

## 1 Systems Theory

### 1.1 Linearization

### 1.2 Discretization

#### Exact

#### Forward-Euler

#### Backward-Euler

### 1.3 Lyapunov Function

$V(0) = 0, x \neq 0 \implies V(x) > 0$ ,  
 $V(g(x(k+1))) - V(x(k+1)) \leq -\alpha(x(k))$   
System asymptotically stable if  $V(x)$  exists. Globally stable iff  
 $\|x\| \rightarrow \infty \implies V(x) \rightarrow \infty$ .  
Check Eig. values of  $(APA - P)$  neg.,  $V(x) = x^T P x$  ?

## 2 Unconstrained Control

### 2.1 Block Approach (used also for $\bar{w}$ substitution)

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{A} \\ \vdots \\ \mathbf{A}^N \end{bmatrix} x(0) + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{AB} & \mathbf{B} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \mathbf{0} \\ \mathbf{A}^{N-1}\mathbf{B} & \cdots & \mathbf{AB} & \mathbf{B} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$$\begin{aligned} x &= \mathbf{S}^x \cdot x(0) + \mathbf{S}^u \cdot u & \text{size}(\mathbf{S}^x) &= [n_{\text{states}} \cdot (N+1), N] \\ & & \text{size}(\mathbf{S}^u) &= [n_{\text{states}} \cdot (N+1), n_{\text{states}}] \\ \bar{\mathbf{Q}} &= \text{diag}(\mathbf{Q}, \dots, \mathbf{Q}, \mathbf{P}) & \text{size}(\bar{\mathbf{Q}}) &= [n_{\text{states}} \cdot (N+1), n_{\text{states}} \cdot (N+1)] \\ \bar{\mathbf{R}} &= \text{diag}(\mathbf{R}, \dots, \mathbf{R}) & \text{size}(\bar{\mathbf{R}}) &= (n_{\text{input}} \cdot N, n_{\text{input}} \cdot N) \\ \mathbf{H} &= \mathbf{S}^{uT} \bar{\mathbf{Q}} \mathbf{S}^u + \mathbf{R} & \mathbf{F} &= \mathbf{S}^{xT} \bar{\mathbf{Q}} \mathbf{S}^u \\ \mathbf{Y} &= \mathbf{S}^{xT} \bar{\mathbf{Q}} \mathbf{S}^x \end{aligned}$$

#### Optimal cost and control

$$\begin{aligned} J^*(x_0) &= -x_0^T \mathbf{F} \mathbf{H} \mathbf{F}^T x_0 + x_0^T \mathbf{Y} x_0 \\ u^*(x_0) &= -\mathbf{H}^{-1} \mathbf{F}^T x_0 = -\left(\mathbf{S}^{uT} \bar{\mathbf{Q}} \mathbf{S}^u + \mathbf{R}\right)^{-1} \mathbf{S}^{uT} \bar{\mathbf{Q}} \mathbf{S}^x x_0 \end{aligned}$$

## 2.2 Recursive Approach

$$J_k^*(x_k) = \min_{u_k} I(x_k, u_k) + J_{k+1}(x_{k+1})$$

Is a feedback controller as opposed to the Batch Approach. For LQR solve via Riccati Difference Equation (RDE).

$$\begin{aligned} \mathbf{F}_k &= -(\mathbf{B}^T \mathbf{P}_{k+1} \mathbf{B} + \mathbf{R})^{-1} \mathbf{B}^T \mathbf{P}_{k+1} \mathbf{A} \\ \mathbf{P}_k &= \mathbf{A}^T \mathbf{P}_{k+1} \mathbf{A} + \mathbf{Q} - \mathbf{A}^T \mathbf{P}_{k+1} \mathbf{B} (\mathbf{B}^T \mathbf{P}_{k+1} \mathbf{B} + \mathbf{R})^{-1} \mathbf{B}^T \mathbf{P}_{k+1} \mathbf{A} \end{aligned}$$

$$u_k^* = \mathbf{F}_k x_k \quad J_k^*(x_k) = x_k^T \mathbf{P}_k x_k \quad \mathbf{P}_N = \mathbf{P}$$

