Swiss Institute of Banking and Finance

University of St.Gallen

2. Fixed Income

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7,150 Financial Markets

Agenda

- **Basic Valuation**
- Term Structure of Interest Rates
- Yields and Pricing
- Duration
- Immunization
- Key-Rate Duration
- Credit Risk

Valuation of Assets under Certainty

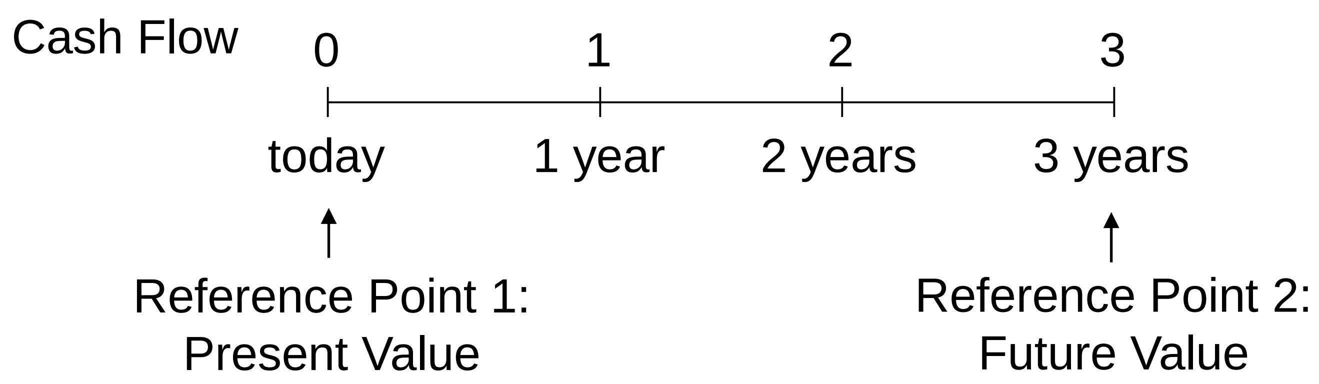
Time value of money

Certainty: Future cash flows of an investment are **certain**.

Valuation principle:

Future cash flows are (usually) less valuable than today‘s cash flows.

⇒ Valuation takes place according to predetermined reference point in time (e.g. Reference Point 1 or 2)

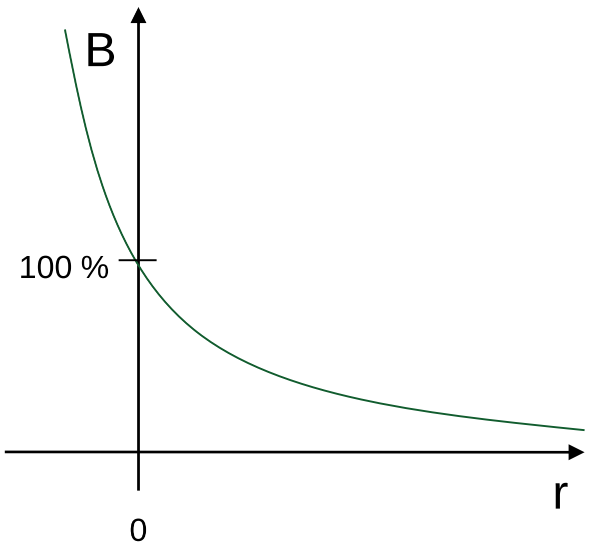


Present Value: Zero Bond

a) Zero Bond

Given: Zero Bond with maturity of 5 years

Discrete Returns (r = 5%)	Continuous Returns (r = 5%)
$B = \frac{100\%}{(1+r)^5} = 78.35\%$	$B = \frac{100\%}{e^{5r}} = 77.88\%$
Discrete discount factor:	Continuous discount factor:
$\frac{1}{(1+r)^5}$	e^{-5r}



Present Value: Coupon Bond

b) Coupon Bond

Given: 10% Coupon, maturity 3 years, 3 coupons

$$B = \frac{10\%}{(1+r)} + \frac{10\%}{(1+r)^2} + \frac{110\%}{(1+r)^3}$$

With r = 5%:

113.62% = 9.52% + 9.07% + 95.02%

Present Value: Annuity

c) Annuity and perpetual annuity (Consol Bond)

Constant discount factor for each period: $q=\frac{1}{1+r}<1$

- Perpetual annuity in arrear

	0	1	2	3	4	n-1	n
Coupon		a	a	a	a	a	a

$$P_0=\lim_{n\to\infty}\sum_{i=1}^{n}(a\cdot q^i)=a\left(\frac{1}{1-q}-1\right)=\frac{a}{r}$$

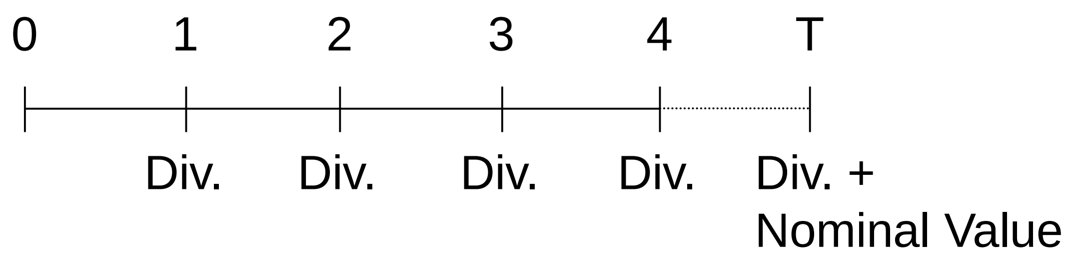
- Perpetual annuity in advance

	0	1	2	3	4	n-1	n
Coupon	a	a	a	a	a	a	

$$P_0=\lim_{n\to\infty}\sum_{i=0}^{n-1}(a\cdot q^i)=a\left(\frac{1}{1-q}\right)=\frac{a(1+r)}{r}$$

Present Value: Stock (1/4)

d) Stock – Classic Dividend Discount Model



Div. = Dividend

Present value of the stock:

$$A_0 = \sum_{t=0}^{T} \frac{D_t}{(1+r)^t} + \frac{N}{(1+r)^T}$$

T = date of liquidation
r = cost of capital

But: assumption that the company will exist forever

$$A_0 = \sum_{t=1}^{\infty} \frac{D_t}{(1+r)^t}$$

→ Infinite geometric series as with an annuity

Present Value: Stock (2/4)

$\Rightarrow$ Infinite geometric series as with an annuity but:

How can the future dividend D_t be reliably determined?

Example:

- The company attains a Return on Investment (ROI) of 12%
- The initial investment is 833.33 CHF
- 40% of the profits are going to be paid as dividends

Year	Profit	Dividend	Reinvested Profit	Total Investment
0	-	-	-	833.33
1	100.00	40.00	60.00	893.33
2	107.20	42.88	64.32	957.65
3	114.92	45.97	68.95	1026.60

...

Present Value: Stock (3/4)

$\Rightarrow$ Dividend is continuously growing

Growth rate g:

$$g = ik$$

Return on invested capital (ROI)

Reinvested part of profit

$1 + ik = 1 + 0.12 * 0.6 = 1.072$

Example: Dividend after 9 years:

$D_9 = D_1(1 + ik)^8 = 40*(1+0.072)^8 = 69.76$

Present Value: Stock (4/4)

Value of the stock at time 1:

$$A_1 = D_1 + D_1 \underbrace{\frac{1+ik}{1+r}}_{q} + D_1 \underbrace{\left(\frac{1+ik}{1+r}\right)^2}_{q^2} + \ldots + D_1 q^n$$

Annuity (in advance):

$$A_1 = \frac{D_1}{1-q} = \frac{D_1}{1-\frac{1+ik}{1+r}} = \frac{D_1\left(1+r\right)}{(1+r)-(1+ik)} = \frac{D_1(1+r)}{r-ik}$$

$$A_0 = \frac{A_1}{1+r} = \frac{D_1}{r-ik}$$ $\Rightarrow$ Pricing Formula of the Dividend Discount Model (**Gordon Growth Model**)

Optimal Payout

Company should invest until $i = r$

$r > (<) i$ indicates that ROI is smaller (larger) than the cost of capital, e.g. a company earns less (more) than it could on the capital market with the same risk.

$\Rightarrow$ If $r < i$ the company should pay out as little as possible.
$\rightarrow$ The higher the payout ratio, the lower the stock price and vice versa.

$\Rightarrow$ If $r > i$ the company should pay out as much as possible.
$\rightarrow$ The higher the payout ratio, the higher the stock price and vice versa.

- At the optimum ($i = r$) the payout ratio is irrelevant for the stock price.
- The optimum is highly sensitive to changes in r and i.
- $ik \rightarrow r \quad \Rightarrow \quad \text{Value} \rightarrow \infty$
- In practice, i is unknown.

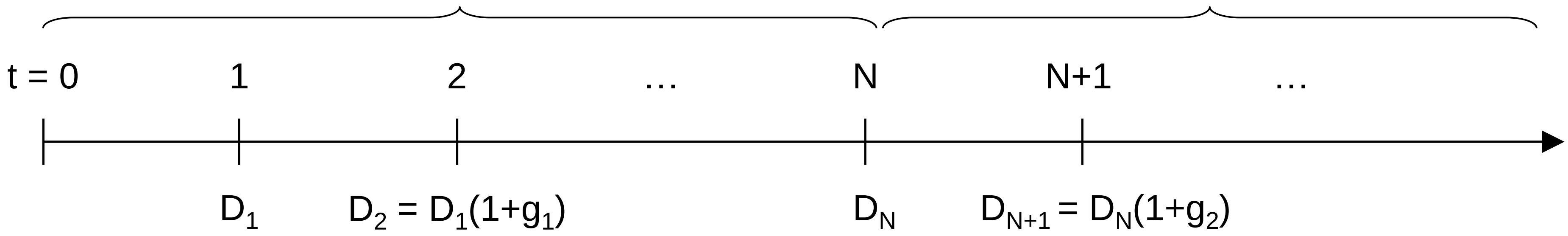
Multistage Dividend Discount Models

The Two-Stage Dividend Discount Model solves the problem of unrealistically high long term dividend growth by assuming that the initial growth rate (g_1) is followed by a sustained terminal growth rate (g_2).

$$A_0 = \sum_{t=1}^{N} \frac{D_1(1+g_1)^{t-1}}{(1+r)^t} + \frac{\dfrac{D_N(1+g_2)}{(r-g_2)}}{(1+r)^N}$$

Value of future dividends before N years discounted to the present

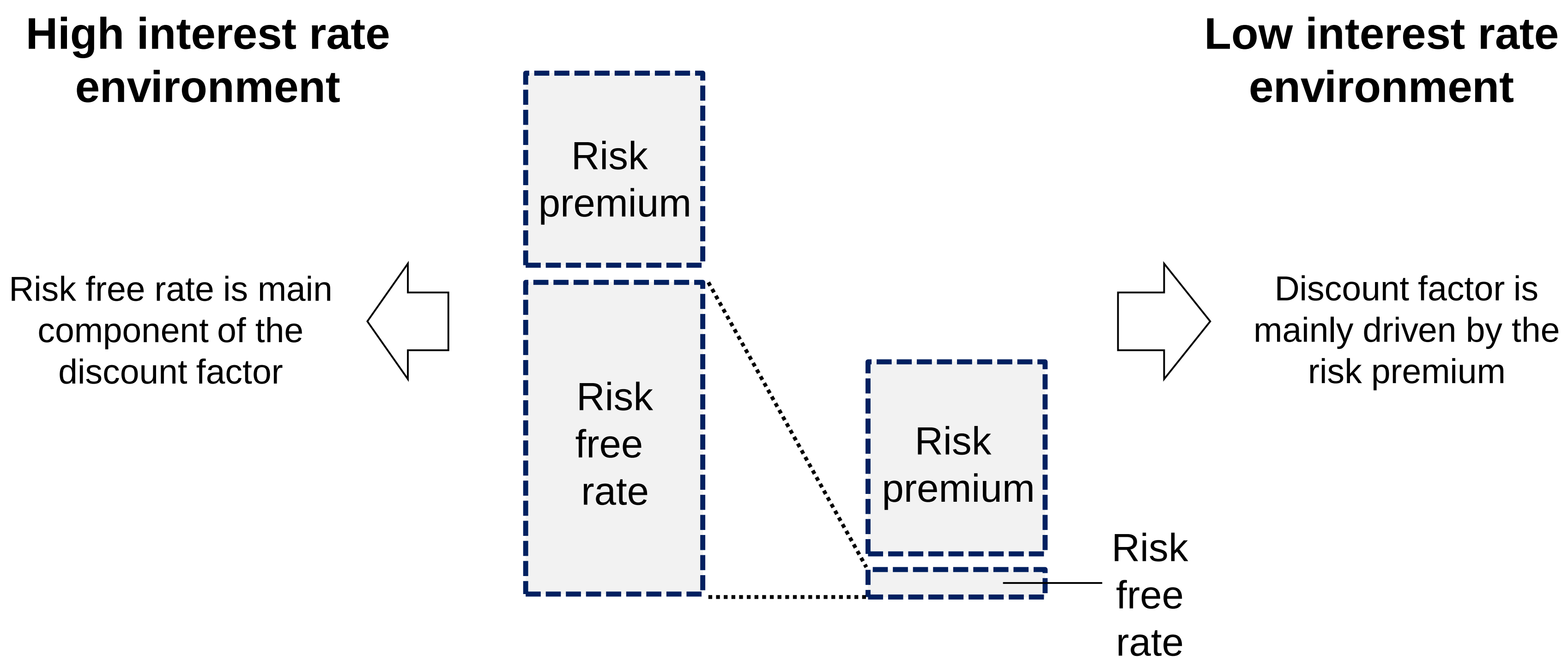
Value of all future dividends after N years discounted to the present



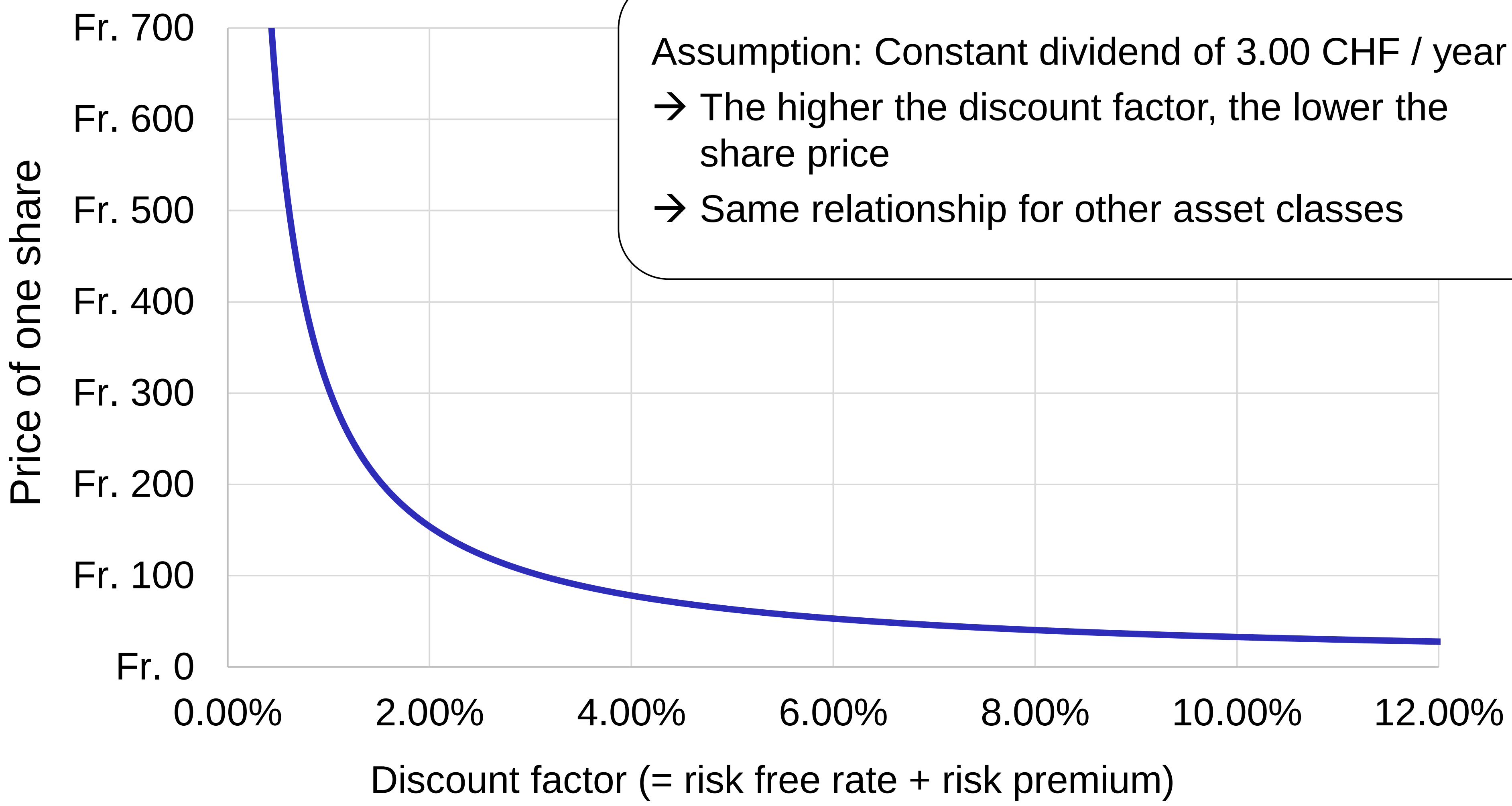
A Three-Stage or even a Multi-Stage Dividend Discount Model would allow for even greater complexity.

