1 System Theory

$$\dot{x}(t) = A^c x(t) + B^c u(t)$$
$$y(t) = C^c x(t) + D^c u(t)$$
$$x(t) = e^{A^c(t-t_0)}x_0 + \int_{t_0}^{t} e^{A^c(t-\tau)}Bu(\tau)d\tau$$
$$e^{A^c t} = \sum_{n=0}^{\infty} \frac{(A^c t)^n}{n!}$$
$$x(k+N) = A^N x(k) + \sum_{i=0}^{N-1} A^i Bu(k+N-1-i)$$

1.1 Nonlinear Systems

we define a system to be stable in the sense of Lyapunov, if it stays in any arbitrarily small neighborhood of the origin when it is disturbed slightly

1.2 Lyapunov Stability

Lyapunov stable if for every $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that

$$\operatorname{norm} x(0) < \delta(\epsilon) \to \operatorname{norm} x(k) < \epsilon, \forall k \geq 0$$

asymptotically stable in $\Omega \subseteq \mathbb{R}^n$ if it is Lyapunov stable and attractive $\lim_{k\to\infty} x(k) = 0, \forall x(0) \in \Omega$

1.3 Lyapunov Function

Consider the equilibrium point $x = 0$. Let $\Omega \subset \mathbb{R}^n$ be a closed and bounded set containing the origin. A function $V : \mathbb{R}^n \to \mathbb{R}$, continous at the origin, finite for every $x \in \Omega$, and such that

$$V(0) = 0 \text{ and } V(x) > 0, \forall x \in \Omega \backslash \{0\}$$
$$V(g(x)) - V(x) \leq -\alpha(x), \forall x \in \Omega \backslash \{0\}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}$ is continuous positive definite
If a system admins a Lyapunov function $V(x)$, then $x = 0$ is **asymptotically stable** in Ω (sufficient but not necessary)
If a system admits a Lyapunov function, which additionally satisfies $\operatorname{norm} x \to \infty \to V(x) \to \infty$, then $x = 0$ is **globally asymptotically stable**

tim: Check Eig. values of $(APA - P)$ neg., $V(x) = x^T Px$?

Linear systems: iff eigenvalues of A inside unit circle (i.e. stable) then $\exists$*unique* $P > 0$ that solves $A_{cl}^T PA_{cl} - P = -Q,\ Q > 0$ and $V = x^T Px$ is a lyapunov function.

1.4 Discretization

Euler: $A = I + T_s A^c,\ B = T_s B^c,\ C = C^c,\ D = D^c$

$$x(k+1) = x(k) + T_s g^c(x(k), u(k)) = g(x(k), u(k))$$
$$y(k) = h^c(x(k), u(k)) = h(x(k), u(k))$$

Exact: (assumption of a constant $u(t)$ during T_s)

$$A = e^{A^c T_s},\ B = \int_0^{T_s} e^{A^c(T_s-\tau')}B^c d\tau$$
$$B = (A^c)^{-1}(A - I)B^c, \text{ if } A^c \text{ invertible}$$

1.5 Controllability (reachability) and observability

$$C = [B\ AB\ ...\ A^{n-1}B]$$
$$O = [C^T\ (CA)^T\ ...\ (CA^{n-1})^T]$$

2 Unconstrained Control

2.1 Block Approach (used also for $\bar{w}$ substition)

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{A} \\ \vdots \\ \boldsymbol{A}^N \end{bmatrix} x(0) + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{AB} & \boldsymbol{B} & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{0} \\ \boldsymbol{A}^{N-1}\boldsymbol{B} & \dots & \boldsymbol{AB} & \boldsymbol{B} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_1} \end{bmatrix}$$

