1 System Theory

1.1 Nonlinear Systems

tim: Put here linearization formulae

1.2 Linear Systems

$$\dot{x}(t) = A^c x(t) + B^c u(t)$$
$$y(t) = C^c x(t) + D^c u(t)$$
$$x(t) = e^{A^c(t-t_0)}x_0 + \int_{t_0}^{t} e^{A^c(t-\tau)}Bu(\tau)d\tau$$
$$e^{A^c t} = \sum_{n=0}^{\infty}\frac{(A^c t)^n}{n!}$$
$$x(k+N) = A^N x(k) + \sum_{i=0}^{N-1} A^i Bu(k+N-1-i)$$

1.3 Lyapunov Stability

we define a system to be stable in the sense of Lyapunov, if it stays in any arbitrarily small neighborhood of the origin when it is disturbed slightly

Lyapunov stable if for every $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that

$$\operatorname{norm} x(0) < \delta(\epsilon) \rightarrow \operatorname{norm} x(k) < \epsilon, \forall k \geq 0$$

asymptotically stable in $\Omega \subseteq \mathbb{R}^n$ if it is Lyapunov stable and attractive $\lim_{k\rightarrow\infty} x(k) = 0, \forall x(0) \in \Omega$

1.4 Lyapunov Function

Consider the equilibrium point $x = 0$. Let $\Omega \subset \mathbb{R}^n$ be a closed and bounded set containing the origin. A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, continous at the origin, finite for every $x \in \Omega$, and such that

$$V(0) = 0 \text{ and } V(x) > 0, \forall x \in \Omega \backslash \{0\}$$
$$V(g(x)) - V(x) \leq -\alpha(x), \forall x \in \Omega \backslash \{0\}$$

where $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous positive definite
If a system admins a Lyapunov function $V(x)$, then $x = 0$ is **asymptotically stable** in Ω (sufficient but not necessary)
If a system admits a Lyapunov function, which additionally satisfies $\operatorname{norm} x \rightarrow \infty \rightarrow V(x) \rightarrow \infty$, then $x = 0$ is **globally asymptotically stable**

tim: Check Eig. values of $(APA - P)$ neg., $V(x) = x^T Px$?

Linear systems: iff eigenvalues of A inside unit circle (i.e. stable) then $\exists$*unique* $P > 0$ that solves $A_{cl}^T P A_{cl} - P = -Q,\ Q > 0$ and $V = x^T Px$ is a lyapunov function.

1.5 Discretization

Euler: $A = I + T_s A^c,\ B = T_s B^c,\ C = C^c,\ D = D^c$

$$x(k+1) = x(k) + T_s g^c(x(k), u(k)) = g(x(k), u(k))$$
$$y(k) = h^c(x(k), u(k)) = h(x(k), u(k))$$

Exact: (assumption of a constant $u(t)$ during T_s)

$$A = e^{A^c T_s},\ B = \int_0^{T_s} e^{A^c(T_s-\tau')}B^c d\tau$$
$$B = (A^c)^{-1}(A - I)B^c, \text{ if } A^c \text{ invertible}$$

1.6 Controllability (reachability) and observability

$$C = [B\ AB\ ...\ A^{n-1}B]$$
$$O = [C^T\ (CA)^T\ ...\ (CA^{n-1})^T]$$

2 Unconstrained Control

2.1 Block Approach (used also for $\bar{w}$ substition)

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{A} \\ \vdots \\ \boldsymbol{A}^N \end{bmatrix} x(0) + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{AB} & \boldsymbol{B} & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{0} \\ \boldsymbol{A}^{N-1}\boldsymbol{B} & \dots & \boldsymbol{AB} & \boldsymbol{B} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_1} \end{bmatrix}$$

