Calculus (II)

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Contents

1. Multiple Integral

1.1. Defining Mass

Assume D is Jordan measurable. Given a density function $f : D \to \mathbb{R}$, the mass of the object is given by the integral of the density function over the region D, denoted by $\int_D f \, \mathrm{d}\sigma$.

How can we define the integral? Like what we learned in the single-variate case, we can use Riemann sums to define the integral.

$$\int_D f \, \mathrm{d}\sigma = \lim_{\|P\| \to 0} \sum \tilde{f}(\xi_i) \ \mathrm{Vol}(P_i), \text{ if the limit exists.}$$

Here, P is a partition of D, and ξ_i is a point in the i-th subregion of the partition. The volume of the i-th subregion is denoted by $\mathrm{Vol}(P_i)$.

1.2. Multiple Integral

The intuition above is "using finite combination of straight polyhedron to approximate the region D". Similarly, introducing Riemann sums gives us the multiple integral, as in the single-variate case.

By the intuition above, we can define the multiple integral of a function over a region D.

Definition 1.2.1 Multiple Integral

Let $D \subset Q$ be a bounded Jordan measurable set, where Q is a bounded straight polyhedron in $\mathbb{R}^n$. Let $f : D \to \mathbb{R}$. For any given straight polyhedron partition P of Q and any choice of points ξ_i in the i-th subregion of the partition, we consider the Riemann sum

$$\sum_{i=1}^{n(P)} \tilde{f}(\xi_i) \ \mathrm{Vol}(P_i)$$

where

$$\tilde{f} = \begin{cases} f(\xi), \text{if } \xi \in D \\ 0, \text{otherwise} \end{cases}.$$

If the limit of the Riemann sum exists as the norm of the partition goes to zero, we say that f is integrable over D, and we define the multiple integral of f over D as

$$\int_D f \, \mathrm{d}\sigma = \lim_{\|P\| \to 0} \sum \tilde{f}(\xi_i) \ \mathrm{Vol}(P_i).$$

Note 1.2.1

(1) f is integral over $D \Rightarrow f$ is bounded on D.

However, the converse is not true. Consider $f(x,y) = \begin{cases} 1, & x,y \in \mathbb{Q} \cap [0,1] \\ 0, & \text{otherwise} \end{cases}$.

(2) f is integral over $D \Leftrightarrow \lim_{\|P\| \to 0} \omega(P_i)\mathrm{Vol}(P_i) = 0$, where

$$\omega(P_i) = \sup_{\xi \in P_i} f(\xi) - \inf_{\xi \in P_i} f(\xi)$$

stands for the oscillation of f on P_i.

(3) If f is continuous on $\bar{D}$, then f is integrable over D.

(4) f is integrable over $D \Leftrightarrow$ The Lebesgue measure of the set of discontinuities of f on D is zero.

The basic properties of multiple integrals are similar to those of single integrals, omitted here.

1.2.1. Calculation of Multiple Integrals - Fubini Theorem

The core idea of the Fubini theorem is "reducing the multiple integral to iterated integrals", which we are able to compute by the single-variable calculus.

Theorem 1.2.1.1 Fubini Theorem

Let f be continuous on $D = [a,b] \times [c,d]$. Then

$$\int_a^b \underbrace{\int_c^d f(x,y)\,\mathrm{d}y}_{I(x)}\,\mathrm{d}x \overset{(I)}{=} \iint_D f(x,y)\,\mathrm{d}\sigma \overset{(J)}{=} \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Corollary 1.2.1.1

(1)

$$\iint_D f(x,y)\,\mathrm{d}\sigma = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

(2)

$$\iint_D f(x,y)\,\mathrm{d}\sigma = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Example 1.2.1.1

$$\int_0^1 \frac{x-1}{\ln x}\,\mathrm{d}x.$$

Solution:

$$\int_0^1 \frac{x-1}{\ln x} \, \mathrm{d}x = \int_0^1 \int_0^1 x^y \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 \int_0^1 x^y \, \mathrm{d}x \, \mathrm{d}y = \dots.$$

□

Example 1.2.1.2 Using Periodicity of the Integrand

$$\iint_D |\cos(x+y)| \, \mathrm{d}\sigma, D = [0, \pi] \times [0, \pi].$$

Solution: Notice

$$\int_0^\pi |\cos(a+x)| \, \mathrm{d}x = \int_0^\pi |\cos x| \, \mathrm{d}x = 2,$$

$$\iint_D |\cos(x+y)| = \int_0^\pi 2 \, \mathrm{d}y = 2\pi.$$

□

1.2.1.1. Fubini Theorem for Triple Integrals

The Fubini theorem can be generalized to triple integrals.

Theorem 1.2.1.1.1 Fubini Theorem for Triple Integrals, (i)

(i) Ω is a cylindrical set, and f is continuous on Ω.

Let the projection of Ω onto the xy-plane be D, and the lower and upper surfaces of Ω be $z = h_1(x, y)$ and $z = h_2(x, y)$, respectively. Then

$$\iiint_\Omega f(x, y, z) \, \mathrm{d}V = \iint_D \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z) \, \mathrm{d}z \right] \mathrm{d}x \, \mathrm{d}y.$$

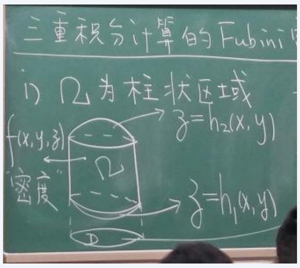

Figure 1: (i)

Example 1.2.1.1.1

Solve for the volume of the solid bounded by the equations

$$\begin{cases} z = x^2 + 3y^2 \\ z = 8 - x^2 - y^2 \end{cases}.$$

Solution: The intersection: $x^2 + 3y^2 = 8 - x^2 - y^2$ gives $\partial D : \frac{x^2}{4} + \frac{y^2}{2} = 1$.

$$V = \iint_D \left(\int_{x^2+3y^2}^{8-x^2-y^2} \mathrm{d}z \right) \mathrm{d}x \, \mathrm{d}y = \iint_D (8 - 2x^2 - 4y^2) \, \mathrm{d}x \, \mathrm{d}y$$

$$= \int_0^2 \mathrm{d}x \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8 - 2x^2 - 4y^2) \, \mathrm{d}y = \dots.$$

$\square$

Theorem 1.2.1.1.2 Fubini Theorem for Triple Integrals, (ii)

(ii) Ω is a laminar set, and f is continuous on Ω.

Let the z-coordinate of the lowermost and uppermost points of Ω be $z = a, b$. Then

$$\iiint_\Omega f(x, y, z) \, \mathrm{d}V = \int_a^b \iint_{D(z)} f(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z.$$

Question 1.2.1.1.1 Volume of Hyperspheres

$B_n(a) = \{(x_1, \dots, x_n) | \sum x_i^2 \leq a^2\}$. Solve for $\mathrm{Vol}\left(B_{n(a)}\right)$.

Solution: $\#_1$: Using recurrence.

