1 System Theory

1.1 Nonlinear Systems

$$\dot{x} = g(x,u) \qquad y = h(x,u)$$
$$\dot{x_s} = g(x_s,u_s) = 0 \qquad y_s = h(x_s,u_s)$$
$$\boldsymbol{A}^c = \frac{\partial g}{\partial x^T}\bigg|_{x=x_s,u=u_s} \qquad \boldsymbol{B}^c = \frac{\partial g}{\partial u^T}\bigg|_{x=x_s,u=u_s}$$
$$\boldsymbol{C}^c = \frac{\partial h}{\partial x^T}\bigg|_{x=x_s,u=u_s} \qquad \boldsymbol{D}^c = \frac{\partial h}{\partial u^T}\bigg|_{x=x_s,u=u_s}$$

1.2 Linear Systems

Continuous

$$\dot{x}(t) = \boldsymbol{A}^c x(t) + \boldsymbol{B}^c u(t)$$
$$x(t) = e^{\boldsymbol{A}^c(t-t_0)}x_0 + \int_{t_0}^{t} e^{\boldsymbol{A}^c(t-\tau)}\boldsymbol{B}u(\tau)d\tau$$
$$e^{\boldsymbol{A}^c t} = \sum_{n=0}^{\infty}\frac{(\boldsymbol{A}^c t)^n}{n!}$$

Discrete

$$x_{k+1} = \boldsymbol{A}x_k + \boldsymbol{B}u_k$$
$$y_k = \boldsymbol{C}x_k + \boldsymbol{D}u_k$$
$$x_{k+N} = \boldsymbol{A}^N x_k + \sum_{i=0}^{N-1}\boldsymbol{A}^i\boldsymbol{B}u_{k+N-1-i}$$

Forward Euler $A = I + T_s A^c,\ B = T_s B^c,\ C = C^c,\ D = D^c$
$$x_{k+1} = x_k + T_s g^c(x_k,u_k) = g(x_k,u_k)$$
$$y_k = h^c(x_k,u_k) = h(x_k,u_k)$$

Exact discretization (assume constant $u(t)$ during T_s)
$$\boldsymbol{A} = e^{\boldsymbol{A}^c T_s},\ \boldsymbol{B} = \int_0^{T_s} e^{\boldsymbol{A}^c(T_s-\tau')}\boldsymbol{B}^c d\tau$$
$$\boldsymbol{B} = (\boldsymbol{A}^c)^{-1}(\boldsymbol{A}-\boldsymbol{I})\boldsymbol{B}^c,\ \text{if } \boldsymbol{A}^c \text{ invertible}$$

1.3 Lyapunov Stability

System is stable in the sense of Lyapunov iff it stays in any arbitrarily small neighborhood of the origin when it is disturbed slightly.

Lyapunov stable iff $\forall \epsilon > 0\ \exists\delta(\epsilon)$ s.t. $\|x_0\| < \delta(\epsilon) \to \|x_k\| < \epsilon, \forall k \geq 0$

asymptotically stable in $\Omega \subseteq \mathbb{R}^n$ if Lyapunov stable and *attractive* $\lim_{k\to\infty} x_k = 0, \forall x_0 \in \Omega$.

Lyapunov Function $V : \mathbb{R}^n \to \mathbb{R}$ continous at the origin, finite $\forall x \in \Omega$,
$$V(0) = 0 \text{ and } V(x) > 0, \forall x \in \Omega\backslash\{0\}$$
$$V(g(x)) - V(x) \leq -\alpha(x), \forall x \in \Omega\backslash\{0\}$$
where $\alpha : \mathbb{R}^n \to \mathbb{R}$ is continuous positive definite, equilibrium at $x = 0$ and $\Omega \subset \mathbb{R}^n$ closed and bounded set containing the origin.

Lyapunov Theorem If system admits Lyapunov function $V(x)$, then $x = 0$ is asymptotically stable in Ω (sufficient but not necessary).
If additionally $\|x\| \to \infty \Rightarrow V(x) \to \infty$ globally asymptotically stable.
To check if $V(x) = x^T\boldsymbol{P}x$ is valid Lyapunov function of system $x_{k+1} = \boldsymbol{A}x_k$ check if $(\boldsymbol{APA} - \boldsymbol{P})$ has neg. eigen values.
In other words: Iff eigenvalues of A inside unit circle (i.e. stable) then $\exists$*unique* $\boldsymbol{P} > 0$ that solves $\boldsymbol{A}_{cl}^T\boldsymbol{P}\boldsymbol{A}_{cl} - \boldsymbol{P} = -\boldsymbol{Q},\ \boldsymbol{Q} > 0$ and $V(x) = x^T\boldsymbol{P}x$ is a lyapunov function.

1.4 Observability ⇒ Detectability, Controllability ⇒ Stabilizability

(A,C) **observable** if $\text{rank}(\boldsymbol{O}) = n$ (full col. rank) for
$$\boldsymbol{O} = \begin{bmatrix} C^T \\ (CA)^T \\ \dots \\ (CA^{n-1})^T \end{bmatrix} \text{ or rank} \begin{bmatrix}\boldsymbol{A}-\lambda\boldsymbol{I} \\ \boldsymbol{C}\end{bmatrix} = n\ \forall\lambda_i \text{ of } \boldsymbol{A} \text{ (PBH-test).}$$

(A,C) **detectable** iff rank $\begin{bmatrix}\boldsymbol{A}-\lambda\boldsymbol{I} \\ \boldsymbol{C}\end{bmatrix} = n \forall\text{unstable}|\lambda_i| \geq 1$ of $\boldsymbol{A}$.

(A,B) **controllable** if rank $\boldsymbol{C} = n$, $\boldsymbol{C} = \begin{bmatrix}\boldsymbol{B} & \boldsymbol{AB} & \dots & \boldsymbol{A}^{n-1}\boldsymbol{B}\end{bmatrix}$ or if rank$\left(\begin{bmatrix}\lambda_j\boldsymbol{I}-\boldsymbol{A} & \boldsymbol{B}\end{bmatrix}\right) = n\ \forall\lambda_i$ of $\boldsymbol{A}$ (PBH-test).
Intuition: Can reach any state in (at most) n steps.

(A,B) **stabilizable** if rank $\begin{bmatrix}\lambda_j\boldsymbol{I}-\boldsymbol{A} & \boldsymbol{B}\end{bmatrix} = n\ \forall\text{unstable}|\lambda_i| \geq 1$ of $\boldsymbol{A}$.
Intuition: Can reach origin in (at most) n steps.

