1 Stochastic

1.1 PDFs

Valid PDF: $\int_{-\infty}^{\infty} f(x)\,dx = 1$ and $f(x) > 0\,\forall x$
Independency: $f(x,y) = f(x)\cdot f(y)$ and $f(x|y) = f(x)$
Marginalization: $f(x) = \int f(x,y)\,dy$
Cumulative Distribution Function: $F_x(x) = \int_{-\infty}^{x} f_x(\bar{x})\,d\bar{x}$

1.2 Normal Distribution

$$w \sim \mathcal{N}(\mu,\sigma^2), \quad f_w = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(w-\mu)^2}{2\sigma^2}\right)$$

$$w \sim \mathcal{N}(\mu,\Sigma), \quad f_w = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp\left(-\frac{1}{2}(w-\mu)\Sigma^{-1}(w-\mu)^T\right)$$

1.3 Expected Value

$$E[x] = \int x\cdot f(x); \qquad E[x] = \sum x\cdot f(x)$$
$$E[ax] = a\cdot E[x]; \qquad E[x+y] = E[x]+E[y]$$
$$E[x|y] = \int x\cdot f(x|y); \qquad E[y] = E[g(x)]; \quad y = g(x)$$

If (x,y) independent: $\qquad E[x\,y] = E[x]\cdot E[y]$

1.4 Variance

$$Var[x] = E[(x-\mu)(x-\mu)^T] \overset{2D}{=} E[x^2] - E[x]^2 \qquad \text{(cont.)}$$
$$Var[x] = \sum_{i=1}^{n} p_i\cdot(x_i-\mu)^2 = \sum_{i=1}^{n}(p_i\cdot x_i^2) - \mu^2 \qquad \text{(discrete)}$$

For x uniformly in $[a,b]$, $E[x] = 0.5(a+b), Var[x] = 12(b-a)^2$.

1.5 Conditioning

$$f(x,y) = f(x|y)\cdot f(y)$$

1.6 Total Probability Theorem

$$f(x) = \sum_y f(x|y)\,f(y) \qquad f(x) = \int f(x|y)\,f(y)\,dy$$

1.7 Bayes' Theorem

$$f(x|y) = \frac{f(y|x)\cdot f(x)}{f(y)} \qquad f(x|z(1:k)) = \frac{f(z(k)|x) f(x|z(1:k-1))}{f(z(k)|z(1:k-1))}$$

1.8 Multivariable Change of Variables

$\mathcal{Y}_i = g(x_j)|x_j$
$z = g(w)$, g diff'able and strictly monotonic, unique solution $w = h(z)$.

$$\text{discr.:} \quad f_x(x_j) = \sum_{y_{j,i}\in\mathcal{Y}_i} f_y(y_{j,i}) \qquad \text{cont.:} \quad f_{z|x} = \frac{f_{w|x}(h(z))}{\left|\frac{\partial g}{\partial w}(h(z)\right|}$$

2 Bayesian Tracking

Process Model:

$$x(k) = q_{k-1}(x(k-1), v(k-1))$$
$$z(k) = h_k(x(k), w(k))$$

Step 1 (Prior Update):

$$f(x(k)|z(1:k-1)) = \sum_{x(k-1)\in\chi} f(x(k)|x(k-1))\,f(x(k-1)|z(1:k-1))$$

Step 2 (Measurement Update):

$$f(x(k)|z(1:k)) = \frac{f(z(k)|x(k))\,f(x(k)|z(1:k-1))}{\sum_x f(z(k)|x(k))\,f(x(k)|z(1:k-1))}$$

3 Extracting Estimates from PDFs

Maximum Likelihood Estimation (MLE), Maximum a Posteriori (MAP) or Minimum Mean Squared Error (MMSE).

$$\hat{x}_{ML} = \arg\max f(z|x)$$
$$\hat{x}_{MAP} = \arg\max f(z|x) f(x)$$
$$\hat{x}_{MMSE} = \arg\min E[(\hat{x}-x)^T\cdot(\hat{x}-x)|z]$$

$\frac{d}{d\hat{x}}E[(\hat{x}-x)^T\cdot(\hat{x}-x)|z] = 0$ (only CRV) gives $\hat{x}_{MMSE} = E[x|z]$

4 Recursive Least Squares (RLS) Algorithm

Observation Model:

$$z(k) = H(k)x + w(k)$$
$$\bar{x} = E[x], \quad E[w(k)] = 0, \quad R(k) = Var[(w(k)]$$

