Disclaimer

I wrote this to my best knowledge, however, no guarantees are given whatsoever.

Sources

If not noted differently, the source is the lecture slides and/or the accompanying book.

1 Approximate Retrieval

Nearest-Neighbor Find $\boldsymbol{x}^* = \operatorname{argmin}_{\boldsymbol{x}\in X}\ d(\boldsymbol{x},\boldsymbol{y})$ given S, $\boldsymbol{y} \in S$, $X \subseteq S$.

Near-Duplicate detection Find all $\boldsymbol{x},\boldsymbol{x}' \in X$ with $d(\boldsymbol{x},\boldsymbol{x}') \leq \epsilon$.

1.1 k-Shingling

Documents (or videos) as set of k-shingles (a. k. a. k-grams). *k-shingle*is consecutive appearance of k chars/words.
Binary *shingle matrix* $M \in \{0,1\}^{C\times N}$ where $M_{i,j}=1$ iff document j contains shingle i, N documents, C k-shingles.

1.2 Distance functions

Def. $d : S \times S \to \mathbb{R}$ is *distance function* iff pos. definite except $d(x,x)=0$ ($d(x,x')>0 \iff x \neq x'$), symmetric ($d(x,x')=d(x',x)$) and triangle inequality holds ($d(x,x'') \leq d(x,x')+d(x',x'')$).

Euclidean L_r $d_r(x,y)=||x-y||_r=(\sum_i |x_i-y_i|^r)^{1/r}$.

Cosine $\operatorname{Sim}_c(A,B)=\dfrac{A\cdot B}{||A||\cdot||B||}$, $d_c(A,B)=\dfrac{\cos^{-1}(\operatorname{Sim}_c(A,B))}{\pi}$.

Jaccard sim., d. $\operatorname{Sim}_J(A,B)=\dfrac{|A\cap B|}{|A\cup B|}$, $d_J(A,B)=1-\operatorname{Sim}_J(A,B)$.

1.3 LSH – local sensitive hashing

Key Idea: Similiar documents have similiar hash.
Note: Trivial for exact duplicates (hash-collision $\to$ candidate pair).

Min-hash-family $h_\pi(C)$ for Jaccard Hash is the *min (i.e. first) non-zero permutated row index*: $h_\pi(C)=\min_{i,C(i)=1}\pi(i)$, bin. vec. C, rand. perm. π.
Note: $\Pr_\pi[h_\pi(C_1)=h_\pi(C_2)]=\operatorname{Sim}_J(C_1,C_2)$ if $\pi \in_{\text{u.a.r.}} S_{|C|}$.

Min-hash L_r-norm: Fix $a \in \mathbb{R}$. Random line $\boldsymbol{w}$ paritioned in buckets of length a. Project $\boldsymbol{x},\boldsymbol{y}$ onto $\boldsymbol{w}$, if in same bucket, $h_{\boldsymbol{w}}(x)=h_{\boldsymbol{w}}(y)$. In 2-dim. forms a $(a/2,2\cdot a,1/2,1/3)$-sensitive hash-family. In d-dim. there exists a $(d1,d2,p1,p2)$-sensitive family $\forall d1<d2$ with $p1>p2$.

Min-hash cos. $h_{\boldsymbol{w}}(\boldsymbol{x})=\operatorname{sgn}(\boldsymbol{w}^T\boldsymbol{x})$. $\Pr_{\boldsymbol{w}}[h_{\boldsymbol{w}}(x)=h_{\boldsymbol{w}}(y)]=1-\frac{\theta_{\boldsymbol{x},\boldsymbol{y}}}{\pi}$.

Min-hash signature matrix $M_S \in [N]^{n\times C}$ with $M_S(i,c)=h_i(C_c)$ given n hash-fns h_i drawn randomly from a universal hash family.

Pseudo permutation h_π with $\pi(i)=(a\cdot i+b) \mod p \mod N$, N number of shingles, $p \geq N$ prime and $a,b \in_{\text{u.a.r.}} [p]$ with $a \neq 0$.
Use as universal hash family. Only store a and b. Much more efficient.

Compute signature matix M_S For column $c\in[C]$, row $r \in [N]$ with $C_c(r)=1$, $M_S(i,c) \leftarrow \min\{h_i(C_c),M_S(i,c)\}$ for all h_i.