2 Unconstrained Control

2.1 Block Approach (used also for $\bar{w}$ substition)

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{A} \\ \vdots \\ \boldsymbol{A}^N \end{bmatrix} x(0) + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{AB} & \boldsymbol{B} & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{0} \\ \boldsymbol{A}^{N-1}\boldsymbol{B} & \dots & \boldsymbol{AB} & \boldsymbol{B} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_1} \end{bmatrix}$$

$x = \boldsymbol{S}^x \cdot x(0) + \boldsymbol{S}^u \cdot u \quad \text{size}(\boldsymbol{S}^x) = [n_{\text{states}} \cdot (N+1), N]$
$\text{size}(\boldsymbol{S}^u) = [n_{\text{states}} \cdot (N+1), n_{\text{states}}]$
$\bar{\boldsymbol{Q}} = \text{diag}(\boldsymbol{Q},\dots,\boldsymbol{Q},\boldsymbol{P}) \quad \text{size}(\bar{\boldsymbol{Q}}) = [n_{\text{states}} \cdot (N+1), n_{\text{states}} \cdot (N+1)]$
$\bar{\boldsymbol{R}} = \text{diag}(\boldsymbol{R},\dots,\boldsymbol{R}) \quad \text{size}(\bar{\boldsymbol{R}}) = [n_{\text{input}} \cdot N, n_{\text{input}} \cdot N]$
$\boldsymbol{H} = \boldsymbol{S}^{uT}\bar{\boldsymbol{Q}}\boldsymbol{S}^u + \boldsymbol{R} \quad \boldsymbol{F} = \boldsymbol{S}^{xT}\bar{\boldsymbol{Q}}\boldsymbol{S}^u$
$\boldsymbol{Y} = \boldsymbol{S}^{xT}\bar{\boldsymbol{Q}}\boldsymbol{S}^x$

Optimal cost and control
$$J^*(x_0) = -x_0^T\boldsymbol{FHF}^T x_0 + x_0^T\boldsymbol{Y}x_0$$
$$u^*(x_0) = -\boldsymbol{H}^{-1}\boldsymbol{F}^T x_0 = -\left(\boldsymbol{S}^{uT}\bar{\boldsymbol{Q}}\boldsymbol{S}^u + \boldsymbol{R}\right)^{-1}\boldsymbol{S}^{uT}\bar{\boldsymbol{Q}}\boldsymbol{S}^x x_0$$

2.2 Recursive Approach

$$J_k^*(x_k) = \min_{u_k} I(x_k,u_k) + J_{k+1}(x_{k+1})$$

Is a feedback controller as opposed to the Batch Approach. For LQR solve via Riccati Difference Equation (RDE).
$$\boldsymbol{F}_k = -(\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{B} + \boldsymbol{R})^{-1}\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{A}$$
$$\boldsymbol{P}_k = \boldsymbol{A}^T\boldsymbol{P}_{k+1}\boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{A}^T\boldsymbol{P}_{k+1}\boldsymbol{B}(\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{B}+\boldsymbol{R})^{-1}\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{A}$$
$$u_k^* = \boldsymbol{F}_k\ x_k \qquad J_k^*(x_k) = x_k^T\boldsymbol{P}_k\ x_k \qquad \boldsymbol{P}_N = \boldsymbol{P}$$
For unconstrained Infinite Horizon Problem, substituting $\boldsymbol{P}_\infty = \boldsymbol{P}_k = \boldsymbol{P}_{k+1}$ into RDE gives DARE. Uniquely solvable, iff (A,B) stabilizable and (A,G) detectable, where $\boldsymbol{GG}^T = \boldsymbol{Q}$. Follows from closed-loop system $x_{k+1} = (\boldsymbol{A}+\boldsymbol{BF}_k)x_k$

3 (Convex) Optimization

General Problem $\min_{x\in\text{dom}(f)} f(x)$ s. t. $g_i(x) \leq 0$ and $h_j(x) = 0$.

Norm $f(x) : \mathbb{R}^n \to \mathbb{R}$
$$f(x) = 0 \Rightarrow x = 0, \qquad f(x) \geq 0$$
$$f(\alpha \cdot x) = |\alpha| \cdot f(x) \qquad \text{for scalar } \alpha$$
$$f(x+y) \leq f(x) + f(y) \qquad \forall x,y \in R^n$$

3.1 Convexity

Convex set $\mathcal{X}$ iff $\forall\lambda \in [0,1]\forall x,y \in \mathcal{X}\ \lambda x + (1-\lambda)y \in \mathcal{X}$. Intersection preserves convexity, union does not.

Affine set $\mathcal{X}$ $= \{x \in \mathbb{R}^n | \boldsymbol{A}x = b\}$ for some $\boldsymbol{A}, b$

Subspace is affine set through origin, i.e. $b = \boldsymbol{0}$, aka Nullspace of $\boldsymbol{A}$.

Hyperplane $\mathcal{X}$ $= \{x \in \mathbb{R}^n | a^T x = b\}$ for some a, b.

Halfspace $\mathcal{X}$ $= \{x \in \mathbb{R}^n | a^T x \leq b\}$ for some a, b.

Polyhedron $\mathcal{P}$ $= \{x | a_i^T x \leq b_i, i = 1,\dots,n\} = \{x | \boldsymbol{A}x \leq b\}$

Cone $\mathcal{X}$ if for all $x \in \mathcal{X}$, and for all $\theta > 0, \theta x \in \mathcal{X}$.

Ellipsoid $\mathcal{E}$ $= \{x | (x-x_c)^T\boldsymbol{A}^{-1}(x-x_c) \leq 1\}$, x_c center point.

Convex function $f : \mathbf{dom}(f) \to \mathbb{R}$ is convex iff $\text{dom}(f)$ is convex and $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y), \forall\lambda \in (0,1), \forall x,y \in \text{dom}(f)$.

Norm ball is convex (for all norms).

Epigraph set $f : \mathbf{dom}(f) \to \mathbb{R}$ is the set
$$\text{epi}(f) := \left\{\begin{bmatrix} x \\ t \end{bmatrix} | x \in \text{dom}(f), f(x) \leq t\right\} \subseteq \text{dom}(f) \times \mathbb{R}$$

Level set L_a of a function f for value a is the set of all $x \in \text{dom}(f)$ for which $f(x) = a$: $L_a = \{x | x \in \text{dom}(f), f(x) = a\}$.

Sublevel set C_a is defined by $C_a = \{x | x \in \text{dom}(f), f(x) \leq a\}$.

3.2 Linear Programming (LP)

Problem statement $\min c^T x$ such that $\boldsymbol{G}x \leq h$ and $\boldsymbol{A}x = b$.

Norm l_∞ $\min_x \|x\|_\infty = \min_{x\in\mathbb{R}^n} [\max\{x,\dots,x_n,-x_1,\dots,-x_n\}]$:
$$\min_{x,t} t \quad \text{subject to} \quad x_i \leq t, -x_i \leq t, \quad \boldsymbol{F}x \leq g$$
$$\Leftrightarrow \min_{x,t} t \quad \text{subject to} \quad -\boldsymbol{1}t \leq x \leq \boldsymbol{1}t, \quad \boldsymbol{F}_x \leq g.$$

Norm l_1 $\min_x \|x\|_1 = \min_x \left[\sum_{i=1}^m \max\{x_i,-x_i\}\right]$:
$$\min_t t_1 + \dots + t_m \quad \text{subject to} \quad x_i \leq t_i, -x_i \leq t_i, \quad \boldsymbol{F}x \leq g$$
$$\Leftrightarrow \min_t \boldsymbol{1}^T t \quad \text{subject to} \quad -t \leq x \leq t, \quad \boldsymbol{F}_x \leq g.$$
Note that for $\dim x = 1$, l_1 and l_∞ are the same. Note also that t is scalar for norm l_∞ and a vector in norm l_1.