$$x = \boldsymbol{S}^x \cdot x(0) + \boldsymbol{S}^u \cdot u \qquad \operatorname{size}(\boldsymbol{S}^x) = [n_{\text{states}} \cdot (N+1), N]$$
$$\operatorname{size}(\boldsymbol{S}^u) = [n_{\text{states}} \cdot (N+1), n_{\text{states}}]$$
$$\bar{\boldsymbol{Q}} = \operatorname{diag}(\boldsymbol{Q}, \dots, \boldsymbol{Q}, \boldsymbol{P}) \qquad \operatorname{size}(\bar{\boldsymbol{Q}}) = [n_{\text{states}} \cdot (N+1), n_{\text{states}} \cdot (N+1)]$$
$$\bar{\boldsymbol{R}} = \operatorname{diag}(\boldsymbol{R}, \dots, \boldsymbol{R}) \qquad \operatorname{size}(\bar{\boldsymbol{R}}) = [n_{\text{input}} \cdot N, n_{\text{input}} \cdot N]$$
$$\boldsymbol{H} = \boldsymbol{S}^{u\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^u + \boldsymbol{R} \qquad \boldsymbol{F} = \boldsymbol{S}^{x\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^u$$
$$\boldsymbol{Y} = \boldsymbol{S}^{x\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^x$$

Optimal cost and control

$$J^*(x_0) = -x_0^T \boldsymbol{FHF}^T x_0 + x_0^T \boldsymbol{Y} x_0$$
$$u^*(x_0) = -\boldsymbol{H}^{-1}\boldsymbol{F}^T x_0 = -\left(\boldsymbol{S}^{u\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^u + \boldsymbol{R}\right)^{-1} \boldsymbol{S}^{u\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^x x_0$$

2.2 Recursive Approach

$$J_k^*(x_k) = \min_{u_k} I(x_k, u_k) + J_{k+1}(x_{k+1})$$

Is a feedback controller as opposed to the Batch Approach. For LQR solve via Riccati Difference Equation (RDE).

$$\boldsymbol{F}_k = -(\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{B} + \boldsymbol{R})^{-1}\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{A}$$
$$\boldsymbol{P}_k = \boldsymbol{A}^T\boldsymbol{P}_{k+1}\boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{A}^T\boldsymbol{P}_{k+1}\boldsymbol{B}(\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{B} + \boldsymbol{R})^{-1}\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{A}$$

$$u_k^* = \boldsymbol{F}_k\ x_k \qquad J_k^*(x_k) = x_k^T\boldsymbol{P}_k\ x_k \qquad \boldsymbol{P}_N = \boldsymbol{P}$$

For unconstrained Infinite Horizon Problem, substituting $\boldsymbol{P}_\infty = \boldsymbol{P}_k = \boldsymbol{P}_{k+1}$ into RDE gives DARE. Uniquely solvable, iff (A, B) stabilizable and (A, G) detectable, where $\boldsymbol{GG}^T = \boldsymbol{Q}$. Follows from closed-loop system $x_{k+1} = (\boldsymbol{A} + \boldsymbol{BF}_k)x_k$

3 (Convex) Optimization

3.1 Convexity

Convex set $\mathcal{X}$ iff $\forall\lambda \in [0, 1]\forall x, y \in \mathcal{X}\ \lambda x + (1 - \lambda)y \in \mathcal{X}$. Intersection preserves convexity, union does not.

Affine set $\mathcal{X}$ $= \{x \in \mathbb{R}^n | \boldsymbol{A}x = b\}$ for some $\boldsymbol{A}, b$

Subspace is affine set through origin, i.e. $b = \boldsymbol{0}$, aka Nullspace of $\boldsymbol{A}$.

Hyperplane $\mathcal{X}$ $= \{x \in \mathbb{R}^n | a^T x = b\}$ for some a, b.

Halfspace $\mathcal{X}$ $= \{x \in \mathbb{R}^n | a^T x \leq b\}$ for some a, b.

Polyhedron $\mathcal{P}$ $= \{x | a_i^T x \leq b_i, i = 1, \dots, n\} = \{x | \boldsymbol{A}x \leq b\}$

Cone

Ellipsoid $\mathcal{E}$ $= \{x | (x - x_c)^T\boldsymbol{A}^{-1}(x - x_c) \leq 1\}$, x_c center point.

Convex function

Norm $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$

$$f(x) = 0 \implies x = 0, \qquad f(x) \geq 0$$
$$f(\alpha \cdot x) = |\alpha| \cdot f(x) \qquad \text{for scalar } \alpha$$
$$f(x + y) \leq f(x) + f(y) \qquad \forall x, y \in R^n$$

tim: Maybe move the above somewhere else?

General Problem $\min_{x\in\operatorname{dom}(f)} f(x)$ s. t. $g_i(x) \leq 0$ and $h_j(x) = 0$.