$\square$

Theorem 1.2.1.1.3 Generalized Fubini Theorem

Let Ω be a Jordan measurable set in $\mathbb{R}^n$, and f be continuous on Ω. Then

$$\int_{X \times Y} f(x_1, \dots, x_m; y_1, \dots, y_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_m \, \mathrm{d}y_1 \dots \mathrm{d}y_n$$

$$= \int_X \left(\int_Y f(x_1, \dots, x_m; y_1, \dots, y_n) \, \mathrm{d}y_1 \dots \mathrm{d}y_n \right) \mathrm{d}x_1 \dots \mathrm{d}x_m.$$

1.3. Transformation of Coordinates

Theorem 1.3.1

Let $O \subset \mathbb{R}^2$ be a bounded open set; $\varphi : O \to \mathbb{R}^2, (u, v) \mapsto (x, y)$ be a C^1 transformation, and $E \subset O$ be a Jordan measurable set. If

- (i) $\det D\varphi(u, v) = \frac{\partial(x,y)}{\partial(u,v)} \neq 0, \forall (u, v) \in E^o$;
- (ii) φ is injective on E^0,

then $\varphi(E)$ is still Jordan measurable, and for any continuous function f on $\varphi(E)$, we have

$$\int_{\varphi(E)} f(x, y) \,\mathrm{d}\sigma = \int_E (f \circ \varphi) \det D\varphi \,\mathrm{d}u \,\mathrm{d}v.$$

1.4. Exercises

Question 1.4.1

1.5. Appendix : Random Integrals in Calculation

Question 1.5.1

$$\int_0^1 \sqrt{\frac{1 - r^2}{1 - r^2}} r^3 \,\mathrm{d}r.$$

Solution:

$$\begin{aligned}
\int_0^1 \sqrt{\frac{1 - r^2}{1 - r^2}} r^3 \,\mathrm{d}r &= \frac{1}{2} \int_0^1 \sqrt{\frac{1 - t}{1 + t}} t \,\mathrm{d}t \quad (t = r^2) \\
&= \frac{1}{2} \int_0^1 \sqrt{\frac{(1 - t)^2}{1 - t^2}} t \,\mathrm{d}t \\
&= \frac{1}{2} \int_0^1 \frac{1 - \sin v}{\cos v} \sin v \,\mathrm{d}(\sin v) \quad (t = \sin v) \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} (\sin v - \sin^2 v) \,\mathrm{d}v = 1 - \frac{\pi}{4}.
\end{aligned}$$

□

2. Curvilinear Integral

2.1. Line Integral : The Second Kind & Green's Theorem

2.1.1. Flow and Divergence

Consider the flow of a vector field F along a curve Γ^+ in $\mathbb{R}^2$. The line integral of F along Γ^+ is defined as

$$\oint_{\Gamma^+} \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}s$$

where $\boldsymbol{n}$ is the unit normal vector of Γ^+. Compute as follows:

$$\begin{aligned}
\oint_{\Gamma^+} \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}s &= \oint_{\Gamma^+} P \, \mathrm{d}x - Q \, \mathrm{d}y \\
&= \iint_D \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)}_{\boldsymbol{\nabla} \cdot \boldsymbol{F} / \operatorname{div} \boldsymbol{F}} \mathrm{d}\sigma.
\end{aligned}$$

Here, D is the region enclosed by Γ^+, and P, Q are the components of F.

Thus $\boldsymbol{\nabla} \cdot \boldsymbol{F}$ is created to measure the "flow" of F.

Now if we consider u as the potential function of F, then $\boldsymbol{F} = \boldsymbol{\nabla} u$. The line integral of $\boldsymbol{F}$ along Γ^+ is then

$$\begin{aligned}
\oint_{\Gamma^+} \frac{\partial u}{\partial \boldsymbol{n}} \, \mathrm{d}s &= \oint_{\Gamma^+} \boldsymbol{\nabla} u \cdot \boldsymbol{n} \, \mathrm{d}s \\
&= \iint_D \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u \, \mathrm{d}\sigma \\
&= \iint_D \Delta u \, \mathrm{d}\sigma.
\end{aligned}$$

2.1.2. Curl and Circulation

Now we see what happens in $\mathbb{R}^3$ when we consider the circulation of a vector field $\boldsymbol{F}$ along a small closed curve Γ in $\mathbb{R}^2$. The line integral of $\boldsymbol{F}$ along Γ is defined as

$$\oint_{\Gamma} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

Consider the projection of Γ onto those planes xOy, yOz, zOx. Then the circulation of $\boldsymbol{F}$ along Γ is

$$\begin{aligned}
\oint_{\Gamma} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} &= \oint_{\Gamma_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_1 + \oint_{\Gamma_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_2 + \oint_{\Gamma_3} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_3 \\
&= \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathrm{d}x \, \mathrm{d}y + \iint_S \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathrm{d}y \, \mathrm{d}z + \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}z \, \mathrm{d}x \\
&= \iint_S \boldsymbol{\nabla} \times \boldsymbol{F} \cdot \mathrm{d}\sigma.
\end{aligned}$$

2.2. Surface Integral : The First Kind

When the surface is expressed by two parameters u, v,

$$\|\boldsymbol{r}_{\boldsymbol{u}} \times \boldsymbol{r}_{\boldsymbol{v}}\| \, \mathrm{d}u \, \mathrm{d}v$$

is called the "surface element" of the surface S. Now we try to evaluate the factor $\|\boldsymbol{r}_{\boldsymbol{u}} \times \boldsymbol{r}_{\boldsymbol{v}}\|$.

Theorem 2.2.1

$$\|\boldsymbol{r_u} \times \boldsymbol{r_v}\| = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \left(\frac{D(y,z)}{D(u,v)}, \frac{D(z,x)}{D(u,v)}, \frac{D(x,y)}{D(u,v)}\right)$$
$$= (A, B, C)$$

After changing the coordinates, we are here with the factor $\|\boldsymbol{r_u} \times \boldsymbol{r_v}\|$. Given the above way of calculating $\sqrt{A^2 + B^2 + C^2}$, are there any simpler ways to calculate this? Consider the following.

Note 2.2.1

We have the identity (which is just equvalent to $\sin^2 + \cos^2 = 1$):

$$(\boldsymbol{r_u} \times \boldsymbol{r_v}) \cdot (\boldsymbol{r_u} \times \boldsymbol{r_v}) = \|\boldsymbol{r_u} \times \boldsymbol{r_v}\|^2 = \|\boldsymbol{r_u}\|^2 \|\boldsymbol{r_v}\|^2 - (\boldsymbol{r_u} \cdot \boldsymbol{r_v})^2.$$

So letting

$$\begin{cases} E = x_u^2 + y_u^2 + z_u^2 \\ F = x_u x_v + y_u y_v + z_u z_v \\ G = x_v^2 + y_v^2 + z_v^2 \end{cases}$$

yields $\|\boldsymbol{r_u} \times \boldsymbol{r_v}\| = \sqrt{EG - F^2}$. This is the so-called "first fundamental form" of the surface S.

Corollary 2.2.1

For sphere surface

$$\begin{cases} x = a \cos\theta \sin\varphi \\ y = a \sin\theta \sin\varphi \\ z = a \cos\varphi \end{cases}$$

we have

$$\boldsymbol{r_\theta} = (-a \sin\theta \sin\varphi, a \sin\varphi \cos\theta, 0),$$
$$\boldsymbol{r_\varphi} = (a \cos\theta \cos\varphi, a \cos\varphi \sin\theta, -a \sin\varphi),$$

Then

$$E = a^2 \sin\varphi, F = 0, G = a^2.$$

2.3. Surface Integral : The Second Kind

Definition 2.3.1 Surface Integral : The Second Kind

Let S be a orientable smooth surface in $\mathbb{R}^3$, and $\boldsymbol{F}$ be a vector field defined on S, $\boldsymbol{n}$ is a given unit normal vector of S. The surface integral of $\boldsymbol{F}$ over S is defined as

$$\oiint_S \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}\sigma$$

2.4. Exterior Differential

We call $\mathrm{d}x, \mathrm{d}y, \ldots$ 1-differential forms. Now we introduce the operation "wedging" of two differential forms, denoted by $\wedge$. It is a bilinear operation, and satisfies the following properties:

(1) $\mathrm{d}x \wedge \mathrm{d}x = 0$ (Direct consequence of the antisymmetry);

(2) Antisymmetry: $\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$;

(3) Associativity: $(\mathrm{d}x \wedge \mathrm{d}y) \wedge \mathrm{d}z = \mathrm{d}x \wedge (\mathrm{d}y \wedge \mathrm{d}z)$;

Definition 2.4.1 k-differential Form

The k-differential form is a linear combination of the form

$$\omega = \sum a_{i_1 i_2 \ldots i_k} \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \ldots \wedge \mathrm{d}x_{i_k},$$

where $a_{i_1 i_2 \ldots i_k}$ are smooth functions.

