1 System Theory

$$\dot{x}(t) = A^c x(t) + B^c u(t)$$
$$y(t) = C^c x(t) + D^c u(t)$$
$$x(t) = e^{A^c(t-t_0)}x_0 + \int_{t_0}^{t} e^{A^c(t-\tau)}Bu(\tau)d\tau$$
$$e^{A^c t} = \sum_{n=0}^{\infty}\frac{(A^c t)^n}{n!}$$
$$x(k+N) = A^N x(k) + \sum_{i=0}^{N-1} A^i Bu(k+N-1-i)$$

1.1 Nonlinear Systems

we define a system to be stable in the sense of Lyapunov, if it stays in any arbitrarily small neighborhood of the origin when it is disturbed slightly

1.2 Lyapunov Stability

Lyapunov stable if for every $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that

$$\operatorname{norm} x(0) < \delta(\epsilon) \to \operatorname{norm} x(k) < \epsilon, \forall k \geq 0$$

asymptotically stable in $\Omega \subseteq \mathbb{R}^n$ if it is Lyapunov stable and attractive $\lim_{k\to\infty} x(k) = 0, \forall x(0) \in \Omega$

1.3 Lyapunov Function

Consider the equilibrium point $x = 0$. Let $\Omega \subset \mathbb{R}^n$ be a closed and bounded set containing the origin. A function $V : \mathbb{R}^n \to \mathbb{R}$, continous at the origin, finite for every $x \in \Omega$, and such that

$$V(0) = 0 \text{ and } V(x) > 0, \forall x \in \Omega\backslash\{0\}$$
$$V(g(x)) - V(x) \leq -\alpha(x), \forall x \in \Omega\backslash\{0\}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}$ is continuous positive definite
If a system admins a Lyapunov function $V(x)$, then $x = 0$ is **asymptotically stable** in Ω (sufficient but not necessary)
If a system admits a Lyapunov function, which additionally satisfies $\operatorname{norm} x \to \infty \to V(x) \to \infty$, then $x = 0$ is **globally asymptotically stable**

tim: Check Eig. values of $(APA - P)$ neg., $V(x) = x^T Px$?

Linear systems: iff eigenvalues of A inside unit circle (i.e. stable) then $\exists$*unique* $P > 0$ that solves $A_{cl}^T PA_{cl} - P = -Q,\ Q > 0$ and $V = x^T Px$ is a lyapunov function.

1.4 Discretization

Euler: $A = I + T_s A^c,\ B = T_s B^c,\ C = C^c,\ D = D^c$

$$x(k+1) = x(k) + T_s g^c(x(k), u(k)) = g(x(k), u(k))$$
$$y(k) = h^c(x(k), u(k)) = h(x(k), u(k))$$

Exact: (assumption of a constant $u(t)$ during T_s)

$$A = e^{A^c T_s},\ B = \int_0^{T_s} e^{A^c(T_s-\tau')}B^c d\tau$$
$$B = (A^c)^{-1}(A - I)B^c,\ \text{if } A^c \text{ invertible}$$

1.5 Controllability (reachability) and observability

$$C = [B\ AB\ \ldots\ A^{n-1}B]$$
$$O = [C^T\ (CA)^T\ \ldots\ (CA^{n-1})^T]$$

2 Unconstrained Control

2.1 Block Approach (used also for $\bar{w}$ substition)

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{A} \\ \vdots \\ \boldsymbol{A}^N \end{bmatrix} x(0) + \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{0} & \ldots & \boldsymbol{0} \\ \boldsymbol{AB} & \boldsymbol{B} & \ldots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{0} \\ \boldsymbol{A}^{N-1}\boldsymbol{B} & \ldots & \boldsymbol{AB} & \boldsymbol{B} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N_1} \end{bmatrix}$$