3.2 Linear Programming (LP)

Problem statement $\min c^T x$ such that $\boldsymbol{G}x \leq h$ and $\boldsymbol{A}x = b$.

Norm l_∞ $\min_x \|x\|_\infty = \min_{x\in\mathbb{R}^n} [\max\{x, \dots, x_n, -x_1, \dots, -x_n\}]$:

$$\min_{x,t} t \quad \text{subject to} \quad x_i \leq t, -x_i \leq t, \qquad \boldsymbol{F}x \leq g$$
$$\iff \min_{x,t} t \quad \text{subject to} \quad -\boldsymbol{1}t \leq x \leq 1t, \qquad \boldsymbol{F}_x \leq g.$$

Norm l_1 $\min_x \|x\|_1 = \min_x \left[\sum_{i=1}^m \max\{x_i, -x_i\}\right]$:

$$\min_t t_1 + \dots + t_m \quad \text{subject to} \quad x_i \leq t_i, -x_i \leq t_i, \qquad \boldsymbol{F}x \leq g$$
$$\iff \min_t \boldsymbol{I}^T t \quad \text{subject to} \quad -t \leq x \leq t, \qquad \boldsymbol{F}_x \leq g.$$

Note that for $\dim x = 1$, l_1 and l_∞ are the same.

MPC with linear cost
$J(x_0, u) = \|\boldsymbol{P}x_N\|_p + \sum_{i=0}^{N-1}\|\boldsymbol{Q}x_i\|_p + \|\boldsymbol{R}u_i\|_p$.

tim: Insert here slide 45, lect 4

Receding Horizon Control – RHC

QP with substitution (see also Batch approach)

$$J^*(x_k) = \min_u \begin{bmatrix} u^T & x_k^T \end{bmatrix} \begin{bmatrix} \boldsymbol{H} & \boldsymbol{F}^T \\ \boldsymbol{F} & \boldsymbol{Y} \end{bmatrix} \begin{bmatrix} u \\ x_k \end{bmatrix}$$
$$\text{s. t. } \boldsymbol{G}\ u \le w + \boldsymbol{E}\ x_k$$

Latter gives three sets (same for without substitution)

$$\mathcal{X} = \{x | A_x\ x \le b_x\}$$
$$\mathcal{U} = \{u | A_u\ u \le b_u\}$$
$$\mathcal{X}_{\{} = \{x | A_f\ x \le b_f\}$$

State equations are in cost matrix, usually
$\boldsymbol{A}_x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, b_x = \begin{bmatrix} b_{\max} \\ -b_{\min} \end{bmatrix}$

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{A}_u & 0 & \dots & 0 \\ 0 & \boldsymbol{A}_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{A}_u \\ 0 & 0 & \dots & 0 \\ \boldsymbol{A}_x\boldsymbol{B} & 0 & \dots & 0 \\ \boldsymbol{A}_x\boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}_x\boldsymbol{B} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}_x\boldsymbol{A}^{N-2}\boldsymbol{B} & \boldsymbol{A}_x\boldsymbol{A}^{N-3}\boldsymbol{B} & \dots & 0 \\ \boldsymbol{A}_f\boldsymbol{A}^{N-1}\boldsymbol{B} & \boldsymbol{A}_f\boldsymbol{A}^{N-2}\boldsymbol{B} & \dots & \boldsymbol{A}_f\boldsymbol{B} \end{bmatrix} \quad \boldsymbol{E} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\boldsymbol{A}_x \\ -\boldsymbol{A}_x\boldsymbol{A} \\ -\boldsymbol{A}_x\boldsymbol{A}^2 \\ \vdots \\ -\boldsymbol{A}_x\boldsymbol{A}^{N-1} \\ -\boldsymbol{A}_f\boldsymbol{A}^N \end{bmatrix} \quad \boldsymbol{W} = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_x \\ b_f \end{bmatrix}$$

QP with out substitution State equations represented in equality constrainst.

$$J^*(x_k) = \min_z \begin{bmatrix} z^T & x_k^T \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{H}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} z \\ x_k \end{bmatrix}$$
$$\text{s. t. } \boldsymbol{G}\ z \le w + \boldsymbol{E}\ x_k$$
$$\boldsymbol{G}_{\text{eq}}\ z = \boldsymbol{E}_{\text{eq}}\ x_k, \quad \text{system dynamics}$$