For unconstrained Infinite Horizon Problem, substituting  $\mathbf{P}_\infty = \mathbf{P}_k = \mathbf{P}_{k+1}$  into RDE gives DARE. Uniquely solvable, iff  $(\mathbf{A}, \mathbf{B})$  stabilizable and  $(\mathbf{A}, \mathbf{G})$  detectable, where  $\mathbf{G} \mathbf{G}^T = \mathbf{Q}$ . Follows from closed-loop system  $x_{k+1} = (\mathbf{A} + \mathbf{B} \mathbf{F}_k) x_k$

## 3 (Convex) Optimization

### 3.1 Convexity

**Convex set**  $\mathcal{X}$  iff  $\forall \lambda \in [0, 1] \forall x, y \in \mathcal{X} \lambda x + (1 - \lambda)y \in \mathcal{X}$ .  
Intersection preserves convexity, union does not.

**Affine set**  $\mathcal{X} = \{x \in \mathbb{R}^n | \mathbf{A}x = b\}$  for some  $\mathbf{A}, b$

**Subspace** is affine set through origin, i.e.  $b = \mathbf{0}$ , aka Nullspace of  $\mathbf{A}$ .

**Hyperplane**  $\mathcal{X} = \{x \in \mathbb{R}^n | a^T x = b\}$  for some  $a, b$ .

**Halfspace**  $\mathcal{X} = \{x \in \mathbb{R}^n | a^T x \leq b\}$  for some  $a, b$ .

**Polyhedron**  $\mathcal{P} = \{x | a_i^T x \leq b_i, i = 1, \dots, n\} = \{x | \mathbf{A}x \leq b\}$

**Cone**

**Ellipsoid**  $\mathcal{E} = \{x | (x - x_c)^T \mathbf{A}^{-1} (x - x_c) \leq 1\}$ ,  $x_c$  center point.

**Convex function**

**Norm**  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$

$$\begin{aligned} f(x) &= 0 \implies x = 0, & f(x) &\geq 0 \\ f(\alpha \cdot x) &= |\alpha| \cdot f(x) & \text{for scalar } \alpha \\ f(x+y) &\leq f(x) + f(y) & \forall x, y \in \mathbb{R}^n \end{aligned}$$

tim: Maybe move the above somewhere else?

**General Problem**  $\min_{x \in \text{dom}(f)} f(x)$  s. t.  $g_i(x) \leq 0$  and  $h_j(x) = 0$ .

### 3.2 Linear Programming (LP)

**Problem statement**  $\min c^T x$  such that  $\mathbf{G}x \leq h$  and  $\mathbf{A}x = b$ .

**Norm**  $l_\infty$   $\min_x \|x\|_\infty = \min_{x \in \mathbb{R}^n} [\max\{x, \dots, x_n, -x_1, \dots, -x_n\}]$ :

$$\begin{aligned} \min_{x,t} t & \quad \text{subject to} \quad x_i \leq t, -x_i \leq t, & \mathbf{F}x \leq g \\ \iff \min_{x,t} t & \quad \text{subject to} \quad -1t \leq x \leq 1t, & \mathbf{F}x \leq g. \end{aligned}$$

**Norm**  $l_1$   $\min_x \|x\|_1 = \min_x [\sum_{i=1}^m \max\{x_i, -x_i\}]$ :

$$\begin{aligned} \min_t t_1 + \dots + t_m & \quad \text{subject to} \quad x_i \leq t_i, -x_i \leq t_i, & \mathbf{F}x \leq g \\ \iff \min_t \mathbf{1}^T t & \quad \text{subject to} \quad -t \leq x \leq t, & \mathbf{F}x \leq g. \end{aligned}$$

Note that for  $\dim x = 1$ ,  $l_1$  and  $l_\infty$  are the same.