Definition 2.4.2 The d -map

The d-map is a linear map from the space of k-differential forms to the space of $(k+1)$-differential forms, denoted by d. It satisfies the following properties:

(1) $\mathrm{d}(f \, \mathrm{d}x) = \mathrm{d}f \wedge \mathrm{d}x$;

Finally, here's the generalized Stokes' theorem, also known as the "Cartan's magic formula".

2.5. Exercises

Question 2.5.1

Evaluate

$$\oint_C xy \, \mathrm{d}s,$$

where C is the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x + y + z = 0$.

Solution: Consider a basis of the plane: $\boldsymbol{e}_1 = \frac{1}{\sqrt{3}}(1,1,1)$, $\boldsymbol{e}_2 = \frac{1}{\sqrt{2}}(1,0,-1)$.

Then $C : \boldsymbol{x} = a\cos\theta \boldsymbol{e}_1 + a\sin\theta \boldsymbol{e}_2$ is a parameterization of C.

Note 2.5.1

It's wiser to take advantage of the rotational symmetry within the curve.

$$\begin{aligned} I &= \frac{1}{3}\oint_C xy + yz + zx \,\mathrm{d}s \\ &= \frac{1}{6}\oint_C (x+y+z)^2 - \left(x^2+y^2+z^2\right)\mathrm{d}s \\ &= -\frac{1}{6}\pi a^2. \end{aligned}$$

□

2.5.1. Harmonic Functions

Before we introduce the concept of harmonic functions, let's see some equalities on the boundary and interior of a domain.

Theorem 2.5.1.1

Let closed region D be bounded by finitely many smooth curves, and $u, v \in C^2(D)$. $\boldsymbol{n}$ is the unit normal vector of ∂D. Then

(1)

$$\iint_D \Delta u \,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial D} \frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}s,$$

(2)

$$\iint_D v\Delta u \,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial D} v\frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}s - \iint_D \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \,\mathrm{d}x\,\mathrm{d}y,$$

(3)

$$\iint_D (u\Delta v - v\Delta u)\,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial D}\left(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}}\right)\mathrm{d}s.$$

Proof: (1) Let $\boldsymbol{t}$ be the unit tangent vector of L^+ with direction cosines $\cos\alpha, \cos\beta$, then $\boldsymbol{n} = (\cos\beta, -\cos\alpha)$. By the definition of directional derivative,

$$
\begin{aligned}
\oint_{L^+} v\frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}s &= \oint_{L^+}\left(v\cos\beta\frac{\partial u}{\partial x} - v\cos\alpha\frac{\partial u}{\partial y}\right)\mathrm{d}s \\
&= \oint_{L^+}\left(v\frac{\partial u}{\partial x}\,\mathrm{d}y - v\frac{\partial u}{\partial y}\,\mathrm{d}x\right) \\
&= \iint_D \frac{\partial\left(v\frac{\partial u}{\partial x}\right)}{\partial x} + \frac{\partial\left(v\frac{\partial u}{\partial y}\right)}{\partial y}\,\mathrm{d}\sigma(\text{by Green's theorem}) \\
&= \iint_D\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2}\right)\mathrm{d}\sigma \\
&= \iint_D v\Delta u\,\mathrm{d}\sigma + \iint_D\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\mathrm{d}\sigma.
\end{aligned}
$$

Rearranging the terms, we obtain the desired result.

(2) Omitted, similar to (1)(3).

(3) The same discussion as in (1),

$$
\begin{aligned}
\int_{L^+}\left(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}}\right)\mathrm{d}s &= \int_{L^+}\left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right)\mathrm{d}x + \left(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}\right)\mathrm{d}y \\
&= \iint_D\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} - v\frac{\partial^2 u}{\partial y^2}\right. \\
&\quad \left.-\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + v\frac{\partial^2 u}{\partial x^2}\right)\mathrm{d}\sigma \\
&= \iint_D (u\Delta v - v\Delta u)\,\mathrm{d}\sigma.
\end{aligned}
$$

□

Note 2.5.1.1

These are all direct consequences of the Green's theorem. However, when expressed in the language of field theory, they are more intuitive.

$$
\oint_{\partial D}\boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}s = \iint_D \nabla\cdot\boldsymbol{F}\,\mathrm{d}\sigma(\text{Green}).
$$

$$
\oiint_{\partial\Omega^+}\boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}S = \iiint_\Omega \nabla\cdot\boldsymbol{F}\,\mathrm{d}V(\text{Gauss}).
$$

$$
\oint_{\partial S}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \iint_S(\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\,\mathrm{d}S(\text{Stokes}).
$$

Based on these, a simpler proof can be given.

Proof: (1)

$$
\text{RHS} = \oint_{\partial D}\nabla u\cdot\boldsymbol{n}\,\mathrm{d}s = \iint_D \nabla\cdot\nabla u\,\mathrm{d}\sigma = \text{LHS}.
$$

(2)

$$\oint_{\partial D} v\nabla u \, \mathrm{d}s = \iint_D \nabla \cdot (v\nabla u) \, \mathrm{d}\sigma$$
$$= \iint_D \nabla u \nabla v + v\nabla \cdot \nabla u.$$

(3) Direct consequence of (2). □

Definition 2.5.1.1

A function u is called harmonic if it satisfies the Laplace equation $\Delta u = 0$.

Theorem 2.5.1.2 Properties of Harmonic Function

(1) Let D be a domain in $\mathbb{R}^2$, and $u \in C^2(D)$. Then u is harmonic iff u satisfies

$$\oint_C \frac{\partial u}{\partial \boldsymbol{n}} \, \mathrm{d}s = 0,$$

where $C \subset D$ is a circle s.t. the interior of C is in D.

(2) Given the same conditions, u is harmonic iff u satisfies the mean value property:

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta) \, \mathrm{d}\theta,$$

for all $P(x_0, y_0) \in D$ and $0 < r < \mathrm{dist}(P, \partial D)$.

(3) Let D be bounded by finitely many smooth curves, and harmonic function u on D. Then $u \equiv 0$ if $u = 0$ on the boundary of D.

(4) Let L be simple closed curve in $\mathbb{R}^2$, and $u \in C^2(D)$ be harmonic on D. Then u reaches its maximum and minimum on L.

Proof: (1) Just a conclusion in Theorem 2.5.1.1.