$$x = \boldsymbol{S}^x \cdot x(0) + \boldsymbol{S}^u \cdot u \qquad \text{size}(\boldsymbol{S}^x) = [n_{\text{states}} \cdot (N+1), N]$$
$$\text{size}(\boldsymbol{S}^u) = [n_{\text{states}} \cdot (N+1), n_{\text{states}}]$$
$$\bar{\boldsymbol{Q}} = \text{diag}(\boldsymbol{Q}, \dots, \boldsymbol{Q}, \boldsymbol{P}) \qquad \text{size}(\bar{\boldsymbol{Q}}) = [n_{\text{states}} \cdot (N+1), n_{\text{states}} \cdot (N+1)]$$
$$\bar{\boldsymbol{R}} = \text{diag}(\boldsymbol{R}, \dots, \boldsymbol{R}) \qquad \text{size}(\bar{\boldsymbol{R}}) = [n_{\text{input}} \cdot N, n_{\text{input}} \cdot N]$$
$$\boldsymbol{H} = \boldsymbol{S}^{u\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^u + \boldsymbol{R} \qquad \boldsymbol{F} = \boldsymbol{S}^{x\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^u$$
$$\boldsymbol{Y} = \boldsymbol{S}^{x\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^x$$

Optimal cost and control

$$J^*(x_0) = -x_0^T \boldsymbol{F}\boldsymbol{H}\boldsymbol{F}^T x_0 + x_0^T \boldsymbol{Y} x_0$$
$$u^*(x_0) = -\boldsymbol{H}^{-1}\boldsymbol{F}^T x_0 = -\left(\boldsymbol{S}^{u\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^u + \boldsymbol{R}\right)^{-1} \boldsymbol{S}^{u\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^x x_0$$

2.2 Recursive Approach

$$J_k^*(x_k) = \min_{u_k} I(x_k, u_k) + J_{k+1}(x_{k+1})$$

Is a feedback controller as opposed to the Batch Approach. For LQR solve via Riccati Difference Equation (RDE).

$$\boldsymbol{F}_k = -(\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{B} + \boldsymbol{R})^{-1}\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{A}$$
$$\boldsymbol{P}_k = \boldsymbol{A}^T\boldsymbol{P}_{k+1}\boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{A}^T\boldsymbol{P}_{k+1}\boldsymbol{B}(\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{B} + \boldsymbol{R})^{-1}\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{A}$$

$$u_k^* = \boldsymbol{F}_k\ x_k \qquad J_k^*(x_k) = x_k^T\boldsymbol{P}_k\ x_k \qquad \boldsymbol{P}_N = \boldsymbol{P}$$

For unconstrained Infinite Horizon Problem, substituting $\boldsymbol{P}_\infty = \boldsymbol{P}_k = \boldsymbol{P}_{k+1}$ into RDE gives DARE. Uniquely solvable, iff (A, B) stabilizable and (A, G) detectable, where $\boldsymbol{G}\boldsymbol{G}^T = \boldsymbol{Q}$. Follows from closed-loop system $x_{k+1} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}_k)x_k$

3 (Convex) Optimization

3.1 Convexity

Convex set $\mathcal{X}$ iff $\forall\lambda \in [0,1] \forall x, y \in \mathcal{X}\ \lambda x + (1-\lambda)y \in \mathcal{X}$.
Intersection preserves convexity, union does not.

Affine set $\mathcal{X}$ $= \{x \in \mathbb{R}^n | \boldsymbol{A}x = b\}$ for some $\boldsymbol{A}, b$

Subspace is affine set through origin, i.e. $b = \boldsymbol{0}$, aka Nullspace of $\boldsymbol{A}$.

Hyperplane $\mathcal{X}$ $= \{x \in \mathbb{R}^n | a^T x = b\}$ for some a, b.

Halfspace $\mathcal{X}$ $= \{x \in \mathbb{R}^n | a^T x \leq b\}$ for some a, b.

Polyhedron $\mathcal{P}$ $= \{x | a_i^T x \leq b_i, i = 1, \dots, n\} = \{x | \boldsymbol{A}x \leq b\}$

Cone

Ellipsoid $\mathcal{E}$ $= \{x | (x - x_c)^T\boldsymbol{A}^{-1}(x - x_c) \leq 1\}$, x_c center point.

Convex function

Norm $f(x) : \mathbb{R}^n \to \mathbb{R}$

$$f(x) = 0 \implies x = 0, \qquad f(x) \geq 0$$
$$f(\alpha \cdot x) = |\alpha| \cdot f(x) \qquad \text{for scalar } \alpha$$
$$f(x + y) \leq f(x) + f(y) \qquad \forall x, y \in R^n$$

tim: Maybe move the above somewhere else?

