

1 Systems Theory

1.1 Linearization

1.2 Discretization

Exact

Forward-Euler

Backward-Euler

1.3 Lyapunov Function

$V(0) = 0, x \neq 0 \implies V(x) > 0, V(g(x(k+1))) - V(x(k+1)) \leq -\alpha(x(k))$

System asymptotically stable if $V(x)$ exists. Globally stable iff $\|x\| \rightarrow \infty \implies V(x) \rightarrow \infty$.

Check Eig. values of $(A - P)$ neg., $V(x) = x^T P x$?

2 Unconstrained Control

2.1 Block Approach (used also for \bar{w} substitution)

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} I \\ A \\ \vdots \\ A^N \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 & \dots & 0 \\ B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ A^{N-1}B & \dots & AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

$$x = S^x \cdot x(0) + S^u \cdot u \quad \text{size}(S^x) = [n_{\text{states}} \cdot (N+1), N]$$

$$\text{size}(S^u) = [n_{\text{states}} \cdot (N+1), n_{\text{states}}]$$

$$\bar{Q} = \text{diag}(Q, \dots, Q, P) \quad \text{size}(\bar{Q}) = [n_{\text{states}} \cdot (N+1), n_{\text{states}} \cdot (N+1)]$$

$$\bar{R} = \text{diag}(R, \dots, R) \quad \text{size}(\bar{R}) = (n_{\text{input}} \cdot N, n_{\text{input}} \cdot N)$$

$$H = S^{uT} \bar{Q} S^u + R \quad F = S^{xT} \bar{Q} S^u$$

$$Y = S^{xT} \bar{Q} S^x$$

Optimal cost and control

$$J^*(x(0)) = -x(0)^T F H F^T x(0) + x(0)^T S^{xT} \bar{Q} S^x x(0)$$

$$u^*(x(0)) = -H^{-1} F^T x(0) = -\left(S^{uT} \bar{Q} S^u + R\right)^{-1} S^{uT} \bar{Q} S^x x(0)$$

tim: check formulae for J^*, u^*

$$U_0^*(x(0)) = -(S^{uT} \bar{Q} S^u + \bar{R})^{-1} S^{uT} \bar{Q} S^x x(0)$$

$$J_0^*(x(0)) = x(0)^T \left(S^{xT} \bar{Q} S^x - \dots \right. \\ \left. \dots S^{xT} \bar{Q} S^u (S^{uT} \bar{Q} S^u - \bar{R})^{-1} S^{uT} \bar{Q} S^x \right) x(0)$$

2.2 Recursive Approach

$$J_k^*(x_k) = \min_{u_k} I(x_k, u_k) + J_{k+1}(x_{k+1})$$

Is a feedback controller as opposed to the Batch Approach. For LQR solve via Riccati Difference Equation (RDE).

$$F_k = -(B^T P_{k+1} B + R)^{-1} B^T P_{k+1} A$$

$$u_k^* = F_k x_k$$

$$P_k = A^T P_{k+1} A + Q - A^T P_{k+1} B (B^T P_{k+1} B + R)^{-1} B^T P_{k+1} A$$

$$J_k^*(x(0)) = x(0)^T P_k x(0)$$

tim: Initialize $P_N = P$, sometimes $F_k = K$.

Infinite Horizon unconstrained, LQR gives P_∞ satisfying equations above. If (A, B) stabilizable and (A, G) detectable, P_∞ unique, where $GG^T = Q$.

$$J_k^*(x(k)) = x(k)^T P_k x(k)$$

$$P_k = A^T P_{k+1} A + Q - \dots$$

$$\dots A^T P_{k+1} B \left(B^T P_{k+1} B + R \right)^{-1} B^T P_{k+1} A$$

3 (Convex) Optimization

General Problem $\min_{x \in \text{dom}(f)} f(x)$ s. t. $g_i(x) \leq 0$ and $h_j(x) = 0$.

RHC

QP with substitution

QP with out substitution

3.1 Duality

Lagrangian Dual Function

$$L(x, \lambda, \nu) = f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{i=1}^p \nu_i h_i(x)$$

$$d(\lambda, \nu) = \inf_{x \in \mathcal{X}} L(x, \lambda, \nu) \quad \text{i.e. } \nabla_x L(x, \lambda, \nu) = 0$$

Dual Problem (always convex) $\max_{\lambda, \nu} d(\lambda, \nu)$ s. t. $\lambda \geq 0$.