(d_1,d_2,p_1,p_2)**-sensitivity** of $F=\{h_1,\ldots,h_n\}$: $\forall x,y \in S : d(x,y) \leq d_1 \implies P[h(x)=h(y)] \geq p_1$ and $d(x,y) \geq d_2 \implies P[h(x)=h(y)] \leq p_2$.

r**-way AND** $h=[h_1,\ldots,h_r]$, $h(x)=h(y) \Leftrightarrow \forall i\ h_i(x)=h_i(y)$ is (d_1,d_2,p_1^r,p_2^r)-sensitive. Decreases FP.

b**-way OR** $h=[h_1,\ldots,h_b]$, $h(x)=h(y) \Leftrightarrow \exists i\ h_i(x)=h_i(y)$ is $(d_1,d_2,1-(1-p_1)^b,1-(1-p_2)^b)$-sensitive. Decreases FN.

Boosting by composition b-way OR after r-way AND. Group sig. matrix into b bands of r rows. CP match in at least one band (check by hashing). Result is $(d_1,d_2,1-(1-p_1^r)^b,1-(1-p_2^r)^b)$-sensitive. r pulls stronger than b. r pulls down.
r-way AND after b-way OR. Result is $(d_1,d_2,(1-(1-p_1)^b)^r,(1-(1-p_2)^b)^r$-sensitive. b pulls stronger than r. b pulls up.

Tradeoff FP/FN Favor FP (work) over FN (wrong). Filter FP by checking signature matrix, shingles or even whole documents.

2 Supervised Learning

Linear classifier $y_i=\operatorname{sgn}(\boldsymbol{w}^T\boldsymbol{x}_i)$ assuming $\boldsymbol{w}$ goes through origin.

Homogeneous transform $\tilde{\boldsymbol{x}}=[\boldsymbol{x},1]$;$\tilde{\boldsymbol{w}}=[\boldsymbol{w},b]$, now $\boldsymbol{w}$ passes origin.

Kernel k is inner product in high-dim. space: $k(\boldsymbol{x},\boldsymbol{y})=\langle\phi(\boldsymbol{x}),\phi(\boldsymbol{y})\rangle$.
shift-invariance $k(\boldsymbol{x},\boldsymbol{y})=k(\boldsymbol{x}-\boldsymbol{y})$.
Gaussian $k(\boldsymbol{x}-\boldsymbol{y})=\exp(-||\boldsymbol{x}-\boldsymbol{y}||_2^2/h^2)$.

Convex function $f : S \to \mathbb{R}$ is convex iff $\forall x,x' \in S, \lambda \in [0,1], \lambda f(x)+(1-\lambda)f(x') \geq f(\lambda x+(1-\lambda)x')$, i. e. every segment lies above function. Equiv. bounded by linear fn. at every point.

H-strongly convex f H-strongly convex iff $f(x') \geq f(x) + \nabla f(x)^T(x'-x)+\frac{H}{2}||x'-x||_2^2$, i. e. bounded by quadratic fn (at every point).

2.1 Support vector machine (SVM)

SVM primal

Quadratic $\min_{\boldsymbol{w}}\boldsymbol{w}^T\boldsymbol{w}+C\sum_i\xi_i$, s.t. $\forall i : y_i\boldsymbol{w}^T\boldsymbol{x}_i \geq 1-\xi_i$, $\forall i : \xi_i \geq 0$.

Hinge loss $\min_{\boldsymbol{w}}\lambda\boldsymbol{w}^T\boldsymbol{w}+\sum_i\max(0,1-y_i\boldsymbol{w}^T\boldsymbol{x}_i)$ with $\lambda=\frac{1}{C}$.

Norm-constrained $\min_{\boldsymbol{w}}\sum_i\max(0,1-y_i\boldsymbol{w}^T\boldsymbol{x}_i)$ s.t. $||\boldsymbol{w}||_2 \leq \frac{1}{\sqrt{\lambda}}$.