General Problem $\min_{x \in \text{dom}(f)} f(x)$ s. t. $g_i(x) \leq 0$ and $h_j(x) = 0$.

3.2 Linear Programming (LP)

Problem statement $\min c^T x$ such that $\boldsymbol{G}x \leq h$ and $\boldsymbol{A}x = b$.

Norm l_∞ $\min_x \|x\|_\infty = \min_{x \in \mathbb{R}^n} [\max\{x, \dots, x_n, -x_1, \dots, -x_n\}]$:

$$\min_{x,t} t \quad \text{subject to} \quad x_i \leq t, -x_i \leq t, \qquad \boldsymbol{F}x \leq g$$
$$\iff \min_{x,t} t \quad \text{subject to} \quad -\boldsymbol{1}t \leq x \leq 1t, \qquad \boldsymbol{F}_x \leq g.$$

Norm l_1 $\min_x \|x\|_1 = \min_x \left[\sum_{i=1}^m \max\{x_i, -x_i\}\right]$:

$$\min_t t_1 + \dots + t_m \quad \text{subject to} \quad x_i \leq t_i, -x_i \leq t_i, \qquad \boldsymbol{F}x \leq g$$
$$\iff \min_t \boldsymbol{I}^T t \quad \text{subject to} \quad -t \leq x \leq t, \qquad \boldsymbol{F}_x \leq g.$$

Note that for $\dim x = 1$, l_1 and l_∞ are the same.

MPC with linear cost
$J(x_0, u) = \|\boldsymbol{P}x_N\|_p + \sum_{i=0}^{N-1}\|\boldsymbol{Q}x_i\|_p + \|\boldsymbol{R}u_i\|_p$.

tim: Insert here slide 45, lect 4

Receding Horizon Control – RHC

QP with substitution (see also Batch approach)

$$J^*(x_k) = \min_u \begin{bmatrix} u^T & x_k^T \end{bmatrix} \begin{bmatrix} \boldsymbol{H} & \boldsymbol{F}^T \\ \boldsymbol{F} & \boldsymbol{Y} \end{bmatrix} \begin{bmatrix} u \\ x_k \end{bmatrix}$$
$$\text{s. t. } \boldsymbol{G}\ u \leq w + \boldsymbol{E}\ x_k$$

Latter gives three sets (same for without substitution)

$$\begin{aligned} \mathcal{X} &= \{x \mid A_x\ x \leq b_x\} \\ \mathcal{U} &= \{u \mid A_u\ u \leq b_u\} \\ \mathcal{X}_{\{} &= \{x \mid A_f\ x \leq b_f\} \end{aligned}$$

State equations are in cost matrix, usually
$\boldsymbol{A}_x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, b_x = \begin{bmatrix} b_{\max} \\ -b_{\min} \end{bmatrix}$

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{A}_u & 0 & \dots & 0 \\ 0 & \boldsymbol{A}_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{A}_u \\ 0 & 0 & \dots & 0 \\ \boldsymbol{A}_x\boldsymbol{B} & 0 & \dots & 0 \\ \boldsymbol{A}_x\boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}_x\boldsymbol{B} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}_x\boldsymbol{A}^{N-2}\boldsymbol{B} & \boldsymbol{A}_x\boldsymbol{A}^{N-3}\boldsymbol{B} & \dots & 0 \\ \boldsymbol{A}_f\boldsymbol{A}^{N-1}\boldsymbol{B} & \boldsymbol{A}_f\boldsymbol{A}^{N-2}\boldsymbol{B} & \dots & \boldsymbol{A}_f\boldsymbol{B} \end{bmatrix} \quad \boldsymbol{E} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\boldsymbol{A}_x \\ -\boldsymbol{A}_x\boldsymbol{A} \\ -\boldsymbol{A}_x\boldsymbol{A}^2 \\ \vdots \\ -\boldsymbol{A}_x\boldsymbol{A}^{N-1} \\ -\boldsymbol{A}_f\boldsymbol{A}^N \end{bmatrix} \quad \boldsymbol{W} = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_x \\ b_f \end{bmatrix}$$

