tau)|_{\tau=0}$$

Problem 4.50

$$R_{XY}(t_1,t_2)=E[X(t_1)Y(t_2)]=E[Y(t_2)X(t_1)]=R_{YX}(t_2,t_1)$$

If we let $\tau=t_1-t_2$, then using the previous result and the fact that $X(t)$, $Y(t)$ are jointly stationary, so that $R_{XY}(t_1,t_2)$ depends only on τ, we obtain

$$R_{XY}(t_1,t_2)=R_{XY}(t_1-t_2)=R_{YX}(t_2-t_1)=R_{YX}(-\tau)$$

Taking the Fourier transform of both sides of the previous relation we obtain

$$\begin{aligned}
\mathcal{S}_{XY}(f) &= \mathcal{F}[R_{XY}(\tau)] = \mathcal{F}[R_{YX}(-\tau)] \\
&= \int_{-\infty}^{\infty} R_{YX}(-\tau)e^{-j2\pi f\tau}d\tau \\
&= \left[\int_{-\infty}^{\infty} R_{YX}(\tau')e^{-j2\pi f\tau'}d\tau'\right]^* = \mathcal{S}^*_{YX}(f)
\end{aligned}$$

Problem 4.51

1) $\mathcal{S}_X(f) = \frac{N_0}{2}$, $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$. The autocorrelation function and the power spectral density of the output are given by

$$R_Y(t) = R_X(\tau) \star h(\tau) \star h(-\tau), \qquad \mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2$$

With $H(f) = \Pi(\frac{f}{2B})$ we have $|H(f)|^2 = \Pi^2(\frac{f}{2B}) = \Pi(\frac{f}{2B})$ so that

$$\mathcal{S}_Y(f) = \frac{N_0}{2}\Pi(\frac{f}{2B})$$

Taking the inverse Fourier transform of the previous we obtain the autocorrelation function of the output

$$R_Y(\tau) = 2B\frac{N_0}{2}\text{sinc}(2B\tau) = BN_0\text{sinc}(2B\tau)$$

2) The output random process $Y(t)$ is a zero mean Gaussian process with variance

$$\sigma^2_{Y(t)} = E[Y^2(t)] = E[Y^2(t+\tau)] = R_Y(0) = BN_0$$

The correlation coefficient of the jointly Gaussian processes $Y(t+\tau)$, $Y(t)$ is

$$\rho_{Y(t+\tau)Y(t)} = \frac{COV(Y(t+\tau)Y(t))}{\sigma_{Y(t+\tau)}\sigma_{Y(t)}} = \frac{E[Y(t+\tau)Y(t)]}{BN_0} = \frac{R_Y(\tau)}{BN_0}$$

With $\tau = \frac{1}{2B}$, we have $R_Y(\frac{1}{2B}) = \text{sinc}(1) = 0$ so that $\rho_{Y(t+\tau)Y(t)} = 0$. Hence the joint probability density function of $Y(t)$ and $Y(t+\tau)$ is

$$f_{Y(t+\tau)Y(t)} = \frac{1}{2\pi BN_0}e^{-\frac{Y^2(t+\tau)+Y^2(t)}{2BN_0}}$$

Since the processes are Gaussian and uncorrelated they are also independent.

Problem 4.52

The impulse response of a delay line that introduces a delay equal to Δ is $h(t) = \delta(t-\Delta)$. The output autocorrelation function is

$$R_Y(\tau) = R_X(\tau) \star h(\tau) \star h(-\tau)$$

But,

$$\begin{aligned}
h(\tau) \star h(-\tau) &= \int_{-\infty}^{\infty} \delta(-(t-\Delta))\delta(\tau-(t-\Delta))dt \\
&= \int_{-\infty}^{\infty} \delta(t-\Delta)\delta(\tau-(t-\Delta))dt \\
&= \int_{-\infty}^{\infty} \delta(t')\delta(\tau-t')dt' = \delta(\tau)
\end{aligned}$$

Hence,

$$R_Y(\tau) = R_X(\tau) \star \delta(\tau) = R_X(\tau)$$

This is to be expected since a delay line does not alter the spectral characteristics of the input process.

Problem 4.53

The converse of the theorem is not true. Consider for example the random process $X(t) = \cos(2\pi f_0 t) + X$ where X is a random variable. Clearly

$$m_X(t) = \cos(2\pi f_0 t) + m_X$$

is a function of time. However, passing this process through the LTI system with transfer function $\Pi(\frac{f}{2W})$ with $W < f_0$ produces the stationary random process $Y(t) = X$.

Problem 4.54

1) Let $Y(t) = \int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$. Then the mean $m_Y(t)$ is

$$\begin{aligned}
m_Y(t) &= E[\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau] = \int_{-\infty}^{\infty} h(\tau)E[X(t-\tau)]d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)m_X(t-\tau)d\tau
\end{aligned}$$

If $X(t)$ is cyclostationary with period T then

$$m_Y(t+T) = \int_{-\infty}^{\infty} h(\tau)m_X(t+T-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)m_X(t-\tau)d\tau = m_Y(t)$$

Thus the mean of the output process is periodic with the same period of the cyclostationary process $X(t)$. The output autocorrelation function is

$$\begin{aligned}
R_Y(t+\tau, t) &= E[Y(t+\tau)Y(t)] \\
&= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)X(t+\tau-s)h(v)X(t-v)dsdv\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(v)R_X(t+\tau-s, t-v)dsdv
\end{aligned}$$

Hence,

$$\begin{aligned}
R_Y(t+T+\tau, t+T) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(v)R_X(t+T+\tau-s, t+T-v)dsdv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(v)R_X(t+T+\tau-s, t+T-v)dsdv \\
&= R_Y(t+\tau, t)
\end{aligned}$$

where we have used the periodicity of $R_X(t+\tau, t)$ for the last equality. Since both $m_Y(t)$, $R_Y(t+\tau, t)$ are periodic with period T, the output process $Y(t)$ is cyclostationary.

2) The crosscorrelation function is

$$\begin{aligned}
R_{XY}(t+\tau, t) &= E[X(t+\tau)Y(t)] \\
&= E\left[X(t+\tau)\int_{-\infty}^{\infty} X(t-s)h(s)ds\right] \\
&= \int_{-\infty}^{\infty} E[X(t+\tau)X(t-s)]h(s)ds = \int_{-\infty}^{\infty} R_X(t+\tau, t-s)h(s)ds
\end{aligned}$$

which is periodic with period T. Integrating the previous over one period, i.e. from $-\frac{T}{2}$ to $\frac{T}{2}$ we obtain

$$\begin{aligned}
\bar{R}_{XY}(\tau) &= \int_{-\infty}^{\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} R_X(t+\tau, t-s)dth(s)ds \\
&= \int_{-\infty}^{\infty} \bar{R}_X(\tau+s)h(s)ds \\
&= \bar{R}_X(\tau) \star h(-\tau)
\end{aligned}$$

Similarly we can show that

$$\bar{R}_Y(\tau) = \bar{R}_X Y(\tau) \star h(\tau)$$

so that by combining the two we obtain

$$\bar{R}_Y(\tau) = \bar{R}_X(\tau) \star h(\tau) \star h(-\tau)$$

3) Taking the Fourier transform of the previous equation we obtain the desired relation among the spectral densities of the input and output.

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2$$

Problem 4.55

1) $Y(t) = \frac{d}{dt}X(t)$ can be considered as the output process of a differentiator which is known to be a LTI system with impulse response $h(t) = \delta'(t)$. Since $X(t)$ is stationary, its mean is constant so that

$$m_Y(t) = m_{X'}(t) = [m_X(t)]' = 0$$

To prove that $X(t)$ and $\frac{d}{dt}X(t)$ are uncorrelated we have to prove that $R_{XX'}(0) - m_X(t)m_{X'}(t) = 0$ or since $m_{X'}(t) = 0$ it suffices to prove that $R_{XX'}(0) = 0$. But,

$$R_{XX'}(\tau) = R_X(\tau) \star \delta'(-\tau) = -R_X(\tau) \star \delta'(\tau) = -R'_X(\tau)$$

and since $R_X(\tau) = R_X(-\tau)$ we obtain

$$R_{XX'}(\tau) = -R'_X(\tau) = R'_X(-\tau) = -R_{XX'}(-\tau)$$

Thus $R_{XX'}(\tau)$ is an odd function and its value at the origin should be equal to zero

$$R_{XX'}(0) = 0$$

The last proves that $X(t)$ and $\frac{d}{dt}X(t)$ are uncorrelated.

2) The autocorrelation function of the sum $Z(t) = X(t) + \frac{d}{dt}X(t)$ is

$$R_Z(\tau) = R_X(\tau) + R_{X'}(\tau) + R_{XX'}(\tau) + R_{X'X}(\tau)$$

If we take the Fourier transform of both sides we obtain

$$\mathcal{S}_Z(f) = \mathcal{S}_X(f) + \mathcal{S}_{X'}(f) + 2\text{Re}[\mathcal{S}_{XX'}(f)]$$

But, $\mathcal{S}_{XX'}(f) = \mathcal{F}[-R_X(\tau) \star \delta'(\tau)] = \mathcal{S}_X(f)(-j2\pi f)$ so that $\text{Re}[\mathcal{S}_{XX'}(f)] = 0$. Thus,

$$\mathcal{S}_Z(f) = \mathcal{S}_X(f) + \mathcal{S}_{X'}(f)$$

Problem 4.56

1) The impulse response of the system is $h(t) = \mathcal{L}[\delta(t)] = \delta'(t) + \delta'(t-T)$. It is a LTI system so that the output process is a stationary. This is true since $Y(t+c) = \mathcal{L}[X(t+c)]$ for all c, so if $X(t)$ and $X(t+c)$ have the same statistical properties, so do the processes $Y(t)$ and $Y(t+c)$.

2) $\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2$. But, $H(f) = j2\pi f + j2\pi f e^{-j2\pi fT}$ so that

$$\begin{aligned}
\mathcal{S}_Y(f) &= \mathcal{S}_X(f)4\pi^2 f^2 \left|1 + e^{-j2\pi fT}\right|^2 \\
&= \mathcal{S}_X(f)4\pi^2 f^2[(1+\cos(2\pi fT))^2 + \sin^2(2\pi fT)] \\
&= \mathcal{S}_X(f)8\pi^2 f^2(1+\cos(2\pi fT))
\end{aligned}$$

3) The frequencies for which $|H(f)|^2 = 0$ will not be present at the output. These frequencies are $f = 0$, for which $f^2 = 0$ and $f = \frac{1}{2T} + \frac{k}{T}$, $k \in \mathcal{Z}$, for which $\cos(2\pi f T) = -1$.

Problem 4.57

1) $Y(t) = X(t) \star (\delta(t) - \delta(t-T))$. Hence,

$$\begin{aligned} \mathcal{S}_Y(f) &= \mathcal{S}_X(f)|H(f)|^2 = \mathcal{S}_X(f)|1 - e^{-j2\pi f T}|^2 \\ &= \mathcal{S}_X(f)2(1 - \cos(2\pi f T)) \end{aligned}$$

2) $Y(t) = X(t) \star (\delta'(t) - \delta(t))$. Hence,

$$\begin{aligned} \mathcal{S}_Y(f) &= \mathcal{S}_X(f)|H(f)|^2 = \mathcal{S}_X(f)|j2\pi f - 1|^2 \\ &= \mathcal{S}_X(f)(1 + 4\pi^2 f^2) \end{aligned}$$

3) $Y(t) = X(t) \star (\delta'(t) - \delta(t-T))$. Hence,

$$\begin{aligned} \mathcal{S}_Y(f) &= \mathcal{S}_X(f)|H(f)|^2 = \mathcal{S}_X(f)|j2\pi f - e^{-j2\pi f T}|^2 \\ &= \mathcal{S}_X(f)(1 + 4\pi^2 f^2 + 4\pi f \sin(2\pi f T)) \end{aligned}$$

Problem 4.58

Using Schwartz's inequality

$$E^2[X(t+\tau)Y(t)] \leq E[X^2(t+\tau)]E[Y^2(t)] = R_X(0)R_Y(0)$$

where equality holds for independent $X(t)$ and $Y(t)$. Thus

$$|R_{XY}(\tau)| = \left(E^2[X(t+\tau)Y(t)]\right)^{\frac{1}{2}} \leq R_X^{1/2}(0)R_Y^{1/2}(0)$$

The second part of the inequality follows from the fact $2ab \leq a^2 + b^2$. Thus, with $a = R_X^{1/2}(0)$ and $b = R_Y^{1/2}(0)$ we obtain

$$R_X^{1/2}(0)R_Y^{1/2}(0) \leq \frac{1}{2}\left[R_X(0) + R_Y(0)\right]$$

Problem 4.59

1)

$$\begin{aligned} R_{XY}(\tau) &= R_X(\tau) \star \delta(-\tau - \Delta) = R_X(\tau) \star \delta(\tau + \Delta) \\ &= e^{-\alpha|\tau|} \star \delta(\tau + \Delta) = e^{-\alpha|\tau + \Delta|} \\ R_Y(\tau) &= R_{XY}(\tau) \star \delta(\tau - \Delta) = e^{-\alpha|\tau+\Delta|} \star \delta(\tau - \Delta) \\ &= e^{-\alpha|\tau|} \end{aligned}$$

2)

$$\begin{aligned} R_{XY}(\tau) &= e^{-\alpha|\tau|} \star (-\frac{1}{\tau}) = -\int_{-\infty}^{\infty} \frac{e^{-\alpha|v|}}{t-v} dv \\ R_Y(\tau) &= R_{XY}(\tau) \star \frac{1}{\tau} = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-\alpha|v|}}{s-v}\frac{1}{\tau - s} ds dv \end{aligned} \tag{3}$$

The case of $R_Y(\tau)$ can be simplified as follows. Note that $R_Y(\tau) = \mathcal{F}^{-1}[\mathcal{S}_Y(f)]$ where $\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2$. In our case, $\mathcal{S}_X(f) = \frac{2\alpha}{\alpha^2+4\pi^2 f^2}$ and $|H(f)|^2 = \pi^2 \mathrm{sgn}^2(f)$. Since $\mathcal{S}_X(f)$ does not contain any impulses at the origin ($f = 0$) for which $|H(f)|^2 = 0$, we obtain

$$R_Y(\tau) = \mathcal{F}^{-1}[\mathcal{S}_Y(f)] = \pi^2 e^{-\alpha|\tau|}$$

3) The system's transfer function is $H(f) = \frac{-1+j2\pi f}{1+j2\pi f}$. Hence,

$$\begin{aligned}
\mathcal{S}_{XY}(f) &= \mathcal{S}_X(f)H^*(f) = \frac{2\alpha}{\alpha^2+4\pi^2 f^2}\frac{-1-j2\pi f}{1-j2\pi f} \\
&= \frac{4\alpha}{1-\alpha^2}\frac{1}{1-j2\pi f} + \frac{\alpha-1}{1+\alpha}\frac{1}{\alpha+j2\pi f} + \frac{1+\alpha}{\alpha-1}\frac{1}{\alpha-j2\pi f}
\end{aligned}$$

Thus,

$$\begin{aligned}
R_{XY}(\tau) &= \mathcal{F}^{-1}[\mathcal{S}_{XY}(f)] \\
&= \frac{4\alpha}{1-\alpha^2}e^{\tau}u_{-1}(-\tau) + \frac{\alpha-1}{1+\alpha}e^{-\alpha\tau}u_{-1}(\tau) + \frac{1+\alpha}{\alpha-1}e^{\alpha\tau}u_{-1}(-\tau)
\end{aligned}$$

For the output power spectral density we have $\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \mathcal{S}_X(f)\frac{1+4\pi^2 f^2}{1+4\pi^2 f^2} = \mathcal{S}_X(f)$. Hence,

$$R_Y(\tau) = \mathcal{F}^{-1}[\mathcal{S}_X(f)] = e^{-\alpha|\tau|}$$

4) The impulse response of the system is $h(t) = \frac{1}{2T}\Pi(\frac{t}{2T})$. Hence,

$$\begin{aligned}
R_{XY}(\tau) &= e^{-\alpha|\tau|} \star \frac{1}{2T}\Pi(\frac{-\tau}{2T}) = e^{-\alpha|\tau|} \star \frac{1}{2T}\Pi(\frac{\tau}{2T}) \\
&= \frac{1}{2T}\int_{\tau-T}^{\tau+T} e^{-\alpha|v|}dv
\end{aligned}$$

If $\tau \geq T$, then

$$R_{XY}(\tau) = -\frac{1}{2T\alpha}e^{-\alpha v}\Big|_{\tau-T}^{\tau+T} = \frac{1}{2T\alpha}\left(e^{-\alpha(\tau-T)} - e^{-\alpha(\tau+T)}\right)$$

If $0 \leq \tau < T$, then

$$\begin{aligned}
R_{XY}(\tau) &= \frac{1}{2T}\int_{\tau-T}^{0} e^{\alpha v}dv + \frac{1}{2T}\int_{0}^{\tau+T} e^{-\alpha v}dv \\
&= \frac{1}{2T\alpha}\left(2 - e^{\alpha(\tau-T)} - e^{-\alpha(\tau+T)}\right)
\end{aligned}$$

The autocorrelation of the output is given by

$$\begin{aligned}
R_Y(\tau) &= e^{-\alpha|\tau|} \star \frac{1}{2T}\Pi(\frac{\tau}{2T}) \star \frac{1}{2T}\Pi(\frac{\tau}{2T}) \\
&= e^{-\alpha|\tau|} \star \frac{1}{2T}\Lambda(\frac{\tau}{2T}) \\
&= \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|x|}{2T}\right)e^{-\alpha|\tau-x|}dx
\end{aligned}$$

If $\tau \geq 2T$, then

$$R_Y(\tau) = \frac{e^{-\alpha\tau}}{2T\alpha^2}\left[e^{2\alpha T} + e^{-2\alpha T} - 2\right]$$

If $0 \leq \tau < 2T$, then

$$R_Y(\tau) = \frac{e^{-2\alpha T}}{4T^2\alpha^2}\left[e^{-\alpha\tau} + e^{\alpha\tau}\right] + \frac{1}{T\alpha} - \frac{\tau}{2T^2\alpha^2} - 2\frac{e^{-\alpha\tau}}{4T^2\alpha^2}$$

Problem 4.60

Consider the random processes $X(t) = Xe^{j2\pi f_0 t}$ and $Y(t) = Ye^{j2\pi f_0 t}$. Clearly

$$R_{XY}(t+\tau, t) = E[X(t+\tau)Y^*(t)] = E[XY]e^{j2\pi f_0 \tau}$$

However, both $X(t)$ and $Y(t)$ are nonstationary for $E[X(t)] = E[X]e^{j2\pi f_0 t}$ and $E[Y(t)] = E[Y]e^{j2\pi f_0 t}$ are not constant.

Problem 4.61

1)

$$\begin{aligned}
E[X(t)] &= \frac{4}{\pi}\int_0^{\frac{\pi}{4}} A\cos(2\pi f_0 t + \theta)d\theta \\
&= \frac{4A}{\pi}\sin(2\pi f_0 t + \theta)\Big|_0^{\frac{\pi}{4}} \\
&= \frac{4A}{\pi}[\sin(2\pi f_0 t + \frac{\pi}{4}) - \sin(2\pi f_0 t)]
\end{aligned}$$

Thus, $E[X(t)]$ is periodic with period $T = \frac{1}{f_0}$.

$$\begin{aligned}
R_X(t+\tau, t) &= E[A^2\cos(2\pi f_0(t+\tau) + \Theta)\cos(2\pi f_0 t + \Theta)] \\
&= \frac{A^2}{2}E[\cos(2\pi f_0(2t+\tau) + \Theta) + \cos(2\pi f_0 \tau)] \\
&= \frac{A^2}{2}\cos(2\pi f_0 \tau) + \frac{A^2}{2}E[\cos(2\pi f_0(2t+\tau) + \Theta)] \\
&= \frac{A^2}{2}\cos(2\pi f_0 \tau) + \frac{A^2}{2}\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\cos(2\pi f_0(2t+\tau) + \theta)d\theta \\
&= \frac{A^2}{2}\cos(2\pi f_0 \tau) + \frac{A^2}{\pi}(\cos(2\pi f_0(2t+\tau)) - \sin(2\pi f_0(2t+\tau)))
\end{aligned}$$

which is periodic with period $T' = \frac{1}{2f_0}$. Thus the process is cyclostationary with period $T = \frac{1}{f_0}$. Using the results of Problem 4.48 we obtain

$$\begin{aligned}
\mathcal{S}_X(f) &= \mathcal{F}[\frac{1}{T}\int_0^T R_X(t+\tau, t)dt] \\
&= \mathcal{F}\left[\frac{A^2}{2}\cos(2\pi f_0 \tau) + \frac{A^2}{T\pi}\int_0^T(\cos(2\pi f_0(2t+\tau)) - \sin(2\pi f_0(2t+\tau))dt\right] \\
&= \mathcal{F}\left[\frac{A^2}{2}\cos(2\pi f_0 \tau)\right] \\
&= \frac{A^2}{4}(\delta(f-f_0) + \delta(f+f_0))
\end{aligned}$$

2)

$$\begin{aligned}
R_X(t+\tau, t) &= E[X(t+\tau)X(t)] = E[(X+Y)(X+Y)] \\
&= E[X^2] + E[Y^2] + E[YX] + E[XY] \\
&= E[X^2] + E[Y^2] + 2E[X][Y]
\end{aligned}$$

where the last equality follows from the independence of X and Y. But, $E[X] = 0$ since X is uniform on $[-1,\ 1]$ so that

$$R_X(t+\tau, t) = E[X^2] + E[Y^2] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

The Fourier transform of $R_X(t+\tau, t)$ is the power spectral density of $X(t)$. Thus

$$\mathcal{S}_X(f) = \mathcal{F}[R_X(t+\tau, t)] = \frac{2}{3}\delta(f)$$

Problem 4.62

$h(t) = e^{-\beta t}u_{-1}(t) \Rightarrow H(f) = \frac{1}{\beta + j2\pi f}$. The power spectral density of the input process is $\mathcal{S}_X(f) = \mathcal{F}[e^{-\alpha|\tau|}] = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$. If $\alpha = \beta$, then

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \frac{2\alpha}{(\alpha^2 + 4\pi^2 f^2)^2}$$

If $\alpha \neq \beta$, then

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \frac{2\alpha}{(\alpha^2 + 4\pi^2 f^2)(\beta^2 + 4\pi^2 f^2)}$$

Problem 4.63

1) Let $Y(t) = X(t)+N(t)$. The process $\hat{X}(t)$ is the response of the system $h(t)$ to the input process $Y(t)$ so that

$$\begin{aligned}
R_{Y\hat{X}}(\tau) &= R_Y(\tau) \star h(-\tau) \\
&= [R_X(\tau) + R_N(\tau) + R_{XN}(\tau) + R_{NX}(\tau)] \star h(-\tau)
\end{aligned}$$

Also by definition

$$\begin{aligned}
R_{Y\hat{X}}(\tau) &= E[(X(t+\tau) + N(t+\tau))\hat{X}(t)] = R_{X\hat{X}}(\tau) + R_{N\hat{X}}(\tau) \\
&= R_{X\hat{X}}(\tau) + R_N(\tau) \star h(-\tau) + R_{NX}(\tau) \star h(-\tau)
\end{aligned}$$

Substituting this expression for $R_{Y\hat{X}}(\tau)$ in the previous one, and cancelling common terms we obtain

$$R_{X\hat{X}}(\tau) = R_X(\tau) \star h(-\tau) + R_{XN}(\tau) \star h(-\tau)$$

2)

$$E\left[(X(t) - \hat{X}(t))^2\right] = R_X(0) + R_{\hat{X}}(0) - R_{X\hat{X}}(0) - R_{\hat{X}X}(0)$$

We can write $E\left[(X(t) - \hat{X}(t))^2\right]$ in terms of the spectral densities as

$$\begin{aligned}
E\left[(X(t) - \hat{X}(t))^2\right] &= \int_{-\infty}^{\infty} (\mathcal{S}_X(f) + \mathcal{S}_{\hat{X}}(f) - 2\mathcal{S}_{X\hat{X}}(f))df \\
&= \int_{-\infty}^{\infty} \Big[\mathcal{S}_X(f) + (\mathcal{S}_X(f) + \mathcal{S}_N(f) + 2\mathrm{Re}[\mathcal{S}_{XN}(f)])|H(f)|^2 \\
&\qquad -2(\mathcal{S}_X(f) + \mathcal{S}_{XN}(f))H^*(f)\Big]df
\end{aligned}$$

To find the $H(f)$ that minimizes $E\left[(X(t) - \hat{X}(t))^2\right]$ we set the derivative of the previous expression, with respect to $H(f)$, to zero. By doing so we obtain

$$H(f) = \frac{\mathcal{S}_X(f) + \mathcal{S}_{XN}(f)}{\mathcal{S}_X(f) + \mathcal{S}_N(f) + 2\mathrm{Re}[\mathcal{S}_{XN}(f)]}$$

3) If $X(t)$ and $N(t)$ are independent, then

$$R_{XN}(\tau) = E[X(t+\tau)N(t)] = E[X(t+\tau)]E[N(t)]$$

Since $E[N(t)] = 0$ we obtain $R_{XN}(\tau) = 0$ and the optimum filter is

$$H(f) = \frac{\mathcal{S}_X(f)}{\mathcal{S}_X(f) + \frac{N_0}{2}}$$

The corresponding value of $E\left[(X(t) - \hat{X}(t))^2\right]$ is

$$E_{\min}\left[(X(t) - \hat{X}(t))^2\right] = \int_{-\infty}^{\infty} \frac{\mathcal{S}_X(f)N_0}{2\mathcal{S}_X(f) + N_0} df$$

4) With $\mathcal{S}_N(f) = 1$, $\mathcal{S}_X(f) = \frac{1}{1+f^2}$ and $\mathcal{S}_{XN}(f) = 0$, then

$$H(f) = \frac{\frac{1}{1+f^2}}{1 + \frac{1}{1+f^2}} = \frac{1}{2+f^2}$$

Problem 4.64

1) Let $\hat{X}(t)$ and $\tilde{X}(t)$ be the outputs of the systems $h(t)$ and $g(t)$ when the input $Z(t)$ is applied. Then,

$$\begin{aligned}
E[(X(t) - \tilde{X}(t))^2] &= E[(X(t) - \hat{X}(t) + \hat{X}(t) - \tilde{X}(t))^2] \\
&= E[(X(t) - \hat{X}(t))^2] + E[(\hat{X}(t) - \tilde{X}(t))^2] \\
&\quad + E[(X(t) - \hat{X}(t)) \cdot (\hat{X}(t) - \tilde{X}(t))]
\end{aligned}$$

But,

$$\begin{aligned}
&E[(X(t) - \hat{X}(t)) \cdot (\hat{X}(t) - \tilde{X}(t))] \\
&= E[(X(t) - \hat{X}(t)) \cdot Z(t) \star (h(t) - g(t))] \\
&= E\left[(X(t) - \hat{X}(t)) \int_{-\infty}^{\infty} (h(\tau) - g(\tau))Z(t-\tau)d\tau\right] \\
&= \int_{-\infty}^{\infty} E\left[(X(t) - \hat{X}(t))Z(t-\tau)\right](h(\tau) - g(\tau))d\tau = 0
\end{aligned}$$

where the last equality follows from the assumption $E\left[(X(t) - \hat{X}(t))Z(t-\tau)\right] = 0$ for all t, τ. Thus,

$$E[(X(t) - \tilde{X}(t))^2] = E[(X(t) - \hat{X}(t))^2] + E[(\hat{X}(t) - \tilde{X}(t))^2]$$

and this proves that

$$E[(X(t) - \hat{X}(t))^2] \leq E[(X(t) - \tilde{X}(t))^2]$$

2)

$$E[(X(t) - \hat{X}(t))Z(t-\tau)] = 0 \Rightarrow E[X(t)Z(t-\tau)] = E[\hat{X}(t)Z(t-\tau)]$$

or in terms of crosscorrelation functions $R_{XZ}(\tau) = R_{\hat{X}Z}(\tau) = R_{Z\hat{X}}(-\tau)$. However, $R_{Z\hat{X}}(-\tau) = R_Z(-\tau) \star h(\tau)$ so that

$$R_{XZ}(\tau) = R_Z(-\tau) \star h(\tau) = R_Z(\tau) \star h(\tau)$$

3) Taking the Fourier of both sides of the previous equation we obtain

$$\mathcal{S}_{XZ}(f) = \mathcal{S}_Z(f)H(f) \quad \text{or} \quad H(f) = \frac{\mathcal{S}_{XZ}(f)}{\mathcal{S}_Z(f)}$$

4)

$$\begin{aligned}
E[\epsilon^2(t)] &= E\left[(X(t)-\hat{X}(t))((X(t)-\hat{X}(t))\right] \\
&= E[X(t)X(t)] - E[\hat{X}(t)X(t)] \\
&= R_X(0) - E\left[\int_{-\infty}^{\infty} Z(t-v)h(v)X(t)dv\right] \\
&= R_X(0) - \int_{-\infty}^{\infty} R_{ZX}(-v)h(v)dv \\
&= R_X(0) - \int_{-\infty}^{\infty} R_{XZ}(v)h(v)dv
\end{aligned}$$

where we have used the fact that $E[(X(t)-\hat{X}(t))\hat{X}(t)] = E[(X(t)-\hat{X}(t))Z(t)\star h(t)] = 0$

Problem 4.65
1) Using the results of Problem 4.45 we obtain

$$\mathcal{S}_X(f) = \frac{|P(f)|^2}{T}\left[R_A(0) + 2\sum_{k=1}^{\infty} R_A(k)\cos(2\pi kfT)\right]$$

Since, A_n's are independent random variables with zero mean $R_A(k) = \sigma^2\delta(k)$ so that

$$\mathcal{S}_X(f) = \frac{1}{T}\left|\frac{1}{2W}\Pi(\frac{f}{2W})\right|^2\sigma^2 = \frac{\sigma^2}{4W^2T}\Pi(\frac{f}{2W})$$

2) If $T = \frac{1}{2W}$ then

$$P_X(f) = \int_{-\infty}^{\infty}\frac{\sigma^2}{2W}\Pi(\frac{f}{2W})df = \frac{\sigma^2}{2W}\int_{-W}^{W} df = \sigma^2$$

3)

$$\mathcal{S}_{X_1}(f) = \frac{N_0}{2}\Pi(\frac{f}{2W}) \Rightarrow R_{X_1}(\tau) = N_0W\text{sinc}(2W\tau)$$

Hence,

$$\begin{aligned}
E[A_kA_j] &= E[X_1(kT)X_1(jT)] = R_{X_1}((k-j)T) \\
&= N_0W\text{sinc}(2W(k-j)T) = N_0W\text{sinc}(k-j) \\
&= \begin{cases} N_0W & k=j \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Thus, we obtain the same conditions as in the first and second part of the problem with $\sigma^2 = N_0W$. The power spectral density and power content of $X(t)$ will be

$$\mathcal{S}_X(f) = \frac{N_0}{2}\Pi(\frac{f}{2W}), \qquad P_X = N_0W$$

$X(t)$ is the random process formed by sampling $X_1(t)$ at the Nyquist rate.

Problem 4.66
the noise equivalent bandwidth of a filter is

$$B_{neq} = \frac{\int_{-\infty}^{\infty}|H(f)|^2df}{2H_{\max}^2}$$

If we have an ideal bandpass filter of bandwidth W, then $H(f) = 1$ for $|f - f_0| < W$ where f_0 is the central frequency of the filter. Hence,

$$B_{neq} = \frac{1}{2}\left[\int_{-f_0-\frac{W}{2}}^{-f_0+\frac{W}{2}} df + \int_{f_0-\frac{W}{2}}^{f_0+\frac{W}{2}} df\right] = W$$

Problem 4.67

In general

$$\mathcal{S}_{X_c}(f) = \mathcal{S}_{X_s}(f) = \begin{cases} \mathcal{S}_X(f - f_0) + \mathcal{S}_X(f + f_0) & |f| < f_0 \\ 0 & \text{otherwise} \end{cases}$$

If $f_0 = f_c - \frac{W}{2}$ in Example 4.6.1, then using the previous formula we obtain

$$\mathcal{S}_{X_c}(f) = \mathcal{S}_{X_s}(f) = \begin{cases} \frac{N_0}{2} & \frac{W}{2} < |f| < \frac{3W}{2} \\ N_0 & |f| < \frac{W}{2} \\ 0 & \text{otherwise} \end{cases}$$

The cross spectral density is given by

$$\mathcal{S}_{X_cX_s}(f) = \begin{cases} j[\mathcal{S}_X(f + f_0) - \mathcal{S}_X(f - f_0)] & |f| < f_0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, with $f_0 = f_c - \frac{W}{2}$ we obtain

$$\mathcal{S}_{X_cX_s}(f) = \begin{cases} -j\frac{N_0}{2} & -\frac{3W}{2} < f < \frac{W}{2} \\ 0 & |f| < \frac{W}{2} \\ j\frac{N_0}{2} & \frac{W}{2} < f < \frac{3W}{2} \\ 0 & \text{otherwise} \end{cases}$$

Problem 4.68

We have $P_{X_c} = P_{X_s} = P_X$. For the process of the Example 4.6.1

$$\mathcal{S}_X(f) = \begin{cases} \frac{N_0}{2} & |f - f_c| < W \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\begin{aligned} P_X &= \int_{-\infty}^{\infty} \mathcal{S}_X(f)df = \int_{-f_c-W}^{-f_c+W} \frac{N_0}{2}df + \int_{f_c-W}^{f_c+W} \frac{N_0}{2}df \\ &= \frac{N_0}{2}(2W + 2W) = 2N_0W \end{aligned}$$

For the first part of the example,

$$\mathcal{S}_{X_c}(f) = \mathcal{S}_{X_s}(f) \begin{cases} N_0 & |f| < W \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$P_{X_c} = P_{X_s} = \int_{-W}^{W} N_0 df = 2N_0W = P_X$$

For the second part of Example 3.6.12

$$\mathcal{S}_{X_c}(f) = \mathcal{S}_{X_s}(f) \begin{cases} \frac{N_0}{2} & 0 < |f| < 2W \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$P_{X_c} = P_{X_s} = \int_{-2W}^{2W} \frac{N_0}{2} df = 2N_0 W = P_X$$

Problem 4.69

1) The power spectral density of the in-phase and quadrature components is given by

$$\mathcal{S}_{n_c}(f) = \mathcal{S}_{n_s}(f) = \begin{cases} \mathcal{S}_n(f - f_0) + \mathcal{S}_n(f + f_0) & |f| < 7 \\ 0 & \text{otherwise} \end{cases}$$

If the passband of the ideal filter extends from 3 to 11 KHz, then f_0 =7 KHz is the mid-band frequency so that

$$\mathcal{S}_{n_c}(f) = \mathcal{S}_{n_s}(f) = \begin{cases} N_0 & |f| < 7 \\ 0 & \text{otherwise} \end{cases}$$

The cross spectral density is given by

$$\mathcal{S}_{n_c n_s}(f) = \begin{cases} j[\mathcal{S}_n(f + f_0) - \mathcal{S}_n(f - f_0) & |f| < 7 \\ 0 & \text{otherwise} \end{cases}$$

However $\mathcal{S}_n(f + f_0) = \mathcal{S}_n(f - f_0)$ for $|f| < 7$ and therefore $\mathcal{S}_{n_c n_s}(f) = 0$. It turns then that the crosscorrelation $R_{n_c n_s}(\tau)$ is zero.

2) With f_0=6 KHz

$$\mathcal{S}_{n_c}(f) = \mathcal{S}_{n_s}(f) = \begin{cases} \frac{N_0}{2} & 3 < |f| < 5 \\ N_0 & |f| < 3 \\ 0 & \text{otherwise} \end{cases}$$

The cross spectral density is given by

$$\mathcal{S}_{n_c n_s}(f) = \begin{cases} -j\frac{N_0}{2} & -5 < f < 3 \\ j\frac{N_0}{2} & 3 < f < 5 \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\begin{aligned} R_{n_c n_s}(\tau) &= \mathcal{F}^{-1}\left[-j\frac{N_0}{2}\Pi(\frac{t+4}{2}) + j\frac{N_0}{2}\Pi(\frac{t-4}{2})\right] \\ &= -j\frac{N_0}{2}2\text{sinc}(2\tau)e^{-j2\pi4\tau} + j\frac{N_0}{2}2\text{sinc}(2\tau)e^{j2\pi4\tau} \\ &= -2N_0\text{sinc}(2\tau)\sin(2\pi4\tau) \end{aligned}$$

Problem 4.70

The in-phase component of $X(t)$ is

$$\begin{aligned} X_c(t) &= X(t)\cos(2\pi f_0 t) + \hat{X}(t)\sin(2\pi f_0 t) \\ &= \sum_{n=-\infty}^{\infty} A_n p(t - nT)\cos(2\pi f_0(t - nT)) \\ &\quad + \sum_{n=-\infty}^{\infty} A_n \hat{p}(t - nT)\sin(2\pi f_0(t - nT)) \\ &= \sum_{n=-\infty}^{\infty} A_n \left(p(t - nT)\cos(2\pi f_0(t - nT)) + \hat{p}(t - nT)\sin(2\pi f_0(t - nT))\right) \\ &= \sum_{n=-\infty}^{\infty} A_n p_c(t - nT) \end{aligned}$$

where we have used the fact $p_c(t) = p(t)\cos(2\pi f_0 t) + \hat{p}(t)\sin(2\pi f_0 t)$. Similarly for the quadrature component

$$\begin{aligned}
X_s(t) &= \hat{X}(t)\cos(2\pi f_0 t) - X(t)\sin(2\pi f_0 t) \\
&= \sum_{n=-\infty}^{\infty} A_n \hat{p}(t-nT)\cos(2\pi f_0(t-nT)) \\
&\quad - \sum_{n=-\infty}^{\infty} A_n p(t-nT)\sin(2\pi f_0(t-nT)) \\
&= \sum_{n=-\infty}^{\infty} A_n \left(\hat{p}(t-nT)\cos(2\pi f_0(t-nT)) - p(t-nT)\sin(2\pi f_0(t-nT))\right) \\
&= \sum_{n=-\infty}^{\infty} A_n p_s(t-nT)
\end{aligned}$$

Problem 4.71
The envelope $V(t)$ of a bandpass process is defined to be

$$V(t) = \sqrt{X_c^2(t) + X_s^2(t)}$$

where $X_c(t)$ and $X_s(t)$ are the in-phase and quadrature components of $X(t)$ respectively. However, both the in-phase and quadrature components are lowpass processes and this makes $V(t)$ a lowpass process independent of the choice of the center frequency f_0.

Problem 4.72
1) The power spectrum of the bandpass signal is

$$\mathcal{S}_n(f) = \begin{cases} \frac{N_0}{2} & |f - f_c| < W \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\mathcal{S}_{n_c}(f) = \mathcal{S}_{n_s}(f) = \begin{cases} N_0 & |f| < W \\ 0 & \text{otherwise} \end{cases}$$

The power content of the in-phase and quadrature components of $n(t)$ is $P_n = \int_{-W}^{W} N_0 df = 2N_0 W$

2) Since $\mathcal{S}_{n_c n_s}(f) = 0$, the processes $N_c(t)$, $N_s(t)$ are independent zero-mean Gaussian with variance $\sigma^2 = P_n = 2N_0W$. Hence, $V(t) = \sqrt{N_c^2(t) + N_s^2(t)}$ is Rayleigh distributed and the PDF is given by

$$f_V(v) = \begin{cases} \frac{v^2}{2N_0W} e^{-\frac{v^2}{4N_0W}} & v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

3) $X(t)$ is given by

$$X(t) = (A + N_c(t))\cos(2\pi f_0 t) - N_S(t)\sin(2\pi f_0 t)$$

The process $A+N_c(t)$ is Gaussian with mean A and variance $2N_0W$. Hence, $V(t) = \sqrt{(A + N_c(t))^2 + N_s^2(t)}$ follows the Rician distribution (see Problem 4.31). The density function of the envelope is given by

$$f_V(v) = \begin{cases} \frac{v}{2N_0W} I_0\left(\frac{Av}{2N_0W}\right) e^{-\frac{v^2+A^2}{4N_0W}} & v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

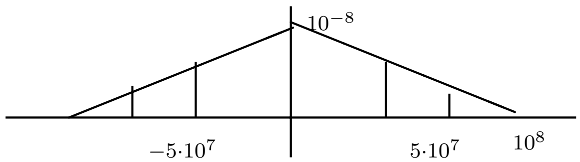
where

$$I_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos u} du$$

Problem 4.73

1) The power spectral density $\mathcal{S}_n(f)$ is depicted in the following figure. The output bandpass process has non-zero power content for frequencies in the band $49 \times 10^6 \leq |f| \leq 51 \times 10^6$. The power content is

$$\begin{aligned}
P &= \int_{-51\times10^6}^{-49\times10^6} 10^{-8}\left(1+\frac{f}{10^8}\right)df + \int_{49\times10^6}^{51\times10^6} 10^{-8}\left(1-\frac{f}{10^8}\right)df \\
&= 10^{-8}x\Big|_{-51\times10^6}^{-49\times10^6} + 10^{-16}\frac{1}{2}x^2\Big|_{-51\times10^6}^{-49\times10^6} + 10^{-8}x\Big|_{49\times10^6}^{51\times10^6} - 10^{-16}\frac{1}{2}x^2\Big|_{49\times10^6}^{51\times10^6} \\
&= 2\times10^{-2}
\end{aligned}$$



2) The output process $N(t)$ can be written as

$$N(t) = N_c(t)\cos(2\pi 50\times10^6 t) - N_s(t)\sin(2\pi50\times10^6 t)$$

where $N_c(t)$ and $N_s(t)$ are the in-phase and quadrature components respectively, given by

$$\begin{aligned}
N_c(t) &= N(t)\cos(2\pi50\times10^6 t) + \hat{N}(t)\sin(2\pi50\times10^6 t) \\
N_s(t) &= \hat{N}(t)\cos(2\pi50\times10^6 t) - N(t)\sin(2\pi50\times10^6 t)
\end{aligned}$$

The power content of the in-phase component is given by

$$\begin{aligned}
E[|N_c(t)|^2] &= E[|N(t)|^2]\cos^2(2\pi50\times10^6 t) + E[|\hat{N}(t)|^2]\sin^2(2\pi50\times10^6 t) \\
&= E[|N(t)|^2] = 2\times10^{-2}
\end{aligned}$$

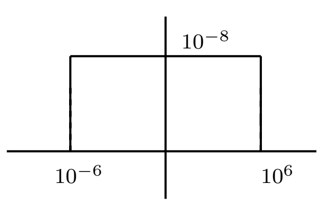
where we have used the fact that $E[|N(t)|^2] = E[|\hat{N}(t)|^2]$. Similarly we find that $E[|N_s(t)|^2] = 2\times10^{-2}$.

3) The power spectral density of $N_c(t)$ and $N_s(t)$ is

$$\mathcal{S}_{N_c}(f) = \mathcal{S}_{N_s}(f) = \begin{cases} \mathcal{S}_N(f-50\times10^6) + \mathcal{S}_N(f+50\times10^6) & |f| \leq 50\times10^6 \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{S}_{N_c}(f)$ is depicted in the next figure. The power content of $\mathcal{S}_{N_c}(f)$ can now be found easily as

$$P_{N_c} = P_{N_s} = \int_{-10^6}^{10^6} 10^{-8} df = 2\times10^{-2}$$



4) The power spectral density of the output is given by

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = (|f| - 49\times10^6)(10^{-8} - 10^{-16}|f|) \quad \text{for } 49\times10^6 \leq |f| \leq 51\times10^6$$

Hence, the power content of the output is

$$
\begin{aligned}
P_Y &= \int_{-51\times 10^6}^{-49\times 10^6} (-f - 49\times 10^6)(10^{-8} + 10^{-16} f)df \\
&\quad + \int_{49\times 10^6}^{51\times 10^6} (f - 49\times 10^6)(10^{-8} - 10^{-16} f)df \\
&= 2\times 10^4 - \frac{4}{3}10^2
\end{aligned}
$$

The power spectral density of the in-phase and quadrature components of the output process is given by

$$
\begin{aligned}
\mathcal{S}_{Y_c}(f) = \mathcal{S}_{Y_s}(f) &= \left((f + 50\times 10^6) - 49\times 10^6\right)\left(10^{-8} - 10^{-16}(f + 50\times 10^6)\right) \\
&\quad + \left(-(f - 50\times 10^6) - 49\times 10^6\right)\left(10^{-8} + 10^{-16}(f - 50\times 10^6)\right) \\
&= -2\times 10^{-16} f^2 + 10^{-2}
\end{aligned}
$$

for $|f| \leq 10^6$ and zero otherwise. The power content of the in-phase and quadrature component is

$$
\begin{aligned}
P_{Y_c} = P_{Y_s} &= \int_{-10^6}^{10^6} (-2\times 10^{-16} f^2 + 10^{-2})df \\
&= -2\times 10^{-16}\frac{1}{3}f^3\Big|_{-10^6}^{10^6} + 10^{-2} f\Big|_{-10^6}^{10^6} \\
&= 2\times 10^4 - \frac{4}{3}10^2 = P_Y
\end{aligned}
$$

Chapter 5

Problem 5.1

The spectrum of the signal at the output of the LPF is $\mathcal{S}_{s,o}(f) = \mathcal{S}_s(f)|\Pi(\frac{f}{2W})|^2$. Hence, the signal power is

$$\begin{aligned} P_{s,o} &= \int_{-\infty}^{\infty} \mathcal{S}_{s,o}(f)df = \int_{-W}^{W} \frac{P_0}{1+(f/B)^2} df \\ &= P_0 B \arctan(\frac{f}{B})\Big|_{-W}^{W} = 2P_0 B \arctan(\frac{W}{B}) \end{aligned}$$

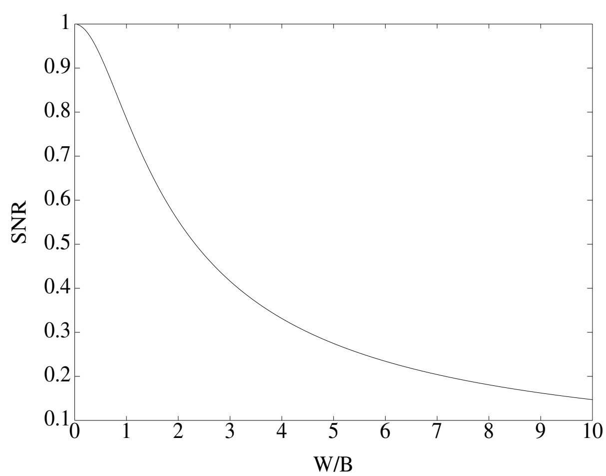
Similarly, noise power at the output of the lowpass filter is

$$P_{n,o} = \int_{-W}^{W} \frac{N_0}{2} df = N_0 W$$

Thus, the SNR is given by

$$\text{SNR} = \frac{2P_0 B \arctan(\frac{W}{B})}{N_0 W} = \frac{2P_0}{N_0} \frac{\arctan(\frac{W}{B})}{\frac{W}{B}}$$

In the next figure we plot SNR as a function of $\frac{W}{B}$ and for $\frac{2P_0}{N_0} = 1$.



Problem 5.2

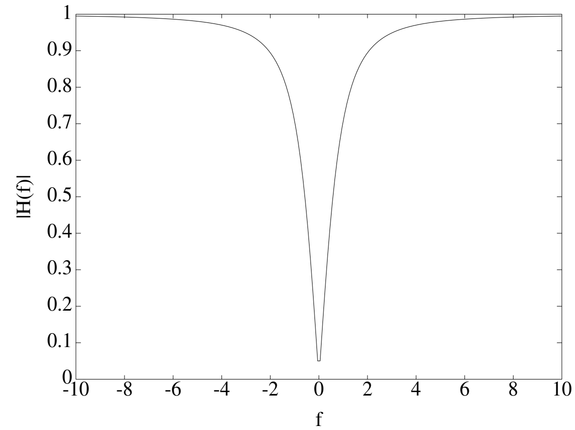
1) The transfer function of the RC filter is

$$H(s) = \frac{R}{\frac{1}{Cs} + R} = \frac{RCs}{1 + RCs}$$

with $s = j2\pi f$. Hence, the magnitude frequency response is

$$|H(f)| = \left(\frac{4\pi^2 (RC)^2 f^2}{1 + 4\pi^2 (RC)^2 f^2} \right)^{\frac{1}{2}}$$

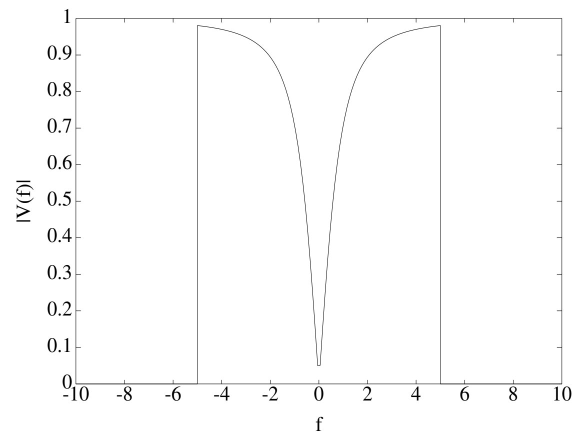
This function is plotted in the next figure for f in $[-10, 10]$ and $4\pi^2(RC)^2 = 1$.



2) The overall system is the cascade of the RC and the LPF filter. If the bandwidth of the LPF is W, then the transfer function of the system is

$$V(f) = \frac{j2\pi RCf}{1 + j2\pi RCf}\Pi(\frac{f}{2W})$$

The next figure depicts $|V(f)|$ for $W = 5$ and $4\pi^2(RC)^2 = 1$.



3) The noise output power is

$$\begin{aligned}
P_n &= \int_{-W}^{W} \frac{4\pi^2(RC)^2 f^2}{1 + 4\pi^2(RC)^2 f^2}\frac{N_0}{2}df \\
&= N_0 W - \frac{N_0}{2}\int_{-W}^{W}\frac{1}{1 + 4\pi^2(RC)^2 f^2}df \\
&= N_0 W - \frac{N_0}{2}\frac{1}{2\pi RC}\arctan(2\pi RCf)\Big|_{-W}^{W} \\
&= N_0 W - \frac{N_0}{2\pi RC}\arctan(2\pi RCW)
\end{aligned}$$

The output signal is a sinusoidal with frequency f_c and amplitude $A|V(f_c)|$. Since $f_c < W$ we conclude that the amplitude of the sinusoidal ouput signal is

$$A|H(f_c)| = A\sqrt{\frac{4\pi^2(RC)^2 f_c^2}{1 + 4\pi^2(RC)^2 f_c^2}}$$

and the output signal power

$$P_s = \frac{A^2}{2}\frac{4\pi^2(RC)^2 f_c^2}{1 + 4\pi^2(RC)^2 f_c^2}$$

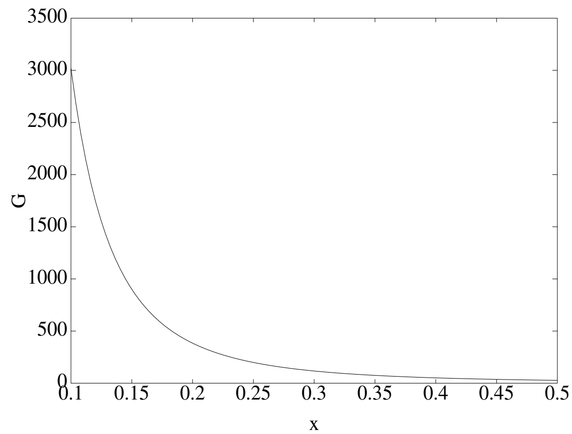
Thus, the SNR at the ouput of the LPF is

$$\text{SNR} = \frac{\frac{A^2}{2}\frac{4\pi^2(RC)^2 f_c^2}{1+4\pi^2(RC)^2 f_c^2}}{N_0 W - \frac{N_0}{2\pi RC}\ \arctan(2\pi RCW)} = \frac{\frac{A^2}{N_0}\frac{\pi RC f_c^2}{1+4\pi^2(RC)^2 f_c^2}}{2\pi RCW -\ \arctan(2\pi RCW)}$$

In the next figure we plot

$$G(W) = \frac{1}{2\pi RCW -\ \arctan(2\pi RCW)}$$

as a function of $x = 2\pi RCW$, when the latter varies from 0.1 to 0.5.



Problem 5.3
The noise power content of the received signal $r(t) = u(t) + n(t)$ is

$$P_n = \int_{-\infty}^{\infty} \mathcal{S}_n(f)df = \frac{N_0}{2} \times 4W = 2N_0 W$$

If we write $n(t)$ as

$$n(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

then,

$$\begin{aligned} n(t)\cos(2\pi f_c t) &= n_c(t)\cos^2(2\pi f_c t) - n_s(t)\cos(2\pi f_c t)\sin(2\pi f_c t) \\ &= \frac{1}{2}n_c(t) + \frac{1}{2}n_c(t)\cos(2\pi 2 f_c t) - n_s(t)\sin(2\pi 2 f_c t) \end{aligned}$$

The noise signal at the output of the LPF is $\frac{1}{2}n_c(t)$ with power content

$$P_{n,o} = \frac{1}{4}P_{n_c} = \frac{1}{4}P_n = \frac{N_0 W}{2}$$

If the DSB modulated signal is $u(t) = m(t)\cos(2\pi f_c t)$, then its autocorrelation function is $\bar{R}_u(\tau) = \frac{1}{2}R_M(\tau)\cos(2\pi f_c \tau)$ and its power

$$P_u = \bar{R}_u(0) = \frac{1}{2}R_M(0) = \int_{-\infty}^{\infty} \mathcal{S}_u(f)df = 2WP_0$$

From this relation we find $R_M(0) = 4WP_0$. The signal at the output of the LPF is $y(t) = \frac{1}{2}m(t)$ with power content

$$P_{s,o} = \frac{1}{4}E[m^2(t)] = \frac{1}{4}R_M(0) = WP_0$$

Hence, the SNR at the output of the demodulator is

$$\text{SNR} = \frac{P_{s,o}}{P_{n,o}} = \frac{WP_0}{\frac{N_0 W}{2}} = \frac{2P_0}{N_0}$$

Problem 5.4

First we determine the baseband signal to noise ratio $(\frac{S}{N})_b$. With $W = 1.5 \times 10^6$, we obtain

$$\left(\frac{S}{N}\right)_b = \frac{P_R}{N_0 W} = \frac{P_R}{2 \times 0.5 \times 10^{-14} \times 1.5 \times 10^6} = \frac{P_R 10^8}{1.5}$$

Since the channel attenuation is 90 db, then

$$10 \log \frac{P_T}{P_R} = 90 \Longrightarrow P_R = 10^{-9} P_T$$

Hence,

$$\left(\frac{S}{N}\right)_b = \frac{P_R 10^8}{1.5} = \frac{10^8 \times 10^{-9} P_T}{1.5} = \frac{P_T}{15}$$

1) If USSB is employed, then

$$\left(\frac{S}{N}\right)_{o,\text{USSB}} = \left(\frac{S}{N}\right)_b = 10^3 \Longrightarrow P_T = 15 \times 10^3 = 15 \text{ KWatts}$$

2) If conventional AM is used, then

$$\left(\frac{S}{N}\right)_{o,\text{AM}} = \eta \left(\frac{S}{N}\right)_b = \eta \frac{P_T}{15}$$

where, $\eta = \frac{\alpha^2 P_{M_n}}{1+\alpha^2 P_{M_n}}$. Since, $\max[|m(t)| = 1$, we have

$$P_{M_n} = P_M = \int_{-1}^{1} \frac{1}{2} x^2 dx = \frac{1}{3}$$

and, therefore

$$\eta = \frac{0.25 \times \frac{1}{3}}{1 + 0.25 \times \frac{1}{3}} = \frac{1}{13}$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\text{AM}} = \frac{1}{13} \frac{P_T}{15} = 10^3 \Longrightarrow P_T = 195 \text{ KWatts}$$

3) For DSB modulation

$$\left(\frac{S}{N}\right)_{o,\text{DSB}} = \left(\frac{S}{N}\right)_b = \frac{P_T}{15} = 10^3 \Longrightarrow P_T = 15 \text{ KWatts}$$

Problem 5.5

1) Since $|H(f)| = 1$ for $f = |f_c \pm f_m|$, the signal at the output of the noise-limiting filter is

$$r(t) = 10^{-3}[1 + \alpha \cos(2\pi f_m t + \phi)] \cos(2\pi f_c t) + n(t)$$

The signal power is

$$\begin{aligned} P_R &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} 10^{-6}[1 + \alpha \cos(2\pi f_m t + \phi)]^2 \cos^2(2\pi f_c t) dt \\ &= \frac{10^{-6}}{2}[1 + \frac{\alpha^2}{2}] = 56.25 \times 10^{-6} \end{aligned}$$

The noise power at the output of the noise-limiting filter is

$$P_{n,o} = \frac{1}{2}P_{n_c} = \frac{1}{2}P_n = \frac{1}{2}\frac{N_0}{2} \times 2 \times 2500 = 25 \times 10^{-10}$$

2) Multiplication of $r(t)$ by $2\cos(2\pi f_c t)$ yields

$$\begin{aligned} y(t) &= \frac{10^{-3}}{2}[1 + \alpha\cos(2\pi f_m t)]2 + \frac{1}{2}n_c(t)2 \\ &\quad + \text{double frequency terms} \end{aligned}$$

The LPF rejects the double frequency components and therefore, the output of the filter is

$$v(t) = 10^{-3}[1 + \alpha\cos(2\pi f_m t)] + n_c(t)$$

If the dc component is blocked, then the signal power at the output of the LPF is

$$P_o = \frac{10^{-6}}{2}0.5^2 = 0.125 \times 10^{-6}$$

whereas, the output noise power is

$$P_{n,o} = P_{n_c} = P_n = 2\frac{N_0}{2}2000 = 40 \times 10^{-10}$$

where we have used the fact that the lowpass filter has a bandwidth of 1000 Hz. Hence, the output SNR is

$$\text{SNR} = \frac{0.125 \times 10^{-6}}{40 \times 10^{-10}} = 31.25 \quad 14.95 \text{ db}$$

Problem 5.6

The one-sided noise equivalent bandwidth is defined as

$$B_{eq} = \frac{\int_0^\infty |H(f)|^2 df}{|H(f)|^2_{\max}}$$

It is usually convenient to substitute $|H(f)|^2_{f=0}$ for $|H(f)|^2_{\max}$ in the denominator, since the peaking of the magnitude transfer function may be high (especially for small ζ) creating in this way anomalies. On the other hand if ζ is less, but close, to one, $|H(f)|^2_{\max}$ can be very well approximated by $|H(f)|^2_{f=0}$. Hence,

$$B_{eq} = \frac{\int_0^\infty |H(f)|^2 df}{|H(f)|^2_{f=0}}$$

and since

$$|H(f)|^2 = \frac{\omega_n^2 + j2\pi f\left(2\zeta\omega_n - \frac{\omega_n^2}{K}\right)}{\omega_n^2 - 4\pi^2 f^2 + j2\pi f 2\zeta\omega_n}$$

we find that $|H(0)| = 1$. Therefore,

$$B_{eq} = \int_0^\infty |H(f)|^2 df$$

For the passive second order filter

$$H(s) = \frac{s(2\zeta\omega_n - \frac{\omega_n^2}{K}) + \omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

$\tau_1 \gg 1$, so that $\frac{\omega_n^2}{K} = \frac{1}{\tau_1} \approx 0$ and

$$H(s) = \frac{s2\zeta\omega_n + \omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

The B_{eq} can be written as

$$B_{eq} = \frac{1}{4\pi j}\int_{-j\infty}^{j\infty} H(s)H(-s)ds$$

Since, $H(s) = \frac{KG(s)/s}{1+KG(s)/s}$ we obtain $\lim_{|s|\to\infty} H(s)H(-s) = 0$. Hence, the integral for B_{eq} can be taken along a contour, which contains the imaginary axis and the left half plane. Furthermore, since $G(s)$ is a rational function of s, the integral is equal to half the sum of the residues of the left half plane poles of $H(s)H(-s)$. Hence,

$$\begin{aligned}
B_{eq} &= \frac{1}{2}\left[(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})H(s)H(-s)\Big|_{s=-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}}\right.\\
&\quad \left.+(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})H(s)H(-s)\Big|_{s=-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}}\right]\\
&= \frac{\omega_n}{8}(4\zeta+\frac{1}{\zeta}) = \frac{1+4\zeta^2}{8\zeta/\omega_n}\\
&= \frac{1+\omega_n^2\tau_2^2+(\frac{\omega_n}{K})^2+2\frac{\omega_n^2\tau_2}{K}}{8\zeta/\omega_n}\\
&\approx \frac{1+\omega_n^2\tau_2^2}{8\zeta/\omega_n}
\end{aligned}$$

where we have used the approximation $\frac{\omega_n}{K} \approx 0$.

Problem 5.7

1) In the case of DSB, the output of the receiver noise-limiting filter is

$$\begin{aligned}
r(t) &= u(t) + n(t)\\
&= A_c m(t)\cos(2\pi f_c t + \phi_c(t))\\
&\quad + n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)
\end{aligned}$$

The power of the received signal is $P_s = \frac{A_c^2}{2}P_m$, whereas the power of the noise

$$P_{n,o} = \frac{1}{2}P_{n_c} + \frac{1}{2}P_{n_s} = P_n$$

Hence, the SNR at the output of the noise-limiting filter is

$$\left(\frac{S}{N}\right)_{o,\lim} = \frac{A_c^2 P_m}{2P_n}$$

Assuming coherent demodulation, the output of the demodulator is

$$y(t) = \frac{1}{2}[A_c m(t) + n_c]$$

The output signal power is $P_o = \frac{1}{4}A_c^2 P_m$ whereas the output noise power

$$P_{n,o} = \frac{1}{4}P_{n_c} = \frac{1}{4}P_n$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\text{dem}} = \frac{A_c^2 P_m}{P_n}$$

and the demodulation gain is given by

$$\text{dem. gain} = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = 2$$

2) In the case of SSB, the output of the receiver noise-limiting filter is

$$r(t) = A_c m(t)\cos(2\pi f_c t) \pm A_c \hat{m}(t)\sin(2\pi f_c t) + n(t)$$

The received signal power is $P_s = A_c^2 P_m$, whereas the received noise power is $P_{n,o} = P_n$. At the output of the demodulator

$$y(t) = \frac{A_c}{2}m(t) + \frac{1}{2}n_c(t)$$

with $P_o = \frac{1}{4}A_c^2 P_m$ and $P_{n,o} = \frac{1}{4}P_{n_c} = \frac{1}{4}P_n$. Therefore,

$$\text{dem. gain} = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{\frac{A_c^2 P_m}{P_n}}{\frac{A_c^2 P_m}{P_n}} = 1$$

3) In the case of conventional AM modulation, the output of the receiver noise-limiting filter is

$$r(t) = [A_c(1 + \alpha m_n(t)) + n_c(t)]\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

The total pre-detection power in the signal is

$$P_s = \frac{A_c^2}{2}(1 + \alpha^2 P_{M_n})$$

In this case, the demodulation gain is given by

$$\text{dem. gain} = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{2\alpha^2 P_{M_n}}{1 + \alpha^2 P_{M_n}}$$

The highest gain is achieved for $\alpha = 1$, that is 100% modulation.

4) For an FM system, the output of the receiver front-end (bandwidth B_c) is

$$\begin{aligned} r(t) &= A_c\cos(2\pi f_c t + \phi(t)) + n(t) \\ &= A_c\cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau) + n(t) \end{aligned}$$

The total signal input power is $P_{s,\text{lim}} = \frac{A_c^2}{2}$, whereas the pre-detection noise power is

$$P_{n,\text{lim}} = \frac{N_0}{2}2B_c = N_0 B_c = N_0 2(\beta_f + 1)W$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\text{lim}} = \frac{A_c^2}{2N_0 2(\beta_f + 1)W}$$

The output (post-detection) signal to noise ratio is

$$\left(\frac{S}{N}\right)_{o,\text{dem}} = \frac{3k_f^2 A_c^2 P_M}{2N_0 W^3}$$

Thus, the demodulation gain is

$$\text{dem. gain} = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{3\beta_f^2 P_M 2(\beta_f+1)}{(\max[|m(t)|])^2} = 6\beta_f^2(\beta_f+1)P_{M_n}$$

5) Similarly for the PM case we find that

$$\left(\frac{S}{N}\right)_{o,\text{lim}} = \frac{A_c^2}{2N_0 2(\beta_p+1)W}$$

and

$$\left(\frac{S}{N}\right)_{o,\text{dem}} = \frac{k_p^2 A_c^2 P_M}{2N_0 W}$$

Thus, the demodulation gain for a PM system is

$$\text{dem. gain} = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{\beta_p^2 P_M 2(\beta_p+1)}{(\max[|m(t)|])^2} = 2\beta_p^2(\beta_p+1)P_{M_n}$$

Problem 5.8

1) Since the channel attenuation is 80 db, then

$$10\log\frac{P_T}{P_R} = 80 \Longrightarrow P_R = 10^{-8}P_T = 10^{-8}\times 40\times 10^3 = 4\times 10^{-4} \text{ Watts}$$

If the noise limiting filter has bandwidth B, then the pre-detection noise power is

$$P_n = 2\int_{f_c-\frac{B}{2}}^{f_c+\frac{B}{2}} \frac{N_0}{2}df = N_0 B = 2\times 10^{-10}B \text{ Watts}$$

In the case of DSB or conventional AM modulation, $B = 2W = 2\times 10^4$ Hz, whereas in SSB modulation $B = W = 10^4$. Thus, the pre-detection signal to noise ratio in DSB and conventional AM is

$$\text{SNR}_{\text{DSB,AM}} = \frac{P_R}{P_n} = \frac{4\times 10^{-4}}{2\times 10^{-10}\times 2\times 10^4} = 10^2$$

and for SSB

$$\text{SNR}_{\text{SSB}} = \frac{4\times 10^{-4}}{2\times 10^{-10}\times 10^4} = 2\times 10^2$$

2) For DSB, the demodulation gain (see Problem 5.7) is 2. Hence,

$$\text{SNR}_{\text{DSB},o} = 2\text{SNR}_{\text{DSB},i} = 2\times 10^2$$

3) The demodulation gain of a SSB system is 1. Thus,

$$\text{SNR}_{\text{SSB},o} = \text{SNR}_{\text{SSB},i} = 2\times 10^2$$

4) For conventional AM with $\alpha = 0.8$ and $P_{M_n} = 0.2$, we have

$$\text{SNR}_{\text{AM},o} = \frac{\alpha^2 P_{M_n}}{1+\alpha^2 P_{M_n}}\text{SNR}_{\text{AM},i} = 0.1135\times 2\times 10^2$$

Problem 5.9

1) For an FM system that utilizes the whole bandwidth $B_c = 2(\beta_f + 1)W$, therefore

$$2(\beta_f + 1) = \frac{100}{4} \Longrightarrow b_f = 11.5$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\text{FM}} = 3\frac{A_c^2}{2}\left(\frac{\beta_f}{\max[|m(t)|]}\right)^2 \frac{P_M}{N_0 W} = 3\frac{A_c^2}{2}\beta_f^2 \frac{P_{M_n}}{N_0 W}$$

For an AM system

$$\left(\frac{S}{N}\right)_{o,\text{AM}} = \frac{A_c^2 \alpha^2 P_{M_n}}{2N_0 W}$$

Hence,

$$\frac{\left(\frac{S}{N}\right)_{o,\text{FM}}}{\left(\frac{S}{N}\right)_{o,\text{AM}}} = \frac{3\beta_f^2}{\alpha^2} = 549.139 \sim 27.3967 \text{ dB}$$

2) Since the PM and FM systems provide the same SNR

$$\left(\frac{S}{N}\right)_{o,\text{PM}} = \frac{k_p^2 A_c^2}{2}\frac{P_M}{N_0 W} = \frac{3k_f^2 A_c^2}{2W^2}\frac{P_M}{N_0 W} = \left(\frac{S}{N}\right)_{o,\text{FM}}$$

or

$$\frac{k_p^2}{3k_f^2} = \frac{1}{W^2} \Longrightarrow \frac{\beta_p^2}{3\beta_f^2 W^2} = \frac{1}{W^2}$$

Hence,

$$\frac{\text{BW}_{\text{PM}}}{\text{BW}_{\text{FM}}} = \frac{2(\beta_p + 1)W}{2(\beta_f + 1)W} = \frac{\sqrt{3}\beta_f + 1}{\beta_f + 1}$$

Problem 5.10

1) The received signal power can be found from

$$10\log\frac{P_T}{P_R} = 80 \Longrightarrow P_R = 10^{-8}P_T = 10^{-4} \text{ Watts}$$

$$\left(\frac{S}{N}\right)_o = \frac{\alpha^2 P_{M_n}}{1 + \alpha^2 P_{M_n}}\left(\frac{S}{N}\right)_b = \frac{\alpha^2 P_{M_n}}{1 + \alpha^2 P_{M_n}}\frac{P_R}{N_0 W}$$

Thus, with $P_R = 10^{-4}$, $P_{M_n} = 0.1$, $\alpha = 0.8$ and

$$N_0 W = 2 \times 0.5 \times 10^{-12} \times 5 \times 10^3 = 5 \times 10^{-9}$$

we find that

$$\left(\frac{S}{N}\right)_o = 1204 \quad 30.806 \text{ db}$$

2) Using Carson's rule, we obtain

$$B_c = 2(\beta + 1)W \Longrightarrow 100 \times 10^3 = 2(\beta + 1)5 \times 10^3 \Longrightarrow \beta = 9$$

We check now if the threshold imposes any restrictions.

$$\left(\frac{S}{N}\right)_{b,th} = \frac{P_R}{N_0 W} = 20(\beta + 1) = \frac{10^{-4}}{10^{-12} \times 5 \times 10^3} \Longrightarrow \beta = 999$$

Since we are limited in bandwidth we choose $\beta = 9$. The output signal to noise ratio is

$$\left(\frac{S}{N}\right)_o = 3\beta^2 0.1 \left(\frac{S}{N}\right)_b = 3 \times 9^2 \times 0.1 \times \frac{10^5}{5} = 486000 \quad 56.866 \text{ db}$$

Problem 5.11

1) First we check whether the threshold or the bandwidth impose a restrictive bound on the modulation index. By Carson's rule

$$B_c = 2(\beta + 1)W \Longrightarrow 60 \times 10^3 = 2(\beta + 1) \times 8 \times 10^3 \Longrightarrow \beta = 2.75$$

Using the relation

$$\left(\frac{S}{N}\right)_o = 60\beta^2(\beta + 1)P_{M_n}$$

with $\left(\frac{S}{N}\right)_o = 10^4$ and $P_{M_n} = \frac{1}{2}$ we find

$$10^4 = 30\beta^2(\beta + 1) \Longrightarrow \beta = 6.6158$$

Since we are limited in bandwidth we choose $\beta = 2.75$. Then,

$$\left(\frac{S}{N}\right)_o = 3\beta^2 P_{M_n} \left(\frac{S}{N}\right)_b \Longrightarrow \left(\frac{S}{N}\right)_b = \frac{2 \times 10^4}{3 \times 2.75^2} = 881.542$$

Thus,

$$\left(\frac{S}{N}\right)_b = \frac{P_R}{N_0 W} = 881.542 \Longrightarrow P_R = 881.542 \times 2 \times 10^{-12} \times 8 \times 10^3 = 1.41 \times 10^{-5}$$

Since the channel attenuation is 40 db, we find

$$P_T = 10^4 P_R = 0.141 \text{ Watts}$$

2) If the minimum required SNR is increased to 60 db, then the β from Carson's rule remains the same, whereas from the relation

$$\left(\frac{S}{N}\right)_o = 60\beta^2(\beta + 1)P_{M_n} = 10^6$$

we find $\beta = 31.8531$. As in part 1) we choose $\beta = 2.75$, and therefore

$$\left(\frac{S}{N}\right)_b = \frac{1}{3\beta^2 P_{M_n}} \left(\frac{S}{N}\right)_o = 8.8154 \times 10^4$$

Thus,

$$P_R = N_0 W 8.8154 \times 10^4 = 2 \times 10^{-12} \times 8 \times 10^3 \times 8.8154 \times 10^4 = 0.0014$$

and

$$P_T = 10^4 P_R = 14 \text{ Watts}$$

3) The frequency response of the receiver (de-emphasis) filter is given by

$$H_d(f) = \frac{1}{1 + j\frac{f}{f_0}}$$

with $f_0 = \frac{1}{2\pi\times 75\times 10^{-6}} = 2100$ Hz. In this case,

$$\left(\frac{S}{N}\right)_{o,\mathrm{PD}} = \frac{(\frac{W}{f_0})^3}{3\left(\frac{W}{f_0} - \arctan\frac{W}{f_0}\right)}\left(\frac{S}{N}\right)_o = 10^6$$

From this relation we find

$$\left(\frac{S}{N}\right)_o = 1.3541\times 10^5 \Longrightarrow P_R = 9.55\times 10^{-5}$$

and therefore,

$$P_T = 10^4 P_R = 0.955 \text{ Watts}$$

Problem 5.12

1. To determine the autocorrelation function of $U(t)$, we have

$$\begin{aligned} R_U(t,t+\tau) &= E[U(t+\tau)U(t)] \\ &= A_c^2 E[\cos(2\pi f_c t + \Phi(t))\cos(2\pi f_c(t+\tau) + \\ &\quad + \Phi(t+\tau))] \end{aligned}$$

Obviously the process $U(t)$ is not stationary.