$\bar{\boldsymbol{H}} = \operatorname{diag}(\boldsymbol{Q}, \dots, \boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{R}, \dots, \boldsymbol{R})$

$$z = \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} \quad \boldsymbol{G}_{\text{eq}} = \left[\begin{array}{cccc|cccc} \boldsymbol{I} & & & & -\boldsymbol{B} & & & \\ -\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B} & & \\ & \ddots & \ddots & & & & \ddots & \\ & & -\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B} \end{array}\right] \quad \boldsymbol{E}_{\text{eq}} = \begin{bmatrix} \boldsymbol{A} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$w = \begin{bmatrix} b_x \\ \vdots \\ b_f \\ b_u \\ \vdots \\ b_u \end{bmatrix} \quad \boldsymbol{G} = \left[\begin{array}{ccc|ccc} 0 & & & & & \\ & \boldsymbol{A}_x & & & & \\ & & \ddots & & & \\ & & \boldsymbol{A}_x & & & \\ \hline & & & \boldsymbol{A}_d & & \\ & & & & \ddots & \\ & & & & & \boldsymbol{A}_d \end{array}\right] \quad \boldsymbol{E} = \begin{bmatrix} -\boldsymbol{A}_x^T \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

3.3 Duality

Lagrangian Dual Function

$$L(x, \lambda, \nu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$
$$d(\lambda, \nu) = \inf_{x \in \mathcal{X}} L(x, \lambda, \nu) \quad \text{i.e. } \nabla_x L(x, \lambda, \nu) = 0$$

Dual Problem (always convex) $\max_{\lambda,\nu} d(\lambda, \nu)$ s. t. $\lambda \ge 0$.
Optimal value is lower bound for primal: $d^* \le p^*$.
If primal convex, *Slater condition* (strict feasibility) implies *strong duality*:

$$\{x \mid Ax = b, f_i(x) < 0, \} \ne \emptyset \Rightarrow d^* = p^*$$

Karush-Kuhn-Tucker (KKT) Conditions are necessary for optimality (and sufficient if primal convex).

- Primal Feasibility:
$$f_i(x^*) \le 0 \quad i = 1, \dots, m$$
$$h_i(x^*) = 0 \quad i = 1, \dots, p$$
- Dual Feasibility: $\lambda^* \ge 0$
- Complementary Slackness:
$$\lambda_i^* \cdot f_i(x^*) = 0 \qquad i = 1, \dots, m$$
- Stationarity:
$$\nabla_x L(x^*, \lambda^*, \nu^*) = 0$$

3.4 Constrained Finite Time Optimal Control (CFTOC)

3.5 Invariance

Def.: $x(k) \in O \implies x(k+1) \in O \forall k$.

$$\text{pre}(S) := \{x | g(x) \in S\} \qquad\qquad = \{x | Ax \in S\}$$

tim: We need more here, poos. inv. set, max. pos.inv O_∞

3.6 Stability and Feasability

Main Idea: Choose $\mathcal{X}_f$ and $\boldsymbol{P}$ to mimic infinite horizon. LQR control law $\kappa(x) = \boldsymbol{F}_\infty x$ from solving DARE. Set terminal weight $\boldsymbol{P} = \boldsymbol{P}_\infty$, terminal set $\mathcal{X}_f$ as maximal invariant set:

$$x_{k+1} = \boldsymbol{A}x_k + \boldsymbol{B}\boldsymbol{F}_\infty\ x_k \in \mathcal{X}_f \quad \forall x_k \in \mathcal{X}_f \text{ terminal set invariant}$$
$$\mathcal{X}_f \subseteq \mathcal{X}, \qquad \boldsymbol{F}_\infty\ x_k \in \mathcal{U} \qquad \forall x_k \in \mathcal{X}_f \text{ constraint satisfied}$$

We get: 1. Positive stage cost function, 2. invariant terminal set by construction, 3. Terminal cost is Lyapunov function with

$$x_{k+1}^T \boldsymbol{P} x_{k+1} - x_k^T \boldsymbol{P} x_k = -x_k^T(\boldsymbol{Q} + \boldsymbol{F}_\infty^T \boldsymbol{R} \boldsymbol{F}_\infty) x_k$$

Extension to non-linear (time-invariant) MPC possible since terminal set and cost do not rely on linearity.

