1 Systems Theory

1.1 Linearization

1.2 Discretization

Exact

Forward-Euler

Backward-Euler

1.3 Lyapunov Function

$V(0) = 0,\ x \neq 0 \implies V(x) > 0,\ V(g(x(k+1))) - V(x(k+1)) \leq -\alpha(x(k))$
System asymptotically stable if $V(x)$ exists. Globally stable iff $\|x\| \to \infty \implies V(x) \to \infty$.
Check Eig. values of $(APA - P)$ neg., $V(x) = x^T P x$?

2 Unconstrained Control

2.1 Block Approach (used also for $\bar{w}$ substition)

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{A} \\ \vdots \\ \boldsymbol{A}^N \end{bmatrix} x(0) + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{AB} & \boldsymbol{B} & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{0} \\ \boldsymbol{A}^{N-1}\boldsymbol{B} & \dots & \boldsymbol{AB} & \boldsymbol{B} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_1} \end{bmatrix}$$

$$\begin{aligned} x &= \boldsymbol{S}^x \cdot x(0) + \boldsymbol{S}^u \cdot u && \mathrm{size}(\boldsymbol{S}^x) = [n_{\text{states}} \cdot (N+1), N] \\ & && \mathrm{size}(\boldsymbol{S}^u) = [n_{\text{states}} \cdot (N+1), n_{\text{states}}] \\ \bar{\boldsymbol{Q}} &= \mathrm{diag}(\boldsymbol{Q}, \dots, \boldsymbol{Q}, \boldsymbol{P}) && \mathrm{size}(\bar{\boldsymbol{Q}}) = [n_{\text{states}} \cdot (N+1), n_{\text{states}} \cdot (N+1)] \\ \bar{\boldsymbol{R}} &= \mathrm{diag}(\boldsymbol{R}, \dots, \boldsymbol{R}) && \mathrm{size}(\bar{\boldsymbol{R}}) = (n_{\text{input}} \cdot N, n_{\text{input}} \cdot N) \\ \boldsymbol{H} &= \boldsymbol{S}^{u\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^u + \boldsymbol{R} && \boldsymbol{F} = \boldsymbol{S}^{x\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^u \\ \boldsymbol{Y} &= \boldsymbol{S}^{x\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^x \end{aligned}$$

Optimal cost and control

$$\begin{aligned} J^*(x_0) &= -x_0^T \boldsymbol{F} \boldsymbol{H} \boldsymbol{F}^T x_0 + x_0^T \boldsymbol{Y} x_0 \\ u^*(x_0) &= -\boldsymbol{H}^{-1} \boldsymbol{F}^T x_0 = -\left(\boldsymbol{S}^{u\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^u + \boldsymbol{R}\right)^{-1} \boldsymbol{S}^{u\,T} \bar{\boldsymbol{Q}} \boldsymbol{S}^x x_0 \end{aligned}$$

tim: check formulae for J*, u*

$$\begin{aligned} U_0^*(x(0)) =& -(\mathcal{S}^{u\,T} \bar{Q} \mathcal{S}^u + \bar{R})^{-1} \mathcal{S}^{u\,T} \bar{Q} \mathcal{S}^x x(0) \\ J_0^*(x(0)) =& x(0)^T \Big(\mathcal{S}^{x\,T} \bar{Q} \mathcal{S}^x - \dots \\ & \dots \mathcal{S}^{x\,T} \bar{Q} \mathcal{S}^u (\mathcal{S}^{u\,T} \bar{Q} \mathcal{S}^u - \bar{R})^{-1} \mathcal{S}^{u\,T} \bar{Q} \mathcal{S}^x \Big) x(0) \end{aligned}$$

2.2 Recursive Approach

$$J_k^*(x_k) = \min_{u_k} I(x_k, u_k) + J_{k+1}(x_{k+1})$$

Is a feedback controller as opposed to the Batch Approach. For LQR solve via Riccati Difference Equation (RDE).

$$\begin{aligned} \boldsymbol{F}_k &= -(\boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{B} + \boldsymbol{R})^{-1} \boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{A} \\ \boldsymbol{P}_k &= \boldsymbol{A}^T \boldsymbol{P}_{k+1} \boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{A}^T \boldsymbol{P}_{k+1} \boldsymbol{B} (\boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{B} + \boldsymbol{R})^{-1} \boldsymbol{B}^T \boldsymbol{P}_{k+1} \boldsymbol{A} \end{aligned}$$

$$u_k^* = \boldsymbol{F}_k\ x_k \qquad J_k^*(x_k) = x_k^T \boldsymbol{P}_k\ x_k \qquad \boldsymbol{P}_N = \boldsymbol{P}$$

For unconstrained Infinite Horizon Problem, substituting $\boldsymbol{P}_\infty = \boldsymbol{P}_k = \boldsymbol{P}_{k+1}$ into RDE gives DARE. Uniquely solvable, iff (A, B) stabilizable and (A, G) detectable, where $\boldsymbol{G}\boldsymbol{G}^T = \boldsymbol{Q}$. Follows from closed-loop system $x_{k+1} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}_k) x_k$

3 (Convex) Optimization

General Problem $\min_{x \in \mathrm{dom}(f)} f(x)$ s. t. $g_i(x) \leq 0$ and $h_j(x) = 0$.