Composition of the Discount Factor under Uncertainty

Investors' expected compensation has two components:
Risk free rate ≙ Compensation for time value of investment
Risk premium ≙ Compensation for the risk (volatility, default risk etc.) of the investment



Impact of the Discount Factor on Asset Prices



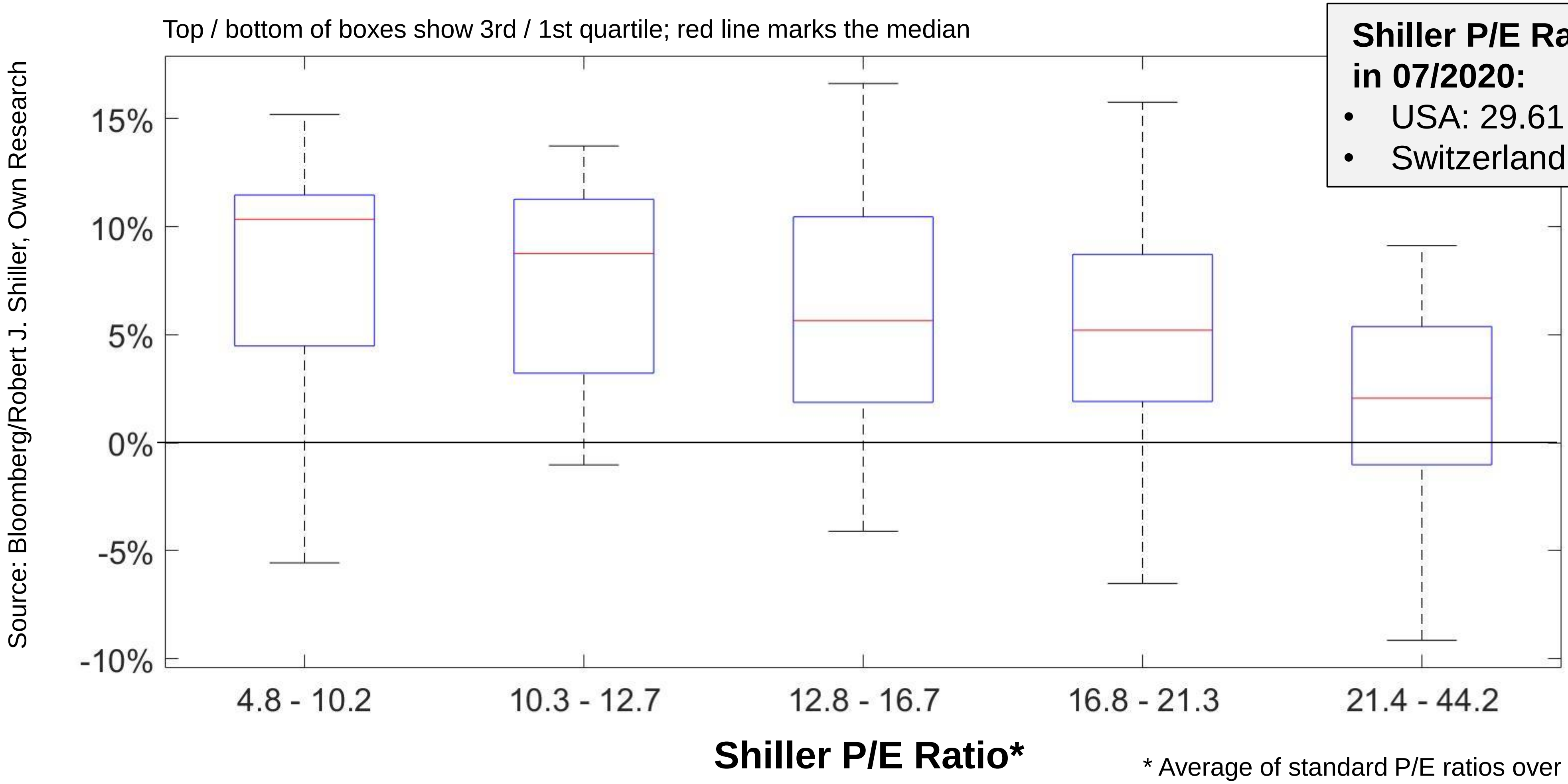
Assumption: Constant dividend of 3.00 CHF / year

→ The higher the discount factor, the lower the share price

→ Same relationship for other asset classes

Empirical Impact of Valuations on Future Returns

Distribution of annualized 10-year returns of S&P 500 index (1910 – 2020)



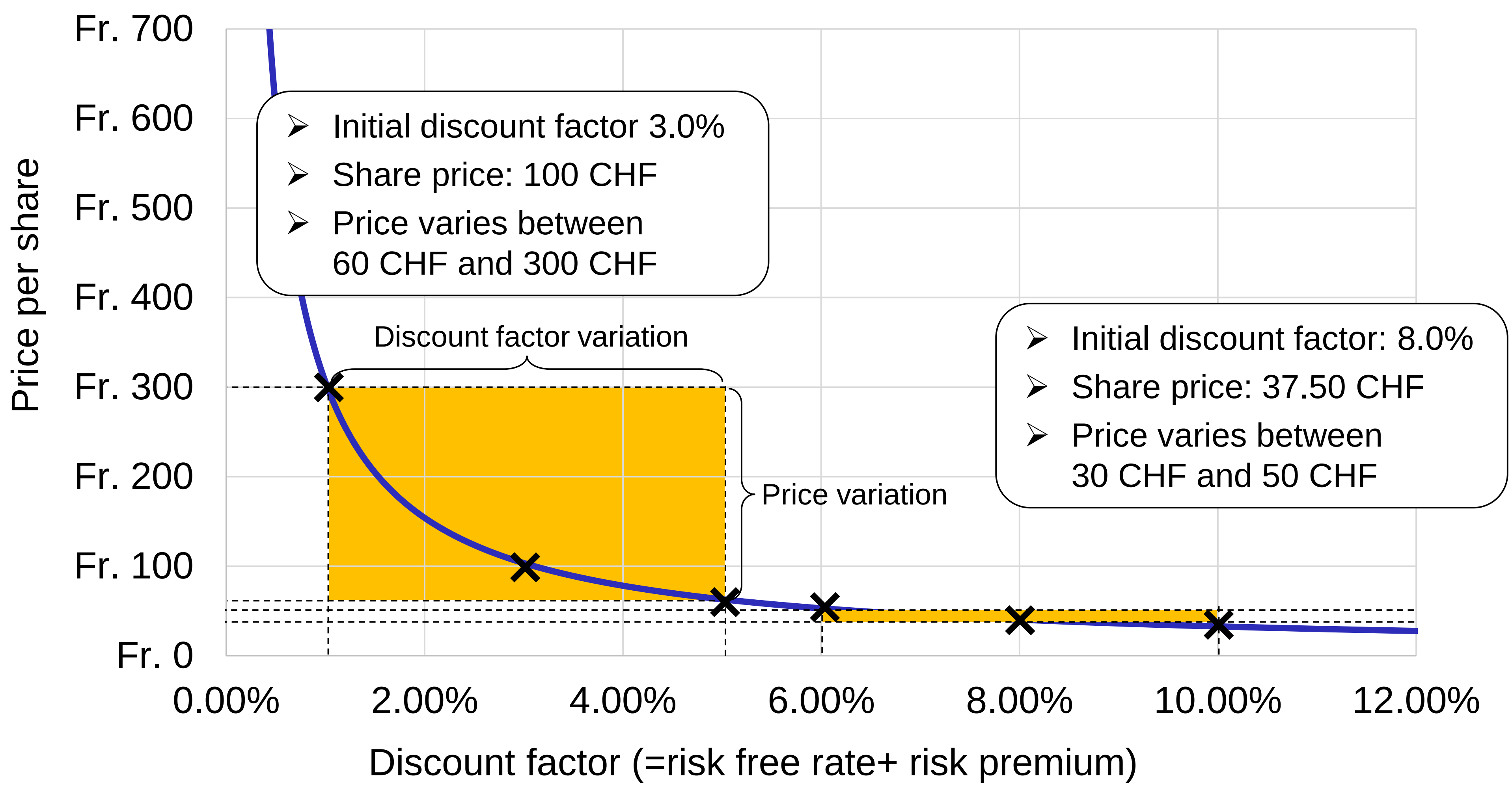
Shiller P/E Ratio in 07/2020:

- USA: 29.61
- Switzerland: 25.67

* Average of standard P/E ratios over 10 years to smooth out cyclical swings

Discount Factors and Price Sensitivity

Assumptions:
- Constant future dividend: 3.00 CHF p.a.
- Uncertainty about the future leads to discount factor variation of ± 2%



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- Immunization
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Term Structure of Interest Rates (1/2)

- Interest rates depend on their life-span. Therefore we have different rates for different maturities.
- The relation of spot rates $\mathbf{r_t}$ to their respective maturities **t** is called the **term structure of interest rates**.

EXAMPLE

Given term structure (spot curve):

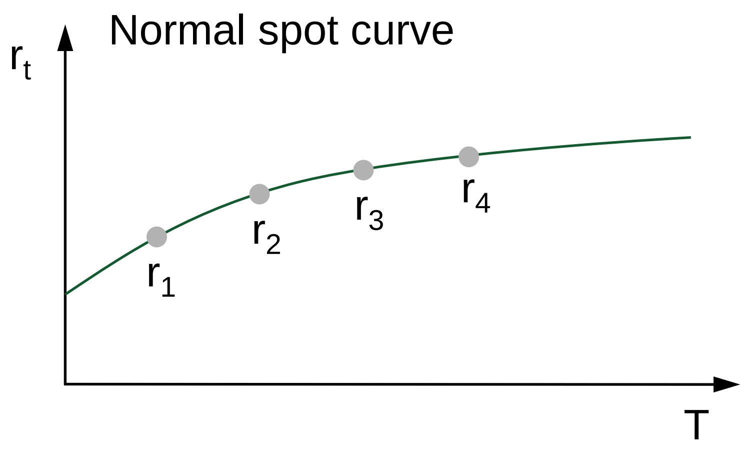
$r_1 = 3.5\%$, $r_2 = 4\%$, $r_3 = 4.5\%$, $r_4 = 5\%$

Which spot rates do we need to value a …

a) 3-year zero coupon bond?

b) 1-year zero coupon bond?

c) 1-year coupon bond?

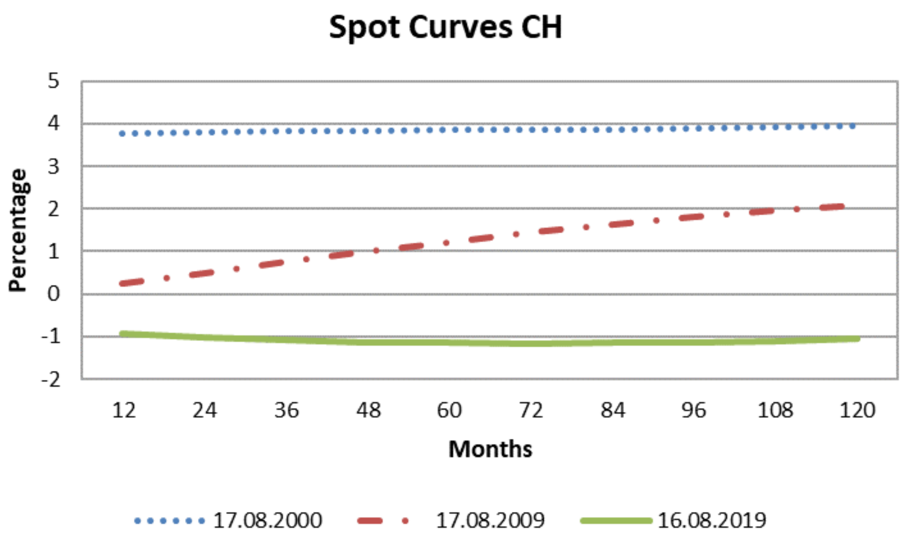


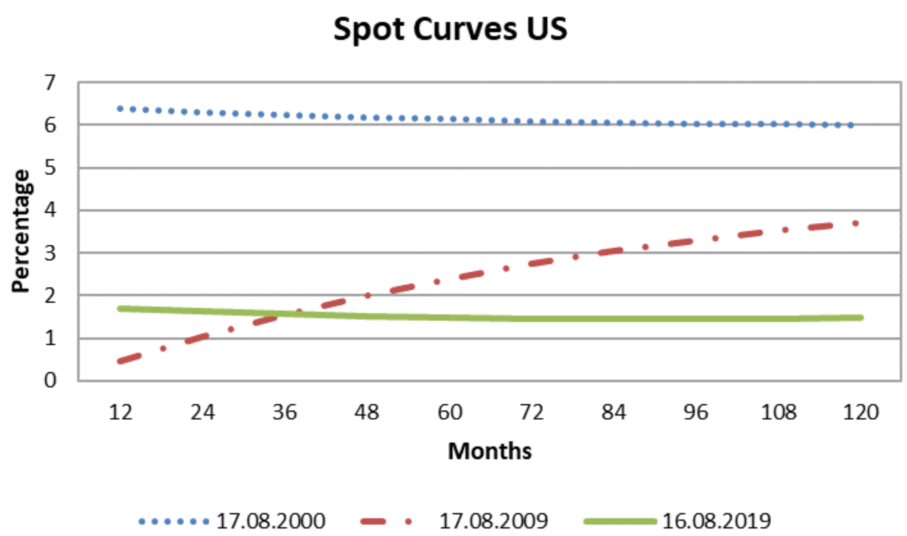
Term Structure of Interest Rates (2/2)