$$x = \boldsymbol{S}^x \cdot x(0) + \boldsymbol{S}^u \cdot u \qquad \text{size}(\boldsymbol{S}^x) = [n_{\text{states}} \cdot (N+1), N]$$
$$\text{size}(\boldsymbol{S}^u) = [n_{\text{states}} \cdot (N+1), n_{\text{states}}]$$
$$\bar{\boldsymbol{Q}} = \text{diag}(\boldsymbol{Q}, \ldots, \boldsymbol{Q}, \boldsymbol{P}) \qquad \text{size}(\bar{\boldsymbol{Q}}) = [n_{\text{states}} \cdot (N+1), n_{\text{states}} \cdot (N+1)]$$
$$\bar{\boldsymbol{R}} = \text{diag}(\boldsymbol{R}, \ldots, \boldsymbol{R}) \qquad \text{size}(\bar{\boldsymbol{R}}) = [n_{\text{input}} \cdot N, n_{\text{input}} \cdot N]$$
$$\boldsymbol{H} = \boldsymbol{S}^{u\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^u + \boldsymbol{R} \qquad \boldsymbol{F} = \boldsymbol{S}^{x\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^u$$
$$\boldsymbol{Y} = \boldsymbol{S}^{x\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^x$$

Optimal cost and control

$$J^*(x_0) = -x_0^T \boldsymbol{FHF}^T x_0 + x_0^T \boldsymbol{Y} x_0$$
$$u^*(x_0) = -\boldsymbol{H}^{-1}\boldsymbol{F}^T x_0 = -\left(\boldsymbol{S}^{u\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^u + \boldsymbol{R}\right)^{-1} \boldsymbol{S}^{u\,T}\bar{\boldsymbol{Q}}\boldsymbol{S}^x x_0$$

2.2 Recursive Approach

$$J_k^*(x_k) = \min_{u_k} I(x_k, u_k) + J_{k+1}(x_{k+1})$$

Is a feedback controller as opposed to the Batch Approach. For LQR solve via Riccati Difference Equation (RDE).

$$\boldsymbol{F}_k = -(\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{B} + \boldsymbol{R})^{-1}\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{A}$$
$$\boldsymbol{P}_k = \boldsymbol{A}^T\boldsymbol{P}_{k+1}\boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{A}^T\boldsymbol{P}_{k+1}\boldsymbol{B}(\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{B} + \boldsymbol{R})^{-1}\boldsymbol{B}^T\boldsymbol{P}_{k+1}\boldsymbol{A}$$

$$u_k^* = \boldsymbol{F}_k\ x_k \qquad J_k^*(x_k) = x_k^T\boldsymbol{P}_k\ x_k \qquad \boldsymbol{P}_N = \boldsymbol{P}$$

For unconstrained Infinite Horizon Problem, substituting $\boldsymbol{P}_\infty = \boldsymbol{P}_k = \boldsymbol{P}_{k+1}$ into RDE gives DARE. Uniquely solvable, iff (A, B) stabilizable and (A, G) detectable, where $\boldsymbol{GG}^T = \boldsymbol{Q}$. Follows from closed-loop system $x_{k+1} = (\boldsymbol{A} + \boldsymbol{BF}_k)x_k$

3 (Convex) Optimization

3.1 Convexity

Convex set $\mathcal{X}$ iff $\forall\lambda \in [0, 1]\forall x, y \in \mathcal{X}\ \lambda x + (1-\lambda)y \in \mathcal{X}$.
Intersection preserves convexity, union does not.

Affine set $\mathcal{X}$ $= \{x \in \mathbb{R}^n | \boldsymbol{A}x = b\}$ for some $\boldsymbol{A}, b$

Subspace is affine set through origin, i.e. $b = \boldsymbol{0}$, aka Nullspace of $\boldsymbol{A}$.

Hyperplane $\mathcal{X}$ $= \{x \in \mathbb{R}^n | a^T x = b\}$ for some a, b.

Halfspace $\mathcal{X}$ $= \{x \in \mathbb{R}^n | a^T x \leq b\}$ for some a, b.

Polyhedron $\mathcal{P}$ $= \{x | a_i^T x \leq b_i, i = 1, \ldots, n\} = \{x | \boldsymbol{A}x \leq b\}$

Cone

Ellipsoid $\mathcal{E}$ $= \{x | (x - x_c)^T \boldsymbol{A}^{-1}(x - x_c) \leq 1\}$, x_c center point.

Convex function

Norm $f(x) : \mathbb{R}^n \to \mathbb{R}$

$$f(x) = 0 \implies x = 0, \qquad f(x) \geq 0$$
$$f(\alpha \cdot x) = |\alpha| \cdot f(x) \qquad \text{for scalar } \alpha$$
$$f(x + y) \leq f(x) + f(y) \qquad \forall x, y \in R^n$$

tim: Maybe move the above somewhere else?

General Problem $\min_{x\in\text{dom}(f)} f(x)$ s. t. $g_i(x) \leq 0$ and $h_j(x) = 0$.

