Modeling and Experimentation: Mass-Spring-Damper System Dynamics

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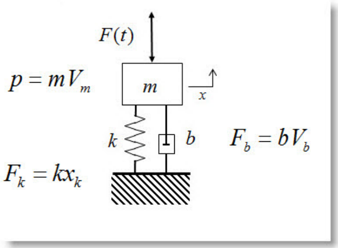
Overview

1. Review two common mass-spring-damper system models and how they are used in practice
2. The standard linear 2nd order ODE will be reviewed, including the natural frequency and damping ratio
3. Show how these models are applied to practical vibration problems, review lab models and objectives
4. Discuss methods that can be helpful in improving estimates of the system parameters, introduce the log decrement method for measuring damping ratio

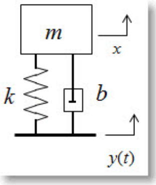
Most common mass-spring-damper models

Fixed-base configuration, spring and damper in parallel.



Ex: structures, buildings, etc.

Base-excited configuration, spring and damper in parallel, motion input at base..



Ex: Vehicle suspensions, seismic sensors

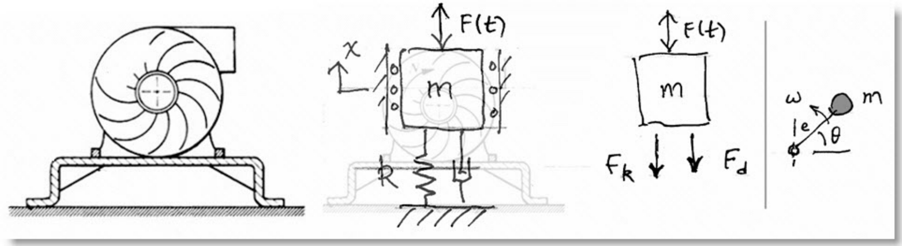
When a parameter like k or b is indicated, it usually implies that a *linear* constitutive law is implied for that model element.

Using mass-spring-damper models

1. Use these models to represent a wide range of practical situations, not just translation but rotation. The models have analogies in all other energy domains.
2. Recognize 'forcing' in each case: force $F(t)$ on mass for fixed-based compared with velocity $\dot{y}(t)$ at the base for base-excited system
3. Unforced response. Some problems are concerned with the system responding to initial conditions and *no forcing* (i.e., $F(t) = 0$, $\dot{y}(t) = 0$), so the transient response and the eventual rest condition is of interest.
4. Forced response. If there is forcing, there may be a need to understand transient changes and then the 'steady' operation under forcing. Types include: step, impulse, harmonic, random

Example: unbalanced fan on support base

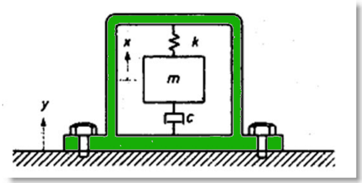
An unbalanced fan is mounted on a support base, and the model to study induced vibration is illustrated below. This is a fixed-base configuration.



An additional modeling issue in this problem is coming up with the forcing function, $F(t)$, which would come from understanding how the eccentricity in the fan impeller induces a force on the total fan mass. Note an assumption can also made that the fan only vibrates in the vertical direction, if the base structure is very stiff laterally. All of these decisions are part of the modeling process.

Example: motion sensor (seismic sensor)

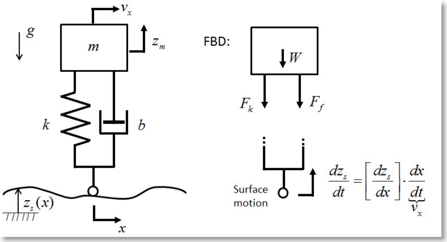
An example of a system that is modeled using the based-excited mass-spring-damper is a class of motion sensors sometimes called seismic sensors. Accelerometers belong to this class of sensors.



The spring and damper elements are in mechanical parallel and support the 'seismic mass' within the case. The case is the base that is excited by the input base motion, $y(t)$.

Example: suspended mass moving over a surface

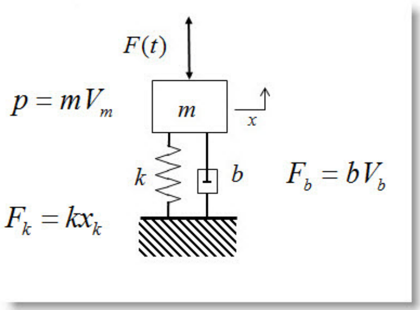
The diagram below is uses a base-excited configuration to model a mass moving over a surface. The spring and damper elements might represent, for example, the tire contacting the ground as the vehicle moves in the x direction. Vibration of the mass is in the z direction.