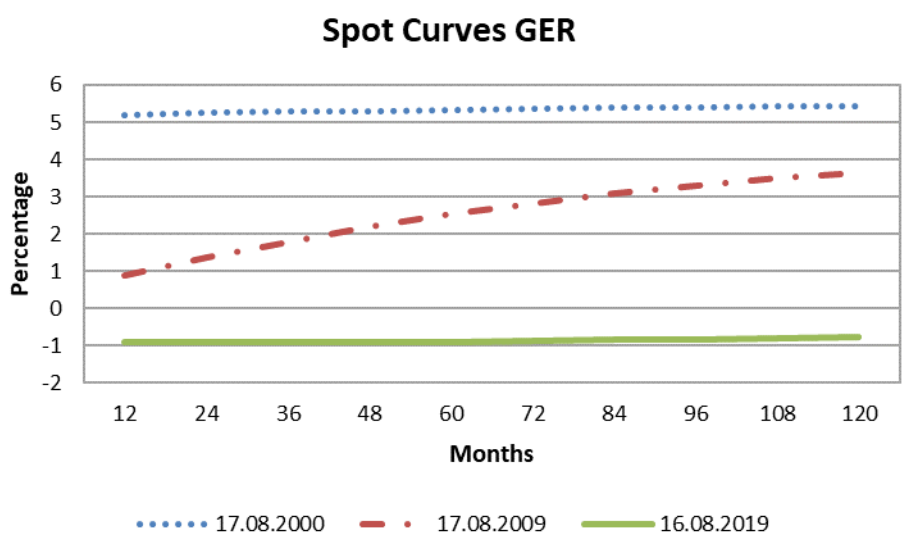
- We can derive the spot curve empirically by evaluating bonds of different maturities.
- Historically, the spot curve has taken three fundamental shapes:

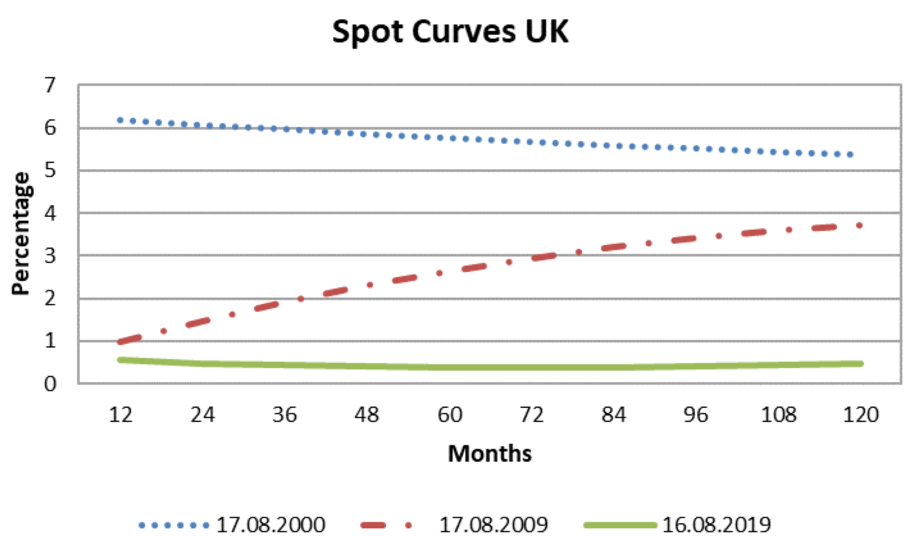
- Note that a flat spot curve does not imply capital market imperfection.
- The term "yield curve" is often used as a colloquial synonym for the spot curve.

Historical Spot Curves

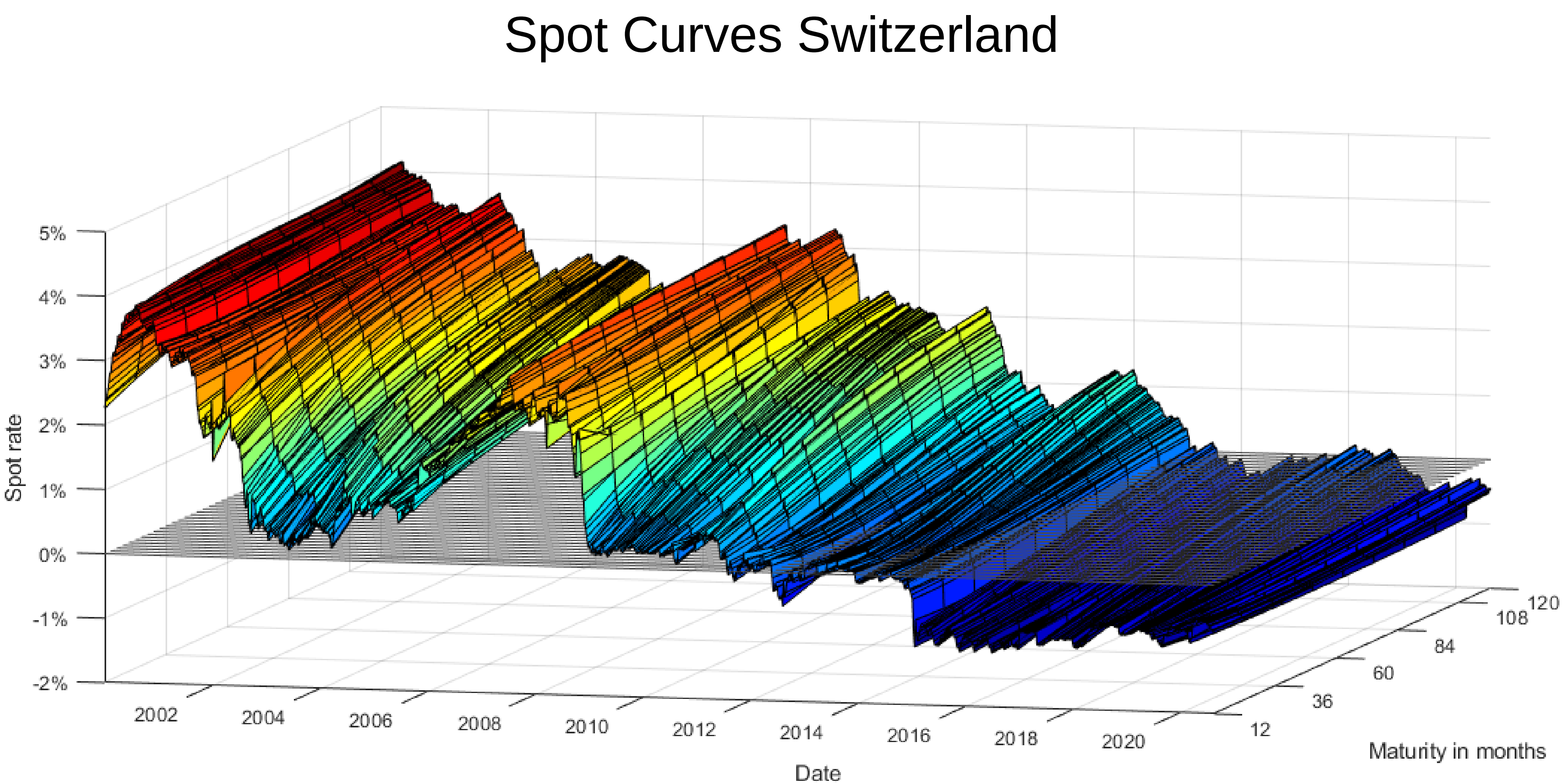








Evolution of Interest Rates and of the Term Structure



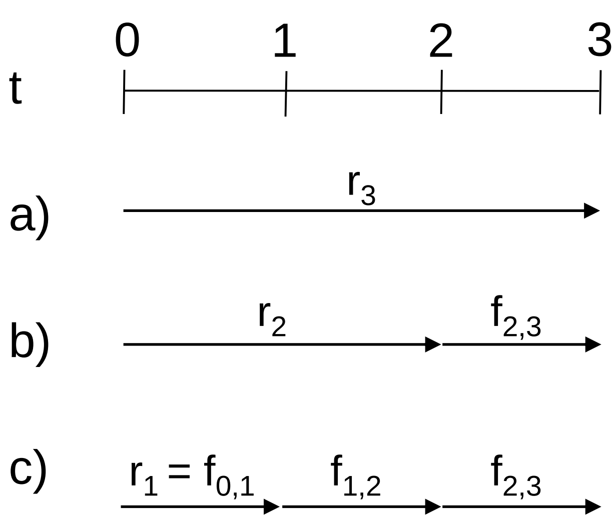
Source: Datastream

Forward Rates

The **forward (interest) rate** is the interest rate for a future time-span starting in the future and **settled today**.

What are the correct forward rates given the current term structure
- for a one-year zero bond starting in one year?
- for a one-year zero bond starting in two years?

Approach:

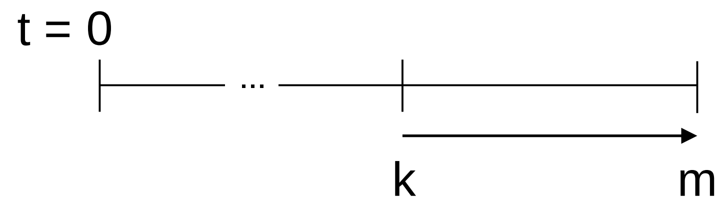


a) One time investment for three years at the current three-year spot rate.

b) Investment for two years at current two-year spot rate and commitment to invest in two years for one year at the current forward rate.

c) Investments for one year each at current spot and forward rates

Spot Rates vs. Forward Rates (1/2)



k = Starting point for investment in years

m = Maturity for investment in years

Arbitrage relation:

- Annually compounding of interest rates:
  $(1+r_m)^m = (1 + r_k)^k(1 + f_{k,m})^{m-k}$

- Continuous compounding of interest rates:
  $$e^{r_m m} = e^{r_k k} \cdot e^{f_{k,m}(m-k)}$$
  $$e^{f_{k,m}(m-k)} = e^{r_m m - r_k k}$$

Spot Rates vs. Forward Rates (2/2)

The above relation yields the general formula to calculate forward rates when compounding annually:

$$f_{k,m} = \sqrt[m-k]{\frac{(1+r_m)^m}{(1+r_k)^k}} - 1 \qquad m > k$$

when continuous compounding

$$f_{k,m} = \frac{\ln \frac{e^{r_m m}}{e^{r_k k}}}{m-k} = \frac{r_m m - r_k k}{m-k}.$$

and with $r_m m - r_k k = r_m(m-k) + r_m k - r_k k$, we have

$$f_{k,m} = r_m + (r_m - r_k)\frac{k}{m-k}.$$

Difference between Coupons and Interest Rates (1/2)

Given: spot curve with

$r_1 = 3.5\%$, $r_2 = 4\%$, $r_3 = 4.5\%$ und $r_4 = 5\%$.

In addition, we have a coupon par bond with a maturity of 2 years.

What is the appropriate coupon?

Difference between Coupons and Interest Rates (2/2)

The appropriate coupon must be less than $r_2 = 4\%$.

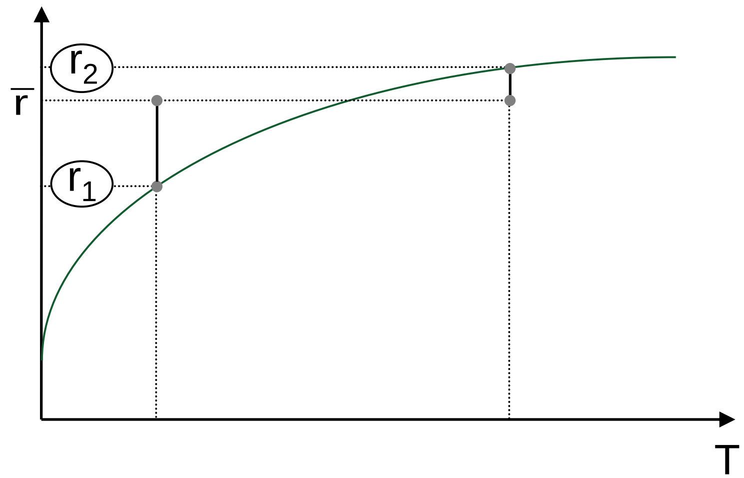
- The reason is that the interest rate is based on zero coupon bonds.
- In contrast coupon bonds pay cash-flows before maturity (coupons).
- The coupon paid at $t = 1$ pays interest according to the forward rate from $t = 1$ to $t = 2$ which is higher than r_2.
- Therefore the coupon needs to be lower than the interest on a 2-year zero coupon bond (4%) in order to arrive at a return of 4% over the entire period.

Yield to Maturity

- In the previous slides we deducted the implied coupon from the spot curve.
- Now we take the opposite view:
    - Given coupon
    - Given price of the bond
    - We deduct the average interest rate

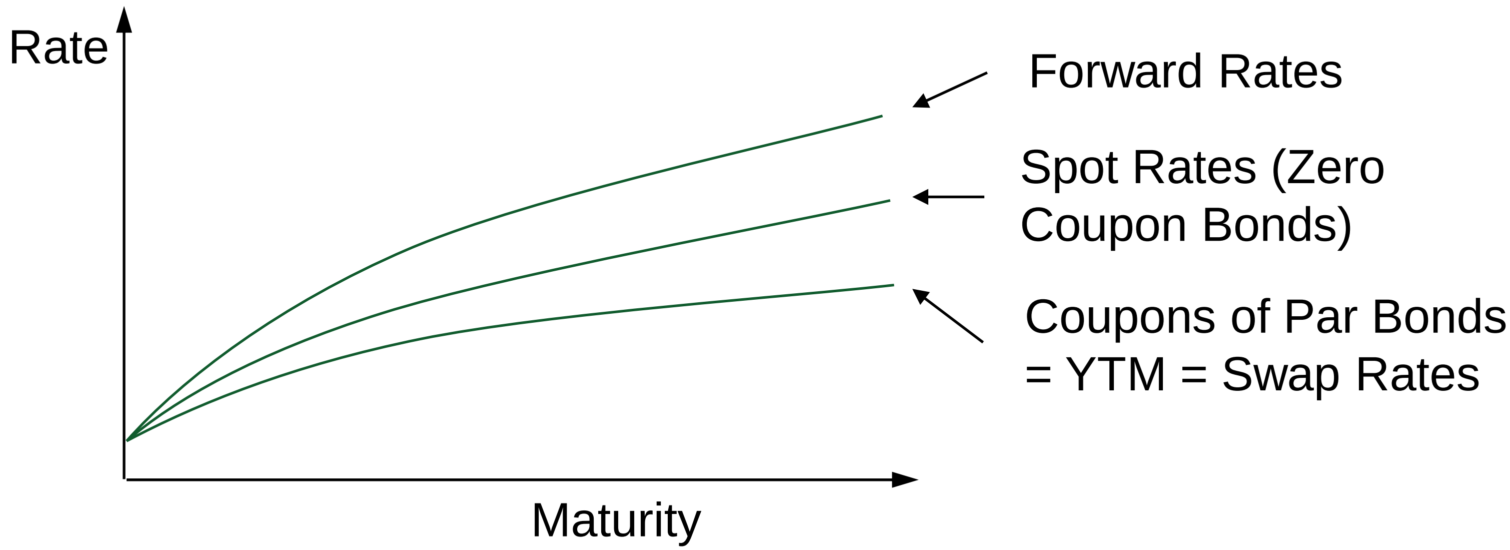
$$\text{price} = \frac{c}{1+r_1} + \frac{c}{(1+r_2)^2} + \ldots + \frac{1+c}{(1+r_n)^n}$$

