Laminar–Turbulent Sheet Conductivity Scaling

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Sheet-flow models

Table 1 lists the different sheet-flow options:

Table 1: Model descriptions

Model	
Laminar	$\mathbf{q} = -k_\mathrm{s} h^3 \nabla\phi$
Turbulent	$\mathbf{q} = -k_\mathrm{s} h^{\alpha_\mathrm{s}} \lvert\nabla\phi\rvert^{-1/2} \nabla\phi$
Transition	$\mathbf{q} + \frac{\omega}{\nu}\left(\frac{h}{h_b}\right)^{3-2\alpha_\mathrm{s}} \lvert\mathbf{q}\rvert\mathbf{q} = -k_\mathrm{s} h^3 \nabla\phi$

Adjusting the sheet conductivity

Note that the units of k_s for the turbulent model depend on the value of α_s and are different from the units for the laminar and transition models. This means we need to find a relationship between the conductivity for the turbulent and laminar/transition models to make them comparable.

There are at least two ways to do this:

1. **Asymptotically match the behaviour.** The transition model asymptotically approaches the laminar model as $\mathrm{Re} = \frac{q}{\nu} \to 0$ by definition. We could also tune the conductivity of the turbulent model so that the turbulent model matches the $\mathrm{Re} \to \infty$ limit of the transition model.

   Taking $\mathrm{Re} \to \infty$, and taking the absolute value to simplify, the transition model becomes

   $$\frac{\omega}{\nu}\left(\frac{h}{h_\mathrm{b}}\right)^{3-2\alpha_\mathrm{s}} q^2 = k_\mathrm{s} h^{3/2} \lvert\nabla\phi\rvert^{1/2}. \tag{1}$$

   This can be written to look like the turbulent model,

   $$q = \left[k_s \frac{\nu}{\omega}(h_\mathrm{b})^{2\alpha_\mathrm{s}-3}\right]^{1/2} h^{\alpha_\mathrm{s}} \lvert\nabla\phi\rvert^{1/2}. \tag{2}$$

   So, we've recovered the turbulent model as the $\mathrm{Re} \to \infty$ limit of the transition model. We can then compute what k_s we should use in the transition/laminar model so that the entire coefficient in Eq. 2 matches the turbulent conductivity we have been using so far.

2. **Match laminar and turbulent models at the transition** Re**.** The idea is to have $\mathbf{q}_\mathrm{laminar} = \mathbf{q}_\mathrm{turbulent}$ at the transition between laminar and turbulent sheet flow. We do this by first computing the sheet thickness that gives $\omega\mathrm{Re} = 1$. It's easiest to do this from the laminar model:

   $$1 = \omega\mathrm{Re} = \omega\frac{q}{\nu} = \frac{\omega}{\nu} k_\mathrm{s} h_\mathrm{crit}^3 \lvert\nabla\phi\rvert, \tag{3}$$

   so that

   $$h_\mathrm{crit}^3 = \frac{\nu}{\omega k_\mathrm{s} \lvert\nabla\phi\rvert}. \tag{4}$$

Now we substitute this expression for h_{crit}^3 into the turbulent model, setting $\omega \text{Re} = \omega \frac{q}{\nu} = 1$. For clarity, write the conductivity parameter for the turbulent model as k_{t}. Then,

$$1 = \frac{\omega}{\nu} k_{\text{t}} h_{\text{crit}}^{\alpha_{\text{s}}} |\nabla \phi|^{1/2}. \tag{5}$$

Simplifying,

$$k_{\text{s}}^{\alpha} = k_{\text{t}}^3 \left(\frac{\omega}{\nu} \right)^{3-\alpha_{\text{s}}} |\nabla \phi|^{\left(\frac{3}{2} - \alpha \right)}. \tag{6}$$

At this point you can go either direction to convert between k_{s} and k_{t}. If you want to continue with a conductivity that matches what you've been using for the turbulent model, use this for your k_{t} and solve for the laminar/transition conductivity k_{s}. Note that the $\nabla \phi$ dependence goes away if (and only if) the turbulent sheet conductivity $\alpha_{\text{s}} = 3/2$, so here you can use any value for $\nabla \phi$. For the turbulent model with $\alpha_{\text{s}} = 5/4$, you will need to assume a value for the hydraulic potential gradient and the result will depend on this value. The sensitivity is very low ($|\nabla \phi|^{1/12}$), so I've just used a taken the difference in potential (assuming $p_{\text{w}} = p_{\text{i}}$) between the terminus and highest point divided by the domain length.

