# 1 System Theory

## 1.1 Nonlinear Systems

## 1.2 Linear Systems

**Continuous**

$$\dot{x}(t) = A^c x(t) + B^c u(t)$$

$$x(t) = e^{A^c(t-t_0)}x_0 + \int_{t_0}^{t} e^{A^c(t-\tau)}Bu(\tau)d\tau$$

$$e^{A^c t} = \sum_{n=0}^{\infty} \frac{(A^c t)^n}{n!}$$

**Discrete**

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k + Du_k$$

$$x_{k+N} = A^N x_k + \sum_{i=0}^{N-1} A^i B u_{k+N-1-i}$$

**Discretization** Euler: $A = I + T_s A^c,\ B = T_s B^c,\ C = C^c,\ D = D^c$

$$x_{k+1} = x_k + T_s g^c(x_k, u_k) = g(x_k, u_k)$$

$$y_k = h^c(x_k, u_k) = h(x_k, u_k)$$

Exact: (assume constant $u(t)$ during $T_s$)

$$A = e^{A^c T_s},\ B = \int_0^{T_s} e^{A^c(T_s-\tau')}B^c d\tau$$

$$B = (A^c)^{-1}(A-I)B^c,\ \text{if } A^c \text{ invertible}$$

## 1.3 Lyapunov Stability

System is stable in the sense of Lyapunov iff it stays in any arbitrarily small neighborhood of the origin when it is disturbed slightly.

**Lyapunov stable** iff $\forall \epsilon > 0\ \exists \delta(\epsilon)$ s.t. $\|x_0\| < \delta(\epsilon) \to \|x_k\| < \epsilon, \forall k \geq 0$

**asymptotically stable** in $\Omega \subseteq \mathbb{R}^n$ if Lyapunov stable and *attractive* $\lim_{k\to\infty} x_k = 0, \forall x_0 \in \Omega$.

**Lyapunov Function** $V : \mathbb{R}^n \to \mathbb{R}$ continous at the origin, finite $\forall x \in \Omega$,

$$V(0) = 0 \text{ and } V(x) > 0, \forall x \in \Omega\backslash\{0\}$$

$$V(g(x)) - V(x) \leq -\alpha(x), \forall x \in \Omega\backslash\{0\}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}$ is continuous positive definite, equilibrium at $x = 0$ and $\Omega \subset \mathbb{R}^n$ closed and bounded set containing the origin.

**Lyapunov Theorem** If system admits Lyapunov function $V(x)$, then $x = 0$ is asymptotically stable in $\Omega$ (sufficient but not necessary).
If additionally $\|x\| \to \infty \Rightarrow V(x) \to \infty$ globally asymptotically stable.
To check if $V(x) = x^T \boldsymbol{P} x$ is valid Lyapunov function of system $x_{k+1} = \boldsymbol{A}x_k$ check if $(\boldsymbol{APA} - \boldsymbol{P})$ has neg. eigen values.
In other words: Iff eigenvalues of $A$ inside unit circle (i.e. stable) then $\exists$*unique* $\boldsymbol{P} > 0$ that solves $\boldsymbol{A}_{cl}^T \boldsymbol{P} \boldsymbol{A}_{cl} - \boldsymbol{P} = -\boldsymbol{Q},\ \boldsymbol{Q} > 0$ and $V(x) = x^T \boldsymbol{P} x$ is a lyapunov function.

## 1.4 Observability ⇒ Detectability, Controllability ⇒ Stabilizability

**$(A, C)$ observable** if rank$(\boldsymbol{O}) = n$ (full col. rank) for

$$\boldsymbol{O} = \begin{bmatrix} C^T \\ (CA)^T \\ \dots \\ (CA^{n-1})^T \end{bmatrix} \text{ or rank} \begin{bmatrix} \boldsymbol{A} - \lambda \boldsymbol{I} \\ \boldsymbol{H} \end{bmatrix} = n\ \forall \lambda_i \text{ of } \boldsymbol{A} \text{ (PBH-test).}$$

**$(A, H)$ detectable** iff rank $\begin{bmatrix} \boldsymbol{A} - \lambda \boldsymbol{I} \\ \boldsymbol{H} \end{bmatrix} = n\ \forall$ unstable $|\lambda_i| \geq 1$ of $\boldsymbol{A}$.

**$(A, B)$ controllable** if rank $\boldsymbol{C} = n$, $\boldsymbol{C} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \dots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix}$ or if rank $\left(\begin{bmatrix} \lambda_j \boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix}\right) = n\ \forall \lambda_i$ of $\boldsymbol{A}$ (PBH-test).
Intuition: Can reach any state in (at most) $n$ steps.

**$(A, B)$ stabilizable** if rank $\begin{bmatrix} \lambda_j \boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix} = n\ \forall$unstable$|\lambda_i| \geq 1$ of $\boldsymbol{A}$.
Intuition: Can reach origin in (at most) $n$ steps.

# 2 Unconstrained Control

## 2.1 Block Approach (used also for $\bar{w}$ substition)

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{A} \\ \vdots \\ \boldsymbol{A}^N \end{bmatrix} x(0) + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{AB} & \boldsymbol{B} & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{0} \\ \boldsymbol{A}^{N-1}\boldsymbol{B} & \dots & \boldsymbol{AB} & \boldsymbol{B} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_1} \end{bmatrix}$$