(2) From (1), □

Question 2.5.1.1 An Exercise on Mean Value Property

Let $u(x, y)$ be continuous on $\mathbb{R}^2$. Prove that

$$u(x, y) = \frac{1}{\pi r^2} \iint_{(x-\xi)^2 + (y-\eta)^2 \le r^2} u(\xi, \eta) \, \mathrm{d}\xi \, \mathrm{d}\eta, \forall r > 0$$

holds iff

$$u(x, y) = \frac{1}{2\pi r} \oint_{C_{r(\xi,\eta)}} u(\xi, \eta) \, \mathrm{d}s, \forall r > 0.$$

Proof: Unify the two equations by change of coordinates.

$$\pi r^2 u(x, y) = \int_0^{2\pi} \mathrm{d}\theta \int_0^r u(x - \rho\cos\theta, y - \rho\sin\theta)\rho \,\mathrm{d}\rho.$$

Take the derivative of the equation above with respect to r.

$$\begin{aligned} 2\pi r u(x, y) &= \int_0^{2\pi} u(x - r\cos\theta, y - r\sin\theta) r \,\mathrm{d}\theta \\ &= \oint_{C_{r(\xi,\eta)}} u(\xi, \eta) \,\mathrm{d}s. \end{aligned}$$

□

2.6. More Exercises

Example 2.6.1 Green Formula with Singularities

Solve for

$$I = \oint_C \frac{e^x}{x^2 + y^2}[(x\sin y - y\cos y)\,\mathrm{d}x + (x\cos y + y\sin y)\,\mathrm{d}y],$$

where C is a close curve enclosing the origin, oriented counterclockwise.

Solution: Note that $\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0$, denote L as a small enough circle with radius r enclosing the origin, oriented clockwise. Then by Green's theorem,

$$\begin{aligned} I &= \oint_{C \cup L} - \oint_L = 0 - \left(-\int_0^{2\pi} e^{r\cos\theta}\cos(r\sin\theta)\,\mathrm{d}\theta\right) \\ &= e^{\theta_0 r\cos\theta_0}\cos(r\sin\theta_0) \cdot \int_0^{2\pi} \mathrm{d}\theta(\text{MVT of Integral}, \theta \in (0,1)) \\ &= 2\pi e^{\theta_0 r\cos\theta_0}\cos(r\sin\theta_0). \end{aligned}$$

Because the integral is independent of r, we just take the limit as $r \to 0$. and $I = 2\pi$. □

2.7. Appendix : Field Theory

Theorem 2.7.1 Gradient and Gauss Theorem

Let M be a closed surface in $\mathbb{R}^3$ satisfying the conditions of the Gauss theorem, and $\boldsymbol{n}$ is the unit normal vector field of ∂M, scalar field $u \in C^1(M, \mathbb{R})$, a point $\boldsymbol{p} \in |(M)$. Then

$$\boldsymbol{\nabla} u(\boldsymbol{p}) = \lim_{M \to \boldsymbol{p}} \frac{1}{\mathrm{Vol}(M)} \oiint_{\partial M} u\boldsymbol{n} \,\mathrm{d}\sigma.$$

Proof: Consider

$$
\begin{aligned}
\oiint_{\partial M} u\boldsymbol{n}\,\mathrm{d}\sigma &= \oiint_{\partial M} (u\cos\alpha, u\cos\beta, u\cos\gamma)\cdot(\cos\alpha,\cos\beta,\cos\gamma)\,\mathrm{d}\sigma \\
&= \oiint_{\partial M} u\,\mathrm{d}y\,\mathrm{d}z + u\,\mathrm{d}z\,\mathrm{d}x + u\,\mathrm{d}x\,\mathrm{d}y \\
&= \iiint_M \boldsymbol{\nabla}\cdot u\,\mathrm{d}\sigma(\text{Gauss}),
\end{aligned}
$$

where α, β, γ are the angles between $\boldsymbol{n}$ and the coordinate axes.

Then

$$
\boldsymbol{\nabla}\cdot u(p) = \lim_{M\to \boldsymbol{p}} \frac{1}{\mathrm{Vol}(M)} \iiint_M \boldsymbol{\nabla}\cdot u\,\mathrm{d}\sigma.
$$

Then the conclusion follows. □

2.7.1. Laplacian, Green Formula and Field Theory

Note 2.7.1.1

For curve integral, let $\boldsymbol{n} = (\cos\alpha, \cos\beta)$ be the unit normal vector of the curve L, then

$$
\cos\alpha\,\mathrm{d}s = \mathrm{d}y, \cos\beta\,\mathrm{d}s = -\,\mathrm{d}x.
$$

Then the curve integral of $\boldsymbol{F}$ along L is

$$
\oint_L \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}s = \oint_L P\,\mathrm{d}y - Q\,\mathrm{d}x.
$$

Corollary 2.7.1.1

$$
\int_U \frac{\partial u}{\partial x_i}\,\mathrm{d}\boldsymbol{x} = \int_{\partial u} u\cos(\boldsymbol{n}_0, \boldsymbol{e}_i)\,\mathrm{d}\boldsymbol{s}.
$$

Theorem 2.7.1.1 Integral by Parts, but Multiple

$$
\iint_D P\cdot\frac{\partial f}{\partial x}\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial D} f\cdot P\,\mathrm{d}y - \iint_D \frac{\partial P}{\partial x} f\,\mathrm{d}x\,\mathrm{d}y.
$$

$$
\iint_D Q\cdot\frac{\partial f}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial D} f\cdot Q\,\mathrm{d}x - \iint_D \frac{\partial Q}{\partial y} f\,\mathrm{d}x\,\mathrm{d}y.
$$

Exercise 2.7.1.1

Let $D : x^2 + y^2 = a^2$, $f(x, y)$ has continuous partial derivative on D, and $f(x, y) = 0$ on ∂D. Prove that

$$I = \left| \iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y \right| \leq \frac{\pi a^3}{3} \max_{(x,y)\in D} |\boldsymbol{\nabla} f|.$$

Proof: First notice similarity to the Cauchy inequality:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \leq \sqrt{x^2+y^2}\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}^2\right)\right)^{\frac{1}{2}}.$$

To get LHS, integrate by parts:

$$\begin{aligned} I = \iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{\partial D} f\,\mathrm{d}x - \iint_D \frac{\partial f}{\partial x} x\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\partial D} f\,\mathrm{d}y - \iint_D \frac{\partial f}{\partial y} y\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

.

Because $f = 0$ on ∂D, we have

$$\int_{\partial D} f\,\mathrm{d}x = 0, \int_{\partial D} f\,\mathrm{d}y = 0.$$

Then

$$I = -\frac{1}{2}\iint_D \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y \max|\boldsymbol{\nabla} f|.$$

□

3. Differential Equations

3.1. Existence and Uniqueness of Solutions

3.1.1. Picard Iteration

Proof: (1) Construct a rectangle area $D = \{(x,y) \mid |x - x_0| \leq a, |y - y_0| \leq b\}$. Because $p(x)$ is continuous, $p(x)$ is bounded on D, i.e. $|p(x)| \leq M$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = p(x)\cos^2 y - \sin^2 y,$$

$$\begin{aligned} \left|\frac{\partial f}{\partial y}\right| &= |-2p(x)\sin y \cos y - 2\sin y \cos y| \\ &\leq 2|p(x)| + 2 \leq 2M + 2. \end{aligned}$$

Then $f(x, y)$ is bounded on D. By Lagrange Mean Value Theorem, for every $y_1, y_2 \in D$,

$$|f(x,y_1) - f(x,y_2)| = \frac{\partial f}{\partial y}(x,\xi)\ |y_1 - y_2| \leq (2M+2)\ |y_1 - y_2|.$$

Then $f(x)$ is Lipschitz continuous on D. By Picard's theorem, the solution in $[x_0 - h, x_0 + h]$ exists and is unique, where $h = \min\left(a, \frac{b}{M}\right), M = \max|f(x, y)|$.

Consider the endpoint $x_0 + h$, which is still in D. Then we can repeatedly apply Picard's theorem to get the solution in a larger interval.

