1 Systems Theory

2 Unconstrained Control

3 (Convex) Optimization

General Problem $\min_{x\in\text{dom}(f)} f(x)$ s. t. $g_i(x) \leq 0$ and $h_j(x) = 0$.

RHC

QP with substitution

QP with out substitution

3.1 Duality

Lagrangian Dual Function

$$L(x,\lambda,\nu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$
$$d(\lambda,\nu) = \inf_{x\in\mathcal{X}} L(x,\lambda,\nu) \quad \text{i.e. } \nabla_x L(x,\lambda,\nu) = 0$$

Dual Problem (always convex) $\max_{\lambda,\nu} d(\lambda,\nu)$ s. t. $\lambda \geq 0$.
Optimal value is lower bound for primal: $d^* \leq p^*$.
If primal convex, *Slater condition* (strict feasibility) implies *strong duality*:

$$\{x \mid Ax = b, f_i(x) < 0, \} \neq \emptyset \Rightarrow d^* = p^*$$

Karush-Kuhn-Tucker (KKT) Conditions are necessary for optimality (and sufficient if primal convex).

- Primal Feasibility:
$$f_i(x^*) \leq 0 \quad i = 1,\dots,m$$
$$h_i(x^*) = 0 \quad i = 1,\dots,p$$
- Dual Feasibility: $\lambda^* \geq 0$
- Complementary Slackness:
$$\lambda_i^* \cdot f_i(x^*) = 0 \qquad i = 1,\dots,m$$
- Stationarity:
$$\nabla_x L(x^*,\lambda^*,\nu^*) = 0$$

3.2 Constrained Finite Time Optimal Control (CFTOC)

3.3 Invariance

3.4 Feasability, Stability

3.5 Practical MPC

3.6 Robust MPC

Tube-MPC

3.7 Explicit MPC

3.8 Hybrid MPC

4 Numerical Optimization

Gradient, Newton, Interior Point

5 Observer Based Control

5.1 LTI Observer

LTI System:

$$x(k) = Ax(k-1) + Bu(k-1) + v(k-1)$$
$$z(k) = Hx(k) + w(k)$$

Linear Static Gain Observer (Luenberger Observer):

$$\hat{x}(k) = A\hat{x}(k-1) + Bu(k-1) + K(z(k) - \hat{z}(k))$$
$$\hat{z}(k) = H(A\hat{x}(k-1) + Bu(k-1))$$
$$e(k) = (I - K\,H)\,A\,e(k-1)$$

$e(k) \to 0$ for $k \to \infty$ if and only if $(I - K\,H)\,A$ is stable.

Steady State:

$$\hat{x}(k) = (I - K_\infty H)\,A\,\hat{x}(k-1) + (I - K_\infty)B\,u(k-1) + K_\infty z(k)$$

The steady-state KF is one way to design the observer gain K (optimal in minimizing the Steady State mean squared error).

(A,H) detectable $\Rightarrow$ K exists such that $(I - K\,H)A$ is stable.

5.2 Static State Feedback Control

Design of a controller without paying attention to the state estimation:

$$x(k) = Ax(k-1) + Bu(k-1) \qquad \text{(Process without noise)}$$
$$z(k) = x(k) \qquad \text{(Perfect State information)}$$
$$u(k) = F\cdot z(k) = F\cdot x(k) \qquad \text{(Control Law)}$$

Closed loop dynamics: $x(k) = (A + BF)$. Hence system is stable if $(A+BF)$ is stable. Such an F exists only if (A,B) is stabilizable.
If (A,B) is stabilizable and (A,G) detectable, then F is given by

$$F = -(B^TPB + \bar{R})^{-1}\cdot B^TPA; \qquad P \geq 0$$

P from DARE: $P = A^TPA + \bar{Q} - A^TPB(B^TPB + \bar{R})^{-1}\cdot B^TPA$

5.3 Separation Principle (Linear Systems only)

Combining Luenberger Observer and Static State Feedback control yields:

$$\begin{bmatrix} x(k) \\ e(k) \end{bmatrix} = \begin{bmatrix} A+BF & -BF \\ 0 & (I-KH)A \end{bmatrix} \cdot \begin{bmatrix} x(k-1) \\ e(k-1) \end{bmatrix}$$

Eigenvalues of closed loop are given bei Eigenvalues of $(I - KH)A$ and $(A+BF)$. System is stable as long as there exists no $|\lambda| \geq 1$.

5.4 Separation Theorem

1. Design steady-state KF which does not depend on $\bar{Q},\bar{R}$. $\Rightarrow \hat{x}(k)$
2. Design state-feedback $u(k) = Fx(k)$ and put both together.