1 Linear Systems

Solution to linear ODE $\dot{x}(t) = A^c x(t) + B^c u(t), x0 := x(t_0)$
$x(t) = e^{A^c(t-t_0)}x_0 + \int_{t_0}^{t} e^{A^c(t-\tau)}B^c u(\tau)d\tau$
Discretization of LTI c-t ss model
$t_0 = t_k, t = t_{k+1}, t_{k+1} - t_k = T_s, u(t) = u(t_k) \forall t \in [t_k, t_{k+1})$
$x(t_{k+1}) = e^{A^c T_s}x(t_k) + \int_0^{T_s} e^{A^c(T_s-\tau')}B^c d\tau' u(t_k)$
$x(t_{k+1}) = Ax(t_k) + Bu(t_k)$
Coordinate Trafo may facilitate system analysis
$\tilde{x} = Tx, det(T) \neq 0$; where $T = [e_1, ..., e_n]^{-1}$
EW: $\det(A - \lambda I) = 0$, EV: $(A - \lambda_i I)e_i = 0$
Asymptotic stability (stays bounded and returns to 0):
$\lim_{k\to\infty} x(k) = 0, \forall x(0) \in \mathbb{R}$ with $u = 0$
Necessary & sufficient cond. on the eigenvalues: $|\lambda_i| < 1, \forall i = 1, ..., n$
Proof by trafo:
$\tilde{x}(k+1) = TAT^{-1}\tilde{x}(k) = diag(\lambda_n)\tilde{x}(k) = \Lambda\tilde{x}(k)$
State $\tilde{x}(k)$ can be expressed as function of $\tilde{x}(0) = Tx(0)$:
$\tilde{x}(k) = diag(\lambda_n^k)\tilde{x}(0) = \Lambda^k\tilde{x}(o)$
$\tilde{x}(k) = \Lambda^k\tilde{x}(0) \Rightarrow |\tilde{x}_i(k)| \leq |\lambda_i|^k|\tilde{x}_i(0)|$
Global Lyapunov Stability (only sufficient)
$x = 0$ is GAS if there is a function $V : \mathbb{R}^n \to \mathbb{R}$:
$\|x| \to \infty \Rightarrow V(x) \to \infty$
$V(0) = 0$ *and* $V(x) > 0, \forall x \neq 0$
$V(f(x(k))) - V(x(k)) < 0, \forall x \neq 0$
Lyapunov Stability for linear system $x(k+1) = Ax(k)$
$V(Ax(k)) - V(x(k)) = x^T(k)(A^TPA - P)x(k) < 0, P > 0$
Lyapunov equation

$$A^TPA - P = -Q, Q > 0$$

Fulfilled if A has all eigenvalues inside the unit circle
Infinite horizon cost-to-go for asymptotically stable system $x(k+1) = Ax(k)$
$\Psi(x(0)) = \sum_{k=0}^{\infty} x(k)^TQx(k) = \sum_{k=0}^{\infty} x(0)^T(A^k)^TQA^kx(0) = x(0)^TPx(0)$
$P = \sum_{k=0}^{\infty}(A^k)^TQA^k = Q + \sum_{k=1}^{\infty}(A^k)^TQA^k =$
$Q + A^T(\sum_{k=0}^{\infty}(A^k)^TQA^k)A = Q + A^TPA$
Controllability: goal state x^* can be reached in finite time: max in n steps

$$x^* = A^Nx(0) + (B\ AB \ldots A^{n-1}B)\begin{pmatrix} u(N-1) \\ u(N-2) \\ \vdots \\ u(0) \end{pmatrix}$$

$= A^Nx(0) + \mathcal{C}U$
Necessary and sufficient cond: $\text{rank}(\mathcal{C}) = n$.
Stabilizability: input sequence exists, that returns the state to the origin asymptotically: A system is stabilizable iff all its uncontrollable modes are stable
if $\text{rank}([\lambda_i I - A|B]) = n\ \forall \lambda_i \in \Lambda_A^+ \Rightarrow (A, B)$ is stabilizable
Λ_A^+ is the set of eigenvalues of A lying on or outside the unit circle.
Observable: measurements $y(0), y(1), ..., y(N-1)$ uniquely distinguish $x(0)$

$$\begin{pmatrix} y(0) \\ y(1) \\ \vdots \\ y((N-1) \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{pmatrix} x(0) = \mathcal{O}x(0)$$

colums of $\mathcal{O}$ linearly independent. Nec & suf: $\text{rank}(O) = n$
Detectability: if all unobservable modes are stable:
if $\text{rank}([A^T - \lambda_i|C]) = n\ \forall \lambda_i \in \Lambda_A^+ \Rightarrow (A, C)$ is stabilizable.