Piecewise Affine
$$\min_x \left[\max_{i=1,\dots,m}\{c_i^T x + d_i\}\right] \quad \text{s.t. } \boldsymbol{G}x \leq h$$
$$\Leftrightarrow \min_{x,t} t \quad \text{s.t. } c_i^T x + d_i \leq t, \boldsymbol{G}x \leq h$$

3.3 Duality

Lagrangian Dual Function
$$L(x,\lambda,\nu) = f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{i=1}^p \nu_i h_i(x)$$
$$d(\lambda,\nu) = \inf_{x\in\mathcal{X}} L(x,\lambda,\nu) \quad \text{i.e. } \nabla_x L(x,\lambda,\nu) = 0$$

Dual Problem (always convex) $\max_{\lambda,\nu} d(\lambda,\nu)$ s. t. $\lambda \geq 0$.
Optimal value is lower bound for primal: $d^* \leq p^*$.
If primal convex, Slater condition (strict feasibility) implies *strong duality*: $\{x|\ Ax = b, f_i(x) < 0,\} \neq \emptyset \Rightarrow d^* = p^*$

Karush-Kuhn-Tucker (KKT) Conditions are necessary for optimality (and sufficient if primal convex).

$$
\begin{array}{lll}
\text{Primal Feasability} & f_i(x^*) \le 0 & i = 1, \dots, m \\
 & h_i(x^*) = 0 & i = 1, \dots, p \\
\text{Dual Feasability} & \lambda^* \ge 0 & \\
\text{Complementary slackness} & \lambda_i^* \cdot f_i(x^*) = 0 & i = 1, \dots, m \\
\text{Stationarity} & \nabla_x L(x^*, \lambda^*, \nu^*) = 0 &
\end{array}
$$

3.4 Dual of LP

$$\min_x c^T x \quad \text{s.t. } \boldsymbol{A}x = b, \boldsymbol{C}x \le e$$

$$\Leftrightarrow \max_{\lambda,\nu} -b^T\nu - e^T\lambda \quad \text{s.t. } \boldsymbol{A}^T\nu + \boldsymbol{C}^T\lambda + c = 0, \lambda \ge 0$$

$\min_x c^T x$ subj. to $\boldsymbol{A}x = b, \boldsymbol{C}x \le e$.

Lagrangian $L(x, \lambda, \nu) = c^T x + \lambda^T(\boldsymbol{A}x - b) + \nu^T(\boldsymbol{C}x - e)$

Dual function

$$
\begin{aligned}
d(\lambda, \nu) &= \min_x L(x, \lambda, \nu) = \min_x (\boldsymbol{A}^T\nu + \boldsymbol{C}^T\lambda + c)^T x - b^T\nu - e^T\lambda \\
&= \begin{cases} -b^T\nu - e^T\lambda & \text{if } \boldsymbol{A}^T\nu + \boldsymbol{C}^T\lambda + c = 0 \\ -\infty & \text{otherwise.} \end{cases}
\end{aligned}
$$

Lower bound $-b^T\nu - e^T\lambda \le p^*$ if $\boldsymbol{A}^T\nu + \boldsymbol{C}^T\lambda + c = 0$ and $\lambda \ge 0$.

3.4.1 Ex. minimize norm

Primal $\min_x \|x\|_2 \quad$ s.t. $\boldsymbol{A}x = b$

Lagrangian $(x, \lambda, \nu) = \|x\|_2 - (\boldsymbol{A}^T\nu)^T x + b^T\nu$

Dual function $d(\nu) = \min_x L(x, \lambda) = \begin{cases} b^T\nu & \text{if } \|\boldsymbol{A}^T\nu\|_2 \le 1 \\ -\infty & \text{otherwise} \end{cases}$

Dual $\max_\nu b^T\nu$ s.t. $\|\boldsymbol{A}^T\nu\|_2 \le 1$.

3.5 Dual of QP

Simple case

$$\min_x \frac{1}{2}x^T\boldsymbol{Q}x + c^T x \quad \text{s.t. } \boldsymbol{C}x \le e$$

$$\Leftrightarrow \min_{\lambda,\nu} \frac{1}{2}\lambda^T\boldsymbol{C}\boldsymbol{Q}^{-1}\boldsymbol{C}^T\lambda + (\boldsymbol{C}\boldsymbol{Q}^{-1}c + e)^T\lambda + \frac{1}{2}c^T\boldsymbol{Q}^{-1}c$$

$$\text{s.t. } \boldsymbol{Q}x + \nu + c^T\lambda = 0, \lambda \ge 0$$

Tim: I'm pretty sure you don't need the constraint, except for $\lambda \ge 0$..

General Case $(Q > 0)$

$$\min_x \frac{1}{2}x^T\boldsymbol{Q}x + c^T x \quad \text{s.t. } \boldsymbol{A}x = b, \boldsymbol{C}x \le e$$

$$\Leftrightarrow \min_{\lambda,\nu} \frac{1}{2}\begin{bmatrix}\nu & \lambda\end{bmatrix}\bar{\boldsymbol{Q}}\begin{bmatrix}\nu \\ \lambda\end{bmatrix} + \bar{c}^T\begin{bmatrix}\nu \\ \lambda\end{bmatrix} + \bar{k} \quad \text{s.t. } \lambda \ge 0.$$

Lagrangian $L(x, \lambda, \nu) = \frac{1}{2}x^T\boldsymbol{Q}x + c^T x + \nu^T(\boldsymbol{A}x - b) + \lambda^T(\boldsymbol{C}x - e)$

Dual function

$d(\lambda, \nu) = \min_x \frac{1}{2}x^T\boldsymbol{Q}x + (\boldsymbol{A}^T\nu + \boldsymbol{C}^T\lambda + c)^T x b^T\nu - e^T\lambda$

Minimize $\Delta_x L(x, \lambda, \nu) = 0$ gives:

$$0 = \boldsymbol{Q}x + \boldsymbol{A}^T\nu + \boldsymbol{C}^T\lambda + c$$

$$\Leftrightarrow x = -\boldsymbol{Q}^{-1}(\boldsymbol{A}^T\nu + \boldsymbol{C}^T\lambda + c)$$