3.2 Linear Programming (LP)

Problem statement $\min c^T x$ such that $\boldsymbol{G}x \leq h$ and $\boldsymbol{A}x = b$.

Norm l_∞ $\min_x \|x\|_\infty = \min_{x\in\mathbb{R}^n} [\max\{x, \ldots, x_n, -x_1, \ldots, -x_n\}]$:

$$\min_{x,t} t \quad \text{subject to} \quad x_i \leq t, -x_i \leq t, \qquad \boldsymbol{F}x \leq g$$
$$\iff \min_{x,t} t \quad \text{subject to} \quad -\boldsymbol{1}t \leq x \leq 1t, \qquad \boldsymbol{F}_x \leq g.$$

Norm l_1 $\min_x \|x\|_1 = \min_x \left[\sum_{i=1}^m \max\{x_i, -x_i\}\right]$:

$$\min_t t_1 + \cdots + t_m \quad \text{subject to} \quad x_i \leq t_i, -x_i \leq t_i, \qquad \boldsymbol{F}x \leq g$$
$$\iff \min_t \boldsymbol{I}^T t \quad \text{subject to} \quad -t \leq x \leq t, \qquad \boldsymbol{F}_x \leq g.$$

Note that for $\dim x = 1$, l_1 and l_∞ are the same.

MPC with linear cost
$J(x_0, u) = \|\boldsymbol{P}x_N\|_p + \sum_{i=0}^{N-1}\|\boldsymbol{Q}x_i\|_p + \|\boldsymbol{R}u_i\|_p$.

tim: Insert here slide 45, lect 4

Receding Horizon Control – RHC

QP with substitution (see also Batch approach)

$$J^*(x_k) = \min_u \begin{bmatrix} u^T & x_k^T \end{bmatrix} \begin{bmatrix} \boldsymbol{H} & \boldsymbol{F}^T \\ \boldsymbol{F} & \boldsymbol{Y} \end{bmatrix} \begin{bmatrix} u \\ x_k \end{bmatrix}$$

$$\text{s. t. } \boldsymbol{G}\ u \leq w + \boldsymbol{E}\ x_k$$

Latter gives three sets (same for without substitution)

$$\begin{aligned} \mathcal{X} &= \{x | A_x\ x \leq b_x\} \\ \mathcal{U} &= \{u | A_u\ u \leq b_u\} \\ \mathcal{X}_{\{} &= \{x | A_f\ x \leq b_f\} \end{aligned}$$

State equations are in cost matrix, usually
$\boldsymbol{A}_x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, b_x = \begin{bmatrix} b_{\max} \\ -b_{\min} \end{bmatrix}$

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{A}_u & 0 & \dots & 0 \\ 0 & \boldsymbol{A}_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{A}_u \\ 0 & 0 & \dots & 0 \\ \boldsymbol{A}_x\boldsymbol{B} & 0 & \dots & 0 \\ \boldsymbol{A}_x\boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}_x\boldsymbol{B} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}_x\boldsymbol{A}^{N-2}\boldsymbol{B} & \boldsymbol{A}_x\boldsymbol{A}^{N-3}\boldsymbol{B} & \dots & 0 \\ \boldsymbol{A}_f\boldsymbol{A}^{N-1}\boldsymbol{B} & \boldsymbol{A}_f\boldsymbol{A}^{N-2}\boldsymbol{B} & \dots & \boldsymbol{A}_f\boldsymbol{B} \end{bmatrix} \quad \boldsymbol{E} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\boldsymbol{A}_x \\ -\boldsymbol{A}_x\boldsymbol{A} \\ -\boldsymbol{A}_x\boldsymbol{A}^2 \\ \vdots \\ -\boldsymbol{A}_x\boldsymbol{A}^{N-1} \\ -\boldsymbol{A}_f\boldsymbol{A}^N \end{bmatrix} \quad \boldsymbol{W} = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_x \\ b_f \end{bmatrix}$$

QP with out substitution State equations represented in equality constrainst.