1.1 Unconstrained finite horizon optimal control problem

Inputs $\mathbf{u} := [u_0^T, ..., u_{N-1}^T]^T$ minimize objective function:
$J_0^*(x(0)) := \min_{\mathbf{u}} J_0(x(0), \mathbf{u}) = \min_{\mathbf{u}} x_N^TPx_N + \sum_{k=0}^{N-1}[x_k^TQx_k + u_k^TRu_k]$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k, k = 0, ..., N-1 \qquad x_0 = x(0)$$

$P \geq 0, P = P^T$ terminal weight
$Q \geq 0, Q = Q^T$ state weight
$R > 0, R = R^T$ input weight
Batch approach: all future states represented in terms of initial condition x_0 and inputs $u_0, ..., u_{N-1}$

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} x(0) + \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ B & 0 & \ldots & 0 \\ AB & B & \ldots & 0 \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & \ldots & AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$\mathbf{x} := \bar{A}x(0) + \bar{B}\mathbf{u}$
Finite horizon cost function:
$J_0(x(0), \mathbf{u}) = \mathbf{x}^T\bar{Q}\mathbf{x} + \mathbf{u}^T\bar{R}\mathbf{u}$
$\bar{Q} := \text{blockdiag}(Q, ..., Q, P)$ and $\bar{R} := \text{blockdiag}(R, ..., R)$
Eliminating $\mathbf{x}$ form J_0 gives
$J_0(x(0), \mathbf{u}) = \mathbf{u}^TH\mathbf{u} + 2x(0)F\mathbf{u} + x(0)^T\bar{A}^T\bar{Q}\bar{A}x(0)$
where $H := \bar{B}^T\bar{Q}\bar{B} + \bar{R}$ and $F := \bar{A}^T\bar{Q}\bar{A}$.
Solution by setting gradient to 0: $\nabla_{\mathbf{u}}J_0(x(0), \mathbf{u}) = 0$
$\mathbf{u}^*(x(0)) = -H^{-1}F^Tx(0)$
Recursive approach: j-step optimal cost-to-go:
$J_j^*(x(j)) := \min_{u_j, ..., u_{N-1}} x_N^TPx_N + \sum_{k=j}^{N-1}[x_k^TQx_k + u_k^TRu_k]$

$$\text{s.t. } x_{k+1} = Ax_k + Bu_k, k = j, ..., N-1 \qquad x_j = x(j)$$

Solution by substituting equ. constr. in objective function and setting gradient of input to 0
Optimal solution for time step k
$u^*(k) = -(B^TP_{k+1}B + R)^{-1}B^TP_{k+1}Ax(k) =: F_kx(k)$
P_k by recursion from $P_N = P$ (Discrete Time Riccati Equ):
$P_k = A^TP_{k+1}A + Q - A^TP_{k+1}B(B^TP_{k+1}B + R)^{-1}B^TP_{k+1}A$
Optimal cost-to-go: $J_k^*(x(0)) = x(k)^TP_kx(k)$
Comparison:
Batch approach: sequence of numerical values
Recursive: dynamic programming, feedback policies $u^*(k) = F_kx(k)$

1.2 Infinite Horizon Control Problem

RDE converges to constant P: $P_k = P_{k+1} = P_\infty$, RDE becomes ARE.
Feedback matrix F_∞: asymptotic form of LQR
If (A, B) stabilizably and $(Q^{1/2}, A)$ detectable, RDE converges to ARE.
Closed-loop system is asymptotically stable with $u(k) = F_\infty x(k)$
Prove by examining cl system $x(k+1) = (A + BF_\infty)x(k)$

State Estimation

Estimate previous/current/future: smoothing/filtering/predictio
Model $x(k+1) = Ax(k) + Bu(k) + \varepsilon_1(k)$; $y(k) = Cx(k) + \varepsilon_2(k)$
ε_1: Process noise; ε_2: Meas. noise. Zero mean: $\text{E}\{\varepsilon_i(k)\} = 0$.
White noise has zero mean. R_1 big: trus meas; R_2 big: meas noisy.
Prediction step: $\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu(k-1)$
Update step: $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K(y(k) - C\hat{x}_{k|k-1})$
State estimation error $\hat{x}_{i|j}^e := x(i) - \hat{x}_{i|j}$
Estimation error dynamics, only depending upon old estimates:
$\hat{x}_{k|k}^e := (A - KCA)\hat{x}_{k-1|k-1}^e + (I - KC)\varepsilon_1(k-1) - K\varepsilon_2(k)$
Stability: iff eigenvalues of $A - KCA$ strictly inside unit circle.
Possible if (CA,A) is observable.

Kalman Filter

1. Compute the a priori estimate and - error covariance matrix
$\hat{x}(k|k-1) = A\hat{x}(k-1|k-1) + Bu(k-1)$
$P(k|k-1) = AP(k-1|k-1)A^T + R_i$
2. Measure $y(k)$
3. Compute gain, new estimate and error covariance matrix
$K(k) = P(k|k-1)C^T(CP(k|k-1)C^T + R_2)^{-1}$
$\hat{x}(k|k) = \hat{x}(k|k-1) + K(k)(y(k) - C\hat{x}(k|k-1))$
$P(k|k) = (I - K(k)C)P(k|k-1)(I - K(k)C)^T + K(k)R_2K(k)^T$

2 General Optimization Problem:

$\min_{x \in \mathcal{X}} f_0(x)$ — f_0: Objective function
s.t. $f_i(x) \leq 0, h_i(x) = 0$ — $\mathcal{X}$: Domain of OF f_0
(Strictly) feasible point:
$x \in \mathcal{X}$, satisf. constr. (strictly), $f_i(x)(<) \leq 0$.

Optimal solution: $x^* \in \mathcal{X}$ such that optimal value $f_0(x^*) \leq f_o(x)$.
Optimal value: $p^* = \inf_{x\in\mathcal{X}}\{f_0(x)|f_i(x) \leq 0, h_i(x) = 0\}$.
Minimizer: Vector x^* that achieves the minimal value.

