The smallest networks on which the Ford-Fulkerson maximum flow procedure may fail to terminate

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Abstract

It is widely known that the Ford-Fulkerson procedure for finding the maximum flow in a network need not terminate if some of the capacities of the network are irrational. Ford and Fulkerson gave as an example a network with 10 vertices and 48 edges on which their procedure may fail to halt. We construct much smaller and simpler networks on which the same may happen. Our smallest network has only 6 vertices and 8 edges. We show that it is the smallest example possible.

1 Introduction

The maximal flow problem is one of the most fundamental combinatorial optimization problems. The Ford-Fulkerson augmenting paths procedure is perhaps the most basic method devised for solving it and many more advanced algorithms are based on it.

Ford and Fulkerson themselves point out that their procedure need not terminate if the network it is applied on has some irrational capacities. In their book [FF62], they describe a network with 10 vertices and 48 edges on which this may happen. Their network is quite complicated and most textbooks (see, e.g., [CLR90],[Eve79],[Gib85],[Law76],[PS82],[Tar83]) that describe their procedure do not present it. A variant of their example appears in [Roc84], it has 14 vertices and 28 edges. We are not aware of any simpler example that had appeared in the literature.

In this note we describe three much smaller and simpler networks, on which the Ford-Fulkerson procedure may fail to terminate. The first two networks contain only 6 vertices and 9 edges each. The third network is yet smaller containing only 6 vertices and 8 edges. All three networks are acyclic. The first two are planar and contain only one edge with an irrational capacity. The third network is layered and it contains only two edges with irrational capacities. We show that the third network is the smallest example of its kind; the Ford-Fulkerson procedure does terminate on every network with at most 5 vertices or at most 7 edges. The networks constructed can be easily presented in an undergraduate course that covers network flow.

In the sequel we assume familiarity with the basic network flow concepts and with the Ford-Fulkerson procedure as described in any one of the textbooks cited earlier.

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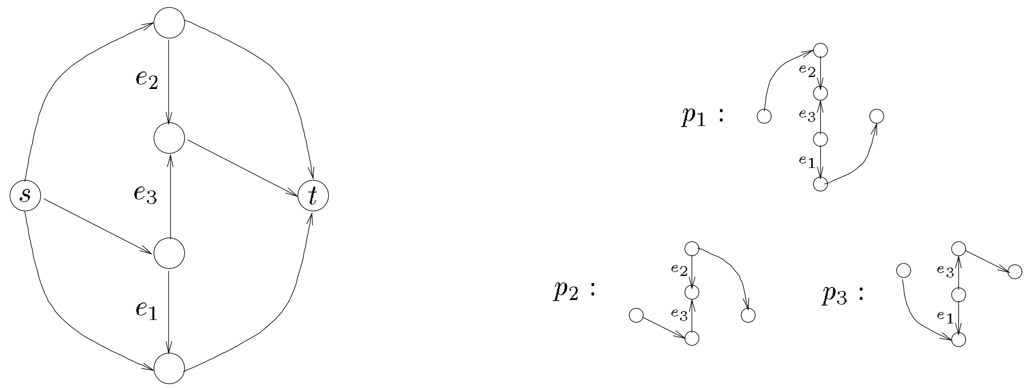

Figure 1: The network N_1

2 The simplest examples

The basis of the example given by Ford and Fulkerson [FF62], as well as of the simplified examples given in this section, is the sequence $\{a_n\}$ that satisfies the recurrence $a_{n+2} = a_n - a_{n+1}$, together with the initial conditions $a_0 = 1$ and $a_1 = r$. It is easy to check that $a_n = r^n$, where $r = \frac{\sqrt{5}-1}{2} \simeq 0.62$.

Ford and Fulkerson observed that on certain network topologies, sequences of augmenting paths can be used to simulate a computation of the sequence $\{a_n\}$. To demonstrate this point, suppose that e_1, e_2 and e_3 are three edges in a network and that their residual capacities are currently a_n, a_{n+1} and 0 respectively. If we can find an augmenting path in this network that contains e_1 and e_2 in their forward direction and e_3 in its backward direction, with e_2 being the critical edge, i.e., the edge on the path with the smallest residual capacity, then a flow augmentation along this path will increase the flow along e_1 and e_2 by a_{n+1} and will decrease the flow along e_3 by a_{n+1}. The resulting residual capacities of e_1, e_2 and e_3 would therefore be $a_n - a_{n+1} = a_{n+2}$, 0 and a_{n+1} respectively. (Note that as e_3 appears in the augmenting path used in its backward direction, it is the residual capacity of the reverse of e_3, and not that of e_3 itself, which is considered when looking for the critical edge along the path.) A similar form of arithmetic can be done on flows. We choose to perform the arithmetic on the residual capacities as this simplifies the setting of the initial conditions.