$$J^*(x_k) = \min_z \begin{bmatrix} z^T & x_k^T \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{H}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} z \\ x_k \end{bmatrix}$$

$$\begin{aligned} \text{s. t. } & \boldsymbol{G}\ z \leq w + \boldsymbol{E}\ x_k \\ & \boldsymbol{G}_{\text{eq}}\ z = \boldsymbol{E}_{\text{eq}}\ x_k, \quad \text{system dynamics} \end{aligned}$$

$$\bar{\boldsymbol{H}} = \text{diag}(\boldsymbol{Q}, \dots, \boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{R}, \dots, \boldsymbol{R})$$

$$z = \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} \quad \boldsymbol{G}_{\text{eq}} = \left[\begin{array}{cccc|cccc} \boldsymbol{I} & & & & -\boldsymbol{B} & & & \\ -\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B} & & \\ & \ddots & \ddots & & & & \ddots & \\ & & -\boldsymbol{A} & \boldsymbol{I} & & & & -\boldsymbol{B} \end{array}\right] \quad \boldsymbol{E}_{\text{eq}} = \begin{bmatrix} \boldsymbol{A} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$w = \begin{bmatrix} b_x \\ \vdots \\ b_f \\ b_u \\ \vdots \\ b_u \end{bmatrix} \quad \boldsymbol{G} = \left[\begin{array}{ccc|ccc} 0 & & & & & \\ & \boldsymbol{A}_x & & & & \\ & & \ddots & & & \\ & & \boldsymbol{A}_x & & & \\ \hline & & & \boldsymbol{A}_d & & \\ & & & & \ddots & \\ & & & & & \boldsymbol{A}_d \end{array}\right] \quad \boldsymbol{E} = \begin{bmatrix} -\boldsymbol{A}_x^T \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

3.3 Duality

Lagrangian Dual Function

$$L(x, \lambda, \nu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

$$d(\lambda, \nu) = \inf_{x \in \mathcal{X}} L(x, \lambda, \nu) \quad \text{i.e. } \nabla_x L(x, \lambda, \nu) = 0$$

Dual Problem (always convex) $\max_{\lambda,\nu} d(\lambda, \nu)$ s. t. $\lambda \geq 0$.
Optimal value is lower bound for primal: $d^* \leq p^*$.
If primal convex, *Slater condition* (strict feasibility) implies *strong duality*:

$$\{x \mid Ax = b, f_i(x) < 0, \} \neq \emptyset \Rightarrow d^* = p^*$$

Karush-Kuhn-Tucker (KKT) Conditions are necessary for optimality (and sufficient if primal convex).

- Primal Feasibility:

$$\begin{aligned} f_i(x^*) \leq 0 \quad & i = 1, \dots, m \\ h_i(x^*) = 0 \quad & i = 1, \dots, p \end{aligned}$$

- Dual Feasibility: $\lambda^* \geq 0$
- Complementary Slackness:

$$\lambda_i^* \cdot f_i(x^*) = 0 \qquad i = 1, \dots, m$$

- Stationarity:

$$\nabla_x L(x^*, \lambda^*, \nu^*) = 0$$

3.4 Constrained Finite Time Optimal Control (CFTOC)

3.5 Invariance

Def.: $x(k) \in O \implies x(k+1) \in O \forall k$.

$$\text{pre}(S) := \{x | g(x) \in S\} \qquad\qquad = \{x | Ax \in S\}$$

tim: We need more here, poos. inv. set, max. pos.inv O_∞

3.6 Stability and Feasability

Main Idea: Choose $\mathcal{X}_f$ and $\boldsymbol{P}$ to mimic infinite horizon. LQR control law $\kappa(x) = \boldsymbol{F}_\infty x$ from solving DARE. Set terminal weight $\boldsymbol{P} = \boldsymbol{P}_\infty$, terminal set $\mathcal{X}_f$ as maximal invariant set:

$$\begin{aligned} x_{k+1} = \boldsymbol{A}x_k + \boldsymbol{B}\boldsymbol{F}_\infty\ x_k \in \mathcal{X}_f \quad & \forall x_k \in \mathcal{X}_f \text{ terminal set invariant} \\ \mathcal{X}_f \subseteq \mathcal{X}, \quad \boldsymbol{F}_\infty\ x_k \in \mathcal{U} \quad & \forall x_k \in \mathcal{X}_f \text{ constrainst satisfied} \end{aligned}$$

We get: 1. Positive stage cost function, 2. invariant terminal set by construction, 3. Terminal cost is Lyapunov function with

$$x_{k+1}^T \boldsymbol{P} x_{k+1} - x_k^T \boldsymbol{P} x_k = -x_k^T (\boldsymbol{Q} + \boldsymbol{F}_\infty^T \boldsymbol{R} \boldsymbol{F}_\infty) x_k$$

Extension to non-linear (time-invariant) MPC possible since terminal set and cost do not rely on linearity.