#### MPC with linear cost

$$J(x_0, u) = \|\mathbf{P}x_N\|_p + \sum_{i=0}^{N-1} \|\mathbf{Q}x_i\|_p + \|\mathbf{R}u_i\|_p.$$

tim: Insert here slide 45, lect 4

#### Receding Horizon Control – RHC

#### QP with substitution (see also Batch approach)

$$\begin{aligned} J^*(x_k) &= \min_u \begin{bmatrix} u^T & x_k^T \end{bmatrix} \begin{bmatrix} \mathbf{H} & \mathbf{F}^T \\ \mathbf{F} & \mathbf{Y} \end{bmatrix} \begin{bmatrix} u \\ x_k \end{bmatrix} \\ \text{s. t. } & \mathbf{G} u \leq w + \mathbf{E} x_k \end{aligned}$$

Latter gives three sets (same for without substitution)

$$\begin{aligned} \mathcal{X} &= \{x | A_x x \leq b_x\} \\ \mathcal{U} &= \{u | A_u u \leq b_u\} \\ \mathcal{X}_f &= \{x | A_f x \leq b_f\} \end{aligned}$$

State equations are in cost matrix, usually

$$\mathbf{A}_x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, b_x = \begin{bmatrix} b_{\max} \\ -b_{\min} \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{A}_u & 0 & \dots & 0 \\ 0 & \mathbf{A}_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{A}_u \\ 0 & 0 & \dots & 0 \\ \mathbf{A}_x \mathbf{B} & 0 & \dots & 0 \\ \mathbf{A}_x \mathbf{A} \mathbf{B} & \mathbf{A}_x \mathbf{B} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_x \mathbf{A}^{N-2} \mathbf{B} & \mathbf{A}_x \mathbf{A}^{N-3} \mathbf{B} & \dots & 0 \\ \mathbf{A}_f \mathbf{A}^{N-1} \mathbf{B} & \mathbf{A}_f \mathbf{A}^{N-2} \mathbf{B} & \dots & \mathbf{A}_f \mathbf{B} \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\mathbf{A}_x \\ -\mathbf{A}_x \mathbf{A} \\ -\mathbf{A}_x \mathbf{A}^2 \\ \vdots \\ -\mathbf{A}_x \mathbf{A}^{N-1} \\ -\mathbf{A}_f \mathbf{A}^N \end{bmatrix} \quad \mathbf{W} = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ \vdots \\ b_x \\ b_x \\ b_f \end{bmatrix}$$

**QP with out substitution** State equations represented in equality constraint.

$$J^*(x_k) = \min_z \begin{bmatrix} z^T & x_k^T \end{bmatrix} \begin{bmatrix} \bar{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} z \\ x_k \end{bmatrix}$$

s. t.  $\mathbf{G} z \leq w + \mathbf{E} x_k$

$\mathbf{G}_{\text{eq}} z = \mathbf{E}_{\text{eq}} x_k$ , system dynamics

$$\bar{\mathbf{H}} = \text{diag}(\mathbf{Q}, \dots, \mathbf{Q}, \mathbf{P}, \mathbf{R}, \dots, \mathbf{R})$$

$$z = \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} \quad \mathbf{G}_{\text{eq}} = \begin{bmatrix} \mathbf{I} & \mathbf{I} & \vdots & -\mathbf{B} & -\mathbf{B} \\ -\mathbf{A} & \mathbf{I} & \vdots & \vdots & \vdots \\ \vdots & \vdots & -\mathbf{A} & \mathbf{I} & \vdots \\ \vdots & \vdots & \vdots & \vdots & -\mathbf{B} \end{bmatrix} \quad \mathbf{E}_{\text{eq}} = \begin{bmatrix} \mathbf{A} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$w = \begin{bmatrix} b_x \\ \vdots \\ b_f \\ b_u \\ \vdots \\ b_u \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} \mathbf{A}_x & \vdots & \vdots \\ \vdots & \mathbf{A}_x & \vdots \\ \vdots & \vdots & \mathbf{A}_d \\ \vdots & \vdots & \vdots & \mathbf{A}_d \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} -\mathbf{A}_x^T \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