An automotive suspension model like this would represent only a quarter of the vehicle, and there would be another *stage* that represents the actual suspension. Note how you model the *base-motion* by factoring how fast you move over a ground profile, $z_s(x)$, a function of distance traveled, x.

Model of fixed-base mass-spring-damper system

Consider the fixed-base system below.



Applying Newton's law to a free-body diagram of the mass,

$$m\ddot{x} = \sum F = -F_b - F_k + F,$$

$$m\ddot{x} = -b\dot{x} - kx + F,$$

Rearrange to derive a 2nd order ODE,

$$m\ddot{x} + b\dot{x} + kx = F.$$

Now, in standard form,

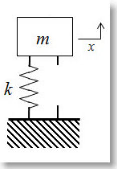
$$\ddot{x} + \underbrace{\left[\frac{b}{m}\right]}_{\triangleq 2\zeta\omega_n}\dot{x} + \underbrace{\left[\frac{k}{m}\right]}_{\triangleq \omega_n^2} x = u$$

where $u = F/m$, ω_n is the *undamped* natural frequency, and ζ is the damping ratio.

From the undamped natural frequency, define the undamped natural period, $T_n = 2\pi/\omega_n$.

Undamped harmonic motion, $\zeta = 0$

For the unforced case, $F = 0$, and undamped, $b = 0$, case,



the model equation becomes, $\ddot{x} + \omega_n^2 x = 0$, with a complete solution for the displacement of the mass,

$$x = x_o \cos(\omega_n t) + \frac{\dot{x}_o}{\omega_n} \sin(\omega_n t)$$

where x_o is the initial displacement and $\dot{x}_o$ the initial velocity of the mass.

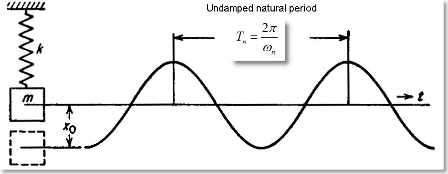
This solution gives x for all time given only the initial conditions and the natural frequency, which in this case is $\omega_n = \sqrt{k/m}$.

NOTE: This ideal case is a 'linear oscillator', a model used to described many other types of systems such as the simple or compound pendulum in small motion. We can see how ω_n in those cases would contain parameters such as length, gravity, mass.

Undamped harmonic motion

If you give the mass an initial displacement x_o and release from rest ($\dot{x}_o = 0$), then this case predicts it would oscillate indefinitely,

$$x = x_o \cos(\omega_n t)$$



Given $x(t)$, the velocity and acceleration are readily found:

$$v = -x_o \omega_n \sin(\omega_n t)$$

$$a = +x_o \omega_n^2 \cos(\omega_n t)$$

NOTE: The undamped natural frequency directly influences peak values of velocity and acceleration.

What about the effect of gravity?

The axis of mass motion may be in line with the gravity vector, so sometimes it is necessary to consider the effect of the force due to gravity, mg.

In vibration problems, however, especially when all the elements follow linear constitutive laws (i.e., spring, $F = kx$, damper, $F = bV$), the effect of gravity will not influence the dynamics. Only the *equilibrium* position of the mass will be changed.

If you had a spring that was not linear, then it is important to consider gravity because vibrational motion will be influenced by the spring behavior in a nonlinear manner.

Bottom line: If system is linear, use gravity to find where your system 'sits' initially, but then the linear solutions discussed here apply directly for motion about that point. If nonlinear, you may need to follow other approaches such as linearizing about the equilibrium, which is found by considering gravity. Direct numerical simulation of the system is often an alternative.

Consider the damped cases now, $\zeta \neq 0$

The special *undamped* case has been described. For systems where $b \neq 0$, the damping ratio will not be zero. Solutions for these cases are classified by ζ, and a system is:

- *underdamped* if $\zeta < 1$,
- *overdamped* if $\zeta > 1$,
- *critically damped* if $\zeta = 1$

The solutions are known for these cases, so it is worthwhile formulating model equations in the standard form,

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u(t)$$

Detailed derivations can be found in system dynamics, vibrations, circuits, etc., type textbooks. Selected excerpts will be posted on the course log for reference.

Underdamped motion, $\zeta < 1$

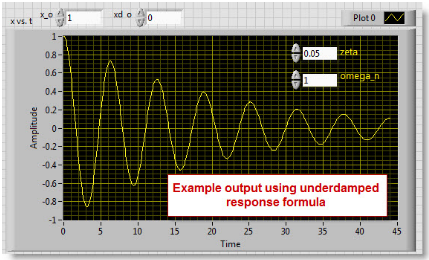
If you give the mass an initial displacement x_o and initial velocity $\dot{x}_o$, the solution for the response, x, is,

$$x = e^{-\zeta\omega_n t}\left[\frac{\dot{x}_o + \zeta\omega_n x_o}{\omega_d}\sin(\omega_d t) + x_o\cos(\omega_d t)\right], 0 < \zeta < 1$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2} = 2\pi/T_d$ is the *damped* natural frequency, and T_d the damped natural period.

