1 Systems Theory

1.1 Linearization

1.2 Discretization

Exact

Forward-Euler

Backward-Euler

1.3 Lyapunov Function

$V(0) = 0,\ x \neq 0 \implies V(x) > 0,$
$V(g(x(k+1))) - V(x(k+1)) \leq -\alpha(x(k))$
System asymptotically stable if $V(x)$ exists. Globally stable iff $\|x\| \to \infty \implies V(x) \to \infty$.
Check Eig. values of $(APA - P)$ neg., $V(x) = x^T P x$?

2 Unconstrained Control

2.1 Block Approach (used also for $\bar{w}$ substition)

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{A} \\ \vdots \\ \boldsymbol{A}^N \end{bmatrix} x(0) + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{AB} & \boldsymbol{B} & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{0} \\ \boldsymbol{A}^{N-1}\boldsymbol{B} & \dots & \boldsymbol{AB} & \boldsymbol{B} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_1} \end{bmatrix}$$

$$\begin{aligned} x &= \boldsymbol{S}^x \cdot x(0) + \boldsymbol{S}^u \cdot u & \text{size}(\boldsymbol{S}^x) &= [n_{\text{states}} \cdot (N+1), N] \\ & & \text{size}(\boldsymbol{S}^u) &= [n_{\text{states}} \cdot (N+1), n_{\text{states}}] \\ \bar{\boldsymbol{Q}} &= \text{diag}(\boldsymbol{Q}, \dots, \boldsymbol{Q}, \boldsymbol{P}) & \text{size}(\bar{\boldsymbol{Q}}) &= [n_{\text{states}} \cdot (N+1), n_{\text{states}} \cdot (N+1)] \\ \bar{\boldsymbol{R}} &= \text{diag}(\boldsymbol{R}, \dots, \boldsymbol{R}) & \text{size}(\bar{\boldsymbol{R}}) &= (n_{\text{input}} \cdot N, n_{\text{input}} \cdot N) \\ \boldsymbol{H} &= \boldsymbol{S}^{uT} \bar{\boldsymbol{Q}} \boldsymbol{S}^u + \boldsymbol{R} & \boldsymbol{F} &= \boldsymbol{S}^{xT} \bar{\boldsymbol{Q}} \boldsymbol{S}^u \\ \boldsymbol{Y} &= \boldsymbol{S}^{xT} \bar{\boldsymbol{Q}} \boldsymbol{S}^x \end{aligned}$$

Optimal cost and control

$$J^*(x_0) = -x_0^T \boldsymbol{F} \boldsymbol{H} \boldsymbol{F}^T x_0 + x_0^T \boldsymbol{Y} x_0$$
$$u^*(x_0) = -\boldsymbol{H}^{-1} \boldsymbol{F}^T x_0 = -\left(\boldsymbol{S}^{uT} \bar{\boldsymbol{Q}} \boldsymbol{S}^u + \boldsymbol{R} \right)^{-1} \boldsymbol{S}^{uT} \bar{\boldsymbol{Q}} \boldsymbol{S}^x x_0$$

2.2 Recursive Approach

$$J_k^*(x_k) = \min_{u_k} I(x_k, u_k) + J_{k+1}(x_{k+1})$$

Is a feedback controller as opposed to the Batch Approach. For LQR solve via Riccati Difference Equation (RDE).

$$\begin{aligned} \boldsymbol{F}_k &= -(\boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{B} + \boldsymbol{R})^{-1} \boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{A} \\ \boldsymbol{P}_k &= \boldsymbol{A}^T \boldsymbol{P}_{k+1} \boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{A}^T \boldsymbol{P}_{k+1} \boldsymbol{B} (\boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{B} + \boldsymbol{R})^{-1} \boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{A} \end{aligned}$$

$$u_k^* = \boldsymbol{F}_k\ x_k \qquad J_k^*(x_k) = x_k^T \boldsymbol{P}_k\ x_k \qquad \boldsymbol{P}_N = \boldsymbol{P}$$

For unconstrained Infinite Horizon Problem, substituting $\boldsymbol{P}_\infty = \boldsymbol{P}_k = \boldsymbol{P}_{k+1}$ into RDE gives DARE. Uniquely solvable, iff (A, B) stabilizable and (A, G) detectable, where $\boldsymbol{G}\boldsymbol{G}^T = \boldsymbol{Q}$. Follows from closed-loop system $x_{k+1} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}_k) x_k$

3 (Convex) Optimization

3.1 Convexity

Convex set $\mathcal{X}$ iff $\forall \lambda \in [0,1] \forall x, y \in \mathcal{X}\ \lambda x + (1-\lambda) y \in \mathcal{X}$.
Intersection preserves convexity, union does not.

