Calculus (II) Problems

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Contents

1. Differential Equations

Question 1.1

Does there exist a differentiable function $f(x, y, y')$ s.t. $y = \sin x$ and $y = x - \frac{1}{6}x^3$ are solutions to the differential equation $y'' = f(x, y, y')$?

Solution: Let $y' = p$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} y \\ p \end{pmatrix} = \begin{pmatrix} p \\ f(x, y, p) \end{pmatrix}.$$

Choose $y(0) = 0, p(0) = 1$, by the uniqueness of the solution, the answer is no. □

Question 1.2 (Hard)

Prove that

$$x''(t) + x(t) + \arctan x(t) = 2\sin t$$

has no nontrivial solution of period 2π.

Proof: Hint. Multiply the equation by $\sin t$ and integrate.

Assume there exists a solution $x(t)$ of period 2π. Then

$$x'' \sin t + x \sin t + \arctan x \sin t = 2\sin^2 t.$$

Integrate by parts,

$$\begin{aligned} \int_0^{2\pi} x'' \sin t \, \mathrm{d}t &= x' \sin t \mid_0^{2\pi} (= 0) - \int_0^{2\pi} x' \cos t \, \mathrm{d}t \\ &= -\int_0^{2\pi} x \sin t \, \mathrm{d}t - x \cos t \mid_0^{2\pi} (= 0) \end{aligned}$$

Then

$$
\begin{aligned}
\int_0^{2\pi} \arctan x \sin t \, \mathrm{d}t = \int_0^{2\pi} & 2 \sin^2 t \, \mathrm{d}t = 2\pi \\
&= \left(\int_0^{\pi} + \int_{\pi}^{2\pi} \right) \arctan x \sin t \, \mathrm{d}t \\
&< \frac{\pi}{2} \int_0^{\pi} \sin t \, \mathrm{d}t - \frac{\pi}{2} \int_{\pi}^{2\pi} \sin t \, \mathrm{d}t = \pi,
\end{aligned}
$$

which is a contradiction. □

2. Series

2.1. 1

Question 2.1.1 1

Evaluate the following series:

$$
\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}, \sum_{n=1}^{\infty} \arctan \frac{1}{n^2+n+1}
$$

Solution:

$$
\begin{aligned}
\arctan \frac{1}{n^2+n+1} &= \arctan \frac{(n+1)-n}{1+n(n+1)} = \arctan(n+1) - \arctan n, \\
S_n &= \arctan(n+1) - \arctan_1, \sum_{n=1}^{\infty} \arctan \frac{1}{n^2+n+1} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.
\end{aligned}
$$

□

Question 2.1.2 2

Let $a_n > 0$, and $\{a_n - a_{n+1}\}$ be a strictly decreasing sequence. Show that if $\sum_{n=1}^{\infty} a_n$ converges, then

$$
\lim_{n \to +\infty} \left(\frac{1}{a_{n+1}} - \frac{1}{a_n} \right) = +\infty.
$$

Proof: $\sum_{n=1}^{\infty} a_n$ converges, so $a_n \to 0$. Then $a_n - a_{n+1} \to 0$, so

$$
a_n - a_{n+1} > 0, a_n > a_{n+1}.
$$

$$
\begin{aligned}
a_n^2 &= \sum_{k=n}^{\infty} (a_k^2 - a_{k+1}^2) = \sum_{k=n}^{\infty} (a_k - a_{k+1})(a_k + a_{k+1}) \\
&< (a_n - a_{n+1}) \sum_{k=n}^{\infty} (a_k + a_{k+1}).
\end{aligned}
$$

$$
\left(\frac{1}{a_{n+1}} - \frac{1}{a_n} \right) = \frac{a_n - a_{n+1}}{a_n a_{n+1}} > \frac{a_n - a_{n+1}}{a_n^2} > \frac{1}{\sum_{k=n}^{\infty} a_k + a_{k+1}} \to +\infty.
$$

□

Question 2.1.3 3

Let $\sum_{n=1}^{\infty} a_n$ be a divergent series of positive terms. Show that

(1) $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges, $\sum_{n=1}^{\infty} \left(\frac{a_n}{1+n^2 a_n}\right)$ converges.

(2) study the convergence of $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n^2}$, $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$.

Solution: $\frac{a_n}{1+n^2 a_n} \leq \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n}$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges, there are two possibilities:

(i) a_n is bounded by M, then $\frac{a_n}{1+a_n} > \frac{a_n}{1+M} > 0$, $\frac{a_n}{1+a_n^2} > \frac{a_n}{1+M^2} > 0$, so $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$, $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n^2}$ diverges.

(ii) a_n is unbounded, then there exists a subsequence $a_{n_k} \to +\infty$. Then the subsequence $\frac{a_{n_k}}{1+a_{n_k}} = \frac{1}{\frac{1}{a_{n_k}}+1} \to \infty$, so $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges.

