## 1 Systems Theory

### 1.1 Linearization

### 1.2 Discretization

#### Exact

#### Forward-Euler

#### Backward-Euler

### 1.3 Lyapunov Function

$V(0) = 0,\ x \neq 0 \implies V(x) > 0,\ V(g(x(k+1))) - V(x(k+1)) \leq -\alpha(x(k))$
System asymptotically stable if $V(x)$ exists. Globally stable iff $\|x\| \to \infty \implies V(x) \to \infty$.
Check Eig. values of $(APA - P)$ neg., $V(x) = x^T P x$ ?

## 2 Unconstrained Control

### 2.1 Block Approach (used also for $\bar{w}$ substition)

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{A} \\ \vdots \\ \boldsymbol{A}^N \end{bmatrix} x(0) + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \boldsymbol{B} & \mathbf{0} & \dots & \mathbf{0} \\ \boldsymbol{AB} & \boldsymbol{B} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \mathbf{0} \\ \boldsymbol{A}^{N-1}\boldsymbol{B} & \dots & \boldsymbol{AB} & \boldsymbol{B} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_1} \end{bmatrix}$$

$$x = \boldsymbol{S}^x \cdot x(0) + \boldsymbol{S}^u \cdot u \qquad \mathrm{size}(\boldsymbol{S}^x) = [n_{\text{states}} \cdot (N+1), N]$$
$$\mathrm{size}(\boldsymbol{S}^u) = [n_{\text{states}} \cdot (N+1), n_{\text{states}}]$$
$$\bar{\boldsymbol{Q}} = \mathrm{diag}(\boldsymbol{Q}, \dots, \boldsymbol{Q}, \boldsymbol{P}) \qquad \mathrm{size}(\bar{\boldsymbol{Q}}) = [n_{\text{states}} \cdot (N+1), n_{\text{states}} \cdot (N+1)]$$
$$\bar{\boldsymbol{R}} = \mathrm{diag}(\boldsymbol{R}, \dots, \boldsymbol{R}) \qquad \mathrm{size}(\bar{\boldsymbol{R}}) = (n_{\text{input}} \cdot N, n_{\text{input}} \cdot N)$$
$$\boldsymbol{H} = \boldsymbol{S}^{u\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^u + \boldsymbol{R} \qquad \boldsymbol{F} = \boldsymbol{S}^{x\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^u$$
$$\boldsymbol{Y} = \boldsymbol{S}^{x\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^x$$

#### Optimal cost and control

$$J^*(x_0) = -x_0^T \boldsymbol{F} \boldsymbol{H} \boldsymbol{F}^T x_0 + x_0^T \boldsymbol{Y} x_0$$
$$u^*(x_0) = -\boldsymbol{H}^{-1} \boldsymbol{F}^T x_0 = -\left( \boldsymbol{S}^{u\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^u + \boldsymbol{R} \right)^{-1} \boldsymbol{S}^{u\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^x x_0$$

### 2.2 Recursive Approach

$$J_k^*(x_k) = \min_{u_k} I(x_k, u_k) + J_{k+1}(x_{k+1})$$

Is a feedback controller as opposed to the Batch Approach. For LQR solve via Riccati Difference Equation (RDE).

$$\boldsymbol{F}_k = -(\boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{B} + \boldsymbol{R})^{-1} \boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{A}$$
$$\boldsymbol{P}_k = \boldsymbol{A}^T \boldsymbol{P}_{k+1} \boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{A}^T \boldsymbol{P}_{k+1} \boldsymbol{B} (\boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{B} + \boldsymbol{R})^{-1} \boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{A}$$

$$u_k^* = \boldsymbol{F}_k\ x_k \qquad J_k^*(x_k) = x_k^T \boldsymbol{P}_k\ x_k \qquad \boldsymbol{P}_N = \boldsymbol{P}$$

For unconstrained Infinite Horizon Problem, substituting $\boldsymbol{P}_\infty = \boldsymbol{P}_k = \boldsymbol{P}_{k+1}$ into RDE gives DARE. Uniquely solvable, iff $(A, B)$ stabilizable and $(A, G)$ detectable, where $\boldsymbol{G}\boldsymbol{G}^T = \boldsymbol{Q}$. Follows from closed-loop system $x_{k+1} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}_k) x_k$