$x = \boldsymbol{S}^x \cdot x(0) + \boldsymbol{S}^u \cdot u$ size$(\boldsymbol{S}^x) = [n_{\text{states}} \cdot (N+1), N]$
size$(\boldsymbol{S}^u) = [n_{\text{states}} \cdot (N+1), n_{\text{states}}]$
$\bar{\boldsymbol{Q}} = \text{diag}(\boldsymbol{Q}, \dots, \boldsymbol{Q}, \boldsymbol{P})$ size$(\bar{\boldsymbol{Q}}) = [n_{\text{states}} \cdot (N+1), n_{\text{states}} \cdot (N+1)]$
$\bar{\boldsymbol{R}} = \text{diag}(\boldsymbol{R}, \dots, \boldsymbol{R})$ size$(\bar{\boldsymbol{R}}) = [n_{\text{input}} \cdot N, n_{\text{input}} \cdot N]$
$\boldsymbol{H} = \boldsymbol{S}^{u\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^u + \boldsymbol{R}$ $\boldsymbol{F} = \boldsymbol{S}^{x\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^u$
$\boldsymbol{Y} = \boldsymbol{S}^{x\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^x$

**Optimal cost and control**

$$J^*(x_0) = -x_0^T \boldsymbol{FHF}^T x_0 + x_0^T \boldsymbol{Y} x_0$$

$$u^*(x_0) = -\boldsymbol{H}^{-1}\boldsymbol{F}^T x_0 = -\left(\boldsymbol{S}^{u\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^u + \boldsymbol{R}\right)^{-1} \boldsymbol{S}^{u\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^x x_0$$

## 2.2 Recursive Approach

$$J_k^*(x_k) = \min_{u_k} I(x_k, u_k) + J_{k+1}(x_{k+1})$$

Is a feedback controller as opposed to the Batch Approach. For LQR solve via Riccati Difference Equation (RDE).

$$\boldsymbol{F}_k = -(\boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{B} + \boldsymbol{R})^{-1} \boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{A}$$

$$\boldsymbol{P}_k = \boldsymbol{A}^T \boldsymbol{P}_{k+1} \boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{A}^T \boldsymbol{P}_{k+1} \boldsymbol{B} (\boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{B} + \boldsymbol{R})^{-1} \boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{A}$$

$$u_k^* = \boldsymbol{F}_k\ x_k \qquad J_k^*(x_k) = x_k^T \boldsymbol{P}_k\ x_k \qquad \boldsymbol{P}_N = \boldsymbol{P}$$

For unconstrained Infinite Horizon Problem, substituting $\boldsymbol{P}_\infty = \boldsymbol{P}_k = \boldsymbol{P}_{k+1}$ into RDE gives DARE. Uniquely solvable, iff $(A, B)$ stabilizable and $(A, G)$ detectable, where $\boldsymbol{GG}^T = \boldsymbol{Q}$. Follows from closed-loop system $x_{k+1} = (\boldsymbol{A} + \boldsymbol{BF}_k)x_k$

# 3 (Convex) Optimization

**General Problem** $\min_{x \in \text{dom}(f)} f(x)$ s. t. $g_i(x) \leq 0$ and $h_j(x) = 0$.

**Norm** $f(x) : \mathbb{R}^n \to \mathbb{R}$

| | |
|---|---|
| $f(x) = 0 \Rightarrow x = 0,$ | $f(x) \geq 0$ |
| $f(\alpha \cdot x) = \lvert\alpha\rvert \cdot f(x)$ | for scalar $\alpha$ |
| $f(x+y) \leq f(x) + f(y)$ | $\forall x, y \in R^n$ |

## 3.1 Convexity

**Convex set** $\mathcal{X}$ iff $\forall \lambda \in [0,1] \forall x, y \in \mathcal{X}\ \lambda x + (1-\lambda)y \in \mathcal{X}$. Intersection preserves convexity, union does not.

**Affine set** $\mathcal{X}$ $= \{x \in \mathbb{R}^n | \boldsymbol{A}x = b\}$ for some $\boldsymbol{A}, b$

**Subspace** is affine set through origin, i.e. $b = \boldsymbol{0}$, aka Nullspace of $\boldsymbol{A}$.

**Hyperplane** $\mathcal{X}$ $= \{x \in \mathbb{R}^n | a^T x = b\}$ for some $a, b$.

**Halfspace** $\mathcal{X}$ $= \{x \in \mathbb{R}^n | a^T x \leq b\}$ for some $a, b$.

**Polyhedron** $\mathcal{P}$ $= \{x | a_i^T x \leq b_i, i = 1, \dots, n\} = \{x | \boldsymbol{A}x \leq b\}$

**Cone** $\mathcal{X}$ if for all $x \in \mathcal{X}$, and for all $\theta > 0, \theta x \in \mathcal{X}$.

**Ellipsoid** $\mathcal{E}$ $= \{x | (x - x_c)^T \boldsymbol{A}^{-1}(x - x_c) \leq 1\}$, $x_c$ center point.

**Convex function** $f : \textbf{dom}(f) \to \mathbb{R}$ is convex iff dom$(f)$ is convex and $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y), \forall \lambda \in (0,1), \forall x, y \in \text{dom}(f)$.

**Norm ball** is convex (for all norms).

**Epigraph set** $f : \textbf{dom}(f) \to \mathbb{R}$ is the set

$$\text{epi}(f) := \left\{ \begin{bmatrix} x \\ t \end{bmatrix} | x \in \text{dom}(f), f(x) \leq t \right\} \subseteq \text{dom}(f) \times \mathbb{R}$$

**Level set** $L_a$ of a function $f$ for value $a$ is the set of all $x \in \text{dom}(f)$ for which $f(x) = a$: $L_a = \{x | x \in \text{dom}(f), f(x) = a\}$.