2.

$$\bar{R}_U(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} R_U(t,t+\tau)\,dt$$

This gives

$$\begin{aligned} \bar{R}_U(\tau) &= A_c^2\lim_{T\to\infty}\frac{1}{T}E\Big[\int_{-T/2}^{T/2}\cos(2\pi f_c t + \Phi(t))\times \\ &\quad \times\cos(2\pi f_c(t+\tau) + \Phi(t+\tau))\,dt\Big] \\ &= \frac{A_c^2}{2}\lim_{T\to\infty}\frac{1}{T}E\Big[\int_{-T/2}^{T/2}[\cos(4\pi f_c t + 2\pi f_c\tau + \Phi(t) + \Phi(t+\tau)) + \\ &\quad + \cos(2\pi f_c\tau + \Phi(t+\tau) - \Phi(t))]\,dt\Big] \\ &\stackrel{a}{=} \frac{A_c^2}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} E\Big[\cos(2\pi f_c\tau + \Phi(t+\tau) - \Phi(t))\Big]\,dt \\ &= \frac{A_c^2}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\mathrm{Re}\Big[e^{j2\pi f_c\tau}E[e^{j(\Phi(t+\tau)-\Phi(t))}]\Big]\,dt \end{aligned}$$

where the equality in (a) follows from the fact that $\cos(4\pi f_c t + 2\pi f_c\tau + \Phi(t) + \Phi(t+\tau))$ is a bandpass signal centered at $2f_c$ and its dc value (as demonstrated by the integral) is zero. Now it remains to find

$$E\Big[[e^{j(\Phi(t+\tau)-\Phi(t))}]\Big]$$

Since $\Phi(t)$ is a zero mean stationary Gaussian process with the autocorrelation function denoted by $R_\Phi(\tau)$, we conclude that for fixed t and τ, the random variable $Z(t,\tau) = \Phi(t+\tau) - \Phi(t)$ is a zero mean Gaussian random variable since it is a linear combination of two jointly Gaussian random variables. The variance of this random variable is easily computed to be

$$\sigma_Z^2 = 2R_\Phi(0) - 2R_\Phi(\tau)$$

Now we have

$$\begin{aligned} E[[e^{j(\Phi(t+\tau)-\Phi(t))}]] &= E[e^{jZ(t,\tau)}] \\ &= e^{-\frac{1}{2}\sigma_Z^2} \\ &= e^{-(R_\Phi(0)-R_\Phi(\tau))} \end{aligned}$$

where we have used the fact that the characteristic function of a zero mean Gaussian random variable is given by

$$E[e^{j\omega X}] = e^{-\frac{1}{2}\omega^2\sigma_X^2}$$

Substituting we obtain

$$\begin{aligned} \bar{R}_U(\tau) &= \frac{A_c^2}{2} \lim_{T\to\infty} \frac{1}{T} \int_{T/2}^{T/2} \mathrm{Re}\left[e^{j2\pi f_c\tau} e^{-(R_\Phi(0)-R_\Phi(\tau))}\right] dt \\ &= \frac{A_c^2}{2} \cos(2\pi f_c\tau) e^{-(R_\Phi(0)-R_\Phi(\tau))} \\ &= \frac{A_c^2}{2} \cos(2\pi f_c\tau) g(\tau) \end{aligned}$$

where, by definition,

$$g(\tau) = e^{-(R_\Phi(0)-R_\Phi(\tau))}$$

3. Now we can obtain the power spectral density of the modulated process $U(t)$ by taking the Fourier transform of $\bar{R}_U(\tau)$.

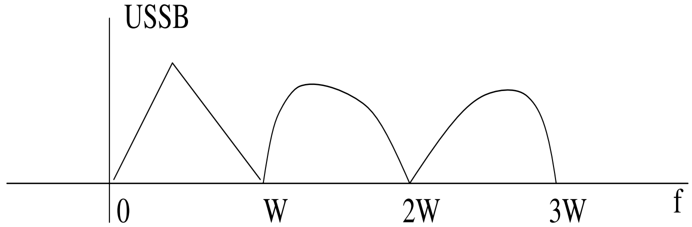
$$\begin{aligned} \mathcal{S}_U(f) &= \mathcal{F}[\frac{A_c^2}{2}\cos(2\pi f_c\tau)g(\tau)] \\ &= \frac{A_c^2}{4}[G(f-f_c)+G(f+f_c)] \end{aligned}$$

where

$$G(f) = e^{-R_\Phi(0)}\mathcal{F}[e^{R_\Phi(\tau)}]$$

Problem 5.13

1) In the next figure we plot a typical USSB spectrum for $K = 3$. Note that only the positive frequency axis is shown.



2) The bandwidth of the signal $m(t)$ is $W_m = KW$.

3) The noise power at the output of the LPF of the FM demodulator is

$$P_{n,o} = \int_{-W_m}^{W_m} \mathcal{S}_{n,o}(f)df = \frac{2N_0W_m^3}{3A_c^2} = \frac{2N_0W^3}{3A_c^2}K^3$$

where A_c is the amplitude of the FM signal. As it is observed the power of the noise that enters the USSB demodulators is proportional to the cube of the number of multiplexed signals.

The i^{th} message USSB signal occupies the frequency band $[(i-1)W, iW]$. Since the power spectral density of the noise at the output of the FM demodulator is $\mathcal{S}_{n,o}(f) = \frac{N_0}{A_c^2} f^2$, we conclude that the noise power at the output of the i^{th} USSB demodulator is

$$P_{n,o_i} = \frac{1}{4} P_{n_i} = \frac{1}{4} 2 \int_{-(i-1)W}^{iW} \frac{N_0}{A_c^2} f^2 df = \frac{N_0}{2A_c^2} \frac{1}{3} f^3 \Big|_{-(i-1)W}^{iW} = \frac{N_0 W^3}{6A_c^2}(3i^2 - 3i + 1)$$

Hence, the noise power at the output of the i^{th} USSB demodulator depends on i.

4) Using the results of the previous part, we obtain

$$\frac{P_{n,o_i}}{P_{n,o_j}} = \frac{3i^2 - 3i + 1}{3j^2 - 3j + 1}$$

5) The output signal power of the i^{th} USSB demodulator is $P_{s_i} = \frac{A_i^2}{4} P_{M_i}$. Hence, the SNR at the output of the i^{th} demodulator is

$$\text{SNR}_i = \frac{\frac{A_i^2}{4} P_{M_i}}{\frac{N_0 W^3}{6A_c^2}(3i^2 - 3i + 1)}$$

Assuming that P_{M_i} is the same for all i, then in order to guarantee a constant SNR_i we have to select A_i^2 proportional to $3i^2 - 3i + 1$.

Problem 5.14

1) The power is given by

$$P = \frac{V^2}{R}$$

Hence, with $R = 50$, $P = 20$, we obtain

$$V^2 = PR = 20 \times 50 = 1000 \Longrightarrow V = 1000^{\frac{1}{2}} = 31.6228 \text{ Volts}$$

2) The current through the load resistance is

$$i = \frac{V}{R} = \frac{31.6228}{50} = 0.6325 \text{ Amp}$$

3) The dBm unit is defined as

$$\text{dBm} = 10 \log \left(\frac{\text{actual power in Watts}}{10^{-3}} \right) = 30 + 10 \log(\text{actual power in Watts})$$

Hence,

$$P = 30 + 10 \log(50) = 46.9897 \text{ dBm}$$

Problem 5.15

1) The overall loss in 200 Km is $200 \times 20 = 400$ dB. Since the line is loaded with the characteristic impedance, the delivered power to the line is twice the power delivered to the load in absence of line loss. Hence, the required power is $20 + 400 = 420$ dBm.

2) Each repeater provides a gain of 20 dB, therefore the spacing between two adjacent receivers can be up to $20/2 = 10$ Km to attain a constant signal level at the input of all repeaters. This means that a total of $200/10 = 20$ repeaters are required.

Problem 5.16

1) Since the noise figure is 2 dB, we have

$$10 \log \left(1 + \frac{\mathcal{T}_e}{290}\right) = 2$$

and therefore $\mathcal{T}_e = 169.62^\circ$ K.

2) To determine the output power we have

$$P_{no} = \mathcal{G} k B_{\text{neq}}(\mathcal{T} + \mathcal{T}_e)$$

where $10 \log \mathcal{G} = 35$, and therefore, $\mathcal{G} = 10^3.5 = 3162$. From this we obtain

$$P_{no} = 3162 \times 1.38 \times 10^{-23} \times 10 \times 10^6 (169.62 + 50) = 9.58 \times 10^{-11} \quad \text{Watts} \sim -161.6 \quad \text{dBm}$$

Problem 5.17

Using the relation $P_{n_o} = \mathcal{G} k B_{neq}(\mathcal{T} + \mathcal{T}_e)$ with $P_{n_o} = 10^8 k T_0$, $B_{neq} = 25 \times 10^3$, $\mathcal{G} = 10^3$ and $\mathcal{T} = T_0$, we obtain

$$(10^8 - 25 \times 10^6) T_0 = 25 \times 10^6 \mathcal{T}_e \Longrightarrow \mathcal{T}_e = 3T_0$$

The noise figure of the amplifier is

$$F = \left(1 + \frac{\mathcal{T}_e}{\mathcal{T}}\right) = 1 + 3 = 4$$

Problem 5.18

The proof is by induction on m, the number of the amplifiers. We assume that the physical temperature $\mathcal{T}$ is the same for all the amplifiers. For $m = 2$, the overall gain of the cascade of the two amplifiers is $\mathcal{G} = \mathcal{G}_1 \mathcal{G}_2$, whereas the total noise at the output of the second amplifier is due to the source noise amplified by two stages, the first stage noise excess noise amplified by the second stage, and the second stage excess noise. Hence,

$$\begin{aligned} P_{n_2} &= \mathcal{G}_1 \mathcal{G}_2 P_{n_s} + \mathcal{G}_2 P_{n_i,1} + P_{n_i,2} \\ &= \mathcal{G}_1 \mathcal{G}_2 k \mathcal{T} B_{neq} + \mathcal{G}_2 (\mathcal{G}_1 k B_{neq} \mathcal{T}_{e_1}) + \mathcal{G}_2 k B_{neq} \mathcal{T}_{e_2} \end{aligned}$$

The noise of a single stage model with effective noise temperature $\mathcal{T}_e$, and gain $\mathcal{G}_1 \mathcal{G}_2$ is

$$P_n = \mathcal{G}_1 \mathcal{G}_2 k B_{neq} (\mathcal{T} + \mathcal{T}_e)$$

Equating the two expressions for the output noise we obtain

$$\mathcal{G}_1 \mathcal{G}_2 (\mathcal{T} + \mathcal{T}_e) = \mathcal{G}_1 \mathcal{G}_2 \mathcal{T} + \mathcal{G}_1 \mathcal{G}_2 \mathcal{T}_{e_1} + \mathcal{G}_2 \mathcal{T}_{e_2}$$

or

$$\mathcal{T}_e = \mathcal{T}_{e_1} + \frac{\mathcal{T}_{e_2}}{\mathcal{G}_1}$$

Assume now that if the number of the amplifiers is $m - 1$, then

$$\mathcal{T}_e' = \mathcal{T}_{e_1} + \frac{\mathcal{T}_{e_2}}{\mathcal{G}_1} + \cdots \frac{\mathcal{T}_{e_{m-1}}}{\mathcal{G}_1 \cdots \mathcal{G}_{m-2}}$$

Then for the cascade of m amplifiers

$$\mathcal{T}_e = \mathcal{T}_e' + \frac{\mathcal{T}_{e_m}}{\mathcal{G}'}$$

where $\mathcal{G}' = \mathcal{G}_1 \cdots \mathcal{G}_{m-1}$ is the gain of the $m - 1$ amplifiers and we have used the results for $m = 2$. Thus,

$$\mathcal{T}_e = \mathcal{T}_{e_1} + \frac{\mathcal{T}_{e_2}}{\mathcal{G}_1} + \cdots \frac{\mathcal{T}_{e_{m-1}}}{\mathcal{G}_1 \cdots \mathcal{G}_{m-2}} + \frac{\mathcal{T}_{e_m}}{\mathcal{G}_1 \cdots \mathcal{G}_{m-1}}$$

Chapter 6

Problem 6.1

$$\begin{aligned} H(X) &= -\sum_{i=1}^{6} p_i \log_2 p_i = -(0.1\log_2 0.1 + 0.2\log_2 0.2 \\ &\quad +0.3\log_2 0.3 + 0.05\log_2 0.05 + 0.15\log_2 0.15 + 0.2\log_2 0.2) \\ &= 2.4087 \quad \text{bits/symbol} \end{aligned}$$

If the source symbols are equiprobable, then $p_i = \frac{1}{6}$ and

$$H_u(X) = -\sum_{i=1}^{6} p_i \log_2 p_i = -\log_2 \frac{1}{6} = \log_2 6 = 2.5850 \quad \text{bits/symbol}$$

As it is observed the entropy of the source is less than that of a uniformly distributed source.

Problem 6.2
If the source is uniformly distributed with size N, then $p_i = \frac{1}{N}$ for $i = 1, \ldots, N$. Hence,

$$\begin{aligned} H(X) &= -\sum_{i=1}^{N} p_i \log_2 p_i = -\sum_{i=1}^{N} \frac{1}{N} \log_2 \frac{1}{N} \\ &= -\frac{1}{N} N \log_2 \frac{1}{N} = \log_2 N \end{aligned}$$

Problem 6.3

$$H(X) = -\sum_i p_i \log p_i = \sum_i p_i \log \frac{1}{p_i}$$

By definition the probabilities p_i satisfy $0 < p_i \leq 1$ so that $\frac{1}{p_i} \geq 1$ and $\log \frac{1}{p_i} \geq 0$. It turns out that each term under summation is positive and thus $H(X) \geq 0$. If X is deterministic, then $p_k = 1$ for some k and $p_i = 0$ for all $i \neq k$. Hence,

$$H(X) = -\sum_i p_i \log p_i = -p_k \log 1 = -p_k 0 = 0$$

Note that $\lim_{x \to 0} x \log x = 0$ so if we allow source symbols with probability zero, they contribute nothing in the entropy.

Problem 6.4
1)

$$\begin{aligned} H(X) &= -\sum_{k=1}^{\infty} p(1-p)^{k-1} \log_2(p(1-p)^{k-1}) \\ &= -p\log_2(p) \sum_{k=1}^{\infty} (1-p)^{k-1} - p\log_2(1-p) \sum_{k=1}^{\infty} (k-1)(1-p)^{k-1} \\ &= -p\log_2(p) \frac{1}{1-(1-p)} - p\log_2(1-p) \frac{1-p}{(1-(1-p))^2} \\ &= -\log_2(p) - \frac{1-p}{p} \log_2(1-p) \end{aligned}$$

2) Clearly $p(X = k|X > K) = 0$ for $k \leq K$. If $k > K$, then

$$p(X = k|X > K) = \frac{p(X = k, X > K)}{p(X > K)} = \frac{p(1-p)^{k-1}}{p(X > K)}$$

But,

$$\begin{aligned}
p(X > K) &= \sum_{k=K+1}^{\infty} p(1-p)^{k-1} = p\left(\sum_{k=1}^{\infty}(1-p)^{k-1} - \sum_{k=1}^{K}(1-p)^{k-1}\right) \\
&= p\left(\frac{1}{1-(1-p)} - \frac{1-(1-p)^K}{1-(1-p)}\right) = (1-p)^K
\end{aligned}$$

so that

$$p(X = k|X > K) = \frac{p(1-p)^{k-1}}{(1-p)^K}$$

If we let $k = K + l$ with $l = 1, 2, \ldots$, then

$$p(X = k|X > K) = \frac{p(1-p)^K(1-p)^{l-1}}{(1-p)^K} = p(1-p)^{l-1}$$

that is $p(X = k|X > K)$ is the geometrically distributed. Hence, using the results of the first part we obtain

$$\begin{aligned}
H(X|X > K) &= -\sum_{l=1}^{\infty} p(1-p)^{l-1}\log_2(p(1-p)^{l-1}) \\
&= -\log_2(p) - \frac{1-p}{p}\log_2(1-p)
\end{aligned}$$

Problem 6.5

$$\begin{aligned}
H(X,Y) &= H(X, g(X)) = H(X) + H(g(X)|X) \\
&= H(g(X)) + H(X|g(X))
\end{aligned}$$

But, $H(g(X)|X) = 0$, since $g(\cdot)$ is deterministic. Therefore,

$$H(X) = H(g(X)) + H(X|g(X))$$

Since each term in the previous equation is non-negative we obtain

$$H(X) \geq H(g(X))$$

Equality holds when $H(X|g(X)) = 0$. This means that the values $g(X)$ uniquely determine X, or that $g(\cdot)$ is a one to one mapping.

Problem 6.6
The entropy of the source is

$$H(X) = -\sum_{i=1}^{6} p_i \log_2 p_i = 2.4087 \quad \text{bits/symbol}$$

The sampling rate is

$$f_s = 2000 + 2 \cdot 6000 = 14000 \quad \text{Hz}$$

This means that 14000 samples are taken per each second. Hence, the entropy of the source in bits per second is given by

$$H(X) = 2.4087 \times 14000 \ \text{(bits/symbol)} \times \text{(symbols/sec)} = 33721.8 \ \text{bits/second}$$

Problem 6.7

Consider the function $f(x) = x - 1 - \ln x$. For $x > 1$,

$$\frac{df(x)}{dx} = 1 - \frac{1}{x} > 0$$

Thus, the function is monotonically increasing. Since, $f(1) = 0$, the latter implies that if $x > 1$ then, $f(x) > f(1) = 0$ or $\ln x < x - 1$. If $0 < x < 1$, then

$$\frac{df(x)}{dx} = 1 - \frac{1}{x} < 0$$

which means that the function is monotonically decreasing. Hence, for $x < 1$, $f(x) > f(1) = 0$ or $\ln x < x - 1$. Therefore, for every $x > 0$,

$$\ln x \leq x - 1$$

with equality if $x = 0$. Applying the inequality with $x = \frac{1/N}{p_i}$, we obtain

$$\ln \frac{1}{N} - \ln p_i \leq \frac{1/N}{p_i} - 1$$

Multiplying the previous by p_i and adding, we obtain

$$\sum_{i=1}^{N} p_i \ln \frac{1}{N} - \sum_{i=1}^{N} p_i \ln p_i \leq \sum_{i=1}^{N} \left(\frac{1}{N} - p_i \right) = 0$$

Hence,

$$H(X) \leq -\sum_{i=1}^{N} p_i \ln \frac{1}{N} = \ln N \sum_{i=1}^{N} p_i = \ln N$$

But, $\ln N$ is the entropy (in nats/symbol) of the source when it is uniformly distributed (see Problem 6.2). Hence, for equiprobable symbols the entropy of the source achieves its maximum.

Problem 6.8

Suppose that q_i is a distribution over $1, 2, 3, \ldots$ and that

$$\sum_{i=1}^{\infty} i q_i = m$$

Let $v_i = \frac{1}{q_i m} \left(1 - \frac{1}{m}\right)^{i-1}$ and apply the inequality $\ln x \leq x - 1$ to v_i. Then,

$$\ln \left[\frac{1}{m} \left(1 - \frac{1}{m}\right)^{i-1} \right] - \ln q_i \leq \frac{1}{q_i m} \left(1 - \frac{1}{m}\right)^{i-1} - 1$$

Multiplying the previous by q_i and adding, we obtain

$$\sum_{i=1}^{\infty} q_i \ln \left[\frac{1}{m} \left(1 - \frac{1}{m}\right)^{i-1} \right] - \sum_{i=1}^{\infty} q_i \ln q_i \leq \sum_{i=1}^{\infty} \frac{1}{m} (1 - \frac{1}{m})^{i-1} - \sum_{i=1}^{\infty} q_i = 0$$

But,

$$\begin{aligned}
\sum_{i=1}^{\infty} q_i \ln\left[\frac{1}{m}\left(1-\frac{1}{m}\right)^{i-1}\right] &= \sum_{i=1}^{\infty} q_i \left[\ln(\frac{1}{m}) + (i-1)\ln(1-\frac{1}{m})\right] \\
&= \ln(\frac{1}{m}) + \ln(1-\frac{1}{m})\sum_{i=1}^{\infty}(i-1)q_i \\
&= \ln(\frac{1}{m}) + \ln(1-\frac{1}{m})\left[\sum_{i=1}^{\infty} iq_i - \sum_{i=1}^{\infty} q_i\right] \\
&= \ln(\frac{1}{m}) + \ln(1-\frac{1}{m})(m-1) = -H(\mathbf{p})
\end{aligned}$$

where $H(\mathbf{p})$ is the entropy of the geometric distribution (see Problem 6.4). Hence,

$$-H(\mathbf{p}) - \sum_{i=1}^{\infty} q_i \ln q_i \leq 0 \Longrightarrow H(\mathbf{q}) \leq H(\mathbf{p})$$

Problem 6.9

The marginal probabilities are given by

$$\begin{aligned}
p(X=0) &= \sum_k p(X=0, Y=k) = p(X=0,Y=0) + p(X=0,Y=1) = \frac{2}{3} \\
p(X=1) &= \sum_k p(X=1, Y=k) = p(X=1,Y=1) = \frac{1}{3} \\
p(Y=0) &= \sum_k p(X=k, Y=0) = p(X=0,Y=0) = \frac{1}{3} \\
p(Y=1) &= \sum_k p(X=k, Y=1) = p(X=0,Y=1) + p(X=1,Y=1) = \frac{2}{3}
\end{aligned}$$

Hence,

$$\begin{aligned}
H(X) &= -\sum_{i=0}^{1} p_i \log_2 p_i = -(\frac{1}{3}\log_2\frac{1}{3} + \frac{1}{3}\log_2\frac{1}{3}) = .9183 \\
H(X) &= -\sum_{i=0}^{1} p_i \log_2 p_i = -(\frac{1}{3}\log_2\frac{1}{3} + \frac{1}{3}\log_2\frac{1}{3}) = .9183 \\
H(X,Y) &= -\sum_{i=0}^{2} \frac{1}{3}\log_2\frac{1}{3} = 1.5850 \\
H(X|Y) &= H(X,Y) - H(Y) = 1.5850 - 0.9183 = 0.6667 \\
H(Y|X) &= H(X,Y) - H(X) = 1.5850 - 0.9183 = 0.6667
\end{aligned}$$

Problem 6.10

$$H(Y|X) = -\sum_{x,y} p(x,y)\log p(y|x)$$

But, $p(y|x) = p(g(x)|x) = 1$. Hence, $\log p(g(x)|x) = 0$ and $H(Y|X) = 0$.

Problem 6.11

1)

$$\begin{aligned}
H(X) &= -(.05\log_2 .05 + .1\log_2 .1 + .1\log_2 .1 + .15\log_2 .15 \\
&\quad +.05\log_2 .05 + .25\log_2 .25 + .3\log_2 .3) = 2.5282
\end{aligned}$$

2) After quantization, the new alphabet is $B = \{-4, 0, 4\}$ and the corresponding symbol probabilities are given by

$$\begin{aligned} p(-4) &= p(-5) + p(-3) = .05 + .1 = .15 \\ p(0) &= p(-1) + p(0) + p(1) = .1 + .15 + .05 = .3 \\ p(4) &= p(3) + p(5) = .25 + .3 = .55 \end{aligned}$$

Hence, $H(Q(X)) = 1.4060$. As it is observed quantization decreases the entropy of the source.

Problem 6.12

Using the first definition of the entropy rate, we have

$$\begin{aligned} H &= \lim_{n\to\infty} H(X_n|X_1, \ldots X_{n-1}) \\ &= \lim_{n\to\infty} \left(H(X_1, X_2, \ldots, X_n) - H(X_1, X_2, \ldots, X_{n-1})\right) \end{aligned}$$

However, X_1, X_2, $\ldots X_n$ are independent, so that

$$H = \lim_{n\to\infty} \left(\sum_{i=1}^{n} H(X_i) - \sum_{i=1}^{n-1} H(X_i) \right) = \lim_{n\to\infty} H(X_n) = H(X)$$

where the last equality follows from the fact that $X_1, \ldots, X_n$ are identically distributed.

Using the second definition of the entropy rate, we obtain

$$\begin{aligned} H &= \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \ldots, X_n) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(X_i) \\ &= \lim_{n\to\infty} \frac{1}{n} n H(X) = H(X) \end{aligned}$$

The second line of the previous relation follows from the independence of X_1, X_2, $\ldots X_n$, whereas the third line from the fact that for a DMS the random variables $X_1, \ldots X_n$ are identically distributed independent of n.

Problem 6.13

$$\begin{aligned} H &= \lim_{n\to\infty} H(X_n|X_1, \ldots, X_{n-1}) \\ &= \lim_{n\to\infty} \left[-\sum_{x_1,\ldots,x_n} p(x_1, \ldots, x_n) \log_2 p(x_n|x_1, \ldots, x_{n-1}) \right] \\ &= \lim_{n\to\infty} \left[-\sum_{x_1,\ldots,x_n} p(x_1, \ldots, x_n) \log_2 p(x_n|x_{n-1}) \right] \\ &= \lim_{n\to\infty} \left[-\sum_{x_n,x_{n-1}} p(x_n, x_{n-1}) \log_2 p(x_n|x_{n-1}) \right] \\ &= \lim_{n\to\infty} H(X_n|X_{n-1}) \end{aligned}$$

However, for a stationary process $p(x_n, x_{n-1})$ and $p(x_n|x_{n-1})$ are independent of n, so that

$$H = \lim_{n\to\infty} H(X_n|X_{n-1}) = H(X_n|X_{n-1})$$

Problem 6.14

$$
\begin{aligned}
H(X|Y) &= -\sum_{x,y} p(x,y)\log p(x|y) = -\sum_{x,y} p(x|y)p(y)\log p(x|y) \\
&= \sum_{y} p(y)\left(-\sum_{x} p(x|y)\log p(x|y)\right) = \sum_{y} p(y)H(X|Y=y)
\end{aligned}
$$

Problem 6.15

1) The marginal distribution $p(x)$ is given by $p(x) = \sum_y p(x,y)$. Hence,

$$
\begin{aligned}
H(X) &= -\sum_{x} p(x)\log p(x) = -\sum_{x}\sum_{y} p(x,y)\log p(x) \\
&= -\sum_{x,y} p(x,y)\log p(x)
\end{aligned}
$$

Similarly it is proved that $H(Y) = -\sum_{x,y} p(x,y)\log p(y)$.

2) Using the inequality $\ln w \le w - 1$ with $w = \frac{p(x)p(y)}{p(x,y)}$, we obtain

$$
\ln\frac{p(x)p(y)}{p(x,y)} \le \frac{p(x)p(y)}{p(x,y)} - 1
$$

Multiplying the previous by $p(x,y)$ and adding over x, y, we obtain

$$
\sum_{x,y} p(x,y)\ln p(x)p(y) - \sum_{x,y} p(x,y)\ln p(x,y) \le \sum_{x,y} p(x)p(y) - \sum_{x,y} p(x,y) = 0
$$

Hence,

$$
\begin{aligned}
H(X,Y) &\le -\sum_{x,y} p(x,y)\ln p(x)p(y) = -\sum_{x,y} p(x,y)(\ln p(x) + \ln p(y)) \\
&= -\sum_{x,y} p(x,y)\ln p(x) - \sum_{x,y} p(x,y)\ln p(y) = H(X) + H(Y)
\end{aligned}
$$

Equality holds when $\frac{p(x)p(y)}{p(x,y)} = 1$, i.e when X, Y are independent.

Problem 6.16

$$
H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)
$$

Also, from Problem 6.15, $H(X,Y) \le H(X) + H(Y)$. Combining the two relations, we obtain

$$
H(Y) + H(X|Y) \le H(X) + H(Y) \Longrightarrow H(X|Y) \le H(X)
$$

Suppose now that the previous relation holds with equality. Then,

$$
-\sum_{x} p(x)\log p(x|y) = -\sum_{x} p(x)\log p(x) \Rightarrow \sum_{x} p(x)\log\left(\frac{p(x)}{p(x|y)}\right) = 0
$$

However, $p(x)$ is always greater or equal to $p(x|y)$, so that $\log(p(x)/p(x|y))$ is non-negative. Since $p(x) > 0$, the above equality holds if and only if $\log(p(x)/p(x|y)) = 0$ or equivalently if and only if $p(x)/p(x|y) = 1$. This implies that $p(x|y) = p(x)$ meaning that X and Y are independent.

Problem 6.17

To show that $\mathbf{q} = \lambda \mathbf{p}_1 + \bar{\lambda}\mathbf{p}_2$ is a legitimate probability vector we have to prove that $0 \leq q_i \leq 1$ and $\sum_i q_i = 1$. Clearly $0 \leq p_{1,i} \leq 1$ and $0 \leq p_{2,i} \leq 1$ so that

$$0 \leq \lambda p_{1,i} \leq \lambda, \qquad 0 \leq \bar{\lambda} p_{2,i} \leq \bar{\lambda}$$

If we add these two inequalities, we obtain

$$0 \leq q_i \leq \lambda + \bar{\lambda} \Longrightarrow 0 \leq q_i \leq 1$$

Also,

$$\sum_i q_i = \sum_i (\lambda p_{1,i} + \bar{\lambda} p_{2,i}) = \lambda \sum_i p_{1,i} + \bar{\lambda} \sum_i p_{2,i} = \lambda + \bar{\lambda} = 1$$

Before we prove that $H(X)$ is a concave function of the probability distribution on $\mathcal{X}$ we show that $\ln x \geq 1 - \frac{1}{x}$. Since $\ln y \leq y - 1$, we set $x = \frac{1}{y}$ so that $-\ln x \leq \frac{1}{x} - 1 \Rightarrow \ln x \geq 1 - \frac{1}{x}$. Equality holds when $y = \frac{1}{x} = 1$ or else if $x = 1$.

$$\begin{aligned}
&H(\lambda \mathbf{p}_1 + \bar{\lambda}\mathbf{p}_2) - \lambda H(\mathbf{p}_1) - \bar{\lambda} H(\mathbf{p}_2) \\
&= \lambda \sum_i p_{1,i} \log \left(\frac{p_{1,i}}{\lambda p_{1,i} + \bar{\lambda} p_{2,i}} \right) + \bar{\lambda} \sum_i p_{2,i} \log \left(\frac{p_{2,i}}{\lambda p_{1,i} + \bar{\lambda} p_{2,i}} \right) \\
&\geq \lambda \sum_i p_{1,i} \left(1 - \frac{\lambda p_{1,i} + \bar{\lambda} p_{2,i}}{p_{1,i}} \right) + \bar{\lambda} \sum_i p_{2,i} \left(1 - \frac{\lambda p_{1,i} + \bar{\lambda} p_{2,i}}{p_{2,i}} \right) \\
&= \lambda(1-1) + \bar{\lambda}(1-1) = 0
\end{aligned}$$

Hence,

$$\lambda H(\mathbf{p}_1) + \bar{\lambda} H(\mathbf{p}_2) \leq H(\lambda \mathbf{p}_1 + \bar{\lambda}\mathbf{p}_2)$$

Problem 6.18

Let $p_i(x_i)$ be the marginal distribution of the random variable X_i. Then,

$$\begin{aligned}
\sum_{i=1}^n H(X_i) &= \sum_{i=1}^n \left[-\sum_{x_i} p_i(x_i) \log p_i(x_i) \right] \\
&= -\sum_{x_1} \sum_{x_2} \cdots \sum_{x_n} p(x_1, x_2, \cdots, x_n) \log \left(\prod_{i=1}^n p_i(x_i) \right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^n H(X_i) - H(X_1, X_2, \cdots X_n) \\
&= \sum_{x_1} \sum_{x_2} \cdots \sum_{x_n} p(x_1, x_2, \cdots, x_n) \log \left(\frac{p(x_1, x_2, \cdots, x_n)}{\prod_{i=1}^n p_i(x_i)} \right) \\
&\geq \sum_{x_1} \sum_{x_2} \cdots \sum_{x_n} p(x_1, x_2, \cdots, x_n) \left(1 - \frac{\prod_{i=1}^n p_i(x_i)}{p(x_1, x_2, \cdots, x_n)} \right) \\
&= \sum_{x_1} \sum_{x_2} \cdots \sum_{x_n} p(x_1, x_2, \cdots, x_n) - \sum_{x_1} \sum_{x_2} \cdots \sum_{x_n} p_1(x_1) p_2(x_2) \cdots p_n(x_n) \\
&= 1 - 1 = 0
\end{aligned}$$

where we have used the inequality $\ln x \geq 1 - \frac{1}{x}$ (see Problem 6.17.) Hence,

$$H(X_1, X_2, \cdots X_n) \leq \sum_{i=1}^{n} H(X_i)$$

with equality if $\prod_{i=1}^{n} p_i(x_i) = p(x_1, \cdots, x_n)$, i.e. a memoryless source.

Problem 6.19

1) The probability of an all zero sequence is

$$p(X_1 = 0, X_2 = 0, \cdots, X_n = 0) = p(X_1 = 0)p(X_2 = 0) \cdots p(X_n = 0) = \left(\frac{1}{2}\right)^n$$

2) Similarly with the previous case

$$p(X_1 = 1, X_2 = 1, \cdots, X_n = 1) = p(X_1 = 1)p(X_2 = 1) \cdots p(X_n = 1) = \left(\frac{1}{2}\right)^n$$

3)

$$\begin{aligned} & p(X_1 = 1, \cdots, X_k = 1, X_{k+1} = 0, \cdots X_n = 0) \\ &= p(X_1 = 1) \cdots p(X_k = 1)p(X_{k+1} = 0) \cdots p(X_n = 0) \\ &= \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} = \left(\frac{1}{2}\right)^n \end{aligned}$$

4) The number of zeros or ones follows the binomial distribution. Hence

$$p(k \text{ ones }) = \left(\begin{array}{c} n \\ k \end{array}\right) \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} = \left(\begin{array}{c} n \\ k \end{array}\right) \left(\frac{1}{2}\right)^n$$

5) In case that $p(X_i = 1) = p$, the answers of the previous questions change as follows

$$\begin{aligned} p(X_1 = 0, X_2 = 0, \cdots, X_n = 0) &= (1-p)^n \\ p(X_1 = 1, X_2 = 1, \cdots, X_n = 1) &= p^n \\ p(\text{first } k \text{ ones, next } n-k \text{ zeros}) &= p^k (1-p)^{n-k} \\ p(k \text{ ones }) &= \left(\begin{array}{c} n \\ k \end{array}\right) p^k (1-p)^{n-k} \end{aligned}$$

Problem 6.20

From the discussion in the beginning of Section 6.2 it follows that the total number of sequences of length n of a binary DMS source producing the symbols 0 and 1 with probability p and $1-p$ respectively is $2^{nH(p)}$. Thus if $p = 0.3$, we will observe sequences having $np = 3000$ zeros and $n(1-p) = 7000$ ones. Therefore,

$$\# \text{ sequences with 3000 zeros } \approx 2^{8813}$$

Another approach to the problem is via the Stirling's approximation. In general the number of binary sequences of length n with k zeros and $n-k$ ones is the binomial coefficient

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{k!(n-k)!}$$

To get an estimate when n and k are large numbers we can use Stirling's approximation

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Hence,

$$\# \text{ sequences with 3000 zeros } = \frac{10000!}{3000!7000!} \approx \frac{1}{21\sqrt{2\pi 30 \cdot 70}} 10^{10000}$$

Problem 6.21

1) The total number of typical sequences is approximately $2^{nH(X)}$ where $n = 1000$ and

$$H(X) = -\sum_i p_i \log_2 p_i = 1.4855$$

Hence,

$$\# \text{ typical sequences } \approx 2^{1485.5}$$

2) The number of all sequences of length n is N^n, where N is the size of the source alphabet. Hence,

$$\frac{\# \text{ typical sequences}}{\# \text{ non-typical sequences}} \approx \frac{2^{nH(X)}}{N^n - 2^{nH(X)}} \approx 1.14510^{-30}$$

3) The typical sequences are almost equiprobable. Thus,

$$p(X = \mathbf{x},\ \mathbf{x} \text{ typical}) \approx 2^{-nH(X)} = 2^{-1485.5}$$

4) Since the number of the total sequences is $2^{nH(X)}$ the number of bits required to represent these sequences is $nH(X) \approx 1486$.

5) The most probable sequence is the one with all a_3's that is $\{a_3, a_3, \ldots, a_3\}$. The probability of this sequence is

$$p(\{a_3, a_3, \ldots, a_3\}) = \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{1000}$$

6) The most probable sequence of the previous question is not a typical sequence. In general in a typical sequence, symbol a_1 is repeated $1000p(a_1) = 200$ times, symbol a_2 is repeated approximately $1000p(a_2) = 300$ times and symbol a_3 is repeated almost $1000p(a_3) = 500$ times.

Problem 6.22

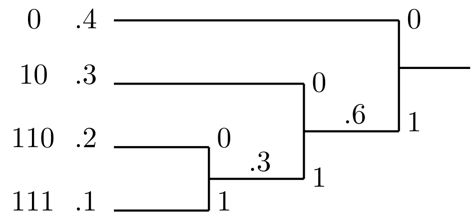
1) The entropy of the source is

$$H(X) = -\sum_{i=1}^{4} p(a_i)\log_2 p(a_i) = 1.8464 \quad \text{bits/output}$$

2) The average codeword length is lower bounded by the entropy of the source for error free reconstruction. Hence, the minimum possible average codeword length is $H(X) = 1.8464$.

3) The following figure depicts the Huffman coding scheme of the source. The average codeword length is

$$\bar{R}(X) = 3 \times (.2 + .1) + 2 \times .3 + .4 = 1.9$$

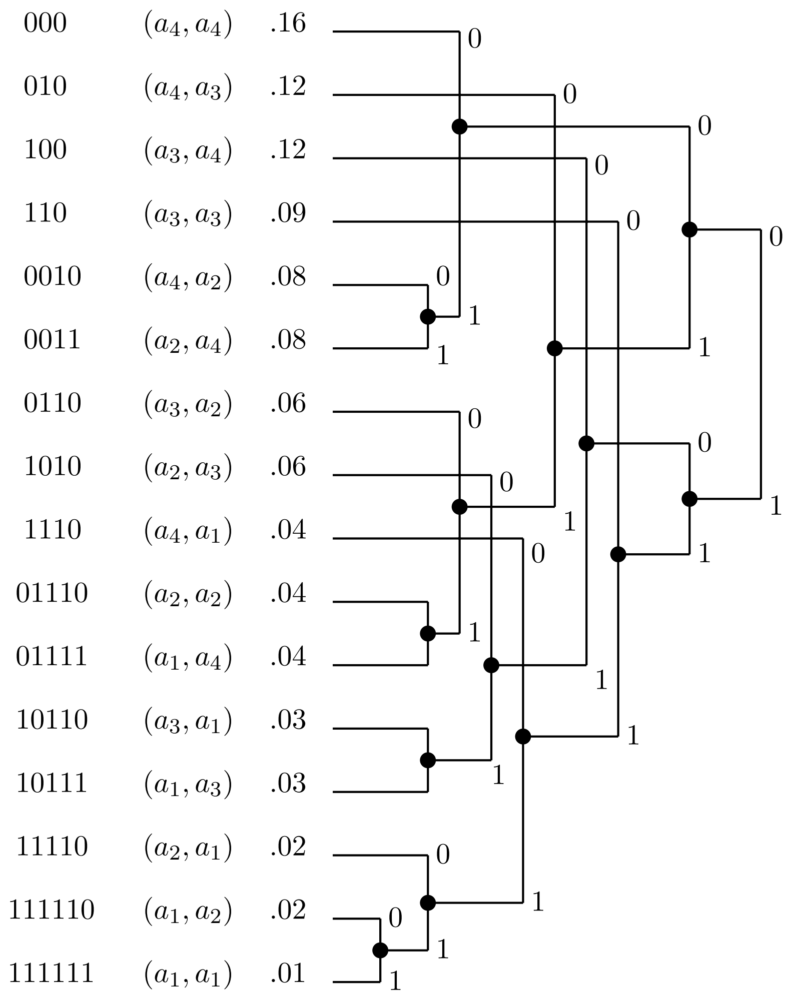


4) For the second extension of the source the alphabet of the source becomes $\mathcal{A}^2 = \{(a_1, a_1), (a_1, a_2), \ldots (a_4, a_4)\}$ and the probability of each pair is the product of the probabilities of each component, i.e. $p((a_1, a_2)) = .2$. A Huffman code for this source is depicted in the next figure. The average codeword length in bits per pair of source output is

$$\bar{R}_2(X) = 3 \times .49 + 4 \times .32 + 5 \times .16 + 6 \times .03 = 3.7300$$

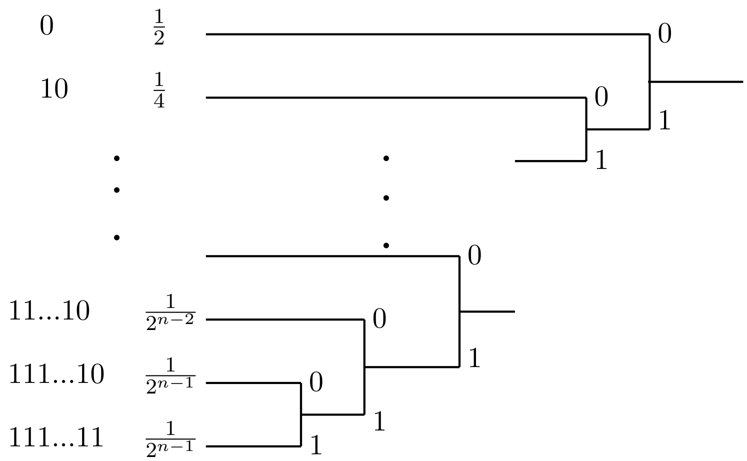
The average codeword length in bits per each source output is $\bar{R}_1(X) = \bar{R}_2(X)/2 = 1.865$.

5) Huffman coding of the original source requires 1.9 bits per source output letter whereas Huffman coding of the second extension of the source requires 1.865 bits per source output letter and thus it is more efficient.



Problem 6.23

The following figure shows the design of the Huffman code. Note that at each step of the algorithm the branches with the lowest probabilities (that merge together) are those at the bottom of the tree.



The entropy of the source is

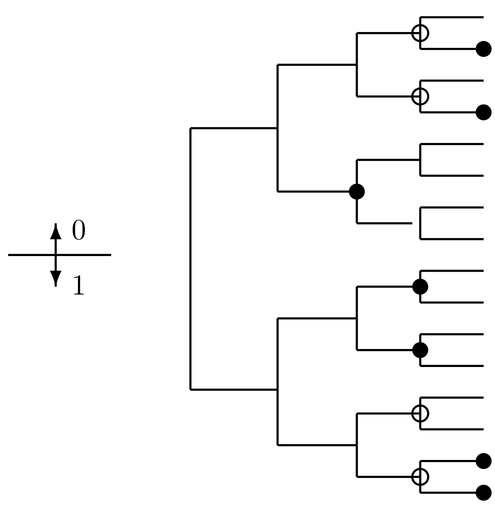
$$
\begin{aligned}
H(X) &= \sum_{i=1}^{n-1} \frac{1}{2^i} \log_2 2^i + \frac{1}{2^{n-1}} \log_2 2^{n-1} \\
&= \sum_{i=1}^{n-1} \frac{1}{2^i} i \log_2 2 + \frac{1}{2^{n-1}} (n-1) \log_2 2 \\
&= \sum_{i=1}^{n-1} \frac{i}{2^i} + \frac{n-1}{2^{n-1}}
\end{aligned}
$$

In the way that the code is constructed, the first codeword (0) has length one, the second codeword (10) has length two and so on until the last two codewords (111...10, 111...11) which have length $n-1$. Thus, the average codeword length is

$$
\begin{aligned}
\bar{R} &= \sum_{x \in \mathcal{X}} p(x) l(x) = \sum_{i=1}^{n-1} \frac{i}{2^i} + \frac{n-1}{2^{n-1}} \\
&= 2\left(1 - (1/2)^{n-1}\right) = H(X)
\end{aligned}
$$

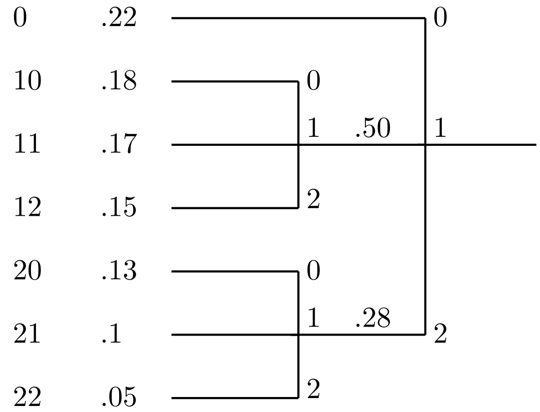
Problem 6.24

The following figure shows the position of the codewords (black filled circles) in a binary tree. Although the prefix condition is not violated the code is not optimum in the sense that it uses more bits that is necessary. For example the upper two codewords in the tree (0001, 0011) can be substituted by the codewords (000, 001) (un-filled circles) reducing in this way the average codeword length. Similarly codewords 1111 and 1110 can be substituted by codewords 111 and 110.



Problem 6.25

The following figure depicts the design of a ternary Huffman code.



The average codeword length is

$$\begin{aligned}\bar{R}(X) &= \sum_x p(x)l(x) = .22 + 2(.18 + .17 + .15 + .13 + .10 + .05)\\ &= 1.78 \quad \text{(ternary symbols/output)}\end{aligned}$$

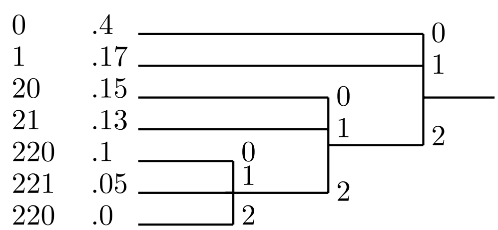
For a fair comparison of the average codeword length with the entropy of the source, we compute the latter with logarithms in base 3. Hence,

$$H(X) = -\sum_x p(x)\log_3 p(x) = 1.7047$$

As it is expected $H(X) \leq \bar{R}(X)$.

Problem 6.26

If D is the size of the code alphabet, then the Huffman coding scheme takes D source outputs and it merges them to 1 symbol. Hence, we have a decrease of output symbols by $D-1$. In K steps of the algorithm the decrease of the source outputs is $K(D-1)$. If the number of the source outputs is $K(D-1)+D$, for some K, then we are in a good position since we will be left with D symbols for which we assign the symbols $0, 1, \ldots, D-1$. To meet the above condition with a ternary code the number of the source outputs should be $2K+3$. In our case that the number of source outputs is six we can add a dummy symbol with zero probability so that $7 = 2 \cdot 2 + 3$. The following figure shows the design of the ternary Huffman code.



Problem 6.27

Parsing the sequence by the rules of the Lempel-Ziv coding scheme we obtain the phrases
0, 00, 1, 001, 000, 0001, 10, 00010, 0000, 0010, 00000, 101, 00001,
000000, 11, 01, 0000000, 110, 0, ...
The number of the phrases is 19. For each phrase we need 5 bits plus an extra bit to represent the new source output.

Dictionary Location	Dictionary Contents	Codeword
1 00001	0	00000 0
2 00010	00	00001 0
3 00011	1	00000 1
4 00100	001	00010 1
5 00101	000	00010 0
6 00110	0001	00101 1
7 00111	10	00011 0
8 01000	00010	00110 0
9 01001	0000	00101 0
10 01010	0010	00100 0
11 01011	00000	01001 0
12 01100	101	00111 1
13 01101	00001	01001 1
14 01110	000000	01011 0
15 01111	11	00011 1
16 10000	01	00001 1
17 10001	0000000	01110 0
18 10010	110	01111 0
19	0	00000

Problem 6.28

$$\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) \\
&= -\sum_x p(x)\log p(x) + \sum_{x,y} p(x,y)\log p(x|y) \\
&= -\sum_{x,y} p(x,y)\log p(x) + \sum_{x,y} p(x,y)\log p(x|y) \\
&= \sum_{x,y} p(x,y)\log\frac{p(x|y)}{p(x)} = \sum_{x,y} p(x,y)\log\frac{p(x,y)}{p(x)p(y)}
\end{aligned}$$

Using the inequality $\ln y \le y-1$ with $y=\frac{1}{x}$, we obtain $\ln x \ge 1-\frac{1}{x}$. Applying this inequality with $x=\frac{p(x,y)}{p(x)p(y)}$ we obtain

$$\begin{aligned}
I(X;Y) &= \sum_{x,y} p(x,y)\log\frac{p(x,y)}{p(x)p(y)} \\
&\ge \sum_{x,y} p(x,y)\left(1-\frac{p(x)p(y)}{p(x,y)}\right) = \sum_{x,y} p(x,y) - \sum_{x,y} p(x)p(y) = 0
\end{aligned}$$

$\ln x \ge 1-\frac{1}{x}$ holds with equality if $x=1$. This means that $I(X;Y)=0$ if $p(x,y)=p(x)p(y)$ or in other words if X and Y are independent.

Problem 6.29

1) $I(X;Y) = H(X) - H(X|Y)$. Since in general, $H(X|Y)\ge 0$, we have $I(X;Y)\le H(X)$. Also (see Problem 6.30), $I(X;Y) = H(Y)-H(Y|X)$ from which we obtain $I(X;Y)\le H(Y)$. Combining the two inequalities, we obtain

$$I(X;Y) \le \min\{H(X), H(Y)\}$$

2) It can be shown (see Problem 6.7), that if X and Z are two random variables over the same set $\mathcal{X}$ and Z is uniformly distributed, then $H(X)\le H(Z)$. Furthermore $H(Z)=\log|\mathcal{X}|$, where $|\mathcal{X}|$ is the size of the set $\mathcal{X}$ (see Problem 6.2). Hence, $H(X) \leq \log|\mathcal{X}|$ and similarly we can prove that $H(Y) \leq \log|\mathcal{Y}|$. Using the result of the first part of the problem, we obtain

$$I(X;Y) \leq \min\{H(X), H(Y)\} \leq \min\{\log|\mathcal{X}|, \log|\mathcal{Y}|\}$$

Problem 6.30

By definition $I(X;Y) = H(X) - H(X|Y)$ and $H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$. Combining the two equations we obtain

$$\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) = H(X) - (H(X,Y) - H(Y)) \\
&= H(X) + H(Y) - H(X,Y) = H(Y) - (H(X,Y) - H(X)) \\
&= H(Y) - H(Y|X) = I(Y;X)
\end{aligned}$$

Problem 6.31

1) The joint probability density is given by

$$\begin{aligned}
p(Y=1, X=0) &= p(Y=1|X=0)p(X=0) = \epsilon p \\
p(Y=0, X=1) &= p(Y=0|X=1)p(X=1) = \epsilon(1-p) \\
p(Y=1, X=1) &= (1-\epsilon)(1-p) \\
p(Y=0, X=0) &= (1-\epsilon)p
\end{aligned}$$

The marginal distribution of Y is

$$\begin{aligned}
p(Y=1) &= \epsilon p + (1-\epsilon)(1-p) = 1 + 2\epsilon p - \epsilon - p \\
p(Y=0) &= \epsilon(1-p) + (1-\epsilon)p = \epsilon + p - 2\epsilon p
\end{aligned}$$

Hence,

$$\begin{aligned}
H(X) &= -p\log_2 p - (1-p)\log_2(1-p) \\
H(Y) &= -(1+2\epsilon p - \epsilon - p)\log_2(1+2\epsilon p - \epsilon - p) \\
&\quad -(\epsilon + p - 2\epsilon p)\log_2(\epsilon + p - 2\epsilon p) \\
H(Y|X) &= -\sum_{x,y} p(x,y)\log_2(p(y|x)) = -\epsilon p\log_2\epsilon - \epsilon(1-p)\log_2\epsilon \\
&\quad -(1-\epsilon)(1-p)\log_2(1-\epsilon) - (1-\epsilon)p\log_2(1-\epsilon) \\
&= -\epsilon\log_2\epsilon - (1-\epsilon)\log_2(1-\epsilon) \\
H(X,Y) &= H(X) + H(Y|X) \\
&= -p\log_2 p - (1-p)\log_2(1-p) - \epsilon\log_2\epsilon - (1-\epsilon)\log_2(1-\epsilon) \\
H(X|Y) &= H(X,Y) - H(Y) \\
&= -p\log_2 p - (1-p)\log_2(1-p) - \epsilon\log_2\epsilon - (1-\epsilon)\log_2(1-\epsilon) \\
&\quad (1+2\epsilon p - \epsilon - p)\log_2(1+2\epsilon p - \epsilon - p) \\
&\quad +(\epsilon + p - 2\epsilon p)\log_2(\epsilon + p - 2\epsilon p) \\
I(X;Y) &= H(X) - H(X|Y) = H(Y) - H(Y|X) \\
&= \epsilon\log_2\epsilon + (1-\epsilon)\log_2(1-\epsilon) \\
&\quad -(1+2\epsilon p - \epsilon - p)\log_2(1+2\epsilon p - \epsilon - p) \\
&\quad -(\epsilon + p - 2\epsilon p)\log_2(\epsilon + p - 2\epsilon p)
\end{aligned}$$

2) The mutual information is $I(X;Y) = H(Y) - H(Y|X)$. As it was shown in the first question $H(Y|X) = -\epsilon\log_2\epsilon - (1-\epsilon)\log_2(1-\epsilon)$ and thus it does not depend on p. Hence, $I(X;Y)$

is maximized when $H(Y)$ is maximized. However, $H(Y)$ is the binary entropy function with probability $q = 1 + 2\epsilon p - \epsilon - p$, that is

$$H(Y) = H_b(q) = H_b(1 + 2\epsilon p - \epsilon - p)$$

$H_b(q)$ achieves its maximum value, which is one, for $q = \frac{1}{2}$. Thus,

$$1 + 2\epsilon p - \epsilon - p = \frac{1}{2} \Longrightarrow p = \frac{1}{2}$$

3) Since $I(X;Y) \geq 0$, the minimum value of $I(X;Y)$ is zero and it is obtained for independent X and Y. In this case

$$p(Y = 1, X = 0) = p(Y = 1)p(X = 0) \Longrightarrow \epsilon p = (1 + 2\epsilon p - \epsilon - p)p$$

or $\epsilon = \frac{1}{2}$. This value of epsilon also satisfies

$$\begin{aligned} p(Y = 0, X = 0) &= p(Y = 0)p(X = 0) \\ p(Y = 1, X = 1) &= p(Y = 1)p(X = 1) \\ p(Y = 0, X = 1) &= p(Y = 0)p(X = 1) \end{aligned}$$

resulting in independent X and Y.

Problem 6.32

$$\begin{aligned} I(X;YZW) &= I(YZW;X) = H(YZW) - H(YZW|X) \\ &= H(Y) + H(Z|Y) + H(W|YZ) \\ &\quad -[H(Y|X) + H(Z|XY) + H(W|XYZ)] \\ &= [H(Y) - H(Y|X)] + [H(Z|Y) - H(Z|YX)] \\ &\quad +[H(W|YZ) - H(W|XYZ)] \\ &= I(X;Y) + I(Z|Y;X) + I(W|ZY;X) \\ &= I(X;Y) + I(X;Z|Y) + I(X;W|ZY) \end{aligned}$$

This result can be interpreted as follows: The information that the triplet of random variables (Y, Z, W) gives about the random variable X is equal to the information that Y gives about X plus the information that Z gives about X, when Y is already known, plus the information that W provides about X when Z, Y are already known.

Problem 6.33

1) Using Bayes rule, we obtain $p(x, y, z) = p(z)p(x|z)p(y|x, z)$. Comparing this form with the one given in the first part of the problem we conclude that $p(y|x, z) = p(y|x)$. This implies that Y and Z are independent given X so that, $I(Y;Z|X) = 0$. Hence,

$$\begin{aligned} I(Y;ZX) &= I(Y;Z) + I(Y;X|Z) \\ &= I(Y;X) + I(Y;Z|X) = I(Y;X) \end{aligned}$$

Since $I(Y;Z) \geq 0$, we have

$$I(Y;X|Z) \leq I(Y;X)$$

2) Comparing $p(x, y, z) = p(x)p(y|x)p(z|x, y)$ with the given form of $p(x, y, z)$ we observe that $p(y|x) = p(y)$ or, in other words, random variables X and Y are independent. Hence,

$$\begin{aligned} I(Y;ZX) &= I(Y;Z) + I(Y;X|Z) \\ &= I(Y;X) + I(Y;Z|X) = I(Y;Z|X) \end{aligned}$$

Since in general $I(Y;X|Z) \geq 0$, we have

$$I(Y;Z) \leq I(Y;Z|X)$$

3) For the first case consider three random variables X, Y and Z, taking the values 0, 1 with equal probability and such that $X = Y = Z$. Then, $I(Y;X|Z) = H(Y|Z) - H(Y|ZX) = 0 - 0 = 0$, whereas $I(Y;X) = H(Y) - H(Y|X) = 1 - 0 = 1$. Hence, $I(Y;X|Z) < I(X;Y)$. For the second case consider two independent random variables X, Y, taking the values 0, 1 with equal probability and a random variable Z which is the sum of X and Y $(Z = X+Y.)$ Then, $I(Y;Z) = H(Y) - H(Y|Z) = 1 - 1 = 0$, whereas $I(Y;Z|X) = H(Y|X) - H(Y|ZX) = 1 - 0 = 1$. Thus, $I(Y;Z) < I(Y;Z|X)$.

Problem 6.34
1)

$$\begin{aligned} I(X;Y) &= H(X) - H(X|Y) \\ &= -\sum_x p(x)\log p(x) + \sum_x \sum_y p(x,y)\log p(x|y) \end{aligned}$$

Using Bayes formula we can write $p(x|y)$ as

$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{p(x)p(y|x)}{\sum_x p(x)p(y|x)}$$

Hence,

$$\begin{aligned} I(X;Y) &= -\sum_x p(x)\log p(x) + \sum_x \sum_y p(x,y)\log p(x|y) \\ &= -\sum_x p(x)\log p(x) + \sum_x \sum_y p(x)p(y|x)\log \frac{p(x)p(y|x)}{\sum_x p(x)p(y|x)} \\ &= \sum_x \sum_y p(x)p(y|x)\log \frac{p(y|x)}{\sum_x p(x)p(y|x)} \end{aligned}$$

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be given on $\mathcal{X}$ and let $\mathbf{p} = \lambda\mathbf{p}_1 + (1-\lambda)\mathbf{p}_2$. Then, $\mathbf{p}$ is a legitimate probability vector, for its elements $p(x) = \lambda p_1(x) + \bar{\lambda}p_2(x)$ are non-negative, less or equal to one and

$$\sum_x p(x) = \sum_x \lambda p_1(x) + \bar{\lambda}p_2(x) = \lambda\sum_x p_1(x) + \bar{\lambda}\sum_x p_2(x) = \lambda + \bar{\lambda} = 1$$

Furthermore,

$$\begin{aligned} &\lambda I(\mathbf{p_1};\mathbf{Q}) + \bar{\lambda} I(\mathbf{p_2};\mathbf{Q}) - I(\lambda\mathbf{p_1} + \bar{\lambda}\mathbf{p_2};\mathbf{Q}) \\ &= \lambda\sum_x \sum_y p_1(x)p(y|x)\log \frac{p(y|x)}{\sum_x p_1(x)p(y|x)} \\ &\quad +\bar{\lambda}\sum_x \sum_y p_2(x)p(y|x)\log \frac{p(y|x)}{\sum_x p_2(x)p(y|x)} \\ &\quad -\sum_x \sum_y (\lambda p_1(x) + \bar{\lambda}p_2(x))p(y|x)\log \frac{p(y|x)}{\sum_x (\lambda p_1(x) + \bar{\lambda}p_2(x))p(y|x)} \\ &= \sum_x \sum_y \lambda p_1(x)p(y|x)\log \frac{(\lambda p_1(x) + \bar{\lambda}p_2(x))p(y|x)}{\sum_x p_1(x)p(y|x)} \\ &\quad +\sum_x \sum_y \bar{\lambda}p_2(x)p(y|x)\log \frac{(\lambda p_1(x) + \bar{\lambda}p_2(x))p(y|x)}{\sum_x p_2(x)p(y|x)} \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_x \sum_y \lambda p_1(x) p(y|x) \left(\frac{(\lambda p_1(x) + \bar{\lambda} p_2(x)) p(y|x)}{\sum_x p_1(x) p(y|x)} - 1 \right) \\
&\quad + \sum_x \sum_y \bar{\lambda} p_2(x) p(y|x) \left(\frac{(\lambda p_1(x) + \bar{\lambda} p_2(x)) p(y|x)}{\sum_x p_2(x) p(y|x)} - 1 \right) \\
&= 0
\end{aligned}
$$

where we have used the inequality $\log x \leq x - 1$. Thus, $I(\mathbf{p}; \mathbf{Q})$ is a concave function in $\mathbf{p}$.

2) The matrix $\mathbf{Q} = \lambda \mathbf{Q}_1 + \bar{\lambda} \mathbf{Q}_2$ is a legitimate conditional probability matrix for its elements $p(y|x) = \lambda p_1(y|x) + \bar{\lambda} p_2(y|x)$ are non-negative, less or equal to one and

$$
\begin{aligned}
\sum_x \sum_y p(y|x) &= \sum_x \sum_y (\lambda p_1(y|x) + \bar{\lambda} p_2(y|x)) \\
&= \lambda \sum_x \sum_y p_1(y|x) + \bar{\lambda} \sum_x \sum_y p_2(y|x) \\
&= \lambda + \bar{\lambda} = \lambda + 1 - \lambda = 1
\end{aligned}
$$

$$
\begin{aligned}
&I(\mathbf{p}; \lambda \mathbf{Q}_1 + \bar{\lambda} \mathbf{Q}_2) - \lambda I(\mathbf{p}; \mathbf{Q}_1) + \bar{\lambda} I(\mathbf{p}; \mathbf{Q}_2) \\
&= \sum_x \sum_y p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x)) \log \frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \\
&\quad - \sum_x \sum_y p(x) \lambda p_1(y|x) \log \frac{p_1(y|x)}{\sum_x p(x) p_1(y|x)} \\
&\quad - \sum_x \sum_y p(x) \bar{\lambda} p_2(y|x) \log \frac{p_2(y|x)}{\sum_x p(x) p_2(y|x)} \\
&= \sum_x \sum_y p(x) \lambda p_1(y|x) \log \left[\frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \frac{\sum_x p(x) p_1(y|x)}{p_1(y|x)} \right] \\
&\quad + \sum_x \sum_y p(x) \bar{\lambda} p_2(y|x) \log \left[\frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \frac{\sum_x p(x) p_2(y|x)}{p_2(y|x)} \right] \\
&\leq \sum_x \sum_y p(x) \lambda p_1(y|x) \left[\frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \frac{\sum_x p(x) p_1(y|x)}{p_1(y|x)} - 1 \right] \\
&\quad + \sum_x \sum_y p(x) \bar{\lambda} p_2(y|x) \left[\frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \frac{\sum_x p(x) p_2(y|x)}{p_2(y|x)} - 1 \right] \\
&= \sum_y \frac{\sum_x p(x) p_1(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \sum_x \lambda p(x) p_1(y|x) \frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))}{p_1(y|x)} \\
&\quad - \lambda \sum_x \sum_y p(x) p_1(y|x) \\
&\quad + \sum_y \frac{\sum_x p(x) p_2(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \sum_x \bar{\lambda} p(x) p_2(y|x) \frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))}{p_2(y|x)} \\
&\quad - \bar{\lambda} \sum_x \sum_y p(x) p_2(y|x) \\
&= 0
\end{aligned}
$$

Hence, $I(\mathbf{p}; \mathbf{Q})$ is a convex function on $\mathbf{Q}$.

Problem 6.35

1) The PDF of the random variable $Y = \alpha X$ is

$$
f_Y(y) = \frac{1}{|\alpha|} f_X(\frac{y}{\alpha})
$$

Hence,

$$\begin{aligned}
h(Y) &= -\int_{-\infty}^{\infty} f_Y(y)\log(f_Y(y))dy \\
&= -\int_{-\infty}^{\infty} \frac{1}{|\alpha|} f_X(\frac{y}{\alpha}) \log\left(\frac{1}{|\alpha|} f_X(\frac{y}{\alpha})\right) dy \\
&= -\log\left(\frac{1}{|\alpha|}\right) \int_{-\infty}^{\infty} \frac{1}{|\alpha|} f_X(\frac{y}{\alpha}) dy - \int_{-\infty}^{\infty} \frac{1}{|\alpha|} f_X(\frac{y}{\alpha}) \log\left(f_X(\frac{y}{\alpha})\right) dy \\
&= -\log\left(\frac{1}{|\alpha|}\right) + h(X) = \log|\alpha| + h(X)
\end{aligned}$$

2) A similar relation does not hold if X is a discrete random variable. Suppose for example that X takes the values $\{x_1, x_2, \ldots, x_n\}$ with probabilities $\{p_1, p_2, \ldots, p_n\}$. Then, $Y = \alpha X$ takes the values $\{\alpha x_1, \alpha x_2, \ldots, \alpha x_n\}$ with probabilities $\{p_1, p_2, \ldots, p_n\}$, so that

$$H(Y) = -\sum_i p_i \log p_i = H(X)$$

Problem 6.36
1)

$$\begin{aligned}
h(X) &= -\int_0^{\infty} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \ln(\frac{1}{\lambda} e^{-\frac{x}{\lambda}}) dx \\
&= -\ln(\frac{1}{\lambda}) \int_0^{\infty} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx + \int_0^{\infty} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \frac{x}{\lambda} dx \\
&= \ln\lambda + \frac{1}{\lambda} \int_0^{\infty} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} x dx \\
&= \ln\lambda + \frac{1}{\lambda}\lambda = 1 + \ln\lambda
\end{aligned}$$

where we have used the fact $\int_0^{\infty} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx = 1$ and $E[x] = \int_0^{\infty} x \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx = \lambda$.

2)

$$\begin{aligned}
h(X) &= -\int_{-\infty}^{\infty} \frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}} \ln(\frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}}) dx \\
&= -\ln(\frac{1}{2\lambda}) \int_{-\infty}^{\infty} \frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}} dx + \frac{1}{\lambda} \int_{-\infty}^{\infty} |x| \frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}} dx \\
&= \ln(2\lambda) + \frac{1}{\lambda}\left[\int_{-\infty}^{0} -x \frac{1}{2\lambda} e^{\frac{x}{\lambda}} dx + \int_0^{\infty} x \frac{1}{2\lambda} e^{-\frac{x}{\lambda}} dx\right] \\
&= \ln(2\lambda) + \frac{1}{2\lambda}\lambda + \frac{1}{2\lambda}\lambda = 1 + \ln(2\lambda)
\end{aligned}$$

3)

$$\begin{aligned}
h(X) &= -\int_{-\lambda}^{0} \frac{x+\lambda}{\lambda^2} \ln\left(\frac{x+\lambda}{\lambda^2}\right) dx - \int_0^{\lambda} \frac{-x+\lambda}{\lambda^2} \ln\left(\frac{-x+\lambda}{\lambda^2}\right) dx \\
&= -\ln\left(\frac{1}{\lambda^2}\right)\left[\int_{-\lambda}^{0} \frac{x+\lambda}{\lambda^2} dx + \int_0^{\lambda} \frac{-x+\lambda}{\lambda^2} dx\right] \\
&\quad -\int_{-\lambda}^{0} \frac{x+\lambda}{\lambda^2} \ln(x+\lambda) dx - \int_0^{\lambda} \frac{-x+\lambda}{\lambda^2} \ln(-x+\lambda) dx
\end{aligned}$$

$$
\begin{aligned}
&= \ln(\lambda^2) - \frac{2}{\lambda^2}\int_0^\lambda z\ln z dz \\
&= \ln(\lambda^2) - \frac{2}{\lambda^2}\left[\frac{z^2\ln z}{2} - \frac{z^2}{4}\right]_0^\lambda \\
&= \ln(\lambda^2) - \ln(\lambda) + \frac{1}{2}
\end{aligned}
$$

Problem 6.37

1) Applying the inequality $\ln z \leq z-1$ to the function $z = \frac{p(x)p(y)}{p(x,y)}$, we obtain

$$
\ln p(x) + \ln p(y) - \ln p(x,y) \leq \frac{p(x)p(y)}{p(x,y)} - 1
$$

Multiplying by $p(x,y)$ and integrating over x, y, we obtain

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\left(\ln p(x) + \ln p(y)\right)dxdy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\ln p(x,y)dxdy \\
\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x)p(y)dxdy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)dxdy \\
= 1-1=0
\end{aligned}
$$

Hence,

$$
\begin{aligned}
h(X,Y) &\leq -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\ln p(x)dxdy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\ln p(y)dxdy \\
&= h(X)+h(Y)
\end{aligned}
$$

Also, $h(X,Y) = h(X|Y) + h(Y)$ so by combining the two, we obtain

$$
h(X|Y) + h(Y) \leq h(X) + h(Y) \Longrightarrow h(X|Y) \leq h(X)
$$

Equality holds if $z = \frac{p(x)p(y)}{p(x,y)} = 1$ or, in other words, if X and Y are independent.

2) By definition $I(X;Y) = h(X) - h(X|Y)$. However, from the first part of the problem $h(X|Y) \leq h(X)$ so that

$$
I(X;Y) \geq 0
$$

Problem 6.38

Let X be the exponential random variable with mean m, that is

$$
f_X(x) = \begin{cases} \frac{1}{m}e^{-\frac{x}{m}} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}
$$

Consider now another random variable Y with PDF $f_Y(x)$, which is non-zero for $x \geq 0$, and such that

$$
E[Y] = \int_0^\infty x f_Y(x)dx = m
$$

Applying the inequality $\ln z \leq z-1$ to the function $x = \frac{f_X(x)}{f_Y(x)}$, we obtain

$$
\ln(f_X(x)) - \ln(f_Y(x)) \leq \frac{f_X(x)}{f_Y(x)} - 1
$$

Multiplying both sides by $f_Y(x)$ and integrating, we obtain

$$
\int_0^\infty f_Y(x)\ln(f_X(x))dx - \int_0^\infty f_Y(x)\ln(f_Y(x))dx \leq \int_0^\infty f_X(x)dx - \int_0^\infty f_Y(x)dx = 0
$$

Hence,

$$\begin{aligned}
h(Y) &\leq -\int_0^\infty f_Y(x)\ln\left(\frac{1}{m}e^{-\frac{x}{m}}\right)dx \\
&= -\ln\left(\frac{1}{m}\right)\int_0^\infty f_Y(x)dx + \frac{1}{m}\int_0^\infty x f_Y(x)dx \\
&= \ln m + \frac{1}{m}m = 1 + \ln m = h(X)
\end{aligned}$$

where we have used the results of Problem 6.36.