3.7 Practical Issues

MPC for tracking Target steady-state conditions $x_s = \boldsymbol{A}x_s + \boldsymbol{B}u_s$ and $y_s = \boldsymbol{C}x_s = r$ and constrainsts give:

$$\min_{x_s, u_s} u_s^T \boldsymbol{R} u_s \text{ subj. to } \begin{bmatrix} \boldsymbol{I} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ r \end{bmatrix}, x_s \in \mathcal{X}, u_s \in \mathcal{U}$$

Usually assume x_s, u_s unique and feasible. If no solution exists, compute closest steady-state ($\min(\boldsymbol{C}x_s - r)^T \boldsymbol{Q}(\boldsymbol{C}x_s - r)$ s. t. $x_s = \boldsymbol{A}x_s + \boldsymbol{B}u_s$).
MPC problem to drive $y \to r$ is:

$$\min_u \|y_N - \boldsymbol{C}x_s\|_{P_y}^2 + \sum_{i=0}^{N-1} \|y_i - \boldsymbol{C}x_s\|_{Q_y}^2 + \|u_i - u_s\|_R^2$$

Delta formulation for reference r $\Delta x_k = x_k - x_s, \Delta u_k = u_k - u_s$:

$$\min V_f(\Delta x_N) + \sum_{i=0}^{N-1} \Delta x_i^T \boldsymbol{Q} \Delta x_i + \Delta u_i^T \boldsymbol{R} \Delta u_i$$

$$\begin{aligned} \text{s.t. } & \Delta x_0 = \Delta x_k \\ & \Delta x_{k+1} = \boldsymbol{A}\Delta x_k + \boldsymbol{B}\Delta u_k \\ \boldsymbol{H}_x x \leq k_x \to & \boldsymbol{H}_x \Delta x \leq k_x - \boldsymbol{H}_x x_s \\ \boldsymbol{H}_u u \leq k_u \to & \boldsymbol{H}_u \Delta u \leq k_u - \boldsymbol{H}_u u_s \\ & \Delta x_N \in \mathcal{X}_f \quad \text{adjusted accordingly} \\ & x_s \oplus \mathcal{X}_f \subseteq \mathcal{X} \\ & \boldsymbol{K}\Delta x + u_s \in \mathcal{U} \end{aligned}$$

Control given by $u_0^* = \Delta u_0^* + u_s$.

Offset free tracking

$$\begin{aligned} x_{k+1} &= \boldsymbol{A}x_k + \boldsymbol{B}u_k + \boldsymbol{B}_d d_k \\ d_{k+1} &= d_k \\ y_k &= \boldsymbol{C}x_k + \boldsymbol{C}_d d_k \end{aligned}$$

$$\begin{bmatrix} \boldsymbol{I} - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_d \hat{d} \\ r - \boldsymbol{C}_d \hat{d} \end{bmatrix}$$

Choice of $\boldsymbol{B}_d, \boldsymbol{C}_d$ requires that $(\boldsymbol{A}, \boldsymbol{C})$ is observable and
$\begin{bmatrix} \boldsymbol{A} - \boldsymbol{I} & \boldsymbol{B}_d \\ \boldsymbol{C} & \boldsymbol{C}_d \end{bmatrix}$ has full $(n_x + n_d)$ column frank (i.e. det $\neq 0$).
Intuition: for fixed y_s at steady-state, d_s is uniquely determined.
If plant has no integrator we can choose $\boldsymbol{B}_d = \boldsymbol{0}$ since $\det(\boldsymbol{A} - \boldsymbol{I}) \neq 0$.

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{d}_{k+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}_d \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{d}_k \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} u_k + \begin{bmatrix} \boldsymbol{L}_x \\ \boldsymbol{L}_d \end{bmatrix} \left(-y_k^m + \boldsymbol{C}\hat{x}_k + \boldsymbol{C}_d \hat{d}_k \right)$$

where y_k^m measured output; choose $\begin{bmatrix} \boldsymbol{L}_x \\ \boldsymbol{L}_d \end{bmatrix}$ s.t. error dynamics stable and converge to zero.

tim: Target condition here

If 1) number of dist. = number of outputs, 2) target steady-state problem feasible and no constraints active at steady-state, 3) closed-loop system converges, then target achieved without offset.