**Sublevel set** $C_a$ is defined by $C_a = \{x | x \in \text{dom}(f), f(x) \leq a\}$.

## 3.2 Linear Programming (LP)

**Problem statement** $\min c^T x$ such that $\boldsymbol{G}x \leq h$ and $\boldsymbol{A}x = b$.

**Norm** $l_\infty$ $\min_x \|x\|_\infty = \min_{x \in \mathbb{R}^n} [\max\{x, \dots, x_n, -x_1, \dots, -x_n\}]$:

$$\min_{x,t} t \quad \text{subject to} \quad x_i \leq t, -x_i \leq t, \qquad \boldsymbol{F}x \leq g$$

$$\iff \min_{x,t} t \quad \text{subject to} \quad -\boldsymbol{1}t \leq x \leq \boldsymbol{1}t, \qquad \boldsymbol{F}_x \leq g.$$

**Norm** $l_1$ $\min_x \|x\|_1 = \min_x \left[\sum_{i=1}^m \max\{x_i, -x_i\}\right]$:

$$\min_t t_1 + \dots + t_m \quad \text{subject to} \quad x_i \leq t_i, -x_i \leq t_i, \qquad \boldsymbol{F}x \leq g$$

$$\iff \min_t \boldsymbol{1}^T t \quad \text{subject to} \quad -t \leq x \leq t, \qquad \boldsymbol{F}_x \leq g.$$

Note that for dim $x = 1$, $l_1$ and $l_\infty$ are the same. Note also that $t$ is scalar for norm $l_\infty$ and a vector in norm $l_1$.

### Piecewise Affine

$$\min_x \left[\max_{i=1,\dots,m} \{c_i^T x + d_i\}\right] \quad \text{s.t. } \boldsymbol{G}x \le h$$
$$\iff \min_{x,t} t \quad \text{s.t. } c_i^T x + d_i \le t, \boldsymbol{G}x \le h$$

## 3.3 Duality

### Lagrangian Dual Function

$$L(x,\lambda,\nu) = f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{i=1}^p \nu_i h_i(x)$$
$$d(\lambda,\nu) = \inf_{x\in\mathcal{X}} L(x,\lambda,\nu) \qquad \text{i.e. } \nabla_x L(x,\lambda,\nu) = 0$$

**Dual Problem (always convex)** $\max_{\lambda,\nu} d(\lambda,\nu)$ s. t. $\lambda \ge 0$.
Optimal value is lower bound for primal: $d^* \le p^*$.

If primal convex, Slater condition (strict feasibility) implies *strong duality*: $\{x \mid Ax = b, f_i(x) < 0, \} \neq \emptyset \Rightarrow d^* = p^*$

**Karush-Kuhn-Tucker (KKT) Conditions** are necessary for optimality (and sufficient if primal convex).

| | | |
|---|---|---|
| Primal Feasability | $f_i(x^*) \le 0$ | $i = 1,\dots,m$ |
| | $h_i(x^*) = 0$ | $i = 1,\dots,p$ |
| Dual Feasability | $\lambda^* \ge 0$ | |
| Complementary slackness | $\lambda_i^* \cdot f_i(x^*) = 0$ | $i = 1,\dots,m$ |
| Stationarity | $\nabla_x L(x^*,\lambda^*,\nu^*) = 0$ | |

### Dual of LP

$$\min_x c^T x \quad \text{s.t. } \boldsymbol{A}x = b, \boldsymbol{C}x \le e$$
$$\iff \max_{\lambda,\nu} -b^T\nu - e^T\lambda \quad \text{s.t. } A^T\nu + C^T\lambda + c = 0, \lambda \ge 0$$

### Dual of QP

$$\min_x \frac{1}{2}x^T\boldsymbol{Q}x + c^T x \quad \text{s.t. } \boldsymbol{C}x \le e$$
$$\iff \min_{\lambda,\nu} \frac{1}{2}\lambda^T\boldsymbol{C}\boldsymbol{Q}^{-1}\boldsymbol{C}^T\lambda + (\boldsymbol{C}\boldsymbol{Q}^{-1}c + e)^T\lambda + \frac{1}{2}c^T\boldsymbol{Q}^{-1}c$$
$$\text{s.t. } \boldsymbol{Q}x + \nu + c^T\lambda = 0, \lambda \ge 0$$

# 4 Constrained Finite Time Optimal Control (CFTOC)

## 4.1 MPC with linear cost

$$J(x_0,u) = \|\boldsymbol{P}x_N\|_p + \sum_{i=0}^{N-1} \|\boldsymbol{Q}x_i\|_p + \|\boldsymbol{R}u_i\|_p.$$

The CFTOC problem can be formulated as an $\infty$-norm LP problem as shown below.

$$\begin{aligned}
\min_z\ & \epsilon_0^x + \dots + \epsilon_N^x + \epsilon_0^u + \dots + \epsilon_{N-1}^u \\
\text{s.t. } & -\boldsymbol{1}_n\epsilon_i^x \le \pm\boldsymbol{Q}\left[\boldsymbol{A}^i x_0 + \sum_{j=0}^{i-1}\boldsymbol{A}^j\boldsymbol{B}u_{i-1-j}\right] \\
& -\boldsymbol{1}_r\epsilon_N^x \le \pm\boldsymbol{P}\left[\boldsymbol{A}^N x_0 + \sum_{j=0}^{N-1}\boldsymbol{A}^j\boldsymbol{B}u_{N-1-j}\right] \\
& -\boldsymbol{1}_m\epsilon_N^u \le \pm\boldsymbol{R}u_i \\
& x_i = \boldsymbol{A}^i x_0 + \sum_{j=0}^{i-1}\boldsymbol{A}^j\boldsymbol{B}u_{i-1-j} \in \mathcal{X} \\
& x_N = \boldsymbol{A}^N x_0 + \sum_{j=1}^{N-1}\boldsymbol{A}^j\boldsymbol{B}u_{N-1-j} \in \mathcal{X} \\
& u_i \in \mathcal{U}
\end{aligned}$$