Optimal value is lower bound for primal: $d^* \leq p^*$.

If primal convex, *Slater condition* (strict feasibility) implies *strong duality*:

$$\{x \mid Ax = b, f_i(x) < 0, \} \neq \emptyset \Rightarrow d^* = p^*$$

Karush-Kuhn-Tucker (KKT) Conditions are necessary for optimality (and sufficient if primal convex).

- Primal Feasibility:

$$f_i(x^*) \leq 0 \quad i = 1, \dots, m$$

$$h_i(x^*) = 0 \quad i = 1, \dots, p$$

- Dual Feasibility: $\lambda^* \geq 0$

- Complementary Slackness:

$$\lambda_i^* \cdot f_i(x^*) = 0 \quad i = 1, \dots, m$$

- Stationarity:

$$\nabla_x L(x^*, \lambda^*, \nu^*) = 0$$

3.2 Constrained Finite Time Optimal Control (CFTOC)

3.3 Invariance

3.4 Feasibility, Stability

3.5 Practical MPC

3.6 Robust MPC

Tube-MPC

3.7 Explicit MPC

3.8 Hybrid MPC

4 Numerical Optimization

Gradient, Newton, Interior Point

5 Observer Based Control

5.1 LTI Observer

LTI System:

$$\begin{aligned}x(k) &= Ax(k-1) + Bu(k-1) + v(k-1) \\z(k) &= Hx(k) + w(k)\end{aligned}$$

Linear Static Gain Observer (Luenberger Observer):

$$\begin{aligned}\hat{x}(k) &= A\hat{x}(k-1) + Bu(k-1) + K(z(k) - \hat{z}(k)) \\ \hat{z}(k) &= H(A\hat{x}(k-1) + Bu(k-1)) \\ e(k) &= (I - KH)Ae(k-1)\end{aligned}$$

$e(k) \rightarrow 0$ for $k \rightarrow \infty$ if and only if $(I - KH)A$ is stable.

Steady State:

$$\hat{x}(k) = (I - K_{\infty}H)A\hat{x}(k-1) + (I - K_{\infty})Bu(k-1) + K_{\infty}z(k)$$

The steady-state KF is one way to design the observer gain K (optimal in minimizing the Steady State mean squared error).

(A, H) detectable $\Rightarrow K$ exists such that $(I - KH)A$ is stable.

5.2 Static State Feedback Control

Design of a controller without paying attention to the state estimation:

$$\begin{aligned}x(k) &= Ax(k-1) + Bu(k-1) && \text{(Process without noise)} \\ z(k) &= x(k) && \text{(Perfect State information)} \\ u(k) &= F \cdot z(k) = F \cdot x(k) && \text{(Control Law)}\end{aligned}$$

Closed loop dynamics: $x(k) = (A + BF)$. Hence system is stable if $(A + BF)$ is stable. Such an F exists only if (A, B) is stabilizable.

If (A, B) is stabilizable and (A, G) detectable, then F is given by

$$F = -(B^T P B + \bar{R})^{-1} \cdot B^T P A; \quad P \geq 0$$

P from DARE: $P = A^T P A + \bar{Q} - A^T P B (B^T P B + \bar{R})^{-1} \cdot B^T P A$

5.3 Separation Principle (Linear Systems only)

Combining Luenberger Observer and Static State Feedback control yields:

$$\begin{bmatrix} x(k) \\ e(k) \end{bmatrix} = \begin{bmatrix} A + BF & -BF \\ 0 & (I - KH)A \end{bmatrix} \cdot \begin{bmatrix} x(k-1) \\ e(k-1) \end{bmatrix}$$

Eigenvalues of closed loop are given by Eigenvalues of $(I - KH)A$ and $(A + BF)$. System is stable as long as there exists no $|\lambda| \geq 1$.

5.4 Separation Theorem

1. Design steady-state KF which does not depend on \bar{Q}, \bar{R} . $\Rightarrow \hat{x}(k)$
2. Design state-feedback $u(k) = Fx(k)$ and put both together.