Soft-constraints via slack variables

3.8 Robust MPC

Enforcing terminal constraints by robust invariance:

$$x \in O^{\mathcal{W}} \implies g(x,w) \in \Omega^W \;\forall w \in \mathcal{W}$$
$$\text{pre}^{\mathcal{W}}(\Omega) = \{x | g(x,w) \in \Omega \;\forall w \in \mathcal{W}\}$$

tim: Maybe an example from exercises to compute O_∞^W

Enforcing sequential constraints for uncertain system ϕ:

$$\phi_i(x_0,u,w) = \left\{ x_i + \sum_{j=0}^{i-1} \boldsymbol{A}^J w_j \,\middle|\, w \in \mathcal{W}^i \right\} \subseteq \mathcal{X}$$

$\phi_N(x_0,u,w) \in \mathcal{X}_f$ as well

tim: One or two words on what is what

$$\boldsymbol{A}_x x \le b_x \text{ becomes } \boldsymbol{A}_x x_i + \boldsymbol{A}_x \sum_{j=0}^{i-1} \boldsymbol{A}^j w_k \le b_x :$$
$$x_i \in \mathcal{X} \ominus \left(\mathcal{W} \oplus \boldsymbol{A}\mathcal{W} \oplus \cdots \oplus \boldsymbol{A}^{i-1}\mathcal{W}\right)$$
$$= \left(\bigoplus_{j=0}^{i-1} \boldsymbol{A}^j \mathcal{W}\right) = \begin{bmatrix}\boldsymbol{A}^0 & \ldots & \boldsymbol{A}^{i-1}\end{bmatrix} \mathcal{W}^i$$

Tube-MPC We want nominal system $z_k = \boldsymbol{A} z_k + \boldsymbol{B} v_k$ with "tracking" controller $u_k = \boldsymbol{K}(x_k - z_k) + v_k$, $\boldsymbol{K}$ found offline.
Step 1: Compute $\mathcal{E} = \bigoplus_{j=1}^{\infty} \boldsymbol{A}^j \mathcal{W}$.
Step 2: Shrink Constraints:

$$\begin{aligned} \{z_i\} \oplus \mathcal{E} \subseteq \mathcal{X} &\implies z_i \in \mathcal{X} \ominus \mathcal{E} \\ u_i \in \boldsymbol{K}\mathcal{E} \oplus \{v_i\} \subset \boldsymbol{U} &\implies v_i \in \mathcal{U} \ominus \boldsymbol{K}\mathcal{E} \end{aligned}$$

Also $z_n \in \mathcal{X}_f \ominus \mathcal{E}$ accordingly.

4 Explicit MPC

$z^*(x_k)$ is continuous and polyhedral piecewise affine over feasible set.

4.1 Quadratic Cost

$J^*(x_k)$ is continuous, convex and polyhedral piecewise quadratic. Using the formulation with substitution,

$$\begin{aligned} J(x_k) &= \min\; z^T H z - x_k^T (Y - F H^{-1} F^T) x_k \\ \text{s.t} &\quad Gz \le w + S x_k \\ z(x_k) &= U + H^{-1} F^T x_k \\ S &= E + G H^{-1} F^T \\ U^* &= z^*(x_k) - H^{-1} F^T x_k \end{aligned}$$

The first solution gives $u^*(x_k) = \kappa(x_k)$, which is continuous and piecewise affine on polyhedra $\kappa(x) = F_j x + g_j$.

4.2 $1/\infty$-norm

$J^*(x_k)$ is continuous, convex and polyhedral piecewise affine over. Optimal solution given by, $u_0^* = 0\,[0 \ldots 0 I_m 0 \ldots 0]\, z^*(x_k)$, and is in the same form as the QP case above.

5 Hybrid MPC

5.1 Piecewise Affine (PWA) Systems

Affine dynamics and output for each region:

$$\begin{cases} x_{k+1} &= A^i x_k + B^i u_k + f^i \\ y_k &= C^i x_k + D^i u_k + g_i \end{cases} \text{ if } x_t \in \mathcal{X}_j$$

Polyhedral partition of the (x,u)-space:

$$\{\mathcal{X}_i\}_{j=1}^s = \{x, u | H_j x + J_j u \le K_j\}$$

5.2 Mixed Logical Dynamical Hybrid Model (MLD)

Idea: associate boolean to binary: $p_j \iff \delta_i = 1, \neg p_j \iff \delta_i = 0$.