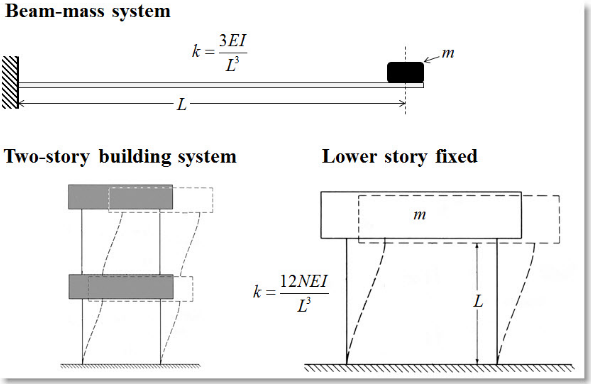
Given $x(t)$, the velocity and acceleration can be found by differentiation.

NOTE: The damped natural frequency is dependent on both the undamped natural frequency and the damping ratio.



Possible lab models

Demonstrating the mass-spring-damper model in the lab can be done in several ways. Simply suspending a mass with a spring-like element is easy, but it can be difficult to get purely one-degree-of-freedom motion. Using beam-mass combinations can provide a simple and reliable way to test the concepts we've discussed.

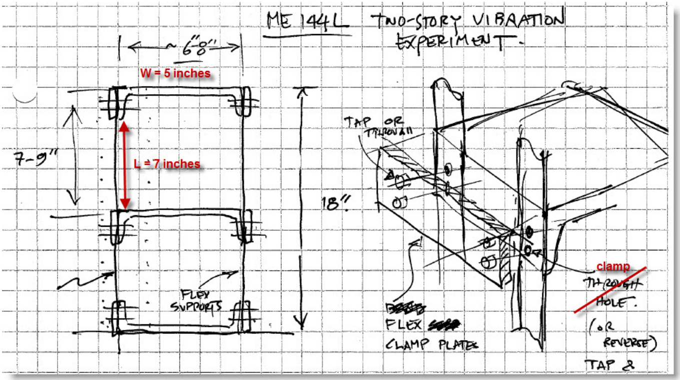


Typical laboratory objectives

1. Estimate the mass and stiffness of the system components before going to lab, and verify or improve once in lab. Try to understand sources of error.
2. Use measurements of motion to estimate the effective system damping ratio, ζ, which can be used to estimate effective damping, b, if the system is linear.
3. Determine the damped and undamped natural frequencies.
4. Assess accuracy of system parameter values.
    - Use the measured natural frequency to estimate the stiffness, assuming confidence in mass estimate.
    - Assess model by comparing theoretical versus measured response. For example, compare measured acceleration to that predicted by model response.
5. Estimating displacement from accelerometer measurement can be challenging. Explore this by integrating the acceleration to get displacement.

Two-story building lab model

The concept for the two-story lab model is shown below. The flex guide lengths, L, are nominally 7 inches, but these can be adjusted. The actual 'floors' are 5 inches in width (and depth), made from aluminum U-channel.



The specific material list is provided on the course log.

How well can you estimate ω_n?

Once you get into the lab, you can measure the mass and stiffness to improve the theoretical estimate of ω_n. How accurate is this value?

A comparison made to a measurement of ω_n is preferred, however remember that we cannot directly measure this value from $T_n = 2\pi/\omega_n$ except in the undamped case. Only the *damped* natural frequency, T_d, can be measured. Since,

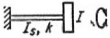
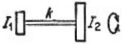
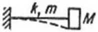
$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}},$$

you can only estimate ω_n if you have an estimate of ζ, which can be done using the log decrement method. Before discussing that method, first consider some sources that could influence estimates of ω_n.

Influence of spring mass on undamped natural frequency

From appendix in Den Hartog [1] (see handout on course log), the undamped natural frequency for several simple 'mass-spring' systems can be estimated *theoretically* (using Rayleigh's method [2]) as follows:

III. Natural Frequencies of Simple Systems

	End mass M; spring mass m, spring stiffness k	$\omega_n = \sqrt{k/(M + m/3)}$	(18)
	End inertia I; shaft inertia I_s, shaft stiffness k	$\omega_n = \sqrt{k/(I + I_s/3)}$	(19)
	Two disks on a shaft	$\omega_n = \sqrt{\dfrac{k(I_1 + I_2)}{I_1 I_2}}$	(20)
	Cantilever; end mass M; beam mass m, stiffness by formula (2)	$\omega_n = \sqrt{\dfrac{k}{M + 0.23m}}$	(21)
	Simply supported beam; central mass M; beam mass m; stiffness by formula (4)	$\omega_n = \sqrt{\dfrac{k}{M + 0.5m}}$	(22)

Note that a *fractional* mass of the spring element is used in each formula.