Initialization: $\hat{x}(0) = \bar{x}, \quad P(0) = P_x = Var[x]$

Recursion: observe $z(k)$ and then update as follows

$$K(k) = P(k-1)H^T(k)\cdot[H(k)\,P(k-1)\,H^T(k) + R(k)]^{-1}$$
$$\hat{x}(k) = \hat{x}(k-1) + K(k)\cdot[z(k) - H(k)\hat{x}(k-1)]$$
$$P(k) = [I - K(k)\,H(k)]\,P(k-1)\,[I - K(k)H(k)]^T + K(k)\,R(k)\,K(k)^T$$
$$e(k) = [I - K(k)\,H(k)]\,e(k-1) - K(k)w(k)$$

5 Kalman Filter

Process Model:

$$x(k) = A(k-1)x(k-1) + B(k-1)u(k-1) + v(k-1)$$
$$z(k) = H(k)\cdot x(k) + w(k)$$

$x(k):$	State	$x(0)\sim\mathcal{N}(x_0,P_0)$
$v(k):$	Process Noise	$v(k)\sim\mathcal{N}(0,Q(k))$
$w(k):$	Sensor Noise	$w(k)\sim\mathcal{N}(0,R(k))$

5.1 Time varying KF

Initialization: $\hat{x}_m(0) = x_0, \quad P_m(0) = P_0$
Step 1 (Prior Update):

$$\hat{x}_p(k) = A(k-1)\,\hat{x}_m(k-1) + B(k-1)\,u(k-1)$$
$$P_p(k) = A(k-1)\,P_m(k-1)\,A^T(k-1) + Q(k-1)$$

Step 2 (Measurement Update):

$$K(k) = P_p(k)H^T(k)\cdot[H(k)P_p(k)H^T(k) + R(k)]^{-1}$$
$$\hat{x}_m(k) = \hat{x}_p(k) + K(k)\cdot(z(k) - H(k)\hat{x}_p(k))$$
$$P_m(k) = [I - K(k)H(k)]P_p(k)$$
$$= [I - K(k)H(k)]P_p(k)[I - K(k)H(k)]^T + K(k)R(k)K^T(k)$$
$$e(k) = [I - K(k)\,H(k)]\,A(k-1)\,e(k-1) + ...$$
$$...[I - K(k)H(k)]\,v(k-1) - K(k)w(k)$$

The time-varying KF is the exact solution to the Bayesian tracking problem for linear systems with Gaussian distributions.

5.2 Kalman Filter as State Observer

A, B, H, Q, R constant, but the KF is still varying:

$$P_p(k) = A\,P_m(k-1)\,A^T + Q$$
$$K(k) = P_p(k)\,H^T\,(H\,P_p(k)\,H^T + R)^{-1}$$
$$P_m(k) = [I - K(k)\,H]\,P_p(k)$$

The variance $P_p(k)$ converges if (A,H) is detectable or observable.
Sometimes the state dynamics can be decoupled (e.g. A is diagonal). Get two independent KFs, simplifies calculations!

5.2.1 Observability

$$\mathcal{O} = \begin{bmatrix} H \\ HA \\ \vdots \\ HA^{n-1} \end{bmatrix}$$

If $\text{rank}(\mathcal{O}) = n$, the pair (A,H) is observable.

Observability $\Rightarrow$ Detectability.

Alternatively, check if $\begin{bmatrix} A-\lambda I \\ H \end{bmatrix}$ has full col. rank for all $\lambda\in\mathbb{C}$.

5.2.2 Detectability (Check observability first)

(A,H) is detectable if $\begin{bmatrix} A-\lambda I \\ H \end{bmatrix}$ has full column rank. It is sufficient to only check the unstable Eigenvalues ($|\lambda_i|\geq 1$) of A. Detectability implies convergence of the variance $P_p(k)$: $\lim_{k\to\infty} P_p(k) = P_\infty$.