2.1 Convex Optimization Problem

Optim prob is cvx, if objective function and feasible set are cvx. $f_0, ..f_m$ are cvx, and h_i are affine.

2.2 Definition of convex set:

$\mathcal{X}$ is convex $\Leftrightarrow \lambda x + (1-\lambda)y \in \mathcal{X}, \forall\lambda \in [0,1], \forall x, y \in \mathcal{X}$
Convex Sets
- Subspace $\mathcal{X} = \{x \in \mathbb{R}^n | Ax = 0\}$. Proof: $\forall\lambda \in [0,1], \forall x, y \in \mathcal{X}$ $A(\lambda a + (1-\lambda)b) = \lambda Aa + (1-\lambda)Ab = \lambda\cdot 0 + (1-\lambda)\cdot 0 = 0$
- Affine space $\mathcal{X} = \{x \in \mathbb{R}^n | Ax = b\}$. Lines and planes.
- Hyperplane $\mathcal{X} = \{x \in \mathbb{R}^n | a^T x = b\}$. $a \neq 0, a \in \mathbb{R}^n$ is the normal vector.
- Halfspace $\mathcal{X} = \{x \in \mathbb{R}^n | a^T x \leq b\}, a \neq 0$,. Can be **open** $(<)$ or **closed** $(\leq)$.
- Set $\mathcal{X}$ is a cone if $\forall x \in \mathcal{X}$, and $\forall\theta > 0, \theta x \in \mathcal{X}$. θ: scaling factor. Pointed if it contains $x = 0$
- Conic combination of two points x_1 and x_2: any point fulfilling $y = \theta_1 x_1 + \theta_2 x_2$
for some $\theta_1, \theta_2 > 0$; Convex if convex cone.
- Polyhedron: intersection of finite number of closed halfspaces: $\mathcal{X} = \{x | a_1^T x \leq b_1, ..., a_m^T x \leq b_m\} = \{x | Ax \leq b\}$
- Polytope: bounded polyhedron. A polyhedron is always cvx.
- Norm: function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying
1.) $f(0) \geq 0$ and $f(x) = 0 \Rightarrow x = 0$.
2.) $f(tx) = |t|f(x), t$ scalar
3.) $f(x+y) \leq f(x) + f(y), \forall x, y \in \mathbb{R}^n$
l_p norm $\|x\|_p := [\sum_{i=1}^n |x_i|^p]^{1/p}$
$p = 1$: Sum of abs values;
$p = \infty$: largest abs value $\|x\|_\infty := max_i |x_i|$
- Norm ball: $\{x | \|x - x_c\| \leq r\}, r \geq 0$.
- Ellipsoid: $\{x | (x - x_c)^T A^{-1}(x - x_c) \leq 1\}, x_c$ center, $A > 0$.
- Euclidean ball: $\{x | \|x - x_c\|_2 \leq r\}$. (Ellipse with $A = r^2 I$).
-Interrsection $\mathcal{X} \bigcap \mathcal{Y}$ of two convex sets is convex:
For any $\lambda \in [0,1], \lambda a + (1-\lambda)b$ is in both $\mathcal{X}$ and $\mathcal{Y}$.
Therefore $\lambda a + (1-\lambda)b \in \mathcal{X} \bigcap \mathcal{Y}, \forall\lambda \in [0,1]$
Set of points $C \triangleq = \{x|...\forall y \in Q\} : C = \bigcap_{y\in Q} C_y$
- Convex hull: set of all convex combinations of points in $\mathcal{X}$.
- Union $\mathcal{X} \bigcup \mathcal{Y}$ is not convex in general.

2.3 Definition of convex function:

A function $f : \text{dom}(f) \to \mathbb{R}$ is convex iff its domain dom(f) is cvx and $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$
$\forall\lambda \in [0,1], \forall x, y \in \text{dom}(f)$,
- f is concave iff $-f$ is convex.

1st-order condition for convexity
Differentiable function $f : \text{dom}(f) \to \mathbb{R}$ with cvx domain is cvx iff $f(y) \geq f(x) + \nabla f(x)^T(y - x), \forall x, y \in \text{dom } (f)$
2nd-order condition for convexity
Function $f : \text{dom}(f) \to \mathbb{R}$ is cvx iff dom f(f) is cvx and Hessian $\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \nabla^2 f(x) \geq 0, \forall x \in \text{dom}(f)$
- Epigraph: $\text{epi}(f) = \{[x\, t]^T] | x \in \text{dom}(f), f(x) \leq t\} \subseteq \text{dom}(f) \times \mathbb{R}$
A function s convex iff its epigraph is cvx
- SubLevel set $\text{L}_\alpha = \{x | x \in \text{dom}(f), f(x) \leq \alpha\}$
f is cvx $\Rightarrow$ sublevel sets of f are cvx $\forall\alpha$. But not $\Leftarrow$!
f is quasi-cvs iff dom(f) is cvx and all sublevel sets of f are cvx.
Convexity-preserving operations
- Non-negative weighted sum: f is cvx $\Rightarrow \alpha f, \forall\alpha \geq 0$ is cvx.
- Composition with affine function:
f is cvx $\Rightarrow f(Ax + b)$ is cvx.
- Pointwise (Pw) max:
If $f_1, .., f_m$ are cvx, $\Rightarrow f(x) = \max\{f_1, .., f_m\}$ is cvx.
- Pw sup: If $f(x, y)$ is cvx in $x, \forall y$, then $g(x) = \sup_{y\in\mathcal{Y}} f(x, y)$ is cvx.
- Min: If $f(x, y)$ is cvx in (x, y) and $\mathcal{C}$ is cvx, then $g(x) = \min_{y\in\mathcal{C}} f(x, y)$ is cvx.

