1 The Lagrange equations with complex mass

In this chapter we start with the classical Lagrangian. By using the Hamilton-Jacobi equations, we will derive the non-linear and linear SchrÃűdinger equations. But there will be one difference with the default derivation. In stead of a real Lagrangian, we assume a *complex* Lagrangian.

1.1 Free particle with complex mass

The starting point of these calculations is the assumption that mass is not a *real* quantity. In addition to its real part, we add an *imaginary* part as in

$$m := m_{\Re} + i m_{\Im}.$$

We will discuss the physical interpretation of this imaginary part in ??. *Todo*

The Lagrangian is defined by the difference of the kinetic energy T and the potential V. We consider a free particle, so our Lagrangian is given by

$$L = \frac{1}{2} m \dot{x}^2. \tag{1.1}$$

Now, putting in the assumed complex mass we get a Lagrangian which also has a real and an imaginary part. L $= \frac{1}{2}(m_{\Re} + i m_{\Im})\dot{x}^2 = \frac{1}{2} m_{\Re} \dot{x}^2 + \frac{1}{2} m_I m \dot{x}^2 =: L_{\Re} + i L_{\Im} [for : complex_l agrangian] Asaconsequence, thesamehappenstotheaction. S = \int_{t_0}^{t_1} L \,\mathrm{d}t = \int_{t_0}^{t_1} L_{\Re} + i L_{\Im} \,\mathrm{d}t = \int_{t_0}^{t_1} L_{\Re} \,\mathrm{d}t + i \int_{t_0}^{t_1} L_{\Im} \,\mathrm{d}t =: S_{\Re} + i S_{\Im} [for : complex_a ction]$

1.1.1 *Lagrangian stress test*

Our task now, is to show that the introduction of our complex mass does not violate any classical laws of physics. The first equations to test are the Euler-Lagrange equations. Our system only depends on x and $\dot{x}$, so we have just one equation to test. $\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} Whichgivesus \frac{\mathrm{d}}{\mathrm{d}t}(m_{\Re} \dot{x} + i m_{\Im} \dot{x}) = 0 m_{\Re} \ddot{x} + i m_{\Im} \ddot{x} = 0 Becauseourmass, complexornot, isnotzero, weget$ẍ $= 0. Thisimpliesthatourparticleisnotaccelerated. Itisnot, becausewearedealingwithafreeparticle.$

The next equations to test are those of Hamilton. With the common definition of the momentum p as p $= \frac{\partial L}{\partial \dot{x}} =$ m$_{\Re} \dot{x} + i m_{\Im} \dot{x} [for : pinxdot] andusing \in [for : complex_l agrangian] theHamiltonianbecomes H = p\dot{x} - L [for : hamiltonian : def] = (m_{\Re} \dot{x} + i m_{\Im} \dot{x})\dot{x} - \frac{1}{2}(m_{\Re} + i m_{\Im})\dot{x}^2$

Now we can check the Hamilton equations. The first one states

$$\frac{\partial H}{\partial x} = -\dot{p}. \tag{1.3}$$

The LHS is zero, since ?? does not depend on x. The RHS can be expressed in terms of $\dot{x}$ by ??. The time derivative of $\dot{x}$ is $\ddot{x}$ and we already stated that our particle is not accelerating. So the RHS is also zero and the first equation is satisfied.

The second Hamilton equations states

$$\frac{\partial H}{\partial p} = \dot{x}.$$

In this case we have to use ??. Derivated to p this gives $\dot{x}$, which is exactly as wanted.

The last (additional) relation between the Hamiltonian and the Lagrangian is

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

Which is easily satisfied because both H and L are not direct functions of time.

1.1.2 *Hello SchrÃűdinger*

The next step towards the non-linear SchrÃűdinger equations is the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} = -H. \tag{1.4}$$

Here S is *Hamilton's principal function*. This is a function of q and $\bar{p}$, a constant of motion. We have the relations:

$$S = S(q, p)$$

Don't know how to write this down. $\bar{p}$ is formed after solving S as

$$S = S(q, \alpha)$$

with α a constant. This is a solution for S in ??. We define

$$\bar{p} := \alpha$$

and we get the relations

$$-\frac{\partial H}{\partial q} = \dot{\bar{p}} = 0$$

This equation states that .... S is called *Hamilton's principal function*.

Let's calculate this principal function. From ?? and ?? we know

$$\frac{\partial S}{\partial t} = -\tfrac{1}{2}(m_{\Re} + im_{\Im})\dot{x}^2$$

Integrating to t gives S = $\int$-$\frac{1}{2}(m_{\Re} + im_{\Im})\dot{x}^2 \mathrm{d}t =$

1.2 A more general approach

Let us now start with the Lagrangian, defined by the difference of the kinetic energy T and the potential V as

$$L\Big(\{q_j(t)\}, \{\dot{q}_j(t)\}, t\Big) := T(\{q_j(t)\}, \{\dot{q}_j(t)\}, t) - V(\{q_j(t)\}, t).$$

Our definition states that L is a function of N generalized coordinates $q_j(t)$, their time derivatives $\dot{q}_j(t)$ and the time t. The notation $\{q_j(t)\}$ implies a set of N coordinates labeled by $j \in \{1 \dots N\}$.

Now we can define the action as the integral of the Lagrangian over time.

$$S(t_0, t_1) := \int_{t_0}^{t_1} \Big(\{q_j(t)\}, \{\dot{q}_j(t)\}, t\Big)\ \mathrm{d}t$$

1.3 SchrÃűdinger for –real–

$$\mathrm{d}S = \frac{\partial S}{\partial x}\dot{x}\mathrm{d}t + \frac{\partial S}{\partial \bar{p}}\dot{\bar{p}} + \frac{\partial S}{\partial t}\mathrm{d}t$$