3.7 Practical Issues

MPC for tracking Target steady-state conditions $x_s = \boldsymbol{A}x_s + \boldsymbol{B}u_s$ and $y_s = \boldsymbol{C}x_s = r$ and constrainsts give:

$$\min_{x_s, u_s} u_s^T \boldsymbol{R} u_s \text{ subj. to } \begin{bmatrix} \boldsymbol{I} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ r \end{bmatrix}, x_s \in \mathcal{X}, u_s \in \mathcal{U}$$

Usually assume x_s, u_s unique and feasible. If no solution exists, compute closest steady-state ($\min(\boldsymbol{C}x_s - r)^T \boldsymbol{Q}(\boldsymbol{C}x_s - r)$ s. t. $x_s = \boldsymbol{A}x_s + \boldsymbol{B}u_s$).
MPC problem to drive $y \to r$ is:

$$\min_u \|y_N - \boldsymbol{C}x_s\|_{P_y}^2 + \sum_{i=0}^{N-1} \|y_i - \boldsymbol{C}x_s\|_{Q_y}^2 + \|u_i - u_s\|_R^2$$

Delta formulation for reference r $\Delta x_k = x_k - x_s, \Delta u_k = u_k - u_s$:

$$\min V_f(\Delta x_N) + \sum_{i=0}^{N-1} \Delta x_i^T \boldsymbol{Q} \Delta x_i + \Delta u_i^T \boldsymbol{R} \Delta u_i$$
$$\text{s.t. } \Delta x_0 = \Delta x_k$$
$$\Delta x_{k+1} = \boldsymbol{A}\Delta x_k + \boldsymbol{B}\Delta u_k$$
$$\boldsymbol{H}_x x \le k_x \rightarrow \boldsymbol{H}_x \Delta x \le k_x - \boldsymbol{H}_x x_s$$
$$\boldsymbol{H}_u u \le k_u \rightarrow \boldsymbol{H}_u \Delta u \le k_u - \boldsymbol{H}_u u_s$$
$$\Delta x_N \in \mathcal{X}_f \quad \text{adjusted accordingly}$$
$$x_s \oplus \mathcal{X}_f \subseteq \mathcal{X}$$
$$\boldsymbol{K}\Delta x + u_s \in \mathcal{U}$$

Control given by $u_0^* = \Delta u_0^* + u_s$.

Offset free tracking

$$x_{k+1} = \boldsymbol{A}x_k + \boldsymbol{B}u_k + \boldsymbol{B}_d d_k$$
$$d_{k+1} = d_k$$
$$y_k = \boldsymbol{C}x_k + \boldsymbol{C}_d d_k$$

$$\begin{bmatrix} \boldsymbol{I} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_d \hat{d} \\ r - \boldsymbol{C}_d \hat{d} \end{bmatrix}$$

Choice of $\boldsymbol{B}_d, \boldsymbol{C}_d$ requires that $(\boldsymbol{A}, \boldsymbol{C})$ is observable and
$\begin{bmatrix} \boldsymbol{A} - \boldsymbol{I} & \boldsymbol{B}_d \\ \boldsymbol{C} & \boldsymbol{C}_d \end{bmatrix}$ has full $(n_x + n_d)$ column frank (i.e. det $\ne 0$).
Intuition: for fixed y_s at steady-state, d_s is uniquely determined.
If plant has no integrator we can choose $\boldsymbol{B}_d = \boldsymbol{0}$ since $\det(\boldsymbol{A} - \boldsymbol{I}) \ne 0$.

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}_d \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} u_k + \begin{bmatrix} \boldsymbol{L}_x \\ \boldsymbol{L}_d \end{bmatrix} \left(-y_k^m + \boldsymbol{C}\hat{x}_k + \boldsymbol{C}_d \hat{d}_k\right)$$

where y_k^m measured output; choose $\begin{bmatrix} \boldsymbol{L}_x \\ \boldsymbol{L}_d \end{bmatrix}$ s.t. error dynamics stable and converge to zero.

tim: Target condition here

If 1) number of dist. = number of outputs, 2) target steady-state problem feasible and no constraints active at steady-state, 3) closed-loop system converges, then target achieved without offset.