Converting to LP form:

$$\begin{aligned}
& \min_z\ c^T z \\
\text{s.t. } & \bar{\boldsymbol{G}}z \le \bar{w} + \bar{s}x_k \\
& z = \begin{bmatrix}\epsilon_0^x & \dots & \epsilon_N^x & \epsilon_0^u & \dots & \epsilon_{N-1}^u & u_0^T & \dots & u_{N-1}^T\end{bmatrix} \\
& c = \begin{bmatrix}1 & \dots & 1 & 1 & \dots & 1 & 0 & \dots & 0\end{bmatrix} \\
& \bar{\boldsymbol{G}} = \begin{bmatrix}\boldsymbol{G}_\epsilon & \boldsymbol{G}_u \\ 0 & \boldsymbol{G}\end{bmatrix}, \quad \bar{w} = \begin{bmatrix}w_\epsilon \\ w\end{bmatrix} \\
& \bar{s} = \begin{bmatrix}s_\epsilon \\ s\end{bmatrix}
\end{aligned}$$

Where $\boldsymbol{G}$ is the normal problem constraints and $[\boldsymbol{G}_\epsilon\boldsymbol{G}_u]$ form the constraints of the newly introduced variable $\epsilon$ as given in the first 3 constraints in the section above. For example, we require:

$$-\epsilon_i^u \le u_i \le \epsilon_i^u$$
$$-\epsilon_0^x \le Ax_0 + Bu_0 \le \epsilon_0^x$$
$$-\epsilon_1^x \le A^2x_0 + Bu_1 + ABu_0 \le \epsilon_1^x$$

## 4.2 QP with substitution (see also Batch approach)

$$J^*(x_k) = \min_u \begin{bmatrix}u^T & x_k^T\end{bmatrix}\begin{bmatrix}\boldsymbol{H} & \boldsymbol{F}^T \\ \boldsymbol{F} & \boldsymbol{Y}\end{bmatrix}\begin{bmatrix}u \\ x_k\end{bmatrix}$$
$$\text{s. t. } \boldsymbol{G}\ u \le w + \boldsymbol{E}\ x_k$$

Latter gives three sets (same for without substitution)

$$\mathcal{X} = \{x \mid A_x\ x \le b_x\}$$
$$\mathcal{U} = \{u \mid A_u\ u \le b_u\}$$
$$\mathcal{X}_f = \{x \mid A_f\ x \le b_f\}$$

State equations are in cost matrix, usually in the form:

$$\boldsymbol{A}_x = \begin{bmatrix}1 \\ -1\end{bmatrix}, b_x = \begin{bmatrix}b_{\max} \\ -b_{\min}\end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix}
\boldsymbol{A}_u & 0 & \dots & 0 \\
0 & \boldsymbol{A}_u & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & \boldsymbol{A}_u \\
0 & 0 & \dots & 0 \\
\boldsymbol{A}_x\boldsymbol{B} & 0 & \dots & 0 \\
\boldsymbol{A}_x\boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}_x\boldsymbol{B} & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\boldsymbol{A}_x\boldsymbol{A}^{N-2}\boldsymbol{B} & \boldsymbol{A}_x\boldsymbol{A}^{N-3}\boldsymbol{B} & \dots & 0 \\
\boldsymbol{A}_f\boldsymbol{A}^{N-1}\boldsymbol{B} & \boldsymbol{A}_f\boldsymbol{A}^{N-2}\boldsymbol{B} & \dots & \boldsymbol{A}_f\boldsymbol{B}
\end{bmatrix} \quad
\boldsymbol{E} = \begin{bmatrix}
0 \\ 0 \\ \vdots \\ 0 \\ -\boldsymbol{A}_x \\ -\boldsymbol{A}_x\boldsymbol{A} \\ -\boldsymbol{A}_x\boldsymbol{A}^2 \\ \vdots \\ -\boldsymbol{A}_x\boldsymbol{A}^{N-1} \\ -\boldsymbol{A}_f\boldsymbol{A}^N
\end{bmatrix} \quad
\boldsymbol{W} = \begin{bmatrix}
b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_x \\ b_f
\end{bmatrix}$$

## 4.3 QP without substitution

State equations represented in equality constraints ($k$ fixed, usually $k = 0$).

$$J^*(x_k) = \min_z \begin{bmatrix}z^T & x_k^T\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{H}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q}\end{bmatrix}\begin{bmatrix}z \\ x_k\end{bmatrix}$$
$$\text{s.t. } \boldsymbol{G}\ z \le w + \boldsymbol{E}\ x_k$$
$$\boldsymbol{G}_{\text{eq}}\ z = \boldsymbol{E}_{\text{eq}}\ x_k, \quad \text{system dynamics}$$

$\bar{\boldsymbol{H}} = \text{diag}(\boldsymbol{Q},\dots,\boldsymbol{Q},\boldsymbol{P},\boldsymbol{R},\dots,\boldsymbol{R})$

$$z = \begin{bmatrix}x_1 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_{N-1}\end{bmatrix} \quad
\boldsymbol{G}_{\text{eq}} = \left[\begin{array}{cccc|cccc}
\boldsymbol{I} & & & & -\boldsymbol{B} & & & \\
-\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B} & & \\
& \ddots & \ddots & & & & \ddots & \\
& & -\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B}
\end{array}\right] \quad
\boldsymbol{E}_{\text{eq}} = \begin{bmatrix}\boldsymbol{A} \\ 0 \\ \vdots \\ 0\end{bmatrix}$$

$$w = \begin{bmatrix}b_x \\ \vdots \\ b_f \\ b_u \\ \vdots \\ b_u\end{bmatrix} \quad
\boldsymbol{G} = \left[\begin{array}{cccc|ccc}
0 & & & & & & \\
& \boldsymbol{A}_x & & & & & \\
& & \ddots & & & & \\
& & & \boldsymbol{A}_x & & & \\
\hline
& & & & \boldsymbol{A}_d & & \\
& & & & & \ddots & \\
& & & & & & \boldsymbol{A}_d
\end{array}\right] \quad
\boldsymbol{E} = \begin{bmatrix}-\boldsymbol{A}_x^T \\ 0 \\ \vdots \\ 0\end{bmatrix}$$