Problem 6.39

Let X be a zero-mean Gaussian random variable with variance σ^2 and Y another zero-mean random variable such that

$$\int_{-\infty}^\infty y^2 f_Y(y)dy = \sigma^2$$

Applying the inequality $\ln z \leq z - 1$ to the function $z = \frac{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}}{f_Y(x)}$, we obtain

$$\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}\right) - \ln f_Y(x) \leq \frac{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}}{f_Y(x)} - 1$$

Multiplying the inequality by $f_Y(x)$ and integrating, we obtain

$$\int_{-\infty}^\infty f_Y(x)\left[\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{x^2}{2\sigma^2}\right]dx + h(Y) \leq 1 - 1 = 0$$

Hence,

$$\begin{aligned}
h(Y) &\leq -\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) + \frac{1}{2\sigma^2}\int_{-\infty}^\infty x^2 f_X(x)dx \\
&= \ln(\sqrt{2\pi\sigma^2}) + \frac{1}{2\sigma^2}\sigma^2 = \ln(e^{\frac{1}{2}}) + \ln(\sqrt{2\pi\sigma^2}) \\
&= h(X)
\end{aligned}$$

Problem 6.40

1) The entropy of the source is

$$H(X) = -.25\log_2 .25 - .75\log_2 .75 = .8113 \text{ bits/symbol}$$

Thus, we can transmit the output of the source using $2000H(X) = 1623$ bits/sec with arbitrarily small probability of error.

2) Since $0 \leq D \leq \min\{p, 1-p\} = .25$ the rate distortion function for the binary memoryless source is

$$R(D) = H_b(p) - H_b(D) = H_b(.25) - H_b(.1) = .8113 - .4690 = .3423$$

Hence, the required number of bits per second is $2000R(D) = 685$.

3) For $D = .25$ the rate is $R(D) = 0$. We can reproduce the source at a distortion of $D = .25$ with no transmission at all by setting the reproduction vector to be the all zero vector.

Problem 6.41

1) For a zero-mean Gaussian source with variance σ^2 and with squared error distortion measure, the rate distortion function is given by

$$R(D) = \begin{cases} \frac{1}{2}\log\frac{\sigma^2}{D} & 0 \leq D \leq \sigma^2 \\ 0 & \text{otherwise} \end{cases}$$

With $R = 1$ and $\sigma^2 = 1$, we obtain

$$2 = \log\frac{1}{D} \Longrightarrow D = 2^{-2} = 0.25$$

2) If we set $D = 0.01$, then

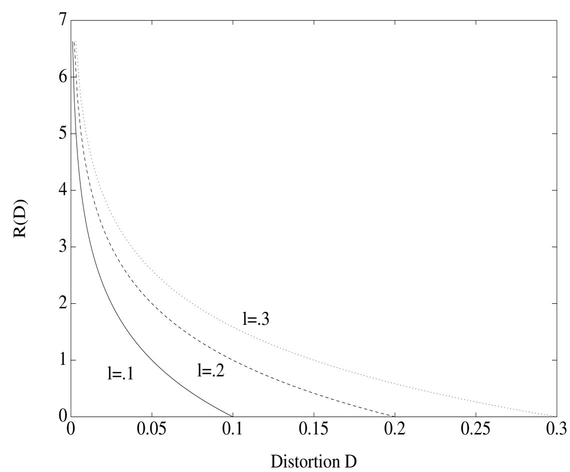
$$R = \frac{1}{2}\log\frac{1}{0.01} = \frac{1}{2}\log 100 = 3.322 \text{ bits/sample}$$

Hence, the required transmission capacity is 3.322 bits per source symbol.

Problem 6.42

1) Since $R(D) = \log\frac{\lambda}{D}$ and $D = \frac{\lambda}{2}$, we obtain $R(D) = \log(\frac{\lambda}{\lambda/2}) = \log(2) = 1$ bit/sample.

2) The following figure depicts $R(D)$ for $\lambda = 0.1$, .2 and .3. As it is observed from the figure, an increase of the parameter λ increases the required rate for a given distortion.



Problem 6.43

1) For a Gaussian random variable of zero mean and variance σ^2 the rate-distortion function is given by $R(D) = \frac{1}{2}\log_2\frac{\sigma^2}{D}$. Hence, the upper bound is satisfied with equality. For the lower bound recall that $h(X) = \frac{1}{2}\log_2(2\pi e\sigma^2)$. Thus,

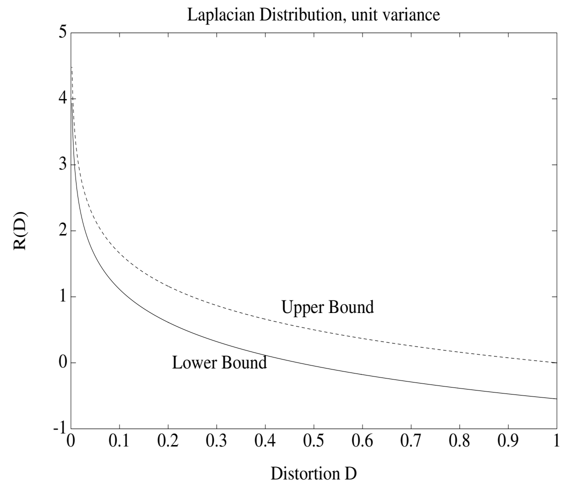
$$\begin{aligned} h(X) - \frac{1}{2}\log_2(2\pi e D) &= \frac{1}{2}\log_2(2\pi e\sigma^2) - \frac{1}{2}\log_2(2\pi e D) \\ &= \frac{1}{2}\log_2\left(\frac{2\pi e\sigma^2}{2\pi e D}\right) = R(D) \end{aligned}$$

As it is observed the upper and the lower bounds coincide.

2) The differential entropy of a Laplacian source with parameter λ is $h(X) = 1 + \ln(2\lambda)$. The variance of the Laplacian distribution is

$$\sigma^2 = \int_{-\infty}^{\infty} x^2\frac{1}{2\lambda}e^{-\frac{|x|}{\lambda}}\,dx = 2\lambda^2$$

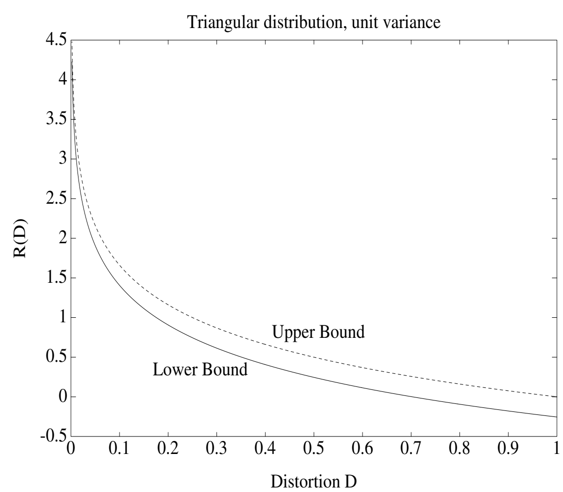
Hence, with $\sigma^2 = 1$, we obtain $\lambda = \sqrt{1/2}$ and $h(X) = 1+\ln(2\lambda) = 1+\ln(\sqrt{2}) = 1.3466$ nats/symbol = 1500 bits/symbol. A plot of the lower and upper bound of $R(D)$ is given in the next figure.



3) The variance of the triangular distribution is given by

$$
\begin{aligned}
\sigma^2 &= \int_{-\lambda}^{0} \left(\frac{x+\lambda}{\lambda^2}\right) x^2 dx + \int_{0}^{\lambda} \left(\frac{-x+\lambda}{\lambda^2}\right) x^2 dx \\
&= \frac{1}{\lambda^2}\left(\frac{1}{4}x^4 + \frac{\lambda}{3}x^3\right)\Big|_{-\lambda}^{0} + \frac{1}{\lambda^2}\left(-\frac{1}{4}x^4 + \frac{\lambda}{3}x^3\right)\Big|_{0}^{\lambda} \\
&= \frac{\lambda^2}{6}
\end{aligned}
$$

Hence, with $\sigma^2 = 1$, we obtain $\lambda = \sqrt{6}$ and $h(X) = \ln(6) - \ln(\sqrt{6}) + 1/2 = 1.7925$ bits /source output. A plot of the lower and upper bound of $R(D)$ is given in the next figure.



Problem 6.44

For a zero-mean Gaussian source of variance σ^2, the rate distortion function is given by $R(D) = \frac{1}{2}\log\frac{\sigma^2}{D}$. Expressing D in terms of R, we obtain $D(R) = \sigma^2 2^{-2R}$. Hence,

$$
\frac{D(R_1)}{D(R_2)} = \frac{\sigma^2 2^{-2R_1}}{\sigma^2 2^{-2R_2}} \Longrightarrow R_2 - R_1 = \frac{1}{2}\log_2\left(\frac{D(R_1)}{D(R_2)}\right)
$$

With $\frac{D(R_1)}{D(R_2)} = 1000$, the number of extra bits needed is $R_2 - R_1 = \frac{1}{2}\log_2 1000 = 5$.

Problem 6.45

1) Consider the memoryless system $Y(t) = \mathcal{Q}(X(t))$. At any given time $t = t_1$, the output $Y(t_1)$ depends only on $X(t_1)$ and not on any other past or future values of $X(t)$. The n^{th} order density $f_{Y(t_1),\ldots,Y(t_n)}(y_1,\ldots,y_n)$ can be determined from the corresponding density $f_{X(t_1),\ldots,X(t_n)}(x_1,\ldots,x_n)$ using

$$f_{Y(t_1),\ldots,Y(t_n)}(y_1,\ldots,y_n) = \sum_{j=1}^{J} \frac{f_{X(t_1),\ldots,X(t_n)}(x_1,\ldots,x_n)}{|J(x_1^j,\ldots,x_n^j)|}$$

where J is the number of solutions to the system

$$y_1 = \mathcal{Q}(x_1), \quad y_2 = \mathcal{Q}(x_2), \quad \cdots, \quad y_n = \mathcal{Q}(x_n)$$

and $J(x_1^j,\ldots,x_n^j)$ is the Jacobian of the transformation system evaluated at the solution $\{x_1^j,\ldots,x_n^j\}$. Note that if the system has a unique solution, then

$$J(x_1,\ldots,x_n) = \mathcal{Q}'(x_1)\cdots\mathcal{Q}'(x_2)$$

From the stationarity of $X(t)$ it follows that the numerator of all the terms under summation, in the expression for $f_{Y(t_1),\ldots,Y(t_n)}(y_1,\ldots,y_n)$, is invariant to a shift of the time origin. Furthermore, the denominators do not depend on t, so that $f_{Y(t_1),\ldots,Y(t_n)}(y_1,\ldots,y_n)$ does not change if t_i is replaced by $t_i + \tau$. Hence, $Y(t)$ is a strictly stationary process.

2) $X(t) - \mathcal{Q}(X(t))$ is a memoryless function of $X(t)$ and since the latter is strictly stationary, we conclude that $\tilde{X}(t) = X(t) - \mathcal{Q}(X(t))$ is strictly stationary. Hence,

$$\text{SQNR} = \frac{E[X^2(t)]}{E[(X(t)-\mathcal{Q}(X(t)))^2]} = \frac{E[X^2(t)]}{E[\tilde{X}^2(t)]} = \frac{R_X(0)}{R_{\tilde{X}}(0)} = \frac{P_X}{P_{\tilde{X}}}$$

Problem 6.46

1) From Table 6.2 we find that for a unit variance Gaussian process, the optimal level spacing for a 16-level uniform quantizer is .3352. This number has to be multiplied by σ to provide the optimal level spacing when the variance of the process is σ^2. In our case $\sigma^2 = 10$ and $\Delta = \sqrt{10}\cdot 0.3352 = 1.060$. The quantization levels are

$$\begin{aligned}
\hat{x}_1 = -\hat{x}_{16} &= -7\cdot 1.060 - \frac{1}{2}\cdot 1.060 = -7.950 \\
\hat{x}_2 = -\hat{x}_{15} &= -6\cdot 1.060 - \frac{1}{2}\cdot 1.060 = -6.890 \\
\hat{x}_3 = -\hat{x}_{14} &= -5\cdot 1.060 - \frac{1}{2}\cdot 1.060 = -5.830 \\
\hat{x}_4 = -\hat{x}_{13} &= -4\cdot 1.060 - \frac{1}{2}\cdot 1.060 = -4.770 \\
\hat{x}_5 = -\hat{x}_{12} &= -3\cdot 1.060 - \frac{1}{2}\cdot 1.060 = -3.710 \\
\hat{x}_6 = -\hat{x}_{11} &= -2\cdot 1.060 - \frac{1}{2}\cdot 1.060 = -2.650 \\
\hat{x}_7 = -\hat{x}_{10} &= -1\cdot 1.060 - \frac{1}{2}\cdot 1.060 = -1.590 \\
\hat{x}_8 = -\hat{x}_9 &= -\frac{1}{2}\cdot 1.060 = -0.530
\end{aligned}$$

The boundaries of the quantization regions are given by

$$a_1 = a_{15} = -7\cdot 1.060 = -7.420$$

$$
\begin{aligned}
a_2 = a_{14} &= -6 \cdot 1.060 = -6.360 \\
a_3 = a_{13} &= -5 \cdot 1.060 = -5.300 \\
a_4 = a_{12} &= -4 \cdot 1.060 = -4.240 \\
a_5 = a_{11} &= -3 \cdot 1.060 = -3.180 \\
a_6 = a_{10} &= -2 \cdot 1.060 = -2.120 \\
a_7 = a_9 &= -1 \cdot 1.060 = -1.060 \\
a_8 &= 0
\end{aligned}
$$

2) The resulting distortion is $D = \sigma^2 \cdot 0.01154 = 0.1154$.

3) The entropy is available from Table 6.2. Nevertheless we will rederive the result here. The probabilities of the 16 outputs are

$$
\begin{aligned}
p(\hat{x}_1) = p(\hat{x}_{16}) &= Q(\frac{a_{15}}{\sqrt{10}}) = 0.0094 \\
p(\hat{x}_2) = p(\hat{x}_{15}) &= Q(\frac{a_{14}}{\sqrt{10}}) - Q(\frac{a_{15}}{\sqrt{10}}) = 0.0127 \\
p(\hat{x}_3) = p(\hat{x}_{14}) &= Q(\frac{a_{13}}{\sqrt{10}}) - Q(\frac{a_{14}}{\sqrt{10}}) = 0.0248 \\
p(\hat{x}_4) = p(\hat{x}_{13}) &= Q(\frac{a_{12}}{\sqrt{10}}) - Q(\frac{a_{13}}{\sqrt{10}}) = 0.0431 \\
p(\hat{x}_5) = p(\hat{x}_{12}) &= Q(\frac{a_{11}}{\sqrt{10}}) - Q(\frac{a_{12}}{\sqrt{10}}) = 0.0674 \\
p(\hat{x}_6) = p(\hat{x}_{11}) &= Q(\frac{a_{10}}{\sqrt{10}}) - Q(\frac{a_{11}}{\sqrt{10}}) = 0.0940 \\
p(\hat{x}_7) = p(\hat{x}_{10}) &= Q(\frac{a_9}{\sqrt{10}}) - Q(\frac{a_{10}}{\sqrt{10}}) = 0.1175 \\
p(\hat{x}_8) = p(\hat{x}_9) &= Q(\frac{a_8}{\sqrt{10}}) - Q(\frac{a_9}{\sqrt{10}}) = 0.1311
\end{aligned}
$$

Hence, the entropy of the quantized source is

$$
H(\hat{X}) = -\sum_{i=1}^{1} 6p(\hat{x}_i) \log_2 p(\hat{x}_i) = 3.6025
$$

This is the minimum number of bits per source symbol required to represent the quantized source.

4) Substituting $\sigma^2 = 10$ and $D = 0.1154$ in the rate-distortion bound, we obtain

$$
R = \frac{1}{2} \log_2 \frac{\sigma^2}{D} = 3.2186
$$

5) The distortion of the 16-level optimal quantizer is $D_{16} = \sigma^2 \cdot 0.01154$ whereas that of the 8-level optimal quantizer is $D_8 = \sigma^2 \cdot 0.03744$. Hence, the amount of increase in SQNR (db) is

$$
10 \log_{10} \frac{\mathrm{SQNR}_{16}}{\mathrm{SQNR}_8} = 10 \cdot \log_{10} \frac{0.03744}{0.01154} = 5.111 \text{ db}
$$

Problem 6.47

With 8 quantization levels and $\sigma^2 = 400$ we obtain

$$
\Delta = \sigma .5860 = 20 \cdot 0.5860 = 11.72
$$

Hence, the quantization levels are

$$\begin{aligned}
\hat{x}_1 = -\hat{x}_8 &= -3 \cdot 11.72 - \frac{1}{2}11.72 = -41.020 \\
\hat{x}_2 = -\hat{x}_7 &= -2 \cdot 11.72 - \frac{1}{2}11.72 = -29.300 \\
\hat{x}_3 = -\hat{x}_6 &= -1 \cdot 11.72 - \frac{1}{2}11.72 = -17.580 \\
\hat{x}_4 = -\hat{x}_5 &= -\frac{1}{2}11.72 = -5.860
\end{aligned}$$

The distortion of the optimum quantizer is

$$D = \sigma^2 \cdot 0.03744 = 14.976$$

As it is observed the distortion of the optimum quantizer is significantly less than that of Example 6.5.1. The informational entropy of the optimum quantizer is found from Table 6.2 to be 2.761.

Problem 6.48

Using Table 6.3 we find the quantization regions and the quantized values for $N = 16$. These values should be multiplied by $\sigma = P_X^{1/2} = \sqrt{10}$, since Table 6.3 provides the optimum values for a unit variance Gaussian source.

$$\begin{aligned}
a_1 = -a_{15} &= -\sqrt{10} \cdot 2.401 = -7.5926 \\
a_2 = -a_{14} &= -\sqrt{10} \cdot 1.844 = -5.8312 \\
a_3 = -a_{13} &= -\sqrt{10} \cdot 1.437 = -4.5442 \\
a_4 = -a_{12} &= -\sqrt{10} \cdot 1.099 = -3.4753 \\
a_5 = -a_{11} &= -\sqrt{10} \cdot 0.7996 = -2.5286 \\
a_6 = -a_{10} &= -\sqrt{10} \cdot 0.5224 = -1.6520 \\
a_7 = -a_9 &= -\sqrt{10} \cdot 0.2582 = -0.8165 \\
a_8 &= 0
\end{aligned}$$

The quantized values are

$$\begin{aligned}
\hat{x}_1 = -\hat{x}_{16} &= -\sqrt{10} \cdot 2.733 = -8.6425 \\
\hat{x}_2 = -\hat{x}_{15} &= -\sqrt{10} \cdot 2.069 = -6.5428 \\
\hat{x}_3 = -\hat{x}_{14} &= -\sqrt{10} \cdot 1.618 = -5.1166 \\
\hat{x}_4 = -\hat{x}_{13} &= -\sqrt{10} \cdot 1.256 = -3.9718 \\
\hat{x}_5 = -\hat{x}_{12} &= -\sqrt{10} \cdot 0.9424 = -2.9801 \\
\hat{x}_6 = -\hat{x}_{11} &= -\sqrt{10} \cdot 0.6568 = -2.0770 \\
\hat{x}_7 = -\hat{x}_{10} &= -\sqrt{10} \cdot 0.3881 = -1.2273 \\
\hat{x}_8 = -\hat{x}_9 &= -\sqrt{10} \cdot 0.1284 = -0.4060
\end{aligned}$$

The resulting distortion is $D = 10 \cdot 0.009494 = 0.09494$. From Table 6.3 we find that the minimum number of bits per source symbol is $H(\hat{X}) = 3.765$. Setting $D = 0.09494$, $\sigma^2 = 10$ in $R = \frac{1}{2}\log_2 \frac{\sigma^2}{D}$ we obtain $R = 3.3594$. Thus, the minimum number of bits per source symbol is slightly larger that the predicted one from the rate-distortion bound.

Problem 6.49

1) The area between the two squares is $4 \times 4 - 2 \times 2 = 12$. Hence, $f_{X,Y}(x, y) = \frac{1}{12}$. The marginal probability $f_X(x)$ is given by $f_X(x) = \int_{-2}^{2} f_{X,Y}(x, y)dy$. If $-2 \leq X < -1$, then

$$f_X(x) = \int_{-2}^{2} f_{X,Y}(x, y)dy = \frac{1}{12}y\Big|_{-2}^{2} = \frac{1}{3}$$

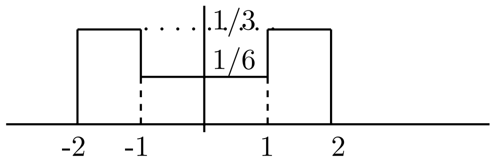
If $-1 \leq X < 1$, then

$$f_X(x) = \int_{-2}^{-1} \frac{1}{12} dy + \int_{1}^{2} \frac{1}{12} dy = \frac{1}{6}$$

Finally, if $1 \leq X \leq 2$, then

$$f_X(x) = \int_{-2}^{2} f_{X,Y}(x,y) dy = \frac{1}{12} y \Big|_{-2}^{2} = \frac{1}{3}$$

The next figure depicts the marginal distribution $f_X(x)$.



Similarly we find that

$$f_Y(y) = \begin{cases} \frac{1}{3} & -2 \leq y < -1 \\ \frac{1}{6} & -1 \leq y < -1 \\ \frac{1}{3} & 1 \leq y \leq 2 \end{cases}$$

2) The quantization levels $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ and $\hat{x}_4$ are set to $-\frac{3}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$ and $\frac{3}{2}$ respectively. The resulting distortion is

$$\begin{aligned} D_X &= 2 \int_{-2}^{-1} (x + \frac{3}{2})^2 f_X(x) dx + 2 \int_{-1}^{0} (x + \frac{1}{2})^2 f_X(x) dx \\ &= \frac{2}{3} \int_{-2}^{-1} (x^2 + 3x + \frac{9}{4}) dx + \frac{2}{6} \int_{-1}^{0} (x^2 + x + \frac{1}{4}) dx \\ &= \frac{2}{3} \left(\frac{1}{3} x^3 + \frac{3}{2} x^2 + \frac{9}{4} x \right) \Big|_{-2}^{-1} + \frac{2}{6} \left(\frac{1}{3} x^3 + \frac{1}{2} x^2 + \frac{1}{4} x \right) \Big|_{-1}^{0} \\ &= \frac{1}{12} \end{aligned}$$

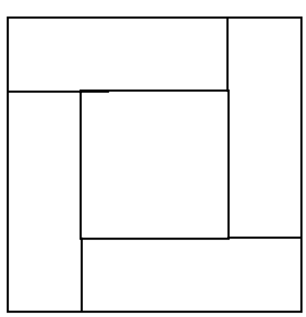
The total distortion is

$$D_{\text{total}} = D_X + D_Y = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

whereas the resulting number of bits per (X, Y) pair

$$R = R_X + R_Y = \log_2 4 + \log_2 4 = 4$$

3) Suppose that we divide the region over which $p(x, y) \neq 0$ into L equal subregions. The case of $L = 4$ is depicted in the next figure.



For each subregion the quantization output vector $(\hat{x}, \hat{y})$ is the centroid of the corresponding rectangle. Since, each subregion has the same shape (uniform quantization), a rectangle with width equal to one and length $12/L$, the distortion of the vector quantizer is

$$
\begin{aligned}
D &= \int_0^1 \int_0^{\frac{12}{L}} [(x,y) - (\frac{1}{2}, \frac{12}{2L})]^2 \frac{L}{12} dx dy \\
&= \frac{L}{12} \int_0^1 \int_0^{\frac{12}{L}} \left[(x - \frac{1}{2})^2 + (y - \frac{12}{2L})^2 \right] dx dy \\
&= \frac{L}{12} \left[\frac{12}{L} \frac{1}{12} + \frac{12^3}{L^3} \frac{1}{12} \right] = \frac{1}{12} + \frac{12}{L^2}
\end{aligned}
$$

If we set $D = \frac{1}{6}$, we obtain

$$
\frac{12}{L^2} = \frac{1}{12} \Longrightarrow L = \sqrt{144} = 12
$$

Thus, we have to divide the area over which $p(x,y) \neq 0$, into 12 equal subregions in order to achieve the same distortion. In this case the resulting number of bits per source output pair (X,Y) is $R = \log_2 12 = 3.585$.

Problem 6.50

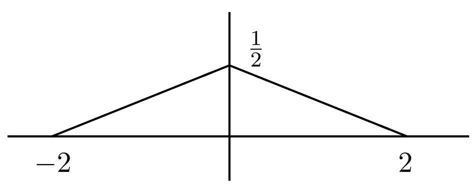
1) The joint probability density function is $f_{XY}(x,y) = \frac{1}{(2\sqrt{2})^2} = \frac{1}{8}$. The marginal distribution $f_X(x)$ is $f_X(x) = \int_y f_{XY}(x,y)dy$. If $-2 \leq x \leq 0$,then

$$
f_X(x) = \int_{-x-2}^{x+2} f_{X,Y}(x,y)dy = \frac{1}{8} y|_{-x-2}^{x+2} = \frac{x+2}{4}
$$

If $0 \leq x \leq 2$,then

$$
f_X(x) = \int_{x-2}^{-x+2} f_{X,Y}(x,y)dy = \frac{1}{8} y|_{x-2}^{-x+2} = \frac{-x+2}{4}
$$

The next figure depicts $f_X(x)$.



From the symmetry of the problem we have

$$
f_Y(y) = \begin{cases} \frac{y+2}{4} & -2 \leq y < 0 \\ \frac{-y+2}{4} & 0 \leq y \leq 2 \end{cases}
$$

2)

$$
\begin{aligned}
D_X &= 2 \int_{-2}^{-1} (x + \frac{3}{2})^2 f_X(x) dx + 2 \int_{-1}^{0} (x + \frac{1}{2})^2 f_X(x) dx \\
&= \frac{1}{2} \int_{-2}^{-1} (x + \frac{3}{2})^2 (x+2) dx + \frac{1}{2} \int_{-1}^{0} (x + \frac{1}{2})^2 (-x+2) dx \\
&= \frac{1}{2} \left(\frac{1}{4} x^4 + \frac{5}{3} x^3 + \frac{33}{8} x^2 + \frac{9}{2} x \right) \Big|_{-2}^{-1} + \frac{1}{2} \left(\frac{1}{4} x^4 + x^3 + \frac{9}{8} x^2 + \frac{1}{2} x \right) \Big|_{-1}^{0} \\
&= \frac{1}{12}
\end{aligned}
$$

The total distortion is

$$
D_{\text{total}} = D_X + D_Y = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}
$$

whereas the required number of bits per source output pair

$$R = R_X + R_Y = \log_2 4 + \log_2 4 = 4$$

3) We divide the square over which $p(x,y) \neq 0$ into $2^4 = 16$ equal square regions. The area of each square is $\frac{1}{2}$ and the resulting distortion

$$\begin{aligned}
D &= \frac{16}{8}\int_0^{\frac{1}{\sqrt{2}}}\int_0^{\frac{1}{\sqrt{2}}}\left[(x-\frac{1}{2\sqrt{2}})^2 + (y-\frac{1}{2\sqrt{2}})^2\right]dxdy \\
&= 4\int_0^{\frac{1}{\sqrt{2}}}\int_0^{\frac{1}{\sqrt{2}}}(x-\frac{1}{2\sqrt{2}})^2 dxdy \\
&= \frac{4}{\sqrt{2}}\int_0^{\frac{1}{\sqrt{2}}}(x^2+\frac{1}{8}-\frac{x}{\sqrt{2}})dx \\
&= \frac{4}{\sqrt{2}}\left(\frac{1}{3}x^3+\frac{1}{8}x-\frac{1}{2\sqrt{2}}x^2\right)\Big|_0^{\frac{1}{\sqrt{2}}} \\
&= \frac{1}{12}
\end{aligned}$$

Hence, using vector quantization and the same rate we obtain half the distortion.

Problem 6.51

$\breve{X} = \frac{X}{x_{\max}} = X/2$. Hence,

$$E[\breve{X}^2] = \frac{1}{4}\int_{-2}^{2}\frac{X^2}{4}dx = \frac{1}{16\cdot 3}x^3\Big|_{-2}^{2} = \frac{1}{3}$$

With $\nu = 8$ and $\overline{\breve{X}^2} = \frac{1}{3}$, we obtain

$$\text{SQNR} = 3\cdot 4^8 \cdot \frac{1}{3} = 4^8 = 48.165(\text{db})$$

Problem 6.52

1)

$$\sigma^2 = E[X^2(t)] = R_X(\tau)|_{\tau=0} = \frac{A^2}{2}$$

Hence,

$$\text{SQNR} = 3\cdot 4^{\nu}\overline{\breve{X}^2} = 3\cdot 4^{\nu}\frac{\overline{X^2}}{x_{\max}^2} = 3\cdot 4^{\nu}\frac{A^2}{2A^2}$$

With SQNR = 60 db, we obtain

$$10\log_{10}\left(\frac{3\cdot 4^q}{2}\right) = 60 \Longrightarrow q = 9.6733$$

The smallest integer larger that q is 10. Hence, the required number of quantization levels is $\nu = 10$.

2) The minimum bandwidth requirement for transmission of a binary PCM signal is $\text{BW} = \nu W$. Since $\nu = 10$, we have $\text{BW} = 10W$.

Problem 6.53

1)

$$
\begin{aligned}
E[X^2(t)] &= \int_{-2}^{0} x^2\left(\frac{x+2}{4}\right)dx + \int_{0}^{2} x^2\left(\frac{-x+2}{4}\right)dx \\
&= \frac{1}{4}\left(\frac{1}{4}x^4+\frac{2}{3}x^3\right)\Big|_{-2}^{0} + \frac{1}{4}\left(-\frac{1}{4}x^4+\frac{2}{3}x^3\right)\Big|_{0}^{2} \\
&= \frac{2}{3}
\end{aligned}
$$

Hence,

$$
\text{SQNR} = \frac{3\times 4^{\nu}\times\frac{2}{3}}{x_{\max}^2} = \frac{3\times 4^5\times\frac{2}{3}}{2^2} = 512 = 27.093(\text{db})
$$

2) If the available bandwidth of the channel is 40 KHz, then the maximum rate of transmission is $\nu = 40/5 = 8$. In this case the highest achievable SQNR is

$$
\text{SQNR} = \frac{3\times 4^8\times\frac{2}{3}}{2^2} = 32768 = 45.154(\text{db})
$$

3) In the case of a guard band of 2 KHz the sampling rate is $f_s = 2W + 2000 = 12$ KHz. The highest achievable rate is $\nu = \frac{2\text{BW}}{f_s} = 6.6667$ and since ν should be an integer we set $\nu = 6$. Thus, the achievable SQNR is

$$
\text{SQNR} = \frac{3\times 4^6\times\frac{2}{3}}{2^2} = 2048 = 33.11(\text{db})
$$

Problem 6.54

1) The probabilities of the quantized source outputs are

$$
\begin{aligned}
p(\hat{x}_1) = p(\hat{x}_4) &= \int_{-2}^{-1}\frac{x+2}{4}dx = \frac{1}{8}x^2\Big|_{-2}^{-1} + \frac{1}{2}x\Big|_{-2}^{-1} = \frac{1}{8} \\
p(\hat{x}_2) = p(\hat{x}_3) &= \int_{0}^{1}\frac{-x+2}{4}dx = -\frac{1}{8}x^2\Big|_{0}^{1} + \frac{1}{2}x\Big|_{0}^{1} = \frac{3}{8}
\end{aligned}
$$

Hence,

$$
H(\hat{X}) = -\sum_{\hat{x}_i} p(\hat{x}_i)\log_2 p(\hat{x}_i) = 1.8113 \text{ bits / output sample}
$$

2) Let $\tilde{X} = X - \mathcal{Q}(X)$. Clearly if $|\tilde{X}| > 0.5$, then $p(\tilde{X}) = 0$. If $|\tilde{X}| \leq 0.5$, then there are four solutions to the equation $\tilde{X} = X - \mathcal{Q}(X)$, which are denoted by x_1, x_2, x_3 and x_4. The solution x_1 corresponds to the case $-2 \leq X \leq -1$, x_2 is the solution for $-1 \leq X \leq 0$ and so on. Hence,

$$
\begin{aligned}
f_X(x_1) &= \frac{x_1+2}{4} = \frac{(\tilde{x}-1.5)+2}{4} \qquad & f_X(x_3) &= \frac{-x_3+2}{4} = \frac{-(\tilde{x}+0.5)+2}{4} \\
f_X(x_2) &= \frac{x_2+2}{4} = \frac{(\tilde{x}-0.5)+2}{4} \qquad & f_X(x_4) &= \frac{-x_4+2}{4} = \frac{-(\tilde{x}+1.5)+2}{4}
\end{aligned}
$$

The absolute value of $(X - \mathcal{Q}(X))'$ is one for $X = x_1, \ldots, x_4$. Thus, for $|\tilde{X}| \leq 0.5$

$$
\begin{aligned}
f_{\tilde{X}}(\tilde{x}) &= \sum_{i=1}^{4}\frac{f_X(x_i)}{|(x_i - \mathcal{Q}(x_i))'|} \\
&= \frac{(\tilde{x}-1.5)+2}{4} + \frac{(\tilde{x}-0.5)+2}{4} + \frac{-(\tilde{x}+0.5)+2}{4} + \frac{-(\tilde{x}+1.5)+2}{4} \\
&= 1
\end{aligned}
$$

Problem 6.55

1)

$$\begin{aligned}R_X(t+\tau,t) &= E[X(t+\tau)X(t)]\\ &= E[Y^2\cos(2\pi f_0(t+\tau)+\Theta)\cos(2\pi f_0 t+\Theta)]\\ &= \frac{1}{2}E[Y^2]E[\cos(2\pi f_0\tau)+\cos(2\pi f_0(2t+\tau)+2\Theta)]\end{aligned}$$

and since

$$E[\cos(2\pi f_0(2t+\tau)+2\Theta)] = \frac{1}{2\pi}\int_0^{2\pi}\cos(2\pi f_0(2t+\tau)+2\theta)d\theta = 0$$

we conclude that

$$R_X(t+\tau,t) = \frac{1}{2}E[Y^2]\cos(2\pi f_0\tau) = \frac{3}{2}\cos(2\pi f_0\tau)$$

2)

$$10\log_{10}\mathrm{SQNR} = 10\log_{10}\left(\frac{3\times 4^\nu\times R_X(0)}{x_{\max}^2}\right) = 40$$

Thus,

$$\log_{10}\left(\frac{4^\nu}{2}\right) = 4 \text{ or } \nu = 8$$

The bandwidth of the process is $W = f_0$, so that the minimum bandwidth requirement of the PCM system is $\mathrm{BW} = 8f_0$.

3) If SQNR = 64 db, then

$$\nu' = \log_4(2\cdot 10^{6.4}) = 12$$

Thus, $\nu'-\nu = 4$ more bits are needed to increase SQNR by 24 db. The new minimum bandwidth requirement is $\mathrm{BW}' = 12f_0$.

Problem 6.56

Suppose that the transmitted sequence is $\mathbf{x}$. If an error occurs at the i^{th} bit of the sequence, then the received sequence $\mathbf{x}'$ is

$$\mathbf{x}' = \mathbf{x} + [0\ldots 010\ldots 0]$$

where addition is modulo 2. Thus the error sequence is $e_i = [0\ldots 010\ldots 0]$, which in natural binary coding has the value 2^{i-1}. If the spacing between levels is Δ, then the error introduced by the channel is $2^{i-1}\Delta$.

2)

$$\begin{aligned}D_{\text{channel}} &= \sum_{i=1}^{\nu} p(\text{error in } i \text{ bit})\cdot(2^{i-1}\Delta)^2\\ &= \sum_{i=1}^{\nu} p_b\Delta^2 4^{i-1} = p_b\Delta^2\frac{1-4^\nu}{1-4}\\ &= p_b\Delta^2\frac{4^\nu-1}{3}\end{aligned}$$

3) The total distortion is

$$\begin{aligned}D_{\text{total}} &= D_{\text{channel}} + D_{\text{quantiz.}} = p_b\Delta^2\frac{4^\nu-1}{3} + \frac{x_{\max}^2}{3\cdot N^2}\\ &= p_b\frac{4\cdot x_{\max}^2}{N^2}\frac{4^\nu-1}{3} + \frac{x_{\max}^2}{3\cdot N^2}\end{aligned}$$

or since $N = 2^\nu$
$$D_{\text{total}} = \frac{x_{\max}^2}{3 \cdot 4^\nu}(1 + 4p_b(4^\nu - 1)) = \frac{x_{\max}^2}{3N^2}(1 + 4p_b(N^2 - 1))$$

4)
$$\text{SNR} = \frac{E[X^2]}{D_{\text{total}}} = \frac{E[X^2]3N^2}{x_{\max}^2(1 + 4p_b(N^2 - 1))}$$

If we let $\breve{X} = \frac{X}{x_{\max}}$, then $\frac{E[X^2]}{x_{\max}^2} = E[\breve{X}^2] = \overline{\breve{X}^2}$. Hence,
$$\text{SNR} = \frac{3N^2\overline{\breve{X}^2}}{1 + 4p_b(N^2 - 1)} = \frac{3 \cdot 4^\nu \overline{\breve{X}^2}}{1 + 4p_b(4^\nu - 1)}$$

Problem 6.57
1)
$$g(x) = \frac{\log(1 + \mu \frac{|x|}{x_{\max}})}{\log(1 + \mu)}\text{sgn}(x)$$

Differentiating the previous using natural logarithms, we obtain
$$g'(x) = \frac{1}{\ln(1 + \mu)} \frac{\mu / x_{\max}}{(1 + \mu \frac{|x|}{x_{\max}})}\text{sgn}^2(x)$$

Since, for the μ-law compander $y_{\max} = g(x_{\max}) = 1$, we obtain
$$\begin{aligned} D &\approx \frac{y_{\max}^2}{3 \times 4^\nu} \int_{-\infty}^{\infty} \frac{f_X(x)}{[g'(x)]^2} dx \\ &= \frac{x_{\max}^2 [\ln(1 + \mu)]^2}{3 \times 4^\nu \mu^2} \int_{-\infty}^{\infty} \left(1 + \mu^2 \frac{|x|^2}{x_{\max}^2} + 2\mu \frac{|x|}{x_{\max}}\right) f_X(x) dx \\ &= \frac{x_{\max}^2 [\ln(1 + \mu)]^2}{3 \times 4^\nu \mu^2} \left[1 + \mu^2 E[\breve{X}^2] + 2\mu E[|\breve{X}|]\right] \\ &= \frac{x_{\max}^2 [\ln(1 + \mu)]^2}{3 \times N^2 \mu^2} \left[1 + \mu^2 E[\breve{X}^2] + 2\mu E[|\breve{X}|]\right] \end{aligned}$$

where $N^2 = 4^\nu$ and $\breve{X} = X / x_{\max}$.

2)
$$\begin{aligned} \text{SQNR} &= \frac{E[X^2]}{D} \\ &= \frac{E[X^2]}{x_{\max}^2} \frac{\mu^2 3 \cdot N^2}{[\ln(1 + \mu)]^2 (\mu^2 E[\breve{X}^2] + 2\mu E[|\breve{X}|] + 1)} \\ &= \frac{3\mu^2 N^2 E[\breve{X}^2]}{[\ln(1 + \mu)]^2 (\mu^2 E[\breve{X}^2] + 2\mu E[|\breve{X}|] + 1)} \end{aligned}$$

3) Since $\text{SQNR}_{\text{unif}} = 3 \cdot N^2 E[\breve{X}^2]$, we have
$$\begin{aligned} \text{SQNR}_{\mu_{\text{law}}} &= \text{SQNR}_{\text{unif}} \frac{\mu^2}{[\ln(1 + \mu)]^2 (\mu^2 E[\breve{X}^2] + 2\mu E[|\breve{X}|] + 1)} \\ &= \text{SQNR}_{\text{unif}} G(\mu, \breve{X}) \end{aligned}$$

where we identify

$$G(\mu, \breve{X}) = \frac{\mu^2}{[\ln(1+\mu)]^2(\mu^2 E[\breve{X}^2] + 2\mu E[|\breve{X}|] + 1)}$$

3) The truncated Gaussian distribution has a PDF given by

$$f_Y(y) = \frac{K}{\sqrt{2\pi}\sigma_x} e^{-\frac{x^2}{2\sigma_x^2}}$$

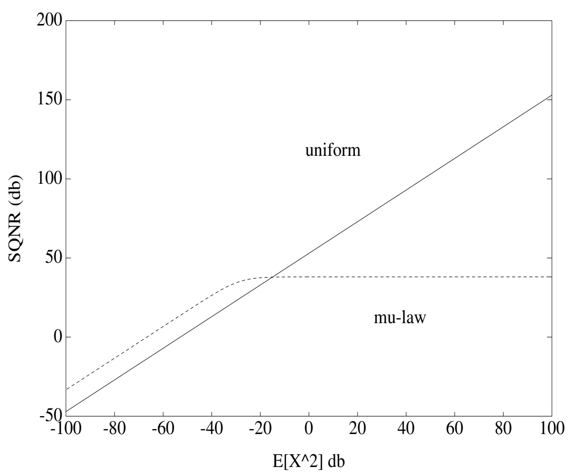
where the constant K is such that

$$K \int_{-4\sigma_x}^{4\sigma_x} \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{x^2}{2\sigma_x^2}} dx = 1 \Longrightarrow K = \frac{1}{1 - 2Q(4)} = 1.0001$$

Hence,

$$\begin{aligned}
E[|\breve{X}|] &= \frac{K}{\sqrt{2\pi}\sigma_x} \int_{-4\sigma_x}^{4\sigma_x} \frac{|x|}{4\sigma_x} e^{-\frac{x^2}{2\sigma_x^2}} dx \\
&= \frac{2K}{4\sqrt{2\pi}\sigma_x^2} \int_0^{4\sigma_x} x e^{-\frac{x^2}{2\sigma_x^2}} dx \\
&= \frac{K}{2\sqrt{2\pi}\sigma_x^2} \left[-\sigma_x^2 e^{-\frac{x^2}{2\sigma_x^2}} \right]_0^{4\sigma_x} \\
&= \frac{K}{2\sqrt{2\pi}} (1 - e^{-2}) = 0.1725
\end{aligned}$$

In the next figure we plot $10\log_{10} \mathrm{SQNR}_{\mathrm{unif}}$ and $10\log_{10} \mathrm{SQNR}_{mu-\mathrm{law}}$ vs. $10\log_{10} E[\breve{X}^2]$ when the latter varies from -100 to 100 db. As it is observed the μ-law compressor is insensitive to the dynamic range of the input signal for $E[\breve{X}^2] > 1$.



Problem 6.58

The optimal compressor has the form

$$g(x) = y_{\max} \left[\frac{2\int_{-\infty}^{x} [f_X(v)]^{\frac{1}{3}} dv}{\int_{-\infty}^{\infty} [f_X(v)]^{\frac{1}{3}} dv} - \right]$$

where $y_{\max} = g(x_{\max}) = g(1)$.

$$\begin{aligned}
\int_{-\infty}^{\infty} [f_X(v)]^{\frac{1}{3}} dv &= \int_{-1}^{1} [f_X(v)]^{\frac{1}{3}} dv = \int_{-1}^{0} (v+1)^{\frac{1}{3}} dv + \int_{0}^{1} (-v+1)^{\frac{1}{3}} dv \\
&= 2\int_0^1 x^{\frac{1}{3}} dx = \frac{3}{2}
\end{aligned}$$

If $x \leq 0$, then

$$\begin{aligned}\int_{-\infty}^{x}[f_X(v)]^{\frac{1}{3}}dv &= \int_{-1}^{x}(v+1)^{\frac{1}{3}}dv = \int_{0}^{x+1} z^{\frac{1}{3}}dz = \frac{3}{4}z^{\frac{4}{3}}\Big|_0^{x+1} \\ &= \frac{3}{4}(x+1)^{\frac{4}{3}}\end{aligned}$$

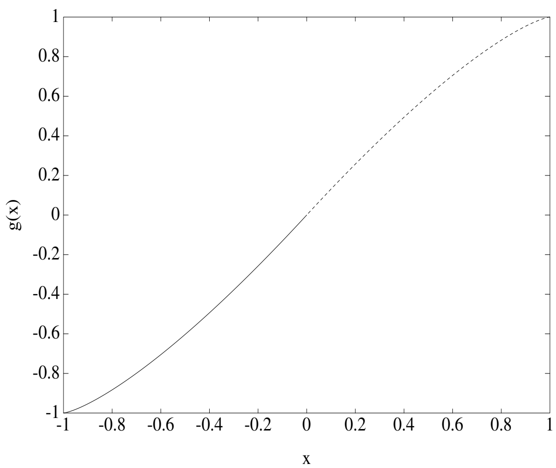
If $x > 0$, then

$$\begin{aligned}\int_{-\infty}^{x}[f_X(v)]^{\frac{1}{3}}dv &= \int_{-1}^{0}(v+1)^{\frac{1}{3}}dv + \int_{0}^{x}(-v+1)^{\frac{1}{3}}dv = \frac{3}{4} + \int_{1-x}^{1} z^{\frac{1}{3}}dz \\ &= \frac{3}{4} + \frac{3}{4}\left(1-(1-x)^{\frac{4}{3}}\right)\end{aligned}$$

Hence,

$$g(x) = \begin{cases} g(1)\left[(x+1)^{\frac{4}{3}} - 1\right] & -1 \leq x < 0 \\ g(1)\left[1-(1-x)^{\frac{4}{3}}\right] & 0 \leq x \leq 1 \end{cases}$$

The next figure depicts $g(x)$ for $g(1) = 1$. Since the resulting distortion is (see Equation 6.6.17)



$$D = \frac{1}{12 \times 4^{\nu}}\left[\int_{-infty}^{\infty}[f_X(x)]^{\frac{1}{3}}dx\right]^3 = \frac{1}{12 \times 4^{\nu}}\left(\frac{3}{2}\right)^3$$

we have

$$\mathrm{SQNR} = \frac{E[X^2]}{D} = \frac{32}{9} \times 4^{\nu}E[X^2] = \frac{32}{9} \times 4^{\nu} \cdot \frac{1}{6} = \frac{16}{27}4^{\nu}$$

Problem 6.59

The sampling rate is $f_s = 44100$ meaning that we take 44100 samples per second. Each sample is quantized using 16 bits so the total number of bits per second is 44100×16. For a music piece of duration 50 min $= 3000$ sec the resulting number of bits per channel (left and right) is

$$44100 \times 16 \times 3000 = 2.1168 \times 10^9$$

and the overall number of bits is

$$2.1168 \times 10^9 \times 2 = 4.2336 \times 10^9$$

Chapter 7

Problem 7.1

The amplitudes A_m take the values

$$A_m = (2m-1-M)\frac{d}{2}, \qquad m = 1, \ldots M$$

Hence, the average energy is

$$\begin{aligned}
\mathcal{E}_{av} &= \frac{1}{M}\sum_{m=1}^{M} s_m^2 = \frac{d^2}{4M}\mathcal{E}_g \sum_{m=1}^{M}(2m-1-M)^2 \\
&= \frac{d^2}{4M}\mathcal{E}_g \sum_{m=1}^{M}[4m^2 + (M+1)^2 - 4m(M+1)] \\
&= \frac{d^2}{4M}\mathcal{E}_g \left(4\sum_{m=1}^{M} m^2 + M(M+1)^2 - 4(M+1)\sum_{m=1}^{M} m\right) \\
&= \frac{d^2}{4M}\mathcal{E}_g \left(4\frac{M(M+1)(2M+1)}{6} + M(M+1)^2 - 4(M+1)\frac{M(M+1)}{2}\right) \\
&= \frac{M^2-1}{3}\frac{d^2}{4}\mathcal{E}_g
\end{aligned}$$

Problem 7.2

The correlation coefficient between the m^{th} and the n^{th} signal points is

$$\gamma_{mn} = \frac{\mathbf{s}_m \cdot \mathbf{s}_n}{|\mathbf{s}_m||\mathbf{s}_n|}$$

where $\mathbf{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN})$ and $s_{mj} = \pm\sqrt{\frac{\mathcal{E}_s}{N}}$. Two adjacent signal points differ in only one coordinate, for which s_{mk} and s_{nk} have opposite signs. Hence,

$$\begin{aligned}
\mathbf{s}_m \cdot \mathbf{s}_n &= \sum_{j=1}^{N} s_{mj}s_{nj} = \sum_{j\neq k} s_{mj}s_{nj} + s_{mk}s_{nk} \\
&= (N-1)\frac{\mathcal{E}_s}{N} - \frac{\mathcal{E}_s}{N} = \frac{N-2}{N}\mathcal{E}_s
\end{aligned}$$

Furthermore, $|\mathbf{s}_m| = |\mathbf{s}_n| = (\mathcal{E}_s)^{\frac{1}{2}}$ so that

$$\gamma_{mn} = \frac{N-2}{N}$$

The Euclidean distance between the two adjacent signal points is

$$d = \sqrt{|\mathbf{s}_m - \mathbf{s}_n|^2} = \sqrt{\left|\pm 2\sqrt{\mathcal{E}_s/N}\right|^2} = \sqrt{4\frac{\mathcal{E}_s}{N}} = 2\sqrt{\frac{\mathcal{E}_s}{N}}$$

Problem 7.3

a) To show that the waveforms $\psi_n(t)$, $n = 1, \ldots, 3$ are orthogonal we have to prove that

$$\int_{-\infty}^{\infty} \psi_m(t)\psi_n(t)dt = 0, \qquad m \neq n$$

Clearly,

$$\begin{aligned}
c_{12} &= \int_{-\infty}^{\infty} \psi_1(t)\psi_2(t)dt = \int_0^4 \psi_1(t)\psi_2(t)dt \\
&= \int_0^2 \psi_1(t)\psi_2(t)dt + \int_2^4 \psi_1(t)\psi_2(t)dt \\
&= \frac{1}{4}\int_0^2 dt - \frac{1}{4}\int_2^4 dt = \frac{1}{4} \times 2 - \frac{1}{4} \times (4-2) \\
&= 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
c_{13} &= \int_{-\infty}^{\infty} \psi_1(t)\psi_3(t)dt = \int_0^4 \psi_1(t)\psi_3(t)dt \\
&= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt - \frac{1}{4}\int_2^3 dt + \frac{1}{4}\int_3^4 dt \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
c_{23} &= \int_{-\infty}^{\infty} \psi_2(t)\psi_3(t)dt = \int_0^4 \psi_2(t)\psi_3(t)dt \\
&= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt + \frac{1}{4}\int_2^3 dt - \frac{1}{4}\int_3^4 dt \\
&= 0
\end{aligned}$$

Thus, the signals $\psi_n(t)$ are orthogonal.

b) We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t)\psi_n(t)dt, \qquad n = 1, 2, 3$$

$$\begin{aligned}
x_1 &= \int_0^4 x(t)\psi_1(t)dt = -\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_1^2 dt - \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0 \\
x_2 &= \int_0^4 x(t)\psi_2(t)dt = \frac{1}{2}\int_0^4 x(t)dt = 0 \\
x_3 &= \int_0^4 x(t)\psi_3(t)dt = -\frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt + \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0
\end{aligned}$$

As it is observed, $x(t)$ is orthogonal to the signal waveforms $\psi_n(t)$, $n = 1, 2, 3$ and thus it can not represented as a linear combination of these functions.

Problem 7.4

a) The expansion coefficients $\{c_n\}$, that minimize the mean square error, satisfy

$$c_n = \int_{-\infty}^{\infty} x(t)\psi_n(t)dt = \int_0^4 \sin\frac{\pi t}{4}\psi_n(t)dt$$

Hence,

$$\begin{aligned}
c_1 &= \int_0^4 \sin\frac{\pi t}{4}\psi_1(t)dt = \frac{1}{2}\int_0^2 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_2^4 \sin\frac{\pi t}{4}dt \\
&= -\frac{2}{\pi}\cos\frac{\pi t}{4}\Big|_0^2 + \frac{2}{\pi}\cos\frac{\pi t}{4}\Big|_2^4 \\
&= -\frac{2}{\pi}(0-1) + \frac{2}{\pi}(-1-0) = 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
c_2 &= \int_0^4 \sin\frac{\pi t}{4}\psi_2(t)dt = \frac{1}{2}\int_0^4 \sin\frac{\pi t}{4}dt \\
&= -\frac{2}{\pi}\cos\frac{\pi t}{4}\Big|_0^4 = -\frac{2}{\pi}(-1-1) = \frac{4}{\pi}
\end{aligned}$$

and

$$\begin{aligned}
c_3 &= \int_0^4 \sin\frac{\pi t}{4}\psi_3(t)dt \\
&= \frac{1}{2}\int_0^1 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_1^2 \sin\frac{\pi t}{4}dt + \frac{1}{2}\int_2^3 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_3^4 \sin\frac{\pi t}{4}dt \\
&= 0
\end{aligned}$$

Note that c_1, c_2 can be found by inspection since $\sin\frac{\pi t}{4}$ is even with respect to the $x = 2$ axis and $\psi_1(t)$, $\psi_3(t)$ are odd with respect to the same axis.

b) The residual mean square error $E_{\min}$ can be found from

$$E_{\min} = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{3} |c_i|^2$$

Thus,

$$\begin{aligned}
E_{\min} &= \int_0^4 \left(\sin\frac{\pi t}{4}\right)^2 dt - \left(\frac{4}{\pi}\right)^2 = \frac{1}{2}\int_0^4 \left(1-\cos\frac{\pi t}{2}\right)dt - \frac{16}{\pi^2} \\
&= 2 - \frac{1}{\pi}\sin\frac{\pi t}{2}\Big|_0^4 - \frac{16}{\pi^2} = 2 - \frac{16}{\pi^2}
\end{aligned}$$

Problem 7.5

a) As an orthonormal set of basis functions we consider the set

$$\psi_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad \psi_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases}$$

$$\psi_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases} \qquad \psi_4(t) = \begin{cases} 1 & 3 \le t < 4 \\ 0 & \text{o.w} \end{cases}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \\ \psi_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

b) The representation vectors are

$$\begin{aligned}
\mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & -1 & -1 \end{bmatrix} \\
\mathbf{s}_2 &= \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix} \\
\mathbf{s}_3 &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\
\mathbf{s}_4 &= \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix}
\end{aligned}$$

c) The distance between the first and the second vector is

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left|\begin{bmatrix} 4 & -2 & -2 & -1 \end{bmatrix}\right|^2} = \sqrt{25}$$

Similarly we find that

$$\begin{aligned}
d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix}\right|^2} = \sqrt{5} \\
d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix}\right|^2} = \sqrt{12} \\
d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} -3 & 2 & 0 & 1 \end{bmatrix}\right|^2} = \sqrt{14} \\
d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} -3 & 3 & 3 & -2 \end{bmatrix}\right|^2} = \sqrt{31} \\
d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 0 & 1 & 3 & -3 \end{bmatrix}\right|^2} = \sqrt{19}
\end{aligned}$$

Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

Problem 7.6

As a set of orthonormal functions we consider the waveforms

$$\psi_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad \psi_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases} \qquad \psi_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned}
\mathbf{s}_1 &= \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \\
\mathbf{s}_2 &= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \\
\mathbf{s}_3 &= \begin{bmatrix} 0 & -2 & -2 \end{bmatrix} \\
\mathbf{s}_4 &= \begin{bmatrix} 2 & 2 & 0 \end{bmatrix}
\end{aligned}$$

Note that $s_3(t) = s_2(t) - s_1(t)$ and that the dimensionality of the waveforms is 3.

Problem 7.7

The energy of the signal waveform $s'_m(t)$ is

$$\begin{aligned}
\mathcal{E}' &= \int_{-\infty}^{\infty} |s'_m(t)|^2\, dt = \int_{-\infty}^{\infty} \left| s_m(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t) \right|^2 dt \\
&= \int_{-\infty}^{\infty} s_m^2(t)dt + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_k(t)s_l(t)dt \\
&\quad -\frac{1}{M}\sum_{k=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_k(t)dt - \frac{1}{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_l(t)dt \\
&= \mathcal{E} + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\mathcal{E}\delta_{kl} - \frac{2}{M}\mathcal{E} \\
&= \mathcal{E} + \frac{1}{M}\mathcal{E} - \frac{2}{M}\mathcal{E} = \left(\frac{M-1}{M}\right)\mathcal{E}
\end{aligned}$$

The correlation coefficient is given by

$$\begin{aligned}
\gamma_{mn} &= \frac{\int_{-\infty}^{\infty} s'_m(t)s'_n(t)dt}{\left[\int_{-\infty}^{\infty}|s'_m(t)|^2dt\right]^{\frac{1}{2}}\left[\int_{-\infty}^{\infty}|s'_n(t)|^2dt\right]^{\frac{1}{2}}} \\
&= \frac{1}{\mathcal{E}'}\int_{-\infty}^{\infty}\left(s_m(t) - \frac{1}{M}\sum_{k=1}^{M}s_k(t)\right)\left(s_n(t) - \frac{1}{M}\sum_{l=1}^{M}s_l(t)\right)dt \\
&= \frac{1}{\mathcal{E}'}\left(\int_{-\infty}^{\infty} s_m(t)s_n(t)dt + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_k(t)s_l(t)dt\right) \\
&\quad -\frac{1}{\mathcal{E}'}\left(\frac{1}{M}\sum_{k=1}^{M}\int_{-\infty}^{\infty} s_n(t)s_k(t)dt + \frac{1}{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_l(t)dt\right) \\
&= \frac{\frac{1}{M^2}M\mathcal{E} - \frac{1}{M}\mathcal{E} - \frac{1}{M}\mathcal{E}}{\frac{M-1}{M}\mathcal{E}} = -\frac{1}{M-1}
\end{aligned}$$

Problem 7.8

Assuming that $E[n^2(t)] = \sigma_n^2$, we obtain

$$\begin{aligned}
E[n_1 n_2] &= E\left[\left(\int_0^T s_1(t)n(t)dt\right)\left(\int_0^T s_2(v)n(v)dv\right)\right] \\
&= \int_0^T\int_0^T s_1(t)s_2(v)E[n(t)n(v)]dtdv \\
&= \sigma_n^2\int_0^T s_1(t)s_2(t)dt \\
&= 0
\end{aligned}$$

where the last equality follows from the orthogonality of the signal waveforms $s_1(t)$ and $s_2(t)$.

Problem 7.9

a) The received signal may be expressed as

$$r(t) = \begin{cases} n(t) & \text{if } s_0(t) \text{ was transmitted} \\ A + n(t) & \text{if } s_1(t) \text{ was transmitted} \end{cases}$$

Assuming that $s(t)$ has unit energy, then the sampled outputs of the crosscorrelators are

$$r = s_m + n, \qquad m = 0, 1$$

where $s_0 = 0$, $s_1 = A\sqrt{T}$ and the noise term n is a zero-mean Gaussian random variable with variance

$$\begin{aligned}
\sigma_n^2 &= E\left[\frac{1}{\sqrt{T}}\int_0^T n(t)dt \frac{1}{\sqrt{T}}\int_0^T n(\tau)d\tau\right] \\
&= \frac{1}{T}\int_0^T\int_0^T E\left[n(t)n(\tau)\right] dtd\tau \\
&= \frac{N_0}{2T}\int_0^T\int_0^T \delta(t-\tau)dtd\tau = \frac{N_0}{2}
\end{aligned}$$

The probability density function for the sampled output is

$$\begin{aligned}
f(r|s_0) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} \\
f(r|s_1) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}}
\end{aligned}$$

Since the signals are equally probable, the optimal detector decides in favor of s_0 if

$$\mathrm{PM}(\mathbf{r}, \mathbf{s}_0) = f(r|s_0) > f(r|s_1) = \mathrm{PM}(\mathbf{r}, \mathbf{s}_1)$$

otherwise it decides in favor of s_1. The decision rule may be expressed as

$$\frac{\mathrm{PM}(\mathbf{r}, \mathbf{s}_0)}{\mathrm{PM}(\mathbf{r}, \mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2 - r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} 1$$

or equivalently

$$r \underset{s_0}{\overset{s_1}{\gtrless}} \frac{1}{2}A\sqrt{T}$$

The optimum threshold is $\frac{1}{2}A\sqrt{T}$.

b) The average probability of error is

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} f(r|s_0)dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} f(r|s_1)dr \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} dr \\
&= \frac{1}{2}\int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + \frac{1}{2}\int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= Q\left[\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}\right] = Q\left[\sqrt{\mathrm{SNR}}\right]
\end{aligned}$$

where

$$\text{SNR} = \frac{\frac{1}{2}A^2 T}{N_0}$$

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

Problem 7.10

Since the rate of transmission is $R = 10^5$ bits/sec, the bit interval T_b is 10^{-5} sec. The probability of error in a binary PAM system is

$$P(e) = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

where the bit energy is $\mathcal{E}_b = A^2 T_b$. With $P(e) = P_2 = 10^{-6}$, we obtain

$$\sqrt{\frac{2\mathcal{E}_b}{N_0}} = 4.75 \Longrightarrow \mathcal{E}_b = \frac{4.75^2 N_0}{2} = 0.112813$$

Thus

$$A^2 T_b = 0.112813 \Longrightarrow A = \sqrt{0.112813 \times 10^5} = 106.21$$

Problem 7.11

a) For a binary PAM system for which the two signals have unequal probability, the optimum detector is

$$r \underset{s_2}{\overset{s_1}{\gtrless}} \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln\frac{1-p}{p} = \eta$$

The average probability of error is

$$\begin{aligned}
P(e) &= P(e|s_1)P(s_1) + P(e|s_2)P(s_2) \\
&= pP(e|s_1) + (1-p)P(e|s_2) \\
&= p\int_{-\infty}^{\eta} f(r|s_1)dr + (1-p)\int_{\eta}^{\infty} f(r|s_1)dr \\
&= p\int_{-\infty}^{\eta} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-\sqrt{\mathcal{E}_b})^2}{N_0}} dr + (1-p)\int_{\eta}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r+\sqrt{\mathcal{E}_b})^2}{N_0}} dr \\
&= p\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\eta_1} e^{-\frac{x^2}{2}} dx + (1-p)\frac{1}{\sqrt{2\pi}}\int_{\eta_2}^{\infty} e^{-\frac{x^2}{2}} dx
\end{aligned}$$

where

$$\eta_1 = -\sqrt{\frac{2\mathcal{E}_b}{N_0}} + \eta\sqrt{\frac{2}{N_0}} \qquad \eta_2 = \sqrt{\frac{2\mathcal{E}_b}{N_0}} + \eta\sqrt{\frac{2}{N_0}}$$

Thus,

$$P(e) = pQ\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \eta\sqrt{\frac{2}{N_0}}\right] + (1-p)Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} + \eta\sqrt{\frac{2}{N_0}}\right]$$

b) If $p = 0.3$ and $\frac{\mathcal{E}_b}{N_0} = 10$, then

$$\begin{aligned}
P(e) &= 0.3Q[4.3774] + 0.7Q[4.5668] = 0.3 \times 6.01 \times 10^{-6} + 0.7 \times 2.48 \times 10^{-6} \\
&= 3.539 \times 10^{-6}
\end{aligned}$$

If the symbols are equiprobable, then

$$P(e) = Q[\sqrt{\frac{2\mathcal{E}_b}{N_0}}] = Q[\sqrt{20}] = 3.88 \times 10^{-6}$$

Problem 7.12

a) The optimum threshold is given by

$$\eta = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{1-p}{p} = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$$

b) The average probability of error is ($\eta = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$)

$$\begin{aligned}
P(e) &= p(a_m = -1) \int_\eta^\infty \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{\mathcal{E}_b})^2/N_0} dr \\
&\quad + p(a_m = 1) \int_{-\infty}^\eta \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{\mathcal{E}_b})^2/N_0} dr \\
&= \frac{2}{3} Q\left[\frac{\eta + \sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}}\right] + \frac{1}{3} Q\left[\frac{\sqrt{\mathcal{E}_b} - \eta}{\sqrt{N_0/2}}\right] \\
&= \frac{2}{3} Q\left[\frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4} + \sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{3} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4}\right]
\end{aligned}$$

Problem 7.13

a) The maximum likelihood criterion selects the maximum of $f(\mathbf{r}|\mathbf{s}_m)$ over the M possible transmitted signals. When $M = 2$, the ML criterion takes the form

$$\frac{f(\mathbf{r}|\mathbf{s}_1)}{f(\mathbf{r}|\mathbf{s}_2)} \underset{s_2}{\overset{s_1}{\gtrless}} 1$$

or, since

$$\begin{aligned}
f(\mathbf{r}|\mathbf{s}_1) &= \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{\mathcal{E}_b})^2/N_0} \\
f(\mathbf{r}|\mathbf{s}_2) &= \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{\mathcal{E}_b})^2/N_0}
\end{aligned}$$

the optimum maximum-likelihood decision rule is

$$r \underset{s_2}{\overset{s_1}{\gtrless}} 0$$

b) The average probability of error is given by

$$P(e) = p \int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{\mathcal{E}_b})^2/N_0} dr + (1-p) \int_{-\infty}^0 \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{\mathcal{E}_b})^2/N_0} dr$$

$$
\begin{aligned}
&= p\int_{\sqrt{2\mathcal{E}_b/N_0}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + (1-p)\int_{-\infty}^{-\sqrt{2\mathcal{E}_b/N_0}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= pQ\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + (1-p)Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] \\
&= Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]
\end{aligned}
$$

Problem 7.14

a) The impulse response of the filter matched to $s(t)$ is

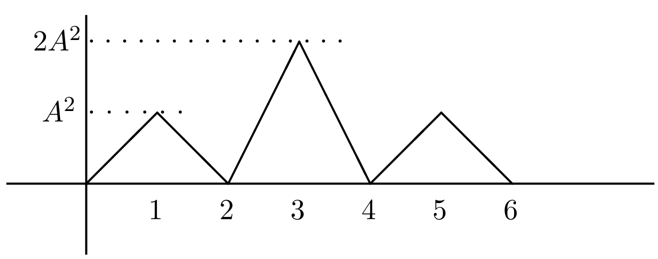
$$h(t) = s(T-t) = s(3-t) = s(t)$$

where we have used the fact that $s(t)$ is even with respect to the $t = \frac{T}{2} = \frac{3}{2}$ axis.

b) The output of the matched filter is

$$
\begin{aligned}
y(t) &= s(t) \star s(t) = \int_0^t s(\tau)s(t-\tau)d\tau \\
&= \begin{cases} 0 & t < 0 \\ A^2 t & 0 \le t < 1 \\ A^2(2-t) & 1 \le t < 2 \\ 2A^2(t-2) & 2 \le t < 3 \\ 2A^2(4-t) & 3 \le t < 4 \\ A^2(t-4) & 4 \le t < 5 \\ A^2(6-t) & 5 \le t < 6 \\ 0 & 6 \le t \end{cases}
\end{aligned}
$$

A scetch of $y(t)$ is depicted in the next figure



c) At the output of the matched filter and for $t = T = 3$ the noise is

$$
\begin{aligned}
n_T &= \int_0^T n(\tau)h(T-\tau)d\tau \\
&= \int_0^T n(\tau)s(T-(T-\tau))d\tau = \int_0^T n(\tau)s(\tau)d\tau
\end{aligned}
$$

The variance of the noise is

$$
\begin{aligned}
\sigma^2_{n_T} &= E\left[\int_0^T \int_0^T n(\tau)n(v)s(\tau)s(v)d\tau dv\right] \\
&= \int_0^T \int_0^T s(\tau)s(v)E[n(\tau)n(v)]d\tau dv
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{N_0}{2}\int_0^T\int_0^T s(\tau)s(v)\delta(\tau - v)d\tau dv \\
&= \frac{N_0}{2}\int_0^T s^2(\tau)d\tau = N_0 A^2
\end{aligned}
$$

d) For antipodal equiprobable signals the probability of error is

$$P(e) = Q\left[\sqrt{\left(\frac{S}{N}\right)_o}\right]$$

where $\left(\frac{S}{N}\right)_o$ is the output SNR from the matched filter. Since

$$\left(\frac{S}{N}\right)_o = \frac{y^2(T)}{E[n_T^2]} = \frac{4A^4}{N_0 A^2}$$

we obtain

$$P(e) = Q\left[\sqrt{\frac{4A^2}{N_0}}\right]$$

Problem 7.15

a) Taking the inverse Fourier transform of $H(f)$, we obtain

$$
\begin{aligned}
h(t) &= \mathcal{F}^{-1}[H(f)] = \mathcal{F}^{-1}\left[\frac{1}{j2\pi f}\right] - \mathcal{F}^{-1}\left[\frac{e^{-j2\pi f T}}{j2\pi f}\right] \\
&= \mathrm{sgn}(t) - \mathrm{sgn}(t-T) = 2\Pi\left(\frac{t-\frac{T}{2}}{T}\right)
\end{aligned}
$$

b) The signal waveform, to which $h(t)$ is matched, is

$$s(t) = h(T-t) = 2\Pi\left(\frac{T-t-\frac{T}{2}}{T}\right) = 2\Pi\left(\frac{\frac{T}{2}-t}{T}\right) = h(t)$$

where we have used the symmetry of $\Pi\left(\frac{t-\frac{T}{2}}{T}\right)$ with respect to the $t = \frac{T}{2}$ axis.

Problem 7.16

If $g_T(t) = \mathrm{sinc}(t)$, then its matched waveform is $h(t) = \mathrm{sinc}(-t) = \mathrm{sinc}(t)$. Since, (see Problem 2.17)

$$\mathrm{sinc}(t) \star \mathrm{sinc}(t) = \mathrm{sinc}(t)$$

the output of the matched filter is the same sinc pulse. If

$$g_T(t) = \mathrm{sinc}(\frac{2}{T}(t-\frac{T}{2}))$$

then the matched waveform is

$$h(t) = g_T(T-t) = \mathrm{sinc}(\frac{2}{T}(\frac{T}{2}-t)) = g_T(t)$$

where the last equality follows from the fact that $g_T(t)$ is even with respect to the $t = \frac{T}{2}$ axis. The output of the matched filter is

$$\begin{aligned}
y(t) &= \mathcal{F}^{-1}[g_T(t) \star g_T(t)] \\
&= \mathcal{F}^{-1}\left[\frac{T^2}{4}\Pi(\frac{T}{2}f)e^{-j2\pi fT}\right] \\
&= \frac{T}{2}\text{sinc}(\frac{2}{T}(t-T)) = \frac{T}{2}g_T(t-\frac{T}{2})
\end{aligned}$$

Thus the output of the matched filter is the same sinc function, scaled by $\frac{T}{2}$ and centered at $t = T$.