## 3 (Convex) Optimization

**General Problem** $\min_{x \in \mathrm{dom}(f)} f(x)$ s. t. $g_i(x) \leq 0$ and $h_j(x) = 0$.

#### RHC

#### QP with substitution (see also Batch approach)

$$J^*(x_k) = \min_u \begin{bmatrix} u^T & x_k^T \end{bmatrix} \begin{bmatrix} \boldsymbol{H} & \boldsymbol{F}^T \\ \boldsymbol{F} & \boldsymbol{Y} \end{bmatrix} \begin{bmatrix} u \\ x_k \end{bmatrix}$$
$$\text{s. t. } \boldsymbol{G}\ u \leq w + \boldsymbol{E}\ x_k$$

Latter gives three sets (same for without substitution)

$$\mathcal{X} = \{x | A_x\ x \leq b_x\}$$
$$\mathcal{U} = \{u | A_u\ u \leq b_u\}$$
$$\mathcal{X}_{\{} = \{x | A_f\ x \leq b_f\}$$

State equations are in cost matrix, usually $\boldsymbol{A}_x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, b_x = \begin{bmatrix} b_{\max} \\ -b_{\min} \end{bmatrix}$

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{A}_u & 0 & \dots & 0 \\ 0 & \boldsymbol{A}_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{A}_u \\ 0 & 0 & \dots & 0 \\ \boldsymbol{A}_x \boldsymbol{B} & 0 & \dots & 0 \\ \boldsymbol{A}_x \boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}_x \boldsymbol{B} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}_x \boldsymbol{A}^{N-2} \boldsymbol{B} & \boldsymbol{A}_x \boldsymbol{A}^{N-3} \boldsymbol{B} & \dots & 0 \\ \boldsymbol{A}_f \boldsymbol{A}^{N-1} \boldsymbol{B} & \boldsymbol{A}_f \boldsymbol{A}^{N-2} \boldsymbol{B} & \dots & \boldsymbol{A}_f \boldsymbol{B} \end{bmatrix} \quad \boldsymbol{E} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\boldsymbol{A}_x \\ -\boldsymbol{A}_x \boldsymbol{A} \\ -\boldsymbol{A}_x \boldsymbol{A}^2 \\ \vdots \\ -\boldsymbol{A}_x \boldsymbol{A}^{N-1} \\ -\boldsymbol{A}_f \boldsymbol{A}^N \end{bmatrix} \quad \boldsymbol{W} = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_x \\ b_f \end{bmatrix}$$

**QP with out substitution** State equations represented in equality constrainst.

$$J^*(x_k) = \min_z \begin{bmatrix} z^T & x_k^T \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{H}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} z \\ x_k \end{bmatrix}$$
$$\text{s. t. } \boldsymbol{G}\ z \leq w + \boldsymbol{E}\ x_k$$
$$\boldsymbol{G}_{\text{eq}}\ z = \boldsymbol{E}_{\text{eq}}\ x_k, \quad \text{system dynamics}$$

$$\bar{\boldsymbol{H}} = \mathrm{diag}(\boldsymbol{Q}, \dots, \boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{R}, \dots, \boldsymbol{R})$$

$$z = \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} \quad \boldsymbol{G}_{\text{eq}} = \left[\begin{array}{cccc|cccc} \boldsymbol{I} & & & & -\boldsymbol{B} & & & \\ -\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B} & & \\ & \ddots & \ddots & & & & \ddots & \\ & & -\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B} \end{array}\right] \quad \boldsymbol{E}_{\text{eq}} = \begin{bmatrix} \boldsymbol{A} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$w = \begin{bmatrix} b_x \\ \vdots \\ b_f \\ b_u \\ \vdots \\ b_u \end{bmatrix} \quad \boldsymbol{G} = \left[\begin{array}{ccc|ccc} 0 & & & & & \\ & \boldsymbol{A}_x & & & & \\ & & \ddots & & & \\ & & \boldsymbol{A}_x & & & \\ \hline & & & \boldsymbol{A}_d & & \\ & & & & \ddots & \\ & & & & & \boldsymbol{A}_d \end{array}\right] \quad \boldsymbol{E} = \begin{bmatrix} -\boldsymbol{A}_x^T \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

### 3.1 Duality

#### Lagrangian Dual Function

$$L(x, \lambda, \nu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$
$$d(\lambda, \nu) = \inf_{x \in \mathcal{X}} L(x, \lambda, \nu) \qquad \text{i.e. } \nabla_x L(x, \lambda, \nu) = 0$$

**Dual Problem (always convex)** $\max_{\lambda, \nu} d(\lambda, \nu)$ s. t. $\lambda \geq 0$.
Optimal value is lower bound for primal: $d^* \leq p^*$.