$$
\begin{aligned}
d(\lambda, \nu) =& \frac{1}{2}(-\boldsymbol{Q}^{-1}(\boldsymbol{A}^T\nu + \boldsymbol{C}^T\lambda + c))^T\boldsymbol{Q}(\boldsymbol{Q}^{-1}(\boldsymbol{A}^T\nu + \boldsymbol{C}^T\lambda + c)) \\
&+ (\boldsymbol{A}^T\nu + \boldsymbol{C}^T\lambda + c)^T(-\boldsymbol{Q}^{-1})(\boldsymbol{A}^T\nu + \boldsymbol{C}^T\lambda + c) - b^T\nu - e^T\lambda \\
=& -\frac{1}{2}(\boldsymbol{A}^T\nu + \boldsymbol{C}^T\lambda + c)^T\boldsymbol{Q}^{-1}(\boldsymbol{A}^T\nu + \boldsymbol{C}^T\lambda + c) - b^T\nu - e^T\lambda \\
=& -\frac{1}{2}(\nu^T\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T\nu + \lambda^T\boldsymbol{C}\boldsymbol{Q}^{-1}\boldsymbol{C}^T\lambda + c^T\boldsymbol{Q}^{-1}c) \\
&- (\boldsymbol{A}\boldsymbol{Q}^{-1}c)^T\nu - \nu^T\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{C}^T\lambda - (\boldsymbol{C}\boldsymbol{Q}^{-1}c)^T\lambda - b^T\nu - e^T\lambda \\
=& -\frac{1}{2}(\nu^T\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T\nu + \lambda^T\boldsymbol{C}\boldsymbol{Q}^{-1}\boldsymbol{C}^T\lambda) - \nu^T\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{C}^T\lambda \\
&- (\boldsymbol{A}\boldsymbol{Q}^{-1}c + b)^T\nu - (\boldsymbol{C}\boldsymbol{Q}^{-1}c + e)^T\lambda - \frac{1}{2}c^T\boldsymbol{Q}^{-1}c \\
=& -\frac{1}{2}\begin{bmatrix}\nu & \lambda\end{bmatrix}\bar{\boldsymbol{Q}}\begin{bmatrix}\nu \\ \lambda\end{bmatrix} - \bar{c}^T\begin{bmatrix}\nu \\ \lambda\end{bmatrix} - \bar{k}
\end{aligned}
$$

$$\bar{\boldsymbol{Q}} = \begin{bmatrix}\boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{A}^T & \boldsymbol{A}\boldsymbol{Q}^{-1}\boldsymbol{C}^T \\ \boldsymbol{C}^T\boldsymbol{Q}^{-1}\boldsymbol{A} & \boldsymbol{C}\boldsymbol{Q}^{-1}\boldsymbol{C}^T\end{bmatrix}$$

$$\bar{c} = \begin{bmatrix}(\boldsymbol{A}\boldsymbol{Q}^{-1}c + b)^T \\ (\boldsymbol{C}\boldsymbol{Q}^{-1}c + e)^T\end{bmatrix} \qquad \bar{k} = \frac{1}{2}c^T\boldsymbol{Q}^{-1}c.$$

Trick: $\max d(\lambda, \nu)$ becomes $\min -d(\lambda, \nu)$.

4 Constrained Finite Time Optimal Control (CFTOC)

4.1 MPC with linear cost

$$J(x_0, u) = \|\boldsymbol{P}x_N\|_p + \sum_{i=0}^{N-1}\|\boldsymbol{Q}x_i\|_p + \|\boldsymbol{R}u_i\|_p.$$

The CFTOC problem can be formulated as an ∞-norm LP problem as shown below.

$$
\begin{aligned}
\min_z\ & \epsilon_0^x + \dots + \epsilon_N^x + \epsilon_0^u + \dots + \epsilon_{N-1}^u \\
\text{s.t. }\ & -\boldsymbol{1}_n\epsilon_i^x \le \pm\boldsymbol{Q}\left[\boldsymbol{A}^i x_0 + \sum_{j=0}^{i-1}\boldsymbol{A}^j\boldsymbol{B}u_{i-1-j}\right] \\
& -\boldsymbol{1}_r\epsilon_N^x \le \pm\boldsymbol{P}\left[\boldsymbol{A}^N x_0 + \sum_{j=0}^{N-1}\boldsymbol{A}^j\boldsymbol{B}u_{N-1-j}\right] \\
& -\boldsymbol{1}_m\epsilon_N^u \le \pm\boldsymbol{R}u_i \\
& x_i = \boldsymbol{A}^i x_0 + \sum_{j=0}^{i-1}\boldsymbol{A}^j\boldsymbol{B}u_{i-1-j} \in \mathcal{X} \\
& x_N = \boldsymbol{A}^N x_0 + \sum_{j=1}^{N-1}\boldsymbol{A}^j\boldsymbol{B}u_{N-1-j} \in \mathcal{X} \\
& u_i \in \mathcal{U}
\end{aligned}
$$

Converting to LP form:

$$
\begin{aligned}
&\min_z\ c^T z \\
&\text{s.t. } \bar{\boldsymbol{G}}z \le \bar{w} + \bar{s}x_k \\
&z = \begin{bmatrix}\epsilon_0^x & \dots & \epsilon_N^x & \epsilon_0^u & \dots & \epsilon_{N-1}^u & u_0^T & \dots & u_{N-1}^T\end{bmatrix} \\
&c = \begin{bmatrix}1 & \dots & 1 & 1 & \dots & 1 & 0 & \dots & 0\end{bmatrix} \\
&\bar{\boldsymbol{G}} = \begin{bmatrix}\boldsymbol{G}_\epsilon & \boldsymbol{G}_u \\ 0 & \boldsymbol{G}\end{bmatrix}, \quad \bar{w} = \begin{bmatrix}w_\epsilon \\ w\end{bmatrix}, \quad \bar{s} = \begin{bmatrix}s_\epsilon \\ s\end{bmatrix}
\end{aligned}
$$

Where $\boldsymbol{G}$ is the normal problem constraints and $[\boldsymbol{G}_\epsilon\boldsymbol{G}_u]$ form the constraints of the newly introduced variable ϵ as given in the first 3 constraints in the section above. For example, we require:

$$
\begin{aligned}
-\epsilon_i^u &\le u_i \le \epsilon_i^u \\
-\epsilon_0^x &\le Ax_0 + Bu_0 \le \epsilon_0^x \\
-\epsilon_1^x &\le A^2x_0 + Bu_1 + ABu_0 \le \epsilon_1^x
\end{aligned}
$$

4.2 QP with substitution (see also Batch approach)

$$J^*(x_k) = \min_u \begin{bmatrix}u^T & x_k^T\end{bmatrix}\begin{bmatrix}\boldsymbol{H} & \boldsymbol{F}^T \\ \boldsymbol{F} & \boldsymbol{Y}\end{bmatrix}\begin{bmatrix}u \\ x_k\end{bmatrix}$$

$$\text{s. t. } \boldsymbol{G}\ u \le w + \boldsymbol{E}\ x_k$$

Latter gives three sets (same for without substitution)

$$
\begin{aligned}
\mathcal{X} &= \{x | A_x\ x \le b_x\} \\
\mathcal{U} &= \{u | A_u\ u \le b_u\} \\
\mathcal{X}_f &= \{x | A_f\ x \le b_f\}
\end{aligned}
$$

State equations are in cost matrix, usually in the form:

$\boldsymbol{A}_x = \begin{bmatrix}1 \\ -1\end{bmatrix}, b_x = \begin{bmatrix}b_{\text{max}} \\ -b_{\text{min}}\end{bmatrix}$

$$
\boldsymbol{G} = \begin{bmatrix}
\boldsymbol{A}_u & 0 & \dots & 0 \\
0 & \boldsymbol{A}_u & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & \boldsymbol{A}_u \\
0 & 0 & \dots & 0 \\
\boldsymbol{A}_x\boldsymbol{B} & 0 & \dots & 0 \\
\boldsymbol{A}_x\boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}_x\boldsymbol{B} & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\boldsymbol{A}_x\boldsymbol{A}^{N-2}\boldsymbol{B} & \boldsymbol{A}_x\boldsymbol{A}^{N-3}\boldsymbol{B} & \dots & 0 \\
\boldsymbol{A}_f\boldsymbol{A}^{N-1}\boldsymbol{B} & \boldsymbol{A}_f\boldsymbol{A}^{N-2}\boldsymbol{B} & \dots & \boldsymbol{A}_f\boldsymbol{B}
\end{bmatrix}
\quad
\boldsymbol{E} = \begin{bmatrix}
0 \\ 0 \\ \vdots \\ 0 \\ -\boldsymbol{A}_x \\ -\boldsymbol{A}_x\boldsymbol{A} \\ -\boldsymbol{A}_x\boldsymbol{A}^2 \\ \vdots \\ -\boldsymbol{A}_x\boldsymbol{A}^{N-1} \\ -\boldsymbol{A}_f\boldsymbol{A}^N
\end{bmatrix}
\quad
\boldsymbol{W} = \begin{bmatrix}
b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_x \\ b_f
\end{bmatrix}
$$