5.3 Steady state KF

A, B, H, Q, R constant; $\lim K(k) = K_\infty$; $\lim P_p(k) = P_\infty$

Discrete Algebraic Riccati Equation (DARE), multi-dim. and scalar:

$$P_\infty = AP_\infty A^T + Q - AP_\infty H^T (HP_\infty H^T + R)^{-1} \cdot HP_\infty A^T$$
$$K_\infty = P_\infty H^T (HP_\infty H^T + R)^{-1}$$
$$p = a^2 p + \sigma_v^2 - \frac{a^2 p^2 h^2}{h^2 p + \sigma_w^2}$$
$$K_\infty = \frac{ph}{h^2 p + \sigma_w^2}$$

Steady State Estimator ($\hat{x}(k) = \hat{x}_m(k)$) with error dynamics ($e_k = x_k - \hat{x}_k$):

$$\hat{x}(k) = (I - K_\infty H)A\hat{x}(k-1) + (I - K_\infty H)Bu(k-1) + K_\infty z(k)$$
$$= \hat{A}\hat{x}(k-1) + \hat{B}u(k) + K_\infty z(k)$$
$$e(k) = (I - K_\infty H)Ae(k-1) + (I - K_\infty H)v(k-1) - K_\infty w(k)$$
$$E[e(k)] = (I - K_\infty H)AE[e(k-1)].$$

DARE only has a unique solution for P_∞ if (A, H) detectable and (A, G) stabilizable (guaranteed if $Q \geq 0$) with $Q = GG^T$. Existence of unique positive semidefinite solution guarantees stable error dynamics.
Check $(I - K_\infty H) \cdot A$ for stability of error danymics (eigenvalues mag. less than 1).

5.4 Extended KF (EKF)

Process Model:

$$x(k) = q_{k-1}(x(k-1), u(k-1), v(k-1))$$
$$z(k) = h_k(x(k), w(k))$$

$$E[x(0)] = x_0 \qquad Var[x(0)] = P_0$$
$$E[v(k-1)] = 0 \qquad Var[v(k-1)] = Q(k-1)$$
$$E[w(k)] = 0 \qquad Var[w(k)] = R(k)$$

Initialization: $\hat{x}_m(0) = x_0$, $P_m(0) = P_0$

Step 1 (Prior Update):

$$A(k-1) = \frac{\partial q_{k-1}}{\partial x(k-1)}(\hat{x}_m(k-1), u(k-1), \textbf{v(k-1)=0})$$

$$L(k-1) = \frac{\partial q_{k-1}}{\partial v(k-1)}(\hat{x}_m(k-1), u(k-1), \textbf{v(k-1)=0})$$

$$\hat{x}_p(k) = q_{k-1}(\hat{x}_m(k-1), u(k-1), \textbf{v(k-1)=0})$$
$$P_p(k) = A(k-1)P_m(k-1)A^T(k-1) + L(k-1)Q(k-1)L^T(k-1)$$

Step 2 (Measurement Update):

$$H(k) = \frac{\partial h_k}{\partial x(k)}(\hat{x}_p(k), \textbf{w(k) = 0})$$

$$M(k) = \frac{\partial h_k}{\partial w(k)}(\hat{x}_p(k), \ \textbf{w(k) = 0})$$

$$K(k) = P_p(k)H^T(k)[H(k)P_p(k)H^T(k) + M(k)R(k)M^T(k)]^{-1}$$
$$\hat{x}_m(k) = \hat{x}_p(k) + K(k)\,(z(k) - h_k(\hat{x}_p(k), 0))$$
$$P_m(k) = (I - K(k)H(k))\,P_p(k)$$

5.5 Hybrid KF (HKF)

System with continuous process model and discrete measurements:

$$\dot{x}(t) = q(x(t), u(t), v(t), t) \qquad E[v(t)] = 0; \quad E[v(t)v(t+T)^T] = Q_c\delta$$
$$z[k] = h_k(x(k), w(k)) \qquad E[w(k)] = 0; \quad Var[w(k)] = R$$

Initialization: $\hat{x}_m(0) = E[x_0] = x_0$; $P_m(0) = Var[x_0] => P_0$

Step 1 (Prior Update):
Solve $\dot{\hat{x}}(t) = q(\hat{x}(t), u(t), \textbf{v=0}, t)$
for $(k-1)T \leq t \leq kT$ and $\hat{x}((k-1)T) = \hat{x}_m(k-1)$
Then $\hat{x}_p(k) = \hat{x}(kT)$

Solve $\dot{P}(t) = A(t)\,P(t) + P(t)\,A^T(t) + L(t)\,Q_c\,L^T(t)$
for $(k-1)T \leq t \leq kT$ and $P((k-1)T) = P_m(k-1)$
where $A(t) = \frac{\partial q}{\partial x}(\hat{x}(t), u(t), 0, t)$ and $L(t) = \frac{\partial q}{\partial v}(\hat{x}(t), u(t), 0, t)$.
Then $P_p(k) = P(kT)$