Thus we proved that the solution exists and is unique.

(2) It's easily □

3.2. Exercises

Exercise 3.2.1

$$(xy - x^3 y^3)\,\mathrm{d}x + (1 + x^2)\,\mathrm{d}y = 0.$$

Solution: y: Divide both sides by y^3,

$$\frac{1}{y^2} x\,\mathrm{d}x - x^3\,\mathrm{d}x + (1 + x^2)\,\mathrm{d}\frac{y}{y^3} = 0$$

$$\frac{1}{y^2} x\,\mathrm{d}x - x^3\,\mathrm{d}x + (1 + x^2)\,\mathrm{d}\left(\frac{1}{y^2}\right) = 0$$

$$vx\,\mathrm{d}x - x^3\,\mathrm{d}x + (1 + x^2)\,\mathrm{d}v = 0, v = \frac{1}{y^2}.$$

Notice $\mathrm{d}\left(\frac{1}{1+x^2}\right) = -\frac{2x}{1+x^2}$, divide both sides by $(1 + x^2)^2$,

$$v\,\mathrm{d}u + u\,\mathrm{d}v = \left(\frac{1}{u} + 1\right)\mathrm{d}u$$

$$\mathrm{d}(uv) = \left(\frac{1}{u} + 1\right)\mathrm{d}u$$

$$v = \ln\frac{|u|}{u} + 1 + \frac{C}{u}.$$

□

3.3. n-th Order Linear ODEs

3.3.1. When Coefficients are Constant

Let's begin from the simplest case where the coefficients are constant.

From the perspective of linear algebra, we can consider the n-th order linear ODE in a elegant way:

Definition 3.3.1.1 Linear ODE

Given the ODE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0,$$

when we denote $\boldsymbol{y} = \left(y, y', \ldots, y^{(n-1)}\right)^T$, we can rewrite the ODE as

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y},$$

where $\boldsymbol{A}$ is the Frobenius matrix

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{pmatrix}.$$

At the first glance, we might want to unify $\boldsymbol{y}'$ and $\boldsymbol{y}$ in both sides. Note that for normal functions, e^{Ax} does the job by $\left(e^{Ax}\right)' = Ae^{Ax}$. So a natural choice is to consider the exponential function of a matrix.

Definition 3.3.1.2 Matrix Exponential

Given a matrix $\boldsymbol{A}$, the exponential of $\boldsymbol{A}$ is defined as

$$e^{\boldsymbol{A}} = I + \boldsymbol{A} + \frac{1}{2!}\boldsymbol{A}^2 + \frac{1}{3!}\boldsymbol{A}^3 + \ldots$$

where I is the identity matrix.

We might wonder why the definition makes sence - after all, matrices and numbers are different things altogether. Using the theory of power series, we can show that the series converges for all matrices, omitting the proof here.

We see that $e^{\boldsymbol{A}x}$ is indeed a solution to the ODE $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$. So the problem reduces to finding a simple form of $e^{\boldsymbol{A}x}$. Recall the theory of operators, $\boldsymbol{A}$ may be better to work with under a change of basis, giving us the Jordan form. Before that, we need the important property of the matrix exponential.

Theorem 3.3.1.1 Properties of Matrix Exponential

The matrix exponential has the following properties:

1. $e^{\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}} = \boldsymbol{P}^{-1} e^{\boldsymbol{A}} \boldsymbol{P}$ for any invertible matrix $\boldsymbol{P}$.

Proof Sketch:

1. We can show that for every polynomial $f(x)$, $f\left(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\right) = \boldsymbol{P}^{-1} f(\boldsymbol{A}) \boldsymbol{P}$. The result follows by considering the power series of $e^{\boldsymbol{A}}$.

□

So we can work with the Jordan form $\boldsymbol{J} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ instead of $\boldsymbol{A}$. The above property allows us to calculate like this:

$$e^{\boldsymbol{A}x} = e^{\boldsymbol{P}\boldsymbol{J}x\boldsymbol{P}^{-1}} \Rightarrow e^{\boldsymbol{A}}\boldsymbol{P} = \boldsymbol{P}e^{\boldsymbol{J}x}.$$

Multiplying $\boldsymbol{P}$ to the right of the equation does not change the result, so the solution has a better form $\boldsymbol{P}e^{\boldsymbol{J}x}$. The Jordan form is a block diagonal matrix, so we can calculate the exponential of each block separately.

Before considering the general case, we can consider the simplest case where $\boldsymbol{A}$ is diagonalizable.

Theorem 3.3.1.2

If $\boldsymbol{J} = \operatorname{diag}(\lambda_1, ..., \lambda_n)$ is diagonal, let $\boldsymbol{P} = (\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_n)$ be the matrix of eigenvectors. Then

$$\boldsymbol{P}e^{\boldsymbol{J}x} = e^{\lambda_1 x}\boldsymbol{\xi}_1 + ... + e^{\lambda_n x}\boldsymbol{\xi}_n.$$

Now $\boldsymbol{P}e^{\boldsymbol{J}x}$ is a basic solution matrix to the ODE. And then we can extract y from the first row of the matrix:

$$y(x) = c_1 e^{\lambda_1 x} + ... + c_n e^{\lambda_n x}.$$

When $\boldsymbol{J}$ is a non-trivial Jordan form, we calculate $e^{\boldsymbol{J}x}$ by considering the Jordan blocks.

Lemma 3.3.1.1 Exponential of Jordan Blocks

Given a Jordan block

$$\boldsymbol{J} = \begin{pmatrix} \lambda & 1 & 0 & ... & 0 \\ 0 & \lambda & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \lambda \end{pmatrix},$$

and a function $f(x)$, if $f(\boldsymbol{J})$ converges, then

$$f(\boldsymbol{J}) = \begin{pmatrix} f(\lambda) & f'(\lambda) & \frac{1}{2!}f''(\lambda) & ... & \frac{1}{(n-1)!}f^{(n-1)}(\lambda) \\ 0 & f(\lambda) & f'(\lambda) & ... & \frac{1}{(n-2)!}f^{(n-2)}(\lambda) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & f(\lambda) \end{pmatrix}.$$

In particular, $e^{\boldsymbol{J}}$ is calculated by the above formula:

$$e^{\boldsymbol{J}} = \begin{pmatrix} e^{\lambda} & e^{\lambda}x & \frac{1}{2!}e^{\lambda}x^2 & ... & \frac{1}{(n-1)!}e^{\lambda}x^{n-1} \\ 0 & e^{\lambda} & e^{\lambda}x & ... & \frac{1}{(n-2)!}e^{\lambda}x^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & e^{\lambda} \end{pmatrix}.$$

Proof Sketch: By virtue of mathematical induction, we can show the fact

$$\boldsymbol{J}^m = \begin{pmatrix} \lambda^m & \binom{m}{1}\lambda^{m-1} & \binom{m}{2}\lambda^{m-2} & ... & \binom{m}{n-1}\lambda^{m-n+1} \\ 0 & \lambda^m & \binom{m}{1}\lambda^{m-1} & ... & \binom{m}{n-2}\lambda^{m-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \lambda^m \end{pmatrix}.$$

Then for polynomial f, the lemma is trivial; for others we might need to consider the power series of f, and the result follows. □

Then

$$e^{\boldsymbol{J}_i(\lambda_i)x} = e^{\lambda_i}\left(I + \boldsymbol{N}_i x + \frac{1}{2!}\boldsymbol{N}_i^2 x^2 + \dots + \frac{1}{(n-1)!}\boldsymbol{N}_i^{n-1} x^{n-1}\right),$$

as a result, for every Jordan block $\boldsymbol{J}_i$ of order n_i, we have a solution matrix block

$$\boldsymbol{P}_i e^{\boldsymbol{J}_i x} = e^{\lambda_i x}\left(\boldsymbol{\xi}_1, x\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2, \dots, \frac{1}{(n_i-1)!}x^{n_i-1}\boldsymbol{\xi}_1 + \dots + \boldsymbol{\xi}_{n_i}\right),$$

corresponding to the solution family of dimension n_i:

$$y(x) = \left(a_0 + a_1 x + \dots + a_{n_i-1}x^{n_i-1}\right)e^{\lambda_i x}.$$

The general solution is the linear combination of all solution families.