2.4 Optimality Criterion for Differentiable f_0

For cvx problem, x is optimal iff it is feasible and $\nabla f_0(x)^T(y - x) \geq 0, \forall$ feasible y
$a^T b = |a||b| \cos\sphericalangle(a, b)$
Angle between gradient and any vector in the set is $< 90°$.
Equivalent Optimization Problems:
Intoducing equality constraints
$\min_x f_0(A_0 x + b_0)$ $\quad\quad \min_x f_0(y_0) \quad\quad i = 0, 1, ..m$
s.t. $f_i(A_i x + b_i) \leq 0, i = 1, .., m$ $\quad$ s.t. $f_i(y_i) \leq 0, A_i x + b_i = y_i$
Intoducing slack variables
$\min_x f_0(x)$ $\quad\quad \min_x f_0(y_0) \quad\quad i = 1, ..m$
s.t. $A_i x \leq b_i, i = 1, .., m$ $\quad$ s.t. $A_i x + s_i = b_i,\ s_i \geq 0$
Example Linear Programs
-Constrained l_∞ minimization:
$\min_x \|Ax - b\|\infty$ $\quad \min_{x,t} t$
s.t. $Fx \leq g$ $\quad$ s.t. $Ax - b \leq t\cdot\mathbf{1}, Ax - b \geq -t\cdot\mathbf{1}, Fx \leq g$
-Constrained l_1 minimization:
$\min_x \|Ax - b\|_1$ $\quad \min_{x,y} \mathbf{1}^T y$
s.t. $Fx \leq g$ $\quad$ s.t. $Ax - b \leq y, Ax - b \geq -y, Fx \leq g$

Lagrangian Function: $L : \mathcal{X} \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$
$L(x, \lambda, \nu) = f_o(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x)$
Lagrange Dual function $g : \mathbb{R}^m \times \mathbb{R}^p$
$g(\lambda, \nu) = \inf_{x\in\mathcal{X}} L(x, \lambda, \nu)$
$g(\lambda, \nu)$ pointwise infimum of affine functions.
dual generates lower bounds for $p^* : g(\lambda, \nu) \leq p^*, \forall(\lambda \geq 0, \nu \in \mathbb{R}^p)$.
Procedure: 1.) Compute Lagrangian $L(x, \lambda, \nu)$
2.) Minimize it by setting gradient to 0: $\nabla L(x, \lambda, \nu) = 0 \Rightarrow x^*$
3.) Substitute x back into L to get dual function $g(\lambda, \nu)$
4.) Lower bound property $p^* : g(\lambda, \nu) \leq p^*, \forall(\lambda \geq 0$
Dual Problem: maximize dual function:
$\max_{\lambda,\nu} g(\lambda, \nu) \quad\quad$ s.t. $\lambda \geq 0$
- Properties: Dual prob is convex, even if primal is not
- Optimal value $d^* \leq p^*$
- Point (λ, ν) is dual feasible if $\lambda \geq 0$ and $(\lambda, \nu) \in$ dom g
Weak duality: it is always true that $d^* \leq p^*$;
duality gap: $d^* - p^*$
Strong duality: $d^* = p^*$ often for cvx problems, check by
Slater condition: if there exists one strictly feasible point inside:
$\{x | Ax = b, f_i(x) < 0, \forall i \in \{1, ..., m\}\} \neq \emptyset$
If strong duality holds: $d^* = p^* \Rightarrow g(\lambda^*, \nu^*) = f_0(x^*)$

2.5 Karush-Kuhn-Tucker Conditions:

1.) Primal feasibility $\quad f_i(x^*) \leq 0, h_i(x^*) = 0$
2.) Dual feasibility $\quad \lambda^* \geq 0$
3.) Complementary Slackness $\quad \lambda_i^* f_i(x^*) = 0$
4.) Stationarity $\quad \nabla_x L(x^*, \lambda^*, \nu^*) = \nabla_x f_o(x^*) + \sum_{i=1}^m \lambda_i^* \nabla_x f_i(x^*) + \sum_{i=1}^p \nu_i^* \nabla_x h_i(x^*) = 0$

3 MPC as QP

with constraints
$E_u u \leq f_u;\ E_x x \leq f_x \quad \Rightarrow \quad \bar{E}_u \mathbf{u} \leq \bar{f}_u;\ \bar{E}_x \mathbf{x} \leq \bar{f}_x$;
$\bar{E}_u = \text{blockdiag}(E_u, ..., E_u)$; $\bar{f}_u = [f_u, ...f_u]^T$
$\bar{f}_x = [f_x, ...f_x]^T$

$$\bar{E}_x = \begin{bmatrix} E_x & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & E_x & 0 \end{bmatrix}$$