QP with out substitution

State equations represented in equality constrainst.

$$J^*(x_k) = \min_z \begin{bmatrix} z^T & x_k^T \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{H}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} z \\ x_k \end{bmatrix}$$
$$\begin{aligned} \text{s. t. } \boldsymbol{G}\ z &\leq w + \boldsymbol{E}\ x_k \\ \boldsymbol{G}_{\text{eq}}\ z &= \boldsymbol{E}_{\text{eq}}\ x_k, \quad \text{system dynamics} \end{aligned}$$

$\bar{\boldsymbol{H}} = \operatorname{diag}(\boldsymbol{Q}, \dots, \boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{R}, \dots, \boldsymbol{R})$

$$z = \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} \quad \boldsymbol{G}_{\text{eq}} = \left[\begin{array}{cccc|cccc} \boldsymbol{I} & & & & -\boldsymbol{B} & & & \\ -\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B} & & \\ & \ddots & \ddots & & & & \ddots & \\ & & -\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B} \end{array}\right] \quad \boldsymbol{E}_{\text{eq}} = \begin{bmatrix} \boldsymbol{A} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$w = \begin{bmatrix} b_x \\ \vdots \\ b_f \\ b_u \\ \vdots \\ b_u \end{bmatrix} \quad \boldsymbol{G} = \left[\begin{array}{cccc|ccc} 0 & & & & & & \\ & \boldsymbol{A}_x & & & & & \\ & & \ddots & & & & \\ & & & \boldsymbol{A}_x & & & \\ \hline & & & & \boldsymbol{A}_d & & \\ & & & & & \ddots & \\ & & & & & & \boldsymbol{A}_d \end{array}\right] \quad \boldsymbol{E} = \begin{bmatrix} -\boldsymbol{A}_x^T \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

3.3 Duality

Lagrangian Dual Function

$$L(x, \lambda, \nu) = f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{i=1}^p \nu_i h_i(x)$$
$$d(\lambda, \nu) = \inf_{x \in \mathcal{X}} L(x, \lambda, \nu) \quad \text{i.e. } \nabla_x L(x, \lambda, \nu) = 0$$

Dual Problem (always convex)

$\max_{\lambda, \nu} d(\lambda, \nu)$ s. t. $\lambda \geq 0$.
Optimal value is lower bound for primal: $d^* \leq p^*$.
If primal convex, *Slater condition* (strict feasibility) implies *strong duality*:

$$\{x \mid Ax = b, f_i(x) < 0, \} \neq \emptyset \Rightarrow d^* = p^*$$

Karush-Kuhn-Tucker (KKT) Conditions

are necessary for optimality (and sufficient if primal convex).

- Primal Feasibility:
$$\begin{aligned} f_i(x^*) &\leq 0 \quad i = 1, \dots, m \\ h_i(x^*) &= 0 \quad i = 1, \dots, p \end{aligned}$$
- Dual Feasibility: $\lambda^* \geq 0$
- Complementary Slackness:
$$\lambda_i^* \cdot f_i(x^*) = 0 \qquad i = 1, \dots, m$$
- Stationarity:
$$\nabla_x L(x^*, \lambda^*, \nu^*) = 0$$

3.4 Constrained Finite Time Optimal Control (CFTOC)

3.5 Invariance

Def.: $x(k) \in O \implies x(k+1) \in O \forall k$.

$$\text{pre}(S) := \{x \mid g(x) \in S\} \qquad = \{x \mid Ax \in S\}$$

tim: We need more here, poos. inv. set, max. pos.inv O_∞

3.6 Stability and Feasability

Main Idea: Choose $\mathcal{X}_f$ and $\boldsymbol{P}$ to mimic infinite horizon. LQR control law $\kappa(x) = \boldsymbol{F}_\infty x$ from solving DARE. Set terminal weight $\boldsymbol{P} = \boldsymbol{P}_\infty$, terminal set $\mathcal{X}_f$ as maximal invariant set:

$$\begin{aligned} x_{k+1} = \boldsymbol{A}x_k + \boldsymbol{B}\boldsymbol{F}_\infty\ x_k \in \mathcal{X}_f \quad & \forall x_k \in \mathcal{X}_f \text{ terminal set invariant} \\ \mathcal{X}_f \subseteq \mathcal{X}, \quad \boldsymbol{F}_\infty\ x_k \in \mathcal{U} \quad & \forall x_k \in \mathcal{X}_f \text{ constrainst satisfied} \end{aligned}$$