Soft-constraints via slack variables

3.8 Robust MPC

Enforcing terminal constraints by robust invariance:

$$x \in O^{\mathcal{W}} \implies g(x,w) \in \Omega^W \ \forall w \in \mathcal{W}$$
$$\mathrm{pre}^{\mathcal{W}}(\Omega) = \{x | g(x,w) \in \Omega \ \forall w \in \mathcal{W}\}$$

tim: Maybe an example from exercises to compute O_∞^W

Enforcing sequential constraints for uncertain system ϕ:

$$\phi_i(x_0,u,w) = \left\{ x_i + \sum_{j=0}^{i-1} \boldsymbol{A}^J w_j \middle| w \in \mathcal{W}^i \right\} \subseteq \mathcal{X}$$
$$\phi_N(x_0,u,w) \in \mathcal{X}_f \quad \text{as well}$$

tim: One or two words on what is what

$$\boldsymbol{A}_x x \le b_x \text{ becomes } \boldsymbol{A}_x x_i + \boldsymbol{A}_x \sum_{j=0}^{i-1} \boldsymbol{A}^j w_k \le b_x:$$
$$x_i \in \mathcal{X} \ominus \left(\mathcal{W} \oplus \boldsymbol{A}\mathcal{W} \oplus \cdots \oplus \boldsymbol{A}^{i-1}\mathcal{W}\right)$$
$$= \left(\bigoplus_{j=0}^{i-1} \boldsymbol{A}^j \mathcal{W}\right) = \begin{bmatrix}\boldsymbol{A}^0 & \dots & \boldsymbol{A}^{i-1}\end{bmatrix} \mathcal{W}^i$$

Tube-MPC We want nominal system $z_k = \boldsymbol{A} z_k + \boldsymbol{B} v_k$ with "tracking" controller $u_k = \boldsymbol{K}(x_k - z_k) + v_k$, $\boldsymbol{K}$ found offline.
Step 1: Compute $\mathcal{E} = \bigoplus_{j=1}^{\infty} \boldsymbol{A}^j \mathcal{W}$.
Step 2: Shrink Constraints:

$$\{z_i\} \oplus \mathcal{E} \subseteq \mathcal{X} \implies z_i \in \mathcal{X} \ominus \mathcal{E}$$
$$u_i \in \boldsymbol{K}\mathcal{E} \oplus \{v_i\} \subset \boldsymbol{U} \implies v_i \in \mathcal{U} \ominus \boldsymbol{K}\mathcal{E}$$

Also $z_n \in \mathcal{X}_f \ominus \mathcal{E}$ accordingly.

4 Explicit MPC

$z^*(x_k)$ is continuous and polyhedral piecewise affine over feasible set.

4.1 Quadratic Cost

$J^*(x_k)$ is continuous, convex and polyhedral piecewise quadratic.
Using the formulation with substitution,

$$J(x_k) = \min z^T H z - x_k^T (Y - F H^{-1} F^T) x_k$$
$$\text{s.t} \quad Gz \le w + Sx_k$$
$$z(x_k) = U + H^{-1} F^T x_k$$
$$S = E + G H^{-1} F^T$$
$$U^* = z^*(x_k) - H^{-1} F^T x_k$$

The first solution gives $u^*(x_k) = \kappa(x_k)$, which is continuous and piecewise affine on polyhedra $\kappa(x) = F_j x + g_j$.

4.2 $1/\infty$-norm

$J^*(x_k)$ is continuous, convex and polyhedral piecewise affine over. Optimal solution given by, $u_0^* = 0\,[0\dots 0 I_m 0 \dots 0]\, z^*(x_k)$, and is in the same form as the QP case above.

5 Hybrid MPC

5.1 Piecewise Affine (PWA) Systems

Affine dynamics and output for each region:

$$\begin{cases} x_{k+1} &= A^i x_k + B^i u_k + f^i \\ y_k &= C^i x_k + D^i u_k + g_i \end{cases} \text{ if } x_t \in \mathcal{X}_j$$

Polyhedral partition of the (x,u)-space:

$$\{\mathcal{X}_i\}_{j=1}^s = \{x, u | H_j x + J_j u \le K_j\}$$

5.2 Mixed Logical Dynamical Hybrid Model (MLD)

Idea: associate boolean to binary: $p_j \iff \delta_i = 1, \neg p_j \iff \delta_i = 0$.

