1 Systems Theory

1.1 Linearization

1.2 Discretization

Exact

Forward-Euler

Backward-Euler

1.3 Lyapunov Function

$V(0) = 0,\ x \neq 0 \implies V(x) > 0,\ V(g(x(k+1))) - V(x(k+1)) \leq -\alpha(x(k))$
System asymptotically stable if $V(x)$ exists. Globally stable iff $\|x\| \to \infty \implies V(x) \to \infty$.
Check Eig. values of $(APA - P)$ neg., $V(x) = x^T P x$?

2 Unconstrained Control

DARE Initialize $\boldsymbol{P}_N = \boldsymbol{P}$, sometimes $\boldsymbol{F}_k = \boldsymbol{K}$.

$$\begin{aligned}
&\boldsymbol{P}_k = \boldsymbol{A}^T \boldsymbol{P}_{k+1} \boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{A}^T \boldsymbol{P}_{k+1} \boldsymbol{B} (\boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{B} + \boldsymbol{R})^{-1} \boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{A} \\
&\boldsymbol{F}_k - (\boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{B} + \boldsymbol{R})^{-1} \boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{A}
\end{aligned}$$

Infinite Horizon unconstrained, LQR gives $\boldsymbol{P}_\infty$ satisfying equations above. If (A, B) stabilizable and (A, G) detectable, $\boldsymbol{P}_\infty$ unique, where $\boldsymbol{G}\boldsymbol{G}^T = \boldsymbol{Q}$.

3 (Convex) Optimization

General Problem $\min_{x \in \mathrm{dom}(f)} f(x)$ s. t. $g_i(x) \leq 0$ and $h_j(x) = 0$.

RHC

QP with substitution

QP with out substitution

3.1 Duality

Lagrangian Dual Function

$$\begin{aligned}
L(x, \lambda, \nu) &= f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \\
d(\lambda, \nu) &= \inf_{x \in \mathcal{X}} L(x, \lambda, \nu) \quad \text{i.e. } \nabla_x L(x, \lambda, \nu) = 0
\end{aligned}$$

Dual Problem (always convex) $\max_{\lambda, \nu} d(\lambda, \nu)$ s. t. $\lambda \geq 0$.
Optimal value is lower bound for primal: $d^* \leq p^*$.
If primal convex, *Slater condition* (strict feasibility) implies *strong duality*:

$$\{x \mid Ax = b, f_i(x) < 0, \} \neq \emptyset \Rightarrow d^* = p^*$$

Karush-Kuhn-Tucker (KKT) Conditions are necessary for optimality (and sufficient if primal convex).

- Primal Feasibility:
$$\begin{aligned}
f_i(x^*) \leq 0 \quad i = 1, \dots, m \\
h_i(x^*) = 0 \quad i = 1, \dots, p
\end{aligned}$$
- Dual Feasibility: $\lambda^* \geq 0$
- Complementary Slackness:
$$\lambda_i^* \cdot f_i(x^*) = 0 \qquad i = 1, \dots, m$$
- Stationarity:
$$\nabla_x L(x^*, \lambda^*, \nu^*) = 0$$

3.2 Constrained Finite Time Optimal Control (CFTOC)

3.3 Invariance

3.4 Feasability, Stability

3.5 Practical MPC

3.6 Robust MPC

Tube-MPC

3.7 Explicit MPC

3.8 Hybrid MPC

4 Numerical Optimization

Gradient, Newton, Interior Point

5 Observer Based Control

5.1 LTI Observer

LTI System:

$$\begin{aligned}
x(k) &= Ax(k-1) + Bu(k-1) + v(k-1) \\
z(k) &= Hx(k) + w(k)
\end{aligned}$$

Linear Static Gain Observer (Luenberger Observer):

$$\begin{aligned}
\hat{x}(k) &= A\hat{x}(k-1) + Bu(k-1) + K(z(k) - \hat{z}(k)) \\
\hat{z}(k) &= H(A\hat{x}(k-1) + Bu(k-1)) \\
e(k) &= (I - K\,H)\,A\,e(k-1)
\end{aligned}$$

$e(k) \to 0$ for $k \to \infty$ if and only if $(I - K\,H)\,A$ is stable.

Steady State:

$$\hat{x}(k) = (I - K_\infty H)\,A\,\hat{x}(k-1) + (I - K_\infty) B\,u(k-1) + K_\infty z(k)$$

The steady-state KF is one way to design the observer gain K (optimal in minimizing the Steady State mean squared error).

(A, H) detectable $\Rightarrow K$ exists such that $(I - K\,H)A$ is stable.

5.2 Static State Feedback Control

Design of a controller without paying attention to the state estimation:

$$\begin{aligned}
x(k) &= Ax(k-1) + Bu(k-1) && \text{(Process without noise)} \\
z(k) &= x(k) && \text{(Perfect State information)} \\
u(k) &= F \cdot z(k) = F \cdot x(k) && \text{(Control Law)}
\end{aligned}$$

Closed loop dynamics: $x(k) = (A + BF)$. Hence system is stable if $(A + BF)$ is stable. Such an F exists only if (A, B) is stabilizable.
If (A, B) is stabilizable and (A, G) detectable, then F is given by

$$F = -(B^T P B + \bar{R})^{-1} \cdot B^T P A; \qquad P \geq 0$$

P from DARE: $P = A^T P A + \bar{Q} - A^T P B (B^T P B + \bar{R})^{-1} \cdot B^T P A$

5.3 Separation Principle (Linear Systems only)

Combining Luenberger Observer and Static State Feedback control yields:

$$\begin{bmatrix} x(k) \\ e(k) \end{bmatrix} = \begin{bmatrix} A + BF & -BF \\ 0 & (I - KH)A \end{bmatrix} \cdot \begin{bmatrix} x(k-1) \\ e(k-1) \end{bmatrix}$$

Eigenvalues of closed loop are given bei Eigenvalues of $(I - KH)A$ and $(A + BF)$. System is stable as long as there exists no $|\lambda| \geq 1$.

5.4 Separation Theorem

1. Design steady-state KF which does not depend on $\bar{Q}, \bar{R}$. $\Rightarrow \hat{x}(k)$
2. Design state-feedback $u(k) = Fx(k)$ and put both together.