The physics and maths of space-like quantum correlations

Exercises (II)

Please send your results to miguel.navascues@oeaw.ac.at with the caption 'Quantum correlations exercises'.

I. TWO-POINT QUANTUM CORRELATORS (50%)

In the $nn22$ Bell scenario, we consider the family of Bell functionals:

$$\mathcal{B}_n(P) := \sum_{i=1}^{n} \langle A_i B_i \rangle + \sum_{i=1}^{n-1} \langle A_{i+1} B_i \rangle - \langle A_1 B_n \rangle, \tag{1}$$

where $\langle A_x B_y \rangle := \sum_{a,b=\pm 1} P(a,b|x,y)ab$. Note that $\mathcal{B}_2$ equals the CHSH parameter.

Compute the maximum quantum value of $\mathcal{B}_n(P)$ for $n = 3, 4, 5, 6$. To solve the corresponding semidefinite program, I advise you to use the python package cvxpy, which you can find here: `https://www.cvxpy.org/`.

II. INFORMATION CAUSALITY (50%)

Consider a Popescu-Rohrlich box, defined by:

$$PR(a,b|x,y) = \frac{1}{2}\delta_{a \oplus b, xy}, \tag{2}$$

with $a, b, x, y = 0, 1$.

We showed in class that a single copy of this box allows carrying out a perfect random access code with $n = 2$ bits of input and $m = 1$ bits of communication. Namely, Alice would input $x_0 \oplus x_1$ in her system and transmit Bob the bit $a' = a \oplus x_0$. Bob's input would be $y = 0, 1$ and his guess on x_y would be $b \oplus a'$. Hence a single copy of a perfect PR box allows us to violate the principle of information causality.

Compute the violation of information causality of the corresponding RAC if the box $P(a,b|x,y)$ used by Alice and Bob is noisy and isotropic, i.e., of the form:

$$P(a,b|x,y) = \lambda PR(a,b|x,y) + (1-\lambda) PR^{-}(a,b|x,y), \tag{3}$$

with $0 \leq \lambda \leq 1$ and

$$PR^{-}(a,b|x,y) = \frac{1}{2}\delta_{a\oplus b\oplus 1, xy}. \tag{4}$$

For which value of λ does the RAC protocol cease to violate information causality? What does this imply, in terms of the maximal violation of the CHSH inequality?

III. BONUS QUESTION: MACROSCOPIC LOCALITY (20%)

Consider the $n_A n_B 22$ Bell scenario. As explained in the lectures, the full experimental behavior is a function of the marginals

$$\begin{aligned} \langle A_x \rangle &= \sum_{a,b=\pm 1} P(a,b|x,y)a, x = 1, ..., n_A \\ \langle B_y \rangle &= \sum_{a,b=\pm 1} P(a,b|x,y)b, y = 1, ..., n_B, \end{aligned} \tag{5}$$

and the two-point correlators

$$\langle A_x B_y \rangle = \sum_{a,b=\pm 1} P(a,b|x,y)ab, x = 1, ..., n_A, y = 1, ..., n_B. \tag{6}$$

1. Prove that any behavior P can be turned, via a wiring with a local distribution, into another behavior P' with the same two-point correlators as P, but vanishing marginals.

   Tip: first find a wiring that keeps the values of the two-point correlators but changes the sign of all the marginals.

2. Let $P \in Q_c$ have vanishing marginals. If we bring a microscopic Bell experiment involving P to the macroscopic scale, we will obtain a set of Gaussian distributions $\{\rho(I^1, I^{-1}, J^1, J^{-1}|x,y)\}_{x,y}$, where I^a (J^b) denotes the intensity fluctuation that Alice (Bob) measures on detector a (b).

   Since $P \in Q_c$, by macroscopic locality we have that $\rho(I^1, I^{-1}, J^1, J^{-1}|x,y) \in L$, and so is any wiring thereof. This includes the wiring which simply drops off the outcomes I^{-1}, J^{-1}. Thus $\rho(I^1, J^1|x,y) \in L$. From the lectures, we know that, for fixed x, y, $\rho(I^1, J^1|x,y)$ is a bi-variate Gaussian distribution with zero displacement vector. If, in addition, the microscopic distribution $P(a,b|x,y)$ has zero marginals, then its covariance matrix reads:

$$\gamma(x,y) = \frac{1}{4}\begin{pmatrix} 1 & \langle A_x B_y \rangle \\ \langle A_x B_y \rangle & 1 \end{pmatrix}. \tag{7}$$

Finally, applying the wiring $I^1 \to \text{sign}(I^1)$, $J^1 \to \text{sign}(J^1)$ to $\rho(I^1, J^1|x,y)$ leads to the classical $n_A n_B 22$ box

$$\tilde{P}(a,b|x,y) := \int_{\text{sign}(I^1)=a} dI^1 \int_{\text{sign}(J^1)=b} dJ^1 \rho(I^1, J^1|x,y). \tag{8}$$

Compute the marginals and two-point correlators of $\tilde{P}(a,b|x,y)$ as a function of the two-point correlators of $P(a,b|x,y)$ (remember: we assume that the marginals of $P(a,b|x,y)$ are zero).

For this, you will need the integral:

$$\int dzdte^{-(t^2+z^2+2atz)}\text{sign}(z)\text{sign}(t) = -2\frac{\arcsin(a)}{\sqrt{1-a^2}}. \tag{9}$$

3. With the results of the previous two points, prove the Tsirelson-Landau-Masanes conditions:

$$\begin{aligned}
&|\arcsin(\langle A_1B_1\rangle) + \arcsin(\langle A_2B_1\rangle) + \arcsin(\langle A_1B_2\rangle) - \arcsin(\langle A_2B_2\rangle)| \leq \pi, \\
&|\arcsin(\langle A_1B_1\rangle) + \arcsin(\langle A_2B_1\rangle) - \arcsin(\langle A_1B_2\rangle) + \arcsin(\langle A_2B_2\rangle)| \leq \pi, \\
&|\arcsin(\langle A_1B_1\rangle) - \arcsin(\langle A_2B_1\rangle) + \arcsin(\langle A_1B_2\rangle) + \arcsin(\langle A_2B_2\rangle)| \leq \pi, \\
&|-\arcsin(\langle A_1B_1\rangle) + \arcsin(\langle A_2B_1\rangle) + \arcsin(\langle A_1B_2\rangle) + \arcsin(\langle A_2B_2\rangle)| \leq \pi.
\end{aligned} \tag{10}$$