For my simulations, it turned out that option (2) provided a more fair comparison between the models. For the same laminar/transition conductivity, option (1) results in a higher turbulent sheet conductivity, such that it produced fewer and smaller channels and extremely low winter water pressure.

In practice

For my simulations, parameters are in Table 2. I chose a laminar/transition sheet conductivity $k_{\text{s}} = 0.05\,\text{Pa}\,\text{s}^{-1}$, and this gave a conductivity for the turbulent 5/4 model of 0.0071 (with appropriate units) using option (2) for scaling (Table 3).

Table 2: Constants (top group) and model parameters (bottom group) for GlaDS simulations.

Symbol	**Description**	**Value**	**Units**
ρ_w	Density of water	1000	kg m^{-3}
ρ_i	Density of ice	910	kg m^{-3}
g	Gravitational acceleration	9.81	m^{3} s^{-1}
c_w	Specific heat capacity of water	4.22×10^{3}	J kg^{-1}
c_t	Pressure melting coefficient	-7.50×10^{-8}	K Pa^{-1}
ν	Kinematic viscosity of water at 0°C	1.793×10^{-6}	m s^{-2}
k_s	Effective laminar sheet conductivity	0.1	Pa s^{-1}
α_s	Sheet-flow exponent	$[\frac{5}{4}, \frac{3}{2}, 3]$	
β_s	Sheet-flow exponent	$[\frac{3}{2}, 2]$	
k_c	Channel conductivity	0.2	m$^{3/2}$s^{-1}
α_c	Channel-flow exponent	5/4	
β_c	Channel-flow exponent	3/2	
h_b	Bed bump height	0.5	m
l_b	Bed bump length	10	m
l_c	Width of sheet-flow contributing to channel	10	m
e_v	Englacial porosity	1×10^{-4}	
ω	Laminar–turbulent transition parameter	1/2000	
u_b	Basal velocity	30	m a^{-1}
$\tilde{A}$[a]	Rheological parameter for creep closure	1.78×10^{-25}	s^{-1} Pa^{-3}
$\tilde{A}_\mathrm{s}$	Rheological parameter for creep when $N < 0$	0	s^{-1} Pa^{-3}
n	Ice-flow exponent	3	
$\dot{m}_\mathrm{s}$	Basal melt rate	0.01	m w.e. a^{-1}

[a] $\tilde{A}$ differs from the canonical rheology parameter A by a factor of $\frac{2}{27}$. The listed value for $\tilde{A}$ corresponds to the recommended value $A = 2.4 \times 10^{-24}$ s^{-1} Pa^{-3} for temperate ice.

Table 3: Comparable model conductivities. Caution, these values are conditioned on $\omega = 1/2000$ (Table 2) and a hydraulic potential gradient of $\nabla\phi = 135\,\mathrm{Pa\,m^{-1}}$

Model	**Sheet conductivity**	**Units**
Turbulent ($\alpha_\mathrm{s} = 5/4$)	0.0071	$\mathrm{m^{7/4}\,kg^{-1/2}}$
Turbulent ($\alpha_\mathrm{s} = 3/2$)	0.0134	$\mathrm{m\,kg^{-1/2}}$
Laminar	0.05	$\mathrm{Pa\,s^{-1}}$
Transition 5/4	0.05	$\mathrm{Pa\,s^{-1}}$
Transition 3/2	0.05	$\mathrm{Pa\,s^{-1}}$