As for $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n^2}$, we only have : when $a_n \to +\infty$, $\frac{a_n}{1+a_n^2} \sim \frac{1}{a_n}$, the convergence is uncertain. Example:

$$a_n = \begin{cases} n, n = 2k-1 \\ 0, n = 2k \end{cases} (\text{diverges}), a_n = \begin{cases} n, n = 2^k \\ 0, n \neq 2^k \end{cases} (\text{converges}).$$

□

Question 2.1.4 4

$$\sum_{n=3}^{\infty} \frac{1}{n^\alpha \ln^\beta n (\ln n^\gamma \ln n)}.$$

Question 2.1.5 5

Let $0 < a_n < 1$. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \ln(1-a_n)$ converges.

Proof: View as function $f(x) = \ln(1-x)$. Then

$$f(x) - f(0) = f'(\theta x)x = \frac{x}{1-\theta x} \leq \frac{x}{1-x}, 0 < \theta < 1.$$

$$\Rightarrow \ln(1-a_n) \leq \frac{a_n}{1-a_n}.$$

Convergence of $\sum_{n=1}^{\infty} a_n$ implies $a_n \to 0$. Then $\exists N, \forall n > N, 1 - a_n > \frac{1}{2}$. So

$$\ln(1-a_n) < 2a_n, n > N.$$

Which means $\sum_{n=1}^{\infty} \ln(1-a_n)$ converges. □

Question 2.1.6 7

Show that $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ converges.

Proof: □

Question 2.1.7 8

If $I = \sum_{n=1}^{\infty} a_n$ is series of positive terms, show that $J = \sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges, and the opposite is not necessarily true.

Proof: Counterexample:

$$a_n = \begin{cases} 1, n = 2k-1 \\ \frac{1}{n^4}, n = 2k \end{cases}$$

Then $J = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but I obviously diverges. □

Question 2.1.8 9

Discuss the convergence of the series

$$a_n = \left(\frac{\sqrt{n + \sqrt{n + \sqrt{n}}} - \sqrt{n}}{n} \right)^p,$$

$$b_n = \left(1 - \sqrt[3]{\frac{n-1}{n+1}} \right)^p (p > 0).$$

Solution: Asymptotically,

$$\sqrt{n + \sqrt{n + \sqrt{n}}} = \sqrt{n} \left(1 + \frac{\sqrt{n + \sqrt{n}}}{n} \right)^{\frac{1}{2}} = \sqrt{n} \left(1 + \frac{1}{2} \frac{\sqrt{n + \sqrt{n}}}{n} + o\left(\frac{1}{n^2} \right) \right)$$

$$= \sqrt{n} + \frac{1}{2} + o\left(\frac{1}{\sqrt{n}} \right).$$

$$a_n = \left(\frac{\frac{1}{2} + o\left(\frac{1}{\sqrt{n}} \right)}{n} \right)^p = \frac{1}{(2n)^p} + o\left(\frac{1}{n^{p+1}} \right).\text{(Needs Recheck)}$$

So $p > 1$ for convergence.

$$
\begin{aligned}
\sqrt[3]{\frac{n-1}{n+1}} &= \left(1-\frac{2}{n+1}\right)^{\frac{1}{3}} \\
&\sim 1-\frac{2}{3n}+o\left(\frac{1}{n}\right),
\end{aligned}
$$

$$
b_n = \left(1-1+\frac{2}{3n}+o\left(\frac{1}{n}\right)\right)^p = \left(\frac{2}{3n}+o\left(\frac{1}{n}\right)\right)^p = \frac{1}{\left(\frac{3n}{2}\right)^p}+o\left(\frac{1}{n^p}\right).
$$

□

3. Improper Integrals

Question 3.1

$$
\int_0^{+\infty} \frac{1}{1+x^4}\,\mathrm{d}x.
$$

Solution: $t := \frac{1}{1+x^4}$.

$$
\begin{aligned}
I &= \frac{1}{4}\int_0^1 t^{-\frac{1}{4}}(1-t)^{-\frac{3}{4}}\,\mathrm{d}t \\
&= \frac{1}{4}\mathrm{B}\left(\frac{1}{4},\frac{3}{4}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)} = \frac{\pi}{2\sqrt{2}}.
\end{aligned}
$$

□

Proposition 3.1

$f(x)$ is monotone decreasing on $[0,+\infty)$, $f(x) \geq 0$, if

$$
\int_0^{+\infty} f(x)\,\mathrm{d}x
$$

converges, then

$$
\lim_{x\to+\infty} xf(x) = 0, \lim_{x\to+\infty} x\ln x f(x) = 0.
$$

Proof: By Cauchy's criterion, for any $\varepsilon > 0$, there exists N, for any $n > N$, we choose $\frac{x}{2} > N$,

$$
\varepsilon > \int_{\frac{x}{2}}^{x} f(t)\,\mathrm{d}t \geq f(x)\frac{x}{2} \Rightarrow \lim_{x\to+\infty} xf(x) = 0.
$$

$$
\begin{aligned}
\varepsilon > \int_{x_1}^{x_2} f(x)\,\mathrm{d}x &= \int_{x_1}^{x_2} x f(x)\,\mathrm{d}\ln x \\
&\geq \left(\min_{x_1 \leq x \leq x_2} x f(x)\right) \int_{x_1}^{x_2} \mathrm{d}\ln x = 0 \\
&\text{Choose } x_1 = \sqrt{x}, x_2 = x. \\
&= \frac{1}{2} x f(x) \ln x.
\end{aligned}
$$

□

Question 3.2

Let $f(x)$ be monotone decreasing on $[1, +\infty)$, $f(x) \geq 0$, $f(x) \to 0$. Show that if

$$
\int_1^{+\infty} \frac{f^{p-1}(x)}{x^{\frac{1}{p}}}\,\mathrm{d}x (p > 1)
$$

converges, then

$$
\int_1^{+\infty} f^p(x)\,\mathrm{d}x
$$

converges.