Lagrangian dual $\max_{\boldsymbol{\alpha}}\sum_i\alpha_i+\frac{1}{2}\sum_{i,j}\alpha_i\alpha_jy_iy_j\boldsymbol{x}_i^T\boldsymbol{x}_j$, $\alpha_i \in [0,C]$.
Apply kernel trick: $\max_{\boldsymbol{\alpha}}\sum_i\alpha_i+\frac{1}{2}\sum_{i,j}\alpha_i\alpha_jy_iy_jk(\boldsymbol{x}_i,\boldsymbol{x}_j)$, $\alpha_i \in [0,C]$, prediction becomes $y=\operatorname{sgn}(\sum_{i=1}^n\alpha_iy_ik(\boldsymbol{x}_i,\boldsymbol{x}))$.

2.2 Convex Programming

Convex program $\min_{\boldsymbol{x}}f(\boldsymbol{x})$, s. t. $\boldsymbol{x} \in S$, f convex.

Online convex program (OCP) $\min_{\boldsymbol{w}}\sum_{t=1}^Tf_t(\boldsymbol{w})$, s. t. $\boldsymbol{w} \in S$.

General regularized form $\min_{\boldsymbol{w}}\sum_{i=1}^nl(\boldsymbol{w};\boldsymbol{x}_i,y_i)+\lambda R(\boldsymbol{w})$, where l is a (convex) loss function and R is the (convex) regularizer.

General norm-constrained form $\min_{\boldsymbol{w}}\sum_{i=1}^nl(\boldsymbol{w};\boldsymbol{x}_i,y_i)$, s. t. $\boldsymbol{w} \in S_\lambda$, l is loss and S_λ some (norm-)constraint. Note: This is an OCP.

Solving OCP *Feasible set* $S \subseteq \mathbb{R}^d$ and *start pt.* $\boldsymbol{w}_0 \in S$, OCP (as above). Round $t \in [T]$: pick feasible pt. $\boldsymbol{w}_t$, get convex fn. f_t, incur $l_t=f_t(\boldsymbol{w}_t)$. Regret $R_T=(\sum_{t=1}^Tl_t)-\min_{\boldsymbol{w}\in S}\sum_{t=1}^Tf_t(\boldsymbol{w})$.

Online SVM $||\boldsymbol{w}||_2 \leq \frac{1}{\lambda}$ (norm-constr.). For new pt. $\boldsymbol{x}_t$ classify $y_t=\operatorname{sgn}(\boldsymbol{w}_t^T\boldsymbol{x}_t)$, incur $l_t=\max(0,1-y_t\boldsymbol{w}_t^T\boldsymbol{x}_t)$, update $\boldsymbol{w}_t$ (see later). Best $L^*=\min_{\boldsymbol{w}}\sum_{t=1}^T\max(0,1-y_t\boldsymbol{w}^T\boldsymbol{x}_t)$, regret $R_t=\sum_{t=1}^Tl_t-L^*$.

Online proj. gradient descent (OPGD) Update for online SVM: $\boldsymbol{w}_{t+1}=\operatorname{Proj}_S(\boldsymbol{w}_t-\eta_t\nabla f_t(\boldsymbol{w}_t))$ with $\operatorname{Proj}_S(\boldsymbol{w})=\operatorname{argmin}_{\boldsymbol{w}'\in S}||\boldsymbol{w}'-\boldsymbol{w}||_2$, gives regret bound $\frac{R_T}{T} \leq \frac{1}{\sqrt{T}}(||\boldsymbol{w}_0-\boldsymbol{w}^*||_2^2+||\nabla f||_2^2)$, $\nabla f_t(\boldsymbol{w}_t)=-y_t\boldsymbol{x}_t$, if $y_t\boldsymbol{w}_t^T\boldsymbol{x}_t \leq 1$, else 0; $\eta_t=\frac{1}{\sqrt{t}}$ and $\operatorname{Proj}_s(\boldsymbol{w}')=\boldsymbol{w}'\cdot\min\Big(1,1/(\sqrt{\lambda}||\boldsymbol{w}'||_2)\Big)$.

For H-strongly convex fn, $\eta_t=\frac{1}{Ht}$ gives $R_t \leq \frac{||\nabla f||^2}{2H}(1+\log T)$.

Stochastic Gradient Descent (SGD) Convex program is unconstrained and decomposes $f(w)=\sum_i^n g(w;i)$. In SVM $g(w;i)=\frac{\lambda w^Tw}{n}+l_h(w;x_i,y_i)$. *Algo:* $w \leftarrow 0$ and choose β_t. until convergence: $(x_i,y_i) \in_{u.a.r} X$. $w \leftarrow w-\beta_t\cdot\nabla_w(g(w;i))$.

