HEP Muography: Theoretical Validation of Simulation

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1 Scintillation Light Yield of BC-408

1.1 BC-408 Stopping Power from Bethe-Bloch Formula

The Bethe-Bloch equation tells us the average stopping power, $\frac{dE}{dx}$, of the material when some charged particle incident to the material travels through it. That is, it tells us the average total energy loss per unit distance, due to electromagnetic interactions. These interactions are ionization and excitation. We use the Bethe-Bloch equation to calculate how much energy we expect a 100GeV positive muon to deposit into BC-408, after traveling 1.27cm through the plastic.

Since BC-408 is 91.512% carbon, we approximate the calculation by treating it as only carbon, for simplicity. This is a reliable approximation because the stopping power of a material combines linearly in the atomic composition of the material.

$$\begin{aligned} -\frac{dE}{dx} &= Kz^2 \frac{Z}{A}\frac{1}{\beta^2}\left[\frac{1}{2}\ln\frac{2m_e c^2\beta^2\gamma^2 T_{max}}{I^2} - \beta^2 - \frac{\delta}{2}\right] \\ &= K\frac{6}{A}\frac{1}{\beta^2}\left[\frac{1}{2}\ln\frac{2(0.511\text{MeV})\beta^2\gamma^2 T_{max}}{I^2} - \beta^2 - \frac{\delta}{2}\right] \end{aligned}$$

A positive muon has charge $z = 1$ in electron charge units.

A positive muon's rest mass $m_0 = 105.7$ MeV$/c^2$. Since we have 100 GeV muons, it follows that $\gamma^2 = 946$ and $\beta^2 \approx 1$.

The mean ionization potential for carbon is 78eV. $T_{\max}$ is the maximum kinetic energy transferred by the muon to a single free electron.

Using relativistic kinematics, we can find that this is $\dfrac{2m_e\beta^2\gamma^2}{1 + 2\dfrac{m_e}{m_\mu} + \dfrac{m_e^2}{m_\mu^2}} = 90137.4$ MeV.

Finally, we have that $K = 0.307075$ MeV g^{-1} cm^2 for $A = 1$ g mol^{-1}. Since the atomic mass of carbon is 12.01 g mol^{-1}, we have $K/A = 0.025589$ MeV g^{-1} cm^2. Thus,

$$-\frac{dE}{dx} = \frac{K\cdot 6}{A}\left[\frac{1}{2}\ln\frac{2(0.511\text{MeV})(895,055)}{I} - 1 - \frac{\delta}{2}\right]$$

$$-\frac{dE}{dx} = \frac{K\cdot 6}{A}\left[\frac{1}{2}(23.185) - 1 - \frac{\delta}{2}\right]$$

$$-\frac{dE}{dx} = 6\cdot 0.025589\cdot\left[\frac{1}{2}(30.237) - 1 - \frac{\delta}{2}\right]$$

We then approximate this without the density effect correction, δ, which is a good approximation at the energy levels we are concerned with, and get

$$-\frac{dE}{dx} = 6\cdot 0.025589\cdot\left[\frac{1}{2}(30.237) - 1\right] = 2.17 \text{ MeV cm}^{-2}.$$

1.2 Scintillation Yield from Birks' Law

The Bethe-Bloch equation tells us nothing about what happens to that energy after it is deposited into the material. This is what Birks' Law tells us.

Birks' law is an empirical equation for calculating the number of scintillation photons produced per unit length as function of the energy loss per unit length. It is usually expressed as

$$\frac{dL}{dx} = S\frac{\dfrac{dE}{dx}}{1 + kB\dfrac{dE}{dx}}$$

The constant k is probability of quenching. The constant B is another constant of proportional which is specific to the material. Together, kB is referred to as Birks' coefficient, and has units of distance per energy. Reference (3) gives that kB=0.155 mm/MeV consistent with the fact that Polyvinyl toluenes have $0.126 \leq kB \leq 0.207$ g MeV / cm^2.

S is the scintillation efficiency, which is generally define to be the number of scintillation photons produced per unit distance traversed by the incident particle. (In certain contexts, it is also sometimes defined as the percentage of deposited energy which is transformed into scintillation light, but not in the context of Birks' Law - we can see this by dimensional analysis of Birks' Law.)

Using the result obtained from the Bethe-Bloch equation, we have an estimate for the total light yield through Birks' Law as follows:

$$\begin{aligned}\frac{dL}{dx} &= S\frac{\frac{dE}{dx}}{1 + kB\frac{dE}{dx}} \\ &= S\frac{2}{1.031} \\ &= 10,000 \cdot 1.94 \\ &= 19,398 \text{ photons cm}^{-1}.\end{aligned}$$