## 4.4 Invariance

Def.: $x(k) \in O \Rightarrow x(k+1) \in O \forall k$.

$$\text{pre}(S) := \{x \mid g(x) \in S\} \qquad = \{x \mid Ax \in S\}$$

Max invariant set calculation: $\Omega_{i+1} \leftarrow \text{pre}(\Omega_i) \cap \Omega_i$, terminating when $\Omega_{i+1} = \Omega_i$.

tim: We need more here, pos. inv. set, max. pos.inv $O_\infty$

## 4.5 Stability and Feasability

Main Idea: Choose $\mathcal{X}_f$ and $\boldsymbol{P}$ to mimic infinite horizon. LQR control law $\kappa(x) = \boldsymbol{F}_\infty x$ from solving DARE. Set terminal weight $\boldsymbol{P} = \boldsymbol{P}_\infty$, terminal set $\mathcal{X}_f$ as maximal invariant set:

$$x_{k+1} = \boldsymbol{A}x_k + \boldsymbol{B}\boldsymbol{F}_\infty\ x_k \in \mathcal{X}_f \qquad \forall x_k \in \mathcal{X}_f \text{ terminal set invariant}$$
$$\mathcal{X}_f \subseteq \mathcal{X}, \qquad \boldsymbol{F}_\infty\ x_k \in \mathcal{U} \qquad \forall x_k \in \mathcal{X}_f \text{ constrainst satisfied}$$

We get: 1. Positive stage cost function, 2. invariant terminal set by construction, 3. Terminal cost is Lyapunov function with

$$x_{k+1}^T\boldsymbol{P}x_{k+1} - x_k^T\boldsymbol{P}x_k = -x_k^T(\boldsymbol{Q} + \boldsymbol{F}_\infty^T\boldsymbol{R}\boldsymbol{F}_\infty)x_k$$

Extension to non-linear (time-invariant) MPC possible since terminal set and cost do not rely on linearity.

## 5 Practical Issues

### 5.1 MPC for tracking

Target steady-state conditions $x_s = \boldsymbol{A}x_s + \boldsymbol{B}u_s$ and $y_s = \boldsymbol{C}x_s = r$ and constrainsts give:

$$\min_{x_s,u_s} u_s^T \boldsymbol{R} u_s \text{ subj. to } \begin{bmatrix} \boldsymbol{I}-\boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ r \end{bmatrix}, x_s \in \mathcal{X}, u_s \in \mathcal{U}$$

Usually assume $x_s, u_s$ unique and feasible. If no solution exists, compute closest steady-state $\min(\boldsymbol{C}x_s - r)^T \boldsymbol{Q}(\boldsymbol{C}x_s - r)$ s. t. $x_s = \boldsymbol{A}x_s + \boldsymbol{B}u_s$.

MPC problem to drive $y \to r$ is:

$$\min_u \|y_N - \boldsymbol{C}x_s\|_{P_y}^2 + \sum_{i=0}^{N-1} \|y_i - \boldsymbol{C}x_s\|_{Q_y}^2 + \|u_i - u_s\|_R^2$$

### 5.2 Delta formulation

Reference $r$, $\Delta x_k = x_k - x_s, \Delta u_k = u_k - u_s$:

$$\begin{aligned}
&\min V_f(\Delta x_N) + \sum_{i=0}^{N-1} \Delta x_i^T \boldsymbol{Q} \Delta x_i + \Delta u_i^T \boldsymbol{R} \Delta u_i \\
&\text{s.t. } \Delta x_0 = \Delta x_k \\
&\quad \Delta x_{k+1} = \boldsymbol{A}\Delta x_k + \boldsymbol{B}\Delta u_k \\
&\boldsymbol{H}_x x \le k_x \Rightarrow \boldsymbol{H}_x \Delta x \le k_x - \boldsymbol{H}_x x_s \\
&\boldsymbol{H}_u u \le k_u \Rightarrow \boldsymbol{H}_u \Delta u \le k_u - \boldsymbol{H}_u u_s \\
&\quad \Delta x_N \in \mathcal{X}_f \quad \text{adjusted accordingly, shift (and scaled)} \\
&\quad x_s \oplus \mathcal{X}_f \subseteq \mathcal{X} \\
&\quad \boldsymbol{K}\Delta x + u_s \in \mathcal{U}
\end{aligned}$$

Control given by $u_0^* = \Delta u_0^* + u_s$.

### 5.3 Offset free tracking

$$\begin{aligned}
x_{k+1} &= \boldsymbol{A}x_k + \boldsymbol{B}u_k + \boldsymbol{B}_d d_k \\
d_{k+1} &= d_k \\
y_k &= \boldsymbol{C}x_k + \boldsymbol{C}_d d_k
\end{aligned}$$

$$\begin{bmatrix} \boldsymbol{I}-\boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_d \hat{d} \\ r - \boldsymbol{C}_d \hat{d} \end{bmatrix}$$

Choice of $\boldsymbol{B}_d, \boldsymbol{C}_d$ requires that $(\boldsymbol{A}, \boldsymbol{C})$ is observable and $\begin{bmatrix} \boldsymbol{A}-\boldsymbol{I} & \boldsymbol{B}_d \\ \boldsymbol{C} & \boldsymbol{C}_d \end{bmatrix}$ has full $(n_x + n_d)$ column frank (i.e. det $\neq 0$). Intuition: for fixed $y_s$ at steady-state, $d_s$ is uniquely determined.

If plant has no integrator we can choose $\boldsymbol{B}_d = \boldsymbol{0}$ since $\det(\boldsymbol{A} - \boldsymbol{I}) \neq 0$.