Asymmetric, as constraints on x normally defined for $k \in \{0, ..., N-1\}$

Vectorized MPC QP problem

$$\min_{\mathbf{u}} \mathbf{u}^T[\bar{R} + \bar{B}^T\bar{Q}\bar{B}]\mathbf{u} + 2\mathbf{u}^T\bar{B}^T\bar{Q}\bar{A}x_0$$
$$\text{s.t. } \left(\begin{smallmatrix}\bar{E}_u \\ \bar{E}_x\bar{B}\end{smallmatrix}\right)\mathbf{u} \leq \left(\begin{smallmatrix}\bar{f}_u \\ \bar{f}_x - \bar{E}_x\bar{A}x_0\end{smallmatrix}\right)$$

MPC, LP $p = 1$: $\min \sum(\|Qx\|_1) + (\|Rx\|_1)$, s.t...
Introduce vectors (Z_x, Z_u) to model abs values of $(\bar{Q}\mathbf{x}, \bar{R}\mathbf{u})$.

$$\min_{\mathbf{u}} \mathbf{1}^T Z_x + \mathbf{1}^T Z_u$$

$$\left(\begin{smallmatrix}\bar{E}_u \\ \bar{E}_x\bar{B}\end{smallmatrix}\right)\mathbf{u} \leq \left(\begin{smallmatrix}\bar{f}_u \\ \bar{f}_x - \bar{E}_x\bar{A}x(0)\end{smallmatrix}\right)$$
$$\text{s.t.}\ -Z_x \leq \bar{Q}(\bar{A}x_0 + \bar{B}\mathbf{u}) \leq Z_x$$
$$-Z_u \leq \bar{R}\mathbf{u} \leq Z_u$$

MPC, LP $p = \infty$: $\min \sum(\|Qx\|_\infty) + (\|Rx\|_\infty)$, s.t...

Intro scalars (Z_x, Z_u) to model largest abs vals of $(\bar{Q}\mathbf{x}, \bar{R}\mathbf{u})$.

$$\min_{\mathbf{u}} Z_x + Z_u$$
$$\left(\begin{smallmatrix}\bar{E}_u \\ \bar{E}_x\bar{B}\end{smallmatrix}\right)\mathbf{u} \leq \left(\begin{smallmatrix}\bar{f}_u \\ \bar{f}_x - \bar{E}_x\bar{A}x(0)\end{smallmatrix}\right)$$
$$\text{s.t.}\ -\mathbf{1}Z_x \leq \bar{Q}(\bar{A}x_0 + \bar{B}\mathbf{u}) \leq \mathbf{1}Z_x$$
$$-\mathbf{1}Z_u \leq \bar{R}\mathbf{u} \leq \mathbf{1}Z_u$$

Linear or Quadratic

Linear: - Easy to compute; Sol may not be unique; Far from origin: conservative; close to origin: discontinuity, dead-beat behaviour, jitter

Quadratic: - Connection to LQR; Sol unique; Far from origin: large inputs; close to origin: smooth action

Reference Tracking

- Regulation: Reject disturbances around fix point
- Tracking: make output follow reference signal
- For prediction , model reference as constants: $r(k+j) = r(k), j \geq 0$

Steady State computation

$$\begin{bmatrix}(I-A) & -B \\ C0 & \end{bmatrix}\begin{pmatrix}x_{ss} \\ u_{ss}\end{pmatrix} = \begin{pmatrix}0 \\ r\end{pmatrix}$$

Resulting optimal control problem

$V(x) := \min \sum_{i=0}^{N-1}(x_i - x_{ss})^T Q(x_i - x_{ss}) + (u_i - u_{ss})^T R(u_i - u_{ss})$ s.t. $x_{i+1} = Ax_i + Bu_i, x_0 = x, x_i \in \mathbb{X}, u_i \in \mathbb{U}, \forall i \in \{0, ..., N-1\}$

3.1 Stability & Feasibility

Stability not guaranteed. Check stab and feas a priori, derive conditions for all Q and R (conservative) or..

Constraints to ensure stability:

- Infinite prediction horizon: $N \to \infty$
- (Relaxed) Terminal state constraint: $x_N = 0$, $(x_N \in \mathbb{X}_N)$

Common idea: Use optimal value function $V^*(x)$ as Lyapunov function.

Assumption: Objective $f_0(x)$ positive definite, radially unbounded, $f_0(0) = 0$

To work out, 1.) terminal state and control constraints hold in terminal set: $\mathbb{X}_n \subseteq \mathbb{X}, x \in \mathbb{X}_n \Rightarrow K(x) \in \mathbb{U}$

2.) Terminal set invariant with controller:

$$x \in \mathbb{X}_n \Rightarrow f(x, K(x)) \in \mathbb{X}_n$$

3.) the infinite horizon cost has to be bounded by a terminal cost: $\sum_{i=N}^{\infty} l(x_i, K(x_i)) \leq \Psi(x_N)$

Recursive feasibility:

1.) Assume feasibility at k with $=[u_0^*, ..., u_{N-1}^*]$

2.) A feas sol at $k+1$ is $[\hat{u}_0, ..., \hat{u}_{N-1}] = [u_1^*, ..., u_{N-1}^*, K(x_N^*)]$

3.) Associated feasible state trajectory:

$$[\hat{x}_0, ..., \hat{x}_N] = [x_1^*, ..., x_{N-1}^*, f(x_N^*, K(x_N^*))]$$

4.) Shiftet states/inputs guaranteed to satisfy all constraints!