We get: 1. Positive stage cost function, 2. invariant terminal set by construction, 3. Terminal cost is Lyapunov function with

$$x_{k+1}^T \boldsymbol{P} x_{k+1} - x_k^T \boldsymbol{P} x_k = -x_k^T (\boldsymbol{Q} + \boldsymbol{F}_\infty^T \boldsymbol{R} \boldsymbol{F}_\infty) x_k$$

Extension to non-linear (time-invariant) MPC possible since terminal set and cost do not rely on linearity.

3.7 Practical Issues

$$\begin{bmatrix} \boldsymbol{I} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ r \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_d \hat{d} \\ r - \boldsymbol{C}_d \hat{d} \end{bmatrix}$$

$\min u_s^T \boldsymbol{R}_s u_s$ else $\min(\boldsymbol{C}x_s - r)^T \boldsymbol{Q}_s (\boldsymbol{C}x_s - r)$ subject to $x_s = \boldsymbol{A}x_s + \boldsymbol{B}u_s$.

MPC for tracking

Target steady-state conditions $x_s = \boldsymbol{A}x_s + \boldsymbol{B}u_s$ and $y_s = \boldsymbol{C}x_s = r$ and constrainsts give:

$$\min_{x_s, u_s} u_s^T \boldsymbol{R} u_s \text{ subj. to } \begin{bmatrix} \boldsymbol{I} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ r \end{bmatrix}, x_s \in \mathcal{X}, u_s \in \mathcal{U}$$

Usually assume x_s, u_s unique and feasible. If no solution exists, compute closest steady-state ($\min(\boldsymbol{C}x_s - r)^T \boldsymbol{Q}(\boldsymbol{C}x_s - r)$ s. t. $x_s = \boldsymbol{A}x_s + \boldsymbol{B}u_s$).
MPC problem to drive $y \to r$ is:

$$\min_u \|y_N - \boldsymbol{C}x_s\|_{P_y}^2 + \sum_{i=0}^{N-1} \|y_i - \boldsymbol{C}x_s\|_{Q_y}^2 + \|u_i - u_s\|_R^2$$

Delta formulation for reference r

$\Delta x_k = x_k - x_s, \Delta u_k = u_k - u_s$:

$$\min V_f(\Delta x_N) + \sum_{i=0}^{N-1} \Delta x_i^T \boldsymbol{Q} \Delta x_i + \Delta u_i^T \boldsymbol{R} \Delta u_i$$
$$\begin{aligned} \text{s.t. } & \Delta x_0 = \Delta x_k \\ & \Delta x_{k+1} = \boldsymbol{A}\Delta x_k + \boldsymbol{B}\Delta u_k \\ \boldsymbol{H}_x x \leq k_x \rightarrow & \boldsymbol{H}_x \Delta x \leq k_x - \boldsymbol{H}_x x_s \\ \boldsymbol{H}_u u \leq k_u \rightarrow & \boldsymbol{H}_u \Delta u \leq k_u - \boldsymbol{H}_u u_s \\ & \Delta x_N \in \mathcal{X}_f \quad \text{adjusted accordingly} \\ & x_s \oplus \mathcal{X}_f \subseteq \mathcal{X} \\ & \boldsymbol{K}\Delta x + u_s \in \mathcal{U} \end{aligned}$$

Control given by $u_0^* = \Delta u_0^* + u_s$

Offset free tracking

$$\begin{aligned} x_{k+1} &= \boldsymbol{A}x_k + \boldsymbol{B}u_k + \boldsymbol{B}_d d_k \\ d_{k+1} &= d_k \\ y_k &= \boldsymbol{C}x_k + \boldsymbol{C}_d d_k \end{aligned}$$