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}_d \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} u_k + \begin{bmatrix} \boldsymbol{L}_x \\ \boldsymbol{L}_d \end{bmatrix} \left(-y_k^m + \boldsymbol{C}\hat{x}_k + \boldsymbol{C}_d \hat{d}_k\right)$$

where $y_k^m$ measured output; choose $\begin{bmatrix} \boldsymbol{L}_x \\ \boldsymbol{L}_d \end{bmatrix}$ s.t. error dynamics stable and converge to zero.

tim: Target condition here

If 1) number of dist. = number of outputs, 2) target steady-state problem feasible and no constraints active at steady-state, 3) closed-loop system converges, then target achieved without offset.

### 5.4 Soft-constraints via slack variables

$$\min_x f(z) + l_\epsilon(\epsilon) \quad \text{s.t. } g(z) \le \epsilon, \epsilon \ge 0$$

Requirement: Softened problem has same minimiser as original problem if feasible.

Quadratic error function $l_\epsilon(\epsilon) = v\epsilon + w\epsilon^2$, $w > 0$ gives smoothness, choose $v > \lambda^* \ge 0$ for exact penalty (above requirement fulfilled).

### 5.5 Move Blocking

Main idea to set a number of inputs as the same, $u_2 = u_3 = \cdots = u_N$, to reduce computational burden, at the slight cost of sub-optimality.

## 6 Robust MPC

**Enforcing terminal constraints** by robust invariance:

$$x \in O^{\mathcal{W}} \Rightarrow g(x, w) \in \Omega^W \; \forall w \in \mathcal{W}$$

$$\text{pre}^{\mathcal{W}}(\Omega) = \{x | g(x, w) \in \Omega \; \forall w \in \mathcal{W}\}$$

**Enforcing sequential constraints** for uncertain system $\phi$:

$$\phi_i(x_0, u, w) = \left\{ x_i + \sum_{j=0}^{i-1} \boldsymbol{A}^j w_j \middle| w \in \mathcal{W}^i \right\} \subseteq \mathcal{X}$$

$$\phi_N(x_0, u, w) \in \mathcal{X}_f \quad \text{as well}$$

The uncertain system evolves with the summation of all the disturbances up to time $i$, hence we have to restrict the open-loop (determine control before disturbance is measured):

$$\boldsymbol{A}_x x \le b_x \text{ becomes } \boldsymbol{A}_x x_i + \boldsymbol{A}_x \sum_{j=0}^{i-1} \boldsymbol{A}^j w_k \le b_x :$$

$$\begin{aligned}
x_i &\in \mathcal{X} \ominus (\mathcal{W} \oplus \boldsymbol{A}\mathcal{W} \oplus \cdots \oplus \boldsymbol{A}^{i-1}\mathcal{W}) \\
&= \left( \bigoplus_{j=0}^{i-1} \boldsymbol{A}^j \mathcal{W} \right) = \begin{bmatrix} \boldsymbol{A}^0 & \ldots & \boldsymbol{A}^{i-1} \end{bmatrix} \mathcal{W}^i
\end{aligned}$$

For example: Robust invariant set calculation of $x_{k+1} = 0.5x_k + w_k$ under $-10 \le x \le 10$ and $-1 \le w \le 1$.

$$\begin{aligned}
\Omega_0 &= [-10, 10] \\
\text{pre}^{\mathcal{W}}(\Omega_0) &= \{x | -10 \le 0.5x + w \le 10 \text{ for } -1 \le w \le 1\} \\
&= \{x | -20 - 2w \le x \le 20 + 2w \text{ for } -1 \le w \le 1\} \\
&= \{x | -18 \le x \le 18\} \\
\Omega_1 &= [-10, 10] \cap [-18, 18] = [-10, 10] = \mathcal{O}_\infty^{\mathcal{W}}
\end{aligned}$$

For example: Terminal set calculation of $x_{k+1} = w_k$, $-1 \le w \le 1$, N= 2:

$$\mathcal{X}_f^{\mathcal{W}} = \mathcal{X}_f \ominus \left( \bigoplus_{j=0}^{1} \boldsymbol{A}^j \mathcal{W} \right) = \mathcal{X}_f \ominus 2\mathcal{W} = [-10, 10] \ominus [-2, 2] = [-8, 8]$$

**Tube-MPC** We want nominal system $z_k = \boldsymbol{A}z_k + \boldsymbol{B}v_k$ with "tracking" controller $u_k = \boldsymbol{K}(x_k - z_k) + v_k$ i.e. closed-loop, $\boldsymbol{K}$ found offline.

Step 1: Compute the minimal robust invariant set $\mathcal{E} = \bigoplus_{j=1}^{\infty} \boldsymbol{A}_{cl}^j \mathcal{W}$.

Step 2: Shrink Constraints:

$$\begin{aligned}
\{z_i\} \oplus \mathcal{E} \subseteq \mathcal{X} \quad &\Rightarrow \{z_i\} \in \mathcal{X} \ominus \mathcal{E} \\
u_i \in \boldsymbol{K}\mathcal{E} \oplus \{v_i\} \subset \boldsymbol{U} \quad &\Rightarrow \{v_i\} \in \mathcal{U} \ominus \boldsymbol{K}\mathcal{E} \\
z_n \in \mathcal{X}_f \ominus \mathcal{E} &
\end{aligned}$$

Also check that the set $\mathcal{X}_f$ is invariant for the nominal system with tightened constraints: $(A + BK)\mathcal{X}_f \subseteq \mathcal{X}_f$, and that it satisfies the constraints: $\mathcal{X}_f \subseteq \mathcal{X} \ominus \mathcal{E}$ and $K\mathcal{X}_f \subseteq \mathcal{U} \ominus K\mathcal{E}$.

## 7 Explicit MPC

$z^*(x_k)$ is continuous and polyhedral piecewise affine over feasible set.