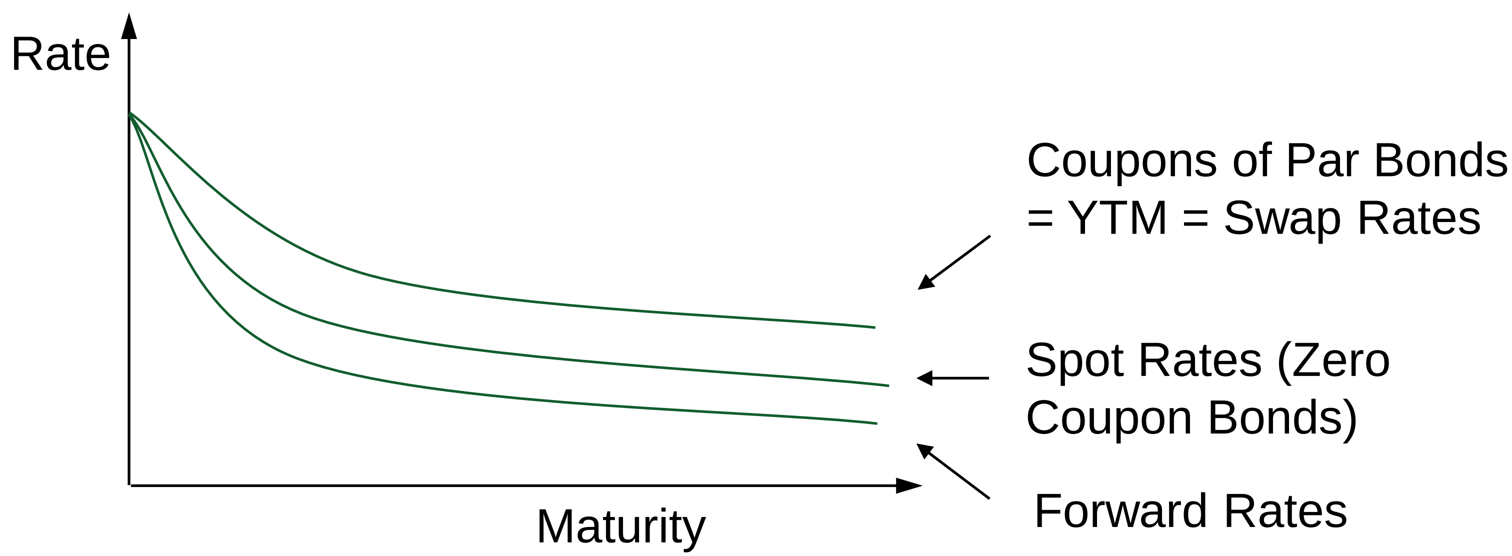
$$= \frac{c}{1+\bar{r}} + \frac{c}{(1+\bar{r})^2} + \ldots + \frac{1+c}{(1+\bar{r})^n}$$



- Solving for $\bar{r}$ we arrive at the payoff-weighted return (Yield to Maturity). Usually solving for $\bar{r}$ requires numerical methods.
- The Internal Rate of Return (IRR) of an investment is an equivalent concept.

Different Interest Rates





Term Structure Theory

Forward rates $f_{t,T}$ as expectations of future interest rates?

Expectation theory: forward rate = expected future spot rate: $f_{t,T} = E[\tilde{r}_{t-T}]$

Empirical Evidence:
On average one-time investments have higher returns than rolling investments.

$$f_{t,T} > E[\tilde{r}_{t-T}]$$

Risk Theory explanation:

Premium for long term investors due to a higher exposure to risk (liquidity, interest rate changes, inflation).

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Yields of Bonds With Different Coupons (1/4)

EXAMPLE

r_1	r_2	r_3	r_4	r_5
5%	6%	7%	8%	9%

Bond A and B differ in the coupon size:

	Bond A	Bond B
Bond Rating	AAA	AAA
Maturity	5 years	5 years
Coupon	4%	10%
Yield (y)	?	?

Yields of Bonds With Different Coupons (2/4)

- *Step 1*: present value calculation

Bond A:

$$PV = \frac{4}{(1+0.05)} + \frac{4}{(1+0.06)^2} + \frac{4}{(1+0.07)^3} + \frac{4}{(1+0.08)^4} + \frac{104}{(1+0.09)^5} = 81.168$$

Bond B:

$$PV = \frac{10}{(1+0.05)} + \frac{10}{(1+0.06)^2} + \frac{10}{(1+0.07)^3} + \frac{10}{(1+0.08)^4} + \frac{110}{(1+0.09)^5} = 105.43$$

Yields of Bonds With Different Coupons (3/4)

• *Step 2*: calculate yield to maturity

Bond A:

$$PV = 81.168 = \sum_{t=1}^{5} \frac{c_t}{(1+y)^t} \quad \Rightarrow \quad y = 8.819\%$$

Bond B:

$$PV = 105.43 = \sum_{t=1}^{5} \frac{c_t}{(1+y)^t} \quad \Rightarrow \quad y = 8.618\%$$

$\Rightarrow$ With a non-flat spot curve, the concept of yield to maturity is misleading.

Yields of Bonds With Different Coupons (4/4)

EXAMPLE Bonds with a maturity of 5 years and varying cash-flows

a) Flat spot curve	8%	8%	8%	8%	8%
b) Normal spot curve	5%	6%	7%	8%	9%
c) Inverse spot curve	9%	8%	7%	6%	5%

Prices

	Zero Bond	4% Cp. Bond	10% Cp. Bond
a) Flat spot curve	68.06	84.03	107.98
b) Normal spot curve	64.99	81.17	105.43
c) Inverse spot curve	78.35	95.02	120.02

Yields

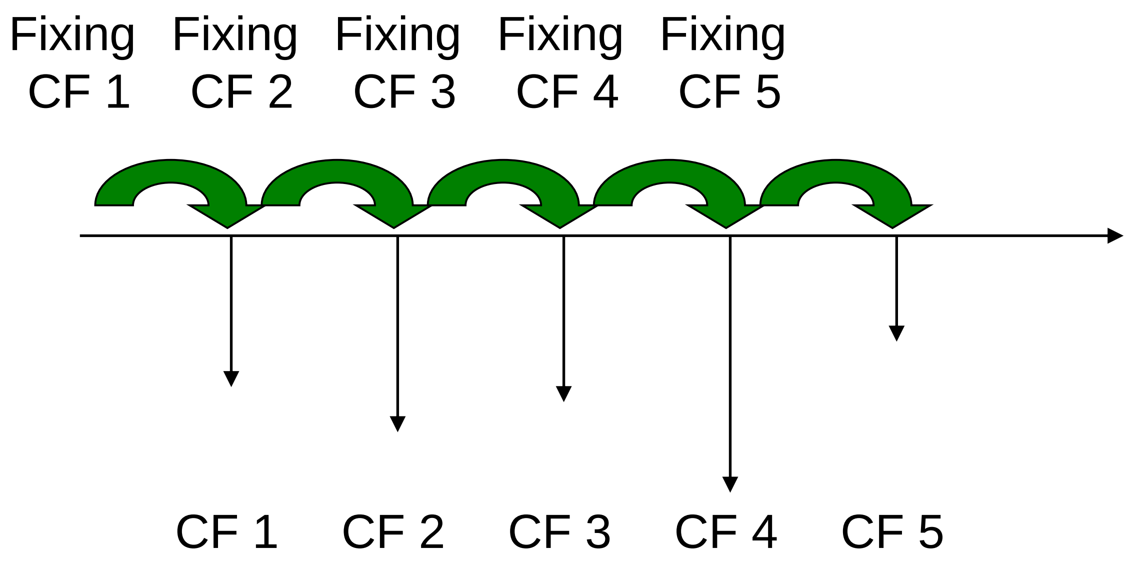
	Zero Bond	4% Cp. Bond	10% Cp. Bond
a) Flat spot curve	8%	8%	8%
b) Normal spot curve	9%	8.818%	8.618%
c) Inverse spot curve	5%	5.155%	5.333%

Valuation of Complex Fixed Income Products

- The cash-flows of complex products can be split into cash-flows of simple instruments.
- Procedure:
  - Stripping of the product into single components.
  - Pricing of each component.
  - The price of the complex product is the sum of the prices of the components.
- The risk exposure of the complex product is the sum of risks of the components.
- Therefore stripping the complex product into simpler components does not only offer insights in the pricing but also in the composition of the risk exposure.

Floating Rate Note

- A floating rate note pays coupons according to a market reference rate, e.g. LIBOR plus 20 basis points.
- The coupon is usually determined at the beginning of each coupon period. The process is called fixing.



- Therefore the exposure to a change in the spot curve is limited. If the reference rate is equal to the discount rate the bond is quoted at par. The main risk results from a change in the credit rating of the issuer.

Pricing of a Floating Rate Note

Basic considerations:

- If a floating rate note pays in every period the interest rate encountered in the market, its value must quote at par. The condition is met at fixing.
- If market conditions change and the previously fixed rate differs from the market rate, the floating rate note will **not** quote at par anymore.

Example: We have a floating rate note with a maturity of 3.5 years. The note pays a yearly coupon according to the one-year US t-bill rate with yearly fixing. According to the last fixing the coupon is set to 2.25%. What is the price of the floater with a fixing in half a year, if the following table describes the current structure.

time t	0.5	1.5	2.5	3.5
Spot Rates: R_t (annualisiert)	2.13%	2.37%	2.74%	3.09%
Forward Rates: $f_{t,t+1}$	2.49%	3.30%	3.96%	-

Pricing of a Floating Rate Note by Forward Rate

Time t	0.5	1.5	2.5	3.5
Spot Rates: R_t (annual)	2.13%	2.37%	2.74%	3.09%
Forward Rates: $f_{t,t+1}$	2.49%	3.30%	3.96%	-
Cash Flow of the Floater (assumption)	**2.25**	2.49	3.30	103.96
Discountrate	0.9895	0.9655	0.9347	0.8990
Present Value	2.2265	2.4056	3.0834	93.465
Value of the Floaters (dirty price)	101.18			
Accrued Interest	1.125			
Clean Price	**100.06**			

Coupon according to last fixing

- Between fixing dates the price can differ from its nominal value.

Note: Excel with corresponding calculation will be uploaded to StudyNet.

Pricing of a Floating Rate Note by Par Value Approach

- At fixing (t=0.5) the interest rates of the bond correspond to the market rates and therefore the bond quotes at par (N = 100) (ignoring credit risk)

- In addition, the bond pays the fixed next coupon of 2.25.

- After discounting the value of the bond yields
  $P = 100 * 0.9895 + 2.25 * 0.9895 = 101.18.$

- Without accrued interest the clean price is $P_{clean} = 101.18 – 1.125 = 100.06.$

Pricing by Replication

EXAMPLE Reverse Floater

We adapt the procedure of stripping to price a reverse floater. The examined reverse floater pays an annual coupon of 15% minus two times LIBOR. The coupon is calculated at the beginning of each period. On redemption date the bond is redeemed at par, i.e. the nominal value. Prices for some plain vanilla bonds are given in the table:

Bond	Maturity	Price
Fix 5 %	5.5 years	91.76
Fix 7.5 %	5.5 years	104.13
Fix 10 %	5.5 years	116.50
LIBOR-Floater	5.5 years	102.47

Reverse Floater

Cash-flows of the introduced bonds

	Price	0.5	…	4.5	5.5	Amount
Fix 5 %	91.76	5	…	5	105	3
Fix 7.5 %	104.13	7.5	…	7.5	107.5	0
Fix 10 %	116.5	10	…	10	110	0
LIBOR-Floater	102.47	LIBOR	…	LIBOR	100+LIBOR	-2
Reverse Floater	?	15-2 LIBOR	…	15-2 LIBOR	115-2 LIBOR	

Cash-flows of the replication portfolio

	Amount	0.5	…	4.5	5.5	Value
Fix 5 %	3	15	…	15	315	275.27
Fix 7.5 %	0	0	…	0	0	0
Fix 10 %	0	0	…	0	0	0
LIBOR-Floater	-2	-2 LIBOR	…	-2 LIBOR	-200-2 LIBOR	-204.94
Portfolio		15-2 LIBOR	…	15-2 LIBOR	315-200-2 LIBOR	70.33

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Interest Rate Risk – the Discrete Case

Interest Rate Risk – in absolute terms

$$P_0 = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_n}{(1+r)^n}$$

$$\frac{\partial P_0}{\partial r} = -\frac{C_1}{(1+r)^2} - 2\frac{C_2}{(1+r)^3} - \ldots - n\frac{C_n}{(1+r)^{n+1}}$$

$$= -\frac{1}{1+r}\sum_{k=1}^{n} k\frac{C_k}{(1+r)^k}$$

P_0 = Value of the bond at time t=0

C_k = Cash-flow at time k

r = Interest rate

k = Point in time

Interest Rate Risk – in relative terms

$$\frac{\frac{\partial P_0}{P_0}}{\partial r} = \frac{\partial P_0}{\partial r}\frac{1}{P_0} = \underbrace{-\frac{1}{1+r}\underbrace{\frac{\sum_{k=1}^{n} k\frac{C_k}{(1+r)^k}}{P_0}}_{\text{Macaulay Duration } (D_{Mac})}}_{\text{Modified Duration } (D_{Mod})}$$

Interest Rate Risk – the Continuous Case

Interest Rate Risk – in absolute terms

$$P_0 = C_1 e^{-r} + C_2 e^{-2r} + \ldots + C_n e^{-nr} = \sum_{k=1}^{n} C_k e^{-kr}$$

$$\frac{\partial P_0}{\partial r} = -C_1 e^{-r} - 2C_2 e^{-2r} - \ldots - nC_n e^{-rn} = \sum_{k=1}^{n} -k \cdot C_k e^{-rk}$$

Interest Rate Risk – in relative terms

$$\frac{\frac{\partial P_0}{P_0}}{\partial r} = \frac{\partial P_0}{\partial r}\frac{1}{P_0} = \frac{\sum_{k=1}^{n} -k \cdot C_k e^{-rk}}{\sum_{k=1}^{n} C_k e^{-rk}} = -\frac{\sum_{k=1}^{n} k \cdot C_k e^{-rk}}{\sum_{k=1}^{n} C_k e^{-rk}}$$