Affine set $\mathcal{X}$ $= \{x \in \mathbb{R}^n | \boldsymbol{A}x = b\}$ for some $\boldsymbol{A}, b$

Subspace is affine set through origin, i.e. $b = \boldsymbol{0}$, aka Nullspace of $\boldsymbol{A}$.

Hyperplane $\mathcal{X}$ $= \{x \in \mathbb{R}^n | a^T x = b\}$ for some a, b.

Halfspace $\mathcal{X}$ $= \{x \in \mathbb{R}^n | a^T x \leq b\}$ for some a, b.

Polyhedron $\mathcal{P}$ $= \{x | a_i^T x \leq b_i, i = 1, \dots, n\} = \{x | \boldsymbol{A}x \leq b\}$

Cone

Ellipsoid $\mathcal{E}$ $= \{x | (x - x_c)^T \boldsymbol{A}^{-1} (x - x_c) \leq 1\}$, x_c center point.

Convex function

Norm $f(x) : \mathbb{R}^n \to \mathbb{R}$

$$\begin{aligned} f(x) = 0 &\implies x = 0, & f(x) &\geq 0 \\ f(\alpha \cdot x) &= |\alpha| \cdot f(x) & &\text{for scalar } \alpha \\ f(x+y) &\leq f(x) + f(y) & &\forall x, y \in R^n \end{aligned}$$

tim: Maybe move the above somewhere else?

General Problem $\min_{x \in \text{dom}(f)} f(x)$ s. t. $g_i(x) \leq 0$ and $h_j(x) = 0$.

3.2 Linear Programming (LP)

Problem statement $\min c^T x$ such that $\boldsymbol{G}x \leq h$ and $\boldsymbol{A}x = b$.

Norm l_∞ $\min_x \|x\|_\infty = \min_{x \in \mathbb{R}^n} [\max\{x, \dots, x_n, -x_1, \dots, -x_n\}]$:

$$\begin{aligned} &\min_{x,t} t && \text{subject to} && x_i \leq t, -x_i \leq t, && \boldsymbol{F}x \leq g \\ \iff &\min_{x,t} t && \text{subject to} && -\boldsymbol{1}t \leq x \leq 1t, && \boldsymbol{F}_x \leq g. \end{aligned}$$

Norm l_1 $\min_x \|x\|_1 = \min_x \left[\sum_{i=1}^m \max\{x_i, -x_i\}\right]$:

$$\begin{aligned} &\min_t t_1 + \dots + t_m && \text{subject to} && x_i \leq t_i, -x_i \leq t_i, && \boldsymbol{F}x \leq g \\ \iff &\min_t \boldsymbol{I}^T t && \text{subject to} && -t \leq x \leq t, && \boldsymbol{F}_x \leq g. \end{aligned}$$

Note that for $\dim x = 1$, l_1 and l_∞ are the same.

MPC with linear cost
$J(x_0, u) = \|\boldsymbol{P}x_N\|_p + \sum_{i=0}^{N-1} \|\boldsymbol{Q}x_i\|_p + \|\boldsymbol{R}u_i\|_p$.

tim: Insert here slide 45, lect 4

Receding Horizon Control – RHC

QP with substitution (see also Batch approach)

$$J^*(x_k) = \min_u \begin{bmatrix} u^T & x_k^T \end{bmatrix} \begin{bmatrix} \boldsymbol{H} & \boldsymbol{F}^T \\ \boldsymbol{F} & \boldsymbol{Y} \end{bmatrix} \begin{bmatrix} u \\ x_k \end{bmatrix}$$
$$\text{s. t. } \boldsymbol{G}\ u \leq w + \boldsymbol{E}\ x_k$$

Latter gives three sets (same for without substitution)

$$\begin{aligned} \mathcal{X} &= \{x | A_x\ x \leq b_x\} \\ \mathcal{U} &= \{u | A_u\ u \leq b_u\} \\ \mathcal{X}_{\{} &= \{x | A_f\ x \leq b_f\} \end{aligned}$$

State equations are in cost matrix, usually

$\boldsymbol{A}_x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, b_x = \begin{bmatrix} b_{\max} \\ -b_{\min} \end{bmatrix}$