Proof: Note all the integrand are positive, so we can apply the comparison test.

$$
\lim_{x \to +\infty}
$$

□

Question 3.3

Let $f(x) \geq 0$ be monotone increasing on $[0, +\infty)$, and $F(x) = \int_0^x f(t)\,\mathrm{d}t$; $\lim_{x \to 0^+} \frac{x}{F(x)} = 1$. Show that

$$
I = \int_0^{+\infty} \frac{1}{f(x)}\,\mathrm{d}x \text{ and } J = \int_0^{+\infty} \frac{x}{F(x)}\,\mathrm{d}x
$$

both converge or both diverge.

Proof:

$$
\begin{aligned}
&\frac{x}{2} f\left(\frac{x}{2}\right) \leq \int_{\frac{x}{2}}^{x} f(t)\,\mathrm{d}t \leq F(x) \leq x f(x) \\
\Rightarrow\ & 2\frac{x}{2F(2x)} \leq \frac{1}{f(x)} \leq \frac{x}{F(x)}
\end{aligned}
$$

□

Question 3.4

Integrate

$$\int_0^{\pi} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} \, \mathrm{d}x.$$

Solution:

$$\begin{aligned}
I &= \frac{1}{ab} \int_0^{\pi} \frac{\frac{a}{b}\, \mathrm{d}(\tan x)}{\left(\frac{a}{b} \tan x\right)^2 + 1} \, \mathrm{d}x \\
&\text{Let } t = \frac{a}{b} \tan x, \\
&= \frac{1}{ab} \left(\int_0^{+\infty} + \int_{-\infty}^{0} \right) \frac{1}{t^2 + 1} \, \mathrm{d}t = \frac{\pi}{ab}.
\end{aligned}$$

Note 3.1

There's a singularity at $x = \frac{\pi}{2}$, so we split the integral into two parts.

□

Lemma 3.1

Question 3.5

Integrate

$$\int_0^1 \frac{\ln x}{1 - x^2} \, \mathrm{d}x.$$

Solution:

$$\begin{aligned}
I &= \int_0^1 \ln x \sum_{k=0}^{\infty} x^{2k} \, \mathrm{d}x \\
&= \sum_{k=0}^{\infty} \int_0^1 x^{2k} \ln x \, \mathrm{d}x \\
&= -\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = -\frac{\pi^2}{8}.
\end{aligned}$$

□

Question 3.6

$$I = \int_0^1 x^\alpha \ln^\beta x \, \mathrm{d}x$$

When is I convergent?

Solution: The singularities are $0, 1$. We consider

$$I_1 = \int_0^{\frac{1}{2}} x^\alpha \ln^\beta x \, \mathrm{d}x, I_2 = \int_{\frac{1}{2}}^1 x^\alpha \ln^\beta x \, \mathrm{d}x.$$

For I_1, use $\int_0^1 x^\gamma \, \mathrm{d}x$. By a ratio test,

$$\frac{x^\alpha \ln^\beta x}{x^{\alpha-\varepsilon}} = x^\varepsilon \ln^\beta x \to 0, x \to 0.$$

(We need the $\varepsilon > 0$ to ensure the convergence of the ratio).

Note that $\int_0^{\frac{1}{2}} \frac{1}{x^{\varepsilon-\alpha}} \, \mathrm{d}x$ converges iff $\varepsilon - \alpha < 1 \Rightarrow \alpha > -1$.

For I_2,

$$x^\alpha \ln^\beta x \sim \frac{x^\alpha}{(1-x)^{-\beta}}, \text{because}$$
$$\ln x = \ln(1-(1-x)) \sim 1 - x (x \to 1^-).$$

So I_2 converges iff $\beta > -1$. □

4. Fourier Series

Question 4.1

$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$, show that

$$\sum_{n=1}^{\infty} \frac{a_n}{n}, \sum_{n=1}^{\infty} \frac{b_n}{n}$$

both converge.

Proof:

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| \frac{a_n}{n} \right| &\le \sum_{n=1}^{\infty} \frac{a_n^2 + \frac{1}{n^2}}{2} \\
&\le \frac{1}{2} \left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) + \sum_{n=1}^{\infty} \frac{1}{n^2} \right) \\
&\le \frac{1}{2} \left(\|f\|^2 + \frac{\pi^2}{6} \right) < +\infty. (\text{Bessel})
\end{aligned}$$

□