Problem 7.17

1) The output of the integrator is

$$\begin{aligned}
y(t) &= \int_0^t r(\tau)d\tau = \int_0^t [s_i(\tau) + n(\tau)]d\tau \\
&= \int_0^t s_i(\tau)d\tau + \int_0^t n(\tau)d\tau
\end{aligned}$$

At time $t = T$ we have

$$y(T) = \int_0^T s_i(\tau)d\tau + \int_0^T n(\tau)d\tau = \pm\sqrt{\frac{\mathcal{E}_b}{T}}T + \int_0^T n(\tau)d\tau$$

The signal energy at the output of the integrator at $t = T$ is

$$\mathcal{E}_s = \left(\pm\sqrt{\frac{\mathcal{E}_b}{T}}T\right)^2 = \mathcal{E}_b T$$

whereas the noise power

$$\begin{aligned}
P_n &= E\left[\int_0^T \int_0^T n(\tau)n(v)d\tau dv\right] \\
&= \int_0^T \int_0^T E[n(\tau)n(v)]d\tau dv \\
&= \frac{N_0}{2}\int_0^T \int_0^T \delta(\tau - v)d\tau dv = \frac{N_0}{2}T
\end{aligned}$$

Hence, the output SNR is

$$\text{SNR} = \frac{\mathcal{E}_s}{P_n} = \frac{2\mathcal{E}_b}{N_0}$$

2) The transfer function of the RC filter is

$$H(f) = \frac{1}{1 + j2\pi RCf}$$

Thus, the impulse response of the filter is

$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u_{-1}(t)$$

and the output signal is given by

$$\begin{aligned}
y(t) &= \frac{1}{RC}\int_{-\infty}^{t} r(\tau)e^{-\frac{t-\tau}{RC}}\,d\tau \\
&= \frac{1}{RC}\int_{-\infty}^{t} (s_i(\tau)+n(\tau))e^{-\frac{t-\tau}{RC}}\,d\tau \\
&= \frac{1}{RC}e^{-\frac{t}{RC}}\int_{0}^{t} s_i(\tau)e^{\frac{\tau}{RC}}\,d\tau + \frac{1}{RC}e^{-\frac{t}{RC}}\int_{-\infty}^{t} n(\tau)e^{\frac{\tau}{RC}}\,d\tau
\end{aligned}$$

At time $t = T$ we obtain

$$y(T) = \frac{1}{RC}e^{-\frac{T}{RC}}\int_{0}^{T} s_i(\tau)e^{\frac{\tau}{RC}}\,d\tau + \frac{1}{RC}e^{-\frac{T}{RC}}\int_{-\infty}^{T} n(\tau)e^{\frac{\tau}{RC}}\,d\tau$$

The signal energy at the output of the filter is

$$\begin{aligned}
\mathcal{E}_s &= \frac{1}{(RC)^2}e^{-\frac{2T}{RC}}\int_0^T\int_0^T s_i(\tau)s_i(v)e^{\frac{\tau}{RC}}e^{\frac{v}{RC}}\,d\tau dv \\
&= \frac{1}{(RC)^2}e^{-\frac{2T}{RC}}\frac{\mathcal{E}_b}{T}\left(\int_0^T e^{\frac{\tau}{RC}}\,d\tau\right)^2 \\
&= e^{-\frac{2T}{RC}}\frac{\mathcal{E}_b}{T}\left(e^{\frac{T}{RC}}-1\right)^2 \\
&= \frac{\mathcal{E}_b}{T}\left(1-e^{-\frac{T}{RC}}\right)^2
\end{aligned}$$

The noise power at the output of the filter is

$$\begin{aligned}
P_n &= \frac{1}{(RC)^2}e^{-\frac{2T}{RC}}\int_{-\infty}^{T}\int_{-\infty}^{T} E[n(\tau)n(v)]d\tau dv \\
&= \frac{1}{(RC)^2}e^{-\frac{2T}{RC}}\int_{-\infty}^{T}\int_{-\infty}^{T} \frac{N_0}{2}\delta(\tau-v)e^{\frac{\tau+v}{RC}}\,d\tau dv \\
&= \frac{1}{(RC)^2}e^{-\frac{2T}{RC}}\int_{-\infty}^{T}\frac{N_0}{2}e^{\frac{2\tau}{RC}}\,d\tau \\
&= \frac{1}{2RC}e^{-\frac{2T}{RC}}\frac{N_0}{2}e^{\frac{2T}{RC}} = \frac{1}{2RC}\frac{N_0}{2}
\end{aligned}$$

Hence,

$$\text{SNR} = \frac{\mathcal{E}_s}{P_n} = \frac{4\mathcal{E}_b RC}{TN_0}\left(1-e^{-\frac{T}{RC}}\right)^2$$

3) The value of RC that maximizes SNR, can be found by setting the partial derivative of SNR with respect to RC equal to zero. Thus, if $a = RC$, then

$$\frac{\vartheta \text{SNR}}{\vartheta a} = 0 = (1-e^{-\frac{T}{a}}) - \frac{T}{a}e^{-\frac{T}{a}} = -e^{-\frac{T}{a}}\left(1+\frac{T}{a}\right)+1$$

Solving this transcendental equation numerically for a, we obtain

$$\frac{T}{a} = 1.26 \Longrightarrow RC = a = \frac{T}{1.26}$$

Problem 7.18

1) The matched filter is

$$h_1(t) = s_1(T-t) = \begin{cases} -\frac{1}{T}t+1, & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

The output of the matched filter is

$$y_1(t) = \int_{-\infty}^{\infty} s_1(\tau) h_1(t-\tau) d\tau$$

If $t \leq 0$, then $y_1(t) = 0$, If $0 < t \leq T$, then

$$\begin{aligned} y_1(t) &= \int_0^{\infty} \frac{\tau}{T}\left(-\frac{1}{T}(t-\tau)+1\right) d\tau \\ &= \int_0^t \tau\left(\frac{1}{T}-\frac{t}{T^2}\right) d\tau + \frac{1}{T^2}\int_0^t \tau^2 d\tau \\ &= -\frac{t^3}{6T^2} + \frac{t^2}{2T} \end{aligned}$$

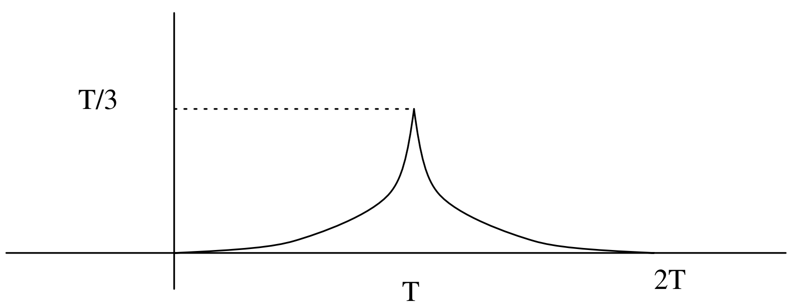
If $T \leq t \leq 2T$, then

$$\begin{aligned} y_1(t) &= \int_{t-\tau}^{T} \frac{\tau}{T}\left(-\frac{1}{T}(t-\tau)+1\right) d\tau \\ &= \int_{t-\tau}^{T} \tau\left(\frac{1}{T}-\frac{t}{T^2}\right) d\tau + \frac{1}{T^2}\int_{t-\tau}^{T} \tau^2 d\tau \\ &= \frac{(t-T)^3}{6T^2} - \frac{t-T}{2} + \frac{T}{3} \end{aligned}$$

For $2T < 0$, we obtain $y_1(t) = 0$. In summary

$$y_1(t) = \begin{cases} 0 & t \leq 0 \\ -\frac{t^3}{6T^2} + \frac{t^2}{2T} & 0 < t \leq T \\ \frac{(t-T)^3}{6T^2} - \frac{t-T}{2} + \frac{T}{3} & T < t \leq 2T \\ 0 & 2T < t \end{cases}$$

A sketch of $y_1(t)$ is given in the next figure. As it is observed the maximum of $y_1(t)$, which is $\frac{T}{3}$, is achieved for $t = T$.



2) The signal waveform matched to $s_2(t)$ is

$$h_2(t) = \begin{cases} -1, & 0 \leq t \leq \frac{T}{2} \\ 2, & \frac{T}{2} < t \leq T \end{cases}$$

The output of the matched filter is

$$y_2(t) = \int_{-\infty}^{\infty} s_2(\tau) h_2(t-\tau) d\tau$$

If $t \leq 0$ or $t \geq 2T$, then $y_2(t) = 0$. If $0 < t \leq \frac{T}{2}$, then $y_2(t) = \int_0^t (-2) d\tau = -2t$. If $\frac{T}{2} < t \leq T$, then

$$y_2(t) = \int_0^{t-\frac{T}{2}} 4 d\tau + \int_{t-\frac{T}{2}}^{\frac{T}{2}} (-2) d\tau + \int_{-\frac{T}{2}}^{t} d\tau = 7t - \frac{9}{2}T$$

If $T < t \leq \frac{3T}{2}$, then

$$y_2(t) = \int_{t-T}^{\frac{T}{2}} 4d\tau + \int_{\frac{T}{2}}^{t-\frac{T}{2}} (-2)d\tau + \int_{t-\frac{T}{2}}^{T} d\tau = \frac{19T}{2} - 7t$$

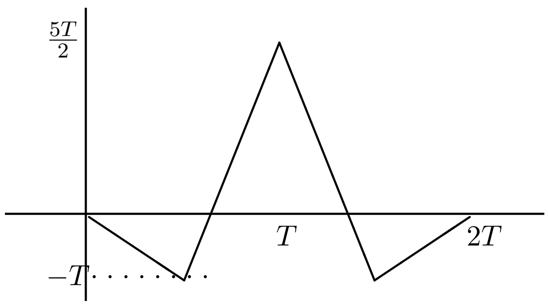
For, $\frac{3T}{2} < t \leq 2T$, we obtain

$$y_2(t) = \int_{t-T}^{T} (-2)d\tau = 2t - 4T$$

In summary

$$y_2(t) = \begin{cases} 0 & t \leq 0 \\ -2t & 0 < t \leq \frac{T}{2} \\ 7t - \frac{9}{2}T & \frac{T}{2} < t \leq T \\ \frac{19T}{2} - 7t & T < t \leq \frac{3T}{2} \\ 2t - 4T & \frac{3T}{2} < t \leq 2T \\ 0 & 2T < t \end{cases}$$

A plot of $y_2(t)$ is shown in the next figure



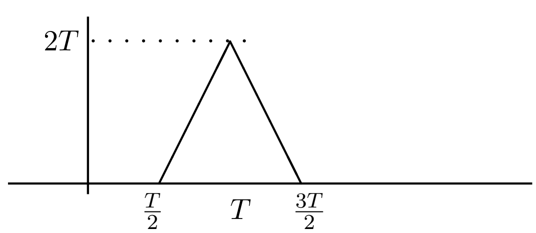
3) The signal waveform matched to $s_3(t)$ is

$$h_3(t) = \begin{cases} 2 & 0 \leq t \leq \frac{T}{2} \\ 0 & \frac{T}{2} < t \leq T \end{cases}$$

The output of the matched filter is

$$y_3(t) = h_3(t) \star s_3(t) = \begin{cases} 4t - 2T & \frac{T}{2} \leq t < T \\ -4t + 6T & T \leq t \leq \frac{3T}{2} \end{cases}$$

In the next figure we have plotted $y_3(t)$.



Problem 7.19

1) Since $m_2(t) = -m_3(t)$ the dimensionality of the signal space is two.

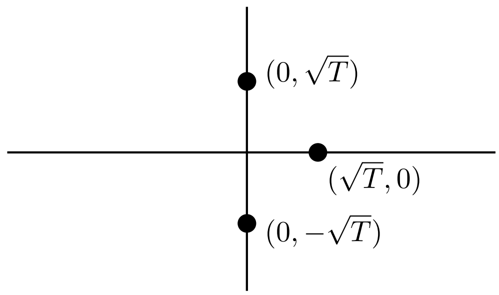
2) As a basis of the signal space we consider the functions

$$\psi_1(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \qquad \psi_2(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t \leq \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} < t \leq T \\ 0 & \text{otherwise} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned}\mathbf{m}_1 &= [\sqrt{T},\ \ 0]\\ \mathbf{m}_2 &= [0,\ \ \sqrt{T}]\\ \mathbf{m}_3 &= [0,\ -\sqrt{T}]\end{aligned}$$

3) The signal constellation is depicted in the next figure



4) The three possible outputs of the matched filters, corresponding to the three possible transmitted signals are $(r_1, r_2) = (\sqrt{T} + n_1, n_2)$, $(n_1, \sqrt{T} + n_2)$ and $(n_1, -\sqrt{T} + n_2)$, where n_1, n_2 are zero-mean Gaussian random variables with variance $\frac{N_0}{2}$. If all the signals are equiprobable the optimum decision rule selects the signal that maximizes the metric

$$C(\mathbf{r} \cdot \mathbf{m}_i) = 2\mathbf{r} \cdot \mathbf{m}_i - |\mathbf{m}_i|^2$$

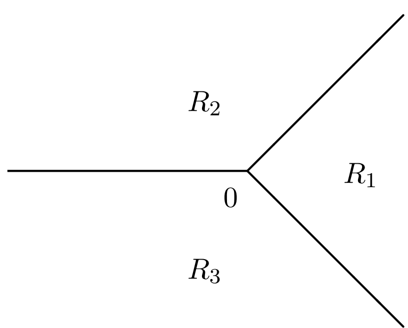
or since $|\mathbf{m}_i|^2$ is the same for all i,

$$C'(\mathbf{r} \cdot \mathbf{m}_i) = \mathbf{r} \cdot \mathbf{m}_i$$

Thus the optimal decision region R_1 for $\mathbf{m}_1$ is the set of points (r_1, r_2), such that $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_2$ and $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_3$. Since $(r_1, r_2) \cdot \mathbf{m}_1 = \sqrt{T} r_1$, $(r_1, r_2) \cdot \mathbf{m}_2 = \sqrt{T} r_2$ and $(r_1, r_2) \cdot \mathbf{m}_3 = -\sqrt{T} r_2$, the previous conditions are written as

$$r_1 > r_2 \quad \text{and} \quad r_1 > -r_2$$

Similarly we find that R_2 is the set of points (r_1, r_2) that satisfy $r_2 > 0$, $r_2 > r_1$ and R_3 is the region such that $r_2 < 0$ and $r_2 < -r_1$. The regions R_1, R_2 and R_3 are shown in the next figure.



5) If the signals are equiprobable then,

$$P(e|\mathbf{m}_1) = P(|\mathbf{r} - \mathbf{m}_1|^2 > |\mathbf{r} - \mathbf{m}_2|^2 | \mathbf{m}_1) + P(|\mathbf{r} - \mathbf{m}_1|^2 > |\mathbf{r} - \mathbf{m}_3|^2 | \mathbf{m}_1)$$

When $\mathbf{m}_1$ is transmitted then $\mathbf{r} = [\sqrt{T} + n_1, n_2]$ and therefore, $P(e|\mathbf{m}_1)$ is written as

$$P(e|\mathbf{m}_1) = P(n_2 - n_1 > \sqrt{T}) + P(n_1 + n_2 < -\sqrt{T})$$

Since, n_1, n_2 are zero-mean statistically independent Gaussian random variables, each with variance $\frac{N_0}{2}$, the random variables $x = n_1 - n_2$ and $y = n_1 + n_2$ are zero-mean Gaussian with variance N_0. Hence,

$$\begin{aligned} P(e|\mathbf{m}_1) &= \frac{1}{\sqrt{2\pi N_0}} \int_{\sqrt{T}}^{\infty} e^{-\frac{x^2}{2N_0}} dx + \frac{1}{\sqrt{2\pi N_0}} \int_{-\infty}^{-\sqrt{T}} e^{-\frac{y^2}{2N_0}} dy \\ &= Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{T}{N_0}}\right] = 2Q\left[\sqrt{\frac{T}{N_0}}\right] \end{aligned}$$

When $\mathbf{m}_2$ is transmitted then $\mathbf{r} = [n_1, n_2 + \sqrt{T}]$ and therefore,

$$\begin{aligned} P(e|\mathbf{m}_2) &= P(n_1 - n_2 > \sqrt{T}) + P(n_2 < -\sqrt{T}) \\ &= Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{2T}{N_0}}\right] \end{aligned}$$

Similarly from the symmetry of the problem, we obtain

$$P(e|\mathbf{m}_2) = P(e|\mathbf{m}_3) = Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{2T}{N_0}}\right]$$

Since $Q[\cdot]$ is momononically decreasing, we obtain

$$Q\left[\sqrt{\frac{2T}{N_0}}\right] < Q\left[\sqrt{\frac{T}{N_0}}\right]$$

and therefore, the probability of error $P(e|\mathbf{m}_1)$ is larger than $P(e|\mathbf{m}_2)$ and $P(e|\mathbf{m}_3)$. Hence, the message $\mathbf{m}_1$ is more vulnerable to errors.

Problem 7.20

The optimal receiver bases its decisions on the metrics

$$\mathrm{PM}(\mathbf{r}, \mathbf{s}_m) = f(\mathbf{r}|\mathbf{s}_m)P(\mathbf{s}_m)$$

For an additive noise channel $\mathbf{r} = \mathbf{s}_m + \mathbf{n}$, so

$$\mathrm{PM}(\mathbf{r}, \mathbf{s}_m) = f(\mathbf{n})P(\mathbf{s}_m)$$

where $f(\mathbf{n})$ is the N-dimensional PDF for the noise channel vector. If the noise is AWG, then

$$f\mathbf{n}) = \frac{1}{(\pi N_0)^{\frac{N}{2}}} e^{-\frac{|\mathbf{r}-\mathbf{s}_m|^2}{N_0}}$$

Maximizing $f(\mathbf{r}|\mathbf{s}_m)P(\mathbf{s}_m)$ is the same as minimizing the reciprocal $e^{\frac{|\mathbf{r}-\mathbf{s}_m|^2}{N_0}}/P(\mathbf{s}_m)$, or by taking the natural logarithm, minimizing the cost

$$D(\mathbf{r}, \mathbf{s}_m) = |\mathbf{r} - \mathbf{s}_m|^2 - N_0 P(\mathbf{s}_m)$$

This is equivalent to the maximization of the quantity

$$C(\mathbf{r}, \mathbf{s}_m) = \mathbf{r} \cdot \mathbf{s}_m - \frac{1}{2}|\mathbf{s}_m|^2 + \frac{N_0}{2} \ln P(\mathbf{s}_m)$$

If the vectors $\mathbf{r}$, $\mathbf{s}_m$ correspond to the waveforms $r(t)$ and $s_m(t)$, where

$$\begin{aligned} r(t) &= \sum_{i=1}^{N} r_i \psi_i(t) \\ s_m(t) &= \sum_{i=1}^{N} s_{m,i} \psi_i(t) \end{aligned}$$

then,

$$\begin{aligned} \int_{-\infty}^{\infty} r(t) s_m(t) dt &= \int_{-\infty}^{\infty} \sum_{i=1}^{N} r_i \psi_i(t) \sum_{j=1}^{N} s_{m,j} \psi_j(t) dt \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} r_i s_{m,j} \int_{-\infty}^{\infty} \psi_i(t) \psi_j(t) dt \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} r_i s_{m,j} \delta_{i,j} = \sum_{i=1}^{N} r_i s_{m,i} \\ &= \mathbf{r} \cdot \mathbf{s}_m \end{aligned}$$

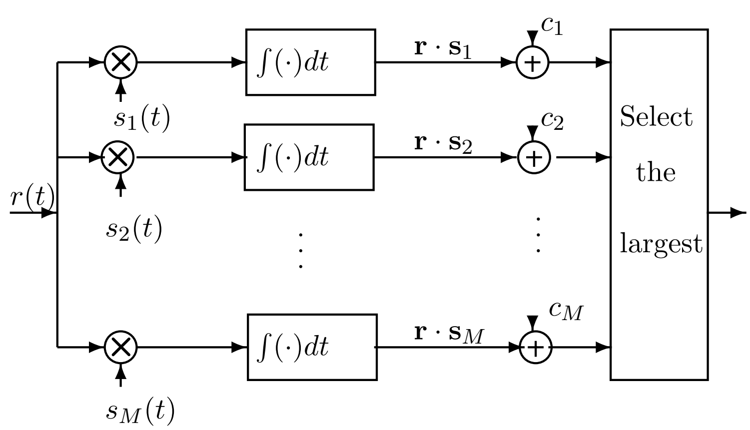
Similarly we obtain

$$\int_{-\infty}^{\infty} |s_m(t)|^2 dt = |\mathbf{s}_m|^2 = \mathcal{E}_{s_m}$$

Therefore, the optimal receiver can use the costs

$$\begin{aligned} C(\mathbf{r}, \mathbf{s}_m) &= \int_{-\infty}^{\infty} r(t) s_m(t) dt - \frac{1}{2} \int_{-\infty}^{\infty} |s_m(t)|^2 dt + \frac{N_0}{2} \ln P(\mathbf{s}_m) \\ &= \int_{-\infty}^{\infty} r(t) s_m(t) dt + c_m \end{aligned}$$

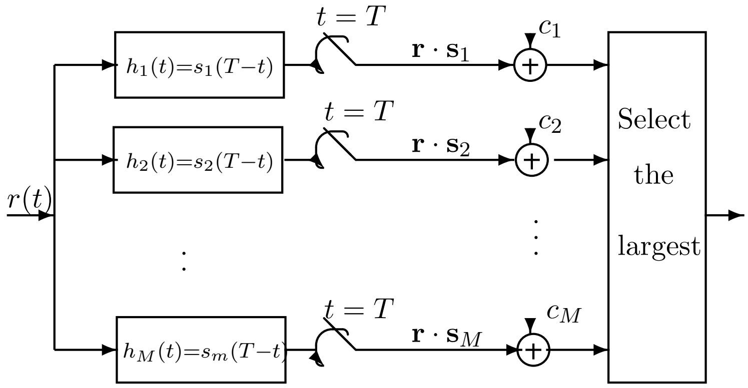
to base its decisions. This receiver can be implemented using M correlators to evaluate $\int_{-\infty}^{\infty} r(t) s_m(t) dt$. The bias constants c_m can be precomputed and added to the output of the correlators. The structure of the receiver is shown in the next figure.



Parallel to the development of the optimal receiver using N filters matched to the orthonormal functions $\psi_i(t)$, $i = 1, \ldots, N$, the M correlators can be replaced by M equivalent filters matched to the signal waveforms $s_m(t)$. The output of the m^{th} matched filter $h_m(t)$, at the time instant T is

$$\begin{aligned} \int_0^T r(\tau) h_m(T - \tau) d\tau &= \int_0^T r(\tau) s_m(T - (T - \tau)) d\tau \\ &= \int_0^T r(\tau) s_m(\tau) d\tau \\ &= \mathbf{r} \cdot \mathbf{s}_m \end{aligned}$$

The structure of this optimal receiver is shown in the next figure. The optimal receivers, derived in this problem, are more costly than those derived in the text, since N is usually less than M, the number of signal waveforms. For example, in an M-ary PAM system, $N = 1$ always less than M.



Problem 7.21

1) The optimal receiver (see Problem 7.20) computes the metrics

$$C(\mathbf{r}, \mathbf{s}_m) = \int_{-\infty}^{\infty} r(t)s_m(t)dt - \frac{1}{2}\int_{-\infty}^{\infty} |s_m(t)|^2 dt + \frac{N_0}{2}\ln P(\mathbf{s}_m)$$

and decides in favor of the signal with the largest $C(\mathbf{r}, \mathbf{s}_m)$. Since $s_1(t) = -s_2(t)$, the energy of the two message signals is the same, and therefore the detection rule is written as

$$\int_{-\infty}^{\infty} r(t)s_1(t)dt \overset{s_1}{\underset{s_2}{\gtrless}} \frac{N_0}{4}\ln\frac{P(\mathbf{s}_2)}{P(\mathbf{s}_1)} = \frac{N_0}{4}\ln\frac{p_2}{p_1}$$

2) If $s_1(t)$ is transmitted, then the output of the correlator is

$$\begin{aligned} \int_{-\infty}^{\infty} r(t)s_1(t)dt &= \int_0^T (s_1(t))^2 dt + \int_0^T n(t)s_1(t)dt \\ &= \mathcal{E}_s + n \end{aligned}$$

where $\mathcal{E}_s$ is the energy of the signal and n is a zero-mean Gaussian random variable with variance

$$\begin{aligned} \sigma_n^2 &= E\left[\int_0^T\int_0^T n(\tau)n(v)s_1(\tau)s_1(v)d\tau dv\right] \\ &= \int_0^T\int_0^T s_1(\tau)s_1(v)E[n(\tau)n(v)]d\tau dv \\ &= \frac{N_0}{2}\int_0^T\int_0^T s_1(\tau)s_1(v)\delta(\tau - v)d\tau dv \\ &= \frac{N_0}{2}\int_0^T |s_1(\tau)|^2 d\tau = \frac{N_0}{2}\mathcal{E}_s \end{aligned}$$

Hence, the probability of error $P(e|\mathbf{s}_1)$ is

$$\begin{aligned} P(e|\mathbf{s}_1) &= \int_{-\infty}^{\frac{N_0}{4}\ln\frac{p_2}{p_1} - \mathcal{E}_s} \frac{1}{\sqrt{\pi N_0 \mathcal{E}_s}} e^{-\frac{x^2}{N_0\mathcal{E}_s}} dx \\ &= Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} - \frac{1}{4}\sqrt{\frac{2N_0}{\mathcal{E}_s}}\ln\frac{p_2}{p_1}\right] \end{aligned}$$

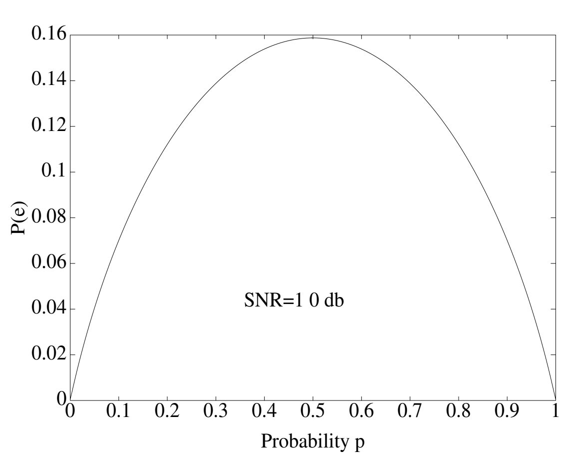
Similarly we find that

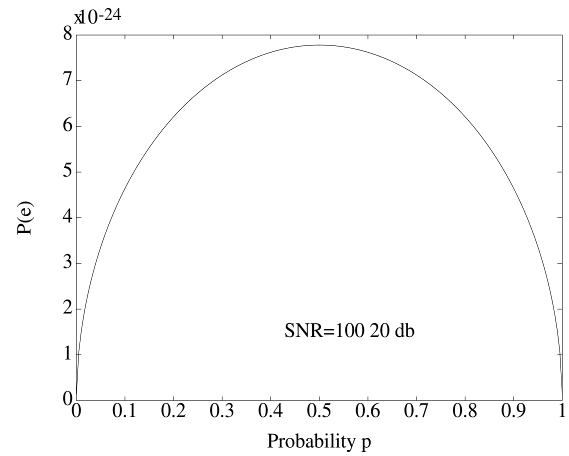
$$P(e|\mathbf{s}_2) = Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} + \frac{1}{4}\sqrt{\frac{2N_0}{\mathcal{E}_s}}\ln\frac{p_2}{p_1}\right]$$

The average probability of error is

$$\begin{aligned} P(e) &= p_1 P(e|\mathbf{s}_1) + p_2 P(e|\mathbf{s}_2) \\ &= p_1 Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} - \frac{1}{4}\sqrt{\frac{2N_0}{\mathcal{E}_s}}\ln\frac{1-p_1}{p_1}\right] + (1-p_1) Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} + \frac{1}{4}\sqrt{\frac{2N_0}{\mathcal{E}_s}}\ln\frac{1-p_1}{p_1}\right] \end{aligned}$$

3) In the next figure we plot the probability of error as a function of p_1, for two values of the $SNR = \frac{2\mathcal{E}_s}{N_0}$. As it is observed the probability of error attains its maximum for equiprobable signals.





Problem 7.22

1) The two equiprobable signals have the same energy and therefore the optimal receiver bases its decisions on the rule

$$\int_{-\infty}^{\infty} r(t)s_1(t)dt \underset{s_2}{\overset{s_1}{\gtrless}} \int_{-\infty}^{\infty} r(t)s_2(t)dt$$

2) If the message signal $s_1(t)$ is transmitted, then $r(t) = s_1(t) + n(t)$ and the decision rule becomes

$$\begin{aligned} &\int_{-\infty}^{\infty} (s_1(t) + n(t))(s_1(t) - s_2(t))dt \\ &= \int_{-\infty}^{\infty} s_1(t)(s_1(t) - s_2(t))dt + \int_{-\infty}^{\infty} n(t)(s_1(t) - s_2(t))dt \\ &= \int_{-\infty}^{\infty} s_1(t)(s_1(t) - s_2(t))dt + n \underset{s_2}{\overset{s_1}{\gtrless}} 0 \end{aligned}$$

where n is a zero mean Gaussian random variable with variance

$$\sigma_n^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (s_1(\tau) - s_2(\tau))(s_1(v) - s_2(v))E[n(\tau)n(v)]d\tau dv$$

$$
\begin{aligned}
&= \int_0^T \int_0^T (s_1(\tau) - s_2(\tau))(s_1(v) - s_2(v)) \frac{N_0}{2} \delta(\tau - v) d\tau dv \\
&= \frac{N_0}{2} \int_0^T (s_1(\tau) - s_2(\tau))^2 d\tau \\
&= \frac{N_0}{2} \int_0^T \int_0^T \left(\frac{2A\tau}{T} - A \right)^2 d\tau \\
&= \frac{N_0}{2} \frac{A^2 T}{3}
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_{-\infty}^{\infty} s_1(t)(s_1(t) - s_2(t)) dt &= \int_0^T \frac{At}{T} \left(\frac{2At}{T} - A \right) dt \\
&= \frac{A^2 T}{6}
\end{aligned}
$$

the probability of error $P(e|\mathbf{s}_1)$ is given by

$$
\begin{aligned}
P(e|\mathbf{s}_1) &= P(\frac{A^2 T}{6} + n < 0) \\
&= \frac{1}{\sqrt{2\pi \frac{A^2 T N_0}{6}}} \int_{-\infty}^{-\frac{A^2 T}{6}} \exp\left(-\frac{x^2}{2 \frac{A^2 T N_0}{6}} \right) dx \\
&= Q\left[\sqrt{\frac{A^2 T}{6 N_0}} \right]
\end{aligned}
$$

Similarly we find that

$$
P(e|\mathbf{s}_2) = Q\left[\sqrt{\frac{A^2 T}{6 N_0}} \right]
$$

and since the two signals are equiprobable, the average probability of error is given by

$$
\begin{aligned}
P(e) &= \frac{1}{2} P(e|\mathbf{s}_1) + \frac{1}{2} P(e|\mathbf{s}_2) \\
&= Q\left[\sqrt{\frac{A^2 T}{6 N_0}} \right] = Q\left[\sqrt{\frac{\mathcal{E}_s}{2 N_0}} \right]
\end{aligned}
$$

where $\mathcal{E}_s$ is the energy of the transmitted signals.

Problem 7.23

a) The PDF of the noise n is

$$
f(n) = \frac{\lambda}{2} e^{-\lambda |n|}
$$

The optimal receiver uses the criterion

$$
\frac{f(r|A)}{f(r|-A)} = e^{-\lambda[|r-A| - |r+A|]} \underset{-A}{\overset{A}{\gtrless}} 1 \Longrightarrow r \underset{-A}{\overset{A}{\gtrless}} 0
$$

The average probability of error is

$$
\begin{aligned}
P(e) &= \frac{1}{2}P(e|A) + \frac{1}{2}P(e|-A) \\
&= \frac{1}{2}\int_{-\infty}^{0} f(r|A)dr + \frac{1}{2}\int_{0}^{\infty} f(r|-A)dr \\
&= \frac{1}{2}\int_{-\infty}^{0} \lambda_2 e^{-\lambda|r-A|}dr + \frac{1}{2}\int_{0}^{\infty} \lambda_2 e^{-\lambda|r+A|}dr \\
&= \frac{\lambda}{4}\int_{-\infty}^{-A} e^{-\lambda|x|}dx + \frac{\lambda}{4}\int_{A}^{\infty} e^{-\lambda|x|}dx \\
&= \frac{\lambda}{4}\frac{1}{\lambda}e^{\lambda x}\Big|_{-\infty}^{-A} + \frac{\lambda}{4}\left(-\frac{1}{\lambda}\right)e^{-\lambda x}\Big|_{A}^{\infty} \\
&= \frac{1}{2}e^{-\lambda A}
\end{aligned}
$$

b) The variance of the noise is

$$
\begin{aligned}
\sigma_n^2 &= \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{-\lambda|x|}x^2 dx \\
&= \lambda\int_{0}^{\infty} e^{-\lambda x}x^2 dx = \lambda\frac{2!}{\lambda^3} = \frac{2}{\lambda^2}
\end{aligned}
$$

Hence, the SNR is

$$
\text{SNR} = \frac{A^2}{\frac{2}{\lambda^2}} = \frac{A^2\lambda^2}{2}
$$

and the probability of error is given by

$$
P(e) = \frac{1}{2}e^{-\sqrt{\lambda^2 A^2}} = \frac{1}{2}e^{-\sqrt{2\text{SNR}}}
$$

For $P(e) = 10^{-5}$ we obtain

$$
\ln(2\times 10^{-5}) = -\sqrt{2\text{SNR}} \Longrightarrow \text{SNR} = 58.534 = 17.6741 \text{ dB}
$$

If the noise was Gaussian, then

$$
P(e) = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\text{SNR}}\right]
$$

where SNR is the signal to noise ratio at the output of the matched filter. With $P(e) = 10^{-5}$ we find $\sqrt{\text{SNR}} = 4.26$ and therefore $\text{SNR} = 18.1476 = 12.594$ dB. Thus the required signal to noise ratio is 5 dB less when the additive noise is Gaussian.

Problem 7.24

The energy of the two signals $s_1(t)$ and $s_2(t)$ is

$$
\mathcal{E}_b = A^2 T
$$

The dimensionality of the signal space is one, and by choosing the basis function as

$$
\psi(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t < \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} \le t \le T \end{cases}
$$

we find the vector representation of the signals as

$$s_{1,2} = \pm A\sqrt{T} + n$$

with n a zero-mean Gaussian random variable of variance $\frac{N_0}{2}$. The probability of error for antipodal signals is given by, where $\mathcal{E}_b = A^2T$. Hence,

$$P(e) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left[\sqrt{\frac{2A^2T}{N_0}}\right]$$

Problem 7.25

The three symbols A, 0 and $-A$ are used with equal probability. Hence, the optimal detector uses two thresholds, which are $\frac{A}{2}$ and $-\frac{A}{2}$, and it bases its decisions on the criterion

$$\begin{array}{ll} A: & r > \frac{A}{2} \\ 0: & -\frac{A}{2} < r < \frac{A}{2} \\ -A: & r < -\frac{A}{2} \end{array}$$

If the variance of the AWG noise is σ_n^2, then the average probability of error is

$$\begin{aligned} P(e) &= \frac{1}{3}\int_{-\infty}^{\frac{A}{2}} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(r-A)^2}{2\sigma_n^2}} dr + \frac{1}{3}\left(1 - \int_{-\frac{A}{2}}^{\frac{A}{2}} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{r^2}{2\sigma_n^2}} dr\right) \\ &\quad + \frac{1}{3}\int_{-\frac{A}{2}}^{\infty} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(r+A)^2}{2\sigma_n^2}} dr \\ &= \frac{1}{3}Q\left[\frac{A}{2\sigma_n}\right] + \frac{1}{3}2Q\left[\frac{A}{2\sigma_n}\right] + \frac{1}{3}Q\left[\frac{A}{2\sigma_n}\right] \\ &= \frac{4}{3}Q\left[\frac{A}{2\sigma_n}\right] \end{aligned}$$

Problem 7.26

The biorthogonal signal set has the form

$$\begin{array}{ll} \mathbf{s}_1 = [\sqrt{\mathcal{E}_s}, 0, 0, 0] & \mathbf{s}_5 = [-\sqrt{\mathcal{E}_s}, 0, 0, 0] \\ \mathbf{s}_2 = [0, \sqrt{\mathcal{E}_s}, 0, 0] & \mathbf{s}_6 = [0, -\sqrt{\mathcal{E}_s}, 0, 0] \\ \mathbf{s}_3 = [0, 0, \sqrt{\mathcal{E}_s}, 0] & \mathbf{s}_7 = [0, 0, -\sqrt{\mathcal{E}_s}, 0] \\ \mathbf{s}_4 = [0, 0, 0, \sqrt{\mathcal{E}_s}] & \mathbf{s}_8 = [0, 0, 0, -\sqrt{\mathcal{E}_s}] \end{array}$$

For each point $\mathbf{s}_i$, there are $M - 2 = 6$ points at a distance

$$d_{i,k} = |\mathbf{s}_i - \mathbf{s}_k| = \sqrt{2\mathcal{E}_s}$$

and one vector $(-\mathbf{s}_i)$ at a distance $d_{i,m} = 2\sqrt{\mathcal{E}_s}$. Hence, the union bound on the probability of error $P(e|\mathbf{s}_i)$ is given by

$$P_{\mathrm{UB}}(e|\mathbf{s}_i) = \sum_{k=1,k\neq i}^{M} Q\left[\frac{d_{i,k}}{\sqrt{2N_0}}\right] = 6Q\left[\sqrt{\frac{\mathcal{E}_s}{N_0}}\right] + Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]$$

Since all the signals are equiprobable, we find that

$$P_{\mathrm{UB}}(e) = 6Q\left[\sqrt{\frac{\mathcal{E}_s}{N_0}}\right] + Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]$$

With $M = 8 = 2^3$, $\mathcal{E}_s = 3\mathcal{E}_b$ and therefore,

$$P_{\mathrm{UB}}(e) = 6Q\left[\sqrt{\frac{3\mathcal{E}_b}{N_0}}\right] + Q\left[\sqrt{\frac{6\mathcal{E}_b}{N_0}}\right]$$

Problem 7.27

It is convenient to find first the probability of a correct decision. Since all signals are equiprobable

$$P(C) = \sum_{i=1}^{M} \frac{1}{M} P(C|\mathbf{s}_i)$$

All the $P(C|\mathbf{s}_i)$, $i = 1, \ldots, M$ are identical because of the symmetry of the constellation. By translating the vector $\mathbf{s}_i$ to the origin we can find the probability of a correct decision, given that $\mathbf{s}_i$ was transmitted, as

$$P(C|\mathbf{s}_i) = \int_{-\frac{d}{2}}^{\infty} f(n_1)dn_1 \int_{-\frac{d}{2}}^{\infty} f(n_2)dn_2 \ldots \int_{-\frac{d}{2}}^{\infty} f(n_N)dn_N$$

where the number of the integrals on the right side of the equation is N, d is the minimum distance between the points and

$$f(n_i) = \frac{1}{\pi N_0} e^{-\frac{n_i^2}{N_0}}$$

Hence,

$$\begin{aligned} P(C|\mathbf{s}_i) &= \left(\int_{-\frac{d}{2}}^{\infty} f(n)dn\right)^N = \left(1 - \int_{-\infty}^{-\frac{d}{2}} f(n)dn\right)^N \\ &= \left(1 - Q\left[\frac{d}{\sqrt{2N_0}}\right]\right)^N \end{aligned}$$

and therefore, the probability of error is given by

$$\begin{aligned} P(e) &= 1 - P(C) = 1 - \sum_{i=1}^{M} \frac{1}{M}\left(1 - Q\left[\frac{d}{\sqrt{2N_0}}\right]\right)^N \\ &= 1 - \left(1 - Q\left[\frac{d}{\sqrt{2N_0}}\right]\right)^N \end{aligned}$$

Note that since

$$\mathcal{E}_s = \sum_{i=1}^{N} s_{m,i}^2 = \sum_{i=1}^{N} (\frac{d}{2})^2 = N\frac{d^2}{4}$$

the probability of error can be written as

$$P(e) = 1 - \left(1 - Q\left[\sqrt{\frac{2\mathcal{E}_s}{NN_0}}\right]\right)^N$$

Problem 7.28

Consider first the signal

$$y(t) = \sum_{k=1}^{n} c_k \delta(t - kT_c)$$

The signal $y(t)$ has duration $T = nT_c$ and its matched filter is

$$\begin{aligned} g(t) &= y(T-t) = y(nT_c - t) = \sum_{k=1}^{n} c_k \delta(nT_c - kT_c - t) \\ &= \sum_{i=1}^{n} c_{n-i+1} \delta((i-1)T_c - t) = \sum_{i=1}^{n} c_{n-i+1} \delta(t - (i-1)T_c) \end{aligned}$$

that is, a sequence of impulses starting at $t = 0$ and weighted by the mirror image sequence of $\{c_i\}$. Since,

$$s(t) = \sum_{k=1}^{n} c_k p(t - kT_c) = p(t) \star \sum_{k=1}^{n} c_k \delta(t - kT_c)$$

the Fourier transform of the signal $s(t)$ is

$$S(f) = P(f) \sum_{k=1}^{n} c_k e^{-j2\pi f k T_c}$$

and therefore, the Fourier transform of the signal matched to $s(t)$ is

$$\begin{aligned} H(f) &= S^*(f) e^{-j2\pi f T} = S^*(f) e^{-j2\pi f n T_c} \\ &= P^*(f) \sum_{k=1}^{n} c_k e^{j2\pi f k T_c} e^{-j2\pi f n T_c} \\ &= P^*(f) \sum_{i=1}^{n} c_{n-i+1} e^{-j2\pi f (i-1) T - c} \\ &= P^*(f) \mathcal{F}[g(t)] \end{aligned}$$

Thus, the matched filter $H(f)$ can be considered as the cascade of a filter,with impulse response $p(-t)$, matched to the pulse $p(t)$ and a filter, with impulse response $g(t)$, matched to the signal $y(t) = \sum_{k=1}^{n} c_k \delta(t - kT_c)$. The output of the matched filter at $t = nT_c$ is

$$\begin{aligned} \int_{-\infty}^{\infty} |s(t)|^2 &= \sum_{k=1}^{n} c_k^2 \int_{-\infty}^{\infty} p^2(t - kT_c) dt \\ &= T_c \sum_{k=1}^{n} c_k^2 \end{aligned}$$

where we have used the fact that $p(t)$ is a rectangular pulse of unit amplitude and duration T_c.

Problem 7.29

The bandwidth required for transmission of an M-ary PAM signal is

$$W = \frac{R_b}{2 \log_2 M} \text{ Hz}$$

Since,

$$R_b = 8 \times 10^3 \frac{\text{samples}}{\text{sec}} \times 8 \frac{\text{bits}}{\text{sample}} = 64 \times 10^3 \frac{\text{bits}}{\text{sec}}$$

we obtain

$$W = \begin{cases} 16 \text{ KHz} & M = 4 \\ 10.667 \text{ KHz} & M = 8 \\ 8 \text{ KHz} & M = 16 \end{cases}$$

Problem 7.30

The vector $\mathbf{r} = [r_1, r_2]$ at the output of the integrators is

$$\mathbf{r} = [r_1, r_2] = [\int_0^{1.5} r(t)dt, \quad \int_1^2 r(t)dt]$$

If $s_1(t)$ is transmitted, then

$$\begin{aligned}
\int_0^{1.5} r(t)dt &= \int_0^{1.5} [s_1(t) + n(t)]dt = 1 + \int_0^{1.5} n(t)dt \\
&= 1 + n_1 \\
\int_1^2 r(t)dt &= \int_1^2 [s_1(t) + n(t)]dt = \int_1^2 n(t)dt \\
&= n_2
\end{aligned}$$

where n_1 is a zero-mean Gaussian random variable with variance

$$\sigma_{n_1}^2 = E\left[\int_0^{1.5}\int_0^{1.5} n(\tau)n(v)d\tau dv\right] = \frac{N_0}{2}\int_0^{1.5} d\tau = 1.5$$

and n_2 is is a zero-mean Gaussian random variable with variance

$$\sigma_{n_2}^2 = E\left[\int_1^2\int_1^2 n(\tau)n(v)d\tau dv\right] = \frac{N_0}{2}\int_1^2 d\tau = 1$$

Thus, the vector representation of the received signal (at the output of the integrators) is

$$\mathbf{r} = [1 + n_1, \ n_2]$$

Similarly we find that if $s_2(t)$ is transmitted, then

$$\mathbf{r} = [0.5 + n_1, \ 1 + n_2]$$

Suppose now that the detector bases its decisions on the rule

$$r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} T$$

The probability of error $P(e|s_1)$ is obtained as

$$\begin{aligned}
P(e|s_1) &= P(r1 - r_2 < T|s_1) \\
&= P(1 + n_1 - n_2 < T) = P(n_1 - n_2 < T - 1) \\
&= P(n < T)
\end{aligned}$$

where the random variable $n = n_1 - n_2$ is zero-mean Gaussian with variance

$$\begin{aligned}
\sigma_n^2 &= \sigma_{n_1}^2 + \sigma_{n_2}^2 - 2E[n_1 n_2] \\
&= \sigma_{n_1}^2 + \sigma_{n_2}^2 - 2\int_1^{1.5} \frac{N_0}{2} d\tau \\
&= 1.5 + 1 - 2 \times 0.5 = 1.5
\end{aligned}$$

Hence,

$$P(e|s_1) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_{-\infty}^{T-1} e^{-\frac{x^2}{2\sigma_n^2}} dx$$

Similarly we find that

$$\begin{aligned}
P(e|s_2) &= P(0.5 + n_1 - 1 - n_2 > T) \\
&= P(n_1 - n_2 > T + 0.5) \\
&= \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_{T+0.5}^{\infty} e^{-\frac{x^2}{2\sigma_n^2}} dx
\end{aligned}$$

The average probability of error is

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) \\
&= \frac{1}{2\sqrt{2\pi\sigma_n^2}} \int_{-\infty}^{T-1} e^{-\frac{x^2}{2\sigma_n^2}} dx + \frac{1}{2\sqrt{2\pi\sigma_n^2}} \int_{T+0.5}^{\infty} e^{-\frac{x^2}{2\sigma_n^2}} dx
\end{aligned}$$

To find the value of T that minimizes the probability of error, we set the derivative of $P(e)$ with respect to T equal to zero. Using the Leibnitz rule for the differentiation of definite integrals, we obtain

$$\frac{\vartheta P(e)}{\vartheta T} = \frac{1}{2\sqrt{2\pi\sigma_n^2}} \left[e^{-\frac{(T-1)^2}{2\sigma_n^2}} - e^{-\frac{(T+0.5)^2}{2\sigma_n^2}} \right] = 0$$

or

$$(T-1)^2 = (T+0.5)^2 \Longrightarrow T = 0.25$$

Thus, the optimal decision rule is

$$r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} 0.25$$

Problem 7.31

a) The inner product of $s_i(t)$ and $s_j(t)$ is

$$\begin{aligned}
\int_{-\infty}^{\infty} s_i(t)s_j(t)dt &= \int_{-\infty}^{\infty} \sum_{k=1}^{n} c_{ik}p(t-kT_c) \sum_{l=1}^{n} c_{jl}p(t-lT_c)dt \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} c_{ik}c_{jl} \int_{-\infty}^{\infty} p(t-kT_c)p(t-lT_c)dt \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} c_{ik}c_{jl}\mathcal{E}_p\delta_{kl} \\
&= \mathcal{E}_p \sum_{k=1}^{n} c_{ik}c_{jk}
\end{aligned}$$

The quantity $\sum_{k=1}^{n} c_{ik}c_{jk}$ is the inner product of the row vectors $\underline{C}_i$ and $\underline{C}_j$. Since the rows of the matrix H_n are orthogonal by construction, we obtain

$$\int_{-\infty}^{\infty} s_i(t)s_j(t)dt = \mathcal{E}_p \sum_{k=1}^{n} c_{ik}^2\delta_{ij} = n\mathcal{E}_p\delta_{ij}$$

Thus, the waveforms $s_i(t)$ and $s_j(t)$ are orthogonal.

b) Using the results of Problem 7.28, we obtain that the filter matched to the waveform

$$s_i(t) = \sum_{k=1}^{n} c_{ik} p(t - kT_c)$$

can be realized as the cascade of a filter matched to $p(t)$ followed by a discrete-time filter matched to the vector $\underline{C}_i = [c_{i1}, \ldots, c_{in}]$. Since the pulse $p(t)$ is common to all the signal waveforms $s_i(t)$, we conclude that the n matched filters can be realized by a filter matched to $p(t)$ followed by n discrete-time filters matched to the vectors $\underline{C}_i$, $i = 1, \ldots, n$.

Problem 7.32

a) The optimal ML detector selects the sequence $\underline{C}_i$ that minimizes the quantity

$$D(\mathbf{r}, \underline{C}_i) = \sum_{k=1}^{n} (r_k - \sqrt{\mathcal{E}_b} C_{ik})^2$$

The metrics of the two possible transmitted sequences are

$$D(\mathbf{r}, \underline{C}_1) = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2$$

and

$$D(\mathbf{r}, \underline{C}_2) = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2$$

Since the first term of the right side is common for the two equations, we conclude that the optimal ML detector can base its decisions only on the last $n - w$ received elements of $\mathbf{r}$. That is

$$\sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2 - \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2 \mathop{\gtrless}_{\underline{C}_1}^{\underline{C}_2} 0$$

or equivalently

$$\sum_{k=w+1}^{n} r_k \mathop{\gtrless}_{\underline{C}_2}^{\underline{C}_1} 0$$

b) Since $r_k = \sqrt{\mathcal{E}_b} C_{ik} + n_k$, the probability of error $P(e|\underline{C}_1)$ is

$$\begin{aligned} P(e|\underline{C}_1) &= P\left(\sqrt{\mathcal{E}_b}(n-w) + \sum_{k=w+1}^{n} n_k < 0\right) \\ &= P\left(\sum_{k=w+1}^{n} n_k < -(n-w)\sqrt{\mathcal{E}_b}\right) \end{aligned}$$

The random variable $u = \sum_{k=w+1}^{n} n_k$ is zero-mean Gaussian with variance $\sigma_u^2 = (n-w)\sigma^2$. Hence

$$P(e|\underline{C}_1) = \frac{1}{\sqrt{2\pi(n-w)\sigma^2}} \int_{-\infty}^{-\sqrt{\mathcal{E}_b}(n-w)} \exp(-\frac{x^2}{2\pi(n-w)\sigma^2}) dx = Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}}\right]$$

Similarly we find that $P(e|\underline{C}_2) = P(e|\underline{C}_1)$ and since the two sequences are equiprobable

$$P(e) = Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}}\right]$$

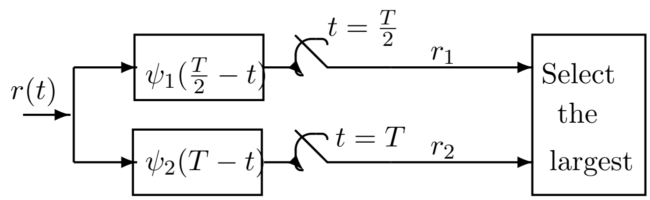
c) The probability of error $P(e)$ is minimized when $\frac{\mathcal{E}_b(n-w)}{\sigma^2}$ is maximized, that is for $w = 0$. This implies that $\underline{C}_1 = -\underline{C}_2$ and thus the distance between the two sequences is the maximum possible.

Problem 7.33

1) The dimensionality of the signal space is two. An orthonormal basis set for the signal space is formed by the signals

$$\psi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \leq t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases} \qquad \psi_2(t) = \begin{cases} \sqrt{\frac{2}{T}}, & \frac{T}{2} \leq t < T \\ 0, & \text{otherwise} \end{cases}$$

2) The optimal receiver is shown in the next figure



3) Assuming that the signal $s_1(t)$ is transmitted, the received vector at the output of the samplers is

$$\mathbf{r} = [\sqrt{\frac{A^2T}{2}} + n_1, n_2]$$

where n_1, n_2 are zero mean Gaussian random variables with variance $\frac{N_0}{2}$. The probability of error $P(e|s_1)$ is

$$\begin{aligned} P(e|s_1) &= P(n-2-n_1 > \sqrt{\frac{A^2T}{2}}) \\ &= \frac{1}{\sqrt{2\pi N_0}} \int_{\frac{A^2T}{2}}^{\infty} e^{-\frac{x^2}{2N_0}} dx = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right] \end{aligned}$$

where we have used the fact the $n = n_2 - n_1$ is a zero-mean Gaussian random variable with variance N_0. Similarly we find that $P(e|s_1) = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right]$, so that

$$P(e) = \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right]$$

4) The signal waveform $\psi_1(\frac{T}{2} - t)$ matched to $\psi_1(t)$ is exactly the same with the signal waveform $\psi_2(T-t)$ matched to $\psi_2(t)$. That is,

$$\psi_1(\frac{T}{2} - t) = \psi_2(T-t) = \psi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \leq t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

Thus, the optimal receiver can be implemented by using just one filter followed by a sampler which samples the output of the matched filter at $t = \frac{T}{2}$ and $t = T$ to produce the random variables r_1 and r_2 respectively.

5) If the signal $s_1(t)$ is transmitted, then the received signal $r(t)$ is

$$r(t) = s_1(t) + \frac{1}{2}s_1(t - \frac{T}{2}) + n(t)$$

The output of the sampler at $t = \frac{T}{2}$ and $t = T$ is given by

$$\begin{aligned}
r_1 &= A\sqrt{\frac{2}{T}\frac{T}{4}} + \frac{3A}{2}\sqrt{\frac{2}{T}\frac{T}{4}} + n_1 = \frac{5}{2}\sqrt{\frac{A^2T}{8}} + n_1 \\
r_2 &= \frac{A}{2}\sqrt{\frac{2}{T}\frac{T}{4}} + n_2 = \frac{1}{2}\sqrt{\frac{A^2T}{8}} + n_2
\end{aligned}$$

If the optimal receiver uses a threshold V to base its decisions, that is

$$r_1 - r_2 \mathop{\gtrless}_{s_2}^{s_1} V$$

then the probability of error $P(e|s_1)$ is

$$P(e|s_1) = P(n_2 - n_1 > 2\sqrt{\frac{A^2T}{8}} - V) = Q\left[2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right]$$

If $s_2(t)$ is transmitted, then

$$r(t) = s_2(t) + \frac{1}{2}s_2(t - \frac{T}{2}) + n(t)$$

The output of the sampler at $t = \frac{T}{2}$ and $t = T$ is given by

$$\begin{aligned}
r_1 &= n_1 \\
r_2 &= A\sqrt{\frac{2}{T}\frac{T}{4}} + \frac{3A}{2}\sqrt{\frac{2}{T}\frac{T}{4}} + n_2 \\
&= \frac{5}{2}\sqrt{\frac{A^2T}{8}} + n_2
\end{aligned}$$

The probability of error $P(e|s_2)$ is

$$P(e|s_2) = P(n_1 - n_2 > \frac{5}{2}\sqrt{\frac{A^2T}{8}} + V) = Q\left[\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right]$$

Thus, the average probability of error is given by

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) \\
&= \frac{1}{2}Q\left[2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right] + \frac{1}{2}Q\left[\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right]
\end{aligned}$$

The optimal value of V can be found by setting $\frac{\vartheta P(e)}{\vartheta V}$ equal to zero. Using Leibnitz rule to differentiate definite integrals, we obtain

$$\frac{\vartheta P(e)}{\vartheta V} = 0 = \left(2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right)^2 - \left(\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right)^2$$

or by solving in terms of V

$$V = -\frac{1}{8}\sqrt{\frac{A^2T}{2}}$$

6) Let a be fixed to some value between 0 and 1. Then, if we argue as in part 5) we obtain

$$\begin{aligned} P(e|s_1, a) &= P(n_2 - n_1 > 2\sqrt{\frac{A^2T}{8}} - V(a)) \\ P(e|s_2, a) &= P(n_1 - n_2 > (a+2)\sqrt{\frac{A^2T}{8}} + V(a)) \end{aligned}$$

and the probability of error is

$$P(e|a) = \frac{1}{2}P(e|s_1, a) + \frac{1}{2}P(e|s_2, a)$$

For a given a, the optimal value of $V(a)$ is found by setting $\frac{\vartheta P(e|a)}{\vartheta V(a)}$ equal to zero. By doing so we find that

$$V(a) = -\frac{a}{4}\sqrt{\frac{A^2T}{2}}$$

The mean square estimation of $V(a)$ is

$$V = \int_0^1 V(a)f(a)da = -\frac{1}{4}\sqrt{\frac{A^2T}{2}}\int_0^1 ada = -\frac{1}{8}\sqrt{\frac{A^2T}{2}}$$

Problem 7.34

For binary phase modulation, the error probability is

$$P_2 = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\frac{A^2T}{N_0}}\right]$$

With $P_2 = 10^{-6}$ we find from tables that

$$\sqrt{\frac{A^2T}{N_0}} = 4.74 \Longrightarrow A^2T = 44.9352 \times 10^{-10}$$

If the data rate is 10 Kbps, then the bit interval is $T = 10^{-4}$ and therefore, the signal amplitude is

$$A = \sqrt{44.9352 \times 10^{-10} \times 10^4} = 6.7034 \times 10^{-3}$$

Similarly we find that when the rate is 10^5 bps and 10^6 bps, the required amplitude of the signal is $A = 2.12 \times 10^{-2}$ and $A = 6.703 \times 10^{-2}$ respectively.

Problem 7.35

1) The impulse response of the matched filter is

$$s(t) = u(T-t) = \begin{cases} \frac{A}{T}(T-t)\cos(2\pi f_c(T-t)) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

2) The output of the matched filter at $t = T$ is

$$\begin{aligned}
g(T) &= u(t) \star s(t)|_{t=T} = \int_0^T u(T-\tau)s(\tau)d\tau \\
&= \frac{A^2}{T^2}\int_0^T (T-\tau)^2\cos^2(2\pi f_c(T-\tau))d\tau \\
&\stackrel{v=T-\tau}{=} \frac{A^2}{T^2}\int_0^T v^2\cos^2(2\pi f_c v)dv \\
&= \frac{A^2}{T^2}\left[\frac{v^3}{6} + \left(\frac{v^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c v) + \frac{v\cos(4\pi f_c v)}{4(2\pi f_c)^2}\right]\Bigg|_0^T \\
&= \frac{A^2}{T^2}\left[\frac{T^3}{6} + \left(\frac{T^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c T) + \frac{T\cos(4\pi f_c T)}{4(2\pi f_c)^2}\right]
\end{aligned}$$

3) The output of the correlator at $t = T$ is

$$\begin{aligned}
q(T) &= \int_0^T u^2(\tau)d\tau \\
&= \frac{A^2}{T^2}\int_0^T \tau^2\cos^2(2\pi f_c\tau)d\tau
\end{aligned}$$

However, this is the same expression with the case of the output of the matched filter sampled at $t = T$. Thus, the correlator can substitute the matched filter in a demodulation system and vise versa.

Problem 7.36

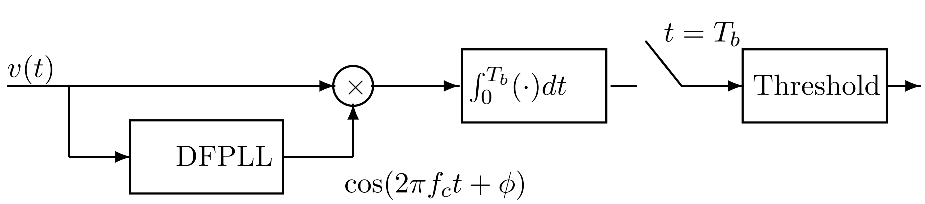
1) The signal $r(t)$ can be written as

$$\begin{aligned}
r(t) &= \pm\sqrt{2P_s}\cos(2\pi f_c t + \phi) + \sqrt{2P_c}\sin(2\pi f_c t + \phi) \\
&= \sqrt{2(P_c + P_s)}\sin\left(2\pi f_c t + \phi + a_n\tan^{-1}\left(\sqrt{\frac{P_s}{P_c}}\right)\right) \\
&= \sqrt{2P_T}\sin\left(2\pi f_c t + \phi + a_n\cos^{-1}\left(\sqrt{\frac{P_c}{P_T}}\right)\right)
\end{aligned}$$

where $a_n = \pm 1$ are the information symbols and P_T is the total transmitted power. As it is observed the signal has the form of a PM signal where

$$\theta_n = a_n\cos^{-1}\left(\sqrt{\frac{P_c}{P_T}}\right)$$

Any method used to extract the carrier phase from the received signal can be employed at the receiver. The following figure shows the structure of a receiver that employs a decision-feedback PLL. The operation of the PLL is described in the next part.



2) At the receiver the signal is demodulated by crosscorrelating the received signal

$$r(t) = \sqrt{2P_T}\sin\left(2\pi f_c t + \phi + a_n\cos^{-1}\left(\sqrt{\frac{P_c}{P_T}}\right)\right) + n(t)$$

with $\cos(2\pi f_c t + \hat{\phi})$ and $\sin(2\pi f_c t + \hat{\phi})$. The sampled values at the output of the correlators are

$$\begin{aligned}
r_1 &= \frac{1}{2}\left[\sqrt{2P_T} - n_s(t)\right]\sin(\phi - \hat{\phi} + \theta_n) + \frac{1}{2}n_c(t)\cos(\phi - \hat{\phi} + \theta_n) \\
r_2 &= \frac{1}{2}\left[\sqrt{2P_T} - n_s(t)\right]\cos(\phi - \hat{\phi} + \theta_n) + \frac{1}{2}n_c(t)\sin(\hat{\phi} - \phi - \theta_n)
\end{aligned}$$

where $n_c(t)$, $n_s(t)$ are the in-phase and quadrature components of the noise $n(t)$. If the detector has made the correct decision on the transmitted point, then by multiplying r_1 by $\cos(\theta_n)$ and r_2 by $\sin(\theta_n)$ and subtracting the results, we obtain (after ignoring the noise)

$$\begin{aligned}
r_1\cos(\theta_n) &= \frac{1}{2}\sqrt{2P_T}\left[\sin(\phi - \hat{\phi})\cos^2(\theta_n) + \cos(\phi - \hat{\phi})\sin(\theta_n)\cos(\theta_n)\right] \\
r_2\sin(\theta_n) &= \frac{1}{2}\sqrt{2P_T}\left[\cos(\phi - \hat{\phi})\cos(\theta_n)\sin(\theta_n) - \sin(\phi - \hat{\phi})\sin^2(\theta_n)\right] \\
e(t) &= r_1\cos(\theta_n) - r_2\sin(\theta_n) = \frac{1}{2}\sqrt{2P_T}\sin(\phi - \hat{\phi})
\end{aligned}$$

The error $e(t)$ is passed to the loop filter of the DFPLL that drives the VCO. As it is seen only the phase θ_n is used to estimate the carrier phase.

3) Having a correct carrier phase estimate, the output of the lowpass filter sampled at $t = T_b$ is

$$\begin{aligned}
r &= \pm\frac{1}{2}\sqrt{2P_T}\sin\cos^{-1}\left(\sqrt{\frac{P_c}{P_T}}\right) + n \\
&= \pm\frac{1}{2}\sqrt{2P_T}\sqrt{1 - \frac{P_c}{P_T}} + n \\
&= \pm\frac{1}{2}\sqrt{2P_T\left(1 - \frac{P_c}{P_T}\right)} + n
\end{aligned}$$

where n is a zero-mean Gaussian random variable with variance

$$\begin{aligned}
\sigma_n^2 &= E\left[\int_0^{T_b}\int_0^{T_b} n(t)n(\tau)\cos(2\pi f_c t + \phi)\cos(2\pi f_c \tau + \phi)dt d\tau\right] \\
&= \frac{N_0}{2}\int_0^{T_b}\cos^2(2\pi f_c t + \phi)dt \\
&= \frac{N_0}{4}
\end{aligned}$$

Note that T_b has been normalized to 1 since the problem has been stated in terms of the power of the involved signals. The probability of error is given by

$$P(\text{error}) = Q\left[\sqrt{\frac{2P_T}{N_0}\left(1 - \frac{P_c}{P_T}\right)}\right]$$

The loss due to the allocation of power to the pilot signal is

$$\text{SNR}_{\text{loss}} = 10\log_{10}\left(1 - \frac{P_c}{P_T}\right)$$

When $P_c/P_T = 0.1$, then

$$\text{SNR}_{\text{loss}} = 10\log_{10}(0.9) = -0.4576 \text{ dB}$$

The negative sign indicates that the SNR is decreased by 0.4576 dB.

Problem 7.37

1) If the received signal is

$$r(t) = \pm g_T(t)\cos(2\pi f_c t + \phi) + n(t)$$

then by crosscorrelating with the signal at the output of the PLL

$$\psi(t) = \sqrt{\frac{2}{\mathcal{E}_g}} g_t(t)\cos(2\pi f_c t + \hat{\phi})$$

we obtain

$$\begin{aligned}
\int_0^T r(t)\psi(t)dt &= \pm\sqrt{\frac{2}{\mathcal{E}_g}}\int_0^T g_T^2(t)\cos(2\pi f_c t + \phi)\cos(2\pi f_c t + \hat{\phi})dt \\
&\quad + \int_0^T n(t)\sqrt{\frac{2}{\mathcal{E}_g}} g_t(t)\cos(2\pi f_c t + \hat{\phi})dt \\
&= \pm\sqrt{\frac{2}{\mathcal{E}_g}}\int_0^T \frac{g_T^2(t)}{2}\left(\cos(2\pi 2 f_c t + \phi + \hat{\phi}) + \cos(\phi - \hat{\phi})\right)dt + n \\
&= \pm\sqrt{\frac{\mathcal{E}_g}{2}}\cos(\phi - \hat{\phi}) + n
\end{aligned}$$

where n is a zero-mean Gaussian random variable with variance $\frac{N_0}{2}$. If we assume that the signal $s_1(t) = g_T(t)\cos(2\pi f_c t + \phi)$ was transmitted, then the probability of error is

$$\begin{aligned}
P(\text{error}|s_1(t)) &= P\left(\sqrt{\frac{\mathcal{E}_g}{2}}\cos(\phi - \hat{\phi}) + n < 0\right) \\
&= Q\left[\sqrt{\frac{\mathcal{E}_g\cos^2(\phi - \hat{\phi})}{N_0}}\right] = Q\left[\sqrt{\frac{2\mathcal{E}_s\cos^2(\phi - \hat{\phi})}{N_0}}\right]
\end{aligned}$$

where $\mathcal{E}_s = \mathcal{E}_g/2$ is the energy of the transmitted signal. As it is observed the phase error $\phi - \hat{\phi}$ reduces the SNR by a factor

$$\text{SNR}_{\text{loss}} = -10\log_{10}\cos^2(\phi - \hat{\phi})$$

2) When $\phi - \hat{\phi} = 45^o$, then the loss due to the phase error is

$$\text{SNR}_{\text{loss}} = -10\log_{10}\cos^2(45^o) = -10\log_{10}\frac{1}{2} = 3.01 \text{ dB}$$

Problem 7.38

1) The closed loop transfer function is

$$H(s) = \frac{G(s)/s}{1 + G(s)/s} = \frac{G(s)}{s + G(s)} = \frac{1}{s^2 + \sqrt{2}s + 1}$$

The poles of the system are the roots of the denominator, that is

$$\rho_{1,2} = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2} = -\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}$$

Since the real part of the roots is negative, the poles lie in the left half plane and therefore, the system is stable.

2) Writing the denominator in the form

$$D = s^2 + 2\zeta\omega_n s + \omega_n^2$$

we identify the natural frequency of the loop as $\omega_n = 1$ and the damping factor as $\zeta = \frac{1}{\sqrt{2}}$.

Problem 7.39

1) The closed loop transfer function is

$$H(s) = \frac{G(s)/s}{1 + G(s)/s} = \frac{G(s)}{s + G(s)} = \frac{K}{\tau_1 s^2 + s + K} = \frac{\frac{K}{\tau_1}}{s^2 + \frac{1}{\tau_1}s + \frac{K}{\tau_1}}$$

The gain of the system at $f = 0$ is

$$|H(0)| = |H(s)|_{s=0} = 1$$

2) The poles of the system are the roots of the denominator, that is

$$\rho_{1,2} = \frac{-1 \pm \sqrt{1 - 4K\tau_1}}{2\tau_1} =$$

In order for the system to be stable the real part of the poles must be negative. Since K is greater than zero, the latter implies that τ_1 is positive. If in addition we require that the damping factor $\zeta = \frac{1}{2\sqrt{\tau_1 K}}$ is less than 1, then the gain K should satisfy the condition

$$K > \frac{1}{4\tau_1}$$

Problem 7.40

The transfer function of the RC circuit is

$$G(s) = \frac{R_2 + \frac{1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}} = \frac{1 + R_2 Cs}{1 + (R_1 + R_2)Cs} = \frac{1 + \tau_2 s}{1 + \tau_1 s}$$

From the last equality we identify the time constants as

$$\tau_2 = R_2 C, \qquad \tau_1 = (R_1 + R_2)C$$

Problem 7.41

Assuming that the input resistance of the operational amplifier is high so that no current flows through it, then the voltage-current equations of the circuit are

$$\begin{aligned} V_2 &= -AV_1 \\ V_1 - V_2 &= \left(R_1 + \frac{1}{Cs}\right) i \\ V_1 - V_0 &= iR \end{aligned}$$

where, V_1, V_2 is the input and output voltage of the amplifier respectively, and V_0 is the signal at the input of the filter. Eliminating i and V_1, we obtain

$$\frac{V_2}{V_1} = \frac{\frac{R_1 + \frac{1}{Cs}}{R}}{1 + \frac{1}{A} - \frac{R_1 + \frac{1}{Cs}}{AR}}$$

If we let $A \to \infty$ (ideal amplifier), then

$$\frac{V_2}{V_1} = \frac{1 + R_1 Cs}{RCs} = \frac{1 + \tau_2 s}{\tau_1 s}$$

Hence, the constants τ_1, τ_2 of the active filter are given by

$$\tau_1 = RC, \qquad \tau_2 = R_1 C$$

Problem 7.42
Using the Pythagorean theorem for the four-phase constellation, we find

$$r_1^2 + r_1^2 = d^2 \Longrightarrow r_1 = \frac{d}{\sqrt{2}}$$

The radius of the 8-PSK constellation is found using the cosine rule. Thus,

$$d^2 = r_2^2 + r_2^2 - 2r_2^2 \cos(45^o) \Longrightarrow r_2 = \frac{d}{\sqrt{2 - \sqrt{2}}}$$

The average transmitted power of the 4-PSK and the 8-PSK constellation is given by

$$P_{4,av} = \frac{d^2}{2}, \qquad P_{8,av} = \frac{d^2}{2 - \sqrt{2}}$$

Thus, the additional transmitted power needed by the 8-PSK signal is

$$P = 10 \log_{10} \frac{2d^2}{(2 - \sqrt{2})d^2} = 5.3329 \text{ dB}$$

We obtain the same results if we use the probability of error given by

$$P_M = 2Q\left[\sqrt{2\rho_s} \sin \frac{\pi}{M}\right]$$

where ρ_s is the SNR per symbol. In this case, equal error probability for the two signaling schemes, implies that

$$\rho_{4,s} \sin^2 \frac{\pi}{4} = \rho_{8,s} \sin^2 \frac{\pi}{8} \Longrightarrow 10 \log_{10} \frac{\rho_{8,s}}{\rho_{4,s}} = 20 \log_{10} \frac{\sin \frac{\pi}{4}}{\sin \frac{\pi}{8}} = 5.3329 \text{ dB}$$

Problem 7.43
The constellation of Fig. P-7.43(a) has four points at a distance $2A$ from the origin and four points at a distance $2\sqrt{2}A$. Thus, the average transmitted power of the constellation is

$$P_a = \frac{1}{8}\left[4 \times (2A)^2 + 4 \times (2\sqrt{2}A)^2\right] = 6A^2$$

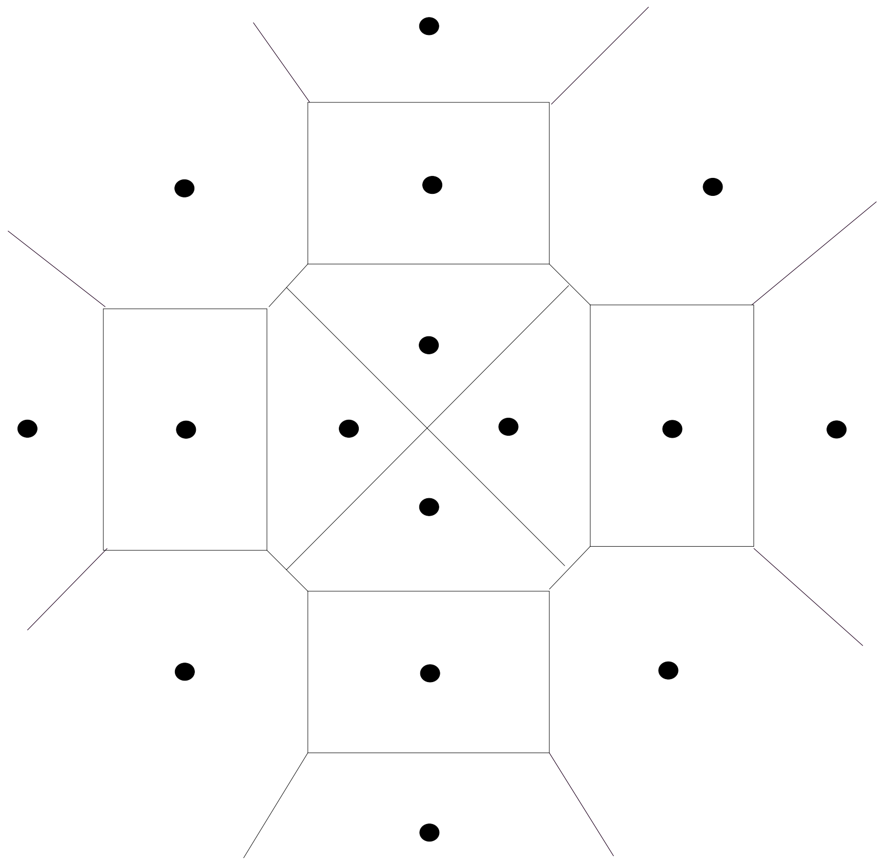
The second constellation has four points at a distance $\sqrt{7}A$ from the origin, two points at a distance $\sqrt{3}A$ and two points at a distance A. Thus, the average transmitted power of the second constellation is

$$P_b = \frac{1}{8}\left[4 \times (\sqrt{7}A)^2 + 2 \times (\sqrt{3}A)^2 + 2A^2\right] = \frac{9}{2}A^2$$

Since $P_b < P_a$ the second constellation is more power efficient.