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{A}_u & 0 & \dots & 0 \\ 0 & \boldsymbol{A}_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{A}_u \\ 0 & 0 & \dots & 0 \\ \boldsymbol{A}_x\boldsymbol{B} & 0 & \dots & 0 \\ \boldsymbol{A}_x\boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}_x\boldsymbol{B} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}_x\boldsymbol{A}^{N-2}\boldsymbol{B} & \boldsymbol{A}_x\boldsymbol{A}^{N-3}\boldsymbol{B} & \dots & 0 \\ \boldsymbol{A}_f\boldsymbol{A}^{N-1}\boldsymbol{B} & \boldsymbol{A}_f\boldsymbol{A}^{N-2}\boldsymbol{B} & \dots & \boldsymbol{A}_f\boldsymbol{B} \end{bmatrix} \quad \boldsymbol{E} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\boldsymbol{A}_x \\ -\boldsymbol{A}_x\boldsymbol{A} \\ -\boldsymbol{A}_x\boldsymbol{A}^2 \\ \vdots \\ -\boldsymbol{A}_x\boldsymbol{A}^{N-1} \\ -\boldsymbol{A}_f\boldsymbol{A}^N \end{bmatrix} \quad \boldsymbol{W} = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_x \\ b_f \end{bmatrix}$$

QP with out substitution State equations represented in equality constrainst.

$$\begin{aligned} J^*(x_k) = \min_z & \begin{bmatrix} z^T & x_k^T \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{H}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} z \\ x_k \end{bmatrix} \\ \text{s. t. } & \boldsymbol{G}\, z \le w + \boldsymbol{E}\, x_k \\ & \boldsymbol{G}_{\text{eq}}\, z = \boldsymbol{E}_{\text{eq}}\, x_k, \quad \text{system dynamics} \end{aligned}$$

$\bar{\boldsymbol{H}} = \operatorname{diag}(\boldsymbol{Q}, \dots, \boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{R}, \dots, \boldsymbol{R})$

$$z = \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} \quad \boldsymbol{G}_{\text{eq}} = \left[\begin{array}{cccc|cccc} \boldsymbol{I} & & & & -\boldsymbol{B} & & & \\ -\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B} & & \\ & \ddots & \ddots & & & & \ddots & \\ & & -\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B} \end{array}\right] \quad \boldsymbol{E}_{\text{eq}} = \begin{bmatrix} \boldsymbol{A} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$w = \begin{bmatrix} b_x \\ \vdots \\ b_f \\ b_u \\ \vdots \\ b_u \end{bmatrix} \quad \boldsymbol{G} = \left[\begin{array}{ccc|ccc} 0 & & & & & \\ & \boldsymbol{A}_x & & & & \\ & & \ddots & & & \\ & & \boldsymbol{A}_x & & & \\ \hline & & & \boldsymbol{A}_d & & \\ & & & & \ddots & \\ & & & & & \boldsymbol{A}_d \end{array}\right] \quad \boldsymbol{E} = \begin{bmatrix} -\boldsymbol{A}_x^T \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

3.3 Duality

Lagrangian Dual Function

$$\begin{aligned} L(x, \lambda, \nu) &= f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \\ d(\lambda, \nu) &= \inf_{x \in \mathcal{X}} L(x, \lambda, \nu) \quad \text{i.e. } \nabla_x L(x, \lambda, \nu) = 0 \end{aligned}$$

Dual Problem (always convex) $\max_{\lambda,\nu} d(\lambda, \nu)$ s. t. $\lambda \ge 0$.
Optimal value is lower bound for primal: $d^* \le p^*$.
If primal convex, *Slater condition* (strict feasibility) implies *strong duality*:

$$\{x \mid Ax = b, f_i(x) < 0, \} \ne \emptyset \Rightarrow d^* = p^*$$

Karush-Kuhn-Tucker (KKT) Conditions are necessary for optimality (and sufficient if primal convex).