But for linear ODEs, things are actually simpler than this, because of the special structure of the matrix $\boldsymbol{A}$, a Frobenius matrix.

Theorem 3.3.1.3 Minimal Polynomial of Frobenius Matrix

Given the Frobenius matrix $\boldsymbol{A}$, the minimal polynomial of $\boldsymbol{A}$ is the characteristic polynomial of $\boldsymbol{A}$.

Proof: See the book **谢启鸿**. □

This nice property promises that every eigenvalue only appears once in the Jordan form, so the approach to solve those ODE can be pretty mechanized.

$$\begin{aligned} &\text{Every eigenvalue only appears once} \\ \Leftrightarrow\ &\text{Eigen root } \lambda_i \text{ with multiplicity } n_i \text{ gives a family of solutions} \\ &\left(a_0 + a_1 x + \dots + a_{n_i-1}x^{n_i-1}\right)e^{\lambda_i x}. \end{aligned}$$

This is the general solution to the ODE, well-known as the characteristic equation method.

3.4. Linear Operator $f(D)$

Denote the vector space of n-times differentiable functions by C^n. The differential operator $D : C^n \to C^0$ is defined as $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Given a polynomial $f(\lambda)$, we can define the linear operator $f(D)$. Then the ODE becomes $f(D)y = 0$. Now it suffice to study the kernel of $f(D)$.

Lemma 3.4.1

$$\ker \operatorname{lcm}(f(x), g(x))(D) = \ker f(D) + \ker g(D).$$

By the primary decomposition lemma, we factor $f(\lambda)$ into irreducible polynomials

$$f(\lambda) = (\lambda - \lambda_1)^{n_1} \dots (\lambda - \lambda_k)^{n_k}.$$

4. Series

4.1. Review

4.1.1. Uniform Convergence

Criteria : M-test, Abel, Dirichlet ($\sum_{n=1}^{\infty} a_n(x)b_n(x)$).

4.1.2. Sufficienct Conditions for Commutative Operations

When is it possible to exchange

$$\lim_{x \to x_0}, \frac{\mathrm{d}}{\mathrm{d}x}, \int_a^b \mathrm{d}x \quad \text{and} \quad \lim_{n \to +\infty}, \sum_0^{\infty}?$$

4.2. Power Series $\sum_0^{\infty} a_n(x - x_0)^n$

Note 4.2.1 Questions of Interest

- Region of convergence: find E s.t. $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges $\Leftrightarrow x \in E$.
- Exchangeability of operations:

$$\lim_{x \to x_0}, \frac{\mathrm{d}}{\mathrm{d}x}, \int_a^b \mathrm{d}x \quad \text{and} \sum_0^{\infty}.$$

- What functions $f(x)$ can be represented by power series?

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}}{n!}(x - x_0)^n, x \in E.$$

4.2.1. Radius of Convergence

Theorem 4.2.1.1 Power Series has a Radius of Convergence

For $\sum_{n=0}^{\infty} a_n x^n$, there exists a number $R \in [0, +\infty]$ such that

i) The series converges absolutely for $|x| < R$.

ii) The series diverges for $|x| > R$.

iii) The series may either converge or diverge for $|x| = R$.

Proof: Let

$$R = \sup\left\{|r| : \sum_{n=0}^{\infty} a_n r^n \quad \text{converges}\right\},$$

then $\forall x, |x| < R, \exists r$ s.t. $\sum_{n=0}^{\infty} a_n r^n$ converges, $|x| < |r| < R$.

Convergence $\Rightarrow$ boundedness, i.e. $|a_n r^n| \leq M, \forall n > N$.

By the **comparison test**,

$$|a_n x^n| = |a_n r^n| \left(\frac{|x|}{|r|}\right)^n \Rightarrow |a_n x^n| \leq M \left(\frac{|x|}{|r|}\right)^n \text{(converges)}.$$

This shows that the series converges absolutely for $|x| < R$. By definition, the series diverges for $|x| > R$. □

Note 4.2.1.1 When $|x| = R$

The series may converge or diverge. E.g.

$$\sum_{n=1}^{\infty} \frac{x^n}{n^p}, R = 1, E = \begin{cases} (-1,1), & p = 0 \\ [-1,1), & p \in (0,1] \\ [-1,1], & p > 1. \end{cases}$$

How to find the radius of convergence? The essence lies in the Cauchy-Hadamard formula.

Theorem 4.2.1.2 Cauchy-Hadamard Formula

Given the power series $\sum_{n=0}^{\infty} a_n x^n$, let

$$R = \frac{1}{\limsup_{n\to+\infty} \sqrt[n]{|a_n|}}.$$

Then the radius of convergence is R.

Example 4.2.1.1

Recall the series $\sum_{n=1}^{\infty} \sqrt[n]{|a_n|}$.

Corollary 4.2.1.1 Ratio Test Equivalent

If $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = l$, then

$$R = \frac{1}{l} = \begin{cases} \frac{1}{l}, l > 0 \\ +\infty, l = 0 \\ 0, l = +\infty. \end{cases}$$

Proof: Fix x, consider $b_n = |a_n x^n|$. Then by a ratio test, we have

$$\lim_{n\to+\infty} \frac{b_{n+1}}{b_n} = |x| \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = l|x|.$$

Compare the above RHS with 1. □

Example 4.2.1.2

$$\sum_{n=1}^{\infty} \frac{1}{n^p} x^n, \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n!}}, \sum_{n=1}^{\infty} n! x^n.$$

All of them have a radius of convergence $R = 1$.

Corollary 4.2.1.2

If $\lim_{n \to +\infty} \sqrt[n]{|a_n|} = l$, then $R = \frac{1}{l}$.

Proof Sketch: Fix x, $\sqrt[n]{|a_n x^n|} = |x| \sqrt[n]{|a_n|} \to l|x|$. □

Example 4.2.1.3

$$\sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{n} x^n, \sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{n} (x+1)^n.$$

4.3. Operations on Power Series

Before we start, let's discuss the properties of power series.

4.3.1. Sum and Product

Theorem 4.3.1.1

Given two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, their sum and product are also power series. Let $R = \min(R_a, R_b)$, then

i) The sum $\sum_{n=0}^{\infty} (a_n \pm b_n) x^n$ has a radius of convergence R.

ii) The product $\sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} c_n x^n$ has a radius of convergence R. $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Proof Sketch (Cauchy): ii) By the property of absolute convergence, reordering the terms of the product series does not change the result. Let $\sum_{n=1}^{\infty} u_n, \sum_{n=1}^{\infty} v_n$ be absolute convergent series, then

$$\sum u_n \sum v_n = \sum c_{i_k} d_{j_k},$$

where all (i_k, j_k) are a permutation of $\mathbb{N} \times \mathbb{N}$.

$$\sum_{0}^{N} \left| c_{i_k} d_{j_k} \right| \le \sum_{n=1}^{\infty} |c_n| \sum_{n=1}^{\infty} |d_n|.$$

□

Example 4.3.1.1

Power Series of $\frac{1}{(1-x)(2-x)}$:

- Sum of two series: $\frac{1}{1-x} - \frac{1}{2-x}$.
- Product of two series: $\frac{1}{1-x} \cdot \frac{1}{2-x}$.