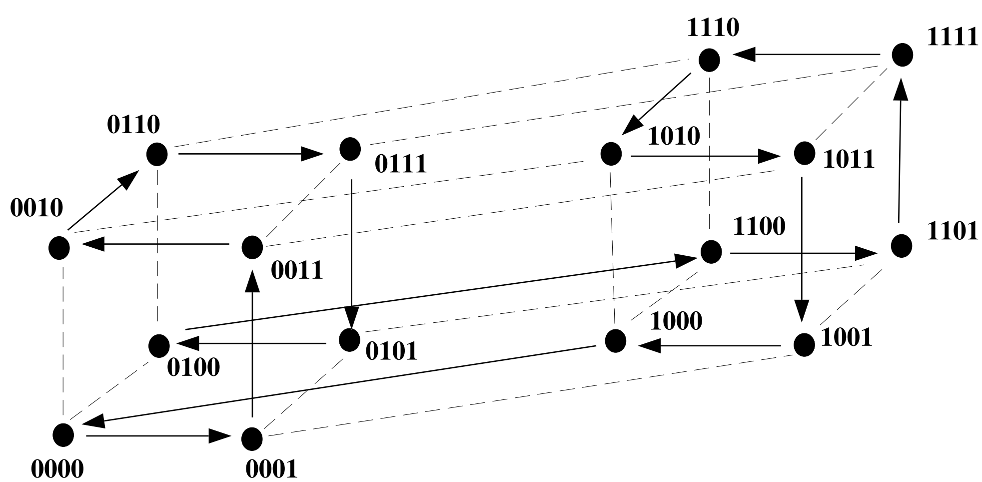
Problem 7.44

The optimum decision boundary of a point is determined by the perpendicular bisectors of each line segment connecting the point with its neighbors. The decision regions for the V.29 constellation are depicted in the next figure.

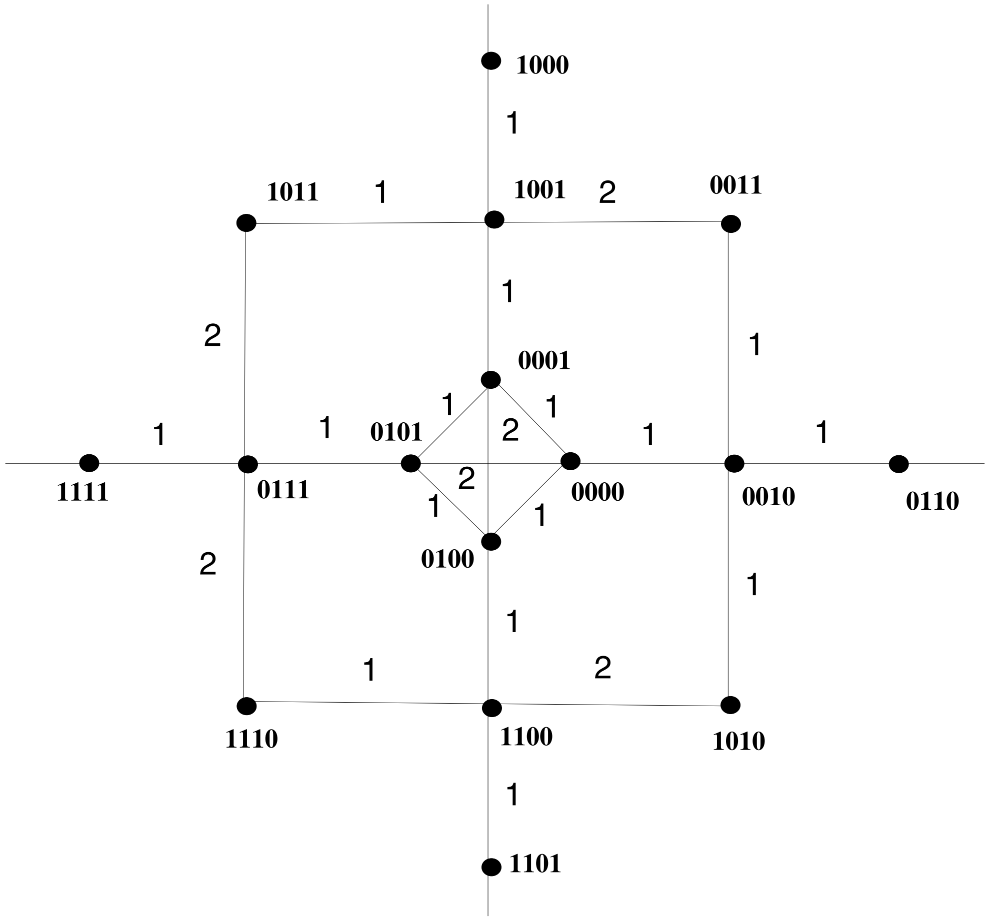


Problem 7.45

The following figure depicts a 4-cube and the way that one can traverse it in Gray-code order (see John F. Wakerly, Digital Design Principles and Practices, Prentice Hall, 1990). Adjacent points are connected with solid or dashed lines.



One way to label the points of the V.29 constellation using the Gray-code is depicted in the next figure. Note that the maximum Hamming distance between points with distance between them as large as 3 is only 2. Having labeled the innermost points, all the adjacent nodes can be found using the previous figure.



Problem 7.46

1) Consider the QAM constellation of Fig. P-7.46. Using the Pythagorean theorem we can find the radius of the inner circle as

$$a^2 + a^2 = A^2 \Longrightarrow a = \frac{1}{\sqrt{2}}A$$

The radius of the outer circle can be found using the cosine rule. Since b is the third side of a triangle with a and A the two other sides and angle between then equal to $\theta = 75^o$, we obtain

$$b^2 = a^2 + A^2 - 2aA\cos 75^o \Longrightarrow b = \frac{1+\sqrt{3}}{2}A$$

2) If we denote by r the radius of the circle, then using the cosine theorem we obtain

$$A^2 = r^2 + r^2 - 2r\cos 45^o \Longrightarrow r = \frac{A}{\sqrt{2-\sqrt{2}}}$$

3) The average transmitted power of the PSK constellation is

$$P_{\text{PSK}} = 8 \times \frac{1}{8} \times \left(\frac{A}{\sqrt{2-\sqrt{2}}}\right)^2 \Longrightarrow P_{\text{PSK}} = \frac{A^2}{2-\sqrt{2}}$$

whereas the average transmitted power of the QAM constellation

$$P_{\text{QAM}} = \frac{1}{8}\left(4\frac{A^2}{2} + 4\frac{(1+\sqrt{3})^2}{4}A^2\right) \Longrightarrow P_{\text{QAM}} = \left[\frac{2+(1+\sqrt{3})^2}{8}\right]A^2$$

The relative power advantage of the PSK constellation over the QAM constellation is

$$\text{gain} = \frac{P_{\text{PSK}}}{P_{\text{QAM}}} = \frac{8}{(2+(1+\sqrt{3})^2)(2-\sqrt{2})} = 1.5927 \text{ dB}$$

Problem 7.47

1) The number of bits per symbol is

$$k = \frac{4800}{R} = \frac{4800}{2400} = 2$$

Thus, a 4-QAM constellation is used for transmission. The probability of error for an M-ary QAM system with $M = 2^k$, is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{\sqrt{M}}\right)Q\left[\sqrt{\frac{3k\mathcal{E}_b}{(M-1)N_0}}\right]\right)^2$$

With $P_M = 10^{-5}$ and $k = 2$ we obtain

$$Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = 5 \times 10^{-6} \Longrightarrow \frac{\mathcal{E}_b}{N_0} = 9.7682$$

2 If the bit rate of transmission is 9600 bps, then

$$k = \frac{9600}{2400} = 4$$

In this case a 16-QAM constellation is used and the probability of error is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{4}\right)Q\left[\sqrt{\frac{3 \times 4 \times \mathcal{E}_b}{15 \times N_0}}\right]\right)^2$$

Thus,

$$Q\left[\sqrt{\frac{3 \times \mathcal{E}_b}{15 \times N_0}}\right] = \frac{1}{3} \times 10^{-5} \Longrightarrow \frac{\mathcal{E}_b}{N_0} = 25.3688$$

3 If the bit rate of transmission is 19200 bps, then

$$k = \frac{19200}{2400} = 8$$

In this case a 256-QAM constellation is used and the probability of error is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{16}\right)Q\left[\sqrt{\frac{3 \times 8 \times \mathcal{E}_b}{255 \times N_0}}\right]\right)^2$$

With $P_M = 10^{-5}$ we obtain

$$\frac{\mathcal{E}_b}{N_0} = 659.8922$$

4) The following table gives the SNR per bit and the corresponding number of bits per symbol for the constellations used in parts a)-c).

k	2	4	8
SNR (db)	9.89	14.04	28.19

As it is observed there is an increase in transmitted power of approximately 3 dB per additional bit per symbol.

Problem 7.48

1) Although it is possible to assign three bits to each point of the 8-PSK signal constellation so that adjacent points differ in only one bit, this is not the case for the 8-QAM constellation of Figure P-7.46. This is because there are fully connected graphs consisted of three points. To see this consider an equilateral triangle with vertices A, B and C. If, without loss of generality, we assign the all zero sequence $\{0, 0, \ldots, 0\}$ to point A, then point B and C should have the form

$$\mathrm{B} = \{0, \ldots, 0, 1, 0, \ldots, 0\} \qquad \mathrm{C} = \{0, \ldots, 0, 1, 0, \ldots, 0\}$$

where the position of the 1 in the sequences is not the same, otherwise B=C. Thus, the sequences of B and C differ in two bits.

2) Since each symbol conveys 3 bits of information, the resulted symbol rate is

$$R_s = \frac{90 \times 10^6}{3} = 30 \times 10^6 \text{ symbols/sec}$$

3) The probability of error for an M-ary PSK signal is

$$P_M = 2Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} \sin\frac{\pi}{M}\right]$$

whereas the probability of error for an M-ary QAM signal is upper bounded by

$$P_M = 4Q\left[\sqrt{\frac{3\mathcal{E}_{av}}{(M-1)N_0}}\right]$$

Since, the probability of error is dominated by the argument of the Q function, the two signals will achieve the same probability of error if

$$\sqrt{2\mathrm{SNR}_{\mathrm{PSK}}} \sin\frac{\pi}{M} = \sqrt{\frac{3\mathrm{SNR}_{\mathrm{QAM}}}{M-1}}$$

With $M = 8$ we obtain

$$\sqrt{2\mathrm{SNR}_{\mathrm{PSK}}} \sin\frac{\pi}{8} = \sqrt{\frac{3\mathrm{SNR}_{\mathrm{QAM}}}{7}} \Longrightarrow \frac{\mathrm{SNR}_{\mathrm{PSK}}}{\mathrm{SNR}_{\mathrm{QAM}}} = \frac{3}{7 \times 2 \times 0.3827^2} = 1.4627$$

4) Assuming that the magnitude of the signal points is detected correctly, then the detector for the 8-PSK signal will make an error if the phase error (magnitude) is greater than 22.5^o. In the case of the 8-QAM constellation an error will be made if the magnitude phase error exceeds 45^o. Hence, the QAM constellation is more immune to phase errors.

Problem 7.49

Consider the following waveforms of the binary FSK signaling:

$$
\begin{aligned}
u_1(t) &= \sqrt{\frac{2\mathcal{E}_b}{T}}\cos(2\pi f_c t) \\
u_2(t) &= \sqrt{\frac{2\mathcal{E}_b}{T}}\cos(2\pi f_c t + 2\pi\Delta f t)
\end{aligned}
$$

The correlation of the two signals is

$$
\begin{aligned}
\gamma_{12} &= \frac{1}{\mathcal{E}_b}\int_0^T u_1(t)u_2(t)dt \\
&= \frac{1}{\mathcal{E}_b}\int_0^T \frac{2\mathcal{E}_b}{T}\cos(2\pi f_c t)\cos(2\pi f_c t + 2\pi\Delta f t)dt \\
&= \frac{1}{T}\int_0^T \cos(2\pi\Delta f t)dt + \frac{1}{T}\int_0^T \cos(2\pi 2 f_c t + 2\pi\Delta f t)dt
\end{aligned}
$$

If $f_c \gg \frac{1}{T}$, then

$$
\gamma_{12} = \frac{1}{T}\int_0^T \cos(2\pi\Delta f t)dt = \frac{\sin(2\pi\Delta f T)}{2\pi\Delta f T}
$$

To find the minimum value of the correlation, we set the derivative of γ_{12} with respect to Δf equal to zero. Thus,

$$
\frac{\vartheta\gamma_{12}}{\vartheta\Delta f} = 0 = \frac{\cos(2\pi\Delta f T)2\pi T}{2\pi\Delta f T} - \frac{\sin(2\pi\Delta f T)}{(2\pi\Delta f T)^2}2\pi T
$$

and therefore,

$$
2\pi\Delta f T = \tan(2\pi\Delta f T)
$$

Solving numerically the equation $x = \tan(x)$, we obtain $x = 4.4934$. Thus,

$$
2\pi\Delta f T = 4.4934 \Longrightarrow \Delta f = \frac{0.7151}{T}
$$

and the value of γ_{12} is -0.2172. Note that when a gradient method like the Gauss-Newton is used to solve the equation $f(x) = x - \tan(x) = 0$, then in order to find the smallest nonzero root, the initial value of the algorithm x_0 should be selected in the range $(\frac{\pi}{2}, \frac{3\pi}{2})$.

The probability of error can be expressed in terms of the distance d_{12} between the signal points, as

$$
p_b = Q\left[\sqrt{\frac{d_{12}^2}{2N_0}}\right]
$$

The two signal vectors $\mathbf{u}_1$, $\mathbf{u}_2$ are of equal energy

$$
\|\mathbf{u}_1\|^2 = \|\mathbf{u}_2\|^2 = \mathcal{E}_b
$$

and the angle θ_{12} between them is such that

$$
\cos(\theta_{12}) = \gamma_{12}
$$

Hence,

$$
d_{12}^2 = \|\mathbf{u}_1\|^2 + \|\mathbf{u}_2\|^2 - 2\|\mathbf{u}_1\|\|\mathbf{u}_2\|\cos(\theta_{12}) = 2\mathcal{E}_s(1-\gamma_{12})
$$

and therefore,

$$
p_b = Q\left[\sqrt{\frac{2\mathcal{E}_s(1-\gamma_{12})}{2N_0}}\right] = Q\left[\sqrt{\frac{\mathcal{E}_s(1+0.2172)}{N_0}}\right]
$$

Problem 7.50

1) The first set represents a 4-PAM signal constellation. The points of the constellation are $\{\pm A, \pm 3A\}$. The second set consists of four orthogonal signals. The geometric representation of the signals is

$$\mathbf{s}_1 = [\ A \quad 0 \quad 0 \quad 0\ ] \qquad \mathbf{s}_3 = [\ 0 \quad 0 \quad A \quad 0\ ]$$
$$\mathbf{s}_2 = [\ 0 \quad A \quad 0 \quad 0\ ] \qquad \mathbf{s}_4 = [\ 0 \quad 0 \quad 0 \quad A\ ]$$

This set can be classified as a 4-FSK signal. The third set can be classified as a 4-QAM signal constellation. The geometric representation of the signals is

$$\mathbf{s}_1 = [\ \tfrac{A}{\sqrt{2}} \quad \tfrac{A}{\sqrt{2}}\ ] \qquad \mathbf{s}_3 = [\ -\tfrac{A}{\sqrt{2}} \quad -\tfrac{A}{\sqrt{2}}\ ]$$
$$\mathbf{s}_2 = [\ \tfrac{A}{\sqrt{2}} \quad -\tfrac{A}{\sqrt{2}}\ ] \qquad \mathbf{s}_4 = [\ -\tfrac{A}{\sqrt{2}} \quad \tfrac{A}{\sqrt{2}}\ ]$$

2) The average transmitted energy for sets I, II and III is

$$\begin{aligned}
\mathcal{E}_{av,I} &= \frac{1}{4}\sum_{i=1}^{4}\|\mathbf{s}_i\|^2 = \frac{1}{4}(A^2+9A^2+9A^2+A^2) = 5A^2 \\
\mathcal{E}_{av,II} &= \frac{1}{4}\sum_{i=1}^{4}\|\mathbf{s}_i\|^2 = \frac{1}{4}(4A^2) = A^2 \\
\mathcal{E}_{av,III} &= \frac{1}{4}\sum_{i=1}^{4}\|\mathbf{s}_i\|^2 = \frac{1}{4}(4\times(\frac{A^2}{2}+\frac{A^2}{2})) = A^2
\end{aligned}$$

3) The probability of error for the 4-PAM signal is given by

$$P_{4,I} = \frac{2(M-1)}{M}Q\left[\sqrt{\frac{6\mathcal{E}_{av,I}}{(M^2-1)N_0}}\right] = \frac{3}{2}Q\left[\sqrt{\frac{6\times 5\times A^2}{15N_0}}\right] = \frac{3}{2}Q\left[\sqrt{\frac{2A^2}{N_0}}\right]$$

4) When coherent detection is employed, then an upper bound on the probability of error is given by

$$P_{4,II,\text{coherent}} \leq (M-1)Q\left[\sqrt{\frac{\mathcal{E}_s}{N_0}}\right] = 3Q\left[\sqrt{\frac{A^2}{N_0}}\right]$$

If the detection is performed noncoherently, then the probability of error is given by

$$\begin{aligned}
P_{4,II,\text{noncoherent}} &= \sum_{n=1}^{M-1}(-1)^{n+1}\binom{M-1}{n}\frac{1}{n+1}e^{-n\rho_s/(n=1)} \\
&= \frac{3}{2}e^{-\frac{\rho_s}{2}} - e^{-\frac{2\rho_s}{3}} + \frac{1}{4}e^{-\frac{3\rho_s}{4}} \\
&= \frac{3}{2}e^{-\frac{A^2}{2N_0}} - e^{-\frac{2A^2}{3N_0}} + \frac{1}{4}e^{-\frac{3A^2}{4N_0}}
\end{aligned}$$

5) It is not possible to use noncoherent detection for the signal set III. This is because all signals have the same square amplitude for every $t \in [0, 2T]$.

6) The following table shows the bit rate to bandwidth ratio for the different types of signaling and the results for $M = 4$.

Type	R/W	$M=4$
PAM	$2\log_2 M$	4
QAM	$\log_2 M$	2
FSK (coherent)	$\frac{2\log_2 M}{M}$	1
FSK (noncoherent)	$\frac{\log_2 M}{M}$	0.5

To achieve a ratio $\frac{R}{W}$ of at least 2, we have to select either the first signal set (PAM) or the second signal set (QAM).

Problem 7.51

1) If the transmitted signal is

$$u_0(t) = \sqrt{\frac{2\mathcal{E}_s}{T}}\cos(2\pi f_c t), \qquad 0 \le t \le T$$

then the received signal is

$$r(t) = \sqrt{\frac{2\mathcal{E}_s}{T}}\cos(2\pi f_c t + \phi) + n(t)$$

In the phase-coherent demodulation of M-ary FSK signals, the received signal is correlated with each of the M-possible received signals $\cos(2\pi f_c t + 2\pi m\Delta f t + \hat{\phi}_m)$, where $\hat{\phi}_m$ are the carrier phase estimates. The output of the m^{th} correlator is

$$\begin{aligned}
r_m &= \int_0^T r(t)\cos(2\pi f_c t + 2\pi m\Delta f t + \hat{\phi}_m)dt \\
&= \int_0^T \sqrt{\frac{2\mathcal{E}_s}{T}}\cos(2\pi f_c t + \phi)\cos(2\pi f_c t + 2\pi m\Delta f t + \hat{\phi}_m)dt \\
&\quad + \int_0^T n(t)\cos(2\pi f_c t + 2\pi m\Delta f t + \hat{\phi}_m)dt \\
&= \sqrt{\frac{2\mathcal{E}_s}{T}}\int_0^T \frac{1}{2}\Big(\cos(2\pi 2 f_c t + 2\pi m\Delta f t + \hat{\phi}_m + \phi) + \cos(2\pi m\Delta f t + \hat{\phi}_m - \phi)\Big) + n \\
&= \sqrt{\frac{2\mathcal{E}_s}{T}}\frac{1}{2}\int_0^T \cos(2\pi m\Delta f t + \hat{\phi}_m - \phi)dt + n
\end{aligned}$$

where n is a zero-mean Gaussian random variable with variance $\frac{N_0}{2}$.

2) In order to obtain orthogonal signals at the demodulator, the expected value of r_m, $E[r_m]$, should be equal to zero for every $m \neq 0$. Since $E[n] = 0$, the latter implies that

$$\int_0^T \cos(2\pi m\Delta f t + \hat{\phi}_m - \phi)dt = 0, \qquad \forall m \neq 0$$

The equality is true when $m\Delta f$ is a multiple of $\frac{1}{T}$. Since the smallest value of m is 1, the necessary condition for orthogonality is

$$\Delta f = \frac{1}{T}$$

Problem 7.52

The noise components in the sampled output of the two correlators for the m^{th} FSK signal, are given by

$$\begin{aligned}
n_{mc} &= \int_0^T n(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t + 2\pi m\Delta f t)dt \\
n_{ms} &= \int_0^T n(t)\sqrt{\frac{2}{T}}\sin(2\pi f_c t + 2\pi m\Delta f t)dt
\end{aligned}$$

Clearly, n_{mc}, n_{ms} are zero-mean random variables since

$$\begin{aligned}
E[n_{mc}] &= E\left[\int_0^T n(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t + 2\pi m\Delta f t)dt\right] \\
&= \int_0^T E[n(t)]\sqrt{\frac{2}{T}}\cos(2\pi f_c t + 2\pi m\Delta f t)dt = 0 \\
E[n_{ms}] &= E\left[\int_0^T n(t)\sqrt{\frac{2}{T}}\sin(2\pi f_c t + 2\pi m\Delta f t)dt\right] \\
&= \int_0^T E[n(t)]\sqrt{\frac{2}{T}}\sin(2\pi f_c t + 2\pi m\Delta f t)dt = 0
\end{aligned}$$

Furthermore,

$$\begin{aligned}
E[n_{mc}n_{kc}] &= E\left[\int_0^T\int_0^T \frac{2}{T}n(t)n(\tau)\cos(2\pi f_c t + 2\pi m\Delta f t)\cos(2\pi f_c t + 2\pi k\Delta f \tau)dtd\tau\right] \\
&= \frac{2}{T}\int_0^T\int_0^T E[n(t)n(\tau)]\cos(2\pi f_c t + 2\pi m\Delta f t)\cos(2\pi f_c t + 2\pi k\Delta f \tau)dtd\tau \\
&= \frac{2}{T}\frac{N_0}{2}\int_0^T \cos(2\pi f_c t + 2\pi m\Delta f t)\cos(2\pi f_c t + 2\pi k\Delta f t)dt \\
&= \frac{2}{T}\frac{N_0}{2}\int_0^T \frac{1}{2}\left(\cos(2\pi 2 f_c t + 2\pi(m+k)\Delta f t) + \cos(2\pi(m-k)\Delta f t)\right)dt \\
&= \frac{2}{T}\frac{N_0}{2}\int_0^T \frac{1}{2}\delta_{mk}dt = \frac{N_0}{2}\delta_{mk}
\end{aligned}$$

where we have used the fact that for $f_c \gg \frac{1}{T}$

$$\int_0^T \cos(2\pi 2 f_c t + 2\pi(m+k)\Delta f t)dt \approx 0$$

and for $\Delta f = \frac{1}{T}$

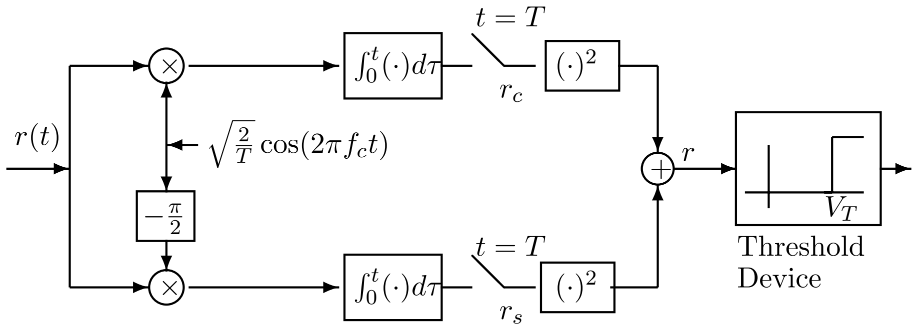
$$\int_0^T \cos(2\pi(m-k)\Delta f t)dt = 0, \qquad m \neq k$$

Thus, n_{mc}, n_{kc} are uncorrelated for $m \neq k$ and since they are zero-mean Gaussian they are independent. Similarly we obtain

$$\begin{aligned}
E[n_{mc}n_{ks}] &= E\left[\int_0^T\int_0^T \frac{2}{T}n(t)n(\tau)\cos(2\pi f_c t + 2\pi m\Delta f t)\sin(2\pi f_c t + 2\pi k\Delta f \tau)dtd\tau\right] \\
&= \frac{2}{T}\int_0^T\int_0^T E[n(t)n(\tau)]\cos(2\pi f_c t + 2\pi m\Delta f t)\sin(2\pi f_c t + 2\pi k\Delta f \tau)dtd\tau \\
&= \frac{2}{T}\frac{N_0}{2}\int_0^T \cos(2\pi f_c t + 2\pi m\Delta f t)\sin(2\pi f_c t + 2\pi k\Delta f t)dt \\
&= \frac{2}{T}\frac{N_0}{2}\int_0^T \frac{1}{2}\left(\sin(2\pi 2 f_c t + 2\pi(m+k)\Delta f t) - \sin(2\pi(m-k)\Delta f t)\right)dt \\
&= 0 \\
E[n_{ms}n_{ks}] &= \frac{N_0}{2}\delta_{mk}
\end{aligned}$$

Problem 7.53

1) The noncoherent envelope detector for the on-off keying signal is depicted in the next figure.



2) If $s_0(t)$ is sent, then the received signal is $r(t) = n(t)$ and therefore the sampled outputs r_c, r_s are zero-mean independent Gaussian random variables with variance $\frac{N_0}{2}$. Hence, the random variable $r = \sqrt{r_c^2 + r_s^2}$ is Rayleigh distributed and the PDF is given by

$$p(r|s_0(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} = \frac{2r}{N_0} e^{-\frac{r^2}{N_0}}$$

If $s_1(t)$ is transmitted, then the received signal is

$$r(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos(2\pi f_c t + \phi) + n(t)$$

Crosscorrelating $r(t)$ by $\sqrt{\frac{2}{T}} \cos(2\pi f_c t)$ and sampling the output at $t = T$, results in

$$\begin{aligned}
r_c &= \int_0^T r(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt \\
&= \int_0^T \frac{2\sqrt{\mathcal{E}_b}}{T_b} \cos(2\pi f_c t + \phi) \cos(2\pi f_c t) dt + \int_0^T n(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt \\
&= \frac{2\sqrt{\mathcal{E}_b}}{T_b} \int_0^T \frac{1}{2} \left(\cos(2\pi 2 f_c t + \phi) + \cos(\phi)\right) dt + n_c \\
&= \sqrt{\mathcal{E}_b} \cos(\phi) + n_c
\end{aligned}$$

where n_c is zero-mean Gaussian random variable with variance $\frac{N_0}{2}$. Similarly, for the quadrature component we have

$$r_s = \sqrt{\mathcal{E}_b} \sin(\phi) + n_s$$

The PDF of the random variable $r = \sqrt{r_c^2 + r_s^2} = \sqrt{\mathcal{E}_b + n_c^2 + n_s^2}$ is (see Problem 4.31)

$$p(r|s_1(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2 + \mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) = \frac{2r}{N_0} e^{-\frac{r^2 + \mathcal{E}_b}{N_0}} I_0\left(\frac{2r\sqrt{\mathcal{E}_b}}{N_0}\right)$$

that is a Rician PDF.

3) For equiprobable signals the probability of error is given by

$$P(\text{error}) = \frac{1}{2} \int_{-\infty}^{V_T} p(r|s_1(t)) dr + \frac{1}{2} \int_{V_T}^{\infty} p(r|s_0(t)) dr$$

Since $r > 0$ the expression for the probability of error takes the form

$$\begin{aligned}
P(\text{error}) &= \frac{1}{2} \int_0^{V_T} p(r|s_1(t)) dr + \frac{1}{2} \int_{V_T}^{\infty} p(r|s_0(t)) dr \\
&= \frac{1}{2} \int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2 + \mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) dr + \frac{1}{2} \int_{V_T}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr
\end{aligned}$$

The optimum threshold level is the value of V_T that minimizes the probability of error. However, when $\frac{\mathcal{E}_b}{N_0} \gg 1$ the optimum value is close to $\frac{\sqrt{\mathcal{E}_b}}{2}$ and we will use this threshold to simplify the analysis. The integral involving the Bessel function cannot be evaluated in closed form. Instead of $I_0(x)$ we will use the approximation

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}$$

which is valid for large x, that is for high SNR. In this case

$$\frac{1}{2}\int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) dr \approx \frac{1}{2}\int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr$$

This integral is further simplified if we observe that for high SNR, the integrand is dominant in the vicinity of $\sqrt{\mathcal{E}_b}$ and therefore, the lower limit can be substituted by $-\infty$. Also

$$\sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} \approx \sqrt{\frac{1}{2\pi\sigma^2}}$$

and therefore,

$$\begin{aligned}
\frac{1}{2}\int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr &\approx \frac{1}{2}\int_{-\infty}^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{1}{2\pi\sigma^2}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr \\
&= \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right]
\end{aligned}$$

Finally

$$\begin{aligned}
P(\text{error}) &= \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2}\int_{\frac{\sqrt{\mathcal{E}_b}}{2}}^{\infty} \frac{2r}{N_0} e^{-\frac{r^2}{N_0}} dr \\
&\leq \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2} e^{-\frac{\mathcal{E}_b}{4N_0}}
\end{aligned}$$

Problem 7.54

(a) Four phase PSK

If we use a pulse shape having a raised cosine spectrum with a rolloff α, the symbol rate is determined from the relation

$$\frac{1}{2T}(1+\alpha) = 50000$$

Hence,

$$\frac{1}{T} = \frac{10^5}{1+\alpha}$$

where $W = 10^5$ Hz is the channel bandwidth. The bit rate is

$$\frac{2}{T} = \frac{2 \times 10^5}{1+\alpha} \quad \text{bps}$$

(b) Binary FSK with noncoherent detection

In this case we select the two frequencies to have a frequency separation of $\frac{1}{T}$, where $\frac{1}{T}$ is the symbol rate. Hence

$$f_1 = f_c - \frac{1}{2T}$$

$$f_2 = f + c + \frac{1}{2T}$$

where f_c is the carrier in the center of the channel band. Thus, we have

$$\frac{1}{2T} = 50000$$

or equivalently

$$\frac{1}{T} = 10^5$$

Hence, the bit rate is 10^5 bps.
(c) $M = 4$ FSK with noncoherent detection
In this case we require four frequencies with adjacent frequencies separation of $\frac{1}{T}$. Hence, we select

$$f_1 = f_c - |\frac{1.5}{T}, \quad f_2 = f_c - \frac{1}{2T}, \quad f_3 = f_c + \frac{1}{2T}, \quad f_4 = f_c + \frac{1.5}{T}$$

where f_c is the carrier frequency and $\frac{1}{2T} = 25000$, or, equivalently,

$$\frac{1}{T} = 50000$$

Since the symbol rate is 50000 symbols per second and each symbol conveys 2 bits, the bit rate is 10^5 bps.

Problem 7.55

a) For n repeaters in cascade, the probability of i out of n repeaters to produce an error is given by the binomial distribution

$$P_i = \binom{n}{i} p^i (1-p)^{n-i}$$

However, there is a bit error at the output of the terminal receiver only when an odd number of repeaters produces an error. Hence, the overall probability of error is

$$P_n = P_{\text{odd}} = \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i}$$

Let P_{even} be the probability that an even number of repeaters produces an error. Then

$$P_{\text{even}} = \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i}$$

and therefore,

$$P_{\text{even}} + P_{\text{odd}} = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = (p+1-p)^n = 1$$

One more relation between P_{even} and P_{odd} can be provided if we consider the difference $P_{\text{even}} - P_{\text{odd}}$. Clearly,

$$\begin{aligned}
P_{\text{even}} - P_{\text{odd}} &= \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i} - \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i} \\
&\overset{a}{=} \sum_{i=\text{even}} \binom{n}{i} (-p)^i (1-p)^{n-i} + \sum_{i=\text{odd}} \binom{n}{i} (-p)^i (1-p)^{n-i} \\
&= (1-p-p)^n = (1-2p)^n
\end{aligned}$$

where the equality (a) follows from the fact that $(-1)^i$ is 1 for i even and -1 when i is odd. Solving the system

$$\begin{aligned}
P_{\text{even}} + P_{\text{odd}} &= 1 \\
P_{\text{even}} - P_{\text{odd}} &= (1-2p)^n
\end{aligned}$$

we obtain

$$P_n = P_{\text{odd}} = \frac{1}{2}(1 - (1 - 2p)^n)$$

b) Expanding the quantity $(1 - 2p)^n$, we obtain

$$(1 - 2p)^n = 1 - n2p + \frac{n(n-1)}{2}(2p)^2 + \cdots$$

Since, $p \ll 1$ we can ignore all the powers of p which are greater than one. Hence,

$$P_n \approx \frac{1}{2}(1 - 1 + n2p) = np = 100 \times 10^{-6} = 10^{-4}$$

Problem 7.56

The overall probability of error is approximated by

$$P(e) = KQ\left[\sqrt{\frac{\mathcal{E}_b}{N_0}}\right]$$

Thus, with $P(e) = 10^{-6}$ and $K = 100$, we obtain the probability of each repeater $P_r = Q\left[\sqrt{\frac{\mathcal{E}_b}{N_0}}\right] = 10^{-8}$. The argument of the function $Q[\cdot]$ that provides a value of 10^{-8} is found from tables to be

$$\sqrt{\frac{\mathcal{E}_b}{N_0}} = 5.61$$

Hence, the required $\frac{\mathcal{E}_b}{N_0}$ is $5.61^2 = 31.47$

Problem 7.57

a) The antenna gain for a parabolic antenna of diameter D is

$$G_R = \eta\left(\frac{\pi D}{\lambda}\right)^2$$

If we assume that the efficiency factor is 0.5, then with

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^9} = 0.3 \text{ m} \qquad D = 3 \times 0.3048 \text{ m}$$

we obtain

$$G_R = G_T = 45.8458 = 16.61 \text{ dB}$$

b) The effective radiated power is

$$\text{EIRP} = P_T G_T = G_T = 16.61 \text{ dB}$$

c) The received power is

$$P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 2.995 \times 10^{-9} = -85.23 \text{ dB} \ = -55.23 \text{ dBm}$$

Note that

$$\text{dBm} = 10\log_{10}\left(\frac{\text{actual power in Watts}}{10^{-3}}\right) = 30 + 10\log_{10}(\text{power in Watts })$$

Problem 7.58

a) The antenna gain for a parabolic antenna of diameter D is

$$G_R = \eta\left(\frac{\pi D}{\lambda}\right)^2$$

If we assume that the efficiency factor is 0.5, then with

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{10^9} = 0.3 \text{ m} \qquad \text{and} \qquad D = 1 \text{ m}$$

we obtain

$$G_R = G_T = 54.83 = 17.39 \text{ dB}$$

b) The effective radiated power is

$$\text{EIRP} = P_T G_T = 0.1 \times 54.83 = 7.39 \text{ dB}$$

c) The received power is

$$P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 1.904 \times 10^{-10} = -97.20 \text{ dB } = -67.20 \text{ dBm}$$

Problem 7.59

The wavelength of the transmitted signal is

$$\lambda = \frac{3\times 10^8}{10\times 10^9} = 0.03 \text{ m}$$

The gain of the parabolic antenna is

$$G_R = \eta\left(\frac{\pi D}{\lambda}\right)^2 = 0.6\left(\frac{\pi 10}{0.03}\right)^2 = 6.58\times 10^5 = 58.18 \text{ dB}$$

The received power at the output of the receiver antenna is

$$P_R = \frac{P_T G_T G_R}{(4\pi\frac{d}{\lambda})^2} = \frac{3\times 10^{1.5}\times 6.58\times 10^5}{(4\times 3.14159\times\frac{4\times 10^7}{0.03})^2} = 2.22\times 10^{-13} = -126.53 \text{ dB}$$

Problem 7.60

a) Since $T = 300^0 K$, it follows that

$$N_0 = kT = 1.38\times 10^{-23}\times 300 = 4.14\times 10^{-21} \text{ W/Hz}$$

If we assume that the receiving antenna has an efficiency $\eta = 0.5$, then its gain is given by

$$G_R = \eta\left(\frac{\pi D}{\lambda}\right)^2 = 0.5\left(\frac{3.14159\times 50}{\frac{3\times 10^8}{2\times 10^9}}\right)^2 = 5.483\times 10^5 = 57.39 \text{ dB}$$

Hence, the received power level is

$$P_R = \frac{P_T G_T G_R}{(4\pi\frac{d}{\lambda})^2} = \frac{10 \times 10 \times 5.483 \times 10^5}{(4 \times 3.14159 \times \frac{10^8}{0.15})^2} = 7.8125 \times 10^{-13} = -121.07 \text{ dB}$$

b) If $\frac{\mathcal{E}_b}{N_0} = 10$ dB $= 10$, then

$$R = \frac{P_R}{N_0}\left(\frac{\mathcal{E}_b}{N_0}\right)^{-1} = \frac{7.8125 \times 10^{-13}}{4.14 \times 10^{-21}} \times 10^{-1} = 1.8871 \times 10^7 = 18.871 \text{ Mbits/sec}$$

Problem 7.61

The overall gain of the system is

$$G_{\text{tot}} = G_{\text{a}_1} + G_{\text{os}} + G_{\text{BPF}} + G_{\text{a}_2} = 10 - 5 - 1 + 25 = 29 \text{ dB}$$

Hence, the power of the signal at the input of the demodulator is

$$P_{s,\text{dem}} = (-113 - 30) + 29 = -114 \text{ dB}$$

The noise-figure for the cascade of the first amplifier and the multiplier is

$$F_1 = F_{\text{a}_1} + \frac{F_{\text{os}} - 1}{G_{\text{a}_1}} = 10^{0.5} + \frac{10^{0.5} - 1}{10} = 3.3785$$

We assume that F_1 is the spot noise-figure and therefore, it measures the ratio of the available PSD out of the two devices to the available PSD out of an ideal device with the same available gain. That is,

$$F_1 = \frac{\mathcal{S}_{n,o}(f)}{\mathcal{S}_{n,i}(f) G_{\text{a}_1} G_{\text{os}}}$$

where $\mathcal{S}_{n,o}(f)$ is the power spectral density of the noise at the input of the bandpass filter and $\mathcal{S}_{n,i}(f)$ is the power spectral density at the input of the overall system. Hence,

$$\mathcal{S}_{n,o}(f) = 10^{\frac{-175-30}{10}} \times 10 \times 10^{-0.5} \times 3.3785 = 3.3785 \times 10^{-20}$$

The noise-figure of the cascade of the bandpass filter and the second amplifier is

$$F_2 = F_{\text{BPF}} + \frac{F_{\text{a}_2} - 1}{G_{\text{BPF}}} = 10^{0.2} + \frac{10^{0.5} - 1}{10^{-0.1}} = 4.307$$

Hence, the power of the noise at the output of the system is

$$P_{n,\text{dem}} = 2\mathcal{S}_{n,o}(f) B G_{\text{BPF}} G_{\text{a}_2} F_2 = 7.31 \times 10^{-12} = -111.36 \text{ dB}$$

The signal to noise ratio at the output of the system (input to the demodulator) is

$$\text{SNR} = \frac{P_{s,\text{dem}}}{P_{n,\text{dem}}} = -114 + 111.36 = -2.64 \text{ dB}$$

Problem 7.62

The wavelength of the transmission is

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{4 \times 10^9} = 0.75 \text{ m}$$

If 1 MHz is the passband bandwidth, then the rate of binary transmission is $R_b = W = 10^6$ bps. Hence, with $N_0 = 4.1 \times 10^{-21}$ W/Hz we obtain

$$\frac{P_R}{N_0} = R_b \frac{\mathcal{E}_b}{N_0} \Longrightarrow 10^6 \times 4.1 \times 10^{-21} \times 10^{1.5} = 1.2965 \times 10^{-13}$$

The transmitted power is related to the received power through the relation

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} \Longrightarrow P_T = \frac{P_R}{G_T G_R} \left(4\pi \frac{d}{\lambda}\right)^2$$

Substituting in this expression the values $G_T = 10^{0.6}$, $G_R = 10^5$, $d = 36 \times 10^6$ and $\lambda = 0.75$ we obtain

$$P_T = 0.1185 = -9.26 \text{ dBW}$$

Problem 7.63

Since $T = 290^0 + 15^0 = 305^0 K$, it follows that

$$N_0 = kT = 1.38 \times 10^{-23} \times 305 = 4.21 \times 10^{-21} \text{ W/Hz}$$

The transmitting wavelength λ is

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{2.3 \times 10^9} = 0.130 \text{ m}$$

Hence, the gain of the receiving antenna is

$$G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2 = 0.55 \left(\frac{3.14159 \times 64}{0.130}\right)^2 = 1.3156 \times 10^6 = 61.19 \text{ dB}$$

and therefore, the received power level is

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} = \frac{17 \times 10^{2.7} \times 1.3156 \times 10^6}{(4 \times 3.14159 \times \frac{1.6 \times 10^{11}}{0.130})^2} = 4.686 \times 10^{-12} = -113.29 \text{ dB}$$

If $\mathcal{E}_b/N_0 = 6$ dB $= 10^{0.6}$, then

$$R = \frac{P_R}{N_0} \left(\frac{\mathcal{E}_b}{N_0}\right)^{-1} = \frac{4.686 \times 10^{-12}}{4.21 \times 10^{-21}} \times 10^{-0.6} = 4.4312 \times 10^9 = 4.4312 \text{ Gbits/sec}$$

Problem 7.64

In the non decision-directed timing recovery method we maximize the function

$$\Lambda_2(\tau) = \sum_m y_m^2(\tau)$$

with respect to τ. Thus, we obtain the condition

$$\frac{d\Lambda_2(\tau)}{d\tau} = 2 \sum_m y_m(\tau) \frac{dy_m(\tau)}{d\tau} = 0$$

Suppose now that we approximate the derivative of the log-likelihood $\Lambda_2(\tau)$ by the finite difference

$$\frac{d\Lambda_2(\tau)}{d\tau} \approx \frac{\Lambda_2(\tau + \delta) - \Lambda_2(\tau - \delta)}{2\delta}$$

Then, if we substitute the expression of $\Lambda_2(\tau)$ in the previous approximation, we obtain

$$\begin{aligned}
&\frac{d\Lambda_2(\tau)}{d\tau} \\
&= \frac{\sum_m y_m^2(\tau+\delta) - \sum_m y_m^2(\tau-\delta)}{2\delta} \\
&= \frac{1}{2\delta}\sum_m \left[\left(\int r(t)u(t-mT-\tau-\delta)dt\right)^2 - \left(\int r(t)u(t-mT-\tau+\delta)dt\right)^2\right]
\end{aligned}$$

where $u(-t) = g_R(t)$ is the impulse response of the matched filter in the receiver. However, this is the expression of the early-late gate synchronizer, where the lowpass filter has been substituted by the summation operator. Thus, the early-late gate synchronizer is a close approximation to the timing recovery system.

Problem 7.65

An on-off keying signal is represented as

$$\begin{aligned}
s_1(t) &= A\cos(2\pi f_c t + \theta_c), \quad 0 \le t \le T \text{ (binary 1)} \\
s_2(t) &= 0, \quad\quad\quad\quad\quad\quad\quad 0 \le t \le T \text{ (binary 0)}
\end{aligned}$$

Let $r(t)$ be the received signal, that is

$$r(t) = s(t;\theta_c) + n(t)$$

where $s(t;\theta_c)$ is either $s_1(t)$ or $s_2(t)$ and $n(t)$ is white Gaussian noise with variance $\frac{N_0}{2}$. The likelihood function, that is to be maximized with respect to θ_c over the interval $[0,T]$, is proportional to

$$\Lambda(\theta_c) = \exp\left[-\frac{2}{N_0}\int_0^T [r(t) - s(t;\theta_c)]^2 dt\right]$$

Maximization of $\Lambda(\theta_c)$ is equivalent to the maximization of the log-likelihood function

$$\begin{aligned}
\Lambda_L(\theta_c) &= -\frac{2}{N_0}\int_0^T [r(t) - s(t;\theta_c)]^2 dt \\
&= -\frac{2}{N_0}\int_0^T r^2(t)dt + \frac{4}{N_0}\int_0^T r(t)s(t;\theta_c)dt - \frac{2}{N_0}\int_0^T s^2(t;\theta_c)dt
\end{aligned}$$

Since the first term does not involve the parameter of interest θ_c and the last term is simply a constant equal to the signal energy of the signal over $[0,T]$ which is independent of the carrier phase, we can carry the maximization over the function

$$V(\theta_c) = \int_0^T r(t)s(t;\theta_c)dt$$

Note that $s(t;\theta_c)$ can take two different values, $s_1(t)$ and $s_2(t)$, depending on the transmission of a binary 1 or 0. Thus, a more appropriate function to maximize is the average log-likelihood

$$\bar{V}(\theta_c) = \frac{1}{2}\int_0^T r(t)s_1(t)dt + \frac{1}{2}\int_0^T r(t)s_2(t)dt$$

Since $s_2(t) = 0$, the function $\bar{V}(\theta_c)$ takes the form

$$\bar{V}(\theta_c) = \frac{1}{2}\int_0^T r(t)A\cos(2\pi f_c t + \theta_c)dt$$

Setting the derivative of $\bar{V}(\theta_c)$ with respect to θ_c equal to zero, we obtain

$$\begin{aligned}
\frac{\vartheta \bar{V}(\theta_c)}{\vartheta \theta_c} = 0 &= \frac{1}{2}\int_0^T r(t)A\sin(2\pi f_c t + \theta_c)dt \\
&= \cos\theta_c \frac{1}{2}\int_0^T r(t)A\sin(2\pi f_c t)dt + \sin\theta_c \frac{1}{2}\int_0^T r(t)A\cos(2\pi f_c t)dt
\end{aligned}$$

Thus, the maximum likelihood estimate of the carrier phase is

$$\theta_{c,ML} = -\arctan\left[\frac{\int_0^T r(t)A\sin(2\pi f_c t)dt}{\int_0^T r(t)A\cos(2\pi f_c t)dt}\right]$$

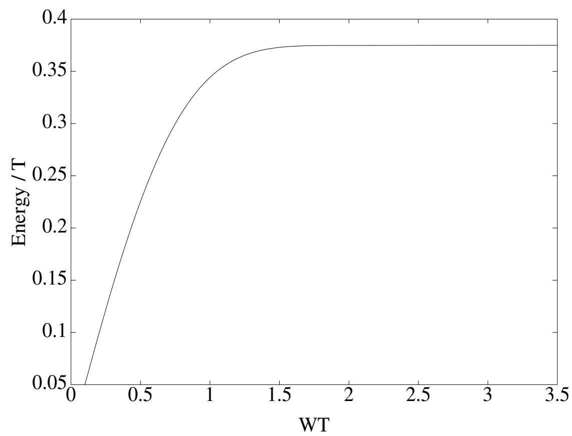
Chapter 8

Problem 8.1

1) The following table shows the values of $\mathcal{E}_h(W)/T$ obtained using an adaptive recursive Newton-Cotes numerical integration rule.

WT	0.5	1.0	1.5	2.0	2.5	3.0
$\mathcal{E}_h(W)/T$	0.2253	0.3442	0.3730	0.3748	0.3479	0.3750

A plot of $\mathcal{E}_h(W)/T$ as a function of WT is given in the next figure



2) The value of $\mathcal{E}_h(W)$ as $W \to \infty$ is

$$\begin{aligned}
\lim_{W\to\infty} \mathcal{E}_h(W) &= \int_{-\infty}^{\infty} g_T^2(t)dt = \int_0^T g_T^2(t)dt \\
&= \frac{1}{4}\int_0^T \left(1+\cos\frac{2\pi}{T}\left(t-\frac{T}{2}\right)\right)^2 dt \\
&= \frac{T}{4} + \frac{1}{2}\int_0^T \cos\frac{2\pi}{T}\left(t-\frac{T}{2}\right)dt \\
&\quad +\frac{1}{8}\int_0^T \left[1+\cos\frac{2\pi}{T}2\left(t-\frac{T}{2}\right)\right]dt \\
&= \frac{T}{4}+\frac{T}{8} = \frac{3T}{8} = 0.3750T
\end{aligned}$$

Problem 8.2

We have

$$y = \begin{cases} a+n-\frac{1}{2} & \text{with Prob. } \frac{1}{4} \\ a+n+\frac{1}{2} & \text{with Prob. } \frac{1}{4} \\ a+n & \text{with Prob. } \frac{1}{2} \end{cases}$$

By symmetry, $P_e = P(e|a=1) = P(e|a=-1)$, hence,

$$\begin{aligned}
P_e = P(e|a=-1) &= \frac{1}{2}P\left(n-1>0\right) + \frac{1}{4}P\left(n-\frac{3}{2}>0\right) + \frac{1}{4}P\left(n-\frac{1}{2}>0\right) \\
&= \frac{1}{2}Q\left(\frac{1}{\sigma_n}\right) + \frac{1}{4}Q\left(\frac{3}{2\sigma_n}\right) + \frac{1}{4}Q\left(\frac{1}{2\sigma_n}\right)
\end{aligned}$$

Problem 8.3

a) If the transmitted signal is

$$r(t) = \sum_{n=-\infty}^{\infty} a_n h(t - nT) + n(t)$$

then the output of the receiving filter is

$$y(t) = \sum_{n=-\infty}^{\infty} a_n x(t - nT) + \nu(t)$$

where $x(t) = h(t) \star h(t)$ and $\nu(t) = n(t) \star h(t)$. If the sampling time is off by 10%, then the samples at the output of the correlator are taken at $t = (m \pm \frac{1}{10})T$. Assuming that $t = (m - \frac{1}{10})T$ without loss of generality, then the sampled sequence is

$$y_m = \sum_{n=-\infty}^{\infty} a_n x((m - \frac{1}{10}T - nT) + \nu((m - \frac{1}{10})T)$$

If the signal pulse is rectangular with amplitude A and duration T, then $\sum_{n=-\infty}^{\infty} a_n x((m-\frac{1}{10}T-nT)$ is nonzero only for $n = m$ and $n = m - 1$ and therefore, the sampled sequence is given by

$$\begin{aligned} y_m &= a_m x(-\frac{1}{10}T) + a_{m-1} x(T - \frac{1}{10}T) + \nu((m - \frac{1}{10})T) \\ &= \frac{9}{10} a_m A^2 T + a_{m-1} \frac{1}{10} A^2 T + \nu((m - \frac{1}{10})T) \end{aligned}$$

The power spectral density of the noise at the output of the correlator is

$$\mathcal{S}_\nu(f) = \mathcal{S}_n(f)|H(f)|^2 = \frac{N_0}{2} A^2 T^2 \text{sinc}^2(fT)$$

Thus, the variance of the noise is

$$\sigma_n u^2 = \int_{-\infty}^{\infty} \frac{N_0}{2} A^2 T^2 \text{sinc}^2(fT) df = \frac{N_0}{2} A^2 T^2 \frac{1}{T} = \frac{N_0}{2} A^2 T$$

and therefore, the SNR is

$$\text{SNR} = \left(\frac{9}{10}\right)^2 \frac{2(A^2T)^2}{N_0 A^2 T} = \frac{81}{100} \frac{2A^2T}{N_0}$$

As it is observed, there is a loss of $10 \log_{10} \frac{81}{100} = -0.9151$ dB due to the mistiming.

b) Recall from part a) that the sampled sequence is

$$y_m = \frac{9}{10} a_m A^2 T + a_{m-1} \frac{1}{10} A^2 T + \nu_m$$

The term $a_{m-1} \frac{A^2 T}{10}$ expresses the ISI introduced to the system. If $a_m = 1$ is transmitted, then the probability of error is

$$\begin{aligned} P(e|a_m = 1) &= \frac{1}{2} P(e|a_m = 1, a_{m-1} = 1) + \frac{1}{2} P(e|a_m = 1, a_{m-1} = -1) \\ &= \frac{1}{2\sqrt{\pi N_0 A^2 T}} \int_{-\infty}^{-A^2 T} e^{-\frac{\nu^2}{N_0 A^2 T}} d\nu + \frac{1}{2\sqrt{\pi N_0 A^2 T}} \int_{-\infty}^{-\frac{8}{10} A^2 T} e^{-\frac{\nu^2}{N_0 A^2 T}} d\nu \\ &= \frac{1}{2} Q\left[\sqrt{\frac{2A^2T}{N_0}}\right] + \frac{1}{2} Q\left[\sqrt{\left(\frac{8}{10}\right)^2 \frac{2A^2T}{N_0}}\right] \end{aligned}$$

Since the symbols of the binary PAM system are equiprobable the previous derived expression is the probability of error when a symbol by symbol detector is employed. Comparing this with the probability of error of a system with no ISI, we observe that there is an increase of the probability of error by

$$P_{\text{diff}}(e) = \frac{1}{2}Q\left[\sqrt{\left(\frac{8}{10}\right)^2 \frac{2A^2T}{N_0}}\right] - \frac{1}{2}Q\left[\sqrt{\frac{2A^2T}{N_0}}\right]$$

Problem 8.4

1) The power spectral density of $X(t)$ is given by

$$\mathcal{S}_x(f) = \frac{1}{T}\mathcal{S}_a(f)|G_T(f)|^2$$

The Fourier transform of $g(t)$ is

$$G_T(f) = \mathcal{F}[g(t)] = AT\frac{\sin \pi f T}{\pi f T}e^{-j\pi f T}$$

Hence,

$$|G_T(f)|^2 = (AT)^2 \text{sinc}^2(fT)$$

and therefore,

$$\mathcal{S}_x(f) = A^2 T \mathcal{S}_a(f)\text{sinc}^2(fT) = A^2 T \text{sinc}^2(fT)$$

2) If $g_1(t)$ is used instead of $g(t)$ and the symbol interval is T, then

$$\begin{aligned} \mathcal{S}_x(f) &= \frac{1}{T}\mathcal{S}_a(f)|G_{2T}(f)|^2 \\ &= \frac{1}{T}(A2T)^2\text{sinc}^2(f2T) = 4A^2T\text{sinc}^2(f2T) \end{aligned}$$

3) If we precode the input sequence as $b_n = a_n + \alpha a_{n-3}$, then

$$R_b(m) = \begin{cases} 1+\alpha^2 & m = 0 \\ \alpha & m = \pm 3 \\ 0 & \text{otherwise} \end{cases}$$

and therefore, the power spectral density $\mathcal{S}_b(f)$ is

$$\mathcal{S}_b(f) = 1 + \alpha^2 + 2\alpha\cos(2\pi f 3T)$$

To obtain a null at $f = \frac{1}{3T}$, the parameter α should be such that

$$1 + \alpha^2 + 2\alpha\cos(2\pi f 3T)|_{f=\frac{1}{3}} = 0 \Longrightarrow \alpha = -1$$

4) The answer to this question is no. This is because $\mathcal{S}_b(f)$ is an analytic function and unless it is identical to zero it can have at most a countable number of zeros. This property of the analytic functions is also referred as the theorem of isolated zeros.

Problem 8.5

1) The power spectral density of $s(t)$ is

$$\mathcal{S}_s(f) = \frac{\sigma_a^2}{T}|G_T(f)|^2 = \frac{1}{T}|G_T(f)|^2$$

The Fourier transform $G_T(f)$ of the signal $g(t)$ is

$$\begin{aligned}
G_T(f) &= \mathcal{F}\left[\Pi\left(\frac{t-\frac{T}{4}}{\frac{T}{2}}\right) - \Pi\left(\frac{t-\frac{3T}{4}}{\frac{T}{2}}\right)\right] \\
&= \frac{T}{2}\text{sinc}(\frac{T}{2}f)e^{-j2\pi f\frac{T}{4}} - \frac{T}{2}\text{sinc}(\frac{T}{2}f)e^{-j2\pi f\frac{3T}{4}} \\
&= \frac{T}{2}\text{sinc}(\frac{T}{2}f)e^{-j2\pi f\frac{T}{2}}\left[e^{j2\pi f\frac{T}{4}} - e^{-j2\pi f\frac{T}{4}}\right] \\
&= \frac{T}{2}\text{sinc}(\frac{T}{2}f)\sin(2\pi f\frac{T}{4})2je^{-j2\pi f\frac{T}{2}}
\end{aligned}$$

Hence,

$$|G_T(f)|^2 = T^2\text{sinc}^2(\frac{T}{2}f)\sin^2(2\pi f\frac{T}{4})$$

and therefore,

$$\mathcal{S}_s(f) = T\text{sinc}^2(\frac{T}{2}f)\sin^2(2\pi f\frac{T}{4})$$

2) If the precoding scheme $b_n = a_n + ka_{n-1}$ is used, then

$$R_b(m) = \begin{cases} 1+k^2 & m=0 \\ k & m=\pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\mathcal{S}_b(f) = 1 + k^2 + 2k\cos(2\pi fT)$$

and therefore the spectrum of $s(t)$ is

$$\mathcal{S}_s(f) = (1 + k^2 + 2k\cos(2\pi fT))T\text{sinc}^2(\frac{T}{2}f)\sin^2(2\pi f\frac{T}{4})$$

In order to produce a frequency null at $f = \frac{1}{T}$ we have to choose k in such a way that

$$1 + k^2 + 2k\cos(2\pi fT)_{|f=1/T} = 1 + k^2 + 2k = 0$$

The appropriate value of k is -1.

3) If the precoding scheme of the previous part is used, then in order to have nulls at frequencies $f = \frac{n}{4T}$, the value of the parameter k should be such that

$$1 + k^2 + 2k\cos(2\pi fT)_{|f=1/4T} = 1 + k^2 = 0$$

As it is observed it is not possible to achieve the desired nulls with real values of k. Instead of the pre-coding scheme of the previous part we suggest pre-coding of the form

$$b_n = a_n + ka_{n-2}$$

In this case

$$R_b(m) = \begin{cases} 1+k^2 & m=0 \\ k & m=\pm 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\mathcal{S}_b(f) = 1 + k^2 + 2k\cos(2\pi 2fT)$$

and therefore $\mathcal{S}_b(\frac{n}{2T}) = 0$ for $k = 1$.

Problem 8.6

a) The power spectral density of the FSK signal may be evaluated by using equation (8.5.32) with $k = 2$ (binary) signals and probabilities $p_0 = p_1 = \frac{1}{2}$. Thus, when the condition that the carrier phase θ_0 and and θ_1 are fixed, we obtain

$$\mathcal{S}(f) = \frac{1}{4T_b^2}\sum_{n=-\infty}^{\infty}|S_0(\frac{n}{T_b}) + S_1(\frac{n}{T_b})|^2\delta(f-\frac{n}{T_b}) + \frac{1}{4T_b}|S_0(f) - S_1(f)|^2$$

where $S_0(f)$ and $S_1(f)$ are the Fourier transforms of $s_0(t)$ and $s_1(t)$. In particular,

$$\begin{aligned} S_0(f) &= \int_0^{T_b} s_0(t)e^{-j2\pi ft}\,dt \\ &= \sqrt{\frac{2\mathcal{E}_b}{T_b}}\int_0^{T_b}\cos(2\pi f_0 t + \theta_0)e^{j2\pi ft}\,dt, \qquad f_0 = f_c - \frac{\Delta f}{2} \\ &= \frac{1}{2}\sqrt{\frac{2\mathcal{E}_b}{T_b}}\left[\frac{\sin\pi T_b(f-f_0)}{\pi(f-f_0)} + \frac{\sin\pi T_b(f+f_0)}{\pi(f+f_0)}\right]e^{-j\pi fT_b}e^{j\theta_0} \end{aligned}$$

Similarly,

$$\begin{aligned} S_1(f) &= \int_0^{T_b} s_1(t)e^{-j2\pi ft}\,dt \\ &= \frac{1}{2}\sqrt{\frac{2\mathcal{E}_b}{T_b}}\left[\frac{\sin\pi T_b(f-f_1)}{\pi(f-f_1)} + \frac{\sin\pi T_b(f+f_1)}{\pi(f+f_1)}\right]e^{-j\pi fT_b}e^{j\theta_1} \end{aligned}$$

where $f_1 = f_c + \frac{\Delta f}{2}$. By expressing $\mathcal{S}(f)$ as

$$\begin{aligned} \mathcal{S}(f) &= \frac{1}{4T_b^2}\sum_{n=-\infty}^{\infty}\left[|S_0(\frac{n}{T_b})|^2 + |S_1(\frac{n}{T_b})|^2 + 2\text{Re}[S_0(\frac{n}{T_b})S_1^*(\frac{n}{T_b})]\delta(f-\frac{n}{T_b})\right] \\ &\quad + \frac{1}{4T_b}\left[|S_0(f)|^2 + |S_1(f)|^2 - 2\text{Re}[S_0(f)S_1^*(f)]\right] \end{aligned}$$

we note that the carrier phases θ_0 and θ_1 affect only the terms $\text{Re}(S_0S_1^*)$. If we average over the random phases, these terms drop out. Hence, we have

$$\begin{aligned} \mathcal{S}(f) &= \frac{1}{4T_b^2}\sum_{n=-\infty}^{\infty}\left[|S_0(\frac{n}{T_b})|^2 + |S_1(\frac{n}{T_b})|^2\right]\delta(f-\frac{n}{T_b})\Big] \\ &\quad + \frac{1}{4T_b}\left[|S_0(f)|^2 + |S_1(f)|^2\right] \end{aligned}$$

where

$$|\mathcal{S}_k(f)|^2 = \frac{T_b\mathcal{E}_b}{2}\left[\frac{\sin\pi T_b(f-f_k)}{\pi(f-f_k)} + \frac{\sin\pi T_b(f+f_k)}{\pi(f+f_k)}\right], \quad k = 0,1$$

Note that the first term in $\mathcal{S}(f)$ consists of a sequence of samples and the second term constitutes the continuous spectrum.

b) It is apparent from $\mathcal{S}(f)$ that the terms $|S_k(f)|^2$ decay proportionally as $\frac{1}{(f-f_k)^2}$. also note that

$$|S_k(f)|^2 = \frac{T_b\mathcal{E}_b}{2}\left[\left(\frac{\sin \pi T_b(f-f_k)}{\pi(f-f_k)}\right)^2 + \left(\frac{\sin \pi T_b(f+f_k)}{\pi(f+f_k)}\right)^2\right]$$

because the product

$$\frac{\sin \pi T_b(f-f_k)}{\pi(f-f_k)} \times \frac{\sin \pi T_b(f+f_k)}{\pi(f+f_k)} \approx 0$$

due to the relation that the carrier frequency $f_c \gg \frac{1}{T_b}$.

Problem 8.7

1) The autocorrelation function of the information symbols $\{a_n\}$ is

$$R_a(k) = E[a_n^* a + n + k] = \frac{1}{4} \times |a_n|^2 \delta(k) = \delta(k)$$

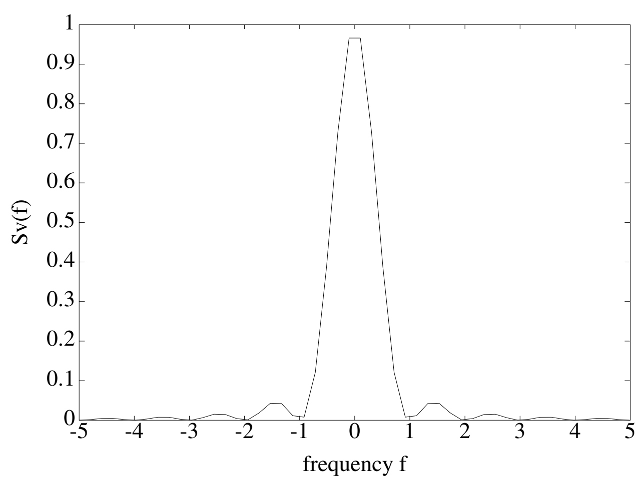
Thus, the power spectral density of $v(t)$ is

$$\mathcal{S}_V(f) = \frac{1}{T}\mathcal{S}_a(f)|G(f)|^2 = \frac{1}{T}|G(f)|^2$$

where $G(f) = \mathcal{F}[g(t)]$. If $g(t) = A\Pi(\frac{t-\frac{T}{2}}{T})$, we obtain $|G(f)|^2 = A^2T^2\text{sinc}^2(fT)$ and therefore,

$$\mathcal{S}_V(f) = A^2T\text{sinc}^2(fT)$$

In the next figure we plot $\mathcal{S}_V(f)$ for $T = A = 1$.



2) If $g(t) = A\sin(\frac{\pi t}{2})\Pi(\frac{t-T/2}{T})$, then

$$\begin{aligned}
G(f) &= A\left[\frac{1}{2j}\delta(f-\frac{1}{4}) - \frac{1}{2j}\delta(f+\frac{1}{4})\right] \star T\text{sinc}(fT)e^{-j2\pi f\frac{T}{2}} \\
&= \frac{AT}{2}[\delta(f-\frac{1}{4}) - \delta(f+\frac{1}{4})] \star \text{sinc}(fT)e^{-j(2\pi f\frac{T}{2}+\frac{\pi}{2})} \\
&= \frac{AT}{2}e^{-j\pi[(f-\frac{1}{4})T+\frac{1}{2}]}\left[\text{sinc}((f-\frac{1}{4})T) - \text{sinc}((f-\frac{1}{4})T)e^{-j\frac{\pi T}{2}}\right]
\end{aligned}$$

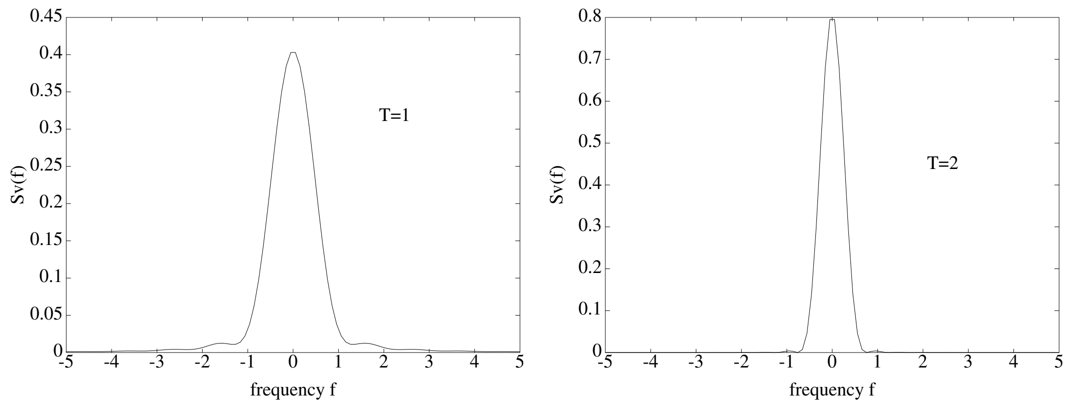
Thus,

$$\begin{aligned}
|G(f)|^2 &= \frac{A^2T^2}{4}\left[\text{sinc}^2((f+\frac{1}{4})T) + \text{sinc}^2((f-\frac{1}{4})T)\right. \\
&\quad \left. -2\text{sinc}((f+\frac{1}{4})T)\text{sinc}((f-\frac{1}{4})T)\cos\frac{\pi T}{2}\right]
\end{aligned}$$

and the power spectral of the transmitted signal is

$$\mathcal{S}_V(f) = \frac{A^2T}{4}\Big[\text{sinc}^2((f+\frac{1}{4})T) + \text{sinc}^2((f-\frac{1}{4})T)$$
$$-2\text{sinc}((f+\frac{1}{4})T)\text{sinc}((f-\frac{1}{4})T)\cos\frac{\pi T}{2}\Big]$$

In the next figure we plot $\mathcal{S}_V(f)$ for two special values of the time interval T. The amplitude of the signal A was set to 1 for both cases.



3) The first spectral null of the power spectrum density in part 1) is at position

$$W_{\text{null}} = \frac{1}{T}$$

The 3-dB bandwidth is specified by solving the equation:

$$S_V(W_{\text{3dB}}) = \frac{1}{2}S_V(0)$$

Since $\text{sinc}^2(0) = 1$, we obtain

$$\text{sinc}^2(W_{\text{3dB}}T) = \frac{1}{2} \Longrightarrow \sin(\pi W_{\text{3dB}}T) = \frac{1}{\sqrt{2}}\pi W_{\text{3dB}}T$$

Solving the latter equation numerically we find that

$$W_{\text{3dB}} = \frac{1.3916}{\pi T} = \frac{0.443}{T}$$

To find the first spectral null and the 3-dB bandwidth for the signal with power spectral density in part 2) we assume that $T = 1$. In this case

$$\mathcal{S}_V(f) = \frac{A^2}{4}\left[\text{sinc}^2((f+\frac{1}{4})) + \text{sinc}^2((f-\frac{1}{4}))\right]$$

and as it is observed there is no value of f that makes $\mathcal{S}_V(f)$ equal to zero. Thus, $W_{\text{null}} = \infty$. To find the 3-dB bandwidth note that

$$\mathcal{S}_V(0) = \frac{A^2}{4}2\text{sinc}(\frac{1}{4}) = \frac{A^2}{4}1.6212$$

Solving numerically the equation

$$S_V(W_{\text{3dB}}) = \frac{1}{2}\frac{A^2}{4}1.6212$$

we find that $W_{\text{3dB}} = 0.5412$. As it is observed the 3-dB bandwidth is more robust as a measure for the bandwidth of the signal.

