Schelling and Voter Model on Random Intersection Graph

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1 Static Random Intersection Graph

Create an auxilary bipartite (multi)-graph with n "individual" vertices on the left hand side and m "group" vertices on the right hand side. Assign independent and identically distributed weights $w_1, w_2, \cdots, w_n$ to the individual vertices. Say that those weights are distributed as the random variable W. Assign independent and identically distributed weights $w'_1, w'_2, \cdots, w'_m$ to the group vertices. Say that those weights are distributed as the random variable W'. Assume that $\mathbb{E}[W] = m\mathbb{E}[W']/n$.

The edges in the auxilary graph are conditioned on the weights of the vertices independently present or absent. An individual vertex of weight w and group vertex with weight w' share a Poisson number of edges with expectation ww'/ℓ'_m, where $\ell'_m := \sum_{i=1}^{m} w'_i$. The bipartite graph can also be obtained by assigning independent degrees to the individual vertices where the degree of a vertex of weight w is Poisson distributed with expectation w. The "group endpoints" of the edges are than chosen independently with replacement and the probability that a group vertex of weight w' is chosen is w'/ℓ'_m.

The random intersection (multi-)graph is then obtained from the auxilary graph as follows. The vertex set are the n individual vertices and two vertices share an edge if and only if there is a group vertex in the auxilary graph that shares an edge with both of the individual vertices. For future modelling we might label edges by the number of the group vertex that was used to create them. If two individual vertices have more than one group vertex as common "neighbour" in the auxilary graph, there are also several edges between them in the random intersection graph. The resulting graph consists of overlapping cliques (fully connected subgraphs) where the edges in a clique all have the same label. A vertex in the random intersection graph of weight w in the auxilary graph is asymptotically part of a Poisson number (expectation w) of cliques, and the clique with edges labeled by a "group vertex" of weight w' contains asymptotically a Poisson number with expectation w' vertices. (See [1] for more on this model).

To start modelling we probably choose $\mathbb{P}(W = \lambda) = \mathbb{P}(W' = \frac{n}{m}\lambda) = 1$ This is called homogeneous random intersection graph with parameters n, m and λ.

2 Schelling model dynamics for Random Intersection Graphs

Start with initial homogeneous random intersection graph with parameters n, m and λ. Assign independently opinions to the vertices. Opinions can be $+1$ or -1. Denote the opinion of vertex i by a_i. We describe the dynamics as dynamics on the auxilary graph. New edges between individual vertex i and group vertex j are created at rate c/m, where without loss of generality the rate can be chosen such that $c = 1$, by rescaling time. An existing edge between individual vertex i and group vertex j is deleted with rate $\beta(a_i, k_j^+(t), k_j^-(t))$, where $k_j^+(t)$ is the number of edges of group vertex j at time t that are shared with an individual vertex with opinion $+1$ and $k_j^-(t)$ is the number of edges of group vertex j at time t that are shared with an individual vertex with opinion -1. Note that if $a_i = +1$, then $k_j^+ \geq 1$ as long as i shares an edge with j.

As a first example set

$$\beta(a_i, k_j^+(t), k_j^-(t)) = g\left(-a_i \frac{k_j^+ - k_j^-}{k_j^+ + k_j^-}\right) = g\left(-a_i \left(2\frac{k_j^+}{k_j^+ + k_j^-} - 1\right)\right),$$

where g is an increasing function, which is basically a function of the fraction of people in the group that agrees with vertex j.

Remark 2.1 *If I am not mistaken, you can analyse the dynamics of the number of +1 and -1 vertices in a given group as an isolated Markov process.*

It might be worth while to extend everything to nonhomogenous random intersection graphs and let β also depend on w_i and w'_j.

Other possible extension is to let rate of joining groups depend on current number of group memberships.

3 Voter model dynamics

We consider two types of voter model in which people can change their opinion. In the first individuals along each pair of neighbours in the random intersection graph independent Poisson clocks tick (at rate γ) and if the clock rings the two neighbours agree on the opinion of the first individual in the pair (or one of the 2 chosen with probability 0.5).

The second voter model is that each edge in the auxilary graph has a Poisson clock with rate γ. If a Poisson clock between individual vertex i and group vertex j rings i changes opinition with probability $\eta((a_i, k_j^+(t), k_j^-(t))$, where η is any function taking values in $[0, 1]$.

4 SIR epidemics and opinion formation

Use SIR model with infectious contact rate between all individuals (or household model) depnding on opinions of contacting individuals (and sepparate within household rate if using household model).

possible opinion dynamics based on own infection status, infections in household or total number of infectious in population.

References

[1] F. Ball, D. Sirl, and P. Trapman, *Epidemics on random intersection graphs*, The Annals of Applied Probability, 24 (2014), p. 1081–1128.