### 7.1 Quadratic Cost

$J^*(x_k)$ is continuous, convex and polyhedral piecewise quadratic.
Using the formulation with substitution,

$$\begin{aligned}
J(x_k) &= \min z^T H z - x_k^T (Y - F H^{-1} F^T) x_k \\
&\text{s.t} \quad Gz \le w + Sx_k \\
z(x_k) &= U + H^{-1} F^T x_k \\
S &= E + G H^{-1} F^T \\
U^* &= z^*(x_k) - H^{-1} F^T x_k
\end{aligned}$$

The first solution gives $u^*(x_k) = \kappa(x_k)$, which is continuous and piecewise affine on polyhedra $\kappa(x) = F_j x + g_j$.

### 7.2 $1/\infty$-norm

$J^*(x_k)$ is continuous, convex and polyhedral piecewise affine. Optimal solution: $u_0^* = \begin{bmatrix} 0 & \ldots & 0 & I_m & 0 & \ldots & 0 \end{bmatrix} z^*(x_k)$, and is in the same form as the QP case above.

### 7.3 Explicit Example

1. Write out KKT conditions and Lagrangian.
2. Determine infeasible regions from primal feasibility constraints. For example, $x1 < 10$.
3. From primal and dual feasibility, and complementary slackness conditions, list out all cases that can occur.

$$\begin{aligned}
\lambda_1 &= 0 & \lambda_1 &\ge 0 \\
g_1(x) &< 0 & g_1(x) &= 0
\end{aligned}$$

4. Solve for each case: $z^*(x_1, x_2)$ and $J^*(x_1, x_2)$, listing the active constraints, and range of validity.

## 8 Hybrid MPC

### 8.1 Piecewise Affine (PWA) Systems

Affine dynamics and output for each region:

$$\begin{cases} x_{k+1} &= A^i x_k + B^i u_k + f^i \\ y_k &= C^i x_k + D^i u_k + g_i \end{cases} \text{ if } x_t \in \mathcal{X}_j$$

Polyhedral partition of the $(x, u)$-space:

$$\{\mathcal{X}_i\}_{j=1}^s = \{x, u | H_j x + J_j u \le K_j\}$$

### 8.2 Mixed Logical Dynamical Hybrid Model (MLD)

Idea: associate boolean to binary: $p_j \iff \delta_i = 1$, $\neg p_j \iff \delta_i = 0$.

### 8.2.1 Translate logic rules to Linear Integer Inequalities

| | | |
|---|---|---|
| AND | $p_1 \wedge p_2$ | $\delta_1 \geq 1, \delta_1 \geq 1$ also $\delta_1 + \delta_2 \geq 2$ |
| OR | $p_1 \vee p_2$ | $\delta_1 + \delta_2 \geq 1$ |
| NOT | $\neg p_1$ | $1 - \delta_1 \geq 1$ also $\delta_1 = 0$ |
| XOR | $p_1 + p_2$ | $\delta_1 + \delta_2 = 1$ |
| IMPLY | $p_1 \rightarrow p_2$ | $\delta_1 - \delta_2 \leq 0$ |
| IFF | $p_1 \leftrightarrow p_2$ | $\delta_1 - \delta_2 = 0$ |
| ASSIGNMENT | $p_3 \leftrightarrow p_1 \wedge p_2$ | $\delta_1 + (1-\delta_3) \geq 1,$ |
| | | $\delta_2 + (1-\delta_3) \geq 1,$ |
| | | $(1-\delta_1) + (1-\delta_2) + \delta_3 \geq 1$ |
| CNF-Clause | $\neg p_1 \vee \neg p_2 \vee p_3$ | $\delta_1 + \delta_2 + \delta_3 \leq 1$ |

#### Logic Equality Rules

$$\neg(A \wedge B) = \neg A \vee \neg B$$
$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$
$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

### 8.2.2 Translate continuous and logical components into Linear Mixed-Integer Relations

Event generator: $\delta_e(k) = f_{EG}(x_c(k), u_c(k), t)$.
Consider: $p \Leftrightarrow a^T x \leq b, \mathcal{X} = \{x | a^T x - b \in [m, M]\}$.
Translated to linear inequalities: $m\delta < a^T x - b \leq M(1-\delta)$, where $[m, M]$ are lower and upper bounds.

#### Representing Switched Affine Dynamics as "IF-THEN-ELSE" relations

IF p THEN $z_k = a_1^T x_k + b_1$ else $z_k = a_2^T x_k + b_2 \Leftrightarrow$

$$(m_2 - M_1)\delta + z_k \leq a_2^T x_k + b_2 \leq -(m_1 - M_2)\delta + z_k$$
$$(m_1 - M_2)(1-\delta) + z_k \leq a_1^T x_k + b_1 \leq -(m_1 - M_2)(1-\delta) + z_k$$

This results in a linear MLD model

$$x_{k+1} = Ax_k + B_1 u_k + B_2 \delta_k + B_3 z_k$$
$$y_k = Cx_k + D_1 u_k + D_2 \delta_k + D_3 z_k$$
$$E_2 \delta_k + E_3 z_k \leq E_4 x_k + E_1 u_k + E_5$$

where the last equation describes the relationship between the continuous and integer variables. Physical constraints on cont. variables: $\{\mathcal{C}\} = \left\{ \begin{bmatrix} x_c \\ u_c \end{bmatrix} \in \mathcal{R}^{n_c + m_c} | F x_c + G u_c \leq H \right\}$

### 8.3 CFTOC for Hybrid Systems

$$J^*(x) = \min_U \; l_N(x_N) + \sum_{k=0}^{N-1} l(x_k, u_k, \delta_k, z_k)$$
$$\text{s.t} \; x_{k+1} = Ax_k + B_1 u_k + B_2 \delta_k + B_3 z_k$$
$$E_2 \delta_k + E_3 z_k \leq E_4 x_k + E_1 u_k + E_5$$
$$x_N \in \mathcal{X}_f, x_0 = x(0)$$

### 8.4 MILP/QP

$$\min \; c_c z_c + c_b z_b + d \quad \text{OR} \quad zHz + qz + d$$
$$\text{s.t} \; G_c z_c + G_b z_b \leq W$$
$$z_c \in R^{s_c}, z_b \in \{0,1\}^{s_b}$$

Branch and bound method can be used to efficiently solve the problem. Explict solution is a time varying fb law for both problems: $u_k^*(x_k) = F_k^j x_k + G_k^j$ if $x_k \in \mathcal{R}_k^j$.