4.3 QP without substitution

State equations represented in equality constraints (k fixed, usually $k = 0$).

$$J^*(x_k) = \min_z \begin{bmatrix}z^T & x_k^T\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{H}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q}\end{bmatrix}\begin{bmatrix}z \\ x_k\end{bmatrix}$$

$$
\begin{aligned}
\text{s.t. } & \boldsymbol{G}\ z \le w + \boldsymbol{E}\ x_k \\
& \boldsymbol{G}_{\text{eq}}\ z = \boldsymbol{E}_{\text{eq}}\ x_k, \quad \text{system dynamics}
\end{aligned}
$$

$\bar{\boldsymbol{H}} = \text{diag}(\boldsymbol{Q}, \dots, \boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{R}, \dots, \boldsymbol{R})$

$$
z = \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}
\quad
\boldsymbol{G}_{\text{eq}} = \left[\begin{array}{cccc|cccc}
\boldsymbol{I} & & & & -\boldsymbol{B} & & & \\
-\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B} & & \\
 & \ddots & \ddots & & & & \ddots & \\
 & & -\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B}
\end{array}\right]
\quad
\boldsymbol{E}_{\text{eq}} = \begin{bmatrix} \boldsymbol{A} \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

$$
w = \begin{bmatrix} b_x \\ \vdots \\ b_f \\ b_u \\ \vdots \\ b_u \end{bmatrix}
\quad
\boldsymbol{G} = \left[\begin{array}{cccc|ccc}
0 & & & & & & \\
 & \boldsymbol{A}_x & & & & & \\
 & & \ddots & & & & \\
 & & & \boldsymbol{A}_x & & & \\
\hline
 & & & & \boldsymbol{A}_d & & \\
 & & & & & \ddots & \\
 & & & & & & \boldsymbol{A}_d
\end{array}\right]
\quad
\boldsymbol{E} = \begin{bmatrix} -\boldsymbol{A}_x^T \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

4.4 Invariance

Pos. Invariant set O iff $x_k \in O \Rightarrow x_{k+1} = g(x_k) \in O\ \forall k$.

Max. Pos. Invariant set $O_\infty \subset \mathcal{X}$ iff $0 \in O_\infty$, O_∞ invariant and contains all invariant sets O with $0 \in O$.

Pre-Set $\text{pre}(S)$ $:= \{x | g(x) \in S\} = \{x | \boldsymbol{A}x \in S\}$
Linear systems: $S = \{x | \boldsymbol{F}x \le f\} \Rightarrow \text{pre}(S) = \{x | \boldsymbol{F}\boldsymbol{A}x \le f\}$.
Note: O invariant $\Leftrightarrow O \subseteq \text{pre}(O) \Leftrightarrow \text{pre}(O) \cap O = O$.

Calculate max. invariant set by $\Omega_{i+1} \leftarrow \text{pre}(\Omega_i) \cap \Omega_i$, terminating when $\Omega_{i+1} = \Omega_i$, starting with $\Omega_0 = \mathcal{X}$.

4.5 Stability and Feasability

Main Idea Choose $\mathcal{X}_f$ and $\boldsymbol{P}$ to mimic infinite horizon. LQR control law $\kappa(x) = \boldsymbol{F}_\infty x$ from solving DARE. Set terminal weight $\boldsymbol{P} = \boldsymbol{P}_\infty$, terminal set $\mathcal{X}_f$ as maximal invariant set:

$$x_{k+1} = \boldsymbol{A}x_k + \boldsymbol{B}\boldsymbol{F}_\infty\, x_k \in \mathcal{X}_f \quad \forall x_k \in \mathcal{X}_f \text{ terminal set invariant}$$
$$\mathcal{X}_f \subseteq \mathcal{X}, \quad \boldsymbol{F}_\infty\, x_k \in \mathcal{U} \quad \forall x_k \in \mathcal{X}_f \text{ constrainst satisfied}$$

Result

1. Positive stage cost function,
2. invariant terminal set by construction and
3. Terminal cost is Lyapunov function with $$x_{k+1}^T \boldsymbol{P} x_{k+1} - x_k^T \boldsymbol{P} x_k = -x_k^T(\boldsymbol{Q} + \boldsymbol{F}_\infty^T \boldsymbol{R} \boldsymbol{F}_\infty)x_k$$

Extension to non-linear (time-invariant) MPC possible since terminal set and cost do not rely on linearity.

5 Practical Issues

5.1 MPC for tracking

Target steady-state conditions $x_s = \boldsymbol{A}x_s + \boldsymbol{B}u_s$ and $y_s = \boldsymbol{C}x_s = r$ and constraints give:

$$\min_{x_s,u_s} u_s^T \boldsymbol{R} u_s \text{ subj. to } \begin{bmatrix} \boldsymbol{I}-\boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ r \end{bmatrix}, x_s \in \mathcal{X}, u_s \in \mathcal{U}$$

Usually assume x_s, u_s unique and feasible. If no solution exists, compute closest steady-state $\min(\boldsymbol{C}x_s - r)^T \boldsymbol{Q}(\boldsymbol{C}x_s - r)$ s. t. $x_s = \boldsymbol{A}x_s + \boldsymbol{B}u_s$.

MPC problem to drive $y \to r$ is:

$$\min_u \|y_N - \boldsymbol{C}x_s\|_{P_y}^2 + \sum_{i=0}^{N-1} \|y_i - \boldsymbol{C}x_s\|_{Q_y}^2 + \|u_i - u_s\|_R^2$$

5.2 Delta formulation

Reference r, $\Delta x_k = x_k - x_s, \Delta u_k = u_k - u_s$:

$$\begin{aligned}
&\min V_f(\Delta x_N) + \sum_{i=0}^{N-1} \Delta x_i^T \boldsymbol{Q} \Delta x_i + \Delta u_i^T \boldsymbol{R} \Delta u_i \\
&\text{s.t. } \Delta x_0 = \Delta x_k \\
&\quad \Delta x_{k+1} = \boldsymbol{A}\Delta x_k + \boldsymbol{B}\Delta u_k \\
&\boldsymbol{H}_x x \le k_x \Rightarrow \boldsymbol{H}_x \Delta x \le k_x - \boldsymbol{H}_x x_s \\
&\boldsymbol{H}_u u \le k_u \Rightarrow \boldsymbol{H}_u \Delta u \le k_u - \boldsymbol{H}_u u_s \\
&\quad \Delta x_N \in \mathcal{X}_f \quad \text{adjusted accordingly, shift (and scaled)} \\
&\quad x_s \oplus \mathcal{X}_f \subseteq \mathcal{X} \\
&\quad \boldsymbol{K}\Delta x + u_s \in \mathcal{U}
\end{aligned}$$

Control given by $u_0^* = \Delta u_0^* + u_s$.