Example 4.3.1.2

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} x^n \cdot \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1}, x \in (-1, 1).$$

Given multiplication, how can we define division? See the following example:

Example 4.3.1.3 Bernoulli Numbers

$$\frac{x}{e^x - 1} \to x = (e^x - 1) \sum_{n=1}^{\infty} B_n \frac{x^n}{n!}, B_0 = 1.$$

4.3.2. Properties of Calculus Operations on Power Series

Theorem 4.3.2.1 Locally Uniform Convergence (-[-]-)

Given a power series $\sum_{n=0}^{\infty} a_n x^n$, the series converges uniformly on every compact subset of the interval of convergence.

i.e. i)

$$\forall [a, b] \text{ s.t.} - R < a < b < R, \sum_{n=0}^{\infty} a_n x^n \rightrightarrows u(x) \text{ on } [a, b].$$

ii) If $\sum_{n=0}^{\infty} a_n R^n$ converges, then $\sum_{n=0}^{\infty} a_n x^n \rightrightarrows u(x)$ on $[0, R]$.

iii) If $\sum_{n=0}^{\infty} a_n (-R)^n$ converges, then $\sum_{n=0}^{\infty} a_n x^n \rightrightarrows u(x)$ on $[-R, 0]$.

Proof Sketch: i)

$$|a_n x^n| \leq |a_n| c^n, c = \max(|a|, |b|).$$

RHS is a convergent geometric series, by M-test, the series converges uniformly on $[a, b]$.

ii)

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \underbrace{a_n R^n}_{\sum_{n=0}^{\infty} a_n R^n \text{ uniformly converges}} \underbrace{\left(\frac{x}{R}\right)^n}_{\substack{\text{monotone} \\ \text{and is uniformly bounded}}}, \left|\frac{x}{R}\right| < 1.$$

The rest follows by Abel test of uniform convergence.

iii) Similar to ii). □

Theorem 4.3.2.2 Continuity of Power Series

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then

i) $\sum_{n=0}^{\infty} a_n x^n$ is continuous on $(-R, R)$.

ii) if $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = R$, then $\sum_{n=0}^{\infty} a_n x^n$ is left-continuous at $x = R$.

iii) if $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = -R$, then $\sum_{n=0}^{\infty} a_n x^n$ is right-continuous at $x = -R$.

Proof Sketch:

□

Theorem 4.3.2.3 Integratability of Power Series

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{is integrable on } (-R, R),$$
$$\int_0^x u(t)\,\mathrm{d}t = \sum_{n=0}^{\infty} \int_0^x a_n t^n\,\mathrm{d}t = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}.\text{(integrate termwise)}$$

Note 4.3.2.1

$\sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$ and $\sum_{n=0}^{\infty} a_n x^n$ have the same radius of convergence. At the endpoints, $\frac{1}{n+1}$ gives us a sense of being smaller, hence being more possibly convergent. We'll discuss this later in the examples.

Proof Sketch:

$$\limsup \sqrt[n]{\left|\frac{a_n}{n+1}\right|} = \limsup \sqrt[n]{|a_n|}.$$

□

Example 4.3.2.1 $\ln(1 + x)$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n, |x| < 1.$$
$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,\mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}, |x| < 1.$$

Solution: Because the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges, $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ is left-continuous at $x = 1$.

At $x = -1$, the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

So $\ln(1 + x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ holds on $(-1, 1]$. □

Example 4.3.2.2 arctan

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}, |x| < 1.$$

$$\forall x \in (-1,1), \arctan x = \sum_{n=0}^{\infty}\int_0^x (-1)^n t^{2n}\,\mathrm{d}t = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}.$$

The series converges at $x = 1, -1$, so $\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ holds on $[-1,1]$.

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

Theorem 4.3.2.4 Differentiating Termwise

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{is differentiable on } (-R,R),$$

$$\frac{\mathrm{d}}{\mathrm{d}x}u(x) = \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}a_n x^n = \sum_{n=0}^{\infty} n a_n x^{n-1}.\text{(differentiate termwise)}$$

$$\forall r < R, \sum_{n=0}^{\infty} a_n x^n \text{ converges in } (-R,R)$$

$$\wedge \sum_{n=0}^{\infty} n a_n x^n \text{ converges uniformly in } [-r,r].$$

The radius of convergence of $\sum_{n=0}^{\infty} n a_n x^n$ is also R.

Corollary 4.3.2.1 Smoothness of Power Series

$$\sum_{n=0}^{\infty} a_n x^n \in C^{\infty}(-R,R)$$

Example 4.3.2.3

Prove that $\forall x \in (-1, 1)$,

$$\sum_{n=0}^{\infty}(n+1)(n+2)x^n = \frac{2}{(1-x)^3}.$$

Reminder: dont be too rigid. Solve for

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)} \left(2 \sum_{n=1}^{\infty} \frac{1}{2n(2n+1)}\right)$$

4.4. Solving for Infinite Sums

4.4.1. Abel Method

Based on Abel's Second Theorem, we can solve for a convergent series $\sum_{n=1}^{\infty} a_n x^n$ by studying the power series $\sum_{n=1}^{\infty} a_n x^n$, whose radius of convergence is at least 1. If we are able to caluclate $S(x)$, then we can find the sum of the series by taking the limit $\lim_{x \to 1^-} S(x) = S(1)$.

Example 4.4.1.1

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n}$$

Solution: By Dirichlet test, the series converges when $x \neq 2k\pi$. Then we only need to calculate the sum on $(0, 2\pi)$. By uniform convergence on compact subsets, $S(x)$ is continuous.

Introduce variable $\alpha \in (-1, 1)$, consider the power series

$$f(\alpha) = \sum_{n=1}^{\infty} \frac{\cos nx}{n} \alpha^n.$$

Differentiate termwise on $(-1, 1)$, we have

$$\begin{aligned} f'(\alpha) &= \sum_{n=1}^{\infty} \alpha^{n-1} \cos nx = \operatorname{Re} \sum_{n=1}^{\infty} \alpha^{n-1} e^{inx} \\ &= \operatorname{Re} \frac{e^{ix}}{1 - \alpha e^{ix}} = \operatorname{Re} \frac{\cos x - \alpha}{1 - 2\alpha \cos x + \alpha^2}. \end{aligned}$$

Integrate on $(-1, 1)$ and notice $f(0) = 0$, we have

$$f(\alpha) = -\frac{1}{2} \ln(1 - 2\alpha \cos x + \alpha^2), \alpha \in (-1, 1).$$

Then

$$S(x) = \lim_{\alpha \to 1^-} f(\alpha) = -\frac{1}{2} \ln(1 - 2\cos x + 1) = -\ln 2 - \ln\left|\sin \frac{x}{2}\right|.$$

Note 4.4.1.1

Informally, we have

$$\begin{aligned}\int \operatorname{Re}\frac{e^{ix}}{1-\alpha e^{ix}}\,\mathrm{d}\alpha &= \frac{1}{2}\left(\int \frac{e^{ix}}{1-\alpha e^{ix}}\,\mathrm{d}\alpha + \text{c.c.}\right)\\ &= -\frac{1}{2}\left(\int \frac{1}{\alpha - e^{-ix}}\,\mathrm{d}\alpha + \text{c.c.}\right)\\ &= -\frac{1}{2}\left(\ln\left(\alpha - e^{-ix}\right) + \ln\left(\alpha - e^{ix}\right)\right)\\ &= -\frac{1}{2}\ln\left(\alpha^2 - \alpha\left(e^{ix}+e^{-ix}\right)+1\right) = -\frac{1}{2}\ln\left(1-2\alpha\cos x+\alpha^2\right).\end{aligned}$$

□

Note 4.4.1.2

From the example above, we see that trig functions are actually the real (imaginary) part of complex exponentials, so we may treat them as geometric series.