5.2.1 Translate logic rules to Linear Integer Inequalities

AND	$p_1 \wedge p_2$	$\delta_1 \ge 1, \delta_1 \ge 1$ also $\delta_1 + \delta_2 \ge 2$
OR	$p_1 \vee p_2$	$\delta_1 + \delta_2 \ge 1$
NOT	$\neg p_1$	$1 - \delta_1 \ge 1$ also $\delta_1 = 0$
XOR	$p_1 + p_2$	$\delta_1 + \delta_2 = 1$
IMPLY	$p_1 \Rightarrow p_2$	$\delta_1 - \delta_2 \le 0$
IIF	$p_1 \Leftrightarrow p_2$	$\delta_1 - \delta_2 = 0$
ASSIGNMENT	$p_3 \Leftrightarrow p_1 \wedge p_2$	$\delta_1 + (1 - \delta_3) \ge 1,$
		$\delta_2 + (1 - \delta_3) \ge 1,$
		$(1 - \delta_1) + (1 - \delta_2) + \delta_3 \ge 1$

Logic Equality Rules:
$\neg(A \wedge B) =\neq A \vee \neg B$
$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$
$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$

5.2.2 Translate continuous and logical components into Linear Mixed-Integer Relations

Event generator: $\delta_e(k) = f_{EG}(x_c(k), u_c(k), t)$.
Consider: $p \Leftrightarrow a^T x \le b, \mathcal{X} = \{x | a^T x - b \in [m, M]\}$.
Translated to linear inequalities: $m\delta < a^T x - b \le M(1 - \delta)$, where $[m, M]$ are lower and upper bounds.

Representing Switched Affine Dynamics as "IF-THEN-ELSE" relations
IF p THEN $z_k = a_1^T x_k + b_1$ else $z_k = a_2^T x_k + b_2 \Leftrightarrow$

$$(m_2 - M_1)\delta + z_k \le a_2^T x_k + b_2 \le -(m_1 - M_2)\delta + z_k$$
$$(m_1 - M_2)(1 - \delta) + z_k \le a_1^T x_k + b_1 \le -(m_1 - M_2)(1 - \delta) + z_k$$

This results in a linear MLD model

$$x_{k+1} = A x_k + B_1 u_k + B_2 \delta_k + B_3 z_k$$
$$y_k = C x_k + D_1 u_k + D_2 \delta_k + D_3 z_k$$
$$E_2 \delta_k + E_3 z_k \le E_4 x_k + E_1 u_k + E_5$$

where the last equation describes the relationship between the continuous and integer variables. Physical constraints on cont. variables: $\{\mathcal{C}\} = \left\{ \begin{bmatrix} x_c \\ u_c \end{bmatrix} \in \mathcal{R}^{n_c + m_c} | F x_c + G u_c \le H \right\}$

5.3 CFTOC for Hybrid Systems

$$J^*(x) = \min_U \ l_N(x_N) + \sum_{k=0}^{N-1} l(x_k, u_k, \delta_k, z_k)$$
$$\text{s.t} \quad x_{k+1} = A x_k + B_1 u_k + B_2 \delta_k + B_3 z_k$$
$$E_2 \delta_k + E_3 z_k \le E_4 x_k + E_1 u_k + E_5$$
$$x_N \in \mathcal{X}_f, x_0 = x(0)$$

5.4 MILP/QP

$$\min \quad c_c z_c + c_b z_b + d \text{ OR } zHz + qz + d$$
$$\text{s.t} \quad G_c z_c + G_b z_b \le W$$
$$z_c \in R^{s_c}, z_b \in \{0,1\}^{s_b}$$

Branch and bound method can be used to efficiently solve the problem. Explict solution is a time varying fb law for both problems: $u_k^*(x_k) = F_k^j x_k + G_k^j$ if $x_k \in \mathcal{R}_k^j$.