→ $\mathbf{D_{Mac} = D_{Mod}}$ in the continuous case

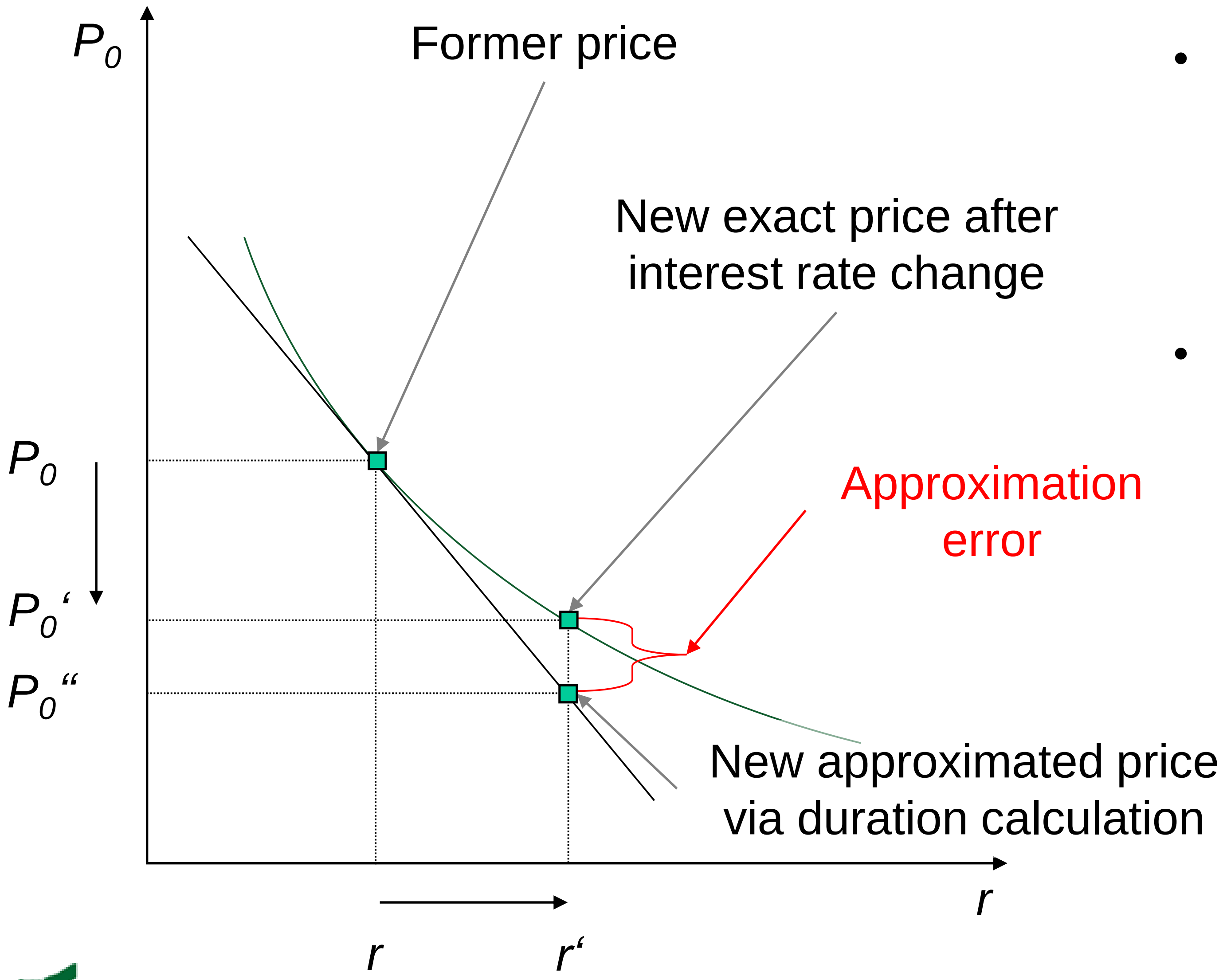
Duration as Risk Measure for Bonds

- The duration as a **relative risk measure** characterizes the **relative change of the bond price** due to an **absolute parallel shift** in the spot curve.

- Duration is extensively used by practitioners to measure interest rate risk of bonds.

- Mathematically, the duration is the first derivative of the bond with respect to the spot rates.

Graphical Interpretation of the Duration Concept

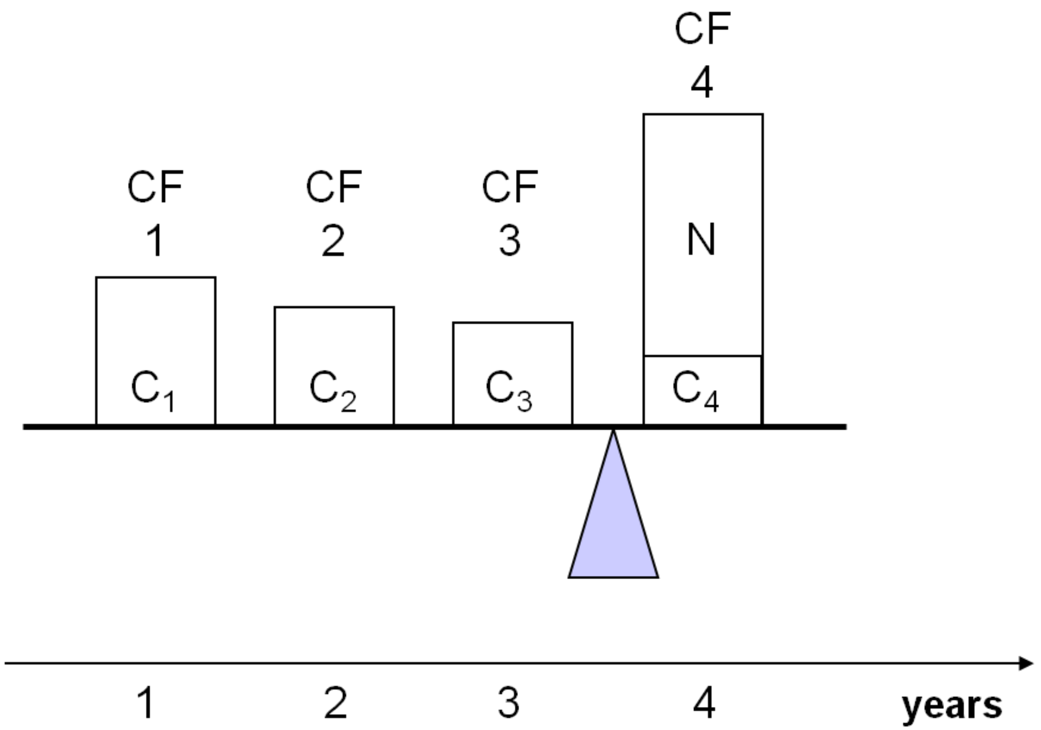


- Duration is the first derivative of the price with respect to the interest rate. The relation is therefore linear.
- After an interest rate increase, duration overestimates the effect (P_0"); after an interest rate decrease duration underestimates the effect.

Macaulay Duration

Non-flat term structure:

$$D_{Mac} = \sum_{k=1}^{n} \underset{\text{Period weight}}{k} \underbrace{\frac{C_k}{(1+r_k)^k \cdot P_0}}_{\text{Fraction of the present value of the cash-flow}}$$



- Dimension is time, i.e. **„years“**
- Interpretation: **Average capital lockup period**

Modified Duration

Non-flat term structure:

$$D_{mod} = \sum_{k=1}^{n} k \frac{C_k}{(1+r_k)^{k+1} \cdot P_0} \approx \frac{1}{1+\underset{\uparrow}{r}} \cdot \underbrace{\sum_{k=1}^{n} k \frac{C_k}{(1+r_k)^{k} \cdot P_0}}_{D_{Mac}}$$

r = Yield to Maturiy (YTM)

- No dimension
- Interpretation: **Approximation of the (negative) bond price change for a given change of the interest rate**

$$\frac{\Delta P_0}{P_0} \approx -D_{Mod} \cdot \Delta r$$

Calculating Duration (1/2)

EXAMPLE

Given:	coupon	$c = 5\%$
	maturity	$n = 3$ years
	interest rates	$r_1 = 3\%$; $r_2 = 4\%$; $r_3 = 4.5\%$

$$P_0 = \frac{5\%}{1+3\%} + \frac{5\%}{(1+4\%)^2} + \frac{1+5\%}{(1+4.5\%)^3} = 101.49\%$$

$$D_{Mac} = \sum_{k=1}^{n} k \frac{C_k}{(1+r_k)^k \cdot P_0} = \frac{1}{101.49\%} \cdot (1 \cdot \frac{5\%}{(1+3\%)^1} + 2 \cdot \frac{5\%}{(1+4\%)^2} + 3 \cdot \frac{1+5\%}{(1+4.5\%)^3}) = 2.859$$

$$D_{Mod} = \sum_{k=1}^{n} k \frac{C_k}{(1+r_k)^{k+1} \cdot P_0} = \frac{1}{101.49\%} \cdot (1 \cdot \frac{5\%}{(1+3\%)^2} + 2 \cdot \frac{5\%}{(1+4\%)^3} + 3 \cdot \frac{1+5\%}{(1+4.5\%)^4}) = 2.737$$

Calculating Duration (2/2)

Alternative:

$$P_0 = \sum_{k=1}^{n} \frac{C_k}{(1+r_k)^k} = \frac{5\%}{(1+3\%)^1} + \frac{5\%}{(1+4\%)^2} + \frac{1+5\%}{(1+4.5\%)^3} = 101.49\% = \sum_{k=1}^{n} \frac{C_k}{(1+r)}$$

$$= \frac{5\%}{(1+r)^1} + \frac{5\%}{(1+r)^2} + \frac{1+5\%}{(1+r)^3} \longrightarrow$$ r = 4.459% (Yield to Maturity, YTM)

$$D_{Mod} \approx \frac{1}{1+r} \cdot D_{Mac} = \frac{1}{1+4.459\%} \cdot 2.859 = 2.737$$

Calculation with Yield to Maturity generates only a slight difference.

$$D_{Mac} = \frac{1 \cdot \frac{5\%}{(1+\underbrace{4.459\%}_{\text{YTM}})^1} + 2 \cdot \frac{5\%}{(1+\underbrace{4.459\%}_{\text{YTM}})^2} + 3 \cdot \frac{1+5\%}{(1+\underbrace{4.459\%}_{\text{YTM}})^3}}{101.49\%} = 2.861$$

Limitations of the Duration Concept

- Duration is a **linear** approximation (for a convex relation).
- For **small** changes in interest rates the approximation is fairly **accurate** but worsens for larger changes.
- The deviation of prices due to duration calculations is called **approximation error**.
- Only **parallel shifts** of the spot curve are considered.
- The interest rates are the only risk factors.

Duration of a Consol Bond

EXAMPLE: For a flat spot curve.

$$P = \frac{C}{r}$$

$$\frac{\partial P}{\partial r}\frac{1}{P} = -\frac{C}{(r)^2}\frac{1}{P} = -\frac{C}{(r)^2}\frac{r}{C} = -\frac{1}{r}$$

$$D_{Consol}^{Mod} = \frac{1}{r}$$

$$D_{Consol}^{Mac} = \frac{1+r}{r}$$

Duration of a Stock

$$A = \frac{Div}{(r - ik)} = \frac{E(1-k)}{(r - ik)}$$

$$\frac{dA}{dr}\frac{1}{A} = -\frac{E(1-k)}{(r-ik)^2}\frac{1}{A} = -\frac{E(1-k)(r-ik)}{(r-ik)^2 E(1-k)} = -\frac{1}{(r-ik)}$$

$$= -\frac{1}{\text{Div.Yield}} = -\frac{\text{P/E Ratio}}{\text{Payout Ratio}}$$

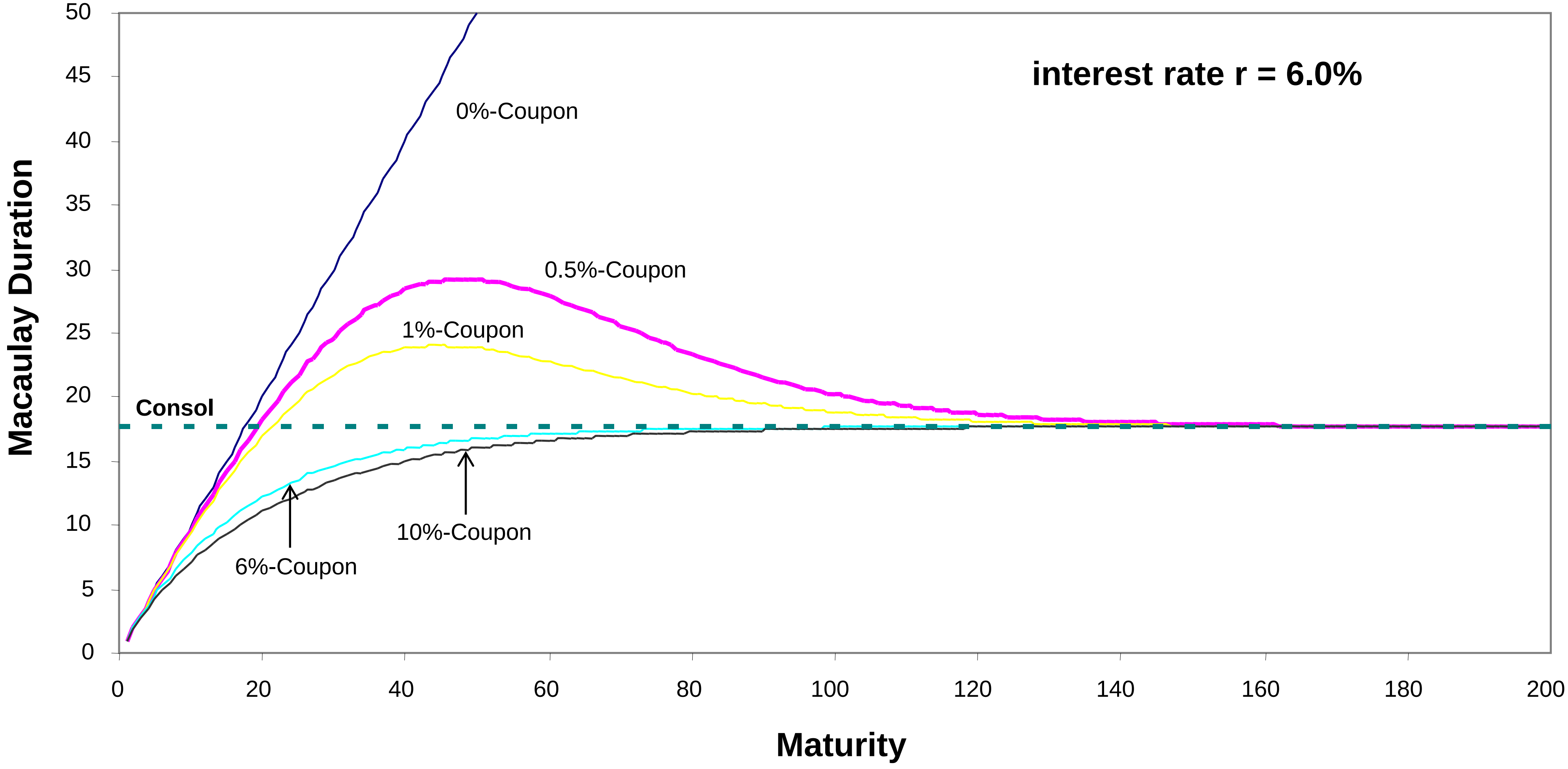
Div = Dividends
r = Cost of Capital, constant
i = Return on invested assets
k = Reinvestment rate
E = Earnings

High dividend yield: → High capital payout ratios

→ Low (average) capital lockup period

Macaulay Duration with Increasing Maturity

Duration as a function of maturity for a flat spot curve.