If primal convex, *Slater condition* (strict feasibility) implies *strong duality*:

$$\{x \mid Ax = b, f_i(x) < 0, \} \neq \emptyset \Rightarrow d^* = p^*$$

**Karush-Kuhn-Tucker (KKT) Conditions** are necessary for optimality (and sufficient if primal convex).

- Primal Feasibility:

$$f_i(x^*) \leq 0 \quad i = 1, \dots, m$$
$$h_i(x^*) = 0 \quad i = 1, \dots, p$$

- Dual Feasibility: $\lambda^* \geq 0$
- Complementary Slackness:

$$\lambda_i^* \cdot f_i(x^*) = 0 \qquad i = 1, \dots, m$$

- Stationarity:

$$\nabla_x L(x^*, \lambda^*, \nu^*) = 0$$

### 3.2 Constrained Finite Time Optimal Control (CFTOC)

### 3.3 Invariance

### 3.4 Feasability, Stability

### 3.5 Practical MPC

### 3.6 Robust MPC

**Tube-MPC**

### 3.7 Explicit MPC

### 3.8 Hybrid MPC

## 4 Numerical Optimization

Gradient, Newton, Interior Point

## 5 Observer Based Control

### 5.1 LTI Observer

LTI System:

$$x(k) = Ax(k-1) + Bu(k-1) + v(k-1)$$
$$z(k) = Hx(k) + w(k)$$

Linear Static Gain Observer (Luenberger Observer):

$$\hat{x}(k) = A\hat{x}(k-1) + Bu(k-1) + K(z(k) - \hat{z}(k))$$
$$\hat{z}(k) = H(A\hat{x}(k-1) + Bu(k-1))$$
$$e(k) = (I - K\,H)\,A\,e(k-1)$$

$e(k) \to 0$ for $k \to \infty$ if and only if $(I - K\,H)\,A$ is stable.

Steady State:

$$\hat{x}(k) = (I - K_\infty H)\,A\,\hat{x}(k-1) + (I - K_\infty)B\,u(k-1) + K_\infty z(k)$$

The steady-state KF is one way to design the observer gain $K$ (optimal in minimizing the Steady State mean squared error).

$(A, H)$ detectable $\Rightarrow$ $K$ exists such that $(I - K\,H)A$ is stable.

### 5.2 Static State Feedback Control

Design of a controller without paying attention to the state estimation:

$$x(k) = Ax(k-1) + Bu(k-1) \qquad \text{(Process without noise)}$$
$$z(k) = x(k) \qquad \text{(Perfect State information)}$$
$$u(k) = F \cdot z(k) = F \cdot x(k) \qquad \text{(Control Law)}$$

Closed loop dynamics: $x(k) = (A + BF)$. Hence system is stable if $(A + BF)$ is stable. Such an $F$ exists only if $(A, B)$ is stabilizable.
If $(A, B)$ is stabilizable and $(A, G)$ detectable, then $F$ is given by

$$F = -(B^T PB + \bar{R})^{-1} \cdot B^T PA; \qquad P \geq 0$$

$P$ from DARE: $P = A^T PA + \bar{Q} - A^T PB(B^T PB + \bar{R})^{-1} \cdot B^T PA$

### 5.3 Separation Principle (Linear Systems only)

Combining Luenberger Observer and Static State Feedback control yields:

$$\begin{bmatrix} x(k) \\ e(k) \end{bmatrix} = \begin{bmatrix} A + BF & -BF \\ 0 & (I - KH)A \end{bmatrix} \cdot \begin{bmatrix} x(k-1) \\ e(k-1) \end{bmatrix}$$

Eigenvalues of closed loop are given bei Eigenvalues of $(I - KH)A$ and $(A + BF)$. System is stable as long as there exists no $|\lambda| \geq 1$.

### 5.4 Separation Theorem

1. Design steady-state KF which does not depend on $\bar{Q}, \bar{R}$. $\Rightarrow \hat{x}(k)$
2. Design state-feedback $u(k) = Fx(k)$ and put both together.