5.2.1 Translate logic rules to Linear Integer Inequalities

AND	$p_1 \wedge p_2$	$\delta_1 \ge 1, \delta_1 \ge 1$ also $\delta_1 + \delta_2 \ge 2$
OR	$p_1 \vee p_2$	$\delta_1 + \delta_2 \ge 1$
NOT	$\neg p_1$	$1 - \delta_1 \ge 1$ also $\delta_1 = 0$
XOR	$p_1 + p_2$	$\delta_1 + \delta_2 = 1$
IMPLY	$p_1 \Rightarrow p_2$	$\delta_1 - \delta_2 \le 0$
IIF	$p_1 \Leftrightarrow p_2$	$\delta_1 - \delta_2 = 0$
ASSIGNMENT	$p_3 \Leftrightarrow p_1 \wedge p_2$	$\delta_1 + (1 - \delta_3) \ge 1,$
		$\delta_2 + (1 - \delta_3) \ge 1,$
		$(1 - \delta_1) + (1 - \delta_2) + \delta_3 \ge 1$

Logic Equality Rules:
$\neg(A \wedge B) = \neq A \vee \neg B$
$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$
$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$

5.2.2 Translate continuous and logical components into Linear Mixed-Integer Relations

Event generator: $\delta_e(k) = f_{EG}(x_c(k), u_c(k), t)$.
Consider: $p \Leftrightarrow a^T x \le b, \mathcal{X} = \{x | a^T x - b \in [m, M]\}$.
Translated to linear inequalities: $m\delta < a^T x - b \le M(1 - \delta)$, where $[m, M]$ are lower and upper bounds.

Representing Switched Affine Dynamics as "IF-THEN-ELSE" relations
IF p THEN $z_k = a_1^T x_k + b_1$ else $z_k = a_2^T x_k + b_2 \Leftrightarrow$

$$(m_2 - M_1)\delta + z_k \le a_2^T x_k + b_2 \le -(m_1 - M_2)\delta + z_k$$
$$(m_1 - M_2)(1 - \delta) + z_k \le a_1^T x_k + b_1 \le -(m_1 - M_2)(1 - \delta) + z_k$$

This results in a linear MLD model

$$\begin{aligned} x_{k+1} &= A x_k + B_1 u_k + B_2 \delta_k + B_3 z_k \\ y_k &= C x_k + D_1 u_k + D_2 \delta_k + D_3 z_k \\ E_2 \delta_k + E_3 z_k &\le E_4 x_k + E_1 u_k + E_5 \end{aligned}$$

where the last equation describes the relationship between the continuous and integer variables. Physical constraints on cont. variables: $\{\mathcal{C}\} = \left\{ \begin{bmatrix} x_c \\ u_c \end{bmatrix} \in \mathcal{R}^{n_c + m_c} | F x_c + G u_c \le H \right\}$

5.3 CFTOC for Hybrid Systems

$$\begin{aligned} J^*(x) = \min_{U}\; & l_N(x_N) + \sum_{k=0}^{N-1} l(x_k, u_k, \delta_k, z_k) \\ \text{s.t}\;\; & x_{k+1} = A x_k + B_1 u_k + B_2 \delta_k + B_3 z_k \\ & E_2 \delta_k + E_3 z_k \le E_4 x_k + E_1 u_k + E_5 \\ & x_N \in \mathcal{X}_f, x_0 = x(0) \end{aligned}$$

5.4 MILP/QP

$$\begin{aligned} \min\;\; & c_c z_c + c_b z_b + d \text{ OR } zHz + qz + d \\ \text{s.t}\;\; & G_c z_c + G_b z_b \le W \\ & z_c \in R^{s_c}, z_b \in \{0,1\}^{s_b} \end{aligned}$$

Branch and bound method can be used to efficiently solve the problem. Explict solution is a time varying fb law for both problems: $u_k^*(x_k) = F_k^j x_k + G_k^j$ if $x_k \in \mathcal{R}_k^j$.