Duration of a Bond Portfolio

The duration of a bond portfolio is the weighted sum of durations of its components (duration is linear).

$$D_{Pf} = \frac{\sum_{k=1}^{n} k \cdot Cash\ Flow_k \cdot e^{-rk}}{\left[w_A P_A + w_B P_B\right] = P_0}$$

w_i is the fraction of value of each component with respect to the overall value (i = A,B).

$$D_{Pf} = \sum_{k=1}^{n} k \cdot (w_A Cash\ Flow_k^A + w_B Cash\ Flow_k^B) \cdot e^{-rk}$$

$$= w_A \sum_{k=1}^{n} k \cdot Cash\ Flow_k^A \cdot e^{-rk} + w_B \sum_{k=1}^{n} k \cdot Cash\ Flow_k^B \cdot e^{-rk}$$

$$= w_A D_A + w_B D_B$$

The general case (more than 2 bonds):

$$\mathrm{D}_{\mathrm{Pf}}^{\mathrm{Mod}} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{w}_{\mathrm{i}} \cdot \mathrm{D}_{\mathrm{i}}^{\mathrm{Mod}}$$

Agenda

- Basic Valuation
- Term Structure of Interest Rates
- Yields and Pricing
- Duration
- **Immunization**
- Key-Rate Duration
- Credit Risk

Immunization

Some initial comments:

- For investors an increase in interest rates has two effects. Prices of **existing bonds drop** in value but at the same time the received coupons can be **invested at higher rates**. The effects are called:
  - **Price Risk (PR)**
  - **Reinvestment Risk (RI)**
- For a momentary view on the portfolio, only market risk is relevant.
- As soon as the investment horizon is larger than 0 reinvestment effects come into effect.
- If the investment horizon matches the duration of the investment both effects offset each other exactly.
- An adequate measure is Holding Period Return (HPR).

Duration and Immunization

- The Macaulay Duration of a bond portfolio corresponds to the investment horizon that is immune to a **parallel shift** of the spot curve.

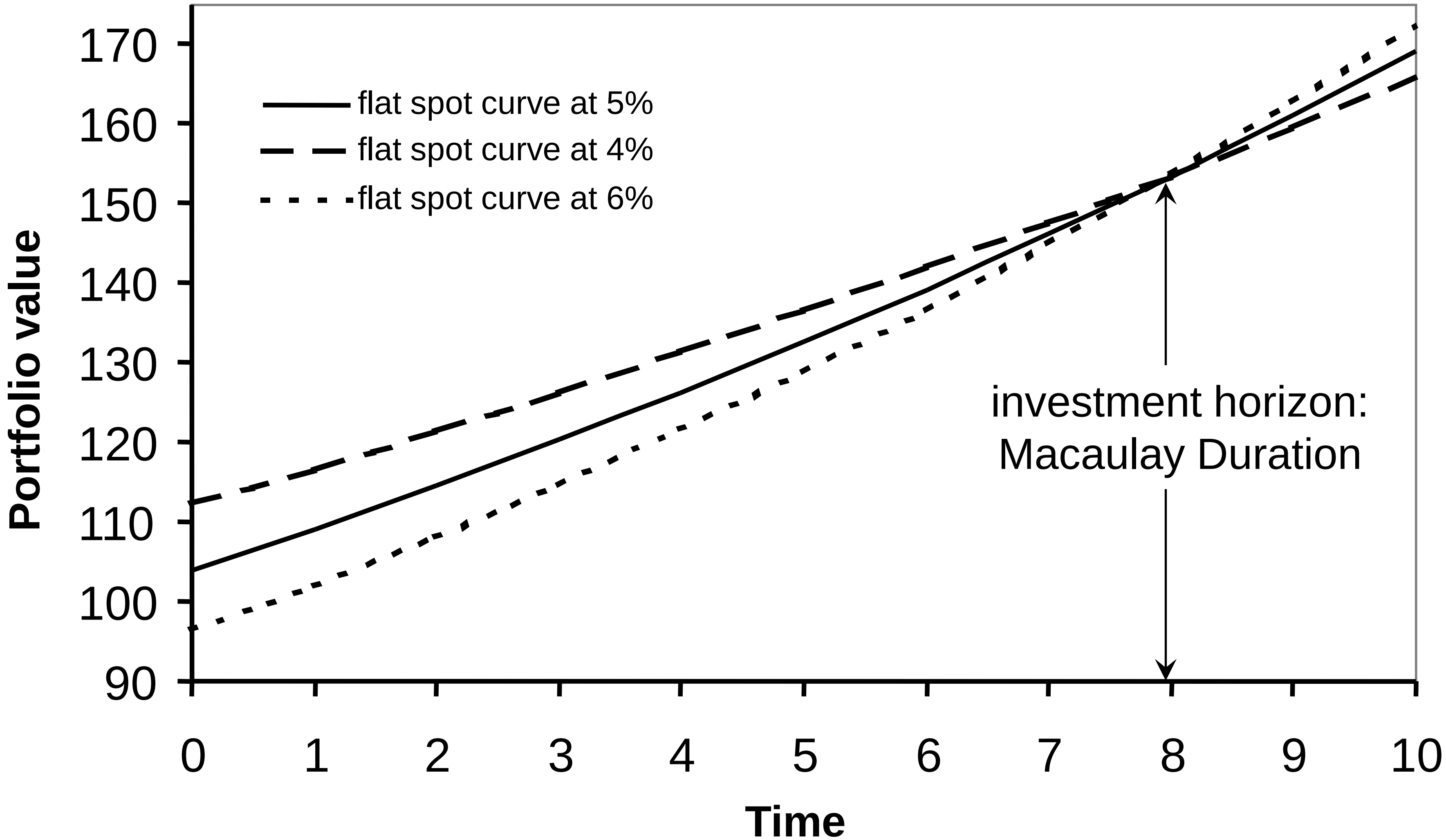
EXAMPLE

- 5.5% coupon bond, 10-year maturity
- spot curve is flat at 5%, it either rises to 6% or drops to 4%

$D^{Mac} = 7.99$ years

Year	**0**	**1**	**2**	**6**	**7**	**8**	**9**	**10**
Bond price (5%)	103.8609	103.5539	103.2316	101.7730	101.3616	100.9297	100.4762	100.0000
Coupon (reinvested)	-	5.5000	11.2750	37.4105	44.7810	52.5201	60.6461	69.1784
Portfolio value	**103.8609**	**109.0539**	**114.5066**	**139.1835**	**146.1427**	**153.4498**	**161.1223**	**169.1784**
Bond price (4%)	112.1663	111.1530	110.0991	105.4448	104.1626	102.8291	101.4423	100.0000
Coupon (reinvested)	-	5.5000	11.2200	36.4814	43.4406	50.6782	58.2054	66.0336
Portfolio value	**112.1663**	**116.6530**	**121.3191**	**141.9262**	**147.6033**	**153.5074**	**159.6477**	**166.0336**
Bond price (6%)	96.3200	96.5992	96.8951	98.2674	98.6635	99.0833	99.5283	100.0000
Coupon (reinvested)	-	5.5000	11.3300	38.3643	46.1661	54.4361	63.2022	72.4944
Portfolio value	**96.3200**	**102.0992**	**108.2251**	**136.6317**	**144.8296**	**153.5194**	**162.7305**	**172.4944**

Investment Horizon and Macaulay Duration



Agenda

- Basic Valuation
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- **Key-Rate Duration**
- Credit Risk

Information Overload vs. Information Aggregation

Listing of every cash-flow with its corresponding spot rate

- **Information overload**
- No loss of information
- Hard to communicate
- High effort
- Enables exact risk management

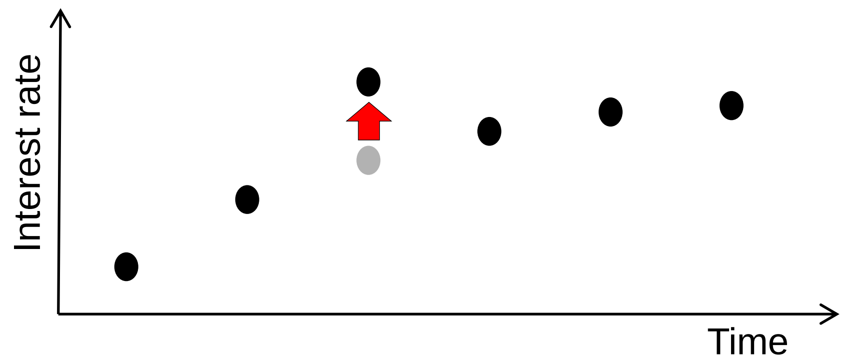
Key measures (Present value, *YTM*, Macaulay Duration)

- **Highly aggregated**
- Loss of information
- Easy to communicate
- Low effort
- Risk management not adequate in a lot of cases

→ Key-rate duration as compromise

The Concept of Key-Rate Duration

New spot curve: One key rate has changed



- The change in value of a bond investment depends on the change of different interest rates of different maturities.
- Instead of a single duration we have multiple key rate durations. Each key-rate duration characterizes the sensitivity of the price with respect to the change in the corresponding spot rate.
- With the concept of key-rate durations we can simulate various changes in spot curve structure.

Calculating KRDs (1/2)

The key-rate duration with maturity k:

$$KRD_k = -\frac{1}{\Delta r_k} \cdot \frac{P_k - P_0}{P_0} = -\frac{1}{\Delta r_k} \cdot \frac{\Delta P_0}{P_0}$$

P_0: Current present value of the instrument

P_k: Present value after a change of + r_k the spot rate with maturity k.

$$KRD_k^{Mod} = -\frac{\partial P_0}{\partial r_k} \cdot \frac{1}{P_0} = \frac{k \cdot C_k}{(1 + r_k)^{k+1}} \cdot \frac{1}{P_0} = \frac{1}{(1 + r_k)} \cdot \underbrace{\frac{k \cdot C_k}{(1 + r_k)^k} \cdot \frac{1}{P_0}}_{KRD_k^{Mac}}$$

Modified KRD

Macaulay KRD $\longrightarrow$ KRD_k^{Mac}

The key-rate duration of maturity k defines the relative change of the value of the instrument after the change of the corresponding key rate of -1%.

Calculating KRDs (2/2)

EXAMPLE

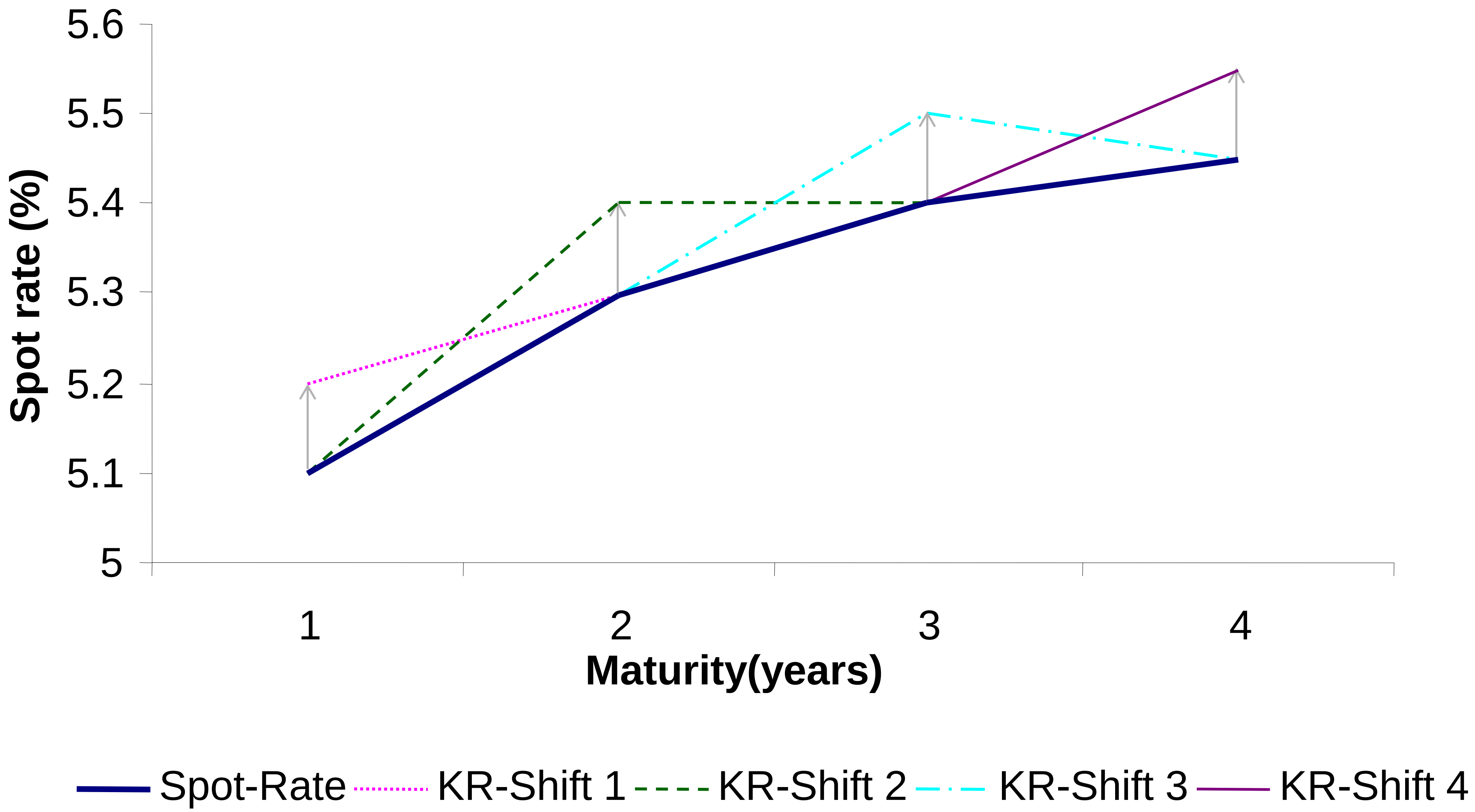
KRD of a 6% coupon bond with a maturity of 4 years

PV = Present Value

period	CF	Spot	PV	Shift 1	PV	Shift 2	PV	Shift 3	PV	Shift 4	PV
1	6	5.10	5.709	5.20	5.703	5.10	5.709	5.10	5.709	5.10	5.709
2	6	5.30	5.411	5.30	5.411	5.40	5.401	5.30	5.411	5.30	5.411
3	6	5.40	5.124	5.40	5.124	5.40	5.124	5.50	5.110	5.40	5.124
4	106	5.45	85.727	5.45	85.727	5.45	85.727	5.45	85.727	5.55	85.403
price			101.972		101.966		101.961		101.957		101.647
KRD					0.053		0.101		0.143		3.181

$$KRD_k = -\frac{1}{\Delta r_k} \cdot \frac{\Delta P_0}{P_0}$$

Interpolation of Key-Rate Shifts



Overall Price Changes with KRDs

- The KRDs of a portfolio are the sum of the value-weighted KRDs of the individual components of the portfolio.
- The KRDs of bonds (long) are always positive. Derivatives may have negative or positive KRDs.
- The sum of the changes for each maturity calculated by the concept of KRD is the overall change in price for the instrument.
- If we define 4 segments (buckets) with spot rates of 1, 2, 3 and 4 years, the following formula describes the overall change:

$$\Delta P_0 = P_0 \cdot \left[KRD_1 \cdot \left(-\Delta r_1\right) + KRD_2 \cdot \left(-\Delta r_2\right) \right.$$
$$\left. + KRD_3 \cdot \left(-\Delta r_3\right) + KRD_4 \cdot \left(-\Delta r_4\right) \right]$$

Calculating Price Movements with KRDs

EXAMPLE

Given: Coupon c = 5%

Maturity n = 3 years

Interest rates $r_1 = 3\%;\ r_2 = 4\%;\ r_3 = 4.5\%$

$$P_0 = \frac{5\%}{1+3\%} + \frac{5\%}{(1+4\%)^2} + \frac{1+5\%}{(1+4.5\%)^3} = 101.49\%$$

$$\left.\begin{aligned} KRD_k^{Mac} &= \frac{k \cdot C_k}{(1+r_k)^k} \cdot \frac{1}{P_0} \\ KRD_k^{Mod} &= \frac{k \cdot C_k}{(1+r_k)^{k+1}} \cdot \frac{1}{P_0} \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} KRD_1^{Mac} &= \frac{1 \cdot 5\%}{(1+3\%)^1} \cdot \frac{1}{101.49\%} = 0.0478 \\ KRD_2^{Mac} &= \frac{2 \cdot 5\%}{(1+4\%)^2} \cdot \frac{1}{101.49\%} = 0.0838 \\ KRD_3^{Mac} &= \frac{3 \cdot 105\%}{(1+4.5\%)^3} \cdot \frac{1}{101.49\%} = 2.7199 \\ KRD_1^{Mod} &= \frac{1 \cdot 5\%}{(1+3\%)^2} \cdot \frac{1}{101.49\%} = 0.0464 \\ KRD_2^{Mod} &= \frac{2 \cdot 5\%}{(1+4\%)^3} \cdot \frac{1}{101.49\%} = 0.0806 \\ KRD_3^{Mod} &= \frac{3 \cdot 105\%}{(1+4.5\%)^4} \cdot \frac{1}{101.49\%} = 2.6027 \end{aligned}\right.$$

$$\sum = D_{Mac} = 2.85$$

$$\sum = D_{Mod} = 2.73$$

Volatility

- D_{Mod} = 2.73 → A change of 1% in interest rates leads to a 2.73% bond price change.
- KRD_1^{Mod} = 0.0464 → A change of 1% in interest rate r_1 leads to a 0.0464% bond price change.
- D_{Mod} and KRD^{Mod} are measures of sensitivity.
- D_{Mod} und KRD^{Mod} are also measures of volatility if the volatility of interest rates is known.