Problem 8.8

The transition probability matrix $\mathbf{P}$ is

$$\mathbf{P} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Hence,

$$\mathbf{P}^2 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{P}^4 = \frac{1}{16}\begin{pmatrix} 2 & 4 & 4 & 6 \\ 4 & 2 & 6 & 4 \\ 4 & 6 & 2 & 4 \\ 6 & 4 & 4 & 2 \end{pmatrix}$$

and therefore,

$$\begin{aligned} \mathbf{P}^4\gamma &= \frac{1}{16}\begin{pmatrix} 2 & 4 & 4 & 6 \\ 4 & 2 & 6 & 4 \\ 4 & 6 & 2 & 4 \\ 6 & 4 & 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{16}\begin{pmatrix} -4 & 0 & 0 & 4 \\ 0 & -4 & 4 & 0 \\ 0 & 4 & -4 & 0 \\ 4 & 0 & 0 & -4 \end{pmatrix} = -\frac{1}{4}\gamma \end{aligned}$$

Thus, $\mathbf{P}^4\gamma = -\frac{1}{4}\gamma$ and by pre-multiplying both sides by $\mathbf{P}^k$, we obtain

$$\mathbf{P}^{k+4}\gamma = -\frac{1}{4}\mathbf{P}^k\gamma$$

Problem 8.9

a) Taking the inverse Fourier transform of $H(f)$, we obtain

$$h(t) = \mathcal{F}^{-1}[H(f)] = \delta(t) + \frac{\alpha}{2}\delta(t-t_0) + \frac{\alpha}{2}\delta(t+t_0)$$

Hence,

$$y(t) = s(t) \star h(t) = s(t) + \frac{\alpha}{2}s(t-t_0) + \frac{\alpha}{2}s(t+t_0)$$

b) If the signal $s(t)$ is used to modulate the sequence $\{a_n\}$, then the transmitted signal is

$$u(t) = \sum_{n=-\infty}^{\infty} a_n s(t-nT)$$

The received signal is the convolution of $u(t)$ with $h(t)$. Hence,

$$\begin{aligned} y(t) &= u(t) \star h(t) = \left(\sum_{n=-\infty}^{\infty} a_n s(t-nT)\right) \star \left(\delta(t) + \frac{\alpha}{2}\delta(t-t_0) + \frac{\alpha}{2}\delta(t+t_0)\right) \\ &= \sum_{n=-\infty}^{\infty} a_n s(t-nT) + \frac{\alpha}{2}\sum_{n=-\infty}^{\infty} a_n s(t-t_0-nT) + \frac{\alpha}{2}\sum_{n=-\infty}^{\infty} a_n s(t+t_0-nT) \end{aligned}$$

Thus, the output of the matched filter $s(-t)$ at the time instant t_1 is

$$\begin{aligned}
w(t_1) &= \sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{\infty} s(\tau - nT)s(\tau - t_1)d\tau \\
&\quad + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{\infty} s(\tau - t_0 - nT)s(\tau - t_1)d\tau \\
&\quad + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{\infty} s(\tau + t_0 - nT)s(\tau - t_1)d\tau
\end{aligned}$$

If we denote the signal $s(t) \star s(t)$ by $x(t)$, then the output of the matched filter at $t_1 = kT$ is

$$\begin{aligned}
w(kT) &= \sum_{n=-\infty}^{\infty} a_n x(kT - nT) \\
&\quad + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} a_n x(kT - t_0 - nT) + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} a_n x(kT + t_0 - nT)
\end{aligned}$$

c) With $t_0 = T$ and $k = n$ in the previous equation, we obtain

$$\begin{aligned}
w_k &= a_k x_0 + \sum_{n \neq k} a_n x_{k-n} \\
&\quad + \frac{\alpha}{2} a_k x_{-1} + \frac{\alpha}{2} \sum_{n \neq k} a_n x_{k-n-1} + \frac{\alpha}{2} a_k x_1 + \frac{\alpha}{2} \sum_{n \neq k} a_n x_{k-n+1} \\
&= a_k \left(x_0 + \frac{\alpha}{2} x_{-1} + \frac{\alpha}{2} x_1 \right) + \sum_{n \neq k} a_n \left[x_{k-n} + \frac{\alpha}{2} x_{k-n-1} + \frac{\alpha}{2} x_{k-n+1} \right]
\end{aligned}$$

The terms under the summation is the ISI introduced by the channel.

Problem 8.10

a) Each segment of the wire-line can be considered as a bandpass filter with bandwidth $W = 1200$ Hz. Thus, the highest bit rate that can be transmitted without ISI by means of binary PAM is

$$R = 2W = 2400 \text{ bps}$$

b) The probability of error for binary PAM transmission is

$$P_2 = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

Hence, using mathematical tables for the function $Q[\cdot]$, we find that $P_2 = 10^{-7}$ is obtained for

$$\sqrt{\frac{2\mathcal{E}_b}{N_0}} = 5.2 \Longrightarrow \frac{\mathcal{E}_b}{N_0} = 13.52 = 11.30 \text{ dB}$$

c) The received power P_R is related to the desired SNR per bit through the relation

$$\frac{P_R}{N_0} = R\frac{\mathcal{E}_b}{N_0}$$

Hence, with $N_0 = 4.1 \times 10^{-21}$ we obtain

$$P_R = 4.1 \times 10^{-21} \times 1200 \times 13.52 = 6.6518 \times 10^{-17} = -161.77 \text{ dBW}$$

Since the power loss of each segment is

$$L_s = 50 \text{ Km } \times 1 \text{ dB/Km } = 50 \text{ dB}$$

the transmitted power at each repeater should be

$$P_T = P_R + L_s = -161.77 + 50 = -111.77 \text{ dBW}$$

Problem 8.11

The pulse $x(t)$ having the raised cosine spectrum is

$$x(t) = \text{sinc}(t/T)\frac{\cos(\pi\alpha t/T)}{1 - 4\alpha^2 t^2/T^2}$$

The function $\text{sinc}(t/T)$ is 1 when $t = 0$ and 0 when $t = nT$. On the other hand

$$g(t) = \frac{\cos(\pi\alpha t/T)}{1 - 4\alpha^2 t^2/T^2} = \begin{cases} 1 & t = 0 \\ \text{bounded} & t \neq 0 \end{cases}$$

The function $g(t)$ needs to be checked only for those values of t such that $4\alpha^2 t^2/T^2 = 1$ or $\alpha t = \frac{T}{2}$. However,

$$\lim_{\alpha t \to \frac{T}{2}} \frac{\cos(\pi\alpha t/T)}{1 - 4\alpha^2 t^2/T^2} = \lim_{x \to 1} \frac{\cos(\frac{\pi}{2}x)}{1 - x}$$

and by using L'Hospital's rule

$$\lim_{x \to 1} \frac{\cos(\frac{\pi}{2}x)}{1 - x} = \lim_{x \to 1} \frac{\pi}{2}\sin(\frac{\pi}{2}x) = \frac{\pi}{2} < \infty$$

Hence,

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

meaning that the pulse $x(t)$ satisfies the Nyquist criterion.

Problem 8.12

Substituting the expression of $X_{rc}(f)$ in the desired integral, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} X_{rc}(f)df &= \int_{-\frac{1+\alpha}{2T}}^{-\frac{1-\alpha}{2T}} \frac{T}{2}\left[1 + \cos\frac{\pi T}{\alpha}(-f - \frac{1-\alpha}{2T})\right]df + \int_{-\frac{1-\alpha}{2T}}^{\frac{1-\alpha}{2T}} T df \\
&\quad + \int_{\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}} \frac{T}{2}\left[1 + \cos\frac{\pi T}{\alpha}(f - \frac{1-\alpha}{2T})\right]df \\
&= \int_{-\frac{1+\alpha}{2T}}^{-\frac{1-\alpha}{2T}} \frac{T}{2}df + T\left(\frac{1-\alpha}{T}\right) + \int_{\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}} \frac{T}{2}df \\
&\quad + \int_{-\frac{1+\alpha}{2T}}^{-\frac{1-\alpha}{2T}} \cos\frac{\pi T}{\alpha}(f + \frac{1-\alpha}{2T})df + \int_{\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}} \cos\frac{\pi T}{\alpha}(f - \frac{1-\alpha}{2T})df \\
&= 1 + \int_{-\frac{\alpha}{T}}^{0} \cos\frac{\pi T}{\alpha}x dx + \int_{0}^{\frac{\alpha}{T}} \cos\frac{\pi T}{\alpha}x dx \\
&= 1 + \int_{-\frac{\alpha}{T}}^{\frac{\alpha}{T}} \cos\frac{\pi T}{\alpha}x dx = 1 + 0 = 1
\end{aligned}$$

Problem 8.13

Let $X(f)$ be such that

$$\mathrm{Re}[X(f)] = \begin{cases} T\Pi(fT) + U(f) & |f| < \frac{1}{T} \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{Im}[X(f)] = \begin{cases} V(f) & |f| < \frac{1}{T} \\ 0 & \text{otherwise} \end{cases}$$

with $U(f)$ even with respect to 0 and odd with respect to $f = \frac{1}{2T}$ Since $x(t)$ is real, $V(f)$ is odd with respect to 0 and by assumption it is even with respect to $f = \frac{1}{2T}$. Then,

$$\begin{aligned}
x(t) &= \mathcal{F}^{-1}[X(f)] \\
&= \int_{-\frac{1}{T}}^{\frac{1}{2T}} X(f)e^{j2\pi ft}df + \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X(f)e^{j2\pi ft}df + \int_{\frac{1}{2T}}^{\frac{1}{T}} X(f)e^{j2\pi ft}df \\
&= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} Te^{j2\pi ft}df + \int_{-\frac{1}{T}}^{\frac{1}{T}} [U(f) + jV(f)]e^{j2\pi ft}df \\
&= \mathrm{sinc}(t/T) + \int_{-\frac{1}{T}}^{\frac{1}{T}} [U(f) + jV(f)]e^{j2\pi ft}df
\end{aligned}$$

Consider first the integral $\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f)e^{j2\pi ft}df$. Clearly,

$$\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f)e^{j2\pi ft}df = \int_{-\frac{1}{T}}^{0} U(f)e^{j2\pi ft}df + \int_{0}^{\frac{1}{T}} U(f)e^{j2\pi ft}df$$

and by using the change of variables $f' = f + \frac{1}{2T}$ and $f' = f - \frac{1}{2T}$ for the two integrals on the right hand side respectively, we obtain

$$\begin{aligned}
&\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f)e^{j2\pi ft}df \\
&= e^{-j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' - \frac{1}{2T})e^{j2\pi f't}df' + e^{j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T})e^{j2\pi f't}df' \\
&\stackrel{a}{=} \left(e^{j\frac{\pi}{T}t} - e^{-j\frac{\pi}{T}t}\right) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T})e^{j2\pi f't}df' \\
&= 2j\sin(\frac{\pi}{T}t) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T})e^{j2\pi f't}df'
\end{aligned}$$

where for step (a) we used the odd symmetry of $U(f')$ with respect to $f' = \frac{1}{2T}$, that is

$$U(f' - \frac{1}{2T}) = -U(f' + \frac{1}{2T})$$

For the integral $\int_{-\frac{1}{T}}^{\frac{1}{T}} V(f)e^{j2\pi ft}df$ we have

$$\begin{aligned}
&\int_{-\frac{1}{T}}^{\frac{1}{T}} V(f)e^{j2\pi ft}df \\
&= \int_{-\frac{1}{T}}^{0} V(f)e^{j2\pi ft}df + \int_{0}^{\frac{1}{T}} V(f)e^{j2\pi ft}df \\
&= e^{-j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' - \frac{1}{2T})e^{j2\pi f't}df' + e^{j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' + \frac{1}{2T})e^{j2\pi f't}df'
\end{aligned}$$

However, $V(f)$ is odd with respect to 0 and since $V(f'+\frac{1}{2T})$ and $V(f'-\frac{1}{2T})$ are even, the translated spectra satisfy

$$\int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f'-\frac{1}{2T})e^{j2\pi f't}df' = -\int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f'+\frac{1}{2T})e^{j2\pi f't}df'$$

Hence,

$$\begin{aligned} x(t) &= \text{sinc}(t/T) + 2j\sin(\frac{\pi}{T}t)\int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f'+\frac{1}{2T})e^{j2\pi f't}df' \\ & \quad -2\sin(\frac{\pi}{T}t)\int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f'+\frac{1}{2T})e^{j2\pi f't}df' \end{aligned}$$

and therefore,

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Thus, the signal $x(t)$ satisfies the Nyquist criterion.

Problem 8.14

The bandwidth of the channel is

$$W = 3000 - 300 = 2700 \text{ Hz}$$

Since the minimum transmission bandwidth required for bandpass signaling is R, where R is the rate of transmission, we conclude that the maximum value of the symbol rate for the given channel is $R_{\max} = 2700$. If an M-ary PAM modulation is used for transmission, then in order to achieve a bit-rate of 9600 bps, with maximum rate of $R_{\max}$, the minimum size of the constellation is $M = 2^k = 16$. In this case, the symbol rate is

$$R = \frac{9600}{k} = 2400 \text{ symbols/sec}$$

and the symbol interval $T = \frac{1}{R} = \frac{1}{2400}$ sec. The roll-off factor α of the raised cosine pulse used for transmission is is determined by noting that $1200(1+\alpha) = 1350$, and hence, $\alpha = 0.125$. Therefore, the squared root raised cosine pulse can have a roll-off of $\alpha = 0.125$.

Problem 8.15

Since the bandwidth of the ideal lowpass channel is $W = 2400$ Hz, the rate of transmission is

$$R = 2 \times 2400 = 4800 \text{ symbols/sec}$$

The number of bits per symbol is

$$k = \frac{14400}{4800} = 3$$

Hence, the number of transmitted symbols is $2^3 = 8$. If a duobinary pulse is used for transmission, then the number of possible transmitted symbols is $2M - 1 = 15$. These symbols have the form

$$b_n = 0, \pm 2d, \pm 4d, \ldots, \pm 12d$$

where $2d$ is the minimum distance between the points of the 8-PAM constellation. The probability mass function of the received symbols is

$$P(b = 2md) = \frac{8 - |m|}{64}, \qquad m = 0, \pm 1, \ldots, \pm 7$$

An upper bound of the probability of error is given by (see (8.4.33))

$$P_M < 2\left(1 - \frac{1}{M^2}\right) Q\left[\sqrt{\left(\frac{\pi}{4}\right)^2 \frac{6}{M^2-1}\frac{k\mathcal{E}_{b,av}}{N_0}}\right]$$

With $P_M = 10^{-6}$ and $M = 8$ we obtain

$$\frac{k\mathcal{E}_{b,av}}{N_0} = 1.3193 \times 10^3 \Longrightarrow \mathcal{E}_{b,av} = 0.088$$

Problem 8.16

a) The spectrum of the baseband signal is

$$\mathcal{S}_V(f) = \frac{1}{T}\mathcal{S}_a(f)|X_{rc}(f)|^2 = \frac{1}{T}|X_{rc}(f)|^2$$

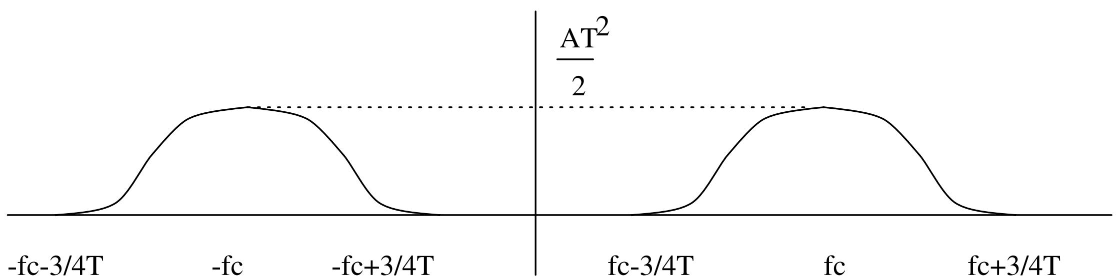
where $T = \frac{1}{2400}$ and

$$X_{rc}(f) = \begin{cases} T & 0 \leq |f| \leq \frac{1}{4T} \\ \frac{T}{2}(1 + \cos(2\pi T(|f| - \frac{1}{4T})) & \frac{1}{4T} \leq |f| \leq \frac{3}{4T} \\ 0 & \text{otherwise} \end{cases}$$

If the carrier signal has the form $c(t) = A\cos(2\pi f_c t)$, then the spectrum of the DSB-SC modulated signal, $\mathcal{S}_U(f)$, is

$$\mathcal{S}_U(f) = \frac{A}{2}[\mathcal{S}_V(f - f_c) + \mathcal{S}_V(f + f_c)]$$

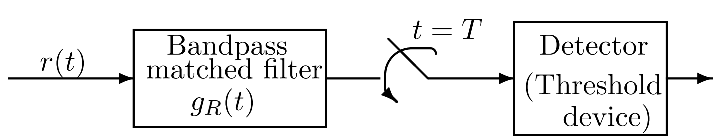
A sketch of $\mathcal{S}_U(f)$ is shown in the next figure.



b) Assuming bandpass coherent demodulation using a matched filter, the received signal $r(t)$ is first passed through a linear filter with impulse response

$$g_R(t) = Ax_{rc}(T - t)\cos(2\pi f_c(T - t))$$

The output of the matched filter is sampled at $t = T$ and the samples are passed to the detector. The detector is a simple threshold device that decides if a binary 1 or 0 was transmitted depending on the sign of the input samples. The following figure shows a block diagram of the optimum bandpass coherent demodulator.



Problem 8.17

a) If the power spectral density of the additive noise is $\mathcal{S}_n(f)$, then the PSD of the noise at the output of the prewhitening filter is

$$\mathcal{S}_\nu(f) = \mathcal{S}_n(f)|H_p(f)|^2$$

In order for $\mathcal{S}_\nu(f)$ to be flat (white noise), $H_p(f)$ should be such that

$$H_p(f) = \frac{1}{\sqrt{\mathcal{S}_n(f)}}$$

2) Let $h_p(t)$ be the impulse response of the prewhitening filter $H_p(f)$. That is, $h_p(t) = \mathcal{F}^{-1}[H_p(f)]$. Then, the input to the matched filter is the signal $\tilde{s}(t) = s(t) \star h_p(t)$. The frequency response of the filter matched to $\tilde{s}(t)$ is

$$\tilde{S}_m(f) = \tilde{S}^*(f)e^{-j2\pi ft_0} == S^*(f)H_p^*(f)e^{-j2\pi ft_0}$$

where t_0 is some nominal time-delay at which we sample the filter output.

3) The frequency response of the overall system, prewhitening filter followed by the matched filter, is

$$G(f) = \tilde{S}_m(f)H_p(f) = S^*(f)|H_p(f)|^2e^{-j2\pi ft_0} = \frac{S^*(f)}{\mathcal{S}_n(f)}e^{-j2\pi ft_0}$$

4) The variance of the noise at the output of the generalized matched filter is

$$\sigma^2 = \int_{-\infty}^{\infty}\mathcal{S}_n(f)|G(f)|^2df = \int_{-\infty}^{\infty}\frac{|S(f)|^2}{\mathcal{S}_n(f)}df$$

At the sampling instant $t = t_0 = T$, the signal component at the output of the matched filter is

$$\begin{aligned} y(T) &= \int_{-\infty}^{\infty}Y(f)e^{j2\pi fT}df = \int_{-\infty}^{\infty}s(\tau)g(T-\tau)d\tau \\ &= \int_{-\infty}^{\infty}S(f)\frac{S^*(f)}{\mathcal{S}_n(f)}df = \int_{-\infty}^{\infty}\frac{|S(f)|^2}{\mathcal{S}_n(f)}df \end{aligned}$$

Hence, the output SNR is

$$\text{SNR} = \frac{y^2(T)}{\sigma^2} = \int_{-\infty}^{\infty}\frac{|S(f)|^2}{\mathcal{S}_n(f)}df$$

Problem 8.18

The bandwidth of the bandpass channel is

$$W = 3300 - 300 = 3000 \text{ Hz}$$

In order to transmit 9600 bps with a symbol rate $R = 2400$ symbols per second, the number of information bits per symbol should be

$$k = \frac{9600}{2400} = 4$$

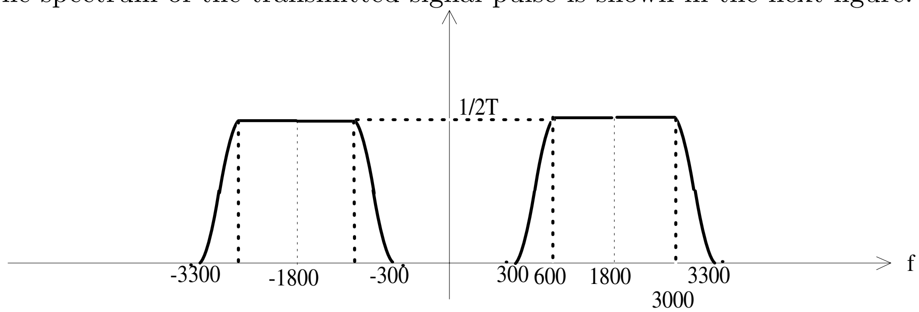
Hence, a $2^4 = 16$ QAM signal constellation is needed. The carrier frequency f_c is set to 1800 Hz, which is the mid-frequency of the frequency band that the bandpass channel occupies. If a pulse with raised cosine spectrum and roll-off factor α is used for spectral shaping, then for the bandpass signal with bandwidth W

$$R = 1200(1+\alpha) = 1500$$

and

$$\alpha = 0.25$$

A sketch of the spectrum of the transmitted signal pulse is shown in the next figure.



Problem 8.19

The channel bandwidth is $W = 4000$ Hz.

(a) Binary PSK with a pulse shape that has $\alpha = \frac{1}{2}$. Hence

$$\frac{1}{2T}(1+\alpha) = 2000$$

and $\frac{1}{T} = 2667$, the bit rate is 2667 bps.

(b) Four-phase PSK with a pulse shape that has $\alpha = \frac{1}{2}$. From (a) the symbol rate is $\frac{1}{T} = 2667$ and the bit rate is 5334 bps.

(c) $M = 8$ QAM with a pulse shape that has $\alpha = \frac{1}{2}$. From (a), the symbol rate is $\frac{1}{T} = 2667$ and hence the bit rate $\frac{3}{T} = 8001$ bps.

(d) Binary FSK with noncoherent detection. Assuming that the frequency separation between the two frequencies is $\Delta f = \frac{1}{T}$, where $\frac{1}{T}$ is the bit rate, the two frequencies are $f_c + \frac{1}{2T}$ and $f_c - \frac{1}{2T}$. Since $W = 4000$ Hz, we may select $\frac{1}{2T} = 1000$, or, equivalently, $\frac{1}{T} = 2000$. Hence, the bit rate is 2000 bps, and the two FSK signals are orthogonal.

(e) Four FSK with noncoherent detection. In this case we need four frequencies with separation of $\frac{1}{T}$ between adjacent frequencies. We select $f_1 = f_c - \frac{1.5}{T}$, $f_2 = f_c - \frac{1}{2T}$, $f_3 = f_c + \frac{1}{2T}$, and $f_4 = f_c + \frac{1.5}{T}$, where $\frac{1}{2T} = 500$ Hz. Hence, the symbol rate is $\frac{1}{T} = 1000$ symbols per second and since each symbol carries two bits of information, the bit rate is 2000 bps.

(f) $M = 8$ FSK with noncoherent detection. In this case we require eight frequencies with frequency separation of $\frac{1}{T} = 500$ Hz for orthogonality. Since each symbol carries 3 bits of information, the bit rate is 1500 bps.

Problem 8.20

1) The bandwidth of the bandpass channel is

$$W = 3000 - 600 = 2400 \text{ Hz}$$

Since each symbol of the QPSK constellation conveys 2 bits of information, the symbol rate of transmission is

$$R = \frac{2400}{2} = 1200 \text{ symbols/sec}$$

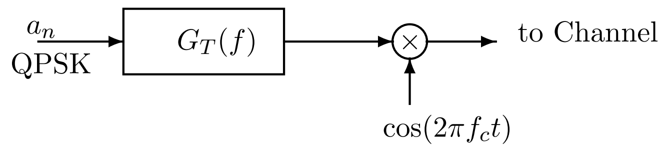
Thus, for spectral shaping we can use a signal pulse with a raised cosine spectrum and roll-off factor $\alpha = 1$, that is

$$X_{rc}(f) = \frac{T}{2}[1 + \cos(\pi T|f|)] = \frac{1}{2400}\cos^2\left(\frac{\pi|f|}{2400}\right)$$

If the desired spectral characteristic is split evenly between the transmitting filter $G_T(f)$ and the receiving filter $G_R(f)$, then

$$G_T(f) = G_R(f) = \sqrt{\frac{1}{1200}} \cos\left(\frac{\pi|f|}{2400}\right), \qquad |f| < \frac{1}{T} = 1200$$

A block diagram of the transmitter is shown in the next figure.



2) If the bit rate is 4800 bps, then the symbol rate is

$$R = \frac{4800}{2} = 2400 \text{ symbols/sec}$$

In order to satisfy the Nyquist criterion, the the signal pulse used for spectral shaping, should have the spectrum

$$X(f) = T\Pi\left(\frac{f}{W}\right)$$

Thus, the frequency response of the transmitting filter is $G_T(f) = \sqrt{T}\Pi\left(\frac{f}{W}\right)$.

Problem 8.21

The bandwidth of the bandpass channel is $W = 4$ KHz. Hence, the rate of transmission should be less or equal to 4000 symbols/sec. If a 8-QAM constellation is employed, then the required symbol rate is $R = 9600/3 = 3200$. If a signal pulse with raised cosine spectrum is used for shaping, the maximum allowable roll-off factor is determined by

$$1600(1+\alpha) = 2000$$

which yields $\alpha = 0.25$. Since α is less than 50%, we consider a larger constellation. With a 16-QAM constellation we obtain

$$R = \frac{9600}{4} = 2400$$

and

$$1200(1+\alpha) = 2000$$

0r $\alpha = 2/3$, which satisfies the required conditions. The probability of error for an M-QAM constellation is given by

$$P_M = 1 - (1 - P_{\sqrt{M}})^2$$

where

$$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left[\sqrt{\frac{3\mathcal{E}_{av}}{(M-1)N_0}}\right]$$

With $P_M = 10^{-6}$ we obtain $P_{\sqrt{M}} = 5 \times 10^{-7}$ and therefore

$$2 \times (1 - \frac{1}{4}) Q\left[\sqrt{\frac{3\mathcal{E}_{av}}{15 \times 2 \times 10^{-10}}}\right] = 5 \times 10^{-7}$$

Using the last equation and the tabulation of the $Q[\cdot]$ function, we find that the average transmitted energy is

$$\mathcal{E}_{av} = 24.70 \times 10^{-9}$$

Note that if the desired spectral characteristic $X_{rc}(f)$ is split evenly between the transmitting and receiving filter, then the energy of the transmitting pulse is

$$\int_{-\infty}^{\infty} g_T^2(t)dt = \int_{-\infty}^{\infty} |G_T(f)|^2 df = \int_{-\infty}^{\infty} X_{rc}(f)df = 1$$

Hence, the energy $\mathcal{E}_{av} = P_{av}T$ depends only on the amplitude of the transmitted points and the symbol interval T. Since $T = \frac{1}{2400}$, the average transmitted power is

$$P_{av} = \frac{\mathcal{E}_{av}}{T} = 24.70 \times 10^{-9} \times 2400 = 592.8 \times 10^{-7}$$

If the points of the 16-QAM constellation are evenly spaced with minimum distance between them equal to d, then there are four points with coordinates $(\pm\frac{d}{2}, \pm\frac{d}{2})$, four points with coordinates $(\pm\frac{3d}{2}, \pm\frac{3d}{2})$, four points with coordinates $(\pm\frac{3d}{2}, \pm\frac{d}{2})$, and four points with coordinates $(\pm\frac{d}{2}, \pm\frac{3d}{2})$. Thus, the average transmitted power is

$$P_{av} = \frac{1}{2 \times 16}\sum_{i=1}^{16}(A_{mc}^2 + A_{ms}^2) = \frac{1}{2}\left[4 \times \frac{d^2}{2} + 4 \times \frac{9d^2}{2} + 8 \times \frac{10d^2}{4}\right] = 20d^2$$

Since $P_{av} = 592.8 \times 10^{-7}$, we obtain

$$d = \sqrt{\frac{P_{av}}{20}} = 0.00172$$

Problem 8.22

The roll-off factor α is related to the bandwidth by the expression $\frac{1+\alpha}{T} = 2W$, or equivalently $R(1+\alpha) = 2W$. The following table shows the symbol rate for the various values of the excess bandwidth and for $W = 1500$ Hz.

α	.25	.33	.50	.67	.75	1.00
R	2400	2256	2000	1796	1714	1500

Problem 8.23

The following table shows the precoded sequence, the transmitted amplitude levels, the received signal levels and the decoded sequence, when the data sequence 10010110010 modulates a duobinary transmitting filter.

Data seq. d_n:		1	0	0	1	0	1	1	0	0	1	0
Precoded seq. p_n:	0	1	1	1	0	0	1	0	0	0	1	1
Transmitted seq. a_n:	-1	1	1	1	-1	-1	1	-1	-1	-1	1	1
Received seq. b_n:		0	2	2	0	-2	0	0	-2	-2	0	2
Decoded seq. d_n:		1	0	0	1	0	1	1	0	0	1	0

Problem 8.24

The following table shows the precoded sequence, the transmitted amplitude levels, the received signal levels and the decoded sequence, when the data sequence 10010110010 modulates a modified duobinary transmitting filter.

Data seq. d_n:			1	0	0	1	0	1	1	0	0	1	0
Precoded seq. p_n:	0	0	1	0	1	1	1	0	0	0	0	1	0
Transmitted seq. a_n:	-1	-1	1	-1	1	1	1	-1	-1	-1	-1	1	-1
Received seq. b_n:			2	0	0	2	0	-2	-2	0	0	2	0
Decoded seq. d_n:			1	0	0	1	0	1	1	0	0	1	0

Problem 8.25

Let $X(z)$ denote the $\mathcal{Z}$-transform of the sequence x_n, that is

$$X(z) = \sum_n x_n z^{-n}$$

Then the precoding operation can be described as

$$P(z) = \frac{D(z)}{X(z)} \quad \text{mod} - M$$

where $D(z)$ and $P(z)$ are the $\mathcal{Z}$-transforms of the data and precoded sequences respectively. For example, if $M = 2$ and $X(z) = 1 + z^{-1}$ (duobinary signaling), then

$$P(z) = \frac{D(z)}{1 + z^{-1}} \Longrightarrow P(z) = D(z) - z^{-1}P(z)$$

which in the time domain is written as

$$p_n = d_n - p_{n-1}$$

and the subtraction is mod-2.

However, the inverse filter $\frac{1}{X(z)}$ exists only if x_0, the first coefficient of $X(z)$ is relatively prime with M. If this is not the case, then the precoded symbols p_n cannot be determined uniquely from the data sequence d_n.

Problem 8.26

In the case of duobinary signaling, the output of the matched filter is

$$x(t) = \text{sinc}(2Wt) + \text{sinc}(2Wt - 1)$$

and the samples x_{n-m} are given by

$$x_{n-m} = x(nT - mT) = \begin{cases} 1 & n - m = 0 \\ 1 & n - m = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the metric $\mu(\mathbf{a})$ in the Viterbi algorithm becomes

$$\begin{aligned} \mu(\mathbf{a}) &= 2\sum_n a_n r_n - \sum_n \sum_m a_n a_m x_{n-m} \\ &= 2\sum_n a_n r_n - \sum_n a_n^2 - \sum_n a_n a_{n-1} \\ &= \sum_n a_n(2r_n - a_n - a_{n-1}) \end{aligned}$$

Problem 8.27

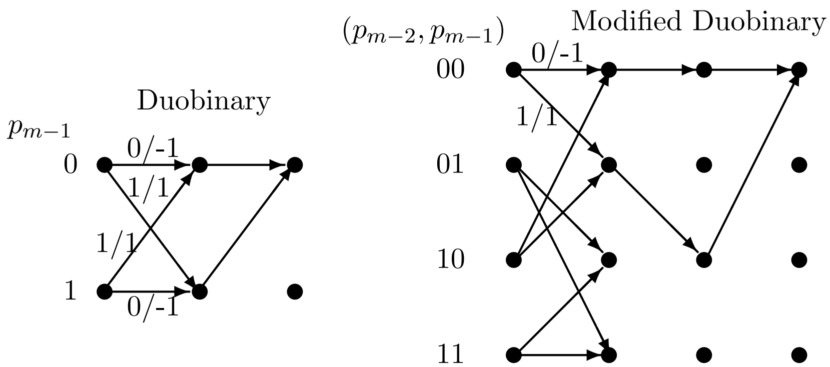
The precoding for the duobinary signaling is given by

$$p_m = d_m \ominus p_{m-1}$$

The corresponding trellis has two states associated with the binary values of the history p_{m-1}. For the modified duobinary signaling the precoding is

$$p_m = d_m \oplus p_{m-2}$$

Hence, the corresponding trellis has four states depending on the values of the pair (p_{m-2}, p_{m-1}). The two trellises are depicted in the next figure. The branches have been labelled as x/y, where x is the binary input data d_m and y is the actual transmitted symbol. Note that the trellis for the modified duobinary signal has more states, but the minimum free distance between the paths is $d_{\text{free}} = 3$, whereas the minimum free distance between paths for the duobinary signal is 2.



Problem 8.28

1) The output of the matched filter demodulator is

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} g_T(\tau - kT_b) g_R(t-\tau) d\tau + \nu(t) \\ &= \sum_{k=-\infty}^{\infty} a_k x(t - kT_b) + \nu(t) \end{aligned}$$

where,

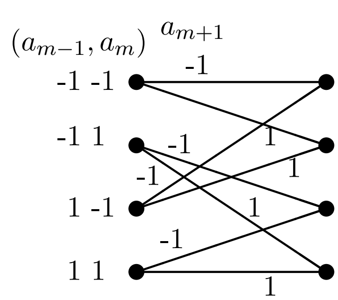
$$x(t) = g_T(t) \star g_R(t) = \frac{\sin \frac{\pi t}{T}}{\frac{\pi t}{T}} \frac{\cos \frac{\pi t}{T}}{1 - 4\frac{t^2}{T^2}}$$

Hence,

$$\begin{aligned} y(mT_b) &= \sum_{k=-\infty}^{\infty} a_k x(mT_b - kT_b) + v(mT_b) \\ &= a_m + \frac{1}{\pi} a_{m-1} + \frac{1}{\pi} a_{m+1} + \nu(mT_b) \end{aligned}$$

The term $\frac{1}{\pi}a_{m-1} + \frac{1}{\pi}a_{m+1}$ represents the ISI introduced by doubling the symbol rate of transmission.

2) In the next figure we show one trellis stage for the ML sequence detector. Since there is postcursor ISI, we delay the received signal, used by the ML decoder to form the metrics, by one sample. Thus, the states of the trellis correspond to the sequence (a_{m-1}, a_m), and the transition labels correspond to the symbol a_{m+1}. Two branches originate from each state. The upper branch is associated with the transmission of -1, whereas the lower branch is associated with the transmission of 1.



Problem 8.29

a) The output of the matched filter at the time instant mT is

$$y_m = \sum_k a_m x_{k-m} + \nu_m = a_m + \frac{1}{4}a_{m-1} + \nu_m$$

The autocorrelation function of the noise samples ν_m is

$$E[\nu_k \nu_j] = \frac{N_0}{2} x_{k-j}$$

Thus, the variance of the noise is

$$\sigma_\nu^2 = \frac{N_0}{2} x_0 = \frac{N_0}{2}$$

If a symbol by symbol detector is employed and we assume that the symbols $a_m = a_{m-1} = \sqrt{\mathcal{E}_b}$ have been transmitted, then the probability of error $P(e|a_m = a_{m-1} = \sqrt{\mathcal{E}_b})$ is

$$\begin{aligned}
P(e|a_m = a_{m-1} = \sqrt{\mathcal{E}_b}) &= P(y_m < 0|a_m = a_{m-1} = \sqrt{\mathcal{E}_b}) \\
&= P(\nu_m < -\frac{5}{4}\sqrt{\mathcal{E}_b}) = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{-\frac{5}{4}\sqrt{\mathcal{E}_b}} e^{-\frac{\nu_m^2}{N_0}} d\nu_m \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{5}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}} e^{-\frac{\nu^2}{2}} d\nu = Q\left[\frac{5}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]
\end{aligned}$$

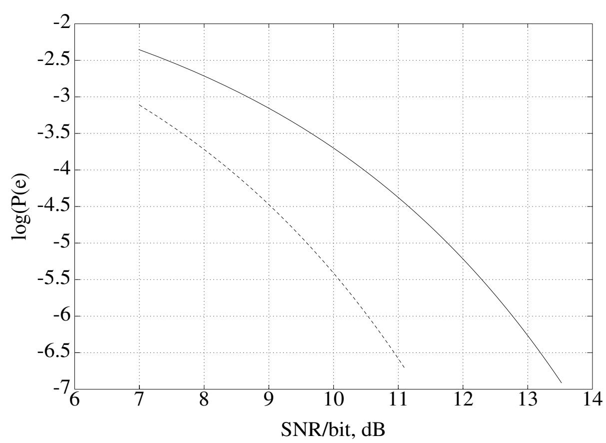
If however $a_{m-1} = -\sqrt{\mathcal{E}_b}$, then

$$P(e|a_m = \sqrt{\mathcal{E}_b}, a_{m-1} = -\sqrt{\mathcal{E}_b}) = P(\frac{3}{4}\sqrt{\mathcal{E}_b} + \nu_m < 0) = Q\left[\frac{3}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

Since the two symbols $\sqrt{\mathcal{E}_b}$, $-\sqrt{\mathcal{E}_b}$ are used with equal probability, we conclude that

$$\begin{aligned}
P(e) &= P(e|a_m = \sqrt{\mathcal{E}_b}) = P(e|a_m = -\sqrt{\mathcal{E}_b}) \\
&= \frac{1}{2}Q\left[\frac{5}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{2}Q\left[\frac{3}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]
\end{aligned}$$

b) In the next figure we plot the error probability obtained in part (a) ($\log_{10}(P(e))$) vs. the SNR per bit and the error probability for the case of no ISI. As it observed from the figure, the relative difference in SNR of the error probability of 10^{-6} is 2 dB.



Problem 8.30

The power spectral density of the noise at the output of the matched filter is

$$\mathcal{S}_\nu(f) = \mathcal{S}_n(f)|G_R(f)|^2 = \frac{N_0}{2}|X(f)| = \frac{N_0}{2}\frac{1}{W}\cos(\frac{\pi f}{2W})$$

Hence, the autocorrelation function of the output noise is

$$\begin{aligned}
R_\nu(\tau) &= \mathcal{F}^{-1}[\mathcal{S}_\nu(f)] = \frac{N_0}{2}\int_{-\infty}^{\infty}\frac{1}{W}\cos(\frac{\pi f}{2W})e^{j2\pi f\tau}df \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty}\frac{1}{W}\cos(\frac{\pi f}{2W})e^{-j\frac{\pi f}{2W}}e^{j2\pi f(\tau+\frac{1}{4W})}df \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty}X(f)e^{j2\pi f(\tau+\frac{1}{4W})}df \\
&= \frac{N_0}{2}x(\tau+\frac{1}{4W})
\end{aligned}$$

and therefore,

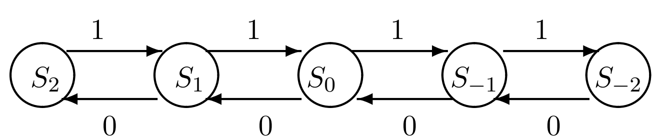
$$\begin{aligned}
R_\nu(0) &= \frac{N_0}{2}x(\frac{1}{4W}) = \frac{N_0}{2}\left(\mathrm{sinc}(\frac{1}{2})+\mathrm{sinc}(-\frac{1}{2})\right) = \frac{2N_0}{\pi} \\
R_\nu(T) &= R_\nu(\frac{1}{2W}) = \frac{N_0}{2}\left(\mathrm{sinc}(\frac{3}{2})+\mathrm{sinc}(\frac{1}{2})\right) = \frac{2N_0}{3\pi}
\end{aligned}$$

Since the noise is of zero mean, the covariance matrix of the noise is given by

$$\mathbf{C} = \begin{pmatrix} R_\nu(0) & R_\nu(T) \\ R_\nu(T) & R_\nu(0) \end{pmatrix} = \frac{2N_0}{\pi}\begin{pmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{pmatrix}$$

Problem 8.31

Let S_i represent the state that the difference between the total number of accumulated zeros and the total number of accumulated ones is i, with $i = -2, \ldots, 2$. The state transition diagram of the corresponding code is depicted in the next figure.



The state transition matrix is

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Setting $\det(D-\lambda I) = 0$, we obtain $\lambda^5 - 4\lambda^3 + 3\lambda = 0$. The roots of the characteristic equation are

$$\lambda = 0, \;\; \pm 1, \;\; \pm\sqrt{3}$$

Thus,

$$C = \log_2 \lambda_{\max} = \log_2\sqrt{3} = .7925$$

Problem 8.32

The state transition matrix of the (0,1) runlength-limited code is

$$D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The eigenvalues of D are the roots of

$$\det(D - \lambda I) = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1$$

The roots of the characteristic equation are

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

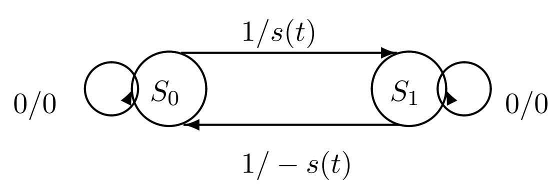
Thus, the capacity of the (0,1) runlength-limited code is

$$C(0,1) = \log_2(\frac{1 \pm \sqrt{5}}{2}) = 0.6942$$

The capacity of a $(1,\infty)$ code is found from Table 8.3 to be 0.6942. As it is observed, the two codes have exactly the same capacity. This result is to be expected since the (0,1) runlength-limited code and the $(1,\infty)$ code produce the same set of code sequences of length n, $N(n)$, with a renaming of the bits from 0 to 1 and vise versa. For example, the (0,1) runlength-limited code with a renaming of the bits, can be described as the code with no minimum number of 1's between 0's in a sequence, and at most one 1 between two 0's. In terms of 0's, this is simply the code with no restrictions on the number of adjacent 0's and no consecutive 1's, that is the $(1,\infty)$ code.

Problem 8.33

Let S_0 represent the state that the running polarity is zero, and S_1 the state that there exists some polarity (dc component). The following figure depicts the transition state diagram of the AMI code



The state transition matrix is

$$D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The eigenvalues of the matrix D can be found from

$$\det(D - \lambda I) = 0 \Longrightarrow (1-\lambda)^2 - 1 = 0 \text{ or } \lambda(2-\lambda) = 0$$

The largest real eigenvalue is $\lambda_{\max} = 2$, so that

$$C = \log_2 \lambda_{\max} = 1$$

Problem 8.34

Let $\{b_k\}$ be a binary sequence, taking the values 1, 0 depending on the existence of polarization at the transmitted sequence up to the time instant k. For the AMI code, b_k is expressed as

$$b_k = a_k \oplus b_{k-1} = a_k \oplus a_{k-1} \oplus a_{k-2} \oplus \ldots$$

where $\oplus$ denotes modulo two addition. Thus, the AMI code can be described as the RDS code, with RDS ($=b_k$) denoting the binary digital sum modulo 2 of the input bits.

Problem 8.35

Defining the efficiency as

$$\text{efficiency} = \frac{k}{n \log_2 3}$$

we obtain

Code	Efficiency
1B1T	0.633
3B2T	0.949
4B3T	0.844
6B4T	0.949

Problem 8.36

a) The characteristic polynomial of D is

$$\det(D - \lambda I) = \det \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1$$

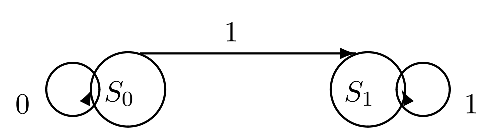
The eigenvalues of D are the roots of the characteristic polynomial, that is

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

Thus, the largest eigenvalue of D is $\lambda_{\max} = \frac{1+\sqrt{5}}{2}$ and therefore

$$C = \log_2 \frac{1+\sqrt{5}}{2} = 0.6942$$

b) The characteristic polynomial is $\det(D - \lambda I) = (1-\lambda)^2$ with roots $\lambda_{1,2} = 1$. Hence, $C = \log_2 1 = 0$. The state diagram of this code is depicted in the next figure.



c) As it is observed the second code has zero capacity. This result is to be expected since with the second code we can have at most $n+1$ different sequences of length n, so that

$$C = \lim_{n\to\infty} \frac{1}{n} \log_2 N(n) = \lim_{n\to\infty} \frac{1}{n} \log_2(n+1) = 0$$

The $n+1$ possible sequences are

$$\underbrace{0\ldots0}_{k}\underbrace{1\ldots1}_{n-k} \qquad (n \text{ sequences})$$

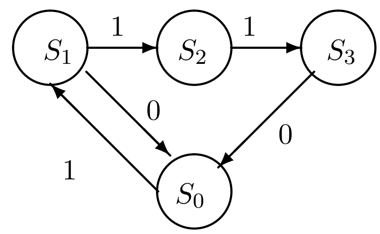
and the sequence $11\ldots1$, which occurs if we start from state S_1.

Problem 8.37

a) The two symbols, dot and dash, can be represented as 10 and 1110 respectively, where 1 denotes line closure and 0 an open line. Hence, the constraints of the code are

- A 0 is always followed by 1.
- Only sequences having one or three repetitions of 1, are allowed.

The next figure depicts the state diagram of the code, where the state S_0 denotes the reception of a dot or a dash, and state S_i denotes the reception of i adjacent 1's.



b) The state transition matrix is

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

c) The characteristic equation of the matrix D is

$$\det(D - \lambda I) = 0 \Longrightarrow \lambda^4 - \lambda^2 - 1 = 0$$

The roots of the characteristic equation are

$$\lambda_{1,2} = \pm\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{2}} \qquad \lambda_{3,4} = \pm\left(\frac{1-\sqrt{5}}{2}\right)^{\frac{1}{2}}$$

Thus, the capacity of the code is

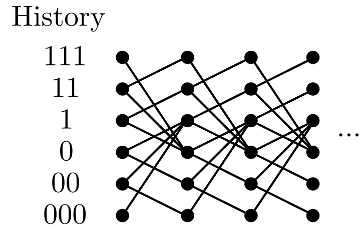
$$C = \log_2 \lambda_{\max} = \log_2 \lambda_1 = \log_2\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{2}} = 0.3471$$

Problem 8.38

The state diagram of Fig. P-8-38 describes a runlength constrained code, that forbids any sequence containing a run of more than three adjacent symbols of the same kind. The state transition matrix is

$$D = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding trellis is shown in the next figure



Problem 8.39

The state transition matrix of the (2,7) runlength-limited code is the 8×8 matrix

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Problem 8.40

The frequency response of the RC filter is

$$C(f) = \frac{\frac{1}{j2\pi RCf}}{R + \frac{1}{j2\pi RCf}} = \frac{1}{1 + j2\pi RCf}$$

The amplitude and the phase spectrum of the filter are

$$|C(f)| = \left(\frac{1}{1 + 4\pi^2(RC)^2 f^2}\right)^{\frac{1}{2}}, \qquad \Theta_c(f) = \arctan(-2\pi RCf)$$

The envelope delay is

$$T_c(f) = -\frac{1}{2\pi}\frac{d\Theta_c(f)}{df} = -\frac{1}{2\pi}\frac{-2\pi RC}{1 + 4\pi^2(RC)^2 f^2} = \frac{RC}{1 + 4\pi^2(RC)^2 f^2}$$

where we have used the formula

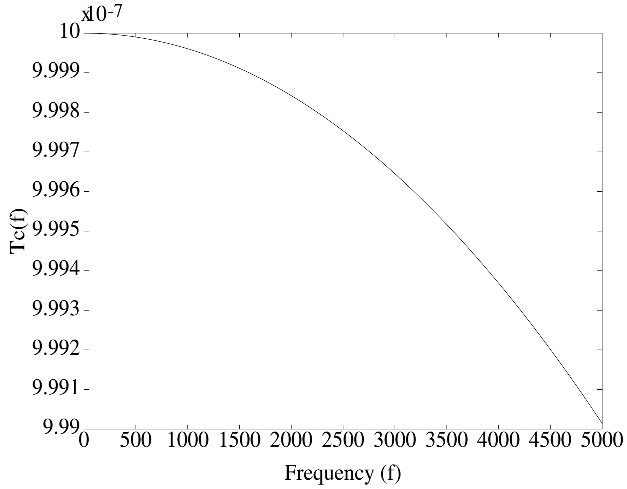
$$\frac{d}{dx}\arctan u = \frac{1}{1+u^2}\frac{du}{dx}$$

Problem 8.41

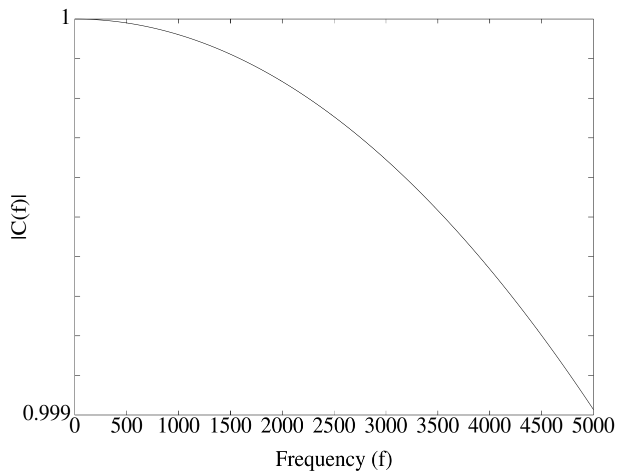
1) The envelope delay of the RC filter is (see Problem 8.40)

$$T_c(f) = \frac{RC}{1 + 4\pi^2(RC)^2 f^2}$$

A plot of $T(f)$ with $RC = 10^{-6}$ is shown in the next figure



2) The following figure is a plot of the amplitude characteristics of the RC filter, $|C(f)|$. The values of the vertical axis indicate that $|C(f)|$ can be considered constant for frequencies up to 2000 Hz. Since the same is true for the envelope delay, we conclude that a lowpass signal of bandwidth $\Delta f = 1$ KHz will not be distorted if it passes the RC filter.



Problem 8.42

Let $G_T(f)$ and $G_R(f)$ be the frequency response of the transmitting and receiving filter. Then, the condition for zero ISI implies

$$G_T(f)C(f)G_R(f) = X_{rc}(f) = \begin{cases} T & 0 \leq |f| \leq \frac{1}{4T} \\ \frac{T}{2}[1+\cos(2\pi T(|f|-\frac{1}{T})] & \frac{1}{4T} \leq |f| \leq \frac{3}{4T} \\ 0 & |f| > \frac{3}{4T} \end{cases}$$

Since the additive noise is white, the optimum transmitting and receiving filter characteristics are given by (see Example 8.6.1)

$$|G_T(f)| = \frac{|X_{rc}(f)|^{\frac{1}{2}}}{|C(f)|^{\frac{1}{2}}}, \qquad |G_R(f)| = \frac{|X_{rc}(f)|^{\frac{1}{2}}}{|C(f)|^{\frac{1}{2}}}$$

Thus,

$$|G_T(f)| = |G_R(f)| = \begin{cases} \left[\frac{T}{1+0.3\cos 2\pi fT}\right]^{\frac{1}{2}} & 0 \leq |f| \leq \frac{1}{4T} \\ \left[\frac{T(1+\cos(2\pi T(|f|-\frac{1}{T})}{2(1+0.3\cos 2\pi fT)}\right]^{\frac{1}{2}} & \frac{1}{4T} \leq |f| \leq \frac{3}{4T} \\ 0 & \text{otherwise} \end{cases}$$

Problem 8.43

A 4-PAM modulation can accommodate $k = 2$ bits per transmitted symbol. Thus, the symbol interval duration is

$$T = \frac{k}{9600} = \frac{1}{4800} \text{ sec}$$

Since, the channel's bandwidth is $W = 2400 = \frac{1}{2T}$, in order to achieve the maximum rate of transmission, $R_{\max} = \frac{1}{2T}$, the spectrum of the signal pulse should be

$$X(f) = T\Pi\left(\frac{f}{2W}\right)$$

Then, the magnitude frequency response of the optimum transmitting and receiving filter is (see Section 8.6.1 and Example 8.6.1)

$$|G_T(f)| = |G_R(f)| = \left[1 + \left(\frac{f}{2400}\right)^2\right]^{\frac{1}{4}} \Pi\left(\frac{f}{2W}\right) = \begin{cases} \left[1 + \left(\frac{f}{2400}\right)^2\right]^{\frac{1}{4}}, & |f| < 2400 \\ 0 & \text{otherwise} \end{cases}$$

Problem 8.44

1) The equivalent discrete-time impulse response of the channel is

$$h(t) = \sum_{n=-1}^{1} h_n \delta(t - nT) = 0.3\delta(t + T) + 0.9\delta(t) + 0.3\delta(t - T)$$

If by $\{c_n\}$ we denote the coefficients of the FIR equalizer, then the equalized signal is

$$q_m = \sum_{n=-1}^{1} c_n h_{m-n}$$

which in matrix notation is written as

$$\begin{pmatrix} 0.9 & 0.3 & 0. \\ 0.3 & 0.9 & 0.3 \\ 0. & 0.3 & 0.9 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The coefficients of the zero-force equalizer can be found by solving the previous matrix equation. Thus,

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} -0.4762 \\ 1.4286 \\ -0.4762 \end{pmatrix}$$

2) The values of q_m for $m = \pm 2, \pm 3$ are given by

$$\begin{aligned} q_2 &= \sum_{n=-1}^{1} c_n h_{2-n} = c_1 h_1 = -0.1429 \\ q_{-2} &= \sum_{n=-1}^{1} c_n h_{-2-n} = c_{-1} h_{-1} = -0.1429 \\ q_3 &= \sum_{n=-1}^{1} c_n h_{3-n} = 0 \\ q_{-3} &= \sum_{n=-1}^{1} c_n h_{-3-n} = 0 \end{aligned}$$

Problem 8.45

1) The output of the zero-force equalizer is

$$q_m = \sum_{n=-1}^{1} c_n x_{m_n}$$

With $q_0 = 1$ and $q_m = 0$ for $m \neq 0$, we obtain the system

$$\begin{pmatrix} 1.0 & 0.1 & -0.5 \\ -0.2 & 1.0 & 0.1 \\ 0.05 & -0.2 & 1.0 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Solving the previous system in terms of the equalizer's coefficients, we obtain

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0.000 \\ 0.980 \\ 0.196 \end{pmatrix}$$

2) The output of the equalizer is

$$q_m = \begin{cases} 0 & m \leq -4 \\ c_{-1}x_{-2} = 0 & m = -3 \\ c_{-1}x_{-1} + c_0 x_{-2} = -0.49 & m = -2 \\ 0 & m = -1 \\ 1 & m = 0 \\ 0 & m = 1 \\ c_0 x_2 + x_1 c_1 = 0.0098 & m = 2 \\ c_1 x_2 = 0.0098 & m = 3 \\ 0 & m \geq 4 \end{cases}$$

Hence, the residual ISI sequence is

$$\text{residual ISI} = \{\ldots, 0, -0.49, 0, 0, 0, 0.0098, 0.0098, 0, \ldots\}$$

and its span is 6 symbols.

Problem 8.46

The MSE performance index at the time instant k is

$$J(\mathbf{c}_k) = E\left[\left|\sum_{n=-N}^{N} c_{k,n} y_{k-n} - a_k\right| 62\right]$$

If we define the gradient vector $\mathbf{g}_k$ as

$$\mathbf{g}_k = \frac{\vartheta J(\mathbf{c}_k)}{2\vartheta \mathbf{c}_k}$$

then its l^{th} element is

$$\begin{aligned} g_{k,l} = \frac{\vartheta J(\mathbf{c}_k)}{2\vartheta c_{k,l}} &= \frac{1}{2} E\left[2\left(\sum_{n=-N}^{N} c_{k,n} y_{k-n} - a_k\right) y_{k-l}\right] \\ &= E\left[-e_k y_{k-l}\right] = -E\left[e_k y_{k-l}\right] \end{aligned}$$

Thus, the vector $\mathbf{g}_k$ is

$$\mathbf{g}_k = \begin{pmatrix} -E[e_k y_{k+N}] \\ \vdots \\ -E[e_k y_{k-N}] \end{pmatrix} = -E[e_k \mathbf{y}_k]$$

where $\mathbf{y}_k$ is the vector $\mathbf{y}_k = [y_{k+N} \cdots y_{k-N}]^T$. Since $\hat{\mathbf{g}}_k = -e_k \mathbf{y}_k$, its expected value is

$$E[\hat{\mathbf{g}}_k] = E[-e_k \mathbf{y}_k] = -E[e_k \mathbf{y}_k] = \mathbf{g}_k$$

Problem 8.47

1) If $\{c_n\}$ denote the coefficients of the zero-force equalizer and $\{q_m\}$ is the sequence of the equalizer's output samples, then

$$q_m = \sum_{n=-1}^{1} c_n x_{m-n}$$

where $\{x_k\}$ is the noise free response of the matched filter demodulator sampled at $t = kT$. With $q_{-1} = 0$, $q_0 = q_1 = \mathcal{E}_b$, we obtain the system

$$\begin{pmatrix} \mathcal{E}_b & 0.9\mathcal{E}_b & 0.1\mathcal{E}_b \\ 0.9\mathcal{E}_b & \mathcal{E}_b & 0.9\mathcal{E}_b \\ 0.1\mathcal{E}_b & 0.9\mathcal{E}_b & \mathcal{E}_b \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{E}_b \\ \mathcal{E}_b \end{pmatrix}$$

The solution to the system is

$$\begin{pmatrix} c_{-1} & c_0 & c_1 \end{pmatrix} = \begin{pmatrix} 0.2137 & -0.3846 & 1.3248 \end{pmatrix}$$

2) The set of noise variables $\{\nu_k\}$ at the output of the sampler is a Gaussian distributed sequence with zero-mean and autocorrelation function

$$R_\nu(k) = \begin{cases} \frac{N_0}{2} x_k & |k| \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the autocorrelation function of the noise at the output of the equalizer is

$$R_n(k) = R_\nu(k) \star c(k) \star c(-k)$$

where $c(k)$ denotes the discrete time impulse response of the equalizer. Therefore, the autocorrelation sequence of the noise at the output of the equalizer is

$$R_n(k) = \frac{N_0 \mathcal{E}_b}{2} \begin{cases} 0.9402 & k = 0 \\ 1.3577 & k = \pm 1 \\ -0.0546 & k = \pm 2 \\ 0.1956 & k = \pm 3 \\ 0.0283 & k = \pm 4 \\ 0 & \text{otherwise} \end{cases}$$

To find an estimate of the error probability for the sequence detector, we ignore the residual interference due to the finite length of the equalizer, and we only consider paths of length two. Thus, if we start at state $a_0 = 1$ and the transmitted symbols are $(a_1, a_2) = (1, 1)$ an error is made by the sequence detector if the path $(-1, 1)$ is more probable, given the received values of r_1 and r_2. The metric for the path $(a_1, a_2) = (1, 1)$ is

$$\mu_2(1,1) = [\ r_1 - 2\mathcal{E}_b \quad r_2 - 2\mathcal{E}_b\ ] \mathbf{C}^{-1} \begin{bmatrix} r_1 - 2\mathcal{E}_b \\ r_2 - 2\mathcal{E}_b \end{bmatrix}$$

where

$$\mathbf{C} = \frac{N_0\mathcal{E}_b}{2}\begin{pmatrix} 0.9402 & 1.3577 \\ 1.3577 & 0.9402 \end{pmatrix}$$

Similarly, the metric of the path $(a_1, a_2) = (-1, 1)$ is

$$\mu_2(-1,1) = [\; r_1 \quad r_2 \;]\mathbf{C}^{-1}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

Hence, the probability of error is

$$P_2 = P(\mu_2(-1,1) < \mu_2(1,1))$$

and upon substitution of $r_1 = 2\mathcal{E}_b + n_1$, $r_2 = 2\mathcal{E}_b + n_2$, we obtain

$$P_2 = P(n_1 + n_2 < -2\mathcal{E}_b)$$

Since n_1 and n_2 are zero-mean Gaussian variables, their sum is also zero-mean Gaussian with variance

$$\sigma_2 = (2 \times 0.9402 + 2 \times 1.3577)\frac{N_0\mathcal{E}_b}{2} = 4.5958\frac{N_0\mathcal{E}_b}{2}$$

and therefore

$$P_2 = Q\left[\sqrt{\frac{8\mathcal{E}_b}{4.5958N_0}}\right]$$

The bit error probability is $\frac{P_2}{2}$.

Problem 8.48

The optimum tap coefficients of the zero-force equalizer can be found by solving the system

$$\begin{pmatrix} 1.0 & 0.3 & 0.0 \\ 0.2 & 1.0 & 0.3 \\ 0.0 & 0.2 & 1.0 \end{pmatrix}\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} -0.3409 \\ 1.1364 \\ -0.2273 \end{pmatrix}$$

b) The output of the equalizer is

$$q_m = \begin{cases} 0 & m \leq -3 \\ c_{-1}x_{-1} = -0.1023 & m = -2 \\ 0 & m = -1 \\ 1 & m = 0 \\ 0 & m = 1 \\ c_1x_1 = -0.0455 & m = 2 \\ 0 & m \geq 3 \end{cases}$$

Hence, the residual ISI sequence is

$$\text{residual ISI} = \{\ldots, 0, -0.1023, 0, 0, 0, -0.0455, 0, \ldots\}$$

Problem 8.49

1) If we assume that the signal pulse has duration T, then the output of the matched filter at the time instant $t = T$ is

$$\begin{aligned} y(T) &= \int_0^T r(\tau)s(\tau)d\tau \\ &= \int_0^T (s(\tau) + \alpha s(\tau - T) + n(\tau))s(\tau)d\tau \\ &= \int_0^T s^2(\tau)d\tau + \int_0^T n(\tau)s(\tau)d\tau \\ &= \mathcal{E}_s + n \end{aligned}$$

where $\mathcal{E}_s$ is the energy of the signal pulse and n is a zero-mean Gaussian random variable with variance $\sigma_n^2 = \frac{N_0 \mathcal{E}_s}{2}$. Similarly, the output of the matched filter at $t = 2T$ is

$$\begin{aligned} y(2T) &= \alpha \int_0^T s^2(\tau)d\tau + \int_0^T n(\tau)s(\tau)d\tau \\ &= \alpha \mathcal{E}_s + n \end{aligned}$$

2) If the transmitted sequence is

$$x(t) = \sum_{n=-\infty}^{\infty} a_n s(t - nT)$$

with a_n taking the values $1, -1$ with equal probability, then the output of the demodulator at the time instant $t = kT$ is

$$y_k = a_k \mathcal{E}_s + \alpha a_{k-1} \mathcal{E}_s + n_k$$

The term $\alpha a_{k-1} \mathcal{E}_s$ expresses the ISI due to the signal reflection. If a symbol by symbol detector is employed and the ISI is ignored, then the probability of error is

$$\begin{aligned} P(e) &= \frac{1}{2} P(\text{error}|a_n = 1, a_{n-1} = 1) + \frac{1}{2} P(\text{error}|a_n = 1, a_{n-1} = -1) \\ &= \frac{1}{2} P((1+\alpha)\mathcal{E}_s + n_k < 0) + \frac{1}{2} P((1-\alpha)\mathcal{E}_s + n_k < 0) \\ &= \frac{1}{2} Q\left[\sqrt{\frac{2(1+\alpha)^2 \mathcal{E}_s}{N_0}}\right] + \frac{1}{2} Q\left[\sqrt{\frac{2(1-\alpha)^2 \mathcal{E}_s}{N_0}}\right] \end{aligned}$$

3) To find the error rate performance of the DFE, we assume that the estimation of the parameter α is correct and that the probability of error at each time instant is the same. Since the transmitted symbols are equiprobable, we obtain

$$\begin{aligned} P(e) &= P(\text{error at } k | a_k = 1) \\ &= P(\text{error at } k-1) P(\text{error at } k | a_k = 1, \text{error at } k-1) \\ &\quad + P(\text{no error at } k-1) P(\text{error at } k | a_k = 1, \text{no error at } k-1) \\ &= P(e) P(\text{error at } k | a_k = 1, \text{error at } k-1) \\ &\quad + (1 - P(e)) P(\text{error at } k | a_k = 1, \text{no error at } k-1) \\ &= P(e)p + (1 - P(e))q \end{aligned}$$

where

$$
\begin{aligned}
p &= P(\text{error at } k|a_k = 1, \text{error at } k-1) \\
&= \frac{1}{2}P(\text{error at } k|a_k = 1, a_{k-1} = 1, \text{error at } k-1) \\
&\quad + \frac{1}{2}P(\text{error at } k|a_k = 1, a_{k-1} = -1, \text{error at } k-1) \\
&= \frac{1}{2}P((1+2\alpha)\mathcal{E}_s + n_k < 0) + \frac{1}{2}P((1-2\alpha)\mathcal{E}_s + n_k < 0) \\
&= \frac{1}{2}Q\left[\sqrt{\frac{2(1+2\alpha)^2\mathcal{E}_s}{N_0}}\right] + \frac{1}{2}Q\left[\sqrt{\frac{2(1-2\alpha)^2\mathcal{E}_s}{N_0}}\right]
\end{aligned}
$$

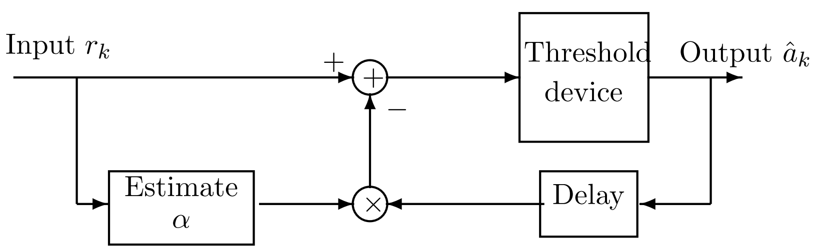
and

$$
\begin{aligned}
q &= P(\text{error at } k|a_k = 1, \text{no error at } k-1) \\
&= P(\mathcal{E}_s + n_k < 0) = Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]
\end{aligned}
$$

Solving for $P(e)$, we obtain

$$
P(e) = \frac{q}{1-p+q} = \frac{Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]}{1 - \frac{1}{2}Q\left[\sqrt{\frac{2(1+2\alpha)^2\mathcal{E}_s}{N_0}}\right] - \frac{1}{2}Q\left[\sqrt{\frac{2(1-2\alpha)^2\mathcal{E}_s}{N_0}}\right] + Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]}
$$

A sketch of the detector structure is shown in the next figure.



Problem 8.50

A discrete time transversal filter equivalent to the cascade of the transmitting filter $g_T(t)$, the channel $c(t)$, the matched filter at the receiver $g_R(t)$ and the sampler, has tap gain coefficients $\{y_m\}$, where

$$
y_m = \begin{cases} 0.9 & m = 0 \\ 0.3 & m = \pm 1 \\ 0 & \text{otherwise} \end{cases}
$$

The noise ν_k, at the output of the sampler, is a zero-mean Gaussian sequence with autocorrelation function

$$
E[\nu_k \nu_l] = \sigma^2 y_{k-l}, \qquad |k-l| \leq 1
$$

If the $\mathcal{Z}$-transform of the sequence $\{y_m\}$, $Y(z)$, assumes the factorization

$$
Y(z) = F(z)F^*(z^{-1})
$$

then the filter $1/F^*(z^{-1})$ can follow the sampler to white the noise sequence ν_k. In this case the output of the whitening filter, and input to the MSE equalizer, is the sequence

$$
u_n = \sum_k a_k f_{n-k} + n_k
$$

where n_k is zero mean Gaussian with variance σ^2. The optimum coefficients of the MSE equalizer, c_k, satisfy (see (8.6.35))

$$\sum_{n=-1}^{1} c_n R_u(n-k) = R_{ua}(k), \qquad k = 0, \pm 1$$

where

$$\begin{aligned} R_u(n-k) &= E[u_{l-k}u_{l-n}] = \sum_{m=0}^{1} f_m f_{m+n-k} + \sigma^2\delta_{n,k} \\ &= \begin{cases} y_{n-k} + \sigma^2\delta_{n,k}, & |n-k| \leq 1 \\ 0 & \text{otherwise} \end{cases} \\ R_{ua}(k) &= E[a_n u_{n-k}] = \begin{cases} f_{-k}, & -1 \leq k \leq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

With

$$Y(z) = 0.3z + 0.9 + 0.3z^{-1} = (f_0 + f_1 z^{-1})(f_0 + f_1 z)$$

we obtain the parameters f_0 and f_1 as

$$f_0 = \begin{cases} \pm\sqrt{0.7854} \\ \pm\sqrt{0.1146} \end{cases}, \qquad f_1 = \begin{cases} \pm\sqrt{0.1146} \\ \pm\sqrt{0.7854} \end{cases}$$

The parameters f_0 and f_1 should have the same sign since $f_0 f_1 = 0.3$. However, the sign itself does not play any role if the data are differentially encoded. To have a stable inverse system $1/F^*(z^{-1})$, we select f_0 and f_1 in such a way that the zero of the system $F^*(z^{-1}) = f_0 + f_1 z$ is inside the unit circle. Thus, we choose $f_0 = \sqrt{0.1146}$ and $f_1 = \sqrt{0.7854}$ and therefore, the desired system for the equalizer's coefficients is

$$\begin{pmatrix} 0.9+0.1 & 0.3 & 0.0 \\ 0.3 & 0.9+0.1 & 0.3 \\ 0.0 & 0.3 & 0.9+0.1 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \sqrt{0.7854} \\ \sqrt{0.1146} \\ 0 \end{pmatrix}$$

Solving this system, we obtain

$$c_{-1} = 0.8596, \quad c_0 = 0.0886, \quad c_1 = -0.0266$$

Problem 8.51

1) The spectrum of the band limited equalized pulse is

$$\begin{aligned} X(f) &= \begin{cases} \frac{1}{2W}\sum_{n=-\infty}^{\infty} x(\frac{n}{2W})e^{-j\frac{\pi n f}{W}} & |f| \leq W \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{2W}\left[2 + 2\cos\frac{\pi f}{W}\right] & |f| \leq W \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{W}\left[1 + 1\cos\frac{\pi f}{W}\right] & |f| \leq W \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $W = \frac{1}{2T_b}$

2) The following table lists the possible transmitted sequences of length 3 and the corresponding output of the detector.

-1	-1	-1	-4
-1	-1	1	-2
-1	1	-1	0
-1	1	1	2
1	-1	-1	-2
1	-1	1	0
1	1	-1	2
1	1	1	4

As it is observed there are 5 possible output levels b_m, with probability $p(b_m = 0) = \frac{1}{4}$, $p(b_m = \pm 2) = \frac{1}{4}$ and $p(b_m = \pm 4) = \frac{1}{8}$.