Bounds on undamped natural frequency estimate

The influence of spring mass suggests one way to calculate upper and lower bounds on the undamped natural frequency is to consider:

Case 1: Assume spring is *massless*. In this case, the undamped natural frequency is,

$$\omega_{n1} = \sqrt{k/m}.$$

Case 2: Assume all the spring mass, m_s, is lumped into main mass. In this case, the undamped natural frequency is,

$$\omega_{n2} = \sqrt{k/(m+m_s)}.$$

The *actual* undamped natural frequency has to lie between these two values.

$$\omega_{n2} \leq \omega_{n,\text{actual}} \leq \omega_{n1}$$

This assumes you know the stiffness well. Of course, one could do a complete uncertainty analysis on both parameters, but if you can assume a good estimate of one or the other these quick bounds can be easily calculated.

Experimental determination of damping ratio (1)

From the model for underdamped response of an *unforced* second order system presented earlier,

$$x = e^{-\zeta\omega_n t}\left[\frac{\dot{x}_o + \zeta\omega_n x_o}{\omega_d}\sin(\omega_d t) + x_o\cos(\omega_d t)\right], 0 < \zeta < 1$$

which can be expressed,

$$x(t) = A_o e^{-\zeta\omega_n t}\cos\left[\omega_d t - \phi\right]$$

where,

$$A_o = \left[\left(\frac{\dot{x}_o + \zeta\omega_n x_o}{\omega_d}\right)^2 + x_o^2\right]^{1/2} \quad \text{and} \quad \tan\phi = \frac{\dot{x}_o + \zeta\omega_n x_o}{\omega_d x_o}$$

The important result here is that the amplitude of the response follows a *decaying exponential* that is a function of initial conditions, ζ, and ω_n.

Experimental determination of damping (2)

The amplitude decay can be determined using peak values measured every T_d seconds every cycle, n. The amplitude ratio between successive peaks is then,

$$\frac{A_n}{A_{n+1}} = \exp\left[\zeta\omega_n\left(t_{n+1} - t_n\right)\right]$$

and the period between peaks is,

$$\left(t_{n+1} - t_n\right) = T_d = 2\pi/\omega_d = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}.$$

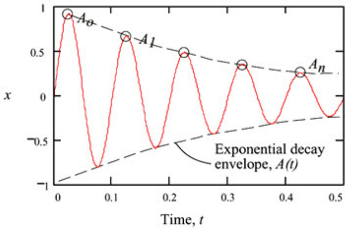
Now, taking log of both sides of the amplitude ratio relation,

$$\ln\left[\frac{A_n}{A_{n+1}}\right] = \left[\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right]\cdot n$$

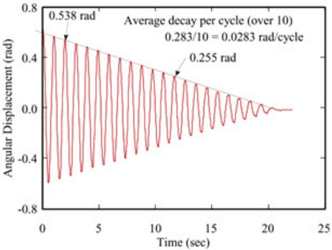
This shows that log of the amplitude ratios–the log decrement–is *linearly* related to the cycle number, n, by a factor that is a function of ζ.

Example decay envelopes

Exponential decay:



Linear decay:

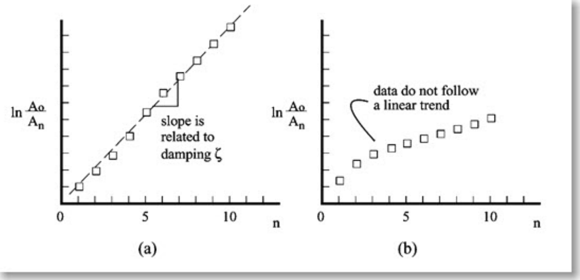


An envelope that takes on an exponential shape may suggest linear damping is dominant, while the envelope on the right with linear decay indicates the system does not have linear damping.

Log decrement quantifies the damping

The relation revealed from the log decrement data will suggest if a system has dominant linear damping.

Case (a) on the left indicates linear damping, while case (b) is for a system with Coulomb (nonlinear) damping, for which a constant ζ is not defined.



Note that the slope of the log decrement plot is, $\beta = \left[\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right]$, from which ζ can be found.

Summary

- Study of mass-spring-damper system models provides insight into a wide range of practical engineering problems.
- You can understand the underlying design of many types of sensors such as accelerometers by understanding 2nd order system dynamics.
- In lab, we will use an accelerometer to measure acceleration. A related lecture reviews practical use of accelerometers.
- The lab work in this first week should provide the basis for building a model for the full two-story system.

References

[1] J.P. Den Hartog, Mechanical Vibrations, Dover Publications, New York, 1934

[2] W.T. Thomson, Theory of Vibrations with Applications, 4th ed., Prentice-Hall, 1993