5.3 Offset free tracking

$$\begin{aligned}
x_{k+1} &= \boldsymbol{A}x_k + \boldsymbol{B}u_k + \boldsymbol{B}_d d_k \\
d_{k+1} &= d_k \\
y_k &= \boldsymbol{C}x_k + \boldsymbol{C}_d d_k
\end{aligned}$$
$$\begin{bmatrix} \boldsymbol{I}-\boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_d \hat{d} \\ r - \boldsymbol{C}_d \hat{d} \end{bmatrix}$$

Choice of $\boldsymbol{B}_d, \boldsymbol{C}_d$ requires that $(\boldsymbol{A}, \boldsymbol{C})$ is observable and $\begin{bmatrix} \boldsymbol{A}-\boldsymbol{I} & \boldsymbol{B}_d \\ \boldsymbol{C} & \boldsymbol{C}_d \end{bmatrix}$ has full $(n_x + n_d)$ column frank (i.e. det $\neq 0$). Intuition: for fixed y_s at steady-state, d_s is uniquely determined.

If plant has no integrator we can choose $\boldsymbol{B}_d = \boldsymbol{0}$ since $\det(\boldsymbol{A} - \boldsymbol{I}) \neq 0$.

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}_d \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} u_k + \begin{bmatrix} \boldsymbol{L}_x \\ \boldsymbol{L}_d \end{bmatrix} \left(-y_k^m + \boldsymbol{C}\hat{x}_k + \boldsymbol{C}_d \hat{d}_k\right)$$

where y_k^m measured output; choose $\begin{bmatrix} \boldsymbol{L}_x \\ \boldsymbol{L}_d \end{bmatrix}$ s.t. error dynamics stable and converge to zero.

If 1) number of dist. = number of outputs, 2) target steady-state problem feasible and no constraints active at steady-state, 3) closed-loop system converges, then target achieved without offset.

Extend *Delta formulation* from above with

$$\begin{aligned}
\Delta x_{k+1} &= \boldsymbol{A}\Delta x_k + \boldsymbol{B}\Delta u_k + \boldsymbol{B}_d \Delta d_k \\
\Delta d_{k+1} &= \Delta d_k
\end{aligned}$$

Algorithm becomes:

1. Estimate state and disturbance $\hat{x}$, $\hat{d}$,
2. Obtain (x_s, u_s) target condition,
3. Solve MPC problem (adapted Delta formulation)

Theorem Case $n_d = n_y$ and RHC is recursively feasible and unconstrained for $k \ge j$ for some $j \in \mathbb{N}$ and closed-loop converges, it converges to reference, i.e. $y_k^m \to r$.

5.4 Soft-constraints via slack variables

$$\min_x f(z) + l_\epsilon(\epsilon) \quad \text{s.t. } g(z) \le \epsilon, \epsilon \ge 0$$

Requirement: Softened problem has same minimiser as original problem if feasible.

Quadratic error function $l_\epsilon(\epsilon) = v\epsilon + w\epsilon^2$, $w > 0$ gives smoothness, choose $v > \lambda^* \ge 0$ for exact penalty (above requirement fulfilled).

5.5 Move Blocking

Main idea to set a number of inputs as the same, $u_2 = u_3 = \cdots = u_N$, to reduce computational burden, at the slight cost of sub-optimality.

6 Robust MPC

Enforcing terminal constraints by robust invariance:

$$x \in O^{\mathcal{W}} \Rightarrow g(x, w) \in \Omega^W \ \forall w \in \mathcal{W}$$
$$\text{pre}^{\mathcal{W}}(\Omega) = \{x | g(x, w) \in \Omega \ \forall w \in \mathcal{W}\}$$

Enforcing sequential constraints for uncertain system ϕ:

$$\phi_i(x_0, u, w) = \left\{ x_i + \sum_{j=0}^{i-1} \boldsymbol{A}^j w_j \middle| w \in \mathcal{W}^i \right\} \subseteq \mathcal{X}$$
$$\phi_N(x_0, u, w) \in \mathcal{X}_f \quad \text{as well}$$

The uncertain system evolves with the summation of all the disturbances up to time i, hence we have to restrict the open-loop (determine control before disturbance is measured):

$$\boldsymbol{A}_x x \le b_x \text{ becomes } \boldsymbol{A}_x x_i + \boldsymbol{A}_x \sum_{j=0}^{i-1} \boldsymbol{A}^j w_k \le b_x :$$
$$x_i \in \mathcal{X} \ominus \left(\mathcal{W} \oplus \boldsymbol{A}\mathcal{W} \oplus \cdots \oplus \boldsymbol{A}^{i-1}\mathcal{W}\right)$$
$$= \left(\bigoplus_{j=0}^{i-1} \boldsymbol{A}^j \mathcal{W}\right) = \begin{bmatrix} \boldsymbol{A}^0 & \dots & \boldsymbol{A}^{i-1} \end{bmatrix} \mathcal{W}^i$$

For example: Robust invariant set calculation of $x_{k+1} = 0.5x_k + w_k$ under $-10 \le x \le 10$ and $-1 \le w \le 1$.

$$\Omega_0 = [-10, 10]$$
$$\begin{aligned}
\text{pre}^{\mathcal{W}}(\Omega_0) &= \{x | -10 \le 0.5x + w \le 10 \text{ for } -1 \le w \le 1\} \\
&= \{x | -20 - 2w \le x \le 20 + 2w \text{ for } -1 \le w \le 1\} \\
&= \{x | -18 \le x \le 18\}
\end{aligned}$$
$$\Omega_1 = [-10, 10] \cap [-18, 18] = [-10, 10] = \mathcal{O}_\infty^{\mathcal{W}}$$

For example: Terminal set calculation of $x_{k+1} = w_k$, $-1 \le w \le 1$, N= 2:

$$\mathcal{X}_f^{\mathcal{W}} = \mathcal{X}_f \ominus \left(\bigoplus_{j=0}^{1} \boldsymbol{A}^j \mathcal{W}\right) = \mathcal{X}_f \ominus 2\mathcal{W} = [-10, 10] \ominus [-2, 2] = [-8, 8]$$

Tube-MPC We want nominal system $z_k = \boldsymbol{A}z_k + \boldsymbol{B}v_k$ with "tracking" controller $u_k = \boldsymbol{K}(x_k - z_k) + v_k$ i.e. closed-loop, $\boldsymbol{K}$ found offline.

Step 1: Compute the minimal robust invariant set $\mathcal{E} = \bigoplus_{j=1}^{\infty} \boldsymbol{A}_{cl}^j \mathcal{W}$.