3) The transmitting filter $G_T(f)$, the receiving filter $G_R(f)$ and the equalizer $G_E(f)$ satisfy the condition

$$G_T(f)G_R(f)G_E(f) = X(f)$$

The power spectral density of the noise at the output of the equalizer is

$$\mathcal{S}_\nu(f) = S_n(f)|G_R(f)G_E(f)|^2 = \sigma^2|G_R(f)G_E(f)|^2$$

With

$$G_T(f) = G_R(f) = P(f) = \frac{\pi T_{50}}{2} e^{-\pi T_{50}|f|}$$

the variance of the output noise is

$$\begin{aligned}
\sigma_\nu^2 &= \sigma^2 \int_{-\infty}^{\infty} |G_R(f)G_E(f)|^2 df = \sigma^2 \int_{-\infty}^{\infty} \left|\frac{X(f)}{G_T(f)}\right|^2 df \\
&= \sigma^2 \int_{-W}^{W} \frac{4}{\pi^2 T_{50}^2 W^2} \frac{|1+\cos\frac{\pi f}{W}|^2}{e^{-2\pi T_{50}|f|}} df \\
&= \frac{8\sigma^2}{\pi^2 T_{50}^2 W^2} \int_0^W \left(1+\cos\frac{\pi f}{W}\right)^2 e^{2\pi T_{50} f} df
\end{aligned}$$

The value of the previous integral can be found using the formula

$$\begin{aligned}
&\int e^{ax}\cos^n bx dx \\
&= \frac{1}{a^2+n^2b^2}\left[(a\cos bx + nb\sin bx)e^a x \cos^{n-1} bx + n(n-1)b^2 \int e^{ax}\cos^{n-2} bx dx\right]
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\sigma_\nu^2 &= \frac{8\sigma^2}{\pi^2 T_{50}^2 W^2} \times \left[\left(e^{2\pi T_{50} W} - 1\right)\left(\frac{1}{2\pi T_{50}} + \frac{2\pi T_{50} + \pi\frac{1}{W^2 T_{50}}}{4\pi^2 T_{50}^2 + 4\frac{\pi^2}{W^2}}\right)\right. \\
&\quad \left. - \frac{4\pi T_{50}}{4\pi^2 T_{50}^2 + \frac{\pi^2}{W^2}}\left(e^{2\pi T_{50} W} + 1\right)\right]
\end{aligned}$$

To find the probability of error using a symbol by symbol detector, we follow the same procedure as in Section 8.4.3. The results are the same with that obtained from a 3-point PAM constellation $(0, \pm 2)$ used with a duobinary signal with output levels having the probability mass function given in part b). An upper bound of the symbol probability of error is

$$\begin{aligned}
P(e) &< P(|y_m| > 1|b_m = 0)P(b_m = 0) + 2P(|y_m - 2| > 1|b_m = 2)P(b_m = 2) \\
&\quad +2P(y_m + 4 > 1|b_m = -4)P(b_m = -4) \\
&= P(|y_m| > 1|b_m = 0)\left[P(b_m = 0) + 2P(b_m = 2) + P(b_m = -4)\right] \\
&= \frac{7}{8}P(|y_m| > 1|b_m = 0)
\end{aligned}$$

But

$$P(|y_m| > 1|b_m = 0) = \frac{2}{\sqrt{2\pi}\sigma_\nu}\int_1^\infty e^{-x^2/2\sigma_\nu^2}dx$$

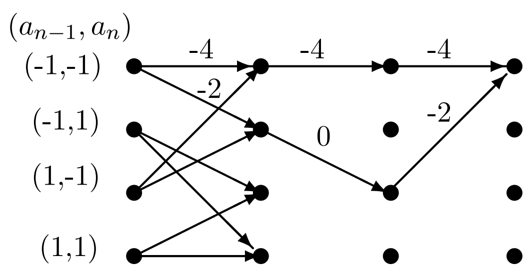
Therefore,

$$P(e) = \frac{14}{8}Q\left[\frac{1}{\sigma_\nu}\right]$$

Problem 8.52

Since the partial response signal has memory length equal to 2, the corresponding trellis has 4 states which we label as (a_{n-1}, a_n). The following figure shows three frames of the trellis. The labels of the branches indicate the output of the partial response system. As it is observed the free distance between merging paths is 3, whereas the Euclidean distance is equal to

$$d_E = 2^2 + 4^2 + 2^2 = 24$$



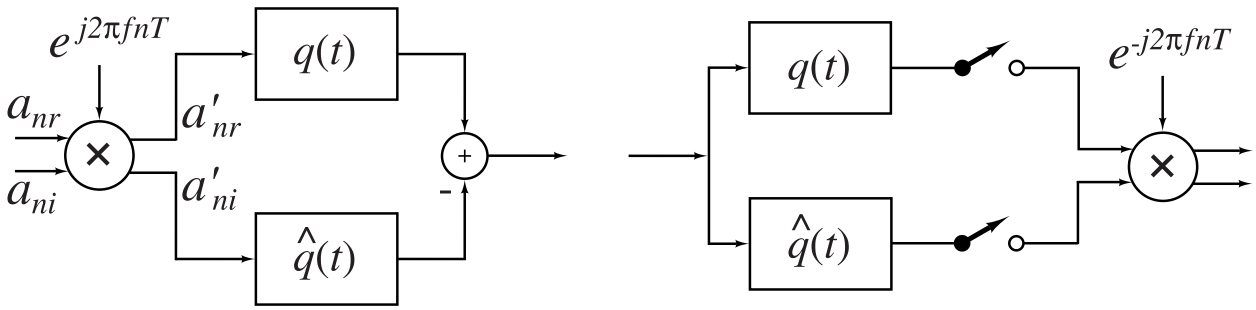
Problem 8.53

a) The alternative expression for $s(t)$ can be rewritten as

$$\begin{aligned}
s(t) &= \text{Re}\left[\sum_n a_n' Q(t - nT)\right] \\
&= \text{Re}\left[\sum_n a_n e^{j2\pi f_c nT} g(t - nT)\left[\cos 2\pi f_c(t - nT) + j\sin 2\pi f_c(t - nT)\right]\right] \\
&= \text{Re}\left[\sum_n a_n g(t - nT)\left[\cos 2\pi f_c nT + j\sin 2\pi f_c nT\right]\left[\cos 2\pi f_c(t - nT) + j\sin 2\pi f_c(t - nT)\right]\right] \\
&= \text{Re}\left[\sum_n a_n g(t - nT)\left[\cos 2\pi f_c nT\cos 2\pi f_c(t - nT) - \sin 2\pi f_c nT\sin 2\pi f_c(t - nT)\right.\right. \\
&\quad \left.\left. +j\sin 2\pi f_c nT\cos 2\pi f_c(t - nT) + j\cos 2\pi f_c nT\sin 2\pi f_c(t - nT)\right]\right] \\
&= \text{Re}\left[\sum_n a_n g(t - nT)\left[\cos 2\pi f_c t + j\sin 2\pi f_c t\right]\right] \\
&= \text{Re}\left[\sum_n a_n g(t - nT)e^{j2\pi f_c t}\right] \\
&= s(t)
\end{aligned}$$

so indeed the alternative expression for $s(t)$ is a valid one.

b)



Modulator
(with phase rotator)

Demodulator
(with phase derotator)

Problem 8.54

a) The impulse response of the pulse having a square-root raised cosine characteristic, is an even function, i.e., $x_{SQ}(t) = x_{SQ}(-t)$, i.e., the pulse $g(t)$ is an even function. We know that the product of an even function times an even function is an even function, while the product of an even function times an odd function is an odd function. Hence $q(t)$ is even while $\hat{q}(t)$ is odd and their product $q(t)\hat{q}(t)$ has odd symmetry. Therefore,

$$\int_{-\infty}^{\infty} q(t)\hat{q}(t)\,dt = \int_{-(1+\beta)/2T}^{(1+\beta)/2T} q(t)\hat{q}(t)\,dt = 0$$

b) We notice that when $f_c = k/T$, where k is an integer, then the rotator/derotator of a carrierless QAM system (described in Problem 8.53) gives a trivial rotation of an integer number of full circles $(2\pi kn)$, and the carrierless QAM/PSK is equivalent to CAP.

Problem 8.55

The analog signal is

$$x(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k e^{j2\pi kt/T}, \quad 0 \le t < T$$

The subcarrier frequencies are: $F_k = k/T$, $k = 0, 1, \ldots, \tilde{N}$, and, hence, the maximum frequency in the analog signal is: $\tilde{N}/T$. If we sample at the Nyquist rate: $2\tilde{N}/T = N/T$, we obtain the discrete-time sequence:

$$x(n) = x(t = nT/N) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k e^{j2\pi k(nT/N)/T} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}, \quad n = 0, 10, \ldots, N-1$$

which is simply the IDFT of the information sequence $\{X_k\}$.

To show that $x(t)$ is a real-valued signal, we make use of the condition: $X_{N-k} = X_k^*$, for $k = 1.2.\ldots, \tilde{N}-1$. By combining the pairs of complex conjugate terms, we obtain for $k = 1, 2, \ldots, \tilde{N}-1$

$$X_k e^{j2\pi kt/T} + X_k^* e^{-j2\pi kt/T} = 2|X_k| \cos\left(\frac{2\pi kt}{T} + \theta_k\right)$$

where $X_k = |X_k|e^{j\theta_k}$. We also note that X_0 and $X_{\tilde{N}}$ are real. Hence, $x(t)$ is a real-valued signal.

Problem 8.56

The filter with system function $H_n(z)$ has the impulse response $h(k) = e^{j2\pi nk/N},\ k = 0, 1, \ldots$. If we pass the sequence $\{X_k,\ k = 0, 1, \ldots, N-1\}$ through such a filter, we obtain the sequence $y_n(m)$, given as

$$\begin{aligned} y_n(m) &= \sum_{k=0}^{m} X_k h(m-k), \quad m = 0, 1, \ldots \\ &= \sum_{k=0}^{m} X_k e^{j2\pi n(m-k)/N} \end{aligned}$$

At $m = N$, where $y_n(N) = \sum_{k=0}^{N} X_k e^{-j2\pi nk/N} = \sum_{k=0}^{N-1} X_k e^{-j2\pi nk/N}$, since $X_N = 0$. Therefore, the IDFT of $\{X_k\}$ can be computed by passing $\{X_k\}$ through the N filters $H_n(z)$ and sampling their outputs at $m = N$.

Chapter 9

Problem 9.1
The capacity of the channel is defined as

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)} [H(Y) - H(Y|X)]$$

The conditional entropy $H(Y|X)$ is

$$H(Y|X) = p(X=a)H(Y|X=a) + p(X=b)H(Y|X=b) + p(X=c)H(Y|X=c)$$

However,

$$\begin{aligned} H(Y|X=a) &= -\sum_k p(Y=k|X=a)\log P(Y=k|X=a) \\ &= -(0.2\log 0.2 + 0.3\log 0.3 + 0.5\log 0.5) \\ &= H(Y|X=b) = H(Y|X=c) = 1.4855 \end{aligned}$$

and therefore,

$$H(Y|X) = \sum_k p(X=k)H(Y|X=k) = 1.4855$$

Thus,

$$I(X;Y) = H(Y) - 1.4855$$

To maximize $I(X;Y)$, it remains to maximize $H(Y)$. However, $H(Y)$ is maximized when Y is a uniformly distributed random variable, if such a distribution can be achieved by an appropriate input distribution. Using the symmetry of the channel, we observe that a uniform input distribution produces a uniform output. Thus, the maximum of $I(X;Y)$ is achieved when $p(X=a) = p(X=b) = p(X=c) = \frac{1}{3}$ and the channel capacity is

$$C = \log_2 3 - H(Y|X) = 0.0995 \text{ bits/transmission}$$

Problem 9.2
The capacity of the channel is defined as

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)} [H(Y) - H(Y|X)]$$

If the probability distribution $p(x)$ that achieves capacity is

$$p(X) = \begin{cases} p & X=0 \\ 1-p & X=1 \end{cases}$$

then,

$$\begin{aligned} H(Y|X) &= pH(Y|X=0) + (1-p)H(Y|X=1) \\ &= ph(\epsilon) + (1-p)h(\epsilon) = h(\epsilon) \end{aligned}$$

where $h(\epsilon)$ is the binary entropy function. As it is seen $H(Y|X)$ is independent on p and therefore $I(X;Y)$ is maximized when $H(Y)$ is maximized. To find the distribution $p(x)$ that maximizes the entropy $H(Y)$ we reduce first the number of possible outputs as follows. Let V be a function of the output defined as

$$V = \begin{cases} 1 & Y = E \\ 0 & \text{otherwise} \end{cases}$$

Clearly $H(V|Y) = 0$ since V is a deterministic function of Y. Therefore,

$$\begin{aligned} H(Y,V) &= H(Y) + H(V|Y) = H(Y) \\ &= H(V) + H(Y|V) \end{aligned}$$

To find $H(V)$ note that $P(V = 1) = P(Y = E) = p\epsilon + (1-p)\epsilon = \epsilon$. Thus, $H(V) = h(\epsilon)$, the binary entropy function at ϵ. To find $H(Y|V)$ we write

$$H(Y|V) = p(V = 0)H(Y|V = 0) + p(V = 1)H(Y|V = 1)$$

But $H(Y|V = 1) = 0$ since there is no ambiguity on the output when $V = 1$, and

$$H(Y|V = 0) = -\sum_{k=0,1} p(Y = k|V = 0)\log_2 p(Y = k|V = 0)$$

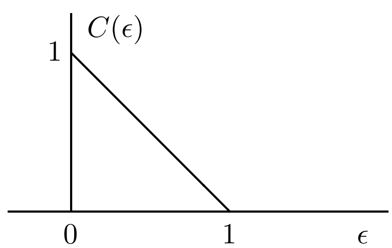
Using Bayes rule, we write the conditional probability $P(Y = 0|V = 0)$ as

$$P(Y = 0|V = 0) = \frac{P(Y = 0, V = 0)}{p(V = 0)} = \frac{p(1-\epsilon)}{(1-\epsilon)} = p$$

Thus, $H(Y|V = 0)$ is $h(p)$ and $H(Y|V) = (1-\epsilon)h(p)$. The capacity is now written as

$$\begin{aligned} C &= \max_{p(x)} [H(V) + H(Y|V) - h(\epsilon)] \\ &= \max_{p(x)} H(Y|V) = \max_{p(x)} (1-\epsilon)h(p) = (1-\epsilon) \end{aligned}$$

and it is achieved for $p = \frac{1}{2}$. The next figure shows the capacity of the channel as a function of ϵ.



Problem 9.3

The overall channel is a binary symmetric channel with crossover probability p. To find p note that an error occurs if an odd number of channels produce an error. Thus,

$$p = \sum_{k=\text{odd}} \binom{n}{k} \epsilon^k (1-\epsilon)^{n-k}$$

Using the results of Problem 7.55, we find that

$$p = \frac{1}{2}\left[1 - (1-2\epsilon)^2\right]$$

and therefore,

$$C = 1 - h(p)$$

If $n \to \infty$, then $(1-2\epsilon)^n \to 0$ and $p \to \frac{1}{2}$. In this case

$$C = \lim_{n\to\infty} C(n) = 1 - h(\frac{1}{2}) = 0$$

Problem 9.4

Denoting $\bar{\epsilon} = 1-\epsilon$, we have $n! \approx \sqrt{2\pi n}n^n e^{-n}$, $(n\epsilon)! \approx \sqrt{2\pi n\epsilon}(n\epsilon)^{n\epsilon}e^{-n\epsilon}$, and $(n\bar{\epsilon})! \approx \sqrt{2\pi n\bar{\epsilon}}(n\bar{\epsilon})^{n\bar{\epsilon}}e^{-n\bar{\epsilon}}$

$$\begin{aligned}
\binom{n}{n\epsilon} &= \frac{n!}{(n\epsilon)!(n\bar{\epsilon})!} \\
&\approx \frac{\sqrt{2\pi n}n^n e^{-n}}{\sqrt{2\pi n\epsilon}(n\epsilon)^{n\epsilon}e^{-n\epsilon}\sqrt{2\pi n\bar{\epsilon}}(n\bar{\epsilon})^{n\bar{\epsilon}}e^{-n\bar{\epsilon}}} \\
&= \frac{1}{\sqrt{2\pi n\epsilon\bar{\epsilon}}\epsilon^{n\epsilon}\bar{\epsilon}^{n\bar{\epsilon}}}
\end{aligned}$$

From above

$$\begin{aligned}
\frac{1}{n}\log_2\binom{n}{n\epsilon} &\approx -\frac{1}{2n}\log_2(2\pi n\epsilon\bar{\epsilon}) - \epsilon\log_2\epsilon - \bar{\epsilon}\log_2\bar{\epsilon} \\
&\to -\epsilon\log_2\epsilon - \bar{\epsilon}\log_2\bar{\epsilon} \quad \text{as } n\to\infty \\
&= H_b(\epsilon)
\end{aligned}$$

This shows that as $n\to\infty$, $\binom{n}{n\epsilon} \approx 2^{nH_b(\epsilon)}$.

Problem 9.5

Due to the symmetry in channel, the capacity is achieved for uniform input distribution, i.e., for $p(X=A) = p(X=-A) = \frac{1}{2}$. For this input distribution, the output distribution is given by

$$p(y) = \frac{1}{2\sqrt{2\pi\sigma^2}}e^{-(y+A)^2/2\sigma^2} + \frac{1}{2\sqrt{2\pi\sigma^2}}e^{-(y-A)^2/2\sigma^2}$$

and the mutual information between the input and the output is

$$\begin{aligned}
I(X;Y) &= \frac{1}{2}\int_{-\infty}^{\infty} p(y \mid X=A)\log_2\frac{p(y\mid X=A)}{p(y)}\,dy \\
&\quad +\frac{1}{2}\int_{-\infty}^{\infty} p(y \mid X=-A)\log_2\frac{p(y\mid X=-A)}{p(y)}\,dy \\
&= \frac{1}{2}I_1 + \frac{1}{2}I_2
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} p(y \mid X=A)\log_2\frac{p(y\mid X=A)}{p(y)}\,dy \\
I_2 &= \int_{-\infty}^{\infty} p(y \mid X=-A)\log_2\frac{p(y\mid X=-A)}{p(y)}\,dy
\end{aligned}$$

Now consider the first term in the above expression. Substituting for $p(y \mid X=A)$ and $p(y)$, we obtain,

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-A)^2}{2\sigma^2}}\log_2\frac{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-A)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-A)^2}{2\sigma^2}}\ \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y+A)^2}{2\sigma^2}}}\,dy \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y/\sigma-A/\sigma)^2}{2}}\log_2\frac{2}{1+e^{-2yA/\sigma^2}}\,dy
\end{aligned}$$

using the change of variable $u = y/\sigma$ and denoting A/σ by a we obtain

$$I_1 = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(u-a)^2}{2}}\log_2\frac{2}{1+e^{-2ua}}\,du$$

A similar approach can be applied to I_2, the second term in the expression for $I(X;Y)$, resulting in

$$I(X;Y) = \frac{1}{2} f\left(\frac{A}{\sigma}\right) + \frac{1}{2} f\left(-\frac{A}{\sigma}\right) \tag{4}$$

where

$$f(a) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(u-a)^2/2} \log_2 \frac{2}{1+e^{-2au}}\, du \tag{5}$$

Problem 9.6

The capacity of the channel is defined as

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)} [H(Y) - H(Y|X)]$$

However,

$$H(Y|X) = \sum_x p(x) H(Y|X=x) = \sum_x p(x) H(R) = H(R)$$

where $H(R)$ is the entropy of a source with symbols having probabilities the elements of a row of the probability transition matrix. The last equality in the previous equation follows from the fact that $H(R)$ is the same for each row since the channel is symmetric. Thus

$$C = \max_{p(x)} H(Y) - H(R)$$

$H(Y)$ is maximized when Y is a uniform random variable. With a symmetric channel we can always find an input distribution that makes Y uniformly distributed, and thus maximize $H(Y)$. To see this, let

$$p(Y=y) = \sum_x p(x) P(Y=y|X=x)$$

If $p(x) = \frac{1}{|\mathcal{X}|}$, where $|\mathcal{X}|$ is the cardinality of $\mathcal{X}$, then

$$p(Y=y) = \frac{1}{|\mathcal{X}|} \sum_x P(Y=y|X=x)$$

But $\sum_x P(Y=y|X=x)$ is the same for each y since the columns of a symmetric channel are permutations of each other. Thus,

$$C = \log|\mathcal{Y}| - H(R)$$

where $|\mathcal{Y}|$ is the cardinality of the output alphabet.

Problem 9.7

a) The capacity of the channel is

$$C_1 = \max_{p(x)} [H(Y) - H(Y|X)]$$

But, $H(Y|X) = 0$ and therefore, $C_1 = \max_{p(x)} H(Y) = 1$ which is achieved for $p(0) = p(1) = \frac{1}{2}$.

b) Let q be the probability of the input symbol 0, and thus $(1-q)$ the probability of the input symbol 1. Then,

$$\begin{aligned}
H(Y|X) &= \sum_x p(x) H(Y|X=x) \\
&= qH(Y|X=0) + (1-q)H(Y|X=1) \\
&= (1-q)H(Y|X=1) = (1-q)h(0.5) = (1-q)
\end{aligned}$$

The probability mass function of the output symbols is

$$\begin{aligned}
P(Y=c) &= qp(Y=c|X=0)+(1-q)p(Y=c|X=1)\\
&= q+(1-q)0.5 = 0.5+0.5q\\
p(Y=d) &= (1-q)0.5 = 0.5-0.5q
\end{aligned}$$

Hence,

$$C_2 = \max_q [h(0.5+0.5q)-(1-q)]$$

To find the probability q that achieves the maximum, we set the derivative of C_2 with respect to q equal to 0. Thus,

$$\begin{aligned}
\frac{\vartheta C_2}{\vartheta q} = 0 &= 1-[0.5\log_2(0.5+0.5q)+(0.5+0.5q)\frac{0.5}{0.5+0.5q}\frac{1}{\ln 2}]\\
&\quad -[-0.5\log_2(0.5-0.5q)+(0.5-0.5q)\frac{-0.5}{0.5-0.5q}\frac{1}{\ln 2}]\\
&= 1+0.5\log_2(0.5-0.5q)-0.5\log_2(0.5+0.5q)
\end{aligned}$$

Therefore,

$$\log_2\frac{0.5-0.5q}{0.5+0.5q} = -2 \Longrightarrow q = \frac{3}{5}$$

and the channel capacity is

$$C_2 = h(\frac{1}{5})-\frac{2}{5} = 0.3219$$

3) The transition probability matrix of the third channel can be written as

$$\mathbf{Q} = \frac{1}{2}\mathbf{Q}_1+\frac{1}{2}\mathbf{Q}_2$$

where $\mathbf{Q}_1$, $\mathbf{Q}_2$ are the transition probability matrices of channel 1 and channel 2 respectively. We have assumed that the output space of both channels has been augmented by adding two new symbols so that the size of the matrices $\mathbf{Q}$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ is the same. The transition probabilities to these newly added output symbols is equal to zero. However, using the results of Problem 6.34 we obtain

$$\begin{aligned}
C &= \max_{\mathbf{p}} I(X;Y) = \max_{\mathbf{p}} I(\mathbf{p};\mathbf{Q})\\
&= \max_{\mathbf{p}} I(\mathbf{p};\frac{1}{2}\mathbf{Q}_1+\frac{1}{2}\mathbf{Q}_2)\\
&\leq \frac{1}{2}\max_{\mathbf{p}} I(\mathbf{p};\mathbf{Q}_1)+\frac{1}{2}\max_{\mathbf{p}} I(\mathbf{p};\mathbf{Q}_2)\\
&= \frac{1}{2}C_1+\frac{1}{2}C_2
\end{aligned}$$

Since $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are different, the inequality is strict. Hence,

$$C < \frac{1}{2}C_1+\frac{1}{2}C_2$$

Problem 9.8

The capacity of a channel is

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)}[H(Y)-H(Y|X)] = \max_{p(x)}[H(X)-H(X|Y)]$$

Since in general $H(X|Y) \geq 0$ and $H(Y|X) \geq 0$, we obtain

$$C \leq \min\{\max[H(Y)], \max[H(X)]\}$$

However, the maximum of $H(X)$ is attained when X is uniformly distributed, in which case $\max[H(X)] = \log|\mathcal{X}|$. Similarly $\max[H(Y)] = \log|\mathcal{Y}|$ and by substituting in the previous inequality, we obtain

$$\begin{aligned} C &\leq \min\{\max[H(Y)], \max[H(X)]\} = \min\{\log|\mathcal{Y}|, \log|\mathcal{X}|\} \\ &= \min\{\log M, \log N\} \end{aligned}$$

Problem 9.9

1) Let q be the probability of the input symbol 0, and therefore $(1-q)$ the probability of the input symbol 1. Then,

$$\begin{aligned} H(Y|X) &= \sum_x p(x)H(Y|X=x) \\ &= qH(Y|X=0) + (1-q)H(Y|X=1) \\ &= (1-q)H(Y|X=1) = (1-q)h(\epsilon) \end{aligned}$$

The probability mass function of the output symbols is

$$\begin{aligned} p(Y=0) &= qp(Y=0|X=0) + (1-q)p(Y=0|X=1) \\ &= q + (1-q)(1-\epsilon) = 1 - \epsilon + q\epsilon \\ p(Y=1) &= (1-q)\epsilon = \epsilon - q\epsilon \end{aligned}$$

Hence,

$$C = \max_q [h(\epsilon - q\epsilon) - (1-q)h(\epsilon)]$$

To find the probability q that achieves the maximum, we set the derivative of C with respect to q equal to 0. Thus,

$$\frac{\vartheta C}{\vartheta q} = 0 = h(\epsilon) + \epsilon \log_2(\epsilon - q\epsilon) - \epsilon \log_2(1 - \epsilon + q\epsilon)$$

Therefore,

$$\log_2 \frac{\epsilon - q\epsilon}{1 - \epsilon + q\epsilon} = -\frac{h(\epsilon)}{\epsilon} \Longrightarrow q = \frac{\epsilon + 2^{-\frac{h(\epsilon)}{\epsilon}}(\epsilon - 1)}{\epsilon(1 + 2^{-\frac{h(\epsilon)}{\epsilon}})}$$

and the channel capacity is

$$C = h\left(\frac{2^{-\frac{h(\epsilon)}{\epsilon}}}{1 + 2^{-\frac{h(\epsilon)}{\epsilon}}}\right) - \frac{h(\epsilon)2^{-\frac{h(\epsilon)}{\epsilon}}}{\epsilon(1 + 2^{-\frac{h(\epsilon)}{\epsilon}})}$$

2) If $\epsilon \to 0$, then using L'Hospital's rule we find that

$$\lim_{\epsilon \to 0} \frac{h(\epsilon)}{\epsilon} = \infty, \qquad \lim_{\epsilon \to 0} \frac{h(\epsilon)}{\epsilon} 2^{-\frac{h(\epsilon)}{\epsilon}} = 0$$

and therefore

$$\lim_{\epsilon \to 0} C(\epsilon) = h(0) = 0$$

If $\epsilon = 0.5$, then $h(\epsilon) = 1$ and $C = h(\frac{1}{5}) - \frac{2}{5} = 0.3219$. In this case the probability of the input symbol 0 is

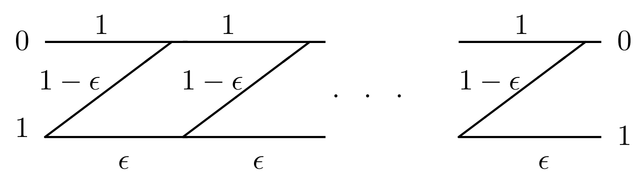
$$q = \frac{\epsilon + 2^{-\frac{h(\epsilon)}{\epsilon}}(\epsilon - 1)}{\epsilon(1 + 2^{-\frac{h(\epsilon)}{\epsilon}})} = \frac{0.5 + 0.25 \times (0.5 - 1)}{0.5 \times (1 + 0.25)} = \frac{3}{5}$$

If $\epsilon = 1$, then $C = h(0.5) = 1$. The input distribution that achieves capacity is $p(0) = p(1) = 0.5$.

3) The following figure shows the topology of the cascade channels. If we start at the input labelled 0, then the output will be 0. If however we transmit a 1, then the output will be zero with probability

$$\begin{aligned} p(Y=0|X=1) &= (1-\epsilon) + \epsilon(1-\epsilon) + \epsilon^2(1-\epsilon) + \cdots \\ &= (1-\epsilon)(1+\epsilon+\epsilon^2+\cdots) \\ &= 1-\epsilon\frac{1-\epsilon^n}{1-\epsilon} = 1-\epsilon^n \end{aligned}$$

Thus, the resulting system is equivalent to a Z channel with $\epsilon_1 = \epsilon^n$.



4) As $n \to \infty$, $\epsilon^n \to 0$ and the capacity of the channel goes to 0.

Problem 9.10
The capacity of Channel A satisfies (see Problem 9.8)

$$C_A \leq \min\{\log_2 M, \log_2 N\}$$

where M, N is the size of the output and input alphabet respectively. Since $M = 2 < 3 = N$, we conclude that $C_A \leq \log_2 2 = 1$. With input distribution $p(A) = p(B) = 0.5$ and $p(C) = 0$, we have a noiseless channel, therefore $C_A = 1$. Similarly, we find that $C_B = 1$, which is achieved when

$$p(a') = p(b') = 0.5,$$

achieved when interpreting B' and C' as a single output. Therefore, the capacity of the cascade channel is $C_{AB} = 1$.

Problem 9.11
The SNR is

$$\text{SNR} = \frac{2P}{N_0 2W} = \frac{P}{2W} = \frac{10}{10^{-9}\times 10^6} = 10^4$$

Thus the capacity of the channel is

$$C = W\log_2(1+\frac{P}{N_0 W}) = 10^6\log_2(1+10000) \approx 13.2879\times 10^6 \text{ bits/sec}$$

Problem 9.12
The capacity of the additive white Gaussian channel is

$$C = \frac{1}{2}\log\left(1+\frac{P}{N_0 W}\right)$$

For the nonwhite Gaussian noise channel, although the noise power is equal to the noise power in the white Gaussian noise channel, the capacity is higher, The reason is that since noise samples are correlated, knowledge of the previous noise samples provides partial information on the future noise samples and therefore reduces their effective variance.

Problem 9.13

1) The capacity of the binary symmetric channel with crossover probability ϵ is

$$C = 1 - h(\epsilon)$$

where $h(\epsilon)$ is the binary entropy function. The rate distortion function of a zero mean Gaussian source with variance σ^2 per sample is

$$R(D) = \begin{cases} \frac{1}{2}\log_2 \frac{\sigma^2}{D} & D \leq \sigma^2 \\ 0 & D > \sigma^2 \end{cases}$$

Since $C > 0$, we obtain

$$\frac{1}{2}\log_2 \frac{\sigma^2}{D} \leq 1 - h(\epsilon) \Longrightarrow \frac{\sigma^2}{2^{2(1-h(\epsilon))}} \leq D$$

and therefore, the minimum value of the distortion attainable at the output of the channel is

$$D_{\min} = \frac{\sigma^2}{2^{2(1-h(\epsilon))}}$$

2) The capacity of the additive Gaussian channel is

$$C = \frac{1}{2}\log_2\left(1 + \frac{P}{\sigma_n^2}\right)$$

Hence,

$$\frac{1}{2}\log_2 \frac{\sigma^2}{D} \leq C \Longrightarrow \frac{\sigma^2}{2^{2C}} \leq D \Longrightarrow \frac{\sigma^2}{1 + \frac{P}{\sigma_n^2}} \leq D$$

The minimum attainable distortion is

$$D_{\min} = \frac{\sigma^2}{1 + \frac{P}{\sigma_n^2}}$$

3) Here the source samples are dependent and therefore one sample provides information about the other samples. This means that we can achieve better results compared to the memoryless case at a given rate. In other words the distortion at a given rate for a source with memory is less than the distortion for a comparable source with memory. Differential coding methods discussed in Chapter 4 are suitable for such sources.

Problem 9.14

The capacity of the channel of the channel is given by

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)}[H(Y) - H(Y|X)]$$

Let the probability of the inputs C, B and A be p, q and $1-p-q$ respectively. From the symmetry of the nodes B, C we expect that the optimum distribution $p(x)$ will satisfy $p(B) = p(C) = p$. The entropy $H(Y|X)$ is given by

$$\begin{aligned} H(Y|X) &= \sum p(x)H(Y|X=x) = (1-2p)H(Y|X=A) + 2pH(Y|X=B) \\ &= 0 + 2ph(0.5) = 2p \end{aligned}$$

The probability mass function of the output is

$$\begin{aligned} p(Y=1) &= \sum p(x)p(Y=1|X=x) = (1-2p) + p = 1-p \\ p(Y=2) &= \sum p(x)p(Y=2|X=x) = 0.5p + 0.5p = p \end{aligned}$$

Therefore,

$$C = \max_p [H(Y) - H(Y|X)] = \max_p (h(p) - 2p)$$

To find the optimum value of p that maximizes $I(X;Y)$, we set the derivative of C with respect to p equal to zero. Thus,

$$\begin{aligned}
\frac{\vartheta C}{\vartheta p} = 0 &= -\log_2(p) - p\frac{1}{p\ln(2)} + \log_2(1-p) - (1-p)\frac{-1}{(1-p)\ln(2)} - 2 \\
&= \log_2(1-p) - \log_2(p) - 2
\end{aligned}$$

and therefore

$$\log_2 \frac{1-p}{p} = 2 \Longrightarrow \frac{1-p}{p} = 4 \Longrightarrow p = \frac{1}{5}$$

The capacity of the channel is

$$C = h(\frac{1}{5}) - \frac{2}{5} = 0.7219 - 0.4 = 0.3219 \text{ bits/transmission}$$

Problem 9.15

The capacity of the "product" channel is given by

$$C = \max_{p(x_1,x_2)} I(X_1X_2;Y_1Y_2)$$

However,

$$\begin{aligned}
I(X_1X_2;Y_1Y_2) &= H(Y_1Y_2) - H(Y_1Y_2|X_1X_2) \\
&= H(Y_1Y_2) - H(Y_1|X_1) - H(Y_2|X_2) \\
&\leq H(Y_1) + H(Y_2) - H(Y_1|X_1) - H(Y_2|X_2) \\
&= I(X_1;Y_1) + I(X_2;Y_2)
\end{aligned}$$

and therefore,

$$\begin{aligned}
C = \max_{p(x_1,x_2)} I(X_1X_2;Y_1Y_2) &\leq \max_{p(x_1,x_2)} [I(X_1;Y_1) + I(X_2;Y_2)] \\
&\leq \max_{p(x_1)} I(X_1;Y_1) + \max_{p(x_2)} I(X_2;Y_2) \\
&= C_1 + C_2
\end{aligned}$$

The upper bound is achievable by choosing the input joint probability density $p(x_1,x_2)$, in such a way that

$$p(x_1,x_2) = \tilde{p}(x_1)\tilde{p}(x_2)$$

where $\tilde{p}(x_1)$, $\tilde{p}(x_2)$ are the input distributions that achieve the capacity of the first and second channel respectively.

Problem 9.16

1) Let $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$, $\mathcal{Y} = \mathcal{Y}_1 + \mathcal{Y}_2$ and

$$p(y|x) = \begin{cases} p(y_1|x_1) & \text{if } x \in \mathcal{X}_1 \\ p(y_2|x_2) & \text{if } x \in \mathcal{X}_2 \end{cases}$$

the conditional probability density function of $\mathcal{Y}$ and $\mathcal{X}$. We define a new random variable M taking the values $1, 2$ depending on the index i of $\mathcal{X}$. Note that M is a function of X or Y. This is because $\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$ and therefore, knowing X we know the channel used for transmission. The capacity of the sum channel is

$$\begin{aligned}
C &= \max_{p(x)} I(X;Y) = \max_{p(x)}[H(Y) - H(Y|X)] = \max_{p(x)}[H(Y) - H(Y|X,M)] \\
&= \max_{p(x)}[H(Y) - p(M=1)H(Y|X,M=1) - p(M=2)H(Y|X,M=2)] \\
&= \max_{p(x)}[H(Y) - \lambda H(Y_1|X_1) - (1-\lambda)H(Y_2|X_2)]
\end{aligned}$$

where $\lambda = p(M=1)$. Also,

$$\begin{aligned}
H(Y) &= H(Y,M) = H(M) + H(Y|M) \\
&= H(\lambda) + \lambda H(Y_1) + (1-\lambda)H(Y_2)
\end{aligned}$$

Substituting $H(Y)$ in the previous expression for the channel capacity, we obtain

$$\begin{aligned}
C &= \max_{p(x)} I(X;Y) \\
&= \max_{p(x)}[H(\lambda) + \lambda H(Y_1) + (1-\lambda)H(Y_2) - \lambda H(Y_1|X_1) - (1-\lambda)H(Y_2|X_2)] \\
&= \max_{p(x)}[H(\lambda) + \lambda I(X_1;Y_1) + (1-\lambda)I(X_2;Y_2)]
\end{aligned}$$

Since $p(x)$ is function of λ, $p(x_1)$ and $p(x_2)$, the maximization over $p(x)$ can be substituted by a joint maximization over λ, $p(x_1)$ and $p(x_2)$. Furthermore, since λ and $1-\lambda$ are nonnegative, we let $p(x_1)$ to maximize $I(X_1;Y_1)$ and $p(x_2)$ to maximize $I(X_2;Y_2)$. Thus,

$$C = \max_{\lambda}[H(\lambda) + \lambda C_1 + (1-\lambda)C_2]$$

To find the value of λ that maximizes C, we set the derivative of C with respect to λ equal to zero. Hence,

$$\frac{dC}{d\lambda} = 0 = -\log_2(\lambda) + \log_2(1-\lambda) + C_1 - C_2 \Longrightarrow \lambda = \frac{2^{C_1}}{2^{C_1} + 2^{C_2}}$$

Substituting this value of λ in the expression for C, we obtain

$$\begin{aligned}
C &= H\left(\frac{2^{C_1}}{2^{C_1}+2^{C_2}}\right) + \frac{2^{C_1}}{2^{C_1}+2^{C_2}}C_1 + \left(1 - \frac{2^{C_1}}{2^{C_1}+2^{C_2}}\right)C_2 \\
&= -\frac{2^{C_1}}{2^{C_1}+2^{C_2}}\log_2\left(\frac{2^{C_1}}{2^{C_1}+2^{C_2}}\right) - \left(1 - \frac{2^{C_1}}{2^{C_1}+2^{C_2}}\right)\log_2\left(\frac{2^{C_1}}{2^{C_1}+2^{C_2}}\right) \\
&\quad + \frac{2^{C_1}}{2^{C_1}+2^{C_2}}C_1 + \left(1 - \frac{2^{C_1}}{2^{C_1}+2^{C_2}}\right)C_2 \\
&= \frac{2^{C_1}}{2^{C_1}+2^{C_2}}\log_2(2^{C_1}+2^{C_2}) + \frac{2^{C_2}}{2^{C_1}+2^{C_2}}\log_2(2^{C_1}+2^{C_2}) \\
&= \log_2(2^{C_1}+2^{C_2})
\end{aligned}$$

Hence

$$C = \log_2(2^{C_1}+2^{C_2}) \Longrightarrow 2^C = 2^{C_1} + 2^{C_2}$$

2)

$$2^C = 2^0 + 2^0 = 2 \Longrightarrow C = 1$$

Thus, the capacity of the sum channel is nonzero although the component channels have zero capacity. In this case the information is transmitted through the process of selecting a channel.

3) The channel can be considered as the sum of two channels. The first channel has capacity $C_1 = \log_2 1 = 0$ and the second channel is BSC with capacity $C_2 = 1 - h(0.5) = 0$. Thus

$$C = \log_2(2^{C_1} + 2^{C_2}) = \log_2(2) = 1$$

Problem 9.17

1) The entropy of the source is

$$H(X) = h(0.3) = 0.8813$$

and the capacity of the channel

$$C = 1 - h(0.1) = 1 - 0.469 = 0.531$$

If the source is directly connected to the channel, then the probability of error at the destination is

$$\begin{aligned} P(\text{error}) &= p(X=0)p(Y=1|X=0) + p(X=1)p(Y=0|X=1) \\ &= 0.3 \times 0.1 + 0.7 \times 0.1 = 0.1 \end{aligned}$$

2) Since $H(X) > C$, some distortion at the output of the channel is inevitable. To find the minimum distortion we set $R(D) = C$. For a Bernoulli type of source

$$R(D) = \begin{cases} h(p) - h(D) & 0 \leq D \leq \min(p, 1-p) \\ 0 & \text{otherwise} \end{cases}$$

and therefore, $R(D) = h(p) - h(D) = h(0.3) - h(D)$. If we let $R(D) = C = 0.531$, we obtain

$$h(D) = 0.3503 \Longrightarrow D = \min(0.07, 0.93) = 0.07$$

The probability of error is

$$P(\text{error}) \leq D = 0.07$$

3) For reliable transmission we must have $H(X) = C = 1 - h(\epsilon)$. Hence, with $H(X) = 0.8813$ we obtain

$$0.8813 = 1 - h(\epsilon) \Longrightarrow \epsilon < 0.016 \text{ or } \epsilon > 0.984$$

Problem 9.18

1) The rate-distortion function of the Gaussian source for $D \leq \sigma^2$ is

$$R(D) = \frac{1}{2}\log_2 \frac{\sigma^2}{D}$$

Hence, with $\sigma^2 = 4$ and $D = 1$, we obtain

$$R(D) = \frac{1}{2}\log_2 4 = 1 \text{ bits/sample } = 8000 \text{ bits/sec}$$

The capacity of the channel is

$$C = W \log_2\left(1 + \frac{P}{N_0 W}\right)$$

In order to accommodate the rate $R = 8000$ bps, the channel capacity should satisfy

$$R(D) \leq C \Longrightarrow R(D) \leq 4000 \log_2(1 + \text{SNR})$$

Therefore,

$$\log_2(1 + \text{SNR}) \geq 2 \Longrightarrow \text{SNR}_{\min} = 3$$

2) The error probability for each bit is

$$p_b = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

and therefore, the capacity of the BSC channel is

$$\begin{aligned} C &= 1 - h(p_b) = 1 - h\left(Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]\right) \text{ bits/transmission} \\ &= 2 \times 4000 \times \left[1 - h\left(Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]\right)\right] \text{ bits/sec} \end{aligned}$$

In this case, the condition $R(D) \leq C$ results in

$$1 \leq 1 - h(p_b) \Longrightarrow Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = 0 \qquad \text{or} \qquad \text{SNR} = \frac{E_b}{N_0} \to \infty$$

Problem 9.19

1) The maximum distortion in the compression of the source is

$$D_{\max} = \sigma^2 = \int_{-\infty}^{\infty} \mathcal{S}_x(f)df = 2\int_{-10}^{10} df = 40$$

2) The rate-distortion function of the source is

$$R(D) = \begin{cases} \frac{1}{2}\log_2 \frac{\sigma^2}{D} & 0 \leq D \leq \sigma^2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2}\log_2 \frac{40}{D} & 0 \leq D \leq 40 \\ 0 & \text{otherwise} \end{cases}$$

3) With $D = 10$, we obtain

$$R = \frac{1}{2}\log_2 \frac{40}{10} = \frac{1}{2}\log_2 4 = 1$$

Thus, the required rate is $R = 1$ bit per sample or, since the source can be sampled at a rate of 20 samples per second, the rate is $R = 20$ bits per second.

4) The capacity-cost function is

$$C(P) = \frac{1}{2}\log_2\left(1 + \frac{P}{N}\right)$$

where,

$$N = \int_{-\infty}^{\infty} \mathcal{S}_n(f)df = \int_{-4}^{4} df = 8$$

Hence,

$$C(P) = \frac{1}{2}\log_2(1 + \frac{P}{8}) \text{ bits/transmission} = 4\log_2(1 + \frac{P}{8}) \text{ bits/sec}$$

The required power such that the source can be transmitted via the channel with a distortion not exceeding 10, is determined by $R(10) \leq C(P)$. Hence,

$$20 \leq 4\log_2(1 + \frac{P}{8}) \Longrightarrow P = 8 \times 31 = 248$$

Problem 9.20

The differential entropy of the Laplacian noise is (see Problem 6.36)

$$h(Z) = 1 + \ln \lambda$$

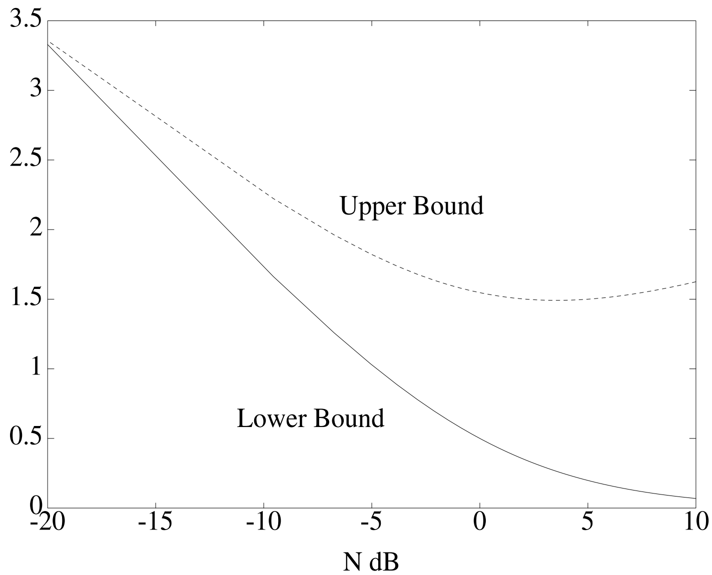
where λ is the mean of the Laplacian distribution, that is

$$E[Z] = \int_0^\infty zp(z)dz = \int_0^\infty z\frac{1}{\lambda}e^{-\frac{z}{\lambda}}dz = \lambda$$

The variance of the noise is

$$N = E[(Z-\lambda)^2] = E[Z^2] - \lambda^2 = \int_0^\infty z^2\frac{1}{\lambda}e^{-\frac{z}{\lambda}}dz - \lambda^2 = 2\lambda^2 - \lambda^2 = \lambda^2$$

In the next figure we plot the lower and upper bound of the capacity of the channel as a function of λ^2 and for $P = 1$. As it is observed the bounds are tight for high SNR, small N, but they become loose as the power of the noise increases.



Problem 9.21

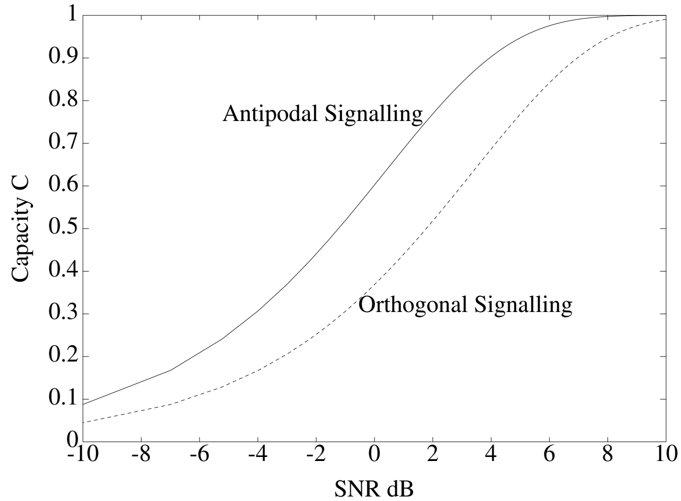
Both channels can be viewed as binary symmetric channels with crossover probability the probability of decoding a bit erroneously. Since,

$$p_b = \begin{cases} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] & \text{antipodal signaling} \\ Q\left[\sqrt{\frac{\mathcal{E}_b}{N_0}}\right] & \text{orthogonal signaling} \end{cases}$$

the capacity of the channel is

$$C = \begin{cases} 1 - h\left(Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]\right) & \text{antipodal signaling} \\ 1 - h\left(Q\left[\sqrt{\frac{\mathcal{E}_b}{N_0}}\right]\right) & \text{orthogonal signaling} \end{cases}$$

In the next figure we plot the capacity of the channel as a function of $\frac{\mathcal{E}_b}{N_0}$ for the two signaling schemes.



Problem 9.22
The codewords of the linear code of Example 9.5.1 are

$$\begin{aligned}\mathbf{c}_1 &= [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ]\\ \mathbf{c}_2 &= [\ 1\ \ 0\ \ 1\ \ 0\ \ 0\ ]\\ \mathbf{c}_3 &= [\ 0\ \ 1\ \ 1\ \ 1\ \ 1\ ]\\ \mathbf{c}_4 &= [\ 1\ \ 1\ \ 0\ \ 1\ \ 1\ ]\end{aligned}$$

Since the code is linear the minimum distance of the code is equal to the minimum weight of the codewords. Thus,

$$d_{\min} = w_{\min} = 2$$

There is only one codeword with weight equal to 2 and this is $\mathbf{c}_2$.

Problem 9.23
The parity check matrix of the code in Example 9.5.3 is

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The codewords of the code are

$$\begin{aligned}\mathbf{c}_1 &= [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ]\\ \mathbf{c}_2 &= [\ 1\ \ 0\ \ 1\ \ 0\ \ 0\ ]\\ \mathbf{c}_3 &= [\ 0\ \ 1\ \ 1\ \ 1\ \ 1\ ]\\ \mathbf{c}_4 &= [\ 1\ \ 1\ \ 0\ \ 1\ \ 1\ ]\end{aligned}$$

Any of the previous codewords when postmultiplied by $\mathbf{H}^t$ produces an all-zero vector of length 3. For example

$$\begin{aligned}\mathbf{c}_2\mathbf{H}^t &= [\ 1 \oplus 1\ \ \ 0\ \ \ 0\ ] = [\ 0\ \ 0\ \ 0\ ]\\ \mathbf{c}_4\mathbf{H}^t &= [\ 1 \oplus 1\ \ \ 1 \oplus 1\ \ \ 1 \oplus 1\ ] = [\ 0\ \ 0\ \ 0\ ]\end{aligned}$$

Problem 9.24

The following table lists all the codewords of the (7,4) Hamming code along with their weight. Since the Hamming codes are linear $d_{\min} = w_{\min}$. As it is observed from the table the minimum weight is 3 and therefore $d_{\min} = 3$.

No.	Codewords	Weight
1	0000000	0
2	1000110	3
3	0100011	3
4	0010101	3
5	0001111	4
6	1100101	4
7	1010011	4
8	1001001	3
9	0110110	4
10	0101100	3
11	0011010	3
12	1110000	3
13	1101010	4
14	1011100	4
15	0111001	4
16	1111111	7

Problem 9.25

The parity check matrix $\mathbf{H}$ of the (15,11) Hamming code consists of all binary sequences of length 4, except the all zero sequence. The systematic form of the matrix $\mathbf{H}$ is

$$\mathbf{H} = [\ \mathbf{P}^t \quad | \quad \mathbf{I}_4\ ] = \left(\begin{array}{ccccccccccc|cccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

The corresponding generator matrix is

$$\mathbf{G} = [\ \mathbf{I}_{11} \quad | \quad \mathbf{P}\ ] = \left(\begin{array}{ccccccccccc|cccc} 1 & & & & & & & & & & & 1 & 1 & 0 & 0 \\ & 1 & & & & & & & & & & 1 & 0 & 1 & 0 \\ & & 1 & & & & & & \mathbf{0} & & & 1 & 0 & 0 & 1 \\ & & & 1 & & & & & & & & 0 & 1 & 1 & 0 \\ & & & & 1 & & & & & & & 0 & 1 & 0 & 1 \\ & & & & & 1 & & & & & & 0 & 0 & 1 & 1 \\ & & & & & & 1 & & & & & 1 & 1 & 1 & 0 \\ & & & & & & & 1 & & & & 1 & 1 & 0 & 1 \\ & & & \mathbf{0} & & & & & 1 & & & 1 & 0 & 1 & 1 \\ & & & & & & & & & 1 & & 0 & 1 & 1 & 1 \\ & & & & & & & & & & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

Problem 9.26

Let C be an (n, k) linear block code with parity check matrix $\mathbf{H}$. We can express the parity check matrix in the form

$$\mathbf{H} = [\ \mathbf{h}_1 \quad \mathbf{h}_2 \quad \cdots \quad \mathbf{h}_n\ ]$$

where $\mathbf{h}_i$ is an $n-k$ dimensional column vector. Let $\mathbf{c} = [c_1 \cdots c_n]$ be a codeword of the code C with l nonzero elements which we denote as c_{i_1}, c_{i_2}, …, c_{i_l}. Clearly $c_{i_1} = c_{i_2} = \ldots = c_{i_l} = 1$ and since $\mathbf{c}$ is a codeword

$$\begin{aligned}\mathbf{c}\mathbf{H}^t = 0 &= c_1\mathbf{h}_1 + c_2\mathbf{h}_2 + \cdots + c_n\mathbf{h}_n \\ &= c_{i_1}\mathbf{h}_{i_1} + c_{i_2}\mathbf{h}_{i_2} + \cdots + c_{i_l}\mathbf{h}_{i_l} \\ &= \mathbf{h}_{i_1} + \mathbf{h}_{i_2} + \cdots + \mathbf{h}_{i_l} = 0\end{aligned}$$

This proves that l column vectors of the matrix $\mathbf{H}$ are linear dependent. Since for a linear code the minimum value of l is $w_{\min}$ and $w_{\min} = d_{\min}$, we conclude that there exist $d_{\min}$ linear dependent column vectors of the matrix $\mathbf{H}$.

Now we assume that the minimum number of column vectors of the matrix $\mathbf{H}$ that are linear dependent is $d_{\min}$ and we will prove that the minimum weight of the code is $d_{\min}$. Let $\mathbf{h}_{i_1}$, $\mathbf{h}_{i_2}$, …, $\mathbf{h}_{d_{\min}}$ be a set of linear dependent column vectors. If we form a vector $\mathbf{c}$ with non-zero components at positions i_1, i_2, …, $i_{d_{\min}}$, then

$$\mathbf{c}\mathbf{H}^t = c_{i_1}\mathbf{h}_{i_1} + \cdots + c_{i_{d_{\min}}} = 0$$

which implies that $\mathbf{c}$ is a codeword with weight $d_{\min}$. Therefore, the minimum distance of a code is equal to the minimum number of columns of its parity check matrix that are linear dependent.

For a Hamming code the columns of the matrix $\mathbf{H}$ are non-zero and distinct. Thus, no two columns $\mathbf{h}_i$, $\mathbf{h}_j$ add to zero and since $\mathbf{H}$ consists of all the $n-k$ tuples as its columns, the sum $\mathbf{h}_i + \mathbf{h}_j = \mathbf{h}_m$ should also be a column of $\mathbf{H}$. Then,

$$\mathbf{h}_i + \mathbf{h}_j + \mathbf{h}_m = 0$$

and therefore the minimum distance of the Hamming code is 3.

Problem 9.27

The generator matrix of the $(n, 1)$ repetition code is a $1 \times n$ matrix, consisted of the non-zero codeword. Thus,

$$\mathbf{G} = \left[\begin{array}{c|ccc} 1 & 1 & \cdots & 1 \end{array}\right]$$

This generator matrix is already in systematic form, so that the parity check matrix is given by

$$\mathbf{H} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{array}\right)$$

Problem 9.28

1) The parity check matrix $\mathbf{H}_e$ of the extended code is an $(n+1-k) \times (n+1)$ matrix. The codewords of the extended code have the form

$$\mathbf{c}_{e,i} = [\ \mathbf{c}_i \ \ | \ \ x\ ]$$

where x is 0 if the weight of $\mathbf{c}_i$ is even and 1 if the weight of $\mathbf{c}_i$ is odd. Since $\mathbf{c}_{e,i}\mathbf{H}_e^t = [\mathbf{c}_i|x]\mathbf{H}_e^t = 0$ and $\mathbf{c}_i\mathbf{H}^t = 0$, the first $n-k$ columns of $\mathbf{H}_e^t$ can be selected as the columns of $\mathbf{H}^t$ with a zero added in the last row. In this way the choice of x is immaterial. The last column of $\mathbf{H}_e^t$ is selected in such a way that the even-parity condition is satisfied for every codeword $\mathbf{c}_{e,i}$. Note that if $\mathbf{c}_{e,i}$ has even weight, then

$$c_{e,i_1} + c_{e,i_2} + \cdots + c_{e,i_{n+1}} = 0 \Longrightarrow \mathbf{c}_{e,i}[\ 1 \ \ 1 \ \ \cdots \ \ 1\ ]^t = 0$$

for every i. Therefore the last column of $\mathbf{H}_e^t$ is the all-one vector and the parity check matrix of the extended code has the form

$$\mathbf{H}_e = \left(\mathbf{H}_e^t\right)^t = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ & & & \\ 0 & 0 & 0 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

2) The original code has minimum distance equal to 3. But for those codewords with weight equal to the minimum distance, a 1 is appended at the end of the codewords to produce even parity. Thus, the minimum weight of the extended code is 4 and since the extended code is linear, the minimum distance is $d_{e,\min} = w_{e,\min} = 4$.

3) The coding gain of the extended code is

$$G_{\text{coding}} = d_{e,min} R_c = 4 \times \frac{3}{7} = 1.7143$$

Problem 9.29

If no coding is employed, we have

$$p_b = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\frac{P}{RN_0}}\right]$$

where

$$\frac{P}{RN_0} = \frac{10^{-6}}{10^4 \times 2 \times 10^{-11}} = 5$$

Thus,

$$p_b = Q[\sqrt{5}] = 1.2682 \times 10^{-2}$$

and therefore, the error probability for 11 bits is

$$P_{\text{error in 11 bits}} = 1 - (1 - p_b)^{11} \approx 0.1310$$

If coding is employed, then since the minimum distance of the $(15, 11)$ Hamming code is 3,

$$p_e \leq (M-1)Q\left[\sqrt{\frac{d_{\min}\mathcal{E}_s}{N_0}}\right] = 10Q\left[\sqrt{\frac{3\mathcal{E}_s}{N_0}}\right]$$

where

$$\frac{\mathcal{E}_s}{N_0} = R_c\frac{\mathcal{E}_b}{N_0} = R_c\frac{P}{RN_0} = \frac{11}{15} \times 5 = 3.6667$$

Thus

$$p_e \leq 10Q\left[\sqrt{3 \times 3.6667}\right] \approx 4.560 \times 10^{-3}$$

As it is observed the probability of error decreases by a factor of 28. If hard decision is employed, then

$$p_e \leq (M-1) \sum_{i=\frac{d_{\min}+1}{2}}^{d_{\min}} \begin{pmatrix} d_{\min} \\ i \end{pmatrix} p_b^i (1-p_b)^{d_{\min}-i}$$

where $M = 10$, $d_{\min} = 3$ and $p_b = Q\left[\sqrt{R_c \frac{P}{RN_0}}\right] = 2.777 \times 10^{-2}$. Hence,

$$p_e = 10 \times (3 \times p_b^2(1-p_b) + p_b^3) = 0.0227$$

In this case coding has decreased the error probability by a factor of 6.

Problem 9.30

The following table shows the standard array for the (7,4) Hamming code.

		$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
		1000000	0100000	0010000	0001000	0000100	0000010	0000001
$\mathbf{c}_1$	0000000	1000000	0100000	0010000	0001000	0000100	0000010	0000001
$\mathbf{c}_2$	1000110	0000110	1100110	1010110	1001110	1000010	1000100	1000111
$\mathbf{c}_3$	0100011	1100011	0000011	0110011	0101011	0100111	0100001	0100010
$\mathbf{c}_4$	0010101	1010101	0110101	0000101	0011101	0010001	0010111	0010100
$\mathbf{c}_5$	0001111	1001111	0101111	0011111	0000111	0001011	0001101	0001110
$\mathbf{c}_6$	1100101	0100101	1000101	1110101	1101101	1100001	1100111	1100100
$\mathbf{c}_7$	1010011	0010011	1110011	1000011	1011011	1010111	1010001	1010010
$\mathbf{c}_8$	1001001	0001001	1101001	1011001	1000001	1001101	1001011	1001000
$\mathbf{c}_9$	0110110	1110110	0010110	0100110	0111110	0110010	0110100	0110111
$\mathbf{c}_{10}$	0101100	1101100	0001100	0111100	0100100	0101000	0101110	0101101
$\mathbf{c}_{11}$	0011010	1011010	0111010	0001010	0010010	0011110	0011000	0011011
$\mathbf{c}_{12}$	1110000	0110000	1010000	1100000	1111000	1110100	1110010	1110001
$\mathbf{c}_{13}$	1101010	0101010	1001010	1111010	1100010	1101110	1101000	1101011
$\mathbf{c}_{14}$	1011100	0011100	1111100	1001100	1010100	1011000	1011110	1011101
$\mathbf{c}_{15}$	0111001	1111001	0011001	0101001	0110001	0111101	0111011	0111000
$\mathbf{c}_{16}$	1111111	0111111	1011111	1101111	1110111	1111011	1111101	1111110

As it is observed the received vector $\mathbf{y} = [1110100]$ is in the 7^{th} column of the table under the error vector $\mathbf{e}_5$. Thus, the received vector will be decoded as

$$\mathbf{c} = \mathbf{y} + \mathbf{e}_5 = [\begin{array}{ccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}] = \mathbf{c}_{12}$$

Problem 9.31

The generator polynomial of degree $m = n - k$ should divide the polynomial $p^6 + 1$. Since the polynomial $p^6 + 1$ assumes the factorization

$$p^6 + 1 = (p+1)^3(p+1)^3 = (p+1)(p+1)(p^2+p+1)(p^2+p+1)$$

we observe that $m = n - k$ can take any value from 1 to 5. Thus, $k = n - m$ can be any number in $[1, \quad 5]$. The following table lists the possible values of k and the corresponding generator polynomial(s).

k	$g(p)$
1	$p^5 + p^4 + p^3 + p^2 + p + 1$
2	$p^4 + p^2 + 1$ or $p^4 + p^3 + p + 1$
3	$p^3 + 1$
4	$p^2 + 1$ or $p^2 + p + 1$
5	$p + 1$

Problem 9.32

To generate a (7,3) cyclic code we need a generator polynomial of degree $7 - 3 = 4$. Since (see Example 9.6.2))

$$\begin{aligned} p^7 + 1 &= (p+1)(p^3+p^2+1)(p^3+p+1) \\ &= (p^4+p^2+p+1)(p^3+p+1) \\ &= (p^3+p^2+1)(p^4+p^3+p^2+1) \end{aligned}$$

either one of the polynomials p^4+p^2+p+1, $p^4+p^3+p^2+1$ can be used as a generator polynomial. With $g(p) = p^4+p^2+p+1$ all the codeword polynomials $c(p)$ can be written as

$$c(p) = X(p)g(p) = X(p)(p^4+p^2+p+1)$$

where $X(p)$ is the message polynomial. The following table shows the input binary sequences used to represent $X(p)$ and the corresponding codewords.

Input	$X(p)$	$c(p) = X(p)g(p)$	Codeword
000	0	0	0000000
001	1	p^4+p^2+p+1	0010111
010	p	$p^5+p^3+p^2+p$	0101110
100	p^2	$p^6+p^4+p^3+p^2$	1011100
011	$p+1$	$p^5+p^4+p^3+1$	0111001
101	p^2+1	p^6+p^3+p+1	1001011
110	p^2+p	$p^6+p^5+p^4+p$	1110010
111	p^2+p+1	$p^6+p^5+p^2+1$	1100101

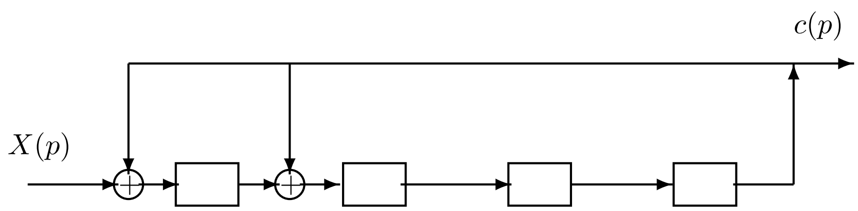
Since the cyclic code is linear and the minimum weight is $w_{\min} = 4$, we conclude that the minimum distance of the (7,3) code is 4.

Problem 9.33

Using Table 9.1 we find that the coefficients of the generator polynomial of the (15,11) code are given in octal form as 23. Since, the binary expansion of 23 is 010011, we conclude that the generator polynomial is

$$g(p) = p^4+p+1$$

The encoder for the (15,11) cyclic code is depicted in the next figure.



Problem 9.34

The i^{th} row of the matrix $\mathbf{G}$ has the form

$$\mathbf{g}_i = [\ 0 \quad \cdots \quad 0 \quad 1 \quad 0\cdots0 \quad p_{i,1} \quad p_{i,2} \quad \cdots \quad p_{i,n-k}\ ], \quad 1 \leq i \leq k$$

where $p_{i,1}, p_{i,2}, \ldots, p_{i,n-k}$ are found by solving the equation

$$p^{n-i} + p_{i,1}p^{n-k-1} + p_{i,2}p^{n-k-2} + \cdots + p_{i,n-k} = p^{n-i} \quad \mod g(p)$$

Thus, with $g(p) = p^4+p+1$ we obtain

$$\begin{aligned}
p^{14} \bmod p^4+p+1 &= (p^4)^3p^2 \bmod p^4+p+1 = (p+1)^3p^2 \bmod p^4+p+1 \\
&= (p^3+p^2+p+1)p^2 \bmod p^4+p+1 \\
&= p^5+p^4+p^3+p^2 \bmod p^4+p+1 \\
&= (p+1)p+p+1+p^3+p^2 \bmod p^4+p+1 \\
&= p^3+1 \\
p^{13} \bmod p^4+p+1 &= (p^3+p^2+p+1)p \bmod p^4+p+1 \\
&= p^4+p^3+p^2+p \bmod p^4+p+1 \\
&= p^3+p^2+1
\end{aligned}$$

$$
\begin{aligned}
p^{12} \bmod p^4+p+1 &= p^3+p^2+p+1\\
p^{11} \bmod p^4+p+1 &= (p^4)^2p^3 \bmod p^4+p+1 = (p+1)^2p^3 \bmod p^4+p+1\\
&= (p^2+1)p^3 \bmod p^4+p+1 = p^5+p^3 \bmod p^4+p+1\\
&= (p+1)p+p^3 \bmod p^4+p+1\\
&= p^3+p^2+p\\
p^{10} \bmod p^4+p+1 &= (p^2+1)p^2 \bmod p^4+p+1 = p^4+p^2 \bmod p^4+p+1\\
&= p^2+p^1\\
p^9 \bmod p^4+p+1 &= (p^2+1)p \bmod p^4+p+1 = p^3+p\\
p^8 \bmod p^4+p+1 &= p^2+1 \bmod p^4+p+1 = p^2+1\\
p^7 \bmod p^4+p+1 &= (p+1)p^3 \bmod p^4+p+1 = p^3+p+1\\
p^6 \bmod p^4+p+1 &= (p+1)p^2 \bmod p^4+p+1 = p^3+p^2\\
p^5 \bmod p^4+p+1 &= (p+1)p \bmod p^4+p+1 = p^2+p\\
p^4 \bmod p^4+p+1 &= p+1 \bmod p^4+p+1 = p+1
\end{aligned}
$$

The generator and the parity check matrix of the code are given by

$$
\mathbf{G} = \left(\begin{array}{ccccccccccc|cccc}
1 & & & & & & & & & & & 1 & 0 & 0 & 1\\
 & 1 & & & & & & & & & & 1 & 1 & 0 & 1\\
 & & 1 & & & & & \mathbf{0} & & & & 1 & 1 & 1 & 1\\
 & & & 1 & & & & & & & & 1 & 1 & 1 & 0\\
 & & & & 1 & & & & & & & 0 & 1 & 1 & 1\\
 & & & & & 1 & & & & & & 1 & 0 & 1 & 0\\
 & & & & & & 1 & & & & & 0 & 1 & 0 & 1\\
 & & & & & & & 1 & & & & 1 & 0 & 1 & 1\\
 & & & \mathbf{0} & & & & & 1 & & & 1 & 1 & 0 & 0\\
 & & & & & & & & & 1 & & 0 & 1 & 1 & 0\\
 & & & & & & & & & & 1 & 0 & 0 & 1 & 1
\end{array}\right)
$$

$$
\mathbf{H} = \left(\begin{array}{ccccccccccc|cccc}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0\\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right)
$$

Problem 9.35

1) Let $g(p) = p^8+p^6+p^4+p^2+1$ be the generator polynomial of an (n,k) cyclic code. Then, $n-k=8$ and the rate of the code is

$$R = \frac{k}{n} = 1-\frac{8}{n}$$

The rate R is minimum when $\frac{8}{n}$ is maximum subject to the constraint that R is positive. Thus, the first choice of n is $n=9$. However, the generator polynomial $g(p)$ does not divide p^9+1 and therefore, it can not generate a $(9,1)$ cyclic code. The next candidate value of n is 10. In this case

$$p^{10}+1 = g(p)(p^2+1)$$

and therefore, $n=10$ is a valid choice. The rate of the code is $R=\frac{k}{n}=\frac{2}{10}=\frac{1}{5}$.

2) In the next table we list the four codewords of the $(10,2)$ cyclic code generated by $g(p)$.

Input	$X(p)$	Codeword
00	0	0000000000
01	1	0101010101
10	p	1010101010
11	$p+1$	1111111111

As it is observed from the table, the minimum weight of the code is 5 and since the code is linear $d_{\min} = w_{\min} = 5$.

3) The coding gain of the $(10, 2)$ cyclic code in part 1) is

$$G_{\text{coding}} = d_{\min} R = 5 \times \frac{2}{10} = 1$$

Problem 9.36

1) For every n

$$p^n + 1 = (p+1)(p^{n-1} + p^{n-2} + \cdots + p + 1)$$

where additions are modulo 2. Since $p+1$ divides p^n+1 it can generate a (n, k) cyclic code, where $k = n-1$.

2) The i^{th} row of the generator matrix has the form

$$\mathbf{g}_i = [\begin{array}{ccccccccc} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & p_{i,1} \end{array}]$$

where $p_{i,1}$, $i = 1, \ldots, n-1$, can be found by solving the equations

$$p^{n-i} + p_{i,1} = p^{n-i} \bmod p + 1, \qquad 1 \leq i \leq n-1$$

Since $p^{n-i} \bmod p + 1 = 1$ for every i, the generator and the parity check matrix are given by

$$\mathbf{G} = \left(\begin{array}{ccc|c} 1 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 1 \end{array}\right), \qquad \mathbf{H} = [\begin{array}{cccc|c} 1 & 1 & \cdots & 1 & 1 \end{array}]$$

3) A vector $\mathbf{c} = [c_1, c_2, \ldots, c_n]$ is a codeword of the $(n, n-1)$ cyclic code if it satisfies the condition $\mathbf{c}\mathbf{H}^t = 0$. But,

$$\mathbf{c}\mathbf{H}^t = 0 = \mathbf{c}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = c_1 + c_2 + \cdots c_n$$

Thus, the vector $\mathbf{c}$ belongs to the code if it has an even weight. Therefore, the cyclic code generated by the polynomial $p+1$ is a simple parity check code.