Step 2 (Measurement Update): Same as EKF $z(k) \to z[k]$

6 Particle Filter

Process model: Same as EKF

Initialization: Draw N samples $x_m^n(0)$ from $f(x(0))$

Step 1 (Prior Update): draw N samples for $v^n(k-1)$

$$x_p^n(k) = q_{k-1}(x_m^n(k-1), v^n(k-1))$$

Step 2 (A Posterior):

$$\beta_n = \alpha\, f_{z|x(k)}; \qquad \alpha = \left(\sum_{n=1}^{N} f_{z_n|x_p}\right)^{-1}$$

Resampling:
Select random number r u.a.r. in $[0, 1]$, pick $\bar{n}$ s.t. $\sum_{n=1}^{\bar{n}} \beta_n \geq r$ and $\sum_{n=1}^{\bar{n}-1} \beta_n < r$. Then draw new particles from old particles as if they were a PDF. New particle subset:

$$x_m^n(k) = x_p^{\bar{n}}$$

For two particles, simply take first particle iff $r \leq \beta_1$.

6.1 Roughening

Perturb particles after resampling: $x_m^n(k) \leftarrow x_m^n(k) + \Delta x^n(k)$

Where $x^n(k)$ is from a zero-mean finite-variance distribution, e.g. $Var[\Delta x_i^n(k)] = \sigma_i^2$ and $\sigma_i = KE_i N^{-1/d}$ where

K:	Tuning Parameter
d:	Dimension of the state space
$N^{-1/d}$:	Space betw. nodes of the corresponding uniform square grid
E_i:	$= \max_{n_1, n_2} \lvert x_{m,i}^{n_1}(k) - x_{m,i}^{n_2}(k)\rvert$

7 Observer Based Control

7.1 LTI Observer

LTI System:

$$x(k) = Ax(k-1) + Bu(k-1) + v(k-1)$$
$$z(k) = Hx(k) + w(k)$$

Linear Static Gain Observer (Luenberger Observer):

$$\hat{x}(k) = A\hat{x}(k-1) + Bu(k-1) + K(z(k) - \hat{z}(k))$$
$$\hat{z}(k) = H(A\hat{x}(k-1) + Bu(k-1))$$
$$e(k) = (I - K\,H)\,A\,e(k-1)$$

$e(k) \to 0$ for $k \to \infty$ if and only if $(I - K\,H)\,A$ is stable.

Steady State:

$$\hat{x}(k) = (I - K_\infty H)\,A\,\hat{x}(k-1) + (I - K_\infty)B\,u(k-1) + K_\infty z(k)$$

The steady-state KF is one way to design the observer gain K (optimal in minimizing the Steady State mean squared error).

(A, H) detectable $\Rightarrow$ K exists such that $(I - K\,H)A$ is stable.

7.2 Static State Feedback Control

Design of a controller without paying attention to the state estimation:

$$x(k) = Ax(k-1) + Bu(k-1) \qquad \text{(Process without noise)}$$
$$z(k) = x(k) \qquad \text{(Perfect State information)}$$
$$u(k) = F \cdot z(k) = F \cdot x(k) \qquad \text{(Control Law)}$$

Closed loop dynamics: $x(k) = (A + BF)$. Hence system is stable if $(A + BF)$ is stable. Such an F exists only if (A, B) is stabilizable.
If (A, B) is stabilizable and (A, G) detectable, then F is given by

$$F = -(B^T PB + \bar{R})^{-1} \cdot B^T PA; \qquad P \geq 0$$

P from DARE: $P = A^T PA + \bar{Q} - A^T PB(B^T PB + \bar{R})^{-1} \cdot B^T PA$

7.3 Separation Principle (Linear Systems only)

Combining Luenberger Observer and Static State Feedback control yields:

$$\begin{bmatrix} x(k) \\ e(k) \end{bmatrix} = \begin{bmatrix} A + BF & -BF \\ 0 & (I - KH)A \end{bmatrix} \cdot \begin{bmatrix} x(k-1) \\ e(k-1) \end{bmatrix}$$

Eigenvalues of closed loop are given bei Eigenvalues of $(I - KH)A$ and $(A + BF)$. System is stable as long as there exists no $|\lambda| \geq 1$.

7.4 Separation Theorem

1. Design steady-state KF which does not depend on $\bar{Q}, \bar{R}$. $\Rightarrow \hat{x}(k)$
2. Design state-feedback $u(k) = Fx(k)$ and put both together.