EXAMPLE

volatility (r) = 0.08

→ volatility (P) ≈ 0.24 (with D_{Mod} = 3)

→ volatility (P) ≈ $D_{Mod} \cdot$ volatility (r_i)

The KRD approximated the contribution in volatility of the corresponding spot rate.

Agenda

- Basic Valuation
- Term Structure of Interest Rates
- Yields and Pricing
- Duration
- Immunization
- Key-Rate Duration
- **Credit Risk**

Credit Risk

- Investing in bonds may bear the risk of default of the issuer.

Expected Loss (EL) = Probability of Default (PD) *
Loss Given Default (LGD) * Exposure at Default (EAD)

- In the presence of default risk, investors demand compensation for the expected loss.

- Rating agencies estimate the probability of default and assign ratings to bonds to facilitate the estimation of the expected loss.

Credit Assessment by Ratings

- Ratings measure the credit quality of a borrower.
- Ratings are assigned to
  - individual securities (issue rating) or
  - the institution (issuer rating).
- Issue ratings refer to the creditworthiness of an obligor with regard to a specific financial obligation.
- Obligations exhibit different repayment priorities (seniority). Therefore obligations of a single issuer may have assigned different issue ratings.
- Issuer ratings refer to the overall financial capacity of an issuer.

Seniority of Bonds

- In case of default the remaining cash of the company is usually spread between the bond holders.
- The seniority of the bond defines the pecking order of the different bond tranches. The junior tranches get repayments only if the cash is sufficient to repay the more senior tranche fully.
- The different categories from highest to lowest repayment are usually
  - Senior secured
  - Senior unsecured
  - Senior subordinated
  - Subordinated
  - Junior subordinated

Example: Credit Rating

- Credit Ratings classify the creditworthiness of the issuer or of the bond itself. The ratings are issued by independent agencies (e.g. Standard & Poor's, Moody's, Fitch Ratings).
- Each bond rated belongs to a class defined below

S&P	Moody’s	Fitch	Level	Investment Type
AAA	Aaa	AAA	Prime	Investment Grade
AA+, AA, AA-	Aa1, Aa2, Aa3	AA+, AA, AA-	High Grade	
A+, A, A-	A1, A2, A3	A+, A, A-	Upper Medium Grade	
BBB+, BBB, BBB-	Baa1, Baa2, Baa3	BBB+, BBB, BBB-	Lower Medium Grade	
BB+, BB, BB-	Ba1, Ba2, Ba3	BB+, BB, BB-	Speculative	Non-Investment Grade
B+, B, B-	B1, B2, B3	B+, B, B-	Highly Speculative	
CCC+	Caa1	CCC	Significant Risk	
CCC	Caa2		Imminent Default	
CCC-, CC, C	Caa3, Ca, C		Default with low recovery expectation	
D		DDD, DD, D	Total Default	

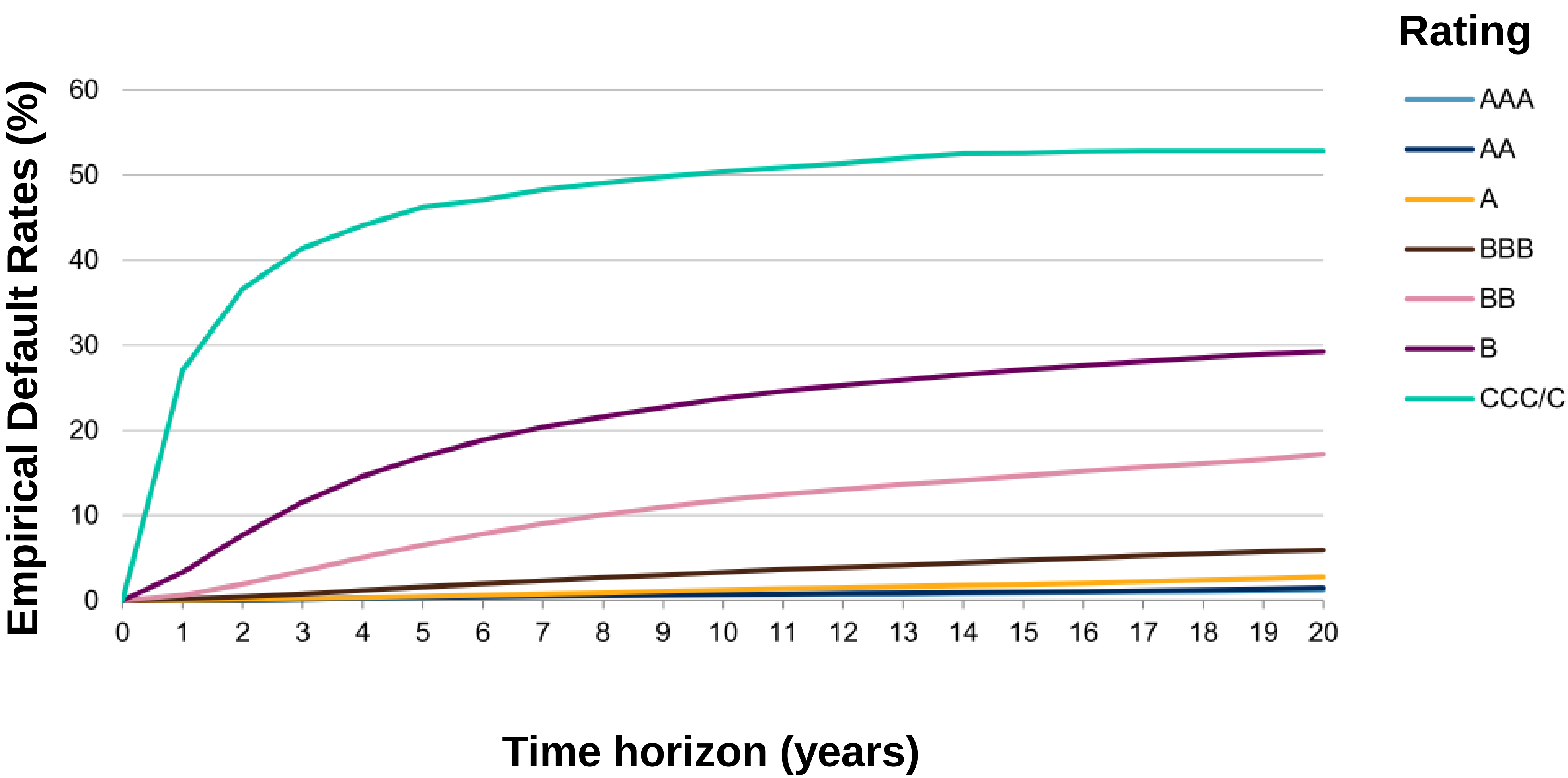
Global Corporate Default Rates

Year	AAA	AA	A	BBB	BB	B	CCC/C
1999	0	0.2	0.2	0.2	1.0	7.3	33.3
2000	0	0	0.3	0.4	1.2	7.7	36.0
2001	0	0	0.3	0.3	**3.0**	**11.5**	**45.9**
2002	0	0	0	1.0	**2.9**	**8.2**	**44.5**
2003	0	0	0	0.2	0.6	4.0	32.8
2004	0	0	0.1	0	0.4	1.5	16.2
2005	0	0	0	0.1	0.3	1.7	9.1
2006	0	0	0	0	0.3	0.8	13.3
2007	0	0	0	0	0.2	0.3	15.2
2008	0	0.4	0.4	0.5	0.8	4.0	27.3
2009	0	0	0.2	0.6	**0.8**	**10.9**	**49.5**
2010	0	0	0	0	0.6	0.9	22.6
2011	0	0	0	0.1	0	1.7	16.3
2012	0	0	0	0	0.3	1.6	27.5
2013	0	0	0	0	0.1	1.6	24.3
2014	0	0	0	0	0	0.8	17.1
2015	0	0	0	0	0.2	2.4	25.9
2016	0	0	0	0	0.5	3.7	32.7
2017	0	0	0	0	0.1	1.0	26.2
2018	0	0	0	0	0	1.0	27.2
2019	0	0	0	0.11	0	1.49	30.1

- Default rates exhibit a strong variation.
- During cyclical downturns default rates are particularly high (1992, 2001, 2002, 2008, 2009)

Source: S&P Gobal Fixed Income Research and S&P CreditPro®

Global Corporate Average Cumulative Default Rates by Rating (1981-2019)



Source: S&P Gobal Fixed Income Research and S&P CreditPro®

US Recovery Rates

Bonds	Recovery Rate
Investment-grade	61%
BB	33%
B	32%
CCC	38%

Source: S&P

- Recovery Rate (RR) = 1 - LGD
- Prices are normalized to a face value of $100.
- E.g. B-rated bond paid on average $32 after default.
- The table shows that the initial rating is a strong indicator for LGD.

Ratings and Yield Spreads

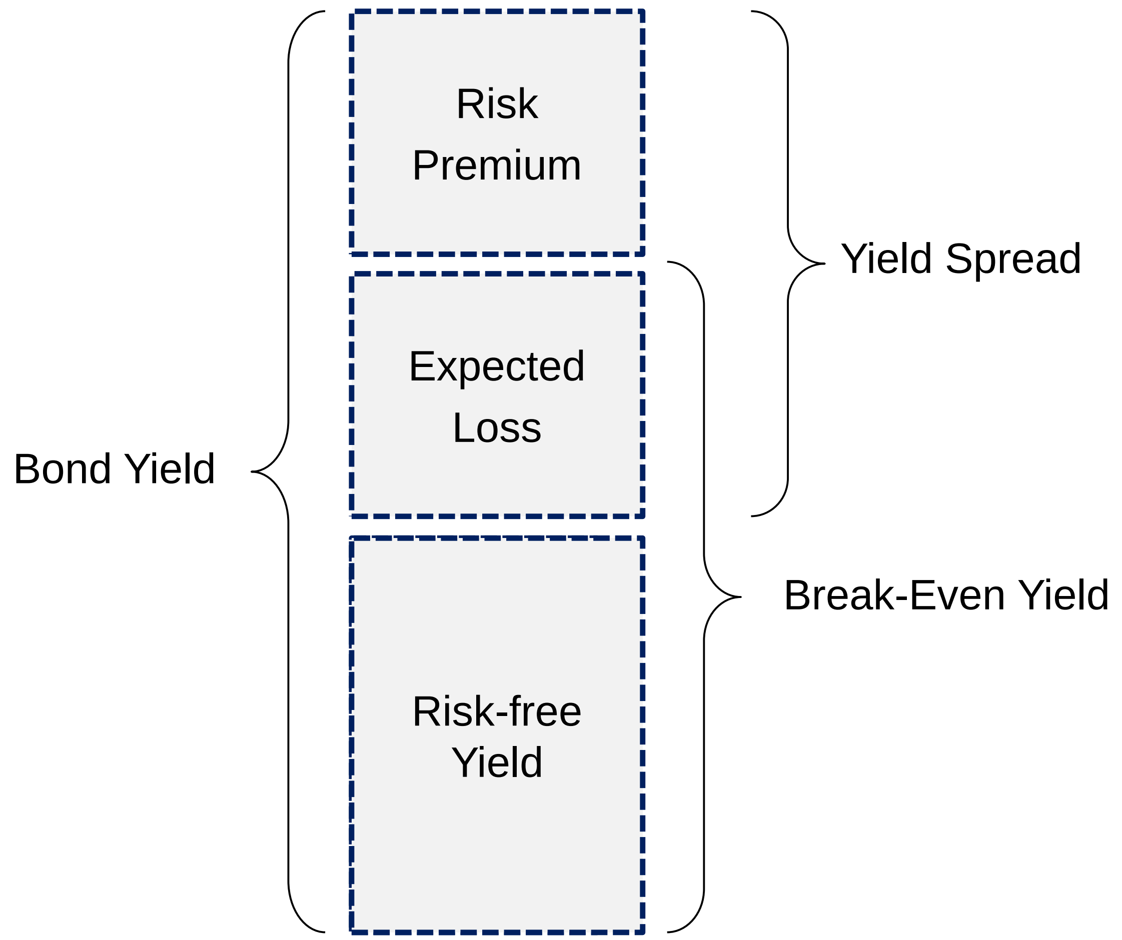
- For taking risk, the investor expects a risk premium leading to a higher return.
- Therefore, bonds with a lower rating generally pay a higher yield than equivalent higher rated bonds.
- The yield spread is the difference in yields between the risk-free (US Treasury Bonds) (i_f) and the risky bond (i) with the same maturity.

EXAMPLE

A zero coupon bond of an issuer with a rating of Aa and a maturity of 4 years pays a yield of 5.78%. The yield of a treasury strips with a matching maturity of 4 years is 5.4%. For the yield spread we have:

Yield spread (s) = $i - i_f = 0.38\%$.