Problem 9.37

1) Using the results of Problem 9.31, we find that the shortest possible generator polynomial of degree 4 is

$$g(p) = p^4 + p^2 + 1$$

The i^{th} row of the generator matrix $\mathbf{G}$ has the form

$$\mathbf{g}_i = \left[\begin{array}{ccccccccccc} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & p_{i,1} & \cdots & p_{i,4} \end{array}\right]$$

where $p_{i,1}, \ldots, p_{i,4}$ are obtained from the relation

$$p^{6-i} + p_{i,1}p^3 + p_{i,2}p^2 p_{i,3}p + p_{i,4} = p^{6-i}(\bmod p^4 + p^2 + 1)$$

Hence,

$$\begin{aligned} p^5 \bmod p^4 + p^2 + 1 &= (p^2+1)p \bmod p^4 + p^2 + 1 = p^3 + p \\ p^4 \bmod p^4 + p^2 + 1 &= p^2 + 1 \bmod p^4 + p^2 + 1 = p^2 + 1 \end{aligned}$$

and therefore,

$$\mathbf{G} = \left(\begin{array}{cc|cccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right)$$

The codewords of the code are

$$\begin{aligned} \mathbf{c}_1 &= [\ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0\ ] \\ \mathbf{c}_2 &= [\ 1 \ \ 0 \ \ 1 \ \ 0 \ \ 1 \ \ 0\ ] \\ \mathbf{c}_3 &= [\ 0 \ \ 1 \ \ 0 \ \ 1 \ \ 0 \ \ 1\ ] \\ \mathbf{c}_4 &= [\ 1 \ \ 1 \ \ 1 \ \ 1 \ \ 1 \ \ 1\ ] \end{aligned}$$

2) The minimum distance of the linear $(6,2)$ cyclic code is $d_{\min} = w_{\min} = 3$. Therefore, the code can correct

$$e_c = \frac{d_{\min} - 1}{2} = 1 \text{ error}$$

3) An upper bound of the block error probability is given by

$$p_e = (M-1)Q\left[\sqrt{\frac{d_{\min}\mathcal{E}_s}{N_0}}\right]$$

With $M = 2$, $d_{\min} = 3$ and

$$\frac{\mathcal{E}_s}{N_0} = R_c\frac{\mathcal{E}_b}{N_0} = R_c\frac{P}{RN_0} = \frac{2}{6} \times \frac{1}{2 \times 6 \times 10^4 \times 2 \times 10^{-6}} = 1.3889$$

we obtain

$$p_e = Q\left[\sqrt{3 \times 1.3889}\right] = 2.063 \times 10^{-2}$$

Problem 9.38

The block generated by the interleaving is a 5×23 block containing 115 binary symbols. Since the Golay code can correct

$$e_c = \frac{d_{\min} - 1}{2} = \frac{7-1}{2} = 3$$

bits per codeword, the resulting block can correct a single burst of errors of duration less or equal to $5 \times 3 = 15$ bits.

Problem 9.39

1-$C_{\max}$ is not in general cyclic, because there is no guarantee that it is linear. For example let $n = 3$ and let $C_1 = \{000, 111\}$ and $C_2 = \{000, 011, 101, 110\}$, then $C_{\max} = C_1 \cup C_2 = \{000, 111, 011, 101, 110\}$, which is obviously nonlinear (for example $111 \oplus 110 = 001 \notin C_{\max}$) and therefore can not be cyclic.

2-$C_{\min}$ is cyclic, the reason is that C_1 and C_2 are both linear therefore any two elements of $C_{\min}$ are both in C_1 and C_2 and therefore their linear combination is also in C_1 and C_2 and therefore in $C_{\min}$. The intersection satisfies the cyclic property because if $\mathbf{c}$ belongs to $C_{\min}$ it belongs to C_1 and C_2 and therefore all cyclic shifts of it belong to C_1 and C_2 and therefore to $C_{\min}$. All codeword polynomials corresponding to the elements of $C_{\min}$ are multiples of $g_1(p)$ and $g_2(p)$ and therefore multiple of $\text{LCM}\{g_1(p), g_2(p)\}$, which in turn divides $p^n + 1$. For any $\mathbf{c} \in C_{\min}$, we have $w(\mathbf{c}) \geq d_1$ and $w(\mathbf{c}) \geq d_2$, therefore the minimum distance of $C_{\min}$ is greater than or equal to $\max\{d_1, d_2\}$.

Problem 9.40

1) Since for each time slot $[mT,(m+1)T]$ we have $\phi_1(t) = \pm\phi_2(t)$, the signals are dependent and thus only one dimension is needed to represent them in the interval $[mT,(m+1)T]$. In this case the dimensionality of the signal space is upper bounded by the number of the different time slots used to transmit the message signals.

2) If $\phi_1(t) \neq \alpha\phi_2(t)$, then the dimensionality of the signal space over each time slot is at most 2. Since there are n slots over which we transmit the message signals, the dimensionality of the signal space is upper bounded by $2n$.

3) Let the decoding rule be that the first codeword is decoded when $\mathbf{r}$ is received if

$$p(\mathbf{r}|\mathbf{x}_1) > p(\mathbf{r}|\mathbf{x}_2)$$

The set of $\mathbf{r}$ that decode into $\mathbf{x}_1$ is

$$R_1 = \{\mathbf{r} : p(\mathbf{r}|\mathbf{x}_1) > p(\mathbf{r}|\mathbf{x}_2)\}$$

The characteristic function of this set $\chi_1(\mathbf{r})$ is by definition equal to 0 if $\mathbf{r} \notin R_1$ and equal to 1 if $\mathbf{r} \in R_1$. The characteristic function can be bounded as

$$1 - \chi_1(\mathbf{r}) \leq \left(\frac{p(\mathbf{r}|\mathbf{x}_2)}{p(\mathbf{r}|\mathbf{x}_1)}\right)^{\frac{1}{2}}$$

This inequality is true if $\chi(\mathbf{r}) = 1$ because the right side is nonnegative. It is also true if $\chi(\mathbf{r}) = 0$ because in this case $p(\mathbf{r}|\mathbf{x}_2) > p(\mathbf{r}|\mathbf{x}_1)$ and therefore,

$$1 \leq \frac{p(\mathbf{r}|\mathbf{x}_2)}{p(\mathbf{r}|\mathbf{x}_1)} \Longrightarrow 1 \leq \left(\frac{p(\mathbf{r}|\mathbf{x}_2)}{p(\mathbf{r}|\mathbf{x}_1)}\right)^{\frac{1}{2}}$$

Given that the first codeword is sent, then the probability of error is

$$\begin{aligned}
P(\text{error}|\mathbf{x}_1) &= \int \cdots \int_{R^N - R_1} p(\mathbf{r}|\mathbf{x}_1) d\mathbf{r} \\
&= \int \cdots \int_{R^N} p(\mathbf{r}|\mathbf{x}_1)[1 - \chi_1(\mathbf{r})] d\mathbf{r} \\
&\leq \int \cdots \int_{R^N} p(\mathbf{r}|\mathbf{x}_1) \left(\frac{p(\mathbf{r}|\mathbf{x}_2)}{p(\mathbf{r}|\mathbf{x}_1)}\right)^{\frac{1}{2}} d\mathbf{r} \\
&= \int \cdots \int_{R^N} \sqrt{p(\mathbf{r}|\mathbf{x}_1)p(\mathbf{r}|\mathbf{x}_2)} d\mathbf{r}
\end{aligned}$$

4) The result follows immediately if we use the union bound on the probability of error. Thus, assuming that $\mathbf{x}_m$ was transmitted, then taking the signals $\mathbf{x}_{m'}$, $m' \neq m$, one at a time and ignoring the presence of the rest, we can write

$$P(\text{error}|\mathbf{x}_m) \leq \sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} \int \cdots \int_{R^N} \sqrt{p(\mathbf{r}|\mathbf{x}_m)p(\mathbf{r}|\mathbf{x}_{m'})} d\mathbf{r}$$

5) Let $\mathbf{r} = \mathbf{x}_m + \mathbf{n}$ with $\mathbf{n}$ an N-dimensional zero-mean Gaussian random variable with variance per dimension equal to $\sigma^2 = \frac{N_0}{2}$. Then,

$$p(\mathbf{r}|\mathbf{x}_m) = p(\mathbf{n}) \qquad \text{and} \qquad p(\mathbf{r}|\mathbf{x}_{m'}) = p(\mathbf{n} + \mathbf{x}_m - \mathbf{x}_{m'})$$

and therefore,

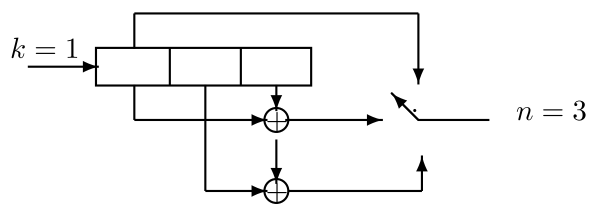
$$
\begin{aligned}
&\int \cdots \int_{R^N} \sqrt{p(\mathbf{r}|\mathbf{x}_m)p(\mathbf{r}|\mathbf{x}_{m'})}d\mathbf{r} \\
&= \int \cdots \int_{R^N} \frac{1}{(\pi N_0)^{\frac{N}{4}}} e^{-\frac{|\mathbf{n}|^2}{2N_0}} \frac{1}{(\pi N_0)^{\frac{N}{4}}} e^{-\frac{|\mathbf{n}+\mathbf{x}_m-\mathbf{x}_{m'}|^2}{2N_0}} d\mathbf{n} \\
&= e^{-\frac{|\mathbf{x}_m-\mathbf{x}_{m'}|^2}{4N_0}} \int \cdots \int_{R^N} \frac{1}{(\pi N_0)^{\frac{N}{2}}} e^{-\frac{2|\mathbf{n}|^2+|\mathbf{x}_m-\mathbf{x}_{m'}|^2/2+2\mathbf{n}\cdot(\mathbf{x}_m-\mathbf{x}_{m'})}{2N_0}} d\mathbf{n} \\
&= e^{-\frac{|\mathbf{x}_m-\mathbf{x}_{m'}|^2}{4N_0}} \int \cdots \int_{R^N} \frac{1}{(\pi N_0)^{\frac{N}{2}}} e^{-\frac{|\mathbf{n}+\frac{\mathbf{x}_m-\mathbf{x}_{m'}}{2}|^2}{N_0}} d\mathbf{n} \\
&= e^{-\frac{|\mathbf{x}_m-\mathbf{x}_{m'}|^2}{4N_0}}
\end{aligned}
$$

Using the union bound in part 4, we obtain

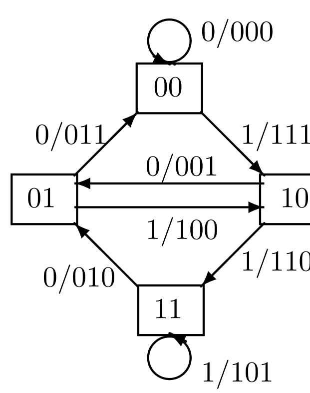
$$
P(\text{error}|x_m(t) \text{ sent}) \leq \sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} e^{-\frac{|\mathbf{x}_m-\mathbf{x}_{m'}|^2}{4N_0}}
$$

Problem 9.41

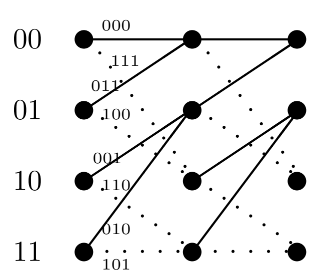
1) The encoder for the $(3,1)$ convolutional code is depicted in the next figure.



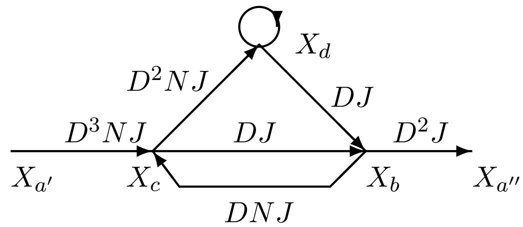
2) The state transition diagram for this code is depicted in the next figure.



3) In the next figure we draw two frames of the trellis associated with the code. Solid lines indicate an input equal to 0, whereas dotted lines correspond to an input equal to 1.



4) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned} X_c &= D^3NJX_{a'} + DNJX_b \\ X_b &= DJX_c + DJX_d \\ X_d &= D^2NJX_c + D^2NJX_d \\ X_{a''} &= D^2JX_b \end{aligned}$$

Eliminating X_b, X_c and X_d results in

$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^6NJ^3}{1 - D^2NJ - D^2NJ^2}$$

To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

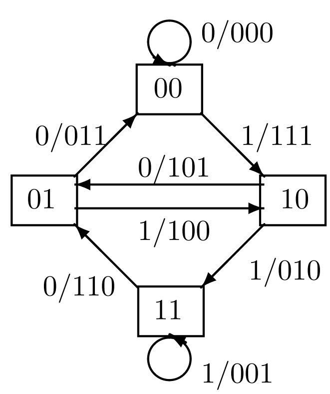
$$T_1(D) = T(D,N,J)|_{N=J=1} = \frac{D^6}{1-2D^2} = D^6 + 2D^8 + 4D^{10} + \cdots$$

Hence, $d_{\text{free}} = 6$

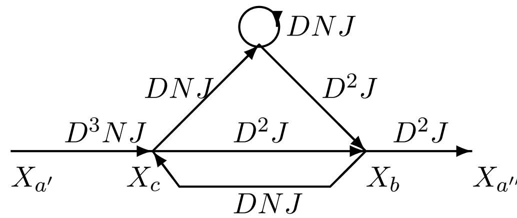
5) Since there is no self loop corresponding to an input equal to 1 such that the output is the all zero sequence, the code is not catastrophic.

Problem 9.42
The number of branches leaving each state correspond to the number possible different inputs to the encoder. Since the encoder at each state takes k binary symbols at its input, the number of branches leaving each state of the trellis is 2^k. The number of branches entering each state is the number of possible kL contents of the encoder shift register that have their first $k(L-1)$ bits corresponding to that particular state (note that the destination state for a branch is determined by the contents of the first $k(L-1)$ bits of the shift register). This means that the number of branches is equal to the number of possible different contents of the last k bits of the encoder, i.e., 2^k.

Problem 9.43
1) The state diagram of the code is depicted in the next figure



2) The diagram used to find the transfer function of the code is depicted in the next figure



Using the flow graph relations we write

$$
\begin{aligned}
X_c &= D^3NJX_{a'} + DNJX_b \\
X_b &= D^2JX_c + D^2JX_d \\
X_d &= DNJX_c + DNJX_d \\
X_{a''} &= D^2JX_b
\end{aligned}
$$

Eliminating X_b, X_c and X_d, we obtain

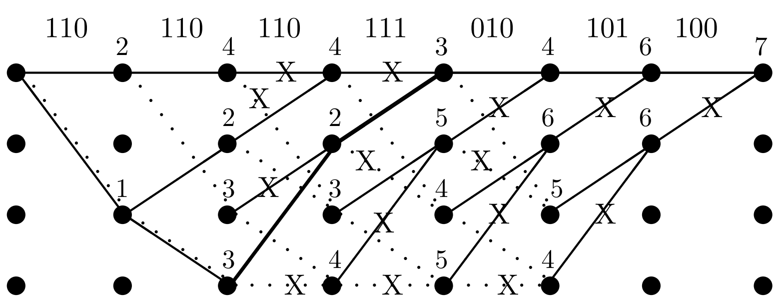
$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^7NJ^3}{1 - DNJ - D^3NJ^2}$$

Thus,

$$T_1(D) = T(D,N,J)|_{N=J=1} = \frac{D^7}{1 - D - D^3} = D^7 + D^8 + D^9 + \cdots$$

3) The minimum free distance of the code is $d_{\text{free}} = 7$

4) The following figure shows 7 frames of the trellis diagram used by the Viterbi decoder. It is assumed that the input sequence is padded by to zeros, so that the actual length of the information sequence is 5. The numbers on the nodes indicate the Hamming distance of the survivor paths. The deleted branches have been marked with an X. In the case of a tie we deleted the lower branch. The survivor path at the end of the decoding is denoted by a thick line.



The information sequence is 11000 and the corresponding codeword 111010110011000...

5) An upper to the bit error probability of the code is given by

$$\bar{p}_b \leq \frac{1}{k} \frac{\vartheta T_2(D,N)}{\vartheta N}\bigg|_{N=1, D=\sqrt{4p(1-p)}}$$

But

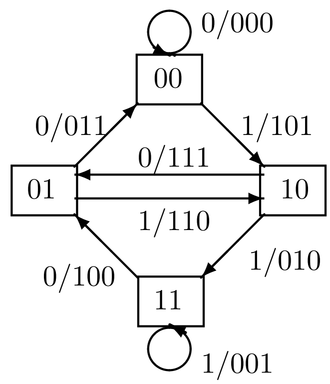
$$\frac{\vartheta T_2(D,N)}{\vartheta N} = \frac{\vartheta}{\vartheta N}\left[\frac{D^7N}{1-(D+D^3)N}\right] = \frac{D^7}{(1-DN-D^3N)^2}$$

and since $k = 1$, $p = 10^{-5}$, we obtain

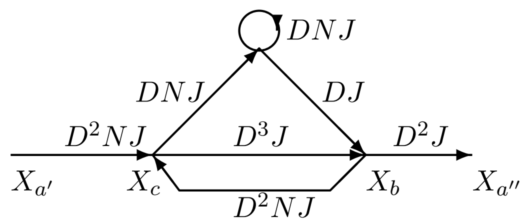
$$\bar{p}_b \leq \frac{D^7}{(1-D-D^3)^2}\bigg|_{D=\sqrt{4p(1-p)}} \approx 4.0993 \times 10^{-16}$$

Problem 9.44

1) The state diagram of the code is depicted in the next figure



2) The diagram used to find the transfer function of the code is depicted in the next figure



Using the flow graph relations we write

$$\begin{aligned}
X_c &= D^2NJX_{a'} + D^2NJX_b \\
X_b &= DJX_d + D^3JX_c \\
X_d &= DNJX_d + DNJX_c \\
X_{a''} &= D^2JX_b
\end{aligned}$$

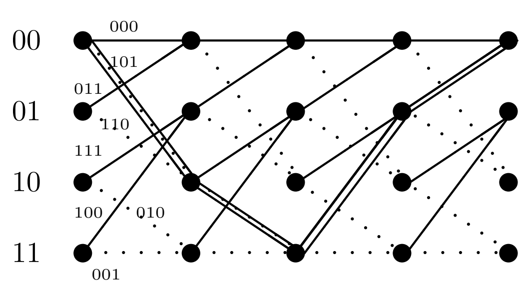
Eliminating X_b, X_c and X_d, we obtain

$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^6N^2J^4 + D^7NJ^3 - D^8N^2J^4}{1 - DNJ - D^4N^2J^3 - D^5NJ^2 + D^6N^2J^3}$$

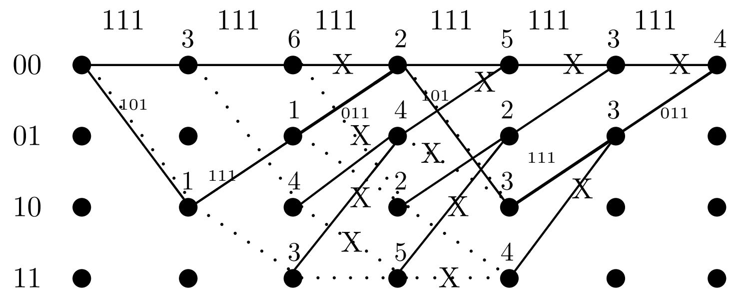
Thus,

$$T_1(D) = T(D,N,J)|_{N=J=1} = \frac{D^6 + D^7 - D^8}{1 - D - D^4 - D^5 + D^6} = D^6 + 2D^7 + D^8 + \cdots$$

3) The minimum free distance of the code is $d_{\text{free}} = 6$. The path, which is at a distance d_{free} from the all zero path, is depicted with a double line in the next figure.

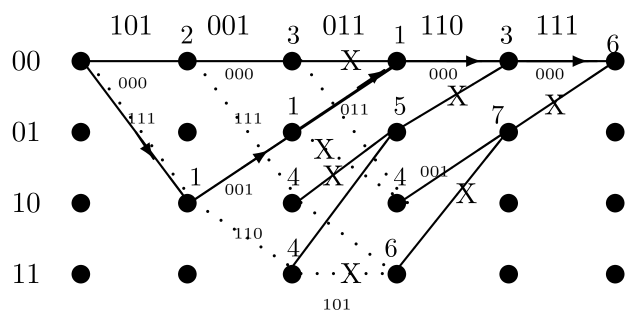


4) The following figure shows 6 frames of the trellis diagram used by the Viterbi algorithm to decode the sequence $\{111, 111, 111, 111, 111, 111\}$. The numbers on the nodes indicate the Hamming distance of the survivor paths from the received sequence. The branches that are dropped by the Viterbi algorithm have been marked with an X. In the case of a tie of two merging paths, we delete the lower path. The decoded sequence is $\{101, 111, 011, 101, 111, 011\}$ which corresponds to the information sequence $\{x_1, x_2, x_3, x_4\} = \{1, 0, 0, 1\}$ followed by two zeros.



Problem 9.45

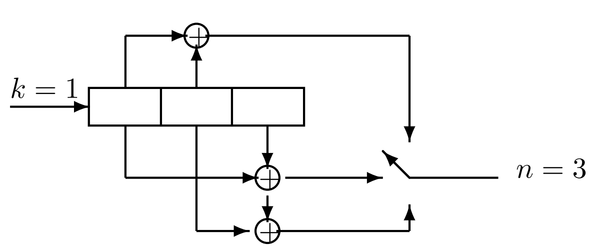
The code of Problem 9.41 is a $(3, 1)$ convolutional code with $L = 3$. The length of the received sequence $\mathbf{y}$ is 15. This means that 5 symbols have been transmitted, and since we assume that the information sequence has been padded by two 0's, the actual length of the information sequence is 3. The following figure depicts 5 frames of the trellis used by the Viterbi decoder. The numbers on the nodes denote the metric (Hamming distance) of the survivor paths. In the case of a tie of two merging paths at a node, we have purged the lower path.



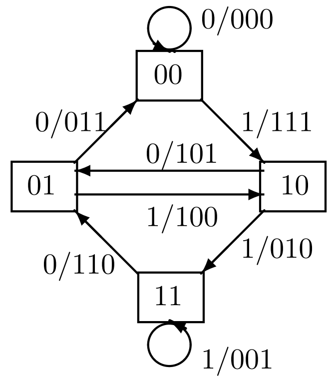
The decoded sequence is $\{111, 001, 011, 000, 000\}$ and corresponds to the information sequence $\{1, 0, 0\}$ followed by two zeros.

Problem 9.46

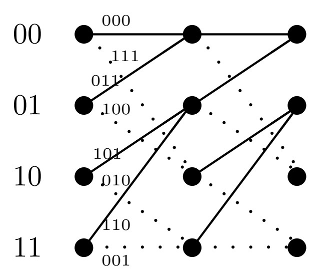
1) The encoder for the $(3, 1)$ convolutional code is depicted in the next figure.



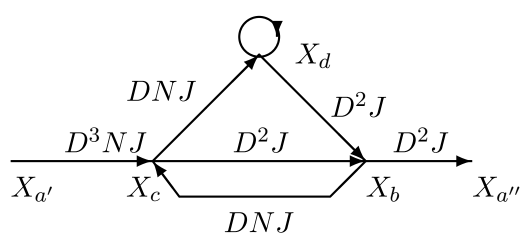
2) The state transition diagram for this code is shown below



3) In the next figure we draw two frames of the trellis associated with the code. Solid lines indicate an input equal to 0, whereas dotted lines correspond to an input equal to 1.



4) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned}
X_c &= D^3NJX_{a'} + DNJX_b \\
X_b &= D^2JX_c + D^2JX_d \\
X_d &= DNJX_c + DNJX_d \\
X_{a''} &= D^2JX_b
\end{aligned}$$

Eliminating X_b, X_c and X_d results in

$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^7NJ^3}{1 - DNJ - D^3NJ^2}$$

To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

$$T_1(D) = T(D,N,J)|_{N=J=1} = \frac{D^7}{1 - D - D^3} = D^7 + D^8 + D^9 + \cdots$$

Hence, $d_{\text{free}} = 7$

5) Since there is no self loop corresponding to an input equal to 1 such that the output is the all zero sequence, the code is not catastrophic.

Problem 9.47

Using the diagram of Figure 9.28, we see that there are only two ways to go from state $X_{a'}$ to state $X_{a''}$ with a total number of ones (sum of the exponents of D) equal to 6. The corresponding transitions are:

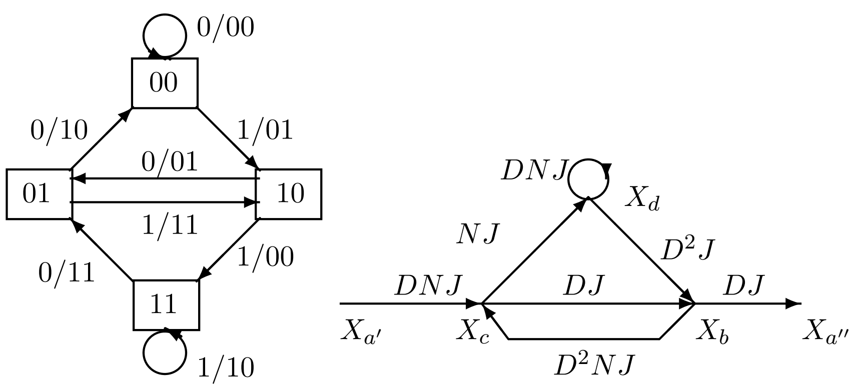
$$\begin{array}{ll} \text{Path 1:} & X_{a'} \xrightarrow{D^2} X_c \xrightarrow{D} X_d \xrightarrow{D} X_b \xrightarrow{D^2} X_{a''} \\ \text{Path 2:} & X_{a'} \xrightarrow{D^2} X_c \xrightarrow{D} X_b \rightarrow X_c \xrightarrow{D} X_b \xrightarrow{D^2} X_{a''} \end{array}$$

These two paths correspond to the codewords

$$\begin{array}{rcl} \mathbf{c}_1 & = & 0,0,\;\; 1,0,\;\; 1,0,\;\; 1,1,\;\; 0,0,\;\; 0,0,\ldots \\ \mathbf{c}_2 & = & 0,0,\;\; 0,1,\;\; 0,0,\;\; 0,1,\;\; 1,1,\;\; 0,0,\ldots \end{array}$$

Problem 9.48

1) The state transition diagram and the flow diagram used to find the transfer function for this code are depicted in the next figure.



Thus,

$$\begin{array}{rcl} X_c & = & DNJX_{a'} + D^2NJX_b \\ X_b & = & DJX_c + D^2JX_d \\ X_d & = & NJX_c + DNJX_d \\ X_{a''} & = & DJX_b \end{array}$$

and by eliminating X_b, X_c and X_d, we obtain

$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^3NJ^3}{1 - DNJ - D^3NJ^2}$$

To find the transfer function of the code in the form $T(D,N)$, we set $J=1$ in $T(D,N,J)$. Hence,

$$T(D,N) = \frac{D^3N}{1 - DN - D^3N}$$

2) To find the free distance of the code we set $N=1$ in the transfer function $T(D,N)$, so that

$$T_1(D) = T(D,N)|_{N=1} = \frac{D^3}{1 - D - D^3} = D^3 + D^4 + D^5 + 2D^6 + \cdots$$

Hence, $d_{\text{free}} = 3$

3) An upper bound on the bit error probability, when hard decision decoding is used, is given by

$$\bar{P}_b \leq \frac{1}{k} \frac{\vartheta T(D,N)}{\vartheta N}\bigg|_{N=1, D=\sqrt{4p(1-p)}}$$

Since

$$\frac{\vartheta T(D,N)}{\vartheta N}\bigg|_{N=1} = \frac{\vartheta}{\vartheta N} \frac{D^3 N}{1-(D+D^3)N}\bigg|_{N=1} = \frac{D^3}{(1-(D+D^3))^2}$$

with $k=1$, $p=10^{-6}$ we obtain

$$\bar{P}_b \leq \frac{D^3}{(1-(D+D^3))^2}\bigg|_{D=\sqrt{4p(1-p)}} = 8.0321 \times 10^{-9}$$

Problem 9.49

1) Let the decoding rule be that the first codeword is decoded when $\mathbf{y}_i$ is received if

$$p(\mathbf{y}_i|\mathbf{x}_1) > p(\mathbf{y}_i|\mathbf{x}_2)$$

The set of $\mathbf{y}_i$ that decode into $\mathbf{x}_1$ is

$$Y_1 = \{\mathbf{y}_i : p(\mathbf{y}_i|\mathbf{x}_1) > p(\mathbf{y}_i|\mathbf{x}_2)\}$$

The characteristic function of this set $\chi_1(\mathbf{y}_i)$ is by definition equal to 0 if $\mathbf{y}_i \notin Y_1$ and equal to 1 if $\mathbf{y}_i \in Y_1$. The characteristic function can be bounded as (see Problem 9.40)

$$1 - \chi_1(\mathbf{y}_i) \leq \left(\frac{p(\mathbf{y}_i|\mathbf{x}_2)}{p(\mathbf{y}_i|\mathbf{x}_1)}\right)^{\frac{1}{2}}$$

Given that the first codeword is sent, then the probability of error is

$$\begin{aligned}
P(\text{error}|\mathbf{x}_1) &= \sum_{\mathbf{y}_i \in Y - Y_1} p(\mathbf{y}_i|\mathbf{x}_1) = \sum_{\mathbf{y}_i \in Y} p(\mathbf{y}_i|\mathbf{x}_1)[1-\chi_1(\mathbf{y}_i)] \\
&\leq \sum_{\mathbf{y}_i \in Y} p(\mathbf{y}_i|\mathbf{x}_1) \left(\frac{p(\mathbf{y}_i|\mathbf{x}_2)}{p(\mathbf{y}_i|\mathbf{x}_1)}\right)^{\frac{1}{2}} = \sum_{\mathbf{y}_i \in Y} \sqrt{p(\mathbf{y}_i|\mathbf{x}_1)p(\mathbf{y}_i|\mathbf{x}_2)} \\
&= \sum_{i=1}^{2^n} \sqrt{p(\mathbf{y}_i|\mathbf{x}_1)p(\mathbf{y}_i|\mathbf{x}_2)}
\end{aligned}$$

where Y denotes the set of all possible sequences $\mathbf{y}_i$. Since, each element of the vector $\mathbf{y}_i$ can take two values, the cardinality of the set Y is 2^n.

2) Using the results of the previous part we have

$$\begin{aligned}
P(\text{error}) &\leq \sum_{i=1}^{2^n} \sqrt{p(\mathbf{y}_i|\mathbf{x}_1)p(\mathbf{y}_i|\mathbf{x}_2)} = \sum_{i=1}^{2^n} p(\mathbf{y}_i) \sqrt{\frac{p(\mathbf{y}_i|\mathbf{x}_1)}{p(\mathbf{y}_i)}} \sqrt{\frac{p(\mathbf{y}_i|\mathbf{x}_2)}{p(\mathbf{y}_i)}} \\
&= \sum_{i=1}^{2^n} p(\mathbf{y}_i) \sqrt{\frac{p(\mathbf{x}_1|\mathbf{y}_i)}{p(\mathbf{x}_1)}} \sqrt{\frac{p(\mathbf{x}_2|\mathbf{y}_i)}{p(\mathbf{x}_2)}} = \sum_{i=1}^{2^n} 2p(\mathbf{y}_i) \sqrt{p(\mathbf{x}_1|\mathbf{y}_i)p(\mathbf{x}_2|\mathbf{y}_i)}
\end{aligned}$$

However, given the vector $\mathbf{y}_i$, the probability of error depends only on those values that $\mathbf{x}_1$ and $\mathbf{x}_2$ are different. In other words, if $x_{1,k} = x_{2,k}$, then no matter what value is the k^{th} element of $\mathbf{y}_i$, it will not produce an error. Thus, if by d we denote the Hamming distance between $\mathbf{x}_1$ and $\mathbf{x}_2$, then

$$p(\mathbf{x}_1|\mathbf{y}_i)p(\mathbf{x}_2|\mathbf{y}_i) = p^d(1-p)^d$$

and since $p(\mathbf{y}_i) = \frac{1}{2^n}$, we obtain

$$P(\text{error}) = P(d) = 2p^{\frac{d}{2}}(1-p)^{\frac{d}{2}} = [4p(1-p)]^{\frac{d}{2}}$$

Problem 9.50
1)

$$\begin{aligned}
Q(x) &= \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{v^2}{2}}dv \\
&\stackrel{v=\sqrt{2}t}{=} \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{2}}}^\infty e^{-t^2}dt \\
&= \frac{1}{2}\frac{2}{\pi}\int_{\frac{x}{\sqrt{2}}}^\infty e^{-t^2}dt \\
&= \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)
\end{aligned}$$

2) The average bit error probability can be bounded as (see (9.7.16))

$$\begin{aligned}
\bar{P}_b &\leq \frac{1}{k}\sum_{d=d_{\text{free}}}^\infty a_d f(d) Q\left[\sqrt{2R_c d\frac{\mathcal{E}_b}{N_0}}\right] = \frac{1}{k}\sum_{d=d_{\text{free}}}^\infty a_d f(d) Q\left[\sqrt{2R_c d\gamma_b}\right] \\
&= \frac{1}{2k}\sum_{d=d_{\text{free}}}^\infty a_d f(d)\text{erfc}(\sqrt{R_c d\gamma_b}) \\
&= \frac{1}{2k}\sum_{d=1}^\infty a_{d+d_{\text{free}}} f(d+d_{\text{free}})\text{erfc}(\sqrt{R_c(d+d_{\text{free}})\gamma_b}) \\
&\leq \frac{1}{2k}\text{erfc}(\sqrt{R_c d_{\text{free}}\gamma_b})\sum_{d=1}^\infty a_{d+d_{\text{free}}} f(d+d_{\text{free}})e^{-R_c d\gamma_b}
\end{aligned}$$

But,

$$T(D,N) = \sum_{d=d_{\text{free}}}^\infty a_d D^d N^{f(d)} = \sum_{d=1}^\infty a_{d+d_{\text{free}}} D^{d+d_{\text{free}}} N^{f(d+d_{\text{free}})}$$

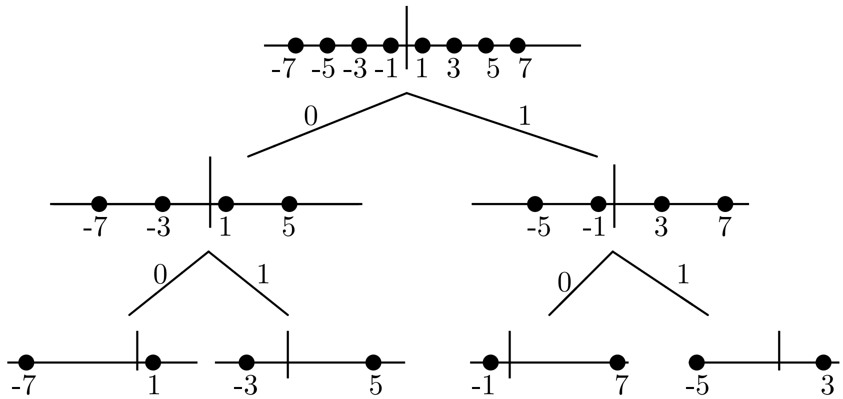
and therefore,

$$\begin{aligned}
\left.\frac{\vartheta T(D,N)}{\vartheta N}\right|_{N=1} &= \sum_{d=1}^\infty a_{d+d_{\text{free}}} D^{d+d_{\text{free}}} f(d+d_{\text{free}}) \\
&= D^{d_{\text{free}}}\sum_{d=1}^\infty a_{d+d_{\text{free}}} D^d f(d+d_{\text{free}})
\end{aligned}$$

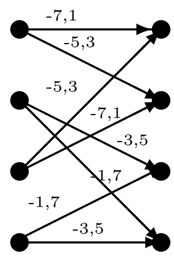
Setting $D = e^{-R_c\gamma_b}$ in the previous and substituting in the expression for the average bit error probability, we obtain

$$\bar{P}_b \leq \frac{1}{2k}\text{erfc}(\sqrt{R_c d_{\text{free}}\gamma_b})e^{R_c d_{\text{free}}\gamma_b}\left.\frac{\vartheta T(D,N)}{\vartheta N}\right|_{N=1, D=e^{-R_c\gamma_b}}$$

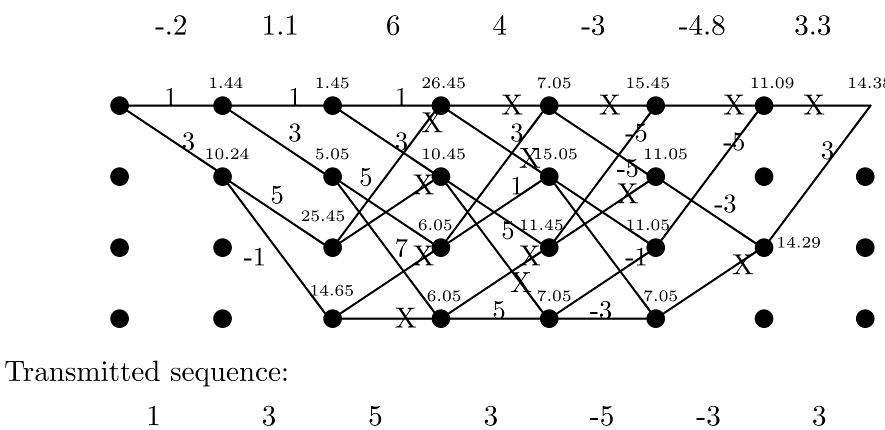
Problem 9.51
The partition of the 8-PAM constellation in four subsets is depicted in the figure below.



2) The next figure shows one frame of the trellis used to decode the received sequence. Each branch consists of two transitions which correspond to elements in the same coset in the final partition level.



The operation of the Viterbi algorithm for the decoding of the sequence $\{-.2, 1.1, 6, 4, -3, -4.8, 3.3\}$ is shown schematically in the next figure. It has been assumed that we start at the all zero state and that a sequence of zeros terminates the input bit stream in order to clear the encoder. The numbers at the nodes indicate the minimum Euclidean distance, and the branches have been marked with the decoded transmitted symbol. The paths that have been purged are marked with an X.



Chapter 10

Problem 10.1

1) The wavelength λ is

$$\lambda = \frac{3 \times 10^8}{10^9} \text{ m } = \frac{3}{10} \text{ m}$$

Hence, the Doppler frequency shift is

$$f_D = \pm \frac{u}{\lambda} = \pm \frac{100 \text{ Km/hr}}{\frac{3}{10} \text{ m}} = \pm \frac{100 \times 10^3 \times 10}{3 \times 3600} \text{Hz} = \pm 92.5926 \text{ Hz}$$

The plus sign holds when the vehicle travels towards the transmitter whereas the minus sign holds when the vehicle moves away from the transmitter.

2) The maximum difference in the Doppler frequency shift, when the vehicle travels at speed 100 km/hr and $f = 1$ GHz, is

$$\Delta f_{D\max} = 2f_D = 185.1852 \text{ Hz}$$

This should be the bandwidth of the Doppler frequency tracking loop.

3) The maximum Doppler frequency shift is obtain when $f = 1$ GHz $+ 1$ MHz and the vehicle moves towards the transmitter. In this case

$$\lambda_{\min} = \frac{3 \times 10^8}{10^9 + 10^6} \text{ m } = 0.2997 \text{ m}$$

and therefore

$$f_{D\max} = \frac{100 \times 10^3}{0.2997 \times 3600} = 92.6853 \text{ Hz}$$

Thus, the Doppler frequency spread is $B_d = 2f_{D\max} = 185.3706$ Hz.

Problem 10.2

1) Since $T_m = 1$ second, the coherence bandwidth

$$B_{cb} = \frac{1}{2T_m} = 0.5 \quad \text{Hz}$$

and with $B_d = 0.01$ Hz, the coherence time is

$$T_{ct} = \frac{1}{2B_d} = 100/2 = 50 \quad \text{seconds}$$

(2) Since the channel bandwidth $W \gg b_{cb}$, the channel is frequency selective.

(3) Since the signal duration $T \ll T_{ct}$, the channel is slowly fading.

(4) The ratio $W/B_{cb} = 10$. Hence, in principle up to tenth order diversity is available by subdividing the channel bandwidth into 10 subchannels, each of width 0.5 Hz. If we employ binary PSK with symbol duration $T = 10$ seconds, then the channel bandwidth can be subdivided into 25 subchannels, each of bandwidth $\frac{2}{T} = 0.2$ Hz. We may choose to have 5^{th} order frequency diversity and for each transmission, thus, have 5 parallel transmissions. Thus, we would have a data rate of 5 bits per signal interval, i.e., a bit rate of 1/2 bps. By reducing the order of diversity, we may increase the data rate, for example, with no diversity, the data rate becomes 2.5 bps.

(5) To answer the question we may use the approximate relation for the error probability given by (10.1.37), or we may use the results in the graph shown in Figure 10.1.10. For example, for binary PSK with $D = 4$, the SNR per bit required to achieve an error probability of 10^{-6} is 18 dB. This the total SNR per bit for the four channels (with maximal ration combining). Hence, the SNR per bit per channel is reduced to 12 dB (a factor of four smaller).

Problem 10.3

The Rayleigh distribution is

$$p(\alpha) = \begin{cases} \frac{\alpha}{\sigma_\alpha^2} e^{-\alpha^2/2\sigma_\alpha^2}, & \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the probability of error for the binary FSK and DPSK with noncoherent detection averaged over all possible values of α is

$$\begin{aligned} P_2 &= \int_0^\infty \frac{1}{2} e^{-c\frac{\alpha^2 \mathcal{E}_b}{N_0}} \frac{\alpha}{\sigma_\alpha^2} e^{-\alpha^2/2\sigma_\alpha^2} d\alpha \\ &= \frac{1}{2\sigma_\alpha^2} \int_0^\infty \alpha e^{-\alpha^2\left[\frac{c\mathcal{E}_b}{N_0} + \frac{1}{2\sigma_\alpha^2}\right]} d\alpha \end{aligned}$$

But,

$$\int_0^\infty x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}}, \quad (a > 0)$$

so that with $n = 0$ we obtain

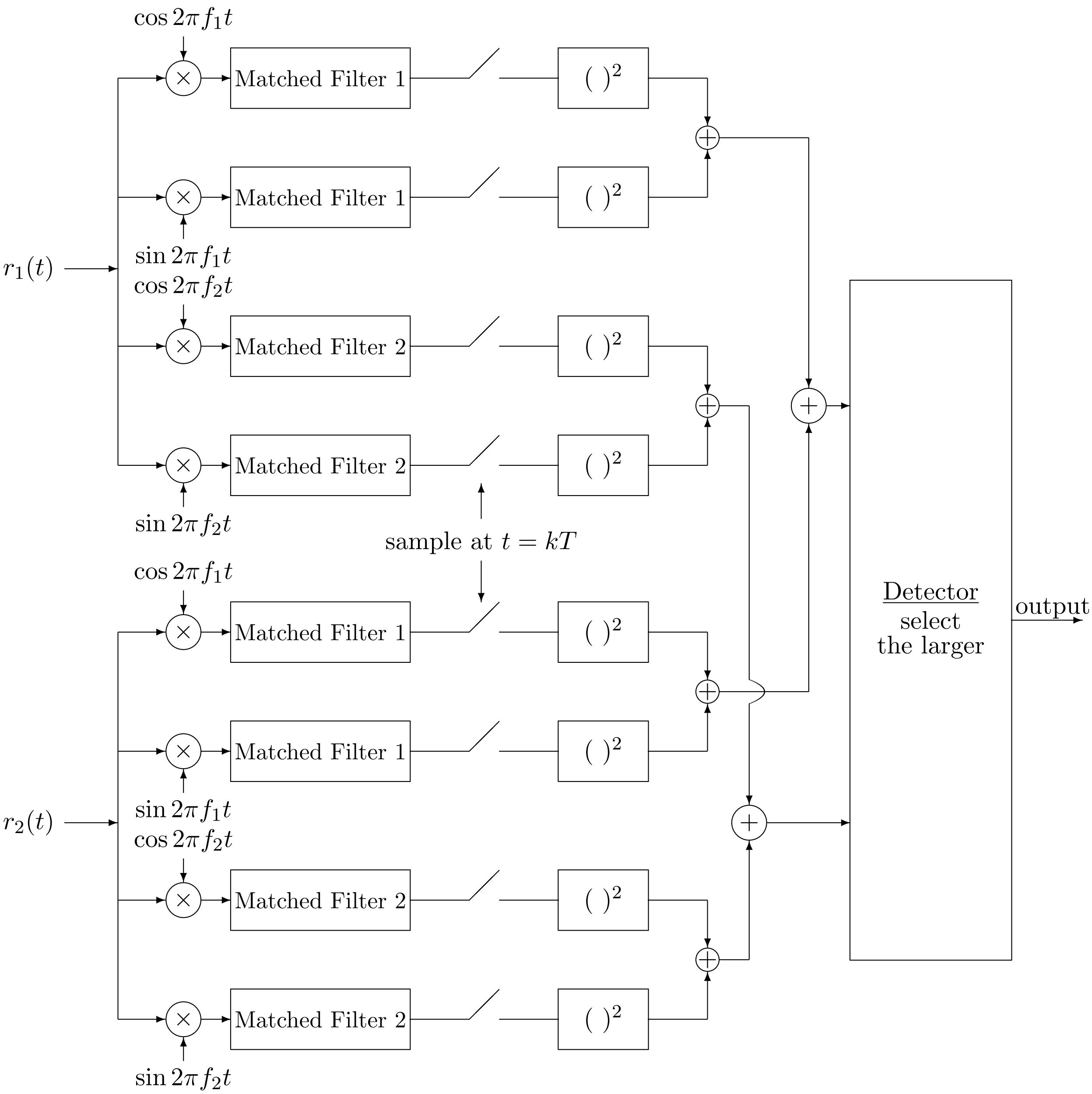
$$\begin{aligned} P_2 &= \frac{1}{2\sigma_\alpha^2} \int_0^\infty \alpha e^{-\alpha^2\left[\frac{c\mathcal{E}_b}{N_0} + \frac{1}{2\sigma_\alpha^2}\right]} d\alpha = \frac{1}{2\sigma_\alpha^2} \frac{1}{2\left[\frac{c\mathcal{E}_b}{N_0} + \frac{1}{2\sigma_\alpha^2}\right]} \\ &= \frac{1}{2\left[c\frac{\mathcal{E}_b 2\sigma_\alpha^2}{N_0} + 1\right]} = \frac{1}{2\left[c\bar{\rho}_b + 1\right]} \end{aligned}$$

where $\bar{\rho}_b = \frac{\mathcal{E}_b 2\sigma_\alpha^2}{N_0}$. With $c = 1$ (DPSK) and $c = \frac{1}{2}$ (FSK) we have

$$P_2 = \begin{cases} \frac{1}{2(1+\bar{\rho}_b)}, & \text{DPSK} \\ \frac{1}{2+\bar{\rho}_b}, & \text{FSK} \end{cases}$$

Problem 10.4

(a)



(b) The probability of error for binary FSK with square-law combining for $D = 2$ is given in Figure 10.1.10. The probability of error for $D = 1$ is also given in Figure 10.1.10. Note that an increase in SNR by a factor of 10 reduces the error probability by a factor of 10 when $D = 1$ and by a factor of 100 when $D = 2$.

Problem 10.5

(a) r is a Gaussian random variable. If $\sqrt{\mathcal{E}_b}$ is the transmitted signal point, then

$$E(r) = E(r_1) + E(r_2) = (1 + k)\sqrt{\mathcal{E}_b} \equiv m_r$$

and the variance is

$$\sigma_r^2 = \sigma_1^2 + k^2\sigma_2^2$$

The probability density function of r is

$$f(r) = \frac{1}{\sqrt{2\pi}\sigma_r} e^{-\frac{(r-m_r)^2}{2\sigma_r^2}}$$

and the probability of error is

$$\begin{aligned} P_2 &= \int_{-\infty}^{0} f(r)\,dr \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{m_r}{\sigma_r}} e^{-\frac{x^2}{2}}\,dx \\ &= Q\left(\sqrt{\frac{m_r^2}{\sigma_r^2}}\right) \end{aligned}$$

where

$$\frac{m_r^2}{\sigma_r^2} = \frac{(1+k)^2\mathcal{E}_b}{\sigma_1^2 + k^2\sigma_2^2}$$

The value of k that maximizes this ratio is obtained by differentiating this expression and solving for the value of k that forces the derivative to zero. Thus, we obtain

$$k = \frac{\sigma_1^2}{\sigma_2^2}$$

Note that if $\sigma_1 > \sigma_2$, then $k > 1$ and r_2 is given greater weight than r_1. On the other hand, if $\sigma_2 > \sigma_1$, then $k < 1$ and r_1 is given greater weight than r_2. When $\sigma_1 = \sigma_2$, $k = 1$. In this case

$$\frac{m_r^2}{\sigma_r^2} = \frac{2\mathcal{E}_b}{\sigma_1^2}$$

(b) When $\sigma_2^2 = 3\sigma_1^2$, $k = \frac{1}{3}$, and

$$\frac{m_r^2}{\sigma_r^2} = \frac{(1+\frac{1}{3})^2\mathcal{E}_b}{\sigma_1^2 + \frac{1}{9}(3\sigma_1^2)} = \frac{4}{3}\left(\frac{\mathcal{E}_b}{\sigma_1^2}\right)$$

On the other hand, if k is set to unity we have

$$\frac{m_r^2}{\sigma_r^2} = \frac{4\mathcal{E}_b}{\sigma_1^2 + 3\sigma_1^2} = \frac{\mathcal{E}_b}{\sigma_1^2}$$

Therefore, the optimum weighting provides a gain of

$$10\log\frac{4}{3} = 1.25 \text{ dB}$$

Problem 10.6

1) The probability of error for a fixed value of a is

$$P_e(a) = Q\left(\sqrt{\frac{2a^2\mathcal{E}}{N_0}}\right)$$

since the given a takes two possible values, namely $a = 0$ and $a = 2$ with probabilities 0.1 and 0.9, respectively, the average probability of error is

$$P_e = \frac{0.1}{2} + Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right) = 0.05 + Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right)$$

(2) As $\frac{\mathcal{E}}{N_0} \to \infty$, $P_e \to 0.05$
(3) The probability of error for fixed values of a_1 and a_2 is

$$P_e(a_1, a_2) = Q\left(\sqrt{\frac{2(a_1^2 + a_2^2)\mathcal{E}}{N_0}}\right)$$

In this case we have four possible values for the pair (a_1, a_2), namely, $(0,0), (0,2), (2,0)$, and $(2,2)$, with corresponding probabilities).01, 0.09, 0.09 and 0.81. Hence, the average probability of error is

$$P_e = \frac{0.01}{2} + 0.18Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right) + 0.81Q\left(\sqrt{\frac{16\mathcal{E}}{N_0}}\right)$$

(4) As $\frac{\mathcal{E}}{N_0} \to \infty$, $P_e \to 0.005$, which is a factor of 10 smaller than in (2).

Problem 10.7
We assume that the input bits 0, 1 are mapped to the symbols -1 and 1 respectively. The terminal phase of an MSK signal at time instant n is given by

$$\theta(n; \mathbf{a}) = \frac{\pi}{2} \sum_{k=0}^{k} a_k + \theta_0$$

where θ_0 is the initial phase and a_k is ± 1 depending on the input bit at the time instant k. The following table shows $\theta(n; \mathbf{a})$ for two different values of θ_0 $(0, \pi)$, and the four input pairs of data: $\{00, 01, 10, 11\}$.

θ_0	b_0	b_1	a_0	a_1	$\theta(n; \mathbf{a})$
0	0	0	-1	-1	$-\pi$
0	0	1	-1	1	0
0	1	0	1	-1	0
0	1	1	1	1	π
π	0	0	-1	-1	0
π	0	1	-1	1	π
π	1	0	1	-1	π
π	1	1	1	1	2π

Problem 10.8
1) The envelope of the signal is

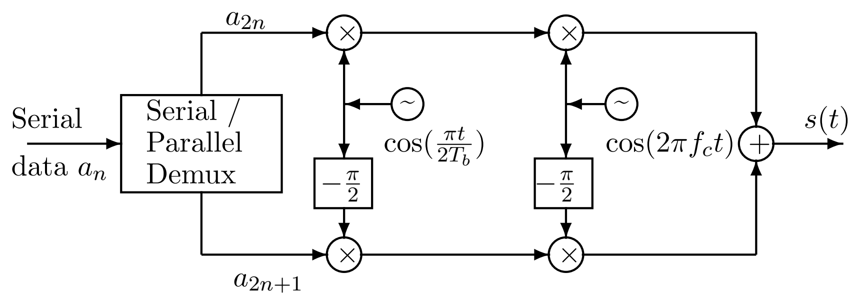
$$\begin{aligned} |s(t)| &= \sqrt{|s_c(t)|^2 + |s_s(t)|^2} \\ &= \sqrt{\frac{2\mathcal{E}_b}{T_b}\cos^2\left(\frac{\pi t}{2T_b}\right) + \frac{2\mathcal{E}_b}{T_b}\sin^2\left(\frac{\pi t}{2T_b}\right)} \\ &= \sqrt{\frac{2\mathcal{E}_b}{T_b}} \end{aligned}$$

Thus, the signal has constant amplitude.

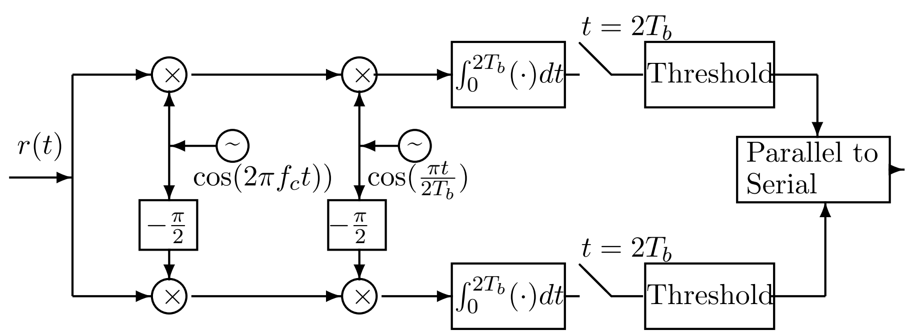
2) The signal $s(t)$ has the form of the four-phase PSK signal with

$$g_T(t) = \cos\left(\frac{\pi t}{2T_b}\right), \qquad 0 \le t \le 2T_b$$

Hence, it is an MSK signal. A block diagram of the modulator for synthesizing the signal is given in the next figure.



3) A sketch of the demodulator is shown in the next figure.



Problem 10.9

Since $p = 2$, m is odd ($m = 1$) and $M = 2$, there are

$$N_s = 2pM = 8$$

phase states, which we denote as $S_n = (\theta_n, a_{n-1})$. The $2p = 4$ phase states corresponding to θ_n are

$$\Theta_s = \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$$

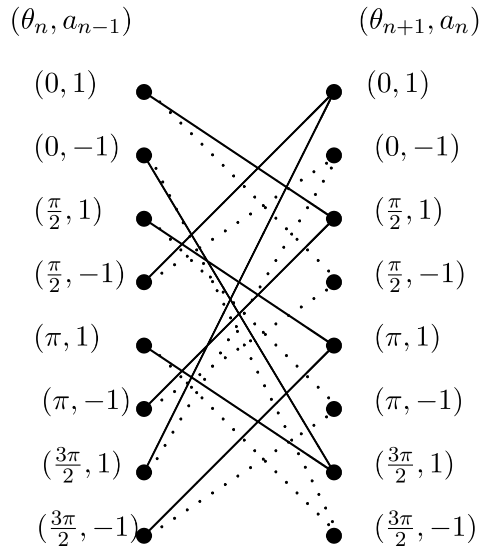
and therefore, the 8 states S_n are

$$\left\{(0,1),\ (0,-1),\ \left(\frac{\pi}{2},1\right),\ \left(\frac{\pi}{2},-1\right),\ (\pi,1),\ (\pi,-1),\ \left(\frac{3\pi}{2},1\right),\ \left(\frac{3\pi}{2},-1\right)\right\}$$

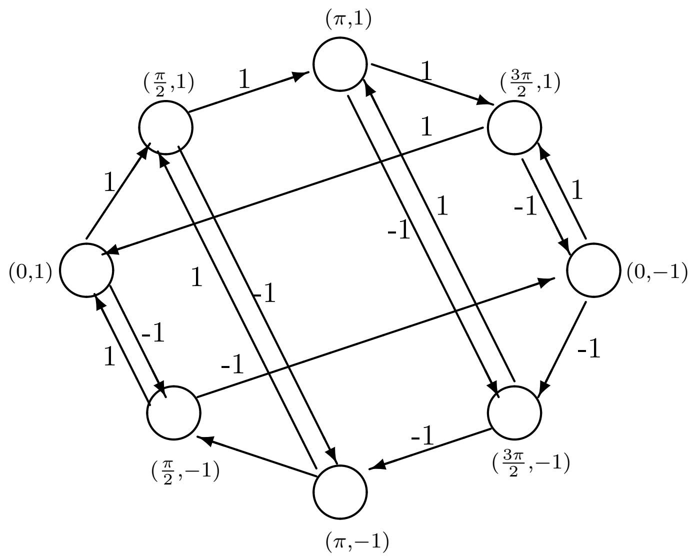
Having at our disposal the state (θ_n, a_{n-1}) and the transmitted symbol a_n, we can find the new phase state as

$$(\theta_n, a_{n-1}) \xrightarrow{a_n} (\theta_n + \frac{\pi}{2}a_{n-1}, a_n) = (\theta_{n+1}, a_n)$$

The following figure shows one frame of the phase-trellis of the partial response CPM signal.



The following is a sketch of the state diagram of the partial response CPM signal.



Problem 10.10

1) For a full response CPFSK signal, L is equal to 1. If $h = \frac{2}{3}$, then since m is even, there are p terminal phase states. If $h = \frac{3}{4}$, the number of states is $N_s = 2p$.

2) With $L = 3$ and $h = \frac{2}{3}$, the number of states is $N_s = p2^2 = 12$. When $L = 3$ and $h = \frac{3}{4}$, the number of states is $N_s = 2p2^2 = 32$.

Problem 10.11

(a) The coding gain is

$$R_c d_{\min}^H = \frac{1}{2} \times 10 = 5\,(7dB)$$

(b) The processing gain is W/R, where $W = 10^7 Hz$ and $R = 2000bps$. Hence,

$$\frac{W}{R} = \frac{10^7}{2 \times 10^3} = 5 \times 10^3\,(37dB)$$

(c) The jamming margin given by (10.3.43) is

$$\begin{aligned}\left(\frac{P_J}{P_s}\right)_{dB} &= \left(\frac{W}{R}\right)_{dB} + (CG)_{dB} - \left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} \\ &= 37 + 7 - 10 = 34dB\end{aligned}$$

Problem 10.12
The probability of error for DS spread spectrum with binary PSK may be expressed as

$$P_2 = Q\left(\sqrt{\frac{2W/R_b}{P_J/P_S}}\right)$$

where W/R is the processing gain and P_J/P_S is the jamming margin. If the jammer is a broadband, WGN jammer, then

$$\begin{aligned}P_J &= WJ_0 \\ P_S &= \mathcal{E}_b/T_b = \mathcal{E}_b R_b\end{aligned}$$

Therefore,

$$P_2 = Q\left(\sqrt{\frac{2\mathcal{E}_b}{J_0}}\right)$$

which is identical to the performance obtained with a non-spread signal.

Problem 10.13
We assume that the interference is characterized as a zero-mean AWGN process with power spectral density J_0. To achieve an error probability of 10^{-5}, the required $\mathcal{E}_b/J_0 = 10$. Then, by using the relation in (10.3.40) and (10.3.44), we have

$$\begin{aligned}\frac{W/R}{P_N/P_S} &= \frac{W/R}{N_u - 1} = \frac{\mathcal{E}_b}{J_0} \\ W/R &= \left(\frac{\mathcal{E}_b}{J_0}\right)(N_u - 1) \\ W &= R\left(\frac{\mathcal{E}_b}{J_0}\right)(N_u - 1)\end{aligned}$$

where $R = 10^4 bps$, $N_u = 30$ and $\mathcal{E}_b/J_0 = 10$. Therefore,

$$W = 2.9 \times 10^6\ Hz$$

The minimum chip rate is $1/T_c = W = 2.9 \times 10^6$ chips/sec.

Problem 10.14
To achieve an error probability of 10^{-6}, we require

$$\left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} = 10.5dB$$

Then, the number of users of the CDMA system is

$$\begin{aligned}N_u &= \frac{W/R_b}{\mathcal{E}_b/J_0} + 1 \\ &= \frac{1000}{11.3} + 1 = 89 \text{ users}\end{aligned}$$

If the processing gain is reduced to $W/R_b = 500$, then

$$N_u = \frac{500}{11.3} + 1 = 45\text{users}$$

Problem 10.15

(a) We are given a system where $(P_J/P_S)_{dB} = 20\ dB, R = 1000\ bps$ and $(\mathcal{E}_b/J_0)_{dB} = 10\ dB$. Hence, using the relation in (10.3.40) we obtain

$$\begin{aligned}
\left(\frac{W}{R}\right)_{dB} &= \left(\frac{P_J}{P_S}\right)_{dB} + \left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} = 30\ dB \\
\frac{W}{R} &= 1000 \\
W &= 1000R = 10^6 Hz
\end{aligned}$$

(b) The duty cycle of a pulse jammer for worst-case jamming is

$$\alpha^* = \frac{0.7}{\mathcal{E}_b/J_0} = \frac{0.7}{10} = 0.07$$

The corresponding probability of error for this worst-case jamming is

$$P_2 = \frac{0.082}{\mathcal{E}_b/J_0} = \frac{0.082}{10} = 8.2 \times 10^{-3}$$

Problem 10.16

The radio signal propagates at the speed of light, $c = 3 \times 10^8 m/\sec$. The difference in propagation delay for a distance of 300 meters is

$$T_d = \frac{300}{3 \times 10^8} = 1\mu\sec$$

The minimum bandwidth of a DS spread spectrum signal required to resolve the propagation paths is $W = 1\ MHz$. Hence, the minimum chip rate is 10^6 chips per second.

Problem 10.17

(a) We have $N_u = 15$ users transmitting at a rate of $10,000\ bps$ each, in a bandwidth of $W = 1\ MHz$. The ϵ_b/J_0 is

$$\begin{aligned}
\frac{\mathcal{E}}{J_0} &= \frac{W/R}{N_u - 1} = \frac{10^6/10^4}{14} = \frac{100}{14} \\
&= 7.14\,(8.54\ dB)
\end{aligned}$$

(b) The processing gain is 100.

(c) With $N_u = 30$ and $\mathcal{E}_b/J_0 = 7.14$, the processing gain should be increased to

$$W/R = (7.14)\,(29) = 207$$

Hence, the bandwidth must be increased to $W = 2.07MHz$.

Problem 10.18

(a) The length of the shift-register sequence is

$$\begin{aligned} L &= 2^m - 1 = 2^{15} - 1 \\ &= 32767 \text{ bits} \end{aligned}$$

For binary FSK modulation, the minimum frequency separation is $2/T$, where $1/T$ is the symbol (bit) rate. The hop rate is 100 $hops/\sec$. Since the shift register has $N = 32767$ states and each state utilizes a bandwidth of $2/T = 200\ Hz$, then the total bandwidth for the FH signal is 6.5534 MHz.

(b) The processing gain is W/R. We have,

$$\frac{W}{R} = \frac{6.5534 \times 10^6}{100} = 6.5534 \times 10^4 \text{ bps}$$

(c) If the noise is AWG with power spectral density N_0, the probability of error expression is

$$P_2 = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{\frac{W/R}{P_N/P_S}}\right)$$

Problem 10.19

(a) If the hopping rate is 2 hops/bit and the bit rate is 100 bits/sec, then, the hop rate is 200 hops/sec. The minimum frequency separation for orthogonality $2/T = 400 Hz$. Since there are $N = 32767$ states of the shift register and for each state we select one of two frequencies separated by 400 Hz, the hopping bandwidth is 13.1068 MHz.

(b) The processing gain is W/R, where $W = 13.1068\ MHz$ and $R = 100bps$. Hence

$$\frac{W}{R} = 0.131068\, MHz$$

(c) The probability of error in the presence of AWGN is given by (10.3.61) with $N = 2$ chips per hop.

Problem 10.20

a) The total SNR for three hops is $20 \sim 13$ dB.Therefore the SNR per hop is 20/3. The probability of a chip error with noncoherent detection is

$$p = \frac{1}{2} e^{-\frac{\mathcal{E}_c}{2N_0}}$$

where $\mathcal{E}_c/N_0 = 20/3$. The probability of a bit error is

$$\begin{aligned} P_b &= 1 - (1-p)^2 \\ &= 1 - (1 - 2p + p^2) \\ &= 2p - p^2 \\ &= e^{-\frac{\mathcal{E}_c}{2N_0}} - \frac{1}{2} e^{-\frac{\mathcal{E}_c}{N_0}} \\ &= 0.0013 \end{aligned}$$

b) In the case of one hop per bit, the SNR per bit is 20, Hence,

$$\begin{aligned} P_b &= \frac{1}{2}e^{-\frac{\mathcal{E}_c}{2N_0}} \\ &= \frac{1}{2}e^{-10} \\ &= 2.27 \times 10^{-5} \end{aligned}$$

Therefore there is a loss in performance of a factor 57 AWGN due to splitting the total signal energy into three chips and, then, using hard decision decoding.

Problem 10.21

(a) We are given a hopping bandwidth of 2 GHz and a bit rate of 10 kbs. Hence,

$$\frac{W}{R} = \frac{2 \times 10^9}{10^4} = 2 \times 10^5 \, (53dB)$$

(b) The bandwidth of the worst partial-band jammer is $\alpha^* W$, where

$$\alpha^* = 2/\left(\mathcal{E}_b/J_0\right) = 0.2$$

Hence

$$\alpha^* W = 0.4GHz$$

(c) The probability of error with worst-case partial-band jamming is

$$\begin{aligned} P_2 &= \frac{e^{-1}}{(\mathcal{E}_b/J_0)} = \frac{e^{-1}}{10} \\ &= 3.68 \times 10^{-2} \end{aligned}$$

Problem 10.22

The processing gain is given as

$$\frac{W}{R_b} = 500 \;\; (27 \; dB)$$

The $(\mathcal{E}_b/J_0)$ required to obtain an error probability of 10^{-5} for binary PSK is 9.5 dB. Hence, the jamming margin is

$$\begin{aligned} \left(\frac{P_J}{P_S}\right)_{dB} &= \left(\frac{W}{R_b}\right)_{dB} - \left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} \\ &= 27 - 9.5 \\ &= 17.5 \; dB \end{aligned}$$

Problem 10.23

If the jammer is a pulse jammer with a duty cycle $\alpha = 0.01$, the probability of error for binary PSK is given as

$$P_2 = \alpha Q\left(\sqrt{\frac{2W/R_b}{P_J/P_S}}\right)$$

For $P_2 = 10^{-5}$, and $\alpha = 0.01$, we have

$$Q\left(\sqrt{\frac{2W/R_b}{P_J/P_S}}\right) = 10^{-3}$$

Then,

$$\frac{W/R_b}{P_J/P_S} = \frac{500}{P_J/P_S} = 5$$

and

$$\frac{P_J}{P_S} = 100\,(20\;dB)$$

Problem 10.24

$$c\,(t) = \sum_{n=-\infty}^{\infty} c_n p\,(t - nT_c)$$

The power spectral density of $c\,(t)$ is given by

$$\mathcal{S}_c\,(f) = \frac{1}{T_c}\mathcal{S}_c\,(f)\,|P\,(f)|^2$$

where

$$|P\,(f)|^2 = (AT_c)^2 \sin c^2\,(fT_c)\,, \quad T_c = 1\mu\,\mathrm{sec}$$

and $\mathcal{S}_c\,(f)$ is the power spectral density of the sequence $\{c_n\}$. Since the autocorrelation of the sequence $\{c_n\}$ is periodic with period L and is given as

$$R_c\,(m) = \begin{cases} L, & m = 0, \pm L, \pm 2L, \ldots \\ \\ -1, & \text{otherwise} \end{cases}$$

then, $R_c\,(m)$ can be represented in a discrete Fourier series as

$$R_c\,(m) = \frac{1}{L}\sum_{k=0}^{L-1} r_C\,(k)\,e^{j2\pi mk/L}, m = 0, 1, \ldots, L-1$$

where $\{r_c\,(k)\}$ are the Fourier series coefficients, which are given as

$$r_c\,(k) = \sum_{m=0}^{L-1} R_c\,(m)\,e^{-j2\pi km/L}, k = 0, 1, \ldots, L-1$$

and $r_c\,(k + nL) = r_c\,(k)$ for $n = 0, \pm 1, \pm 2, \ldots$. The latter can be evaluated to yield

$$\begin{aligned} r_c\,(k) &= L + 1 - \textstyle\sum_{m=0}^{L-1} e^{-j2\pi km/L} \\ &= \begin{cases} 1, & k = 0, \pm L, \pm 2L, \ldots \\ L+1, & \text{otherwise} \end{cases} \end{aligned}$$

The power spectral density of the sequence $\{c_n\}$ may be expressed in terms of $\{r_c\,(k)\}$. These coefficients represent the power in the spectral components at the frequencies $f = k/L$. Therefore, we have

$$\mathcal{S}_c\,(f) = \frac{1}{L}\sum_{k=-\infty}^{\infty} r_c\,(k)\,\delta\left(f - \frac{k}{LT_c}\right)$$

Finally, we have

$$\mathcal{S}_c\,(f) = \frac{1}{LT_c}\sum_{k=-\infty}^{\infty} r_c\,(k)\left|P\left(\frac{k}{LT_c}\right)\right|^2 \delta\left(f - \frac{k}{LT_c}\right)$$

Problem 10.25

Without loss of generality, let us assume that $L_1 < L_2$. Then, the period of the sequence obtained by forming the modulo-2 sum of the two periodic sequences is

$$L_3 = kL_2$$

where k is the smallest integer multiple of L_2 such that kL_2/L_1 is an integer. For example, suppose that $L_1 = 15$ and $L_2 = 63$. Then, we find the smallest multiple of 63 which is divisible by $L_1 = 15$, without a remainder. Clearly, if we take $k = 5$ periods of L_2, which yields a sequence of $L_3 = 315$, and divide L_3 by L_1, the result is 21. Hence, if we take $21L_1$ and $5L_2$, and modulo-2 add the resulting sequences, we obtain a single period of length $L_3 = 21L, = 5L_2$ of the new sequence.

Problem 10.26

(a) The period of the maximum length shift register sequence is

$$L = 2^{10} - 1 = 1023$$

Since $T_b = LT_c$, then the processing gain is

$$L = \frac{T_b}{T_c} = 1023\,(30dB)$$

(b) The jamming margin is

$$\begin{aligned}\left(\frac{P_J}{P_S}\right)_{dB} &= \left(\frac{W}{R_b}\right)_{dB} - \left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} \\ &= 30 - 10 \\ &= 20dB\end{aligned}$$

where $J_{av} = J_0 W \approx J_0/T_c = J_0 \times 10^6$

Problem 10.27

At the bit rate of 270.8 Kbps, the bit interval is

$$T_b = \frac{10^{-6}}{.2708} = 3.69\mu\text{sec}$$

a) For the suburban channel model, the delay spread is 7 μsec. Therefore, the number of bits affected by intersymbol interference is at least 2. The number may be greater than 2 if the signal pulse extends over more than one bit interval, as in the case of partial response signals, such as CPM.

b) For the hilly terrain channel model, the delay spread is approximately 20 μ sec. Therefore, the number of bits affected by ISI is at least 6. The number may be greater than 6 if the signal pulse extends over more than one bit interval.

Problem 10.28

In the case of the urban channel model, the number of RAKE receiver taps will be at least 2. If the signal pulse extends over more than one bit interval, the number of RAKE taps must be further increased to account for the ISI over the time span of the signal pulse. For the hilly terrain channel model, the minimum number of RAKE taps is at least 6 but only three will be active, one for the first arriving signal and 2 for the delayed arrivals.

If the signal pulse extends over more than one bit interval, the number of RAKE taps must be further increased to account for the ISI over the same span of the signal pulse. For this channel, in which the multipath delay characteristic is zero in the range of 2 μsec to 15 μsec, as many as 3 RAKE taps between the first signal arrival and the delayed signal arrivals will contain no signal components.

Problem 10.29

For an automobile travelling at a speed of 100 Km/hr,

$$f_m = \frac{v f_0}{c} = \frac{10^5}{3600} \times \frac{9 \times 10^8}{3^8} = 83.3\text{Hz}$$

For a train travelling at a speed of 200 Km/hr,

$$f_m = 166.6\text{Hz}$$

The corresponding spread factors are

$$T_m B_d = T_m f_m = \begin{cases} 5.83 \times 10^{-4}, & \text{automobile} \\ 1.166 \times 10^{-3}, & \text{train} \end{cases}$$

The plots of the power spectral density for the automobile and the train are shown below