Stability proof:

$V^*(x(k+1)) \leq ..$

$\sum_{i=0}^{N-1} l(x_i^*, u_i^*) + \Psi(x_N^*)$ Previous optimal cost

$-\Psi(x_N^*) - l(x_0^*, u_0^*)$ Terms dropped by shifting

$l(x_N^*, K(x_N^*)) + \Psi(f(x_N^*, K(x_N^*)))$ Terms added

It follows: $V^*(x(k+1)) \leq V^*(x(k)) - l(x_0^*, u_0^*)$

(with the Lyapunov condition:)

$$\Psi(f(x, K(x))) - \Psi(x) \leq -l(x, K(x)) \forall x \in \mathbb{X}_N$$

This tends to zero, so cost tends to zero, so $x(k) \to 0$.

Choice of terminal weight P: $\Psi(x) = x^T P x$

1.) Linear, unconstrained and stable system with quadratic cost: Controller: $K(x) = 0$, terminal constraint $\mathbb{X}_N = \mathbb{R}^n$. Determine P from Lyapunov equation.

2.) Linear, constrained, unstable sys. with quadratic cost: Controller $K(x) = Kx$ (any stabilizing contr), terminal set $\mathbb{X}_n$: invariant set for $x(k+1) = (A + BK)x$. Make P equal optimal cost to go from N to ∞ by solution of ARE. Assumes no constraints are active after N.

3.) Desire of state and input = zero at end. No P but constraint $x_{k+N} = 0$.

3.2 Linear Integer Inequalities

Linear Integer Inequalities can represent logical propositions

Idea: associate to each boolean variable p_i a binary integer variable δ_i: $p_i \Leftrightarrow \{\delta_i = 1\}, \neg p_i \Leftrightarrow \{\delta_i = 0\}$

For a logic proposition $\Omega(p_i)$ it is always possible to define a set of linear inequalities: $A\delta \leq B, \delta \in \{0,1\}^n$

Analytic approach

Conjunctive normal form (CNF)

$\Omega(p_i) = \bigwedge_j [\bigvee_j p_i]$

Logic proposition in CNF into algebraic inequalities

$\neg p_i$, $1 - \delta_i$, Not

$p_i \vee p_j$, $\delta_i + \delta_j \geq 1$, Or

$p_i \wedge p_j$, $\delta_i + \delta_j \geq 2$, And

$p_i \Rightarrow p_j$, $\delta_i - \delta_j \geq 0$, if p_i then p_j

$p_i \Leftrightarrow p_j$, $\delta_i - \delta_j = 0$, logic equality

Rules

$\neg(A \wedge B) =\neq A \vee \neg B$

$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$

$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$

Geometric approach

Idea: Polytope $\mathcal{P} = \{\delta \in \{0,1\}^n | A\delta \leq B\}$ is the convex hull of the rows of the truth table of proposition $\Omega(p_i)$.

Combining logic rules and continuous dynamics

e.g. $p \Leftrightarrow a^T x \leq b, \mathcal{X} = \{x | a^T x - b \in [m, M]\}$.

Bounds m, M have to be specified. Translated to linear inequalities:

$$a^T x - b \leq M(1 - \delta)$$
$$a^T x - b > m\delta$$

MPC for Hybrid Systems Quadratic norm constrained finite time optimal control problem

$J^*(x_t) = \min_U \|x_{t+N}\|_P + \sum_{k=0}^{N-1} \|x_{t+k}\|_{Qx}^2 + \|u_{t+k}\|_{Qu}^2 + \|\delta_{t+k}\|_{Q\delta}^2 + \|z_{t+k}\|_{Qz}^2$,

$$s.t. \begin{cases} x_{t+k+1} = Ax_{t+k} + B_1 u_{t+k} + B_2\delta_{t+k} + B_3 z_{t+k} \\ E_2\delta_{t+k} + E_3 z_{t+k} \leq E_4 x_{t+k} + E_1 u_{t+k} + E_5 \\ x_{t+N} \in \mathcal{X} \end{cases}$$

Mixed Logical Dynamical (MLD) Model

see optimization problem before, with $k = 0$.

Piecewise Affine (PWA) System

- polyhedral partition of the (x, u)-space:

$$\{\mathcal{D}^i\}_{i=1}^D := \{\left(\begin{smallmatrix}x_t \\ u_t\end{smallmatrix}\right) | P_x^i x_t + +P_u^i u_t \leq P_c^i\}$$

- affine dynamics in each region:

$$\left\{\begin{matrix} x_{t+1} = A^i x_t + B^i u_t + f^i \\ y_t = C^i x_t + D^i u_t + g^i \end{matrix}\right\} \text{ if } x_t \in \mathcal{D}^i$$

- any well-posed (for a given $[x_t^T u_t^T]^T \Rightarrow x_{t+1}, y_t$ are uniquely determined) PWA system can be represented by an MLD system, assuming that the set of feasible states and inputs is bounded.
- a completely well-posed (well-posed + uniquely determined $\delta_t, z_t \forall [x_t^T u_t^T]^T$) MLD can be written as a PWA system.