### 3.3 Duality

#### Lagrangian Dual Function

$$L(x, \lambda, \nu) = f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{i=1}^p \nu_i h_i(x)$$

$$d(\lambda, \nu) = \inf_{x \in \mathcal{X}} L(x, \lambda, \nu) \quad \text{i.e.} \quad \nabla_x L(x, \lambda, \nu) = 0$$

**Dual Problem (always convex)**  $\max_{\lambda, \nu} d(\lambda, \nu)$  s. t.  $\lambda \geq 0$ .

Optimal value is lower bound for primal:  $d^* \leq p^*$ .

If primal convex, *Slater condition* (strict feasibility) implies *strong duality*:

$$\{x \mid Ax = b, f_i(x) < 0, \} \neq \emptyset \Rightarrow d^* = p^*$$

**Karush-Kuhn-Tucker (KKT) Conditions** are necessary for optimality (and sufficient if primal convex).

▪ Primal Feasibility:

$$f_i(x^*) \leq 0 \quad i = 1, \dots, m$$

$$h_i(x^*) = 0 \quad i = 1, \dots, p$$

▪ Dual Feasibility:  $\lambda^* \geq 0$

▪ Complementary Slackness:

$$\lambda_i^* \cdot f_i(x^*) = 0 \quad i = 1, \dots, m$$

▪ Stationarity:

$$\nabla_x L(x^*, \lambda^*, \nu^*) = 0$$

### 3.4 Constrained Finite Time Optimal Control (CFTOC)

#### 3.5 Invariance

Def.:  $x(k) \in O \implies x(k+1) \in O \forall k$ .

$$\text{pre}(S) := \{x \mid g(x) \in S\} = \{x \mid Ax \in S\}$$

tim: We need more here, pos. inv. set, max. pos.inv  $O_\infty$

#### 3.6 Stability and Feasability

Recursive Stability, optimal cost is Lyapunov function.

tim: What is meant with that

Main Idea: Choose  $\mathcal{X}_f$  and  $\mathbf{P}$  to mimic inf horizon, terminal cost ist Lyapunov function:  $x_{k+1}^T \mathbf{P} x_{k+1} - x_k^T \mathbf{P} x_k = -x_k^T (\mathbf{Q} + \mathbf{F}_\infty^T \mathbf{R} \mathbf{F}_\infty) x_k$ , such that:

$$x_{k+1} = \mathbf{A} x_k + \mathbf{B} \mathbf{F}_\infty x_k \in \mathcal{X}_f \quad \forall x_k \in \mathcal{X}_f \text{ terminal set invariant}$$

$$\mathcal{X}_f \subseteq \mathcal{X}, \quad \mathbf{F}_\infty x_k \in \mathcal{U} \quad \forall x_k \in \mathcal{X}_f \text{ constraint satisfied}$$

And stage cost is PD-function  $\implies$  Extension to non-linear (time-invariant) MPC possible since terminal set and cost do not rely on linearity.

### 3.7 Practical Issues

$$\begin{bmatrix} \mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ r \end{bmatrix} = \begin{bmatrix} \mathbf{B}_d \hat{d} \\ r - \mathbf{C}_d \hat{d} \end{bmatrix}$$

$\min u_s^T \mathbf{R}_s u_s$  else  $\min (\mathbf{C} x_s - r)^T \mathbf{Q}_s (\mathbf{C} x_s - r)$  subject to  $x_s = \mathbf{A} x_s + \mathbf{B} u_s$ .