4.4.2. Γ and B Functions

Example 4.4.2.1

Prove that

$$\pi = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n+1)!}2^{n+1}.$$

Proof: Note that

$$\mathrm{B}(n+1,n+1) = \frac{(n!)^2}{(2n+1)!} = 2\int_0^{\frac{\pi}{2}} (\cos\theta\sin\theta)^{2n+1}\,\mathrm{d}\theta.$$

We have

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n+1)!}2^{n+1} = 2\sum_{n=0}^{\infty}\mathrm{B}(n+1,n+1)2^{n+1} = 4\sum_{n=0}^{\infty}\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\left(2\cos^2\theta\sin^2\theta\right)^n\,\mathrm{d}\theta.$$

Let $u_n = \left(2\cos^2\theta\sin^2\theta\right)^n = \frac{(1-\cos 2\theta)^n}{4^n}$, then $0 \le u_n \le \frac{1}{2^n}$. By the M-test, the series $\sum_{n=0}^{\infty} u_n$ converges uniformly on $\left[0,\frac{\pi}{2}\right]$, so by continuity we can exchange the sum and integral.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n+1)!}2^{n+1} &= 4\int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\cos\theta\sin\theta\left(2\sin^2\theta\cos^2\theta\right)^n\,\mathrm{d}\theta = 4\int_0^{\frac{\pi}{2}}\frac{\sin\theta\cos\theta}{1-2\sin^2\theta\cos^2\theta}\,\mathrm{d}\theta\\ &= 2\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\sin^2\theta}{2\sin^4\theta-2\sin^2\theta+1} = 2\int_{-1}^{1}\frac{\mathrm{d}u}{u^2+1} = \pi.\end{aligned}$$

□

5. Improper Integrals

5.1. Exercises

Theorem 5.1.1 Frullani

Let $f(x)$ be a continuous function on $(0, +\infty)$, then

$$\int_0^{+\infty} \frac{f(ax) - f(bx)}{x} \,\mathrm{d}x = (f(0^+) - f(+\infty)) \ln\left(\frac{b}{a}\right) \quad (a > 0, b > 0).$$

Even if $f(0^+)$(or $f(+\infty)$) does not exist, we may replace them with 0 if

$$\int_k^{+\infty} \frac{f(x)}{x} \,\mathrm{d}x \text{ converges for some } k \geq 0,$$
$$\left(\text{or } \int_0^{k} \frac{f(x)}{x} \,\mathrm{d}x \text{ converges for some } k > 0\right).$$

Proof:

$$\begin{aligned}
\lim_{\substack{t \to 0 \\ T \to +\infty}} \int_t^T \frac{f(ax) - f(bx)}{x} \,\mathrm{d}x &\overset{u=ax}{=} \lim \int_{at}^{aT} \frac{f(u)}{u} \,\mathrm{d}u - \lim \int_{bt}^{bT} \frac{f(u)}{u} \,\mathrm{d}u \\
&= \lim \int_{at}^{bt} \frac{f(u)}{u} \,\mathrm{d}u - \lim \int_{aT}^{bT} \frac{f(u)}{u} \,\mathrm{d}u \\
&= \lim f(\xi_1) \int_{at}^{bt} \frac{1}{u} \,\mathrm{d}u - f(\xi_2) \int_{aT}^{bT} \frac{1}{u} \,\mathrm{d}u (\text{MVT}) \\
&= (f(0^+) - f(+\infty)) \ln\left(\frac{b}{a}\right) (\xi_1 \to 0^+, \xi_2 \to +\infty).
\end{aligned}$$

Suppose $f(+\infty)$ does not exist, if $\int_k^{+\infty} \frac{f(x)}{x} \,\mathrm{d}x$ converges, by Cauchy criterion, we have

$$\forall \varepsilon > 0, \exists A \text{ s.t. } \forall T > \frac{A}{a}, \left|\int_{aT}^{bT} \frac{f(u)}{u} \,\mathrm{d}u\right| < \varepsilon.$$

Then we can show that the limit

$$\lim_{T \to +\infty} \int_{aT}^{bT} \frac{f(u)}{u} \,\mathrm{d}u = 0.$$

□

Example 5.1.1

$$\int_0^{+\infty} \frac{\sin ax \sin bx}{x} \,\mathrm{d}x.$$

Solution: StackExchange: https://math.stackexchange.com/questions/4436436/calculate-int-0-infty-frac-sinax-sinbxx-dx

Note that $\int_1^{+\infty} \frac{\cos(x)}{x}\,\mathrm{d}x$ converges by Dirichlet's test. Then

$$\begin{aligned} I &= \frac{1}{2}\int_0^{+\infty} \frac{\cos(a-b)x - \cos(a+b)x}{x}\,\mathrm{d}x \\ &= \frac{1}{2} f(0^+) \ln\frac{a+b}{a-b} = \frac{1}{2}\ln\frac{a+b}{a-b}. \end{aligned}$$

□

6. Fourier Series

6.1. Exercises

Question 6.1.1

Let $f(x)$ be a 2π-periodic function with Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

then what's the Fourier series of

$$F(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) f(x+t)\,\mathrm{d}t?$$

Solution: Note that

$$F(x) = \langle f(t) | f(x+t) \rangle,$$

and because

$$g(t) = f(x+t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \cos nt + (b_n \cos nx - a_n \sin nx) \sin nt,$$

by Parseval's identity, we have

$$\begin{aligned} F(x) &= \langle f(t) | g(t) \rangle \\ &= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n (a_n \cos nx + b_n \sin nx) + b_n (b_n \cos nx - a_n \sin nx)) \\ &= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \cos nx. \end{aligned}$$

□

Theorem 6.1.1

$f \in C^k[-\pi, \pi]$, and $f^{(k+1)}$ is integrable or absolute integrable. Then the Fourier coefficients of f satisfy

$$a_n, b_n = o\left(\frac{1}{n^{k+1}}\right).$$

7. Useful Integrals

7.1. Trig Functions

Theorem 7.1.1

$$2\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{A + B\cos\theta} = \int_0^{\pi} \frac{\mathrm{d}\theta}{A + B\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{\mathrm{d}\theta}{A + B\cos\theta} = \frac{\pi}{\sqrt{A^2 - B^2}}, A > B > 0.$$

Proof:

$$\begin{aligned} I_1 &= \int_0^{\pi} \frac{(A - B\cos\theta)\,\mathrm{d}\theta}{A^2 - B^2\cos^2\theta} = \int_0^{\pi} \frac{A\,\mathrm{d}\theta}{A^2 - B^2\cos^2\theta}(\text{odd}) \\ &= \left(\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi}\right) \frac{A(\tan^2 + 1)\,\mathrm{d}\theta}{A^2\tan^2 + (A^2 - B^2)} = \int_{-\infty}^{+\infty} \frac{A\,\mathrm{d}t}{A^2t^2 + (A^2 - B^2)} \\ &= \frac{1}{\sqrt{A^2 - B^2}} \arctan u|_{-\infty}^{+\infty} = \frac{\pi}{\sqrt{A^2 - B^2}}. \end{aligned}$$

$\square$