- Primal Feasibility:
$$\begin{aligned} f_i(x^*) \le 0 \quad i = 1, \dots, m \\ h_i(x^*) = 0 \quad i = 1, \dots, p \end{aligned}$$
- Dual Feasibility: $\lambda^* \ge 0$
- Complementary Slackness:
$$\lambda_i^* \cdot f_i(x^*) = 0 \qquad i = 1, \dots, m$$
- Stationarity:
$$\nabla_x L(x^*, \lambda^*, \nu^*) = 0$$

3.4 Constrained Finite Time Optimal Control (CFTOC)

3.5 Invariance

Def.: $x(k) \in O \implies x(k+1) \in O \forall k$.

$$pre(S) := \{x | g(x) \in S\} \qquad\qquad = \{x | Ax \in S\}$$

tim: We need more here, poos. inv. set, max. pos.inv O_∞

3.6 Stability and Feasability

Recursive Stability, optimal cost is Lyapunov function.

tim: What is meant with that

Main Idea: Choose $\mathcal{X}_f$ and $\boldsymbol{P}$ to mimic inf horizon, terminal cost ist Lyapunov function: $x_{k+1}^T \boldsymbol{P} x_{k+1} - x_k^T \boldsymbol{P} x_k = -x_k^T(\boldsymbol{Q} + \boldsymbol{F}_\infty^T \boldsymbol{R} \boldsymbol{F}_\infty) x_k$, such that:

$$\begin{aligned} x_{k+1} = \boldsymbol{A} x_k + \boldsymbol{B} \boldsymbol{F}_\infty\, x_k \in \mathcal{X}_f \quad & \forall x_k \in \mathcal{X}_f \text{ terminal set invariant} \\ \mathcal{X}_f \subseteq \mathcal{X}, \quad \boldsymbol{F}_\infty\, x_k \in \mathcal{U} \quad & \forall x_k \in \mathcal{X}_f \text{ constrainst satisfied} \end{aligned}$$

And stage cost is PD-function $\implies$ Extension to non-linear (time-invariant) MPC possible since terminal set and cost do not rely on linearity.

3.7 Practical MPC

3.8 Robust MPC

Enforcing terminal constraints by robust invariance:

$$\begin{aligned} x \in O^W \implies g(x, w) \in \Omega^W \ \forall w \in W\} \\ \text{pre}^W(\Omega) = \{x | g(x, w) \in \Omega \ \forall w \in W\} \end{aligned}$$

tim: Maybe an example from exercises to compute O_∞^W

Enforcing sequential constraints for uncertain system ϕ:

$$\begin{aligned} \phi_i(x_0, u, w) &= \left\{ x_i + \sum_{j=0}^{i-1} \boldsymbol{A}^J w_j \,\middle|\, w \in \mathcal{W}^i \right\} \subseteq \mathcal{X} \\ \phi_N(x_0, u, w) &\in \mathcal{X}_f \quad \text{as well} \end{aligned}$$

tim: One or two words on what is what

$$\begin{aligned} x_i &\in \mathcal{X} \ominus (\mathcal{W} \oplus \boldsymbol{A}\mathcal{W} \oplus \dots \oplus \boldsymbol{A}^{i-1}\mathcal{W}) \\ &= \left(\bigoplus_{j=0}^{i-1} \boldsymbol{A}^j \mathcal{W} \right) = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{A}^0 & \dots & \boldsymbol{A}^{i-1} \end{bmatrix} \mathcal{W}^i \\ \boldsymbol{A}_x x &\le b_x \\ \boldsymbol{A}_x x_i + \boldsymbol{A}_x \sum_{j=0}^{i-1} \boldsymbol{A}^j w_k &\le b_x \end{aligned}$$

Tube-MPC We want nominal system $z_k = \boldsymbol{A} z_k + \boldsymbol{B} v_k$ with "tracking" controller $u_k = \boldsymbol{K}(x_k - z_k) + v_k$, $\boldsymbol{K}$ found offline.
Step 1: Compute $\mathcal{E} = \bigoplus_{j=1}^{\infty} \boldsymbol{A}^j \mathcal{W}$.
Step 2: Shrink Constraints:

$$\begin{aligned} \{z_i\} \oplus \mathcal{E} \subseteq \mathcal{X} \qquad & \implies z_i \in \mathcal{X} \ominus \mathcal{E} \\ u_i \in \boldsymbol{K}\mathcal{E} \oplus \{v_i\} \subset \boldsymbol{U} \qquad & \implies v_i \in \mathcal{U} \ominus \boldsymbol{K}\mathcal{E} \end{aligned}$$

Also $z_n \in \mathcal{X}_f \ominus \mathcal{E}$ accordingly.

3.9 Explicit MPC

3.10 Hybrid MPC

4 Numerical Optimization

Gradient, Newton, Interior Point