Stochastic PGD (SPGD) Online-to-batch. Compute $\tilde{\boldsymbol{w}}=\frac{1}{T}\sum_{t=1}^T\boldsymbol{w}_t$. If data i. i. d.: exp. *error (risk)* $\mathbb{E}[L(\tilde{\boldsymbol{w}})] \leq L(\boldsymbol{w}^*)+R_T/T$, $L(\boldsymbol{w}^*)$ is best error (risk) possible.

PEGASOS OPGD w/ mini-batches on strongly convex SVM form. $\min_w\sum_{t=1}^Tg_t(\boldsymbol{w})$, s.t. $||\boldsymbol{w}||_2 \leq \frac{1}{\sqrt{\lambda}}$, $g_t(\boldsymbol{w})=\frac{\lambda}{2}||\boldsymbol{w}||_2^2+f_t(\boldsymbol{w})$.
g_t is λ-strongly convex, $\nabla g_t(\boldsymbol{w})=\lambda\boldsymbol{w}+\nabla f_t(\boldsymbol{w})$.
Performance ϵ-accurate sol. with prob. $\geq 1-\delta$ in runtime $O^*(\frac{d\cdot\log\frac{1}{\delta}}{\lambda\epsilon})$.

ADAGrad Adapt to geometry. *Mahalanobis norm* $||\boldsymbol{w}||_{\boldsymbol{G}}=||\boldsymbol{G}\boldsymbol{w}||_2$. $\boldsymbol{w}_{t+1}=\operatorname{argmin}_{\boldsymbol{w}\in S}||\boldsymbol{w}-(\boldsymbol{w}_t-\eta\boldsymbol{G}_t^{-1}\nabla f_t(\boldsymbol{w}))||_{\boldsymbol{G}_t}$. Min. regret with $G_t=(\sum_{\tau=1}^t\nabla f_\tau(\boldsymbol{w}_\tau)\nabla f_\tau(\boldsymbol{w}_\tau)^T)^{1/2}$. Easily inv'able matrix with $G_t=\operatorname{diag}(\ldots)$. $R_t \in O(\frac{||\boldsymbol{w}^*||_\infty}{\sqrt{T}}\sqrt{d})$, even better for sparse data.

ADAM Add 'momentum' term: $\boldsymbol{w}_{t+1}=\boldsymbol{w}_t-\mu\bar{g}_t$, $g_t=\nabla f_t(\boldsymbol{w})$, $\bar{g}_t=(1-\beta)g_t+\beta\bar{g}_{t-1}$, $\bar{g}_0=0$. Helps for dense gradients.

Parallel SGD (PSGD) Randomly partition to k (indep.) machines. Comp. $\boldsymbol{w}=\frac{1}{k}\sum_{i=1}^k\boldsymbol{w}_i$. $\mathbb{E}[\text{err}] \in O(\epsilon(\frac{1}{k\sqrt{\lambda}}+1))$ if $T \in \Omega(\frac{\log\frac{k\lambda}{\epsilon}}{\epsilon\lambda})$. Suitable for MapReduce cluster, multi. passes possible.

Hogwild! Shared mem., no sync., sparse data. [...]

Implicit kernel trick Map $x \in \mathbb{R}^d \to \phi(x) \in \mathbb{R}^D \to z(x) \in \mathbb{R}^m$, $d \ll D, m \ll D$. Where $\phi(x)$ corresponds to a kernel $k(x,x')=\phi(x)^T\phi(x')$.