#### MPC for tracking

#### Delta formulation

#### Offset free tracking

$$x_{k+1} = \mathbf{A} x_k + \mathbf{B} u_k + \mathbf{B}_d d_k$$

$$d_{k+1} = d_k$$

$$y_k = \mathbf{C} x_k + \mathbf{C}_d d_k$$

Choice of  $\mathbf{B}_d, \mathbf{C}_d$  requires that  $(\mathbf{A}, \mathbf{C})$  is observable and

$\begin{bmatrix} \mathbf{A} - \mathbf{I} & \mathbf{B}_d \\ \mathbf{C} & \mathbf{C}_d \end{bmatrix}$  has full column rank (i.e.  $\det \neq 0$ ).

If plant has no integrator we can choose  $\mathbf{B}_d = \mathbf{0}$  since  $\det(\mathbf{A} - \mathbf{I}) \neq 0$ .

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_d \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u_k + \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_d \end{bmatrix} (-y_m(k) + \mathbf{C} \hat{x}_k + \mathbf{C}_d \hat{d}_k)$$

where  $\begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_d \end{bmatrix}$  stable and causes error dynamics to converge.

??? Will always converge if RHC is recursively feasible unconstrained for  $k \geq j$ .

#### Soft-constraints via slack variables

### 3.8 Robust MPC

**Enforcing terminal constraints** by robust invariance:

$$x \in O^W \implies g(x, w) \in \Omega^W \quad \forall w \in \mathcal{W}$$

$$\text{pre}^W(\Omega) = \{x \mid g(x, w) \in \Omega \quad \forall w \in \mathcal{W}\}$$

tim: Maybe an example from exercises to compute  $O_\infty^W$

**Enforcing sequential constraints** for uncertain system  $\phi$ :

$$\phi_i(x_0, u, w) = \left\{ x_i + \sum_{j=0}^{i-1} \mathbf{A}^j w_j \middle| w \in \mathcal{W}^i \right\} \subseteq \mathcal{X}$$

$$\phi_N(x_0, u, w) \in \mathcal{X}_f \quad \text{as well}$$

tim: One or two words on what is what

$$x_i \in \mathcal{X} \ominus (\mathcal{W} \oplus \mathbf{A}\mathcal{W} \oplus \dots \oplus \mathbf{A}^{i-1}\mathcal{W})$$

$$= \left( \bigoplus_{j=0}^{i-1} \mathbf{A}^j \mathcal{W} \right) = [\textcolor{red}{\mathbf{I}} \quad \textcolor{red}{\mathbf{A}^0} \quad \dots \quad \textcolor{red}{\mathbf{A}^{i-1}}] \mathcal{W}^i$$

$$\mathbf{A}_x x \leq b_x$$

$$\mathbf{A}_x x_i + \mathbf{A}_x \sum_{j=0}^{i-1} \mathbf{A}^j w_k \leq b_x$$

**Tube-MPC** We want nominal system  $z_k = \mathbf{A}z_k + \mathbf{B}v_k$  with “tracking” controller  $u_k = \mathbf{K}(x_k - z_k) + v_k$ ,  $\mathbf{K}$  found offline.

Step 1: Compute  $\mathcal{E} = \bigoplus_{j=1}^{\infty} \mathbf{A}^j \mathcal{W}$ .

Step 2: Shrink Constraints:

$$\{z_i\} \oplus \mathcal{E} \subseteq \mathcal{X} \quad \implies z_i \in \mathcal{X} \ominus \mathcal{E}$$

$$u_i \in \mathbf{K}\mathcal{E} \oplus \{v_i\} \subset \mathcal{U} \quad \implies v_i \in \mathcal{U} \ominus \mathbf{K}\mathcal{E}$$

Also  $z_n \in \mathcal{X}_f \ominus \mathcal{E}$  accordingly.

### 3.9 Explicit MPC

### 3.10 Hybrid MPC

## 4 Numerical Optimization

Gradient, Newton, Interior Point