Composition of the Bond Yield



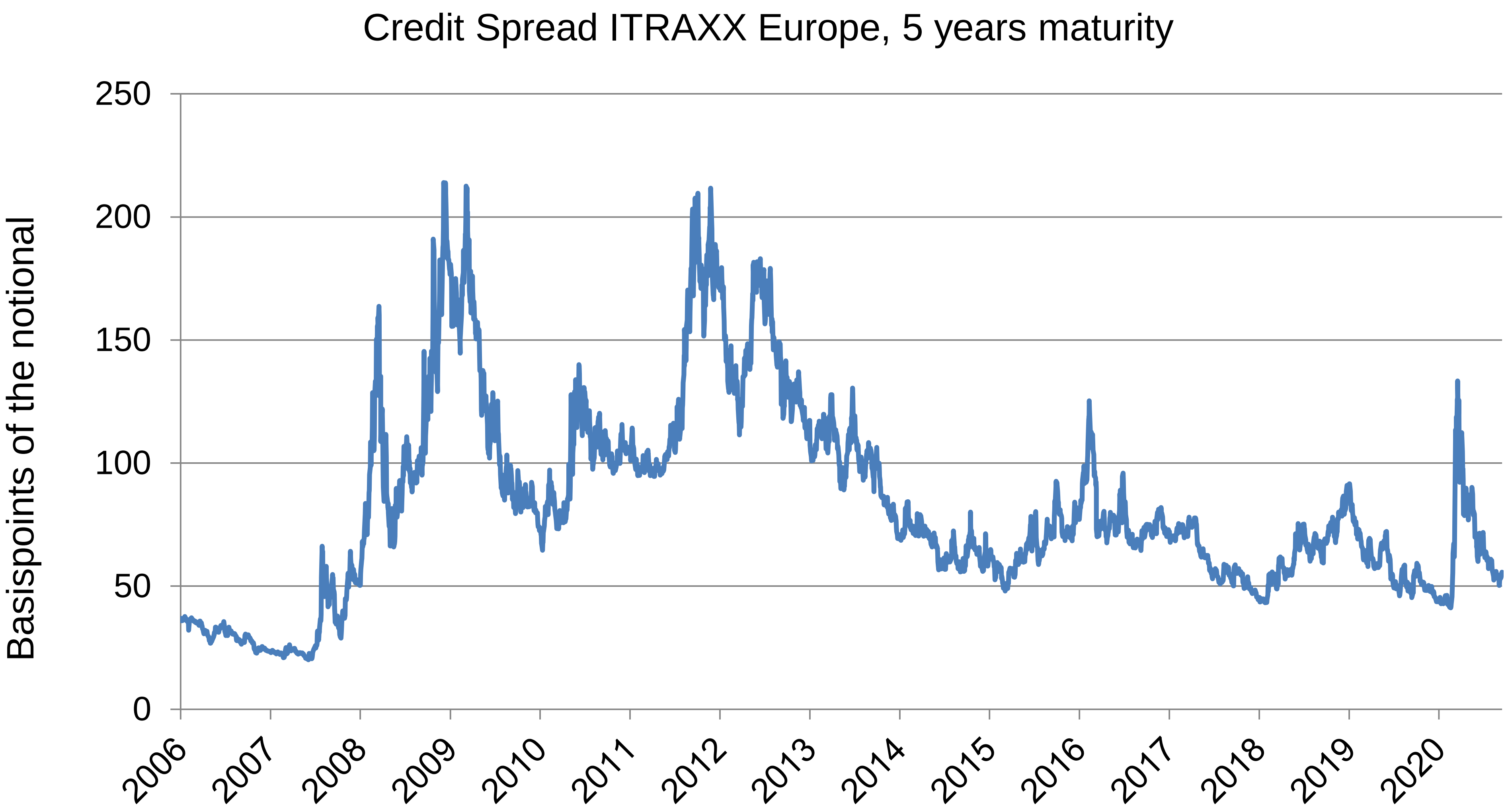
US Credit Risk Premia

- With the known recovery rates the Break-Even Yields can be calculated.
- The market pays premia for credit risk.

Year	High-Yield (average)	Risk-free yield	Average coupon	Break-even Yield (BEY)*	Risk premia
2019	4.1%	2.1%	5.9%	4.3%	-1.7%
2018	3.6%	2.9%	5.8%	5.2%	-1.6%
2017	3.8%	2.3%	6.4%	4.5%	-0.7%
2016	6.0%	1.8%	6.4%	4.6%	1.4%
2015	7.9%	2.1%	6.7%	5.3%	2.6%
2014	6.9%	2.5%	6.6%	5.2%	1.7%
2013	6.2%	2.4%	7.5%	5.1%	1.1%
2012	6.9%	1.8%	8.1%	4.8%	2.1%
2011	8.8%	2.8%	8.3%	4.7%	4.1%
2010	8.4%	3.2%	8.4%	5.8%	2.6%
2009	11.1%	3.3%	8.1%	12.3%	-1.2%
2008	11.6%	3.7%	8.1%	7.9%	3.7%
2007	9.0%	4.6%	7.9%	6.2%	2.8%
2006	8.5%	4.8%	7.9%	6.3%	2.2%
2005	7.7%	4.3%	8.0%	5.8%	1.9%
2004	7.8%	4.3%	8.3%	6.4%	1.4%
Average	***8.8%***	***3.5%***	***7.8%***	***7.5%***	***1.3%***

*BEY=$\frac{r_f + PD \times LGD + PD \times C/2}{1-PD}$, where i_f=risk free yield, PD=probability of default, LGD=loss given default, C=Coupon

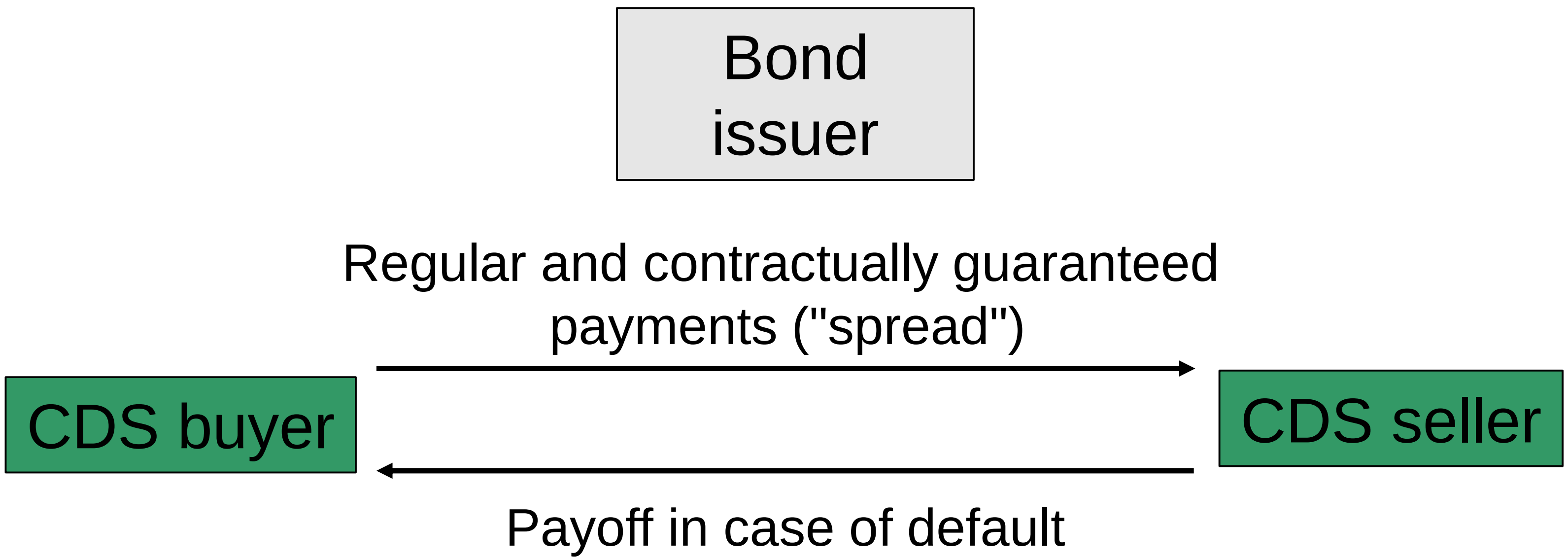
High Spreads in Times of Crisis



Source: Datastream

Credit Default Swaps (CDS)

- A CDS is an over-the-counter contract which offers protection against the default ("credit event") of a regarded corporate bond.



From CDS Spreads to Default Probabilities

Implied default probability?

- Expected payoff from the CDS: **LGD * PD**
- Link between the spread and the expected payoff: **s = LGD * PD**
- Transforming gives: **PD = s / LGD**

Example: Calculation of the Implied Default Probability of UBS AG

From market data we can observe the following input parameters:

1-year Senior CDS spread for UBS AG (21st of August 2019): **3.42 bps**

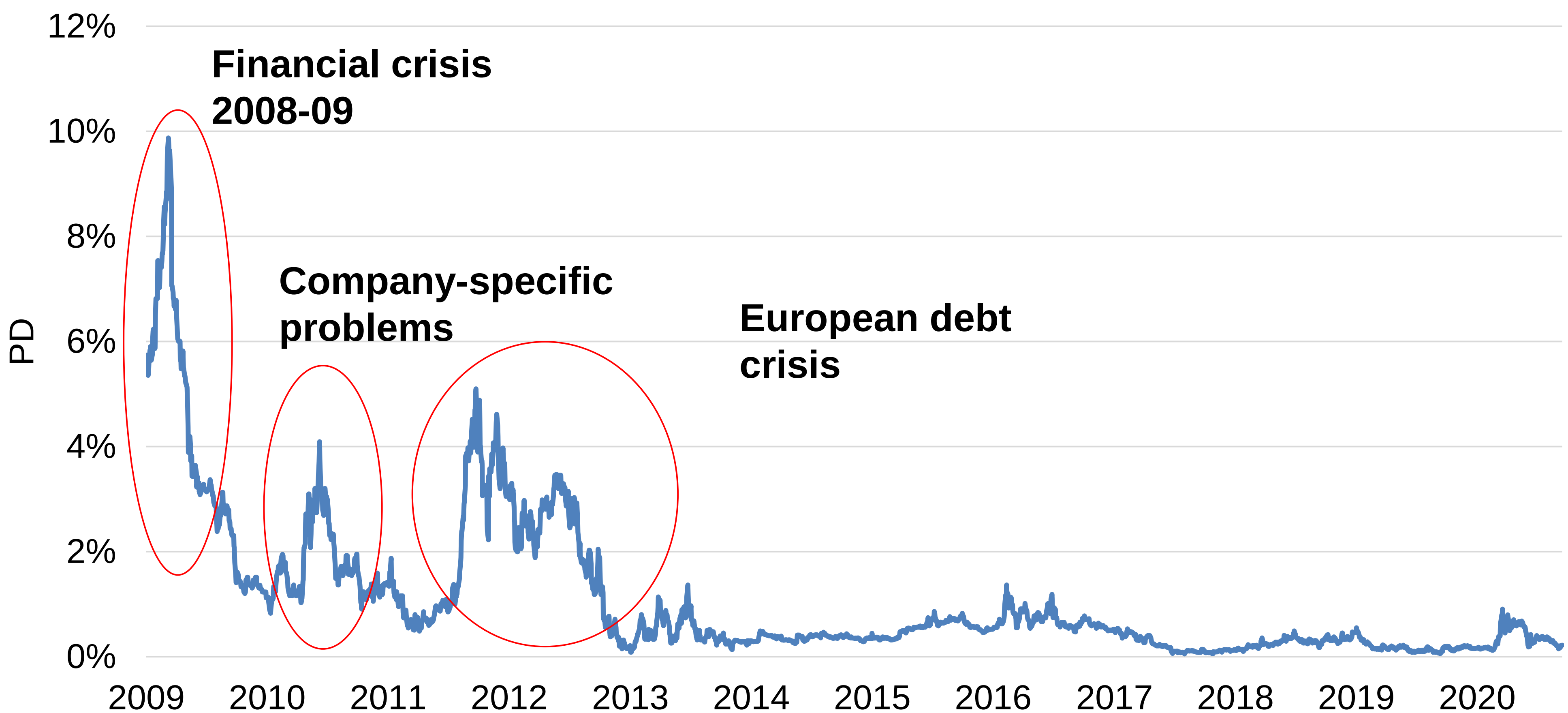
Recovery Rate (investment grade): **61%**

Bonds	Recovery Rate (RR)
Investment-grade	61%
BB	33%
B	32%
CCC	38%

$$PD_{1\ year} = 0.000342\ /\ (1 - 0.61) = \mathbf{0.09\%}$$

Historical UBS Spread and Implied PD

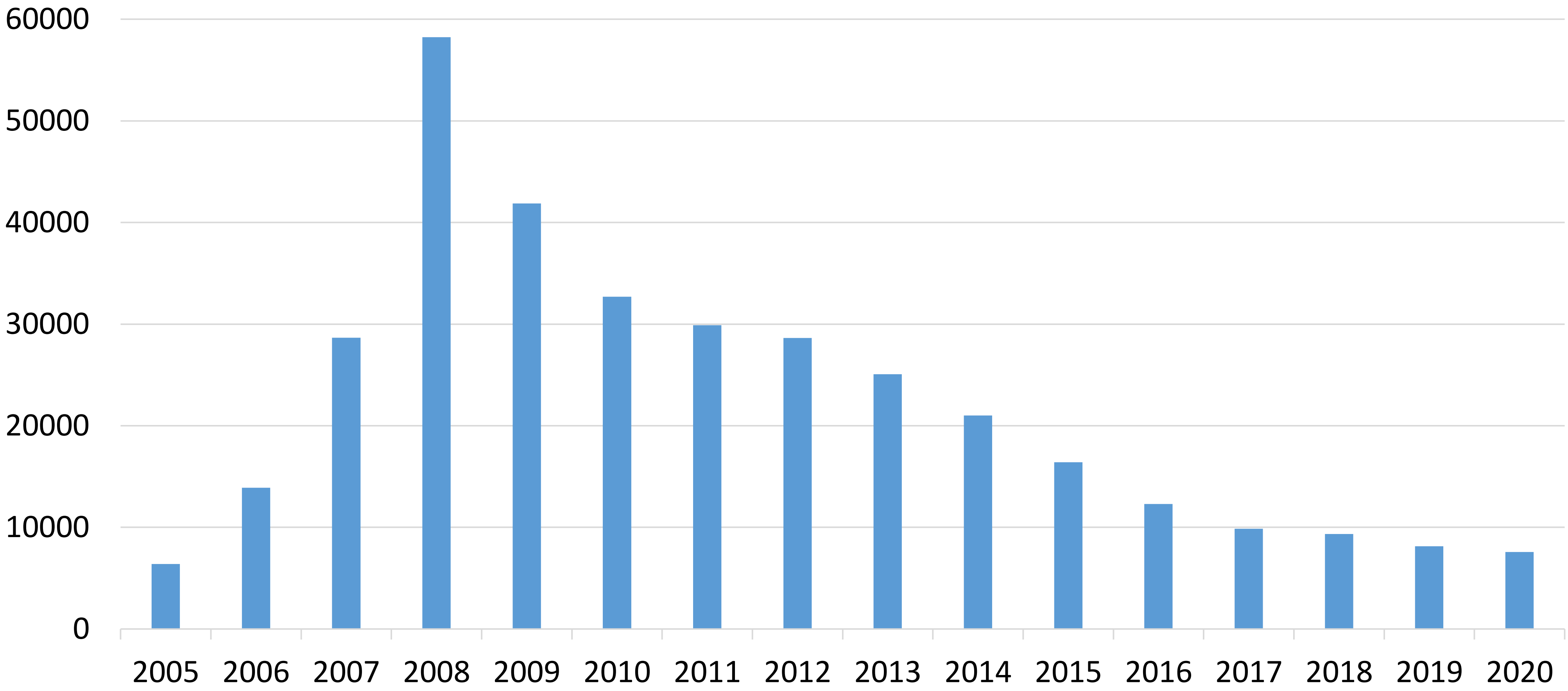
Example: 1-year Senior CDS spread for UBS AG



Source: Datastream

Overall Development of the CDS Market

Credit Default Swap notional (Bn. USD, beginning of the year)



Source: BIS