Random fourier features Given shift-invariant kernel k.
$p(\omega)=\frac{1}{2\pi}\int e^{-j\omega'\delta}k(\delta)\mathrm{d}\Delta$
$\omega_i \sim p=$ eg Gaussian, $b_i \sim U(0,2\pi)$
$\boldsymbol{z}(\boldsymbol{x}) \equiv \sqrt{2/m}[cos(\omega_1'\boldsymbol{x}+b_1)\ldots cos(\omega_m'\boldsymbol{x}+b_m)]$

Nyström features (need entire dataset) In practice: pick random samples $S=\{\hat{\boldsymbol{x}}_1\ldots\hat{\boldsymbol{x}}_n\} \subseteq X$
$\boldsymbol{K}_{SX\,i,j}=k(\hat{\boldsymbol{x}}_i,\hat{\boldsymbol{x}}_j)$, $\boldsymbol{K}_{SS\,i,j}=k(\hat{\boldsymbol{x}}_i,\hat{\boldsymbol{x}}_j)$
approximate $\boldsymbol{K}=\boldsymbol{K}_{XS}\boldsymbol{K}_{SS}^{-1}\boldsymbol{K}_{SX}$,$\boldsymbol{K}_{SS}=\boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^T$.
new point $\boldsymbol{x}'$: $\boldsymbol{z}(\boldsymbol{x}')=\boldsymbol{D}^{-1/2}\boldsymbol{V}^T[k(\boldsymbol{x}',\hat{\boldsymbol{x}}_1),\ldots,k(\boldsymbol{x}',\hat{\boldsymbol{x}}_m)]$

3 Pool-based active Learning (semi-supervised)

Uncertainty sampl. Until all labels can be inferred (or until a *label budget*); $U_t(x)=U(x|x_{1:t-1},y_{1:t-1})$, request y_t for $x_t=\operatorname{argmax}_xU_t(x)$. **SVM:** $x_t=\operatorname{argmin}_{x_i}|\boldsymbol{w}^T\boldsymbol{x}_i|$, i.e. $U_t(\boldsymbol{x})=\frac{1}{|\boldsymbol{w}_t^T\boldsymbol{x}|}$.

Sub-linear time w/ LSH $|\boldsymbol{w}^T\boldsymbol{x}_i|$ small if $\angle_{\boldsymbol{w},\boldsymbol{x}_i}$ close to $\frac{\pi}{2}$ (90°). Hash hyperplane: $h_{\boldsymbol{u},\boldsymbol{v}}(\boldsymbol{a},\boldsymbol{b})=[h_{\boldsymbol{u}}(\boldsymbol{a}),h_{\boldsymbol{v}}(\boldsymbol{b})]=[\text{sgn}(\boldsymbol{u}^T\boldsymbol{a}),\text{sgn}(\boldsymbol{v}^T\boldsymbol{b})]$. LSH hash family: $h_H(z)=h_{u,v}(\boldsymbol{z},\boldsymbol{z})$ if z datapoint, $h_H(z)=h_{u,v}(\boldsymbol{z},-\boldsymbol{z})$ if z query hyperplane. $\Pr[h_H(\boldsymbol{w})=h_H(\boldsymbol{x})]=\Pr[h_{\boldsymbol{u}}(\boldsymbol{w})=h_{\boldsymbol{u}}(\boldsymbol{x})]\Pr[h_{\boldsymbol{v}}(-\boldsymbol{w})=h_{\boldsymbol{v}}(\boldsymbol{x})]=\frac{1}{4}-\frac{1}{\pi^2}(\angle_{\boldsymbol{x},\boldsymbol{w}}-\frac{\pi}{2})^2$. Hash all unlabeled. Loop: Hash $\boldsymbol{w}$, req. labels for hash-coll., update.

Informativeness Metric of "information" gainable; $\neq$ uncertainty.

Version Space $\mathcal{V}(D)=\{\boldsymbol{w} \mid \forall(\boldsymbol{x},y)\in D\ \text{sgn}(\boldsymbol{w}^T\boldsymbol{x})=y)\}$

Relevant version space given unlabeled pool $U=\{x_1',\dots,x_n'\}$. $\tilde{\mathcal{V}}(D;U)=\{h:U\to\{\pm1\} \mid \exists\boldsymbol{w}\in\mathcal{V}(D)\ \forall x\in U\ \text{sgn}(\boldsymbol{w}^T\boldsymbol{x})=h(\boldsymbol{x})\}$.

Generalized binary search Init $D\leftarrow\{\}$. While $|\tilde{\mathcal{V}}(D;U)|>1$, comp. $v^{\pm}(x)=|\tilde{\mathcal{V}}(D\cup\{(x,\pm)\};U)|$, label of $\text{argmin}_x\max\{v^-(x),v^+(x)\}$.