Choice of $\boldsymbol{B}_d, \boldsymbol{C}_d$ requires that $(\boldsymbol{A}, \boldsymbol{C})$ is observable and $\begin{bmatrix} \boldsymbol{A} - \boldsymbol{I} & \boldsymbol{B}_d \\ \boldsymbol{C} & \boldsymbol{C}_d \end{bmatrix}$ has full column frank (i.e. det $\neq 0$).
If plant has no integrator we can choose $\boldsymbol{B}_d = \boldsymbol{0}$ since $\det(\boldsymbol{A} - \boldsymbol{I}) \neq 0$.

tim: What is y_m?

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}_d \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} u_k + \begin{bmatrix} \boldsymbol{L}_x \\ \boldsymbol{L}_d \end{bmatrix} \left(-y_m(k) + \boldsymbol{C}\hat{x}_k + \boldsymbol{C}_d \hat{d}_k\right)$$

where $\begin{bmatrix} \boldsymbol{L}_x \\ \boldsymbol{L}_d \end{bmatrix}$ stable and causes error dynamics to converge.
??? Will always converge if RHC is recursively feasible unconstrained for $k \geq j$.

Soft-constraints via slack variables

3.8 Robust MPC

Enforcing terminal constraints by robust invariance:

$$x \in O^{\mathcal{W}} \implies g(x,w) \in \Omega^{W}\ \forall w \in \mathcal{W}$$
$$\mathrm{pre}^{\mathcal{W}}(\Omega) = \{x | g(x,w) \in \Omega\ \forall w \in \mathcal{W}\}$$

tim: Maybe an example from exercises to compute O_∞^W

Enforcing sequential constraints for uncertain system ϕ:

$$\phi_i(x_0,u,w) = \left\{ x_i + \sum_{j=0}^{i-1} \boldsymbol{A}^J w_j \middle| w \in \mathcal{W}^i \right\} \subseteq \mathcal{X}$$
$$\phi_N(x_0,u,w) \in \mathcal{X}_f \quad \text{as well}$$

tim: One or two words on what is what

$$\boldsymbol{A}_x x \leq b_x \text{ becomes } \boldsymbol{A}_x x_i + \boldsymbol{A}_x \sum_{j=0}^{i-1} \boldsymbol{A}^j w_k \leq b_x :$$
$$\begin{aligned} x_i &\in \mathcal{X} \ominus \left(\mathcal{W} \oplus \boldsymbol{A}\mathcal{W} \oplus \cdots \oplus \boldsymbol{A}^{i-1}\mathcal{W}\right) \\ &= \left(\bigoplus_{j=0}^{i-1} \boldsymbol{A}^j \mathcal{W}\right) = \begin{bmatrix} \boldsymbol{A}^0 & \dots & \boldsymbol{A}^{i-1} \end{bmatrix} \mathcal{W}^i \end{aligned}$$

Tube-MPC We want nominal system $z_k = \boldsymbol{A} z_k + \boldsymbol{B} v_k$ with "tracking" controller $u_k = \boldsymbol{K}(x_k - z_k) + v_k$, $\boldsymbol{K}$ found offline.
Step 1: Compute $\mathcal{E} = \bigoplus_{j=1}^{\infty} \boldsymbol{A}^j \mathcal{W}$.
Step 2: Shrink Constraints:

$$\begin{aligned} \{z_i\} \oplus \mathcal{E} \subseteq \mathcal{X} \quad &\implies z_i \in \mathcal{X} \ominus \mathcal{E} \\ u_i \in \boldsymbol{K}\mathcal{E} \oplus \{v_i\} \subset \boldsymbol{U} \quad &\implies v_i \in \mathcal{U} \ominus \boldsymbol{K}\mathcal{E} \end{aligned}$$

Also $z_n \in \mathcal{X}_f \ominus \mathcal{E}$ accordingly.

3.9 Explicit MPC

3.10 Hybrid MPC

4 Numerical Optimization

Gradient, Newton, Interior Point