Our first network N_1 is given in Fig. 1. It has three special edges e_1, e_2 and e_3 whose capacities respectively are $a_0 = 1, a_1 = r$ and 1. The capacity of all the other edges in the network is M, where $M \geq 4$ is some large integer. The maximum flow in the network N is clearly $2M + 1$.

The important property of the network N_1 is that it contains the three paths shown on the right of Fig. 1. The first path contains e_1 and e_2 in their forward direction and e_3 in its backward direction, as in the example above. The second path contains e_2 in its backward direction and e_3 in its forward direction; it will be used to transfer flow from e_2 to e_3. The third path contains e_1 in its backward direction and e_3 in its forward direction and it will be used to transfer flow from e_1 to e_3.

Starting from the all zero flow in N_1, we use the augmenting path composed of the edge from s to the tail of e_3, of e_3 in its forward direction and of the edge from the head of e_3 to t. A flow of 1 is sent along this path and e_3 becomes saturated. The residual flows of e_1, e_2 and e_3 are now a_0, a_1

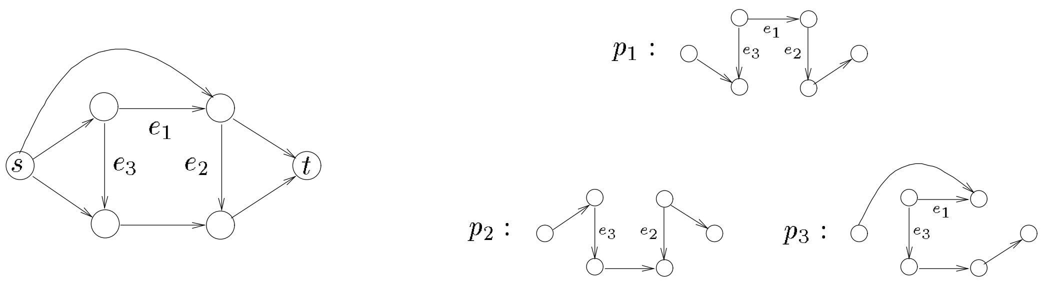

Figure 2: The network N_2

and 0, respectively.

Suppose that residual capacities of the three special edges e_1, e_2 and e_3 are currently a_n, a_{n+1} and 0, respectively, for some $n \geq 0$, and that the residual capacities of all the other edges is at least, say, 1. Note that this is satisfied, with $n = 0$, after the augmentation that saturated e_3. Clearly, the critical edge in any augmenting path in N_1 that includes at least one of the special edges in its forward direction is one of these included special edges.

We now apply, in sequence, the augmenting paths p_1, p_2, p_1, p_3. The residual capacities of e_1, e_2 and e_3 as a result of the these augmentations are as follows:

$$(a_n, a_{n+1}, 0) \stackrel{p_1}{\to} (a_{n+2}, 0, a_{n+1}) \stackrel{p_2}{\to} (a_{n+2}, a_{n+1}, 0) \stackrel{p_1}{\to} (0, a_{n+3}, a_{n+2}) \stackrel{p_3}{\to} (a_{n+2}, a_{n+3}, 0).$$

To verify this note that the critical edge along p_1 is e_2 and its residual capacity is a_{n+1}. The critical edge along p_2 is then e_3 and its residual capacity is again a_{n+1}. Next e_1 is the residual capacity along p_1 and its residual capacity is a_{n+2} and finally, e_3 is the residual capacity along p_3 and its residual capacity is again a_{n+2}. The flow in N_1 is therefore increased as a result of these four augmentations by $2a_n + 2a_{n+1}$. The residual capacities of e_1, e_2 and e_3 after these four augmentations are again of the form in which these augmentations can be applied.

This yields an infinite sequence of flow augmentations. The obtained sequence of flows does not converge to the maximum flow of N_1, whose value is $2M + 1$, but rather to a smaller flow whose value is only $1 + 2\sum_{n=2}^{\infty} a_n = 3$. As the total flow in the network at any stage is at most 3, the residual capacity of each non-special edge in N_1 is at least 1, as required. This completes the description of the first example.

The second example is obtained by using the network N_2 shown in Fig. 2. Again, there are three special edges e_1, e_2 and e_3 whose capacities are $1, r$ and 1 respectively. The residual capacities of all the other edges are again M, where $M \geq 4$ is a large integer. The maximum flow in N_2 is clearly $2M$.