Approx. $|\mathcal{V}|$ Margins of SVM $m^{\pm}(x)$ for labels $\{+,-\},\forall x$. *Max-min* $\max_x\min\{m^+(x),m^-(x))\}$ or *ratio* $\max_x\min\{\frac{m^+(x)}{m^-(x)},\frac{m^-(x)}{m^+(x)}\}$.

4 Model-based clustering – Unsupervised learning

k-means problem $\min_\mu L(\mu)$ with $L(\mu)=\sum_{i=1}^N\min_j||\boldsymbol{x}_i-\mu_j||_2^2$ and *cluster centers* $\mu=\mu_1,\dots,\mu_k$. Non-convex! NP-hard in general!

LLoyd's Init $\mu^{(0)}$ (somehow/randomly). *Assign* all $\boldsymbol{x}_i$ to closest center $z_i\leftarrow\text{argmin}_{j\in[k]}||\boldsymbol{x}_i-\mu_j^{(t-1)}||_2^2$, *Update* to mean: $\mu_j^{(t)}\leftarrow\frac{1}{n_j}\sum_{i:z_i=j}\boldsymbol{x}_i$. Always converge to *local minimum.*

Online k-means Init μ somehow/randomly. For $t\in[n]$ find $c=\text{argmin}_j||\mu_j-\boldsymbol{x}_t||_2$, set $\mu_c\leftarrow\mu_c+\eta_t(\boldsymbol{x}_t-\mu_c)$. For local optimum: $\sum_t\eta_t=\infty\wedge\sum_t\eta_t^2<\infty$ suffices, e.g. $\eta_t=\frac{c}{t}$, $c\in\mathbb{R}$.

Weighted rep. C $L_k(\mu;C)=\sum_{(w,\boldsymbol{x})\in C}w\cdot\min_j||\mu_j-\boldsymbol{x}||_2^2$.

(k,ϵ)-**coreset** iff $\forall\mu:(1-\epsilon)L_k(\mu;D)\le L_k(\mu;C)\le(1+\epsilon)L_k(\mu;D)$.

D^2-**sampling** Sample prob. $p(x)=\frac{d(x,B)^2}{\sum_{x'\in X}d(x',B)^2}$.

Merge coresets union of (k,ϵ)-coreset is also (k,ϵ)-coreset.

Compress a (k,δ)-coreset of a (k,ϵ)-coreset is a $(k,\epsilon+\delta+\epsilon\delta)$-coreset.

Coresets on streams Bin. tree of merge-compress. Error $\propto$ height.

Mapreduce k-means Construct (k,ϵ)-coreset C, solve k-means (w/ many restarts) on coreset. (Repeat.) Near-optimal solution.

5 k-armed bandits as recommender systems

k-**armed bandit** k arms with diff. prob. dist. For $t\in[T]$ rounds, pick $i_t\in[k]$, sample $y_t\in P_i$ (indep. of other rounds). Max. $\sum_{t=1}^T y_t$.

Regret μ_i mean of P_i (arm i), $\mu^*=\max_i\mu_i$. Regret $r_t=\mu^*-\mu_{i_t}$. Total regret $R_T=\sum_{t=1}^T r_t$.

ϵ-**greedy** *Explore* u.a.r. with prob. ϵ_t, *exploit* with prob. $1-\epsilon_t$: choose $\text{argmax}_i\hat{\mu}_i$. Suitable $\epsilon_t\in O(1/t)$ gives $R_T\in O(k\log T)$. Clearly unoptimal.

UCB1 Init $\hat{\mu}_i\leftarrow 0$; try all arms once. Following $t\in[T-k]$ rounds: $UCB(i)\leftarrow\hat{\mu}_i+\sqrt{\frac{2\text{log}t}{n_i}}$, pick $i_t\leftarrow\text{argmax}_i UCB(i)$, observe y_t. Update $n_{i_t}\leftarrow n_{i_t}+1,\hat{\mu}_{i_t}\leftarrow\hat{\mu}_{i_t}+\frac{y_t-\hat{\mu}_{i_t}}{n_{i_t}}$.