6 Numerical Optimization – Iterative Methods

6.1 Gradient descent

$x_{i+1} = x_i - h_i \nabla f(x_i)$ with step-size $h_i = \dfrac{1}{L}$ for L-smooth $f(x)$:

$$\begin{aligned} & \exists L : \|\nabla f(x) - \nabla f(y)\| \le L \|x - y\| \;\forall x, y, \in \mathbb{R}^n \\ \iff & \nabla f \text{ is Lipschitz continuous} \\ \iff & f \text{ can be upperbounded by a quadratic function} \end{aligned}$$

6.2 Newton's Method

$x_{i+1} = x_i - h_i \delta x_{nt} := x_i - h_i (\nabla^2 f(x_i))^{-1} \nabla f(x_i)$
Choose $h_i > 0$ s.t. $f(x_i + h_i \delta x_{nt} \le f(x_i)$ Line-search.
For given equality constraint $\boldsymbol{A} x = b$ solve:

$$\begin{bmatrix} \nabla^2 f(x_i) & \boldsymbol{A}^T \\ \boldsymbol{A} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \delta x \\ \lambda \end{bmatrix} = \begin{bmatrix} -\nabla f(x_i) \\ \boldsymbol{0} \end{bmatrix}$$

6.3 Constrained optimization

$g_i(x) \le 0$ with

Gradient method $x_{i+1} = \pi_Q(x_i - h_i \nabla f(x_i))$ where π_Q is a projection $\pi_q = \arg\min_x \frac{1}{2}\|x - y\|_2^2$. Projection can be solved directly if simple enough, else solve the dual.

6.4 Interior-Point methods

Assumptions $f(x^*) < \infty$, $\tilde{x} \in \text{dom}(f)$.

Barrier method $\min f(x) + \kappa\phi(x)$. Approximate ϕ using diff'able log barrier(instead of indicator function):

$$\phi(x) = \sum_{i=1}^{m} ?? = -\sum_{i=1}^{m} \log(-g_i(x))$$
$$\lim_{\kappa \to 0} x^*(\kappa) = x^*$$

Analytic center: $\arg\min_x \phi(x)$, central path $\{x^*(\kappa) | \kappa > 0\}$.

Path following method

1. Centering $x^*(\kappa) = \arg\min_x f(x) + \kappa\phi(x)$ with newton's method:
1.1. $\Delta x_{\text{nt}} = \left[\nabla^2 f(x) + \kappa\nabla^2\phi(x)\right]^{-1} (-\nabla f(x) - \kappa\nabla\phi(x))$.
1.2. Line search:

$$\text{retain feasability: } \operatorname{argmax}_{h>0} \{h | g_i(x + h\Delta x) < 0\}$$
$$\text{Find } h^* = \operatorname{argmin}_{h \in [0, h_{\max}]} \{f(x + h\Delta x) + \kappa\phi(x + h\Delta x)\}$$

2. Update step $x_i = x^*(\kappa_i)$
3. Stop if $m\kappa_i \leq \epsilon$
4. Decrease $\kappa_{i+1} = \kappa_i/\mu$, $\mu > 1$.

Centering step with equality constraints

$$\begin{bmatrix} \nabla^2 f(x) + \kappa\nabla^2\phi(x) & c^T \\ c & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta x_{\text{nt}} \\ \nu \end{bmatrix} = -\begin{bmatrix} \nabla f(x) + \kappa\nabla\phi(x) \\ 0 \end{bmatrix}$$

Relaxed KKT

$$\begin{aligned} Cx^* &= d & g_i(x^*) + s_i^* &= 0 \\ \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + c^T\nu &= 0 & \lambda_i^* = \kappa\frac{\partial\phi}{\partial g_i} &= -\frac{\kappa}{g_i} \\ \lambda_i^* g_i(x^*) &= -\kappa & \lambda_i^*, s_i^* &\geq 0 \end{aligned}$$

Primal Dual Search Direction Computation

$$\begin{bmatrix} H(x,\lambda) & c^T & G(x)^T & 0 \\ c & 0 & 0 & 0 \\ G(x) & 0 & 0 & \boldsymbol{I} \\ 0 & 0 & \boldsymbol{S} & \boldsymbol{\Lambda} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta u \\ \Delta\lambda \\ \Delta s \end{bmatrix} = -\begin{bmatrix} \nabla f(x) + c^T\nu + G(x)^T\lambda \\ cx - d \\ g(x) + s \\ s\lambda - \nu \end{bmatrix}$$

$\boldsymbol{S} = \operatorname{diag}(s_1, \ldots, s_m), \boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ and
ν is a vector for choosing centering parameters.