4 Numerical Optimization Methods

4.1 Unconstrained Optimization

Algorithm performance Measurements • Convergence: is m finite for every $\epsilon, \delta > 0$

• Convergence speed: dependence of errors $f(x_m) - f(x^*)$ and $dist(x_m, \mathbb{Q})$

• Feasability: For some methods $\delta = 0$, but in general $\delta \neq 0$

• Numerical robustness: Robustness in presence of finite presicion arithmetics

• Warm starting: Improve performance by initializing Algorithm x_0 near x^*

• Preconditioning: Transform problem P into transformed Problem $\tilde{P}$

Gradient Method

Set	x_0
Repeat	$x_{i+1} = x_i - \frac{1}{L} \cdot \nabla f(x_i)$
until	$f(x^m) - f(x^*) \leq \epsilon_1 or \|x_m - x_{m-1}\| \leq \epsilon_2$

Convergence $m \sim \mathcal{O}(L\|x_o - x^*\|^2/\epsilon_1)$

Assumptions:

- ∇f is Lipschitz-continous with Lipschitz constant L: $\|\nabla f(x) - f(y)\| \leq L\|x-y\| \forall x, y \in \mathbb{R}^n$
- f can be upper bounded by a quadractic function: $f(x) \leq f(y) + \nabla f(y)^T(x-y) + \frac{L}{2}\|x-y\|^2 = \bar{f}(x,y), \forall x, y$

Convergence:

$f(x_{i-1}) \leq \bar{f}(x_{i-1}, x_i) \leq \bar{f}(x_i, x_i) = f(x_i)$

Strong convexity: $f(x)$ upper and lower bounded

$\bar{f}(x,y)|_{L=\mu} \leq f(x) \leq \bar{f}(x,y)|_{L=L}$

For $f(x) = x^T Hx$: $L = \max(\text{eig}(H))$, $\mu = \min(\text{eig}(H)$.

Condition number: $\kappa \triangleq= \frac{L}{\mu}$.

problem badly conditioned (slow convergence): $\kappa \geq 1$

Fast Gradient Method

Set	$x_0, y_0 = x_0$ and $\alpha_0 = (\sqrt{5}-1)/2$
Repeat	$x_{i+1} = y_i - \frac{1}{L} \cdot \nabla f(x_i)$
	$\alpha_{i+1} = \alpha_i(\sqrt{\alpha_i^2+4} - \alpha_i)/2$
	$\beta_i = \frac{\alpha_i(1-\alpha_i)}{\alpha_i^2 + \alpha_{i+1}}$
	$y_{i+1} = x_{i+1} + \beta_i(x_{i+1} - x_i)$ for $i = 0, ..., m$
until	$f(x^m) - f(x^*) \leq \epsilon_1 or \|x_m - x_{m-1}\| \leq \epsilon_2$

Convergence $m \sim \mathcal{O}(\sqrt{L\|x_o - x^*\|^2/\epsilon_1})$

Newtons Method:

Min of 2nd-order approx of f at point x_i

$x_{i+1} = \min f(x_i) + \nabla f(x_i)^T v + 1/2 v^T \nabla^2 f(x_i) v; v = (x - x_i)$

$x_{i+1} = x_i + h_i \Delta x_{nt}$

Newton direction: $\Delta x_{nt} = -(\nabla^2 f(x_i))^{-1} \nabla f(x_i)$

Finding h_i is not as easy as in other cases:

Exact way: Compute optimal h_i: $h_i^* = \underset{h>0}{\text{argmin}} f(x_i + h_i \Delta x_{nt})$

Inexact: Find h_i that decreases f by some %, for example:

Backtracking: $\alpha \in (0, 0.05)$, $\beta \in (0,1)$

while $f(x_i + h_i \Delta x_{nt}) > f(x_i) + \alpha h_i \nabla f(x_i)^T \Delta x_{nt}$ **do** $h_i \leftarrow \beta h_i$

4.2 Constrained Optimization

Constraint GM = GM + projection $\pi_{\mathbb{Q}}(.)$ on feasible set.

Gradient method: $x_{i+1} = \pi_{\mathbb{Q}}(x_i - \frac{1}{L} \cdot \nabla f(x_i))$

Fast gradient method with: $x_{i+1} = \pi_{\mathbb{Q}}(y_i - \frac{1}{L} \cdot \nabla f(x_i))$

- If $\pi_{\mathbb{Q}}(.)$ easy to compute: very fast algo.
- If $\pi_{\mathbb{Q}}(.)$ only computed numerically: solve also dual -> slower.

Cheap sets: hyperplane, affine set, halfspace, 1&2-norm ball...

Interior Point Method Unconstrained problem, where constraints are moved to objective via indicator functions Φ.

$\min\ f(x)$ s.t. $g_i(x) \leq 0, i = 1, ..., m$ $\Leftrightarrow$ $x^*(\kappa) = \min f(x) + \kappa\Phi(x)$

With the **barrier function**: $\Phi(x) = -\sum_{i=1}^{m} \log(-g_i(x))$

Central path: $\{x^*(\kappa)|\kappa > 0\}$: optimal solution for all $\kappa > 0$.

Approximation improves as $\kappa \to 0$

Barrier interior-point method: follow central path to optimal solution .

Start with: strictly feasible $x_0, \kappa_0, \mu > 1$

repeat

1. Centering step: compute $x^*(\kappa_i)$
2. update $x_i = x^*(\kappa_i)$
3. stop if $m\kappa_i < 0$ (duality gap)
4. Decrease barrier parameter : $\kappa_{i+1} = \kappa_i/\mu$

5 Parametric Programming

Parametric programming is a way to evade solving a MPC programm online, but rather moving the computational effort offline. For this approach, the feasible set is divided in subsets for which the same constraints are active: **Critical Regions**: Subset of paramet set, where local optimality conditions (KKT) do not change (same constraints are active).