6 Numerical Optimization – Iterative Methods

6.1 Gradient descent

$x_{i+1} = x_i - h_i \nabla f(x_i)$ with step-size $h_i = \frac{1}{L}$ for L-smooth $f(x)$:

$$\exists L : \|\nabla f(x) - \nabla f(y)\| \le L\|x - y\| \ \forall x, y, \in \mathbb{R}^n$$
$$\iff \nabla f \text{ is Lipschitz continuous}$$
$$\iff f \text{ can be upperbounded by a quadratic function}$$

6.2 Newton's Method

$x_{i+1} = x_i - h_i \delta x_{nt} := x_i - h_i (\nabla^2 f(x_i))^{-1} \nabla f(x_i)$
Choose $h_i > 0$ s.t. $f(x_i + h_i \delta x_{nt} \le f(x_i)$ Line-search.
For given equality constraint $\boldsymbol{A}x = b$ solve:

$$\begin{bmatrix} \nabla^2 f(x_i) & \boldsymbol{A}^T \\ \boldsymbol{A} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \delta x \\ \lambda \end{bmatrix} = \begin{bmatrix} -\nabla f(x_i) \\ \boldsymbol{0} \end{bmatrix}$$

6.3 Constrained optimization

$g_i(x) \le 0$ with

Gradient method $x_{i+1} = \pi_Q(x_i - h_i \nabla f(x_i))$ where π_Q is a projection $\pi_q = \arg\min_x \frac{1}{2}\|x - y\|_2^2$. Projection can be solved directly if simple enough, else solve the dual.

6.4 Interior-Point methods

Assumptions $f(x^*) < \infty, \tilde{x} \in \mathrm{dom}(f)$.

Barrier method $\min f(x) + \kappa\phi(x)$. Approximate ϕ using diff'able log barrier(instead of indicator function):

$$\phi(x) = \sum_{i=1}^{m} ?? = -\sum_{i=1}^{m} \log(-g_i(x))$$
$$\lim_{\kappa \to 0} x^*(\kappa) = x^*$$

Analytic center: $\arg\min_x \phi(x)$, central path $\{x^*(\kappa) | \kappa > 0\}$.

Path following method

1. Centering $x^*(\kappa) = \arg\min_x f(x) + \kappa\phi(x)$ with newton's method:
1.1. $\Delta x_{\text{nt}} = \left[\nabla^2 f(x) + \kappa \nabla^2 \phi(x)\right]^{-1} (-\nabla f(x) - \kappa \nabla \phi(x))$.
1.2. Line search:

$$\text{retain feasability: } \operatorname{argmax}_{h>0} \{h | g_i(x + h\Delta x) < 0\}$$
$$\text{Find } h^* = \operatorname{argmin}_{h \in [0, h_{\max}]} \{f(x + h\Delta x) + \kappa\phi(x + h\Delta x)\}$$

2. Update step $x_i = x^*(\kappa_i)$
3. Stop if $m\kappa_i \leq \epsilon$
4. Decrease $\kappa_{i+1} = \kappa_i/\mu$, $\mu > 1$.

Centering step with equality constraints

$$\begin{bmatrix} \nabla^2 f(x) + \kappa \nabla^2 \phi(x) & c^T \\ c & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta x_{\text{nt}} \\ \nu \end{bmatrix} = -\begin{bmatrix} \nabla f(x) + \kappa \nabla \phi(x) \\ 0 \end{bmatrix}$$

Relaxed KKT

$$\begin{aligned} Cx^* &= d & g_i(x^*) + s_i^* &= 0 \\ \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + c^T \nu &= 0 & \lambda_i^* = \kappa \frac{\partial \phi}{\partial g_i} &= -\frac{\kappa}{g_i} \\ \lambda_i^* g_i(x^*) &= -\kappa & \lambda_i^*, s_i^* &\geq 0 \end{aligned}$$

Primal Dual Search Direction Computation

$$\begin{bmatrix} H(x,\lambda) & c^T & G(x)^T & 0 \\ c & 0 & 0 & 0 \\ G(x) & 0 & 0 & \boldsymbol{I} \\ 0 & 0 & \boldsymbol{S} & \boldsymbol{\Lambda} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta u \\ \Delta \lambda \\ \Delta s \end{bmatrix} = -\begin{bmatrix} \nabla f(x) + c^T \nu + G(x)^T \lambda \\ cx - d \\ g(x) + s \\ s\lambda - \nu \end{bmatrix}$$

$\boldsymbol{S} = \text{diag}(s_1, \ldots, s_m), \boldsymbol{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_m)$ and
ν is a vector for choosing centering parameters.