Contextual bandits Round t: Obs. *context* $\boldsymbol{z}_t\in\mathcal{Z}\subseteq\mathbb{R}^d$; *recommend* $\boldsymbol{x}_t\in\mathcal{A}_t$. Reward $y_t=f(\boldsymbol{x}_t,\boldsymbol{z}_t)+\epsilon_t$. Regret $r_t=\max_{\boldsymbol{x}}f(\boldsymbol{x},\boldsymbol{z}_t)-f(\boldsymbol{x}_t,\boldsymbol{z}_t)$. Often $f(\boldsymbol{x},\boldsymbol{z})=\boldsymbol{w}_{\boldsymbol{x}}^T\boldsymbol{z}$ linear.

Idea behind LinUCB Estimate $\hat{\boldsymbol{w}}_i=\text{argmin}_{\boldsymbol{w}}\sum_{t=1}^m(y_t-\boldsymbol{w}^T\boldsymbol{z}_t)+||\boldsymbol{w}||_2^2$. Closed form: $\hat{\boldsymbol{w}}_i=M_i^{-1}D_i^T\boldsymbol{y}_i$, $M_i=D_i^TD_i+I$, $D_i=[\boldsymbol{z}_1|\dots|\boldsymbol{z}_m],\boldsymbol{y}_i=(y_1|\dots|y_m)^T$.

Confidence If $\alpha=1+\sqrt{\ln(\frac{2}{\delta})/2}$:

$$\Pr\left[|\hat{\boldsymbol{w}}_i^T\boldsymbol{z}_t-\boldsymbol{w}_i^T\boldsymbol{z}_t|\le\alpha\sqrt{\boldsymbol{z}_t^TM_i^{-1}\boldsymbol{z}_t}\right]\ge1-\delta.$$

LinUCB (Algorithm) For $t=[T]$ receive *action set* A_t and features $\boldsymbol{z}_t$. For all $\boldsymbol{x}\in A_t$: if $\boldsymbol{x}$ new, set $M_{\boldsymbol{x}}\leftarrow\mathbb{I}$ and $b_x\leftarrow 0$; set $\hat{w}_{\boldsymbol{x}}\leftarrow M_{\boldsymbol{x}}^{-1}b_{\boldsymbol{x}}$; set $UCB_{\boldsymbol{x}}\leftarrow\hat{w}_{\boldsymbol{x}}^T\boldsymbol{z}_t+\alpha\sqrt{\boldsymbol{z}_t^TM_{\boldsymbol{x}}^{-1}\boldsymbol{z}_t}$. Recommend action $\boldsymbol{x}_t=\text{argmax}_{\boldsymbol{x}\in A_t}UCB_{\boldsymbol{x}}$; observe y_t. Set $M_{\boldsymbol{x}}\leftarrow M_{\boldsymbol{x}}+\boldsymbol{z}_t\boldsymbol{z}_t^T$ and $b_{\boldsymbol{x}}\leftarrow b_{\boldsymbol{x}}+y_t\boldsymbol{z}_t$.

Hybrid Model $y_t=\boldsymbol{w}_i^T\boldsymbol{z}_t+\beta^T\phi(\boldsymbol{x}_i,\boldsymbol{z}_t)+\epsilon_t$ captures sep. and shared effects.

Rejection Sampling Evaluate bandit: For $t\in\mathbb{N}$ read log $(\boldsymbol{x}_1^{(t)},\dots,\boldsymbol{x}_k^{(t)},\boldsymbol{z}_t,a_t,y_t)$. Pick a_t' by algo. If $a_t'=a_t$ feed y_t to algo., else ignore line. Stop after T feedbacks.

6 Submodularity

$F:2^V\to\mathbb{R}$ *subm.* iff $F(A\cup\{s\})-F(A)\ge F(B\cup\{s\})-F(B)$, $\forall A\subseteq B\subseteq V,s\in V$ with $s\notin B$.

Closedness of non-negative linear combination: If $F_{1,\dots,m}(A)$ subm. then $\lambda_i\ge 0$: $F'(A):=\sum_i\lambda_iF_i(A)$ subm.

Other closedness: If $F(S)$ is subm. on V and $W\subseteq V$, then *Restriction* $F(S\cap W)$, *Conditioning* $F(S\cup W)$, *Reflection* $F(V\setminus S)$ are submodular.