The augmenting paths shown on the right of Fig. 2 are completely analogous to the augmenting paths of Fig. 1 in the sense that they include the same special edges and in the same directions. The order of the special edges along the paths may differ but this is of no consequence. The sequence of augmentations used for N_1 can be used without change for N_2. We do not repeat the details.

Both N_1 and N_2 have 6 vertices and 9 edges, they are planar, acyclic and only one edge in each one of them has an irrational capacity.

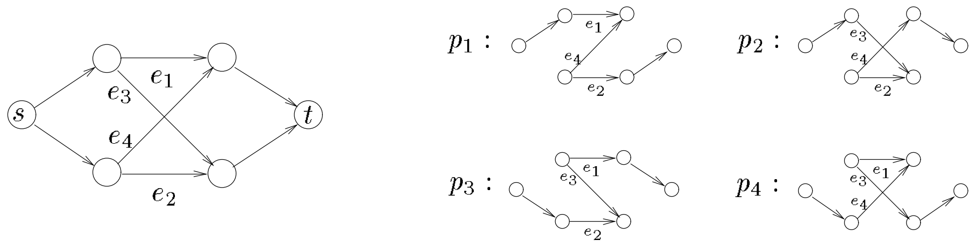

Figure 3: The network N_3

3 The smallest example

Consider the network N_3 shown in Fig. 3. There are four special edges this time e_1, e_2, e_3 and e_4 with capacities $1, r, r^2$ and 1, respectively, where $r = \frac{1+\sqrt{1-4\lambda}}{2} \simeq 0.682378$ and $\lambda \simeq 0.216757$ is the unique real root of the equation $1 - 5x + 2x^2 - x^3 = 0$. The residual capacities of all the other edges is again M, where $M \geq 3$ is a some integer. The maximum flow in N_3 is of size $2 + r + r^2 \simeq 3.147899$.

We begin by using an augmenting path that uses e_4 but none of the other special edges. This saturates e_4 and the residual capacities of the four special edges are now $(1, r, r^2, 0)$.

We henceforth use the four augmenting paths shown on the right of Fig. 3. Note that for each special edge there is a unique path that contains it in its backward direction.

Suppose that the residual capacities of e_1, e_2, e_3 and e_4 are currently $(x, y, z, 0)$ and that $x > y > z > x - y > y - z$. We apply in sequence the augmenting paths p_1, p_2, p_3 and p_4 given in Fig. 3. The resulting residual capacities are

$$\begin{array}{cccccccccc}
 & (& x & , & y & , & z & , & 0 &) \\
\xrightarrow{p_1} & (& x-y & , & 0 & , & z & , & y &) \\
\xrightarrow{p_2} & (& x-y & , & z & , & 0 & , & y-z &) \\
\xrightarrow{p_3} & (& 0 & , & z-(x-y) & , & x-y & , & y-z &) \\
\xrightarrow{p_4} & (& y-z & , & z-(x-y) & , & (x-y)-(y-z) & , & 0 &)
\end{array}$$

The new capacities (x', y', z') of e_1, e_2 and e_3 after these four augmentations satisfy

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

It is easy to check that $1 - 5x + 2x^2 - x^3$ is the characteristic polynomial of the matrix appearing in the equation above. Thus $\lambda \simeq 0.216757$ is an eigenvalue of this matrix. It is also easy to check that $(1, r, r^2)$ is an eigenvector that corresponds to λ.

Starting with e_1, e_2, e_3 and e_4 having residual capacities $(1, r, r^2, 0)$ we can therefore get an infinite sequence of augmenting paths. The residual capacities of e_1, e_2, e_3 and e_4 after using the subsequence p_1, p_2, p_3, p_4 repeatedly n times would be $\lambda^n \cdot (1, r, r^2, 0)$. The n-th application of this subsequence increases the flow in N_3 by $\lambda^{n-1}(2 + r)$. The obtained flows converge therefore to a flow whose value is $1 + \frac{2+r}{1-\lambda} = 2 + r + r^2$ which is therefore the maximum flow.

4 Termination on smaller networks

It can be checked that the Ford-Fulkerson procedure does terminate on every network with at most five vertices, no matter what the (finite) capacities of the edges are. This then immediately implies the same for networks with at most seven edges. It is assumed here, as standard, that the Ford-Fulkerson procedure uses only augmenting paths that are simple, i.e., paths that do not pass through a vertex more than once. The proof of this fact is not difficult but a bit technical. It is based on the fact that every augmenting path in such a network includes at most two edges that do not touch the source and the sink. To keep this note concise, we do not include the exact details.

The example presented in the previous section is therefore the smallest example possible.

References

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