RHC

QP with substitution

QP with out substitution

3.1 Duality

Lagrangian Dual Function

$$\begin{aligned} L(x, \lambda, \nu) &= f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{i=1}^p \nu_i h_i(x) \\ d(\lambda, \nu) &= \inf_{x \in \mathcal{X}} L(x, \lambda, \nu) \quad \text{ i.e. } \nabla_x L(x, \lambda, \nu) = 0 \end{aligned}$$

Dual Problem (always convex) $\max_{\lambda, \nu} d(\lambda, \nu)$ s. t. $\lambda \geq 0$.
Optimal value is lower bound for primal: $d^* \leq p^*$.
If primal convex, *Slater condition* (strict feasibility) implies *strong duality*:

$$\{x \mid Ax = b, f_i(x) < 0, \} \neq \emptyset \Rightarrow d^* = p^*$$

Karush-Kuhn-Tucker (KKT) Conditions are necessary for optimality (and sufficient if primal convex).

- Primal Feasibility:

$$\begin{aligned} f_i(x^*) &\leq 0 \quad i = 1, \dots, m \\ h_i(x^*) &= 0 \quad i = 1, \dots, p \end{aligned}$$

- Dual Feasibility: $\lambda^* \geq 0$
- Complementary Slackness:

$$\lambda_i^* \cdot f_i(x^*) = 0 \qquad i = 1, \dots, m$$

- Stationarity:

$$\nabla_x L(x^*, \lambda^*, \nu^*) = 0$$

3.2 Constrained Finite Time Optimal Control (CFTOC)

3.3 Invariance

3.4 Feasability, Stability

3.5 Practical MPC

3.6 Robust MPC

Tube-MPC

3.7 Explicit MPC

3.8 Hybrid MPC

4 Numerical Optimization

Gradient, Newton, Interior Point

5 Observer Based Control

5.1 LTI Observer

LTI System:

$$x(k) = Ax(k-1) + Bu(k-1) + v(k-1)$$
$$z(k) = Hx(k) + w(k)$$

Linear Static Gain Observer (Luenberger Observer):

$$\hat{x}(k) = A\hat{x}(k-1) + Bu(k-1) + K(z(k) - \hat{z}(k))$$
$$\hat{z}(k) = H(A\hat{x}(k-1) + Bu(k-1))$$
$$e(k) = (I - K\,H)\,A\,e(k-1)$$

$e(k) \to 0$ for $k \to \infty$ if and only if $(I - K\,H)\,A$ is stable.

Steady State:

$$\hat{x}(k) = (I - K_\infty H)\,A\,\hat{x}(k-1) + (I - K_\infty)B\,u(k-1) + K_\infty z(k)$$

The steady-state KF is one way to design the observer gain K (optimal in minimizing the Steady State mean squared error).

(A, H) detectable $\Rightarrow$ K exists such that $(I - K\,H)A$ is stable.

5.2 Static State Feedback Control

Design of a controller without paying attention to the state estimation:

$$x(k) = Ax(k-1) + Bu(k-1) \qquad \text{(Process without noise)}$$
$$z(k) = x(k) \qquad \text{(Perfect State information)}$$
$$u(k) = F \cdot z(k) = F \cdot x(k) \qquad \text{(Control Law)}$$

Closed loop dynamics: $x(k) = (A + BF)$. Hence system is stable if $(A + BF)$ is stable. Such an F exists only if (A, B) is stabilizable.
If (A, B) is stabilizable and (A, G) detectable, then F is given by

$$F = -(B^T PB + \bar{R})^{-1} \cdot B^T PA; \qquad P \geq 0$$

P from DARE: $P = A^T PA + \bar{Q} - A^T PB(B^T PB + \bar{R})^{-1} \cdot B^T PA$

5.3 Separation Principle (Linear Systems only)

Combining Luenberger Observer and Static State Feedback control yields:

$$\begin{bmatrix} x(k) \\ e(k) \end{bmatrix} = \begin{bmatrix} A + BF & -BF \\ 0 & (I - KH)A \end{bmatrix} \cdot \begin{bmatrix} x(k-1) \\ e(k-1) \end{bmatrix}$$

Eigenvalues of closed loop are given bei Eigenvalues of $(I - KH)A$ and $(A + BF)$. System is stable as long as there exists no $|\lambda| \geq 1$.

5.4 Separation Theorem

1. Design steady-state KF which does not depend on $\bar{Q}, \bar{R}$. $\Rightarrow \hat{x}(k)$
2. Design state-feedback $u(k) = Fx(k)$ and put both together.