Marginal gain: $\Delta_F(s|A)=F(\{s\}\cup A)-F(A)$

Greedy algo: In round $i+1$, previously picked $A_i=\{s_1,\dots,s_i\}$; pick $s_{i+1}=\text{argmax}_s\Delta_F(s|A_i)=\text{argmax}_s(F(\{s\}\cup A_i)-F(A_i))$. This gives a $(1-1/e)\approx0.63$-approximation.

Lazy Greedy: *Observation:* Submodularity implies $\Delta(s|A_i)\ge\Delta(s|A_{i+1})$. Algo.: $A_0\leftarrow\{\}$; first iteration as usual. Then keep ordered list of Δ_i from prev. iteration. For $i\in[k]$ do: $\Delta_i=F(A_{i-1}\cup\{s^*\})$, $s^*=\text{argmax}_s\Delta_F(s|A_{i-1})$ (*top element*). If s^* is still top, then $A_i\leftarrow A_{i-1}\cup\{s^*\}$ else resort and pick top element (as in Greedy).

7 Tips

$\sum||y_i-w^Tx_i||^2=\sum(y_i-w^Tx_i)^T(y_i-w^Tx_i)=(y-Xw)^T(y-Xw)$

Showing active learning needs n labels Assume points are all distinct and algo has returned -1 for the first $n-1$ labels. Choose x s.t. algo can not distinguish between $-1,+1$.

Sublinear strategy approach Sort and rename points. Cleverly search witnesses (points determining interval).

Deriving Dual Start with quadratic. Rewrite constraints $c_{1,i},c_{2,i}$ s.t. $c_i(w)\ge0$. Get Lagrangian $L(w,\xi,\alpha,\gamma)$: $f(w)-\sum_i\alpha_ic_{1,i}-\sum_i\gamma_ic_{2,i}$. Set derivatives $\frac{\partial L}{\partial w}$ and $\frac{\partial L}{\partial \xi_i}$ to 0 and get w, C. Insert into $L(w)$. Result is max. qp with constraint $\alpha_i\in[0,C]$

8 Probability

$$\begin{aligned}
\mathbb{E}[X]&=\sum_x\Pr(x)\cdot x\\
\mathbb{E}[aX+aY]&=a\cdot\mathbb{E}[x]+b\cdot\mathbb{E}[y]\\
\text{Var}[X]&=\sum_x p(x)\cdot(x-\mathbb{E}[X])^2\\
&=\mathbb{E}[(X-\mathbb{E}[X])^2]=\mathbb{E}[X^2]-(\mathbb{E}[X])^2\\
\text{Cov}[X,Y]&=\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]=\mathbb{E}[X\cdot Y]-\mathbb{E}[X]\cdot\mathbb{E}[Y]\\
\text{Var}[\alpha X+\beta Y]&=\alpha\text{Var}[X]+2\alpha\beta Cov[X,Y]+\beta\text{Cov}[Y]\\
\text{Cov}[X,X]&=\text{Var}[X]\\
\text{Cov}[A+B,X]&=\text{Cov}[A,X]+\text{Cov}[B,X]\\
\Pr(A\cup B)&=\Pr(A)+P(B)-\Pr(A\cap B)\\
\Pr(A\cup B)&=\Pr(A)+P(B)\ \text{ if A, B mutually exclusive}\\
\Pr(A\cap B)&=\Pr(A|B)P(B)=\Pr(B|A)P(A)\\
\Pr(A\cap B)&=\Pr(A)P(B)\ \text{ if A, B independent}\\
\Pr(A\,|\,B)&=\frac{P(A\cap B)}{P(B)}=\frac{P(B|A)P(A)}{P(B)}
\end{aligned}$$

Hoeffding's Inequality

$$\Pr\left(\overline{X}-\text{E}\left[\overline{X}\right]\ge t\right)\le\exp\left(-\frac{2n^2t^2}{\sum_{i=1}^n(b_i-a_i)^2}\right)$$

$$\Pr\left(\left|\overline{X}-\text{E}\left[\overline{X}\right]\right|\ge t\right)\le2\exp\left(-\frac{2n^2t^2}{\sum_{i=1}^n(b_i-a_i)^2}\right)$$

Markov's Inequality $\mathbb{P}(X\ge a)\le\frac{\mathbb{E}(X)}{a}$.

Gaussian $\mathcal{N}(\mu,\sigma^2)$: $\frac{1}{\sqrt{2\sigma^2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$