Step 2: Shrink Constraints:

$$\begin{aligned}
\{z_i\} \oplus \mathcal{E} \subseteq \mathcal{X} \quad &\Rightarrow \{z_i\} \in \mathcal{X} \ominus \mathcal{E} \\
u_i \in \boldsymbol{K}\mathcal{E} \oplus \{v_i\} \subset \boldsymbol{U} \quad &\Rightarrow \{v_i\} \in \mathcal{U} \ominus \boldsymbol{K}\mathcal{E} \\
z_n \in \mathcal{X}_f \ominus \mathcal{E} &
\end{aligned}$$

Also check that the set $\mathcal{X}_f$ is invariant for the nominal system with tightened constraints: $(A + BK)\mathcal{X}_f \subseteq \mathcal{X}_f$, and that it satisfies the constraints: $\mathcal{X}_f \subseteq \mathcal{X} \ominus \mathcal{E}$ and $K\mathcal{X}_f \subseteq \mathcal{U} \ominus K\mathcal{E}$.

7 Explicit MPC

$z^*(x_k)$ is continuous and polyhedral piecewise affine over feasible set.

7.1 Quadratic Cost

$J^*(x_k)$ is continuous, convex and polyhedral piecewise quadratic. Using the formulation with substitution,

$$\begin{aligned}
J(x_k) &= \min z^T Hz - x_k^T(Y - FH^{-1}F^T)x_k \\
&\text{s.t} \quad Gz \le w + Sx_k \\
z(x_k) &= U + H^{-1}F^T x_k \\
S &= E + GH^{-1}F^T \\
U^* &= z^*(x_k) - H^{-1}F^T x_k
\end{aligned}$$

The first solution gives $u^*(x_k) = \kappa(x_k)$, which is continuous and piecewise affine on polyhedra $\kappa(x) = F_j x + g_j$.

7.2 $1/\infty$-norm

$J^*(x_k)$ is continuous, convex and polyhedral piecewise affine. Optimal solution: $u_0^* = \begin{bmatrix} 0 & \dots & 0 & I_m & 0 & \dots & 0 \end{bmatrix} z^*(x_k)$, and is in the same form as the QP case above.

7.3 Explicit Example

1. Write out KKT conditions and Lagrangian.
2. Determine infeasible regions from primal feasibility constraints. For example, $x1 < 10$.
3. From primal and dual feasibility, and complementary slackness conditions, list out all cases that can occur.
$$\begin{aligned} &\lambda_1 = 0 &\quad &\lambda_1 \ge 0 \\ &g_1(x) < 0 &\quad &g_1(x) = 0 \end{aligned}$$
4. Solve for each case: $z^*(x_1, x_2)$ and $J^*(x_1, x_2)$, listing the active constraints, and range of validity.

8 Hybrid MPC

8.1 Piecewise Affine (PWA) Systems

Affine dynamics and output for each region:

$$\begin{cases} x_{k+1} &= A^i x_k + B^i u_k + f^i \\ y_k &= C^i x_k + D^i u_k + g_i \end{cases} \text{ if } x_t \in \mathcal{X}_j$$

Polyhedral partition of the (x, u)-space:

$$\{\mathcal{X}_i\}_{j=1}^s = \{x, u | H_j x + J_j u \leq K_j\}$$

8.2 Mixed Logical Dynamical Hybrid Model (MLD)

Idea associate boolean to binary: $p_j \Leftrightarrow \delta_i = 1, \neg p_j \Leftrightarrow \delta_i = 0$.

Goal Given a boolean formula $F(p_1, \dots, p_n)$ define polyhedral set P s.t. set of binary values $\{\delta_1, \dots, \delta_n\}$ satisfies Boolean formula F in P

$$F(p_1, \dots, p_n) \Leftrightarrow \boldsymbol{A}\delta \leq b, \delta \in \{0,1\}^n.$$

8.3 Analytical Approach

1. Transform into Conjunctive Normal Form (CNF), i.e. $F(p_1, \dots, p_n) = \bigvee_m \left[\bigwedge_j p_j\right]$.
2. Translate CNF into algebraic inequalities.

Translate logic rules to Linear Integer Inequalities

AND	$p_1 \wedge p_2$	$\delta_1 \geq 1, \delta_1 \geq 1$ also $\delta_1 + \delta_2 \geq 2$
OR	$p_1 \vee p_2$	$\delta_1 + \delta_2 \geq 1$
NOT	$\neg p_1$	$1 - \delta_1 \geq 1$ also $\delta_1 = 0$
XOR	$p_1 + p_2$	$\delta_1 + \delta_2 = 1$
IMPLY	$p_1 \rightarrow p_2$	$\delta_1 - \delta_2 \leq 0$
IFF	$p_1 \leftrightarrow p_2$	$\delta_1 - \delta_2 = 0$
ASSIGN	$p_3 \leftrightarrow p_1 \wedge p_2$	$\delta_1 + (1 - \delta_3) \geq 1$ and
$p_3 = p_1 \wedge p_2$		$\delta_2 + (1 - \delta_3) \geq 1$ and
		$(1 - \delta_1) + (1 - \delta_2) + \delta_3 \geq 1$
CNF-Clause 0	$p_1 \vee p_2 \vee p_3$	$\delta_1 + \delta_2 + \delta_3 \geq 1$
CNF-Clause 1	$\neg p_1 \vee p_2 \vee p_3$	$\delta_1 - \delta_2 - \delta_3 \leq 0$
CNF-Clause 2	$\neg p_1 \vee \neg p_2 \vee p_3$	$\delta_1 + \delta_2 - \delta_3 \leq 1$
CNF-Clause 3	$\neg p_1 \vee \neg p_2 \vee \neg p_3$	$\delta_1 + \delta_2 + \delta_3 \leq 2$

Logic Equality Rules (for Jenwei)

$$\neg(A \wedge B) = \neg A \vee \neg B$$

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

8.3.1 Translate continuous and logical components into Linear Mixed-Integer Relations

Event generator: $\delta_e(k) = f_{EG}(x_c(k), u_c(k), t)$.
Consider: $p \Leftrightarrow a^T x \leq b, \mathcal{X} = \{x | a^T x - b \in [m, M]\}$.
Translated to linear inequalities: $m\delta < a^T x - b \leq M(1 - \delta)$, where $[m, M]$ are lower and upper bounds.

Representing Switched Affine Dynamics as "IF-THEN-ELSE" relations

IF p THEN $z_k = a_1^T x_k + b_1$ else $z_k = a_2^T x_k + b_2 \Leftrightarrow$

$$(m_2 - M_1)\delta + z_k \leq a_2^T x_k + b_2 \leq -(m_1 - M_2)\delta + z_k$$

$$(m_1 - M_2)(1 - \delta) + z_k \leq a_1^T x_k + b_1 \leq -(m_1 - M_2)(1 - \delta) + z_k$$

This results in a linear MLD model

$$x_{k+1} = A x_k + B_1 u_k + B_2 \delta_k + B_3 z_k$$

$$y_k = C x_k + D_1 u_k + D_2 \delta_k + D_3 z_k$$

$$E_2 \delta_k + E_3 z_k \leq E_4 x_k + E_1 u_k + E_5$$

where the last equation describes the relationship between the continuous and integer variables. Physical constraints on cont. variables: $\{\mathcal{C}\} = \left\{ \begin{bmatrix} x_c \\ u_c \end{bmatrix} \in \mathcal{R}^{n_c + m_c} | F x_c + G u_c \leq H \right\}$

