EASC 605 Project 2: Ice-flow modelling
Due: 12 November 2021

Introduction

This project is based on a project given by Thomas Schuler to graduate-level glaciology students and the University of Oslo. It makes use of the basic governing equations of ice flow to develop a one-dimensional flow-line model that can be used to explore the influence of the surface mass balance (ELA and mass-balance gradient), glacier geometry (ice thickness, bed slope) and ice rheology through the flow-law coefficient.

Governing equations

The continuity equation for ice is written in one dimension x as

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = \dot{b}, \tag{1}$$

with ice thickness h, time t, flux $q = \bar{u}\, h$ and net surface mass balance $\dot{b}$, which can be assumed to follow

$$\dot{b} = \Gamma\,(z_s - z_{\mathrm{ELA}}), \tag{2}$$

where Γ is the mass-balance gradient, z_s is the ice-surface elevation and z_{ELA} is the ELA. The bed elevation of the glacier resting on an inclined plane is

$$z_b = b_0 - \frac{\partial z_b}{\partial x} x, \tag{3}$$

for datum b_0, while

$$h = z_s - z_b. \tag{4}$$

We derived the solution for the vertical (z) velocity profile under a set of assumptions to be

$$u(z) = \frac{2A}{n+1}(\rho\, g\, \sin\theta)^n (h^{n+1} - (h - (z - z_b))^{n+1}), \tag{5}$$

with Glen's flow-law coefficient and exponent A and n, respectively, ice density ρ, gravitational acceleration g, bed slope θ, ice thickness h and sliding neglected. Let us make the small angle assumption such that

$$\tan\theta \approx \sin\theta, \tag{6}$$

and assume surface and bed slopes are similar so we can write

$$\frac{\partial z_{\mathrm{s}}}{\partial x} \approx \frac{\partial z_{\mathrm{b}}}{\partial x} = \tan\theta. \tag{7}$$

With these assumptions, the above equations can be combined in the form of a diffusion equation:

$$\frac{\partial z_{\mathrm{s}}}{\partial t} = -\frac{\partial}{\partial x}\left(D\frac{\partial z_{\mathrm{s}}}{\partial x}\right) + \dot{b}, \tag{8}$$

where D is the diffusivity. Standard methods exist to solve the diffusion equation, making this an appealing formulation.

Consider a domain of length L discretized into N gridcells of size $\Delta x = L/n$. The governing equations can be discretized onto the grid using finite-differences to approximate derivatives. For example, the diffusion equation above may be evaluated at gridcell i using the discrete form

$$\frac{\Delta z_{\mathrm{s_i}}}{\Delta t} = -\frac{1}{\Delta x}\left(D_{i+\frac{1}{2}}\frac{(z_{\mathrm{s_{i+1}}} - z_{\mathrm{s_i}})}{\Delta x} - D_{i-\frac{1}{2}}\frac{(z_{\mathrm{s_i}} - z_{\mathrm{s_{i-1}}})}{\Delta x}\right) + \dot{b}_i. \tag{9}$$

Methodology

- Determine the analytical expression for D that allows Equations (1), (5) and (6) to be rewritten as (8)
- Define a domain and use a finite-difference scheme to discretize the diffusion equation on a staggered grid with appropriate boundary conditions. Ice thickness, surface- and bed elevation, and mass balance should all be evaluated at gridcell centres (i), while flux should be evaluated at gridcell interfaces ($i+1/2$, $i-1/2$). D will have to be evaluated at gridcell centres D_i, and then values at interfaces computed as averages, e.g. $D_{i+1/2} = (D_{i+1} + D_i)/2$ and $D_{i-1/2} = (D_i + D_{i-1/2})/2$. The quantity $D\frac{\partial z_{\mathrm{s}}}{\partial x}$ should be evaluated on the staggered grid (at gridcell interfaces). Then the right-hand side of the prognostic

Table 1: Useful parameters

Symbol	Constant	Value
g	Acceleration due to gravity	$9.81\ \mathrm{m\,s^{-2}}$
ρ	Density of ice	$910\ \mathrm{kg\,m^{-3}}$
n	Flow-law exponent	3
Symbol	**Variable**	**Sensible starting value**
L	Domain length	$50\,\mathrm{km}$
A	Flow-law coefficient	$2.4 \times 10^{-24}\,\mathrm{Pa^{-3}\,s^{-1}}$
$\frac{\partial z_b}{\partial x}$	Bed slope	0.05
b_0	Max bed height	$1600\,\mathrm{m}$ a.s.l.
Γ	Mass-balance gradient	$0.007\,\mathrm{a^{-1}}$
z_{ELA}	ELA	$1400\,\mathrm{m}$ a.s.l.

equation can be evaluated at gridcell centres and the incremental change in thickness determined for a given timestep Δt. Use centred differences, except at the boundaries where you should use forward- and backward differences. No-flux boundary conditions would be a reasonable choice at both ends, provided that the domain encompasses the glacier terminus and does not truncate it. Present the discretized equations in your report.

- Test your numerical model against the analytical solution for $h(x)$ under the simplifying assumptions of a flat bed ($\frac{\partial z_b}{\partial x} = 0$) and a constant and uniform mass balance, i.e. $\dot{b} = \gamma_0$. Plot your numerical and analytical solutions together. Make sure the numerical solution reaches steady state. Repeat this comparison for difference gridcell sizes to examine the grid dependence. Note that you may need to adjust your timestep as you adjust the gridcell size. "Explicit" models, such as this one, are prone to large errors with increases in timestep.

- Once you are satisfied with the model performance, explore the parameter space to determine the model sensitivity to mass balance, bed slope and the flow-law coefficient.

Results

Write up your results as a report that includes your computer code (the calculations, not the plotting) in an appendix. Structure the report like a research paper with an abstract, short introduction, methods, results, discussion, conclusion and references.