## 9 Numerical Optimization – Iterative Methods

### 9.1 Gradient descent

$x_{i+1} = x_i - h_i \nabla f(x_i)$ with step-size $h_i = \frac{1}{L}$ for $L$-smooth $f(x)$:

$$\exists L : \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \; \forall x, y, \in \mathbb{R}^n$$
$$\iff \nabla f \text{ is Lipschitz continuous}$$
$$\iff f \text{ can be upperbounded by a quadratic function:}$$
$$f(x) \leq f(y) + \nabla f(y)^T (x - y) + 0.5L \|x - y\|^2 \; \forall x, y \in \mathbb{R}^n$$

### 9.2 Newton's Method

$$x_{i+1} = x_i - h_i \Delta x_{nt} := x_i - h_i (\nabla^2 f(x_i))^{-1} \nabla f(x_i)$$

Line search problem: choose $h_i > 0$ s.t. $f(x_i + h_i \Delta x_{nt}) \leq f(x_i)$.
Either compute exact and best $h_i$ using:

$$h_i^* = \text{argmin } x_i - h_i \Delta x_{nt}$$

Or use the backtracking search method:

For $\alpha \in (0, 0.5)$ and $\beta \in (0, 1)$ :

initialise $h_i = 1$;

while $f(x_i + h_i \Delta x_{nt}) > f(x_i) + \alpha h_i \nabla f(x_i)^T \Delta x_{nt}$ do $h_i \leftarrow \beta h_i$

For given equality constraint $\boldsymbol{A}x = b$ solve:

$$\begin{bmatrix} \nabla^2 f(x_i) & \boldsymbol{A}^T \\ \boldsymbol{A} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \delta x \\ \lambda \end{bmatrix} = \begin{bmatrix} -\nabla f(x_i) \\ \boldsymbol{0} \end{bmatrix}$$

### 9.3 Constrained optimization with $g_i(x) \leq 0$

**Gradient method** $x_{i+1} = \pi_Q(x_i - h_i \nabla f(x_i))$ where $\pi_Q$ is a projection $\pi_q = \arg\min_x \frac{1}{2}\|x - y\|_2^2$. Projection can be solved directly if simple enough, else solve the dual.

### 9.4 Interior-Point methods

Assumptions $f(x^*) < \infty$, $\tilde{x} \in \text{dom}(f)$.

**Barrier method** $\min f(x) + \kappa\phi(x)$. Approximate $\phi$ using diff'able log barrier(instead of indicator function):

$$\phi(x) = \sum_{i=1}^m I_-(g_i(x)) = -\sum_{i=1}^m \log(-g_i(x))$$
$$\lim_{\kappa \to 0} x^*(\kappa) = x^*$$

Analytic center: $\arg\min_x \phi(x)$, central path $\{x^*(\kappa) | \kappa > 0\}$.

**Path following method**

1. Centering $x^*(\kappa) = \arg\min_x f(x) + \kappa\phi(x)$ with newton's method:
1.1. $\Delta x_{\text{nt}} = \left[\nabla^2 f(x) + \kappa \nabla^2 \phi(x)\right]^{-1} (-\nabla f(x) - \kappa \nabla \phi(x))$.
1.2. Line search:
retain feasability: $\text{argmax}_{h>0} \{h | g_i(x + h\Delta x) < 0\}$
Find $h^* = \text{argmin}_{h \in [0, h_{\max}]} \{f(x + h\Delta x) + \kappa\phi(x + h\Delta x)\}$
2. Update step $x_i = x^*(\kappa_i)$
3. Stop if $m\kappa_i \leq \epsilon$
4. Decrease $\kappa_{i+1} = \kappa_i / \mu$, $\mu > 1$.

**Centering step with equality constraints**

$$\begin{bmatrix} \nabla^2 f(x) + \kappa \nabla^2 \phi(x) & c^T \\ c & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \Delta x_{\text{nt}} \\ \nu \end{bmatrix} = -\begin{bmatrix} \nabla f(x) + \kappa \nabla \phi(x) \\ 0 \end{bmatrix}$$

**Relaxed KKT**

$$Cx^* = d \qquad g_i(x^*) + s_i^* = 0$$
$$\nabla f(x^*) + \sum_{i=1}^m \lambda_i^* \nabla g_i(x^*) + c^T \nu = 0 \qquad \lambda_i^* = \kappa \frac{\partial \phi}{\partial g_i} = -\frac{\kappa}{g_i}$$
$$\lambda_i^* g_i(x^*) = -\kappa \qquad \lambda_i^*, s_i^* \geq 0$$

**Primal Dual Search Direction Computation**

$$\begin{bmatrix} H(x, \lambda) & c^T & G(x)^T & 0 \\ c & 0 & 0 & 0 \\ G(x) & 0 & 0 & \boldsymbol{I} \\ 0 & 0 & \boldsymbol{S} & \boldsymbol{\Lambda} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta u \\ \Delta \lambda \\ \Delta s \end{bmatrix} = -\begin{bmatrix} \nabla f(x) + c^T \nu + G(x)^T \lambda \\ cx - d \\ g(x) + s \\ s\lambda - \nu \end{bmatrix}$$

$\boldsymbol{S} = \text{diag}(s_1, \ldots, s_m), \boldsymbol{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_m)$ and $\nu$ is a vector for choosing centering parameters.