8.4 CFTOC for Hybrid Systems

$$J^*(x) = \min_U \; l_N(x_N) + \sum_{k=0}^{N-1} l(x_k, u_k, \delta_k, z_k)$$

$$\begin{aligned} \text{s.t } & x_{k+1} = A x_k + B_1 u_k + B_2 \delta_k + B_3 z_k \\ & E_2 \delta_k + E_3 z_k \leq E_4 x_k + E_1 u_k + E_5 \\ & x_N \in \mathcal{X}_f, x_0 = x(0) \end{aligned}$$

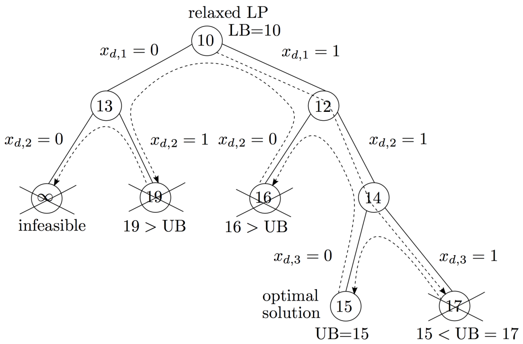
8.5 MILP/MLQP

$$\begin{aligned} \min \; & c_c z_c + c_b z_b + d \quad \text{OR} \quad [z_c z_b] H [z_c z_b] + q[z_c z_b] + d \\ \text{s.t } & G_c z_c + G_b z_b \leq W \\ & z_c \in R^{s_c}, z_b \in \{0,1\}^{s_b} \end{aligned}$$

Explicit solution is a time varying fb law for both problems: $u_k^*(x_k) = F_k^j x_k + G_k^j$ if $x_k \in \mathcal{R}_k^j$.

Brute force: enumerating all the 2^{sb} integer values of the variable z_b and solve the corresponding problem.

Branch and Bound: relaxation of binaries: $\{0,1\} \rightarrow [0,1]$. Lower bound on the optimal solution of the modified problem is found. Any feasible solution to original problem is upper bound on optimal cost.



9 Numerical Optimization – Iterative Methods

9.1 Gradient descent

$$x_{i+1} = x_i - h_i \nabla f(x_i) \text{ with step-size } h_i = \frac{1}{L} \text{ for } L\text{-smooth } f(x):$$

$$\exists L : \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \; \forall x, y, \in \mathbb{R}^n$$

$\Leftrightarrow \nabla f$ is Lipschitz continuous

$\Leftrightarrow f$ can be upperbounded by a quadratic function:

$$f(x) \leq f(y) + \nabla f(y)^T (x - y) + 0.5L \|x - y\|^2 \; \forall x, y \in \mathbb{R}^n$$

9.2 Newton's Method

$$x_{i+1} = x_i + h_i \Delta x_{nt} := x_i - h_i (\nabla^2 f(x_i))^{-1} \nabla f(x_i)$$

Line search problem: choose $h_i > 0$ s.t. $f(x_i + h_i \Delta x_{nt}) \leq f(x_i)$.
Either compute exact and best h_i using:

$$h_i^* = \text{argmin } x_i + h_i \Delta x_{nt}$$

Or use the backtracking search method:
For $\alpha \in (0, 0.5)$ and $\beta \in (0, 1)$:
Initialise $h_i = 1$;
while $f(x_i + h_i \Delta x_{nt}) > f(x_i) + \alpha h_i \nabla f(x_i)^T \Delta x_{nt}$ do $h_i \leftarrow \beta h_i$
For given equality constraint $\boldsymbol{A}x = b$ solve:

$$\begin{bmatrix} \nabla^2 f(x_i) & \boldsymbol{A}^T \\ \boldsymbol{A} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \Delta x \\ \lambda \end{bmatrix} = \begin{bmatrix} -\nabla f(x_i) \\ \boldsymbol{0} \end{bmatrix}$$

9.3 Constrained optimization with $g_i(x) \leq 0$

Gradient method $x_{i+1} = \pi_Q(x_i - h_i \nabla f(x_i))$ where π_Q is a projection $\pi_q = \arg\min_x \frac{1}{2}\|x - y\|_2^2$. Projection can be solved directly if simple enough, else solve the dual.

9.4 Interior-Point methods

Assumptions $f(x^*) < \infty$, $\tilde{x} \in \text{dom}(f)$.

Barrier method $\min f(x) + \kappa\phi(x)$. Approximate ϕ using diff'able log barrier(instead of indicator function):

$$\phi(x) = \sum_{i=1}^m I\_(g_i(x)) = -\sum_{i=1}^m \log(-g_i(x))$$

$$\lim_{\kappa \to 0} x^*(\kappa) = x^*$$

Analytic center: $\arg\min_x \phi(x)$, central path $\{x^*(\kappa) | \kappa > 0\}$.

Path following method

1. Centering $x^*(\kappa) = \arg\min_x f(x) + \kappa\phi(x)$ with newton's method:
1.1. $\Delta x_{\text{nt}} = \left[\nabla^2 f(x) + \kappa \nabla^2 \phi(x)\right]^{-1} (-\nabla f(x) - \kappa \nabla \phi(x))$.
1.2. Line search:
retain feasability: $\text{argmax}_{h>0} \{h | g_i(x + h\Delta x) < 0\}$
Find $h^* = \text{argmin}_{h \in [0, h_{\max}]} \{f(x + h\Delta x) + \kappa\phi(x + h\Delta x)\}$
2. Update step $x_i = x^*(\kappa_i)$
3. Stop if $m\kappa_i \leq \epsilon$
4. Decrease $\kappa_{i+1} = \kappa_i / \mu$, $\mu > 1$.

Centering step with equality constraints

$$\begin{bmatrix} \nabla^2 f(x) + \kappa \nabla^2 \phi(x) & c^T \\ c & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \Delta x_{\text{nt}} \\ \nu \end{bmatrix} = -\begin{bmatrix} \nabla f(x) + \kappa \nabla \phi(x) \\ 0 \end{bmatrix}$$

Relaxed KKT

$$Cx^* = d \qquad g_i(x^*) + s_i^* = 0$$

$$\nabla f(x^*) + \sum_{i=1}^m \lambda_i^* \nabla g_i(x^*) + c^T \nu = 0 \qquad \lambda_i^* = \kappa \frac{\partial \phi}{\partial g_i} = -\frac{\kappa}{g_i}$$

$$\lambda_i^* g_i(x^*) = -\kappa \qquad \lambda_i^*, s_i^* \geq 0$$

Primal Dual Search Direction Computation

$$\begin{bmatrix} H(x, \lambda) & c^T & G(x)^T & 0 \\ c & 0 & 0 & 0 \\ G(x) & 0 & 0 & \boldsymbol{I} \\ 0 & 0 & \boldsymbol{S} & \boldsymbol{\Lambda} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta u \\ \Delta \lambda \\ \Delta s \end{bmatrix} = -\begin{bmatrix} \nabla f(x) + c^T \nu + G(x)^T \lambda \\ cx - d \\ g(x) + s \\ s\lambda - \nu \end{bmatrix}$$

$\boldsymbol{S} = \text{diag}(s_1, \dots, s_m)$, $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_m)$ and ν is a vector for choosing centering parameters.