$\theta \in \mathcal{X} \in \mathbb{R}^n$: vector of parameters.

5.1 Multi-parametric Linear Problem (mpLP)

For linear problems of the following form.

Primal Problem:

$J^*(\theta) = \min_z c^T z$ s.t. $Gz \leq W + S\theta$

Dual Problem

$\max_\pi (W + S\theta)^T \pi$ s.t. $G^T\pi = c$, $\pi \leq 0$

Active and inactive sets are defined as follows:

Active indices: $\mathcal{A}(\theta) = \{i \in \mathcal{I} | \forall z : J(z,\theta) = J^*(\theta) \Rightarrow G_i z - S_i\theta - W_i = 0\}$

Inactive indices: $\mathcal{N}(\theta) = \{i \in \mathcal{I} | \forall z : J(z,\theta) = J^*(\theta) \wedge G_i z - S_i\theta - W_i < 0\}$

The algorithm for finding the regions contains 3 steps:

1. Step

For an inital vector θ_0 solve Primal and Dual problem: Find z^* and π^*

Obtain sets of inactive and active constraints:

Obtain sets $\mathcal{A}(\theta_0)$ and $\mathcal{N}(\theta_0)$

$G_\mathcal{A}, S_\mathcal{A}, W_\mathcal{A} = \{G_i, S_i, W_i | i \in \mathcal{A}\}$

$G_\mathcal{N}, S_\mathcal{N}, W_\mathcal{N} = \{G_i, S_i, W_i | i \in \mathcal{N}\}$

2. Step

Compute optimizer: $z^*(\theta) = G_\mathcal{A}^{-1} S_\mathcal{A}\theta + G_\mathcal{A}^{-1} W_\mathcal{A} = F_0\theta + g_0$

Critical Region: $\mathcal{CR}_0 = \{\theta | (G_\mathcal{N} F_0 - S_\mathcal{N}) < W_\mathcal{N} - G_\mathcal{N} \cdot g_0$

3. Step

Explore rest of $\mathcal{X}$ for example by reversing inequalities (may yield overlapping or artificially splitted Regions) or by just going to a boundary and crossing it by going in direction of gradient by some (empirical) delta.

5.2 Multi Parametric Quadratic Problem mpQP

Primal Problem:

$J^*(\theta) = \min_z\ 1/2 \cdot z^T Hz$ s.t. $Gz \leq W + S\theta$

1. Step

- Solve QP for one θ_0, obtain optimizer $z*(\theta)$
- Identify indices of active $\mathcal{A}(\theta)$ and inactive $\mathcal{N}(\theta)$ sets and corresponding matrices $\{G_i, S_i, W_i\}_\mathcal{A}$ and $\{G_i, S_i, W_i\}_\mathcal{N}$.

2. Step

Compute $z^*(\theta)$, $\lambda^*(\theta)$ and $\mathcal{CR}_0$:

$z^*(\theta) = H^{-1} G_\mathcal{A}^T (G_\mathcal{A} H^{-1} G_\mathcal{A}^T)^{-1} (W_\mathcal{A} + S_\mathcal{A}\theta)$

$\lambda^*(\theta) = -(G_\mathcal{A} H^{-1} G_\mathcal{A}^T)^{-1} (W_\mathcal{A} + S_\mathcal{A}\theta)$

$\mathcal{CR}_0 = \{\theta | A\theta \leq b\}$

$A = \begin{bmatrix} GH^{-1}G_\mathcal{A}^T(G_\mathcal{A}H^{-1}G_\mathcal{A}^T)^{-1}S_\mathcal{A} - S \\ (G_\mathcal{A}H^{-1}G_\mathcal{A}^T)^{-1}S_\mathcal{A} \end{bmatrix}$

$b = \begin{bmatrix} W + GH^{-1}G_\mathcal{A}^T(G_\mathcal{A}H^{-1}G_\mathcal{A}^T)^{-1}W_\mathcal{A} \\ (G_\mathcal{A}H^{-1}G_\mathcal{A}^T)^{-1}W_\mathcal{A} \end{bmatrix}$

3. Step

Explore Rest of $\mathcal{X}$ analogously to mpLP.

6 Calculation Rules:

$$(A \cdot b)^T = b^T \cdot A^T$$
$$(A \cdot b)^{-1} = b^{-1} \cdot A^{-1}$$
$$(A^{-1})^T = (A^T)^{-1}$$
$$\nabla_x(x^T \cdot b) = \nabla_x(b^T \cdot x) = b$$
$$\nabla_x(b \cdot x) = b^T$$
$$\nabla_x(x^T \cdot Q \cdot x) = 2 \cdot Q \cdot x, \quad Q \succeq 0$$
$$x^T \cdot A^T \cdot Q \cdot A \cdot x = (x^T \cdot A^T \cdot Q \cdot A \cdot x)^T, \quad Q \succeq 0$$
$$Q = Q^T, \quad Q \succeq 0$$
$$det(A \cdot B) = det(A) \cdot det(B)$$
$$det(A^T) = det(A)$$
$$det(A^{-1}) = 1/det(A)$$

$\{-1, 1\}$: either -1, or